Supplemental Information: Modified Linear Failure Rate Distribution for Bathtub Hazard Data

1 Modified Linear Failure Rate Odds Ratio Generator of Distribution

$$F_{\text{RT-EOR-H}}(x;\alpha,\beta) = Q\left[\alpha R^{\beta}(x)\right] \tag{1}$$

$$R(x) = \frac{G(x,\Theta)}{\overline{G}(x,\Theta)} \tag{2}$$

$$\begin{aligned}
F_{M-LFRD-G}(x) \quad &= 1-\exp\left\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\right\} \\
&= 1-\exp\left\{-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\} \qquad (3)\\
f_{M-LFRD-G}(x) \quad &= \beta r(x)R^{\beta-1}(k+2\lambda R^{\beta}(x))\exp\left\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\right\} \\
&= \beta g(x,\boldsymbol{\Theta})\frac{G(x,\boldsymbol{\Theta})^{\beta-1}}{\overline{G}(x,\boldsymbol{\Theta})^{\beta+1}}\left[k+2\lambda\left[\frac{G(x,\boldsymbol{\Theta})}{\overline{G}(x,\boldsymbol{\Theta})}\right]^{\beta}\right] \\
&\times\exp\left\{-k\left[\frac{G(x,\boldsymbol{\Theta})}{\overline{G}(x,\boldsymbol{\Theta})}\right]^{\beta}-\lambda\left[\frac{G(x,\boldsymbol{\Theta})}{\overline{G}(x,\boldsymbol{\Theta})}\right]^{2\beta}\right\} \qquad (4)
\end{aligned}$$

2 Mathematical and Statistical Properties

2.1 Expansion of the Probability Density function

$$f_{M-LFRD-G}(x) = \sum_{i,j,m,n=0}^{\infty} c_{i,j,m,n}u_{n+\beta(i+j+m+1)-1}(x,\Theta) \tag{5}$$

Let $r(x) = \frac{g(x;\Theta)}{\left(\overline{G}(x;\Theta)\right)^2}$

$$\exp\left\{-kR^{\beta}(x) - \lambda R^{2\beta}(x)\right\} = \sum_{i}^{\infty} \frac{(-1)^i}{i!}(kR^{\beta}(x) + \lambda R^{2\beta}(x))^i$$

$$= \sum_{i}^{\infty} \frac{(-1)^i}{i!} \sum_{j}^{i} k^{i-j}\lambda^j R^{\beta(i+j)}(x) \tag{6}$$

$$(k + 2\lambda R^{\beta}(x)) = \sum_{m}^{1} \binom{1}{m} k^{1-m} 2^m \lambda^m R^{\beta m}(x) \tag{7}$$

$$f(x) = \beta r(x) \sum_{i,j,m} \binom{i}{j}\binom{1}{m} \frac{(-1)^i}{i!} k^{i-j+1-m} 2^m \lambda^{m+j} R^{\beta(1+m+i+j)-1}(x) \tag{8}$$

$$c_{i,j,m} = \binom{i}{j}\binom{1}{m} \frac{\beta(-1)^i k^{i-j+1-m} 2^m \lambda^{m+j}}{\beta(m+i+j)} \tag{9}$$

$$u_{\beta(m+i+j)-1} = \beta(m+i+j) r(x) R^{\beta(m+i+j)-1} \tag{10}$$

Since $r(x) = R'(x) = \frac{g(x;\Theta)}{\overline{G}(x;\Theta)}$ where coefficients $c_{i,j,m,n}$ are specified by

$$c_{i,j,m,n} = \binom{i}{j}\binom{1}{m}\binom{-\beta(i+j+m+1)-1}{n} \frac{\beta(-1)^{i+n} k^{i-j-m+1} 2^m \lambda^{m+j}}{i!(n+\beta(i+j+m+1))} \tag{11}$$

$$u_{n+\beta(i+j+m+1)-1} = (n+\beta(i+j+m+1)) g(x,\Theta) \left[G(x,\Theta)\right]^{n+\beta(i+j+m+1)} \tag{12}$$

2.2 Renyi Entropy

The Rényi Entropy is applied to P-LFRD-G as below,

$$
\begin{aligned}
I_R(\omega) &= (1-\omega)^{-1}\log\left[\int_{-\infty}^{\infty} f^{\omega}(x)dx\right] \\
&= (1-\omega)^{-1}\Bigg[\omega\log\beta + \log\int_{-\infty}^{\infty} r^{\omega}(x)R^{\omega(\beta-1)}(x)(k+2\lambda R^{\beta}(x))^{\omega} \\
&\times \exp\{-k\omega R^{\beta}(x) - 2\lambda\omega R^{2\beta}(x)\}dx\Bigg] \qquad (13)\\
&= (1-\omega)^{-1}\log\left[\int_{-\infty}^{\infty}\left(\beta g(x,\Theta)\frac{G(x,\Theta)^{\beta-1}}{\overline{G}(x,\Theta)^{\beta+1}}\left[k+2\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}\right]\right.\right. \\
&\left.\left.\times\exp\left\{-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\}\right)^{\omega} dx\right] \\
&= (1-\omega)^{-1}\left\{\omega\log\beta + \log\int_{-\infty}^{\infty} g^{\omega}(x,\Theta)\frac{\left[G(x,\Theta)\right]^{\omega(\beta-1)}}{\left[\overline{G}(x,\Theta)\right]^{\omega(\beta+1)}}\right. \\
&\left.\times\left(k+2\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}\right)^{\omega}\exp\left\{-k\omega\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta} - \lambda\omega\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\}dx\right\} \qquad (14)
\end{aligned}
$$

where $\omega > 0$ and $\omega \neq 1$. Apply the same expansion technique for the pdf, i.e. exponential series and binomial series,

$$
\exp\left\{-k\omega\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta} - \lambda\omega\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\} \qquad (15)
$$

We can generalize the Rényi Entropy as below

$$
\begin{aligned}
I_R(\omega) &= (1-\omega)^{-1}\left\{\omega\log\beta + \log\left[\sum_{a,b,l,u=0}^{\infty}\frac{(-1)^{a+u}\omega^a}{a!}2^l k^{\omega+a-b-l}\lambda^{b+l}\binom{a}{b}\right.\right.\\
&\times\binom{-\omega(\beta+1)-\beta(a+b+l)}{u}\left(\frac{\omega}{\beta(\omega+a+b+l)+u}\right)^{\omega}\\
&\times\left.\left.\int_{-\infty}^{\infty}\left(\frac{\beta(\omega+a+b+l)+u}{\omega}\right)^{\omega} g^{\omega}(x,\Theta)\left[G(x,\Theta)\right]^{\omega(\beta+1)\beta(a+b+l)}dx\right]\right\}\\
&= (1-\omega)^{-1}\left\{\omega\log\beta + \log\left[\sum_{a,b,l,u}^{\infty}\frac{(-1)^{a+u}\omega^a}{a!}2^l k^{\omega+a-b-l}\lambda^{b+l}\binom{a}{b}\right.\right.\\
&\times\binom{-\omega(\beta+1)-\beta(a+b+l)}{u}\left(\frac{\omega}{\beta(\omega+a+b+l)+u}\right)^{\omega}\\
&\times\left.\left.\int_{-\infty}^{\infty}\left[\left(\frac{\beta(\omega+a+b+l)+u}{\omega}\right)g(x,\Theta)\left[G(x,\Theta)\right]^{\frac{\omega(\beta+1)-\beta(a+b+l)}{\omega}}\right]^{\omega}dx\right]\right\}\\
&= (1-\omega)^{-1}\left\{\omega\log\beta + \log\left[\sum_{a,b,l,u}^{\infty}\frac{(-1)^{a+u}\omega^a}{a!}2^l k^{\omega+a-b-l}\lambda^{b+l}\binom{a}{b}\right.\right.\\
&\times\left.\left.\binom{-\omega(\beta+1)-\beta(a+b+l)}{u}\left(\frac{\omega}{\beta(\omega+a+b+l)+u}\right)^{\omega} exp^{(1-\omega)I_{REG}}\right]\right\}
\end{aligned}
\tag{16}
$$

where $\beta(\omega+a+b+l)+u = \omega(\beta-1)+\beta(a+b+l)+u+\omega$

where I_{REG}, as indicated below, is the Rényi Entropy of the exponentiated generalized distribution of the parameter $\beta^* = \frac{\beta(\omega+a+b+l)+u}{\omega}$

$$
I_{REG} = \frac{1}{1-\omega}\int_{-\infty}^{\infty}\left(\frac{\beta(\omega+a+b+l)+u}{\omega}g(x,\Theta)\left[G(x,\Theta)\right]^{\frac{\omega(\beta-1)+\beta(a+b+l)+u}{\omega}}\right)^{\omega}dx
\tag{17}
$$

2.3 Order Statistics

Let $X_1, X_2, ..., X_n$ be independently distributed random variables distributed by the polynomial PDF. The pdf of the i^{th} order statistics $f_{i:n}(x)$ if given by

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f_{P-LFRD-G}(x) \left[F_{P-LFRD-G}(x)\right]^{i-1} \left[\overline{F}_{P-LFRD-G}(x)\right]^{n-i} \\
&= \frac{n! f_{P-LFRD-G}(x)}{(i-1)!(n-i)!} \left[F_{P-LFRD-G}(x)\right]^{i-1} \left[1 - F_{P-LFRD-G}(x)\right]^{n-i} \\
&= \frac{n! f_{P-LFRD-G}(x)}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \left[F_{P-LFRD-G}\right]^{i-1+m} \\
&= \frac{n! f_{P-LFRD-G}(x)}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \\
&\times \left[1 - \exp\left\{-k \left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta}\right]^{\beta} - \lambda \left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\}\right]^{i+m-1} \qquad (18)
\end{aligned}
$$

replace $f_{P-LFRD-G}(x)$ from (5) and apply binomial expansion to

$\left[1-\exp\left\{-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\}\right]^{i+m-1}$, we have

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\beta g(x,\Theta)\frac{G(x,\Theta)^{\beta-1}}{\overline{G}(x,\Theta)^{\beta+1}}\left[k+2\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}\right] \\
&\times \exp\left\{-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m \\
&\times \sum_{s=0}^{\infty}\binom{i+m-1}{s}(-1)^s\exp\left\{s\left[-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right]\right\} \\
&= \frac{n!}{(i-1)!(n-i)!}\beta g(x,\Theta)\frac{G(x,\Theta)^{\beta-1}}{\overline{G}(x,\Theta)^{\beta+1}}\left[k+2\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}\right] \\
&\times \sum_{m=0}^{n-i}\sum_{s=0}^{\infty}\binom{n-i}{m}\binom{i+m-1}{s}(-1)^{m+s}\exp\left\{(s+1)\left[-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right]\right\} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\sum_{s=0}^{\infty}\binom{n-i}{m}\binom{i+m-1}{s}(-1)^{m+s} \\
&\times \beta g(x,\Theta)\frac{G(x,\Theta)^{\beta-1}}{\overline{G}(x,\Theta)^{\beta+1}}\left[\frac{s+1}{s+1}\right]\left[k+2\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}\right] \\
&\times \exp\left\{(s+1)\left[-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right]\right\} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\sum_{s=0}^{\infty}\binom{n-i}{m}\binom{i+m-1}{s}\frac{(-1)^{m+s}}{s+1} \\
&\times f_{P-LFRD-G}(x;(s+1)k,(s+1)\lambda,\beta) \qquad (19)
\end{aligned}
$$

We can present $f_{i:n}(x)$ as a linear combination of the P-LFRD-G with parameter $k^{*}=(s+1)k$ and $\lambda^{*}=(s+1)\lambda$.

2.4 Mean Residual Life

Calculate the time length from a chosen time to the time the subject fail to function (could be the number of years a machine is used to the time it stopped functioning).

Survival function:

$$
\begin{aligned}
S(x;\Theta) &= 1 - F_{M-LFRD-G}(x,\Theta) \\
&= \exp\{-kR^{\beta}(x) - 2\lambda R^{2\beta}(x) \qquad (20) \\
&= \exp\left\{-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\} \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
MRL(x,\Theta) &= E[T-t|T>t] \\
&= \frac{1}{S(t,\Theta)}\int_t^{\infty} S(x)dx \qquad (22) \\
&= \frac{1}{S(t,\Theta)}\int_0^{\infty} S(t+x)dx \\
&= \frac{1}{S(t,\Theta)}\int_t^{\infty} e^{-k\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{\beta} - \lambda\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{2\beta}} dx \\
&= \frac{1}{S(t,\Theta)}\int_t^{\infty} \sum_p^{\infty} \frac{\left(-k\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{\beta} - \lambda\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{2\beta}\right)^{p}}{p!} dx \\
&= \frac{1}{S(t,\Theta)}\sum_p^{\infty}\frac{(-1)^p}{p!}\int_t^{\infty}\sum_i^{p}\frac{p!}{i!(p-i)!}\left(k\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{\beta}\right)^{i}\left(\lambda\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{2\beta}\right)^{p-i} dx \\
&= \frac{1}{S(t,\Theta)}\sum_p^{\infty}\sum_i^{p}\frac{(-1)^p k^i\lambda^{p-i}}{i!(p-i)!}\int_t^{\infty}\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{2\beta p-\beta i} dx \qquad (23)
\end{aligned}
$$

Addtionally, we can calculate the average time to failure, i.e. Mean Time to Failure, from the Mean Residual Life

$$
\begin{aligned}
MTTF &= \int_0^{\infty} S(x)dx \\
&= \sum_p^{\infty}\sum_i^{p}\frac{(-1)^p k^i\lambda^{p-i}}{i!(p-i)!}\int_0^{\infty} R^{2\beta p-\beta i}(x) \qquad (24) \\
&= \sum_p^{\infty}\sum_i^{p}\frac{(-1)^p k^i\lambda^{p-i}}{i!(p-i)!}\int_0^{\infty}\left(\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right)^{2\beta p-\beta i} dx \qquad (25)
\end{aligned}
$$

2.5 Probability Weighted Function

For p, q, and $\tau \geq 0$

$$\begin{aligned} PWM(x;p,q,\tau) &= E\left[x^{\tau}(F_{P-LFRD-G}(x))^{p}(1-F_{P-LFRD-G}(x))^{q}\right] \\ &= \int_{-\infty}^{\infty} x^{\tau}(F_{P-LFRD-G}(x))^{p}(1-F_{P-LFRD-G}(x))^{q} f_{P-LFRD-G}(x)dx \end{aligned} \tag{26}$$

Let $\tau = 0$

$$PWM(x;p,q) = \int_{-\infty}^{\infty}(F_{P-LFRD-G}(x))^{p}(1-F_{P-LFRD-G}(x))^{q} f_{P-LFRD-G}(x)dx \tag{27}$$

Set $p = 1$ and $q = 0$ for lower tail sensitivity to measure the behavior near the beginning of the distribution

$$\begin{aligned} PWM(x) &= \int_{-\infty}^{\infty}(F_{P-LFRD-G}(x)) f_{P-LFRD-G}(x)dx \\ &= \int_{-\infty}^{\infty}(1-\exp\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\})\beta r(x)R^{\beta-1}(x) \\ &\times (k+2\lambda R^{\beta}(x))\exp\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\}dx \qquad (28) \\ &= \int_{-\infty}^{\infty}\left[1-\exp\left\{-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\}\right]\beta g(x,\Theta)\frac{G(x,\Theta)^{\beta-1}}{\overline{G}(x,\Theta)^{\beta+1}} \\ &\times\left[k+2\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}\right]\exp\left\{-k\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x,\Theta)}{\overline{G}(x,\Theta)}\right]^{2\beta}\right\}dx \end{aligned} \tag{29}$$

3 Methods of Parameter Estimation

3.1 MLE for Uncensored Data

Let $\mathit{\Delta} = (k, \lambda, \beta, \mathit{\Theta})^T$, then the likelihood function is as follow,

$$\begin{aligned}
\mathit{\Delta} &= \prod_{i=1}^{n} f_{P-LFRD-G}(x_i; k, \lambda, \beta, \mathit{\Theta}) \\
&= \prod_{i=1}^{n} \beta r_i(x) R^{\beta-1}(x_i)(k + 2\lambda R^{\beta}(x_i)) \exp\left\{-kR^{\beta}(x_i) - \lambda R^{2\beta}(x_i)\right\} \qquad (30) \\
&= \prod_{i=1}^{n} \beta g(x_i, \mathit{\Theta}) \frac{G(x_i, \mathit{\Theta})^{\beta-1}}{\overline{G}(x_i, \mathit{\Theta})^{\beta+1}} \left[k + 2\lambda \left[\frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})}\right]^{\beta}\right] \\
&\times \exp\left\{-k \left[\frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})}\right]^{\beta} - \lambda \left[\frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})}\right]^{2\beta}\right\} \qquad (31)
\end{aligned}$$

Denote $Z(x_i, \mathit{\Theta}) = \frac{G(x_i,\mathit{\Theta})}{\overline{G}(x_i,\mathit{\Theta})}$ ln-likelihood function of $\mathit{\Delta}$ is as follow,

$$\begin{aligned}
\ell(\mathit{\Delta}) &= n \ln \beta + \sum_{i=1}^{n} \ln g(x_i, \mathit{\Theta}) + (\beta - 1) \sum_{i=1}^{n} \ln G(x_i, \mathit{\Theta}) - (\beta + 1) \sum_{i=1}^{n} \ln(1 - G(x_i, \mathit{\Theta})) \\
&+ \sum_{i=1}^{n} \ln \left(k + 2\lambda \left[\frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})}\right]^{\beta}\right) - \sum_{i=1}^{n} k \left[\frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})}\right]^{\beta} - \sum_{i=1}^{n} \lambda \left[\frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})}\right]^{2\beta} \qquad (32)
\end{aligned}$$

The first derivative of the ln-likelihood function with respect to $\mathit{\Delta}$ is as follow,

$$\frac{\partial \ell}{\partial k} = \sum_{i=1}^{n} \frac{1}{\left(k + 2\lambda \left[\frac{G(x_i,\mathit{\Theta})}{\overline{G}(x_i,\mathit{\Theta})}\right]^{\beta}\right)} - \sum_{i=1}^{n} \left[\frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})}\right]^{\beta} \qquad (33)$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\left[\frac{G(x_i,\mathit{\Theta})}{1-G(x_i,\mathit{\Theta})}\right]^{\beta}}{\left(k + 2\lambda \left[\frac{G(x_i,\mathit{\Theta})}{\overline{G}(x_i,\mathit{\Theta})}\right]^{\beta}\right)} - \sum_{i=1}^{n} \left[\frac{G(x_i, \mathit{\Theta})}{1 - G(x_i, \mathit{\Theta})}\right]^{2\beta} \qquad (34)$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \ln \frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})} + \sum_{i=1}^{n} \frac{2\lambda \left[\frac{G(x_i,\mathit{\Theta})}{1-G(x_i,\mathit{\Theta})}\right]^{\beta}}{\left(k + 2\lambda \left[\frac{G(x_i,\mathit{\Theta})}{1-G(x_i,\mathit{\Theta})}\right]^{\beta}\right)} \ln \frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})} \\
&- k \sum_{i=1}^{n} \left[\frac{G(x_i, \mathit{\Theta})}{1 - G(x_i, \mathit{\Theta})}\right]^{\beta} \ln \frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})} - 2\lambda \sum_{i=1}^{n} \left[\frac{G(x_i, \mathit{\Theta})}{1 - G(x_i, \mathit{\Theta})}\right]^{2\beta} \ln \frac{G(x_i, \mathit{\Theta})}{\overline{G}(x_i, \mathit{\Theta})} \qquad (35)
\end{aligned}$$

with Θ_t is the t^{th} element of vector Θ

$$\begin{aligned}\frac{\partial \ell}{\partial \Theta_t} &= \sum_{i=1}^{n} \frac{1}{g(x_i,\Theta)} \frac{\partial g(x_i,\Theta)}{\partial \Theta_t} + (\beta-1)\sum_{i=1}^{n} \frac{1}{G(x_i,\Theta)} \frac{\partial G(x_i,\Theta)}{\partial \Theta_k} \\ &+ (\beta+1)\sum_{i=1}^{n} \frac{1}{1-G(x_i,\Theta)} \frac{\partial G(x_i,\Theta)}{\partial \Theta_t} \\ &+ \sum_{i=1}^{n} \frac{2\lambda\beta}{\left(k+2\lambda\left[\frac{G(x_i,\Theta)}{1-G(x_i,\Theta)}\right]^{\beta}\right)} \frac{G(x_i,\Theta)^{\beta-1}}{[1-G(x_i,\Theta)]^{\beta+1}} \frac{\partial G(x_i,\Theta)}{\partial \Theta_t} \\ &- k\beta\sum_{i=1}^{n} \frac{G(x_i,\Theta)^{\beta-1}}{[1-G(x_i,\Theta)]^{\beta+1}} \frac{\partial G(x_i,\Theta)}{\partial \Theta_t} - 2\lambda\beta\sum_{i=1}^{n} \frac{G(x_i,\Theta)^{2\beta-1}}{[1-G(x_i,\Theta)]^{2\beta+1}} \frac{\partial G(x_i,\Theta)}{\partial \Theta_t}\end{aligned} \tag{36}$$

We can maximize the ln-likelihood function $\ell(\Delta)$ by solving the nonlinear equations $\left(\frac{\partial \ell}{\partial k}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \Theta_t}\right) = 0$ with numerical methods such as Newton-Raphson approach.

3.1.1 Asymptotic Confidence Intervals

We base our calculation on the asymptotic normality distributions for $k, \lambda, \beta, \Theta$, as the sample size approaches infinity. Specifically,

$$\sqrt{n}(\Delta - \hat{\Delta}) \sim N_4(0, I^{-1})$$

where

$$I(\Delta) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial k^2} & \frac{\partial^2 \ell}{\partial k \partial \lambda} & \frac{\partial^2 \ell}{\partial k \partial \beta} & \frac{\partial^2 \ell}{\partial k \partial \Theta} \\ . & \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} & \frac{\partial^2 \ell}{\partial \lambda \partial \Theta} \\ . & . & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \Theta} \\ . & . & . & \frac{\partial^2 \ell}{\partial \Theta^2} \end{bmatrix}$$

Based on this matrix, we can have the approximate variance-covariance matrix evaluated at $\hat{\Delta} = (\hat{k}, \hat{\lambda}, \hat{\beta}, \hat{\Theta})'$ the MLE of $(k, \lambda, \beta, \Theta)'$ as

$$I^{-1}(\hat{\Delta}) = \begin{bmatrix} var(\hat{k}) & cov(\hat{k},\hat{\lambda}) & cov(\hat{k},\hat{\beta}) & cov(\hat{k},\hat{\Theta}) \\ . & var(\hat{\lambda}) & cov(\hat{\lambda},\hat{\beta}) & cov(\hat{\lambda},\hat{\Theta}) \\ . & . & var(\hat{\beta}) & cov(\hat{\beta},\hat{\Theta}) \\ . & . & . & var(\hat{\Theta}) \end{bmatrix} \tag{37}$$

Hence, the $100(1-\delta)\%$ asymptotic confidence intervals (ACIs) for each $\varDelta_t$ is given by

$$ACI_t = \left[\hat{\varDelta}_t - Z_{\frac{\delta}{2}}\sqrt{\hat{I}_{tt}}, \hat{\varDelta}_t + Z_{\frac{\delta}{2}}\sqrt{\hat{I}_{tt}}\right], \tag{38}$$

where $\hat{I}_{tt}$ is the (t,t) diagonal elements of $I_n(\hat{\varDelta})^{-1}$ for t = 1,2,3,4 and $Z_{\frac{\delta}{2}}$ is the upper δ^{th} percentile of the standard normal distribution.

3.2 MLE for Right-Censored Data

Let (y_i, δ_i), where i ranges from i = 1, 2, . . . , n, be a random sample that has been censored. For $\delta_i = 1$, y_i is a failure or survival time, and for $\delta_i = 0$, y_i is a censored time. We can express the log-likelihood function of the P-LFRD-G model in this case as:

$$\ell_C(\varDelta) = \sum_{i=1}^{n} \delta_i \log f(y_i) + \sum_{i=1}^{n} (1-\delta_i) \log S(y_i) \tag{39}$$

$$\begin{aligned} &= \sum_{i=1}^{n} \delta_i \log \beta + \sum_{i=1}^{n} \delta_i \log g(y_i, \varTheta) + (\beta - 1) \sum_{i=1}^{n} \delta_i G(y_i, \varTheta) \\ &+ (\beta + 1) \sum_{i=1}^{n} \delta_i \overline{G}(y_i, \varTheta) + \sum_{i=1}^{n} \delta_i \log \left[k + 2\lambda \left(\frac{G(y_i, \varTheta)}{\overline{G}(y_i, \varTheta)} \right)^{\beta} \right] \\ &- k \sum_{i=1}^{n} \left(\frac{G(y_i, \varTheta)}{\overline{G}(y_i, \varTheta)} \right)^{\beta} - \lambda \sum_{i=1}^{n} \left(\frac{G(y_i, \varTheta)}{\overline{G}(y_i, \varTheta)} \right)^{2\beta} \end{aligned} \tag{40}$$

where f(.) and S(.) are the PDF (**??**) and the survival function (). The estimate $\hat{\varDelta} = (\hat{k}, \hat{\lambda}, \hat{\beta}, \hat{\varTheta})'$ of $(k, \lambda, \beta, \varTheta)'$ can be obtained using the log-likelihood function 32 via a similar approach with the non-censored case in Section 4.1.

3.3 Least Square and Weighted Least Square Estimation

The most common form of least squares analysis is linear least squares, which focuses on finding the best-fitting straight line through points in two-dimensional space.

$\varDelta = (k, \lambda, \beta, \varTheta)^T$

$$
\begin{aligned}
LS(\boldsymbol{\Delta}) &= \sum_{i=1}^{n} \left(F(x_i, \boldsymbol{\Delta}) - \frac{i}{n+1} \right)^2 \\
&= \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right)^2
\end{aligned} \tag{41}
$$

Take derivative of $LS(\varDelta)$ with respect to $\varDelta$,

$$
\begin{aligned}
\frac{\partial LS}{\partial k} &= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\times \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\}
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\frac{\partial LS}{\partial \lambda} &= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\times \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\}
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\frac{\partial LS}{\partial \beta} &= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\times \left[k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} \ln\left(\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right) + 2\lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \ln\left(\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right) \right] \\
&\times \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\} \\
&= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\times \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} \ln\left(\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right) \left(k + 2\lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} \right) \\
&\times \exp\left\{ -k \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{\beta} - \lambda \left[\frac{G(x_i, \varTheta)}{\overline{G}(x_i, \varTheta)} \right]^{2\beta} \right\}
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
\frac{\partial LS}{\partial \Theta_t} &= 2\sum_{i=1}^{n}\left(1-\exp\left\{-k\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{2\beta}\right\}-\frac{i}{n+1}\right)\\
&\times\left[k\beta\frac{(G(x_i,\Theta))^{\beta-1}}{(\overline{G}(x_i,\Theta))^{\beta+1}}\frac{\partial G(x_i,\Theta)}{\partial \Theta_t}+2\lambda\beta\frac{(G(x_i,\Theta))^{2\beta-1}}{(\overline{G}(x_i,\Theta))^{2\beta+1}}\frac{\partial G(x_i,\Theta)}{\partial \Theta_t}\right]\\
&\times\exp\left\{-k\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{2\beta}\right\}\\
&= 2\beta\sum_{i=1}^{n}\left(1-\exp\left\{-k\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{2\beta}\right\}-\frac{i}{n+1}\right)\\
&\times\frac{\partial G(x_i,\Theta)}{\partial \Theta_t}\frac{(G(x_i,\Theta))^{\beta-1}}{(\overline{G}(x_i,\Theta))^{\beta+1}}\left[k+2\lambda\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{\beta}\right]\\
&\times\exp\left\{-k\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{2\beta}\right\} \qquad (45)
\end{aligned}
$$

$$
WLS(\varDelta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F_{P-LRFD-G}(x_i,\varDelta)-\frac{i}{n+1}\right)^2. \qquad (46)
$$

3.4 Maximum Product Spacing Approach of Estimation

$\varDelta=(k,\lambda,\beta,\boldsymbol{\psi})^T$

$$
MPS(\boldsymbol{\Delta})=\left\{\prod_{i=1}^{n+1}D_i(x_i,\boldsymbol{\Delta})\right\}^{\frac{1}{n+1}} \qquad (47)
$$

where

$$
D_i=\begin{cases}F(x_1,\boldsymbol{\Delta}), i=1\\ F(x_i,\boldsymbol{\Delta})-F(x_{i-1},\boldsymbol{\Delta}), i=2,3,...,n\\ 1-F(x_n,\boldsymbol{\Delta}), i=n+1\end{cases}
$$

As we can maximize

$$
F=1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\} \qquad (48)
$$

$$
\begin{aligned}
MPS(\boldsymbol{\Delta}) &= \left\{ F(x_1,\boldsymbol{\Delta})\left(1-F(x_n,\boldsymbol{\Delta})\right)\prod_{i=2}^{n}\left(F(x_i,\boldsymbol{\Delta})-F(x_{i-1},\boldsymbol{\Delta})\right)\right\}^{\frac{1}{n+1}} \\
&= \left\{ \left[1-\exp\left\{-k\left[\frac{G(x_1,\boldsymbol{\psi})}{\overline{G}(x_1,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_1,\boldsymbol{\psi})}{\overline{G}(x_1,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right] \right. \\
&\times\left(\exp\left\{-k\left[\frac{G(x_n,\boldsymbol{\psi})}{\overline{G}(x_n,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_n,\boldsymbol{\psi})}{\overline{G}(x_n,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right) \\
&\times\prod_{i=2}^{n}\left[\exp\left\{-k\left[\frac{G(x_{i-1},\boldsymbol{\psi})}{\overline{G}(x_{i-1},\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_{i-1},\boldsymbol{\psi})}{\overline{G}(x_{i-1},\boldsymbol{\psi})}\right]^{2\beta}\right\}\right. \\
&\left.\left.-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right]\right\}^{\frac{1}{n+1}} \qquad (49)
\end{aligned}
$$

Let $\left(-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right)=X_i$ and $\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}=A_i$ will make the following functions format better. Then,

$$
\frac{\partial X_i}{\partial k}=-\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}=-A_i^{\beta} \qquad (50)
$$

$$
\frac{\partial X_i}{\partial \lambda}=-\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}=-A_i^{2\beta} \qquad (51)
$$

$$
\begin{aligned}
\frac{\partial X_i}{\partial \beta} &= -k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}\ln\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}-2\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\ln\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})} \\
&= -\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}\left(k+2\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}\right)\ln\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})} \\
&= -A_i^{\beta}\left(k+2\lambda A_i^{\beta}\right)\ln A_i \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial X_i}{\partial \boldsymbol{\psi}_t} &= -k\beta\frac{[G(x_i,\boldsymbol{\psi})]^{\beta-1}}{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{\beta+1}}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t}-2\lambda\beta\frac{(G(x_i,\boldsymbol{\psi}))^{2\beta-1}}{(\overline{G}(x_i,\boldsymbol{\psi}))^{2\beta+1}}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t} \\
&= -\beta\frac{[G(x_i,\boldsymbol{\psi})]^{\beta-1}}{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{\beta+1}}\left(k+2\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}\right)\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t} \\
&= -\frac{\beta A_i^{\beta-1}}{\overline{G}(x,\Theta)^2}(k+2\lambda A_i^{\beta})\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t} \qquad (53)
\end{aligned}
$$

$$MPS(\mathbf{\Delta}) = \left\{ \left[1 - e^{X_1}\right] \left(e^{X_n}\right) \prod_{i=2}^{n} \left[e^{X_{i-1}} - e^{X_i}\right] \right\}^{\frac{1}{n+1}} \tag{54}$$

We can also maximize $H = \ln(MPS)$, then

$$\begin{aligned} H(\mathbf{\Delta}) &= \frac{1}{n+1} \sum_{i=1}^{n+1} \ln D_i(x_i, \mathbf{\Delta}) \\ &= \frac{1}{n+1} \left[\ln(1 - e^{X_1}) + \ln e^{X_n} + \sum_{i=2}^{n} \ln(e^{X_{i-1}} - e^{X_i}) \right] \\ &= \frac{1}{n+1} \left[\ln(1 - e^{X_1}) + X_n + \sum_{i=2}^{n} \ln(e^{X_{i-1}} - e^{X_i}) \right] \end{aligned} \tag{55}$$

$$\frac{\partial H}{\partial X_i} = \frac{1}{n+1} \left[\frac{-e^{X_1} X_1'}{1 - e^{X_1}} + X_n' + \sum_{i=2}^{n} \frac{e^{X_{i-1}} X_{i-1}' - e^{X_i} X_i'}{e^{X_{i-1}} - e^{X_i}} \right] \tag{56}$$

$$\frac{\partial H}{\partial k} = \frac{1}{n+1} \left[\frac{e^{X_1} \left[\frac{G(x_1,\boldsymbol{\psi})}{\overline{G}(x_1,\boldsymbol{\psi})}\right]^{\beta}}{1 - e^{X_1}} - \left[\frac{G(x_n, \boldsymbol{\psi})}{\overline{G}(x_n, \boldsymbol{\psi})}\right]^{\beta} + \sum_{i=2}^{n} \frac{- \left[\frac{G(x_{i-1},\boldsymbol{\psi})}{\overline{G}(x_{i-1},\boldsymbol{\psi})}\right]^{\beta} + \left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}}{e^{X_{i-1}} - e^{X_i}} \right] \tag{57}$$

$$\frac{\partial H}{\partial \lambda} = \frac{1}{n+1} \left[\frac{e^{X_1} \left[\frac{G(x_1,\boldsymbol{\psi})}{\overline{G}(x_1,\boldsymbol{\psi})}\right]^{2\beta}}{1 - e^{X_1}} - \left[\frac{G(x_n, \boldsymbol{\psi})}{\overline{G}(x_n, \boldsymbol{\psi})}\right]^{2\beta} + \sum_{i=2}^{n} \frac{- \left[\frac{G(x_{i-1},\boldsymbol{\psi})}{\overline{G}(x_{i-1},\boldsymbol{\psi})}\right]^{2\beta} + \left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}}{e^{X_{i-1}} - e^{X_i}} \right] \tag{58}$$

$$\frac{\partial H}{\partial k} = \frac{1}{n+1} \left[\frac{e^{X_1} \left[\frac{G(x_1,\boldsymbol{\psi})}{\overline{G}(x_1,\boldsymbol{\psi})}\right]^{\beta}}{1 - e^{X_1}} - \left[\frac{G(x_n, \boldsymbol{\psi})}{\overline{G}(x_n, \boldsymbol{\psi})}\right]^{\beta} + \sum_{i=2}^{n} \frac{- \left[\frac{G(x_{i-1},\boldsymbol{\psi})}{\overline{G}(x_{i-1},\boldsymbol{\psi})}\right]^{\beta} e^{X_{i-1}} + \left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta} e^{X_i}}{e^{X_{i-1}} - e^{X_i}} \right] \tag{59}$$

$$\frac{\partial H}{\partial \lambda} = \frac{1}{n+1} \left[\frac{e^{X_1} \left[\frac{G(x_1,\boldsymbol{\psi})}{\overline{G}(x_1,\boldsymbol{\psi})}\right]^{2\beta}}{1 - e^{X_1}} - \left[\frac{G(x_n, \boldsymbol{\psi})}{\overline{G}(x_n, \boldsymbol{\psi})}\right]^{2\beta} + \sum_{i=2}^{n} \frac{- \left[\frac{G(x_{i-1},\boldsymbol{\psi})}{\overline{G}(x_{i-1},\boldsymbol{\psi})}\right]^{2\beta} e^{X_{i-1}} + \left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta} e^{X_i}}{e^{X_{i-1}} - e^{X_i}} \right] \tag{60}$$

$$\begin{aligned}
\frac{\partial H}{\partial \beta} &= \frac{\partial H}{\partial X_i}\frac{\partial X_i}{\partial \beta} \\
&= \frac{1}{n+1}\left[\frac{e^{X_1}A_1^{\beta}\left(k+2\lambda A_1^{\beta}\right)\ln A_1}{1-e^{X_1}} - A_n^{\beta}\left(k+2\lambda A_n^{\beta}\right)\ln A_n\right. \\
&\left. + \sum_{i=2}^{n}\frac{-e^{X_{i-1}}A_{i-1}^{\beta}\left(k+2\lambda A_{i-1}^{\beta}\right)\ln A_{i-1} + e^{X_i}A_i^{\beta}\left(k+2\lambda A_i^{\beta}\right)\ln A_i}{e^{X_{i-1}}-e^{X_i}}\right]
\end{aligned} \tag{61}$$

$$- \frac{\beta A_i^{\beta-1}}{\overline{G}(x,\Theta)^2}(k+2\lambda A_i^{\beta})\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t} \tag{62}$$

$$\begin{aligned}
\frac{\partial H}{\partial \boldsymbol{\psi}_t} &= \frac{\partial H}{\partial X}\frac{\partial X}{\partial \boldsymbol{\psi}_t} \\
&= \frac{1}{n+1}\left[\frac{e^{X_1}\frac{\beta A_1^{\beta-1}}{\overline{G}(x_1,\boldsymbol{\psi})^2}(k+2\lambda A_1^{\beta})\frac{\partial G(x_1,\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t}}{1-e^{X_1}} - \frac{\beta A_n^{\beta-1}}{\overline{G}(x_n,\boldsymbol{\psi})^2}(k+2\lambda A_n^{\beta})\frac{\partial G(x_n,\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t}\right. \\
&\left. + \sum_{i=2}^{n}\frac{-\frac{\beta A_{i-1}^{\beta-1}e^{X_{i-1}}}{\overline{G}(x_{i-1},\boldsymbol{\psi})^2}(k+2\lambda A_{i-1}^{\beta})\frac{\partial G(x_{i-1},\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t} + \frac{\beta A_{i-1}^{\beta-1}e^{X_i}}{\overline{G}(x_{i-1},\boldsymbol{\psi})^2}(k+2\lambda A_{i-1}^{\beta})\frac{\partial G(x_{i-1},\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_t}}{e^{X_{i-1}}-e^{X_i}}\right]
\end{aligned} \tag{63}$$

3.5 Cramér-von Mises Approach of Estimation

Will the theoretical distribution function $F(x,\Theta)$ fits an empirical distribution function?

$\boldsymbol{\Delta} = (k,\lambda,\beta,\boldsymbol{\psi})^T$

$$\begin{aligned}
CM(x;\boldsymbol{\Delta}) &= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}\left(F(x_i;\boldsymbol{\Delta}) - \frac{2i-1}{2n}\right)^2 \\
&= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}\left(1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta} - \lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\} - \frac{2i-1}{2n}\right)^2 \\
&= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2(n-i)+1}{2n} - \exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta} - \lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right)^2 \\
&= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}A_i^2
\end{aligned} \tag{64}$$

Take the partial derivative of $CM(x;\mathbf{\Delta})$

$$\frac{\partial CM}{\partial k}=\frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{\beta}\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right) \tag{65}$$

$$\frac{\partial CM}{\partial \lambda}=\frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right]^{2\beta}\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right) \tag{66}$$

$$\begin{aligned}\frac{\partial CM}{\partial \beta}=&\frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}\ln\left(\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right)+2\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\ln\left(\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right)\right]\right.\\&\left.\times\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right)\end{aligned} \tag{67}$$

$$\begin{aligned}\frac{\partial CM}{\partial \Theta_t}=&\frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[k\beta\frac{G_(x_i,\Theta)^{\beta-1}}{\overline{G}(x_i,\Theta)^{\beta+1}}\frac{\partial G(x_i,\Theta)}{\partial\Theta_t}+2\lambda\beta\frac{G_(x_i,\Theta)^{2\beta-1}}{\overline{G}(x_i,\Theta)^{2\beta+1}}\frac{\partial G(x_i,\Theta)}{\partial\Theta_t}\right]\right.\\&\left.\times\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right)\end{aligned} \tag{68}$$

3.6 Anderson and Darling Approach of Estimation

AD is used to check if a sample comes from a population with a specific distribution.
$\mathit{\Delta}=(k,\lambda,\beta,\boldsymbol{\psi})^T$

$$\begin{aligned}AD(\mathbf{\Delta})=&-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_i,\mathbf{\Delta})+\log\overline{F}(x_{n+1-i},\mathbf{\Delta})]\\=&-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln\left(1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right)\right.\\&\left.+\ln\left(\exp\left\{-k\left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{2\beta}\right\}\right)\right]\\=&-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln\left(1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}\right)\right.\\&\left.-k\left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{2\beta}\right]\end{aligned} \tag{69}$$

Now take the derivative of $AD(\boldsymbol{\Delta})$ by

$$\frac{\partial AD(\boldsymbol{\Delta})}{\partial k} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta} + \left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{\beta}\right] \tag{70}$$

$$\frac{\partial AD(\boldsymbol{\Delta})}{\partial \lambda} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta} + \left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{2\beta}\right] \tag{71}$$

$$\begin{aligned}\frac{\partial AD(\boldsymbol{\Delta})}{\partial \beta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)&\left[\frac{\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}\right.\\ &\times\left(k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}+2\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right)\ln\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\\ &\left.+k\left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{\beta}\ln\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}+2\lambda\left[\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]^{2\beta}\ln\frac{G(x_{n+1-i},\boldsymbol{\psi})}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}\right]\end{aligned} \tag{72}$$

$$\frac{\partial AD(\boldsymbol{\Delta})}{\partial \Theta_t} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{\beta}-\lambda\left[\frac{G(x_i,\boldsymbol{\psi})}{\overline{G}(x_i,\boldsymbol{\psi})}\right]^{2\beta}\right\}}\right.$$
$$\times\left(k\beta\frac{G_(x_i,\Theta)^{\beta-1}}{\overline{G}(x_i,\Theta)^{\beta+1}}\frac{\partial G(x_i,\Theta)}{\partial\Theta_t}+2\lambda\beta\frac{G(x_i,\Theta)^{2\beta-1}}{\overline{G}(x_i,\Theta)^{2\beta+1}}\frac{\partial G(x_i,\Theta)}{\partial\Theta_t}\right)$$
$$+k\beta\frac{G_(x_{n+1-i},\boldsymbol{\psi})^{\beta-1}}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})^{\beta+1}}\frac{\partial G(x_{n+1-i},\boldsymbol{\psi})}{\partial\Theta_t}$$
$$\left.+2\lambda\beta\frac{G_(x_{n+1-i},\boldsymbol{\psi})^{2\beta-1}}{\overline{G}(x_{n+1-i},\boldsymbol{\psi})^{2\beta+1}}\frac{\partial G(x_{n+1-i},\boldsymbol{\psi})}{\partial\Theta_t}\right] \tag{73}$$

3.7 Bayesian Estimation

The Bayesian model is constructed by the product of the likelihood function and prior distribution $\pi(\Delta)$ for $\Delta = (k, \lambda, \beta, \boldsymbol{\psi})'$, which derived the posterior distribution of Δ represented by $\phi(\Delta|\,\mathcal{D})$. $\pi(\Delta)$ denoted the distribution of Δ before observing the data $D : t_1, t_2, ..., t_n$. Based on the Bayes theorem, we can obtain the distribution of the posterior as $\Delta|\,\mathcal{D}$ defined:

$$\pi(\Delta|\,\mathcal{D}) = \frac{\mathcal{L}(\,\mathcal{D}|\Delta)\pi(\Delta)}{\int_{\Theta}\mathcal{L}(\,\mathcal{D}|\Delta)\pi(\Delta)d\Delta} \tag{74}$$
$$\propto \mathcal{L}(\,\mathcal{D}|\Delta)\pi(\Delta) \tag{75}$$

where $\int_{\Theta}\mathcal{L}(\,\mathcal{D}|\Delta)\pi(\Delta)d\Delta$ is the normalizing constant of the posterior distribution of Δ, Θ is the parameter space, and $\mathcal{L}(\,\mathcal{D}|\Delta)$ represents the likelihood function of P-LFRD-G distribution, and is given by

$$\mathcal{L}(\,\mathcal{D}|\Delta) = \beta^n\prod_{i=1}^{n}g(x_i,\Theta)\frac{G(x_i,\Theta)^{\beta-1}}{\overline{G}(x_i,\Theta)^{\beta+1}}\left[k+2\lambda\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right]$$
$$\times\exp\left[-k\sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}-\lambda\sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{2\beta}\right] \tag{76}$$

In this study, the parameters $k, \lambda, \beta, \boldsymbol{\psi}$ are supposed to have gamma priors following the works of Kundu Howlader [48], Soliman et al. [49], and Abba and Wang [29]. Let $\pi(\Delta_t)$ denote the gamma prior having (ν_t, γ_t) as the hyperparameters, with PDF given by

$$\pi(\Delta_t) = \frac{\nu_t^{\gamma_t}}{\Gamma(\gamma_t)}\Delta_t^{\gamma_t-1}e^{-\nu_t\Delta_t}, \tag{77}$$

where $\nu_t > 0, \gamma_t > 0, t = 1, 2, 3, 4$

Thus, $k \sim \pi(k|\nu_1, \gamma_1)$,$\lambda \sim \pi(\lambda|\nu_2, \gamma_2)$, $\beta \sim \pi(\beta|\nu_3, \gamma_3)$, $\boldsymbol{\psi} \sim \pi(\boldsymbol{\psi}|\nu_4, \gamma_4)$. The joint posterior distribution of $\Delta|\,\mathcal{D}$ is therefore obtained after substituting Eq.[76]-[77] as

$$\begin{aligned}
\pi(\Delta|\,\mathcal{D}) \propto\ & \frac{k^{\gamma_1-1}\lambda^{\gamma_2-1}\beta^{n+\gamma_3-1}}{\boldsymbol{\psi}^{-\gamma_4+1}}\prod_{i=1}^{n} g(x_i,\Theta)\frac{G(x_i,\Theta)^{\beta-1}}{\overline{G}(x_i,\Theta)^{\beta+1}} \\
& \times \left[k + 2\lambda\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right] \\
& \times \exp\left(-\left[\alpha_1 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right]k - \left[\alpha_2 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{2\beta}\right]\lambda - \alpha_3\beta - \alpha_4\Theta\right)
\end{aligned} \tag{78}$$

We can determine the marginal posterior densities from 78 for the $k, \lambda, \beta, \boldsymbol{\psi}$ as

$$\begin{aligned}
\pi(k|\,\mathcal{D}) &\propto k^{\gamma_1-1}\prod_{i=1}^{n}\left[k + 2\lambda\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right]\exp\left(-\left[\alpha_1 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right]k\right) \\
\pi(\lambda|\,\mathcal{D}) &\propto \lambda^{\gamma_2-1}\prod_{i=1}^{n}\left[k + 2\lambda\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right]\exp\left(-\left[\alpha_2 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{2\beta}\right]\lambda\right) \\
\pi(\beta|\,\mathcal{D}) &\propto \beta^{n+\gamma_3-1}\prod_{i=1}^{n}\frac{G(x_i,\Theta)^{\beta-1}}{\overline{G}(x_i,\Theta)^{\beta+1}}\left[k + 2\lambda\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right] \\
&\quad \times \exp\left(-\left[\alpha_1 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right]k - \left[\alpha_2 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{2\beta}\right]\lambda - \alpha_3\beta\right) \\
\pi(\boldsymbol{\psi}|\,\mathcal{D}) &\propto \boldsymbol{\psi}^{\gamma_4-1}\prod_{i=1}^{n} g(x_i,\Theta)\frac{G(x_i,\Theta)^{\beta-1}}{\overline{G}(x_i,\Theta)^{\beta+1}}\left[k + 2\lambda\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right] \\
&\quad \times \exp\left(-\left[\alpha_1 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{\beta}\right]k - \left[\alpha_2 + \sum_{i=1}^{n}\left(\frac{G(x_i,\Theta)}{\overline{G}(x_i,\Theta)}\right)^{2\beta}\right]\lambda - \alpha_4\boldsymbol{\psi}\right)
\end{aligned} \tag{79}$$

Applying the square error loss function, the Bayes estimators of $k, \lambda, \beta, \boldsymbol{\psi}$, survival function S(t), and HRF h(t), are derived as follows:

$$\begin{aligned}
\hat{k}_* &= \mathbb{E}(k|\,\mathcal{D}) = \int_{\Theta} k\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{\lambda}_* &= \mathbb{E}(\lambda|\,\mathcal{D}) = \int_{\Theta} \lambda\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{\beta}_* &= \mathbb{E}(\beta|\,\mathcal{D}) = \int_{\Theta} \beta\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{\boldsymbol{\psi}}_* &= \mathbb{E}(\boldsymbol{\psi}|\,\mathcal{D}) = \int_{\Theta} \boldsymbol{\psi}\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{S}_* &= \mathbb{E}(S(t;\Delta)|\,\mathcal{D}) = \int_{\Theta} S(t;\Delta)\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{h}_* &= \mathbb{E}(h(t;\Delta)|\,\mathcal{D}) = \int_{\Theta} h(t;\Delta)\pi(\Delta|\,\mathcal{D})d\Delta
\end{aligned} \tag{80}$$

Calculating the Bayes estimates via the posterior means may be infeasible with no closed-form solutions of the marginal posterior densities $\pi(k|\,\mathcal{D}), \pi(\lambda|\,\mathcal{D}), \pi(\beta|\,\mathcal{D})$, and $\pi(\boldsymbol{\psi}|\,\mathcal{D})$.

Hence, for a posterior sample $\Delta_s, s = 1, 2, ..., N$ generated from $\pi(\Delta|\,\mathcal{D})$, the approximate Bayes estimates of P-LFRD-G parameters $k, \lambda, \beta, \boldsymbol{\psi}$, as well as the reliability function S(t) and HRF h(t) are calculated as

$$\begin{aligned}
\hat{k}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} k_s \\
\hat{\lambda}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} \lambda_s \\
\hat{\beta}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} \beta_s \\
\hat{\boldsymbol{\psi}}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} \boldsymbol{\psi}_s \\
\hat{S}_*(t) &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} S(t;\Delta_s) \\
\hat{h}_*(t) &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} h(t;\Delta_s),
\end{aligned} \tag{81}$$

where $\wp$ represents the number of burn-in observations/ iterations prior to stationarity of the samples. It is highly advisable to run the simulation for m parallel chains (m = 3, 4 or 5) for better assessment of the sampler convergence. Therefore, we can proceed to compute the posterior means for m parallel chains as follows:

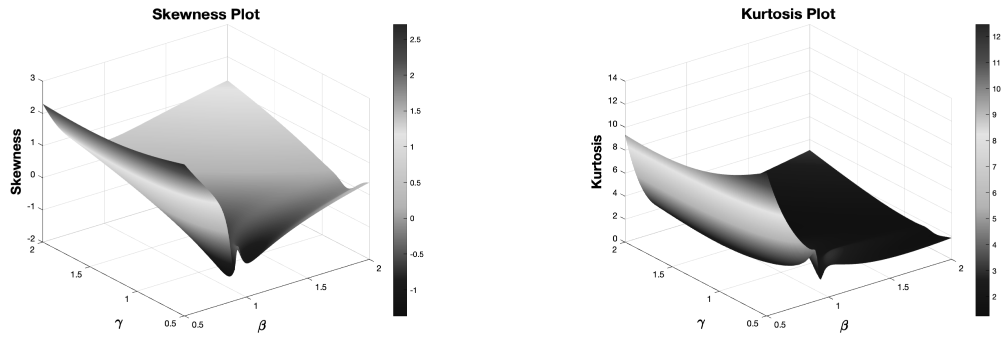
$$\begin{aligned}
\hat{k}_* &\approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} k_{(s,b)} \\
\hat{\lambda}_* &\approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} \lambda_{(s,b)} \\
\hat{\beta}_* &\approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} \beta_{(s,b)} \\
\hat{\boldsymbol{\psi}}_* &\approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} \boldsymbol{\psi}_{(s,b)} \\
\hat{S}_*(t) &\approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} S(t;\Delta_{(s,b)}) \\
\hat{h}_*(t) &\approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} h(t;\Delta_{(s,b)}),
\end{aligned} \tag{82}$$

4 Special Cases

4.1 M-LFRD-Exponential Distribution

4.2 M-LFRD-Uniformed Distribution

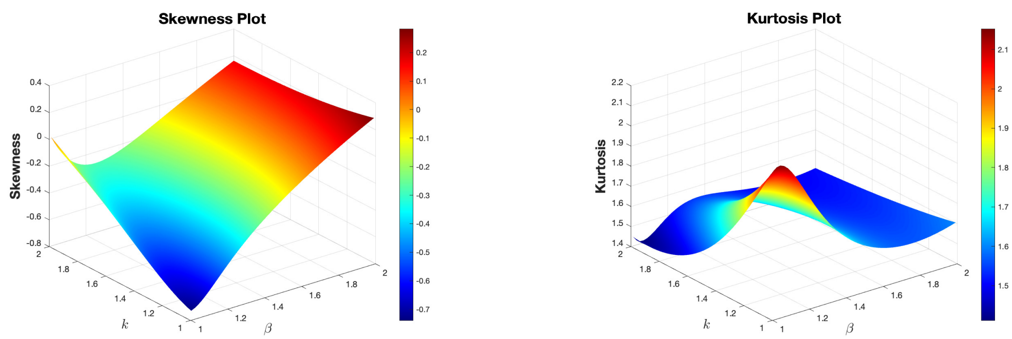
4.3 M-LFRD-Pareto Distribution



(a) Skewness: β vs. γ

(b) Kurtosis: β vs. γ

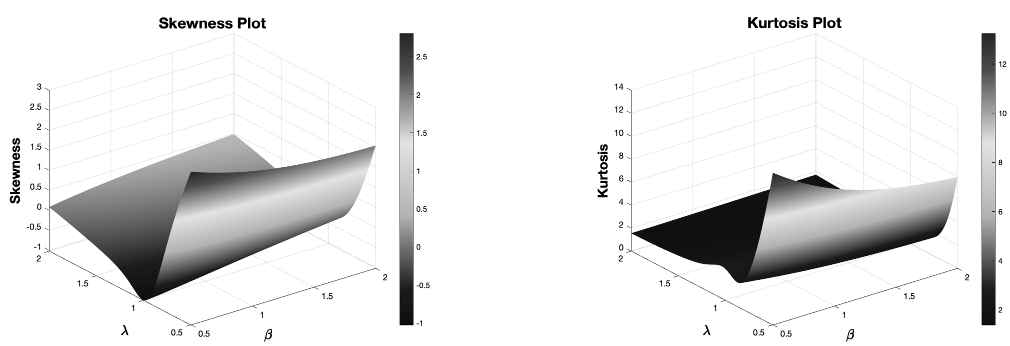
Figure 1: **LFRE: Comparison of Skewness and Kurtosis for β and γ parameters.**



(a) Skewness: β vs. k

(b) Kurtosis: β vs. k

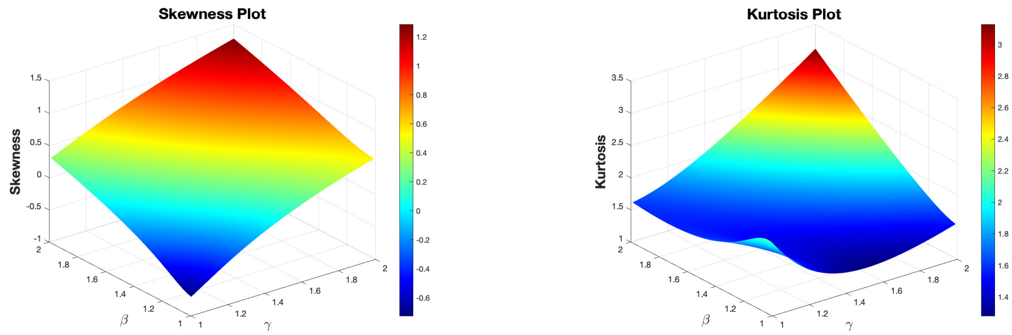
Figure 2: **LFRE: Comparison of Skewness and Kurtosis for β and k parameters.**



(a) Skewness: β vs. λ

(b) Kurtosis: β vs. λ

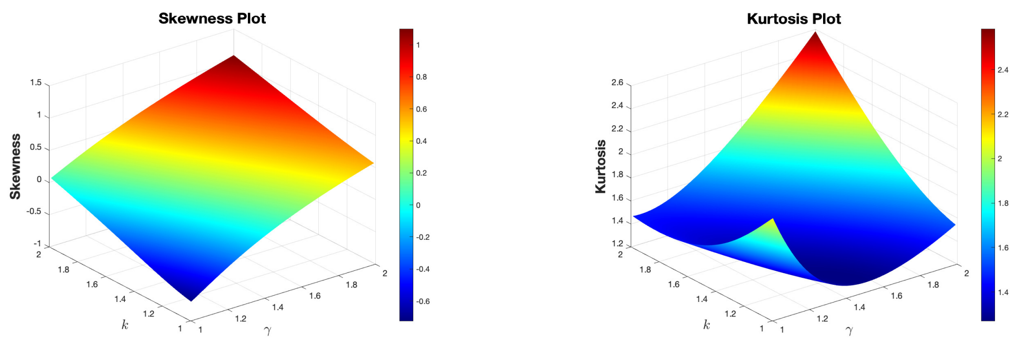
Figure 3: **LFRE: Comparison of Skewness and Kurtosis for β and λ parameters.**



(a) Skewness: γ vs. β

(b) Kurtosis: γ vs. β

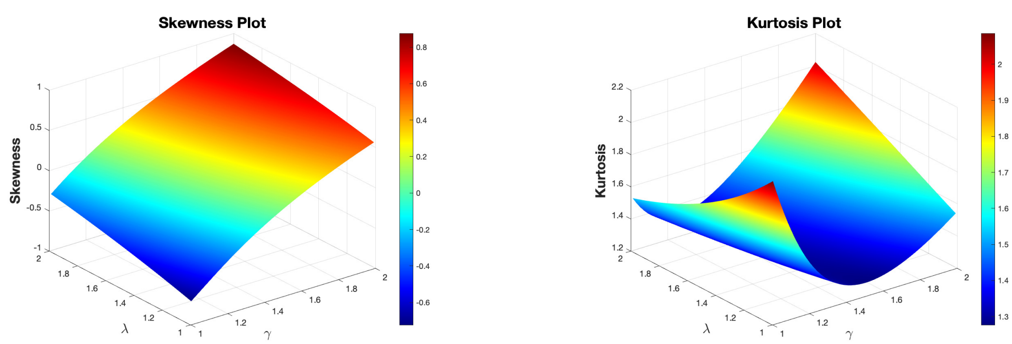
Figure 4: **LFRE: Comparison of Skewness and Kurtosis for γ and β parameters.**



(a) Skewness: γ vs. k

(b) Kurtosis: γ vs. k

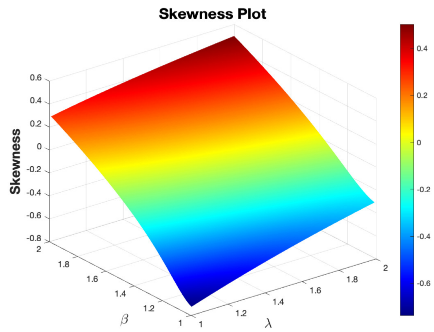
Figure 5: **LFRE: Comparison of Skewness and Kurtosis for γ and k parameters.**



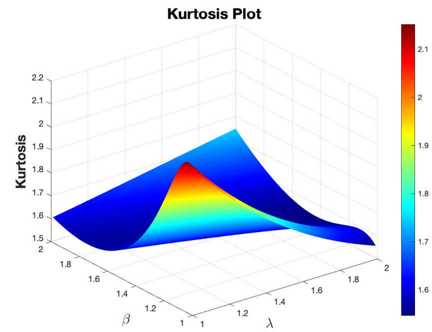
(a) Skewness: γ vs. λ

(b) Kurtosis: γ vs. λ

Figure 6: **LFRE: Comparison of Skewness and Kurtosis for γ and λ parameters.**

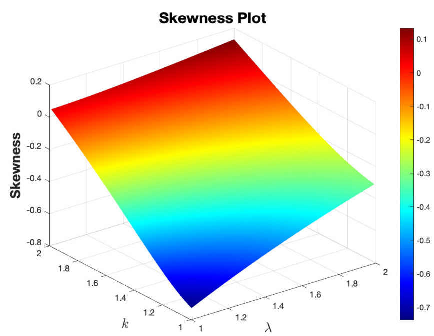


(a) Skewness: λ vs. β

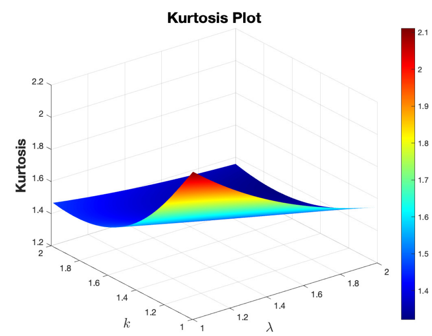


(b) Kurtosis: λ vs. β

Figure 7: **LFRE: Comparison of Skewness and Kurtosis for λ and β parameters.**

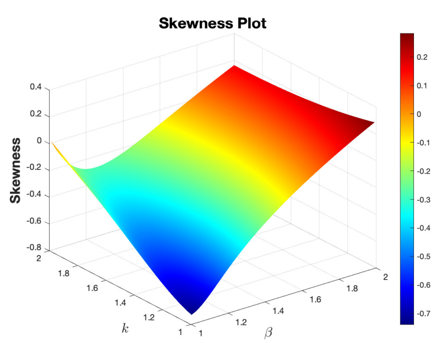


(a) Skewness: λ vs. k

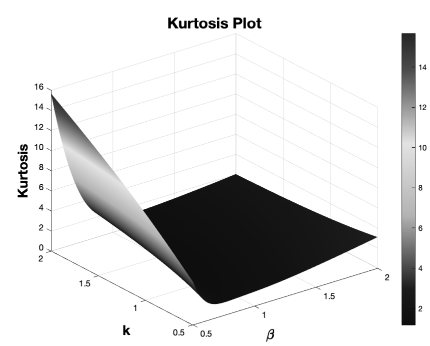


(b) Kurtosis: λ vs. k

Figure 8: **LFRE: Comparison of Skewness and Kurtosis for λ and k parameters.**

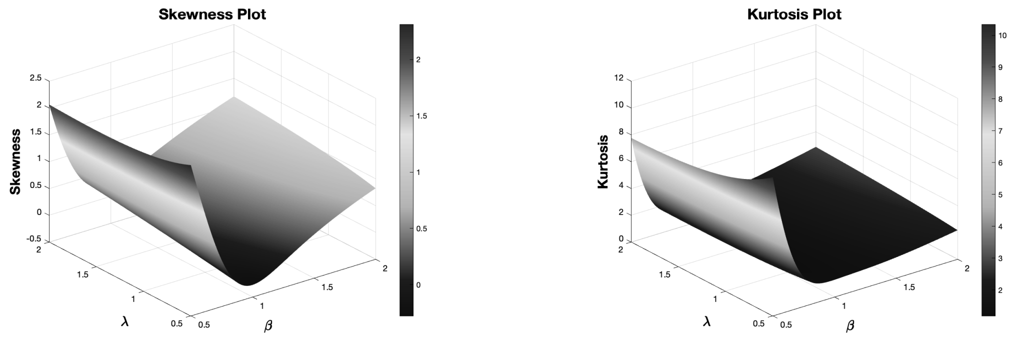


(a) Skewness: β vs. k



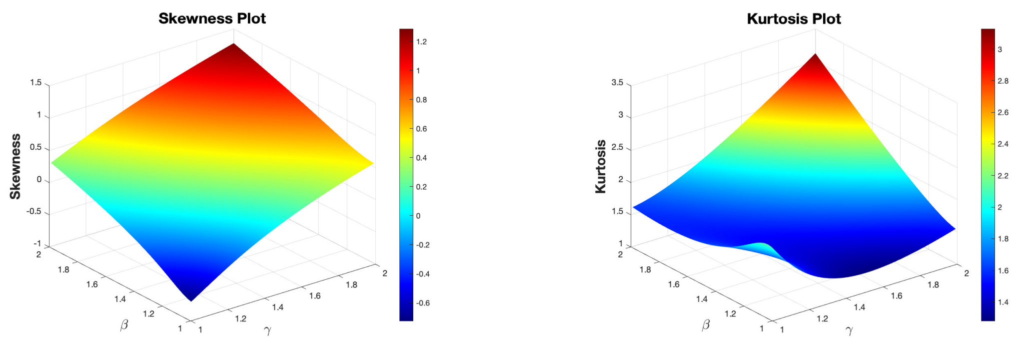
(b) Kurtosis: β vs. k

Figure 9: **LFRU: Comparison of Skewness and Kurtosis for β and k parameters.**



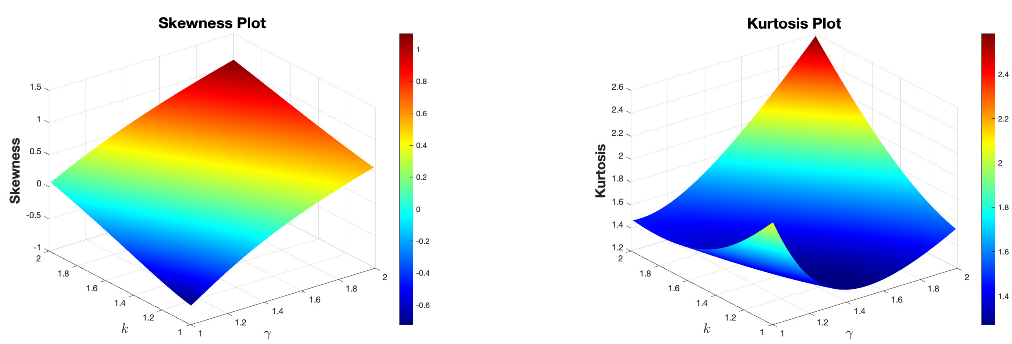
(a) Skewness: β vs. λ (b) Kurtosis: β vs. λ

Figure 10: **LFRU: Comparison of Skewness and Kurtosis for β and λ parameters.**



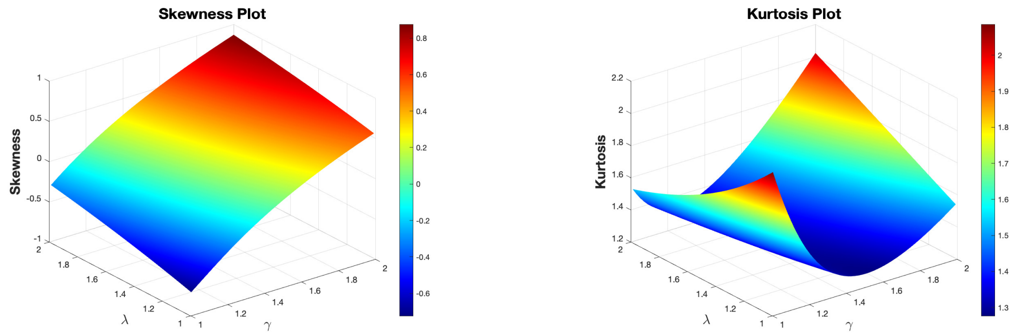
(a) Skewness: γ vs. β (b) Kurtosis: γ vs. β

Figure 11: **LFRU: Comparison of Skewness and Kurtosis for γ and β parameters.**



(a) Skewness: γ vs. k (b) Kurtosis: γ vs. k

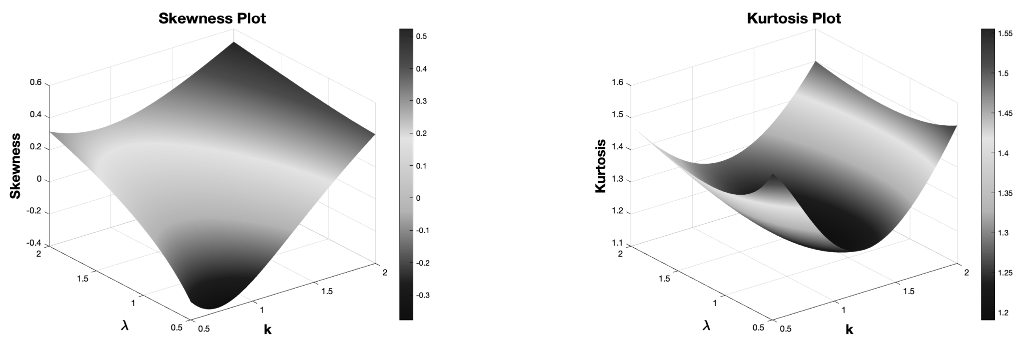
Figure 12: **LFRU: Comparison of Skewness and Kurtosis for γ and k parameters.**



(a) Skewness: γ vs. λ

(b) Kurtosis: γ vs. λ

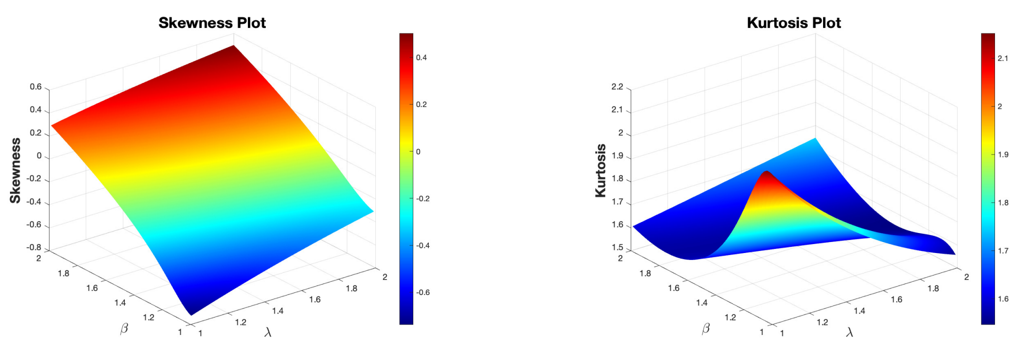
Figure 13: **LFRU: Comparison of Skewness and Kurtosis for γ and λ parameters.**



(a) Skewness: k vs. λ

(b) Kurtosis: k vs. λ

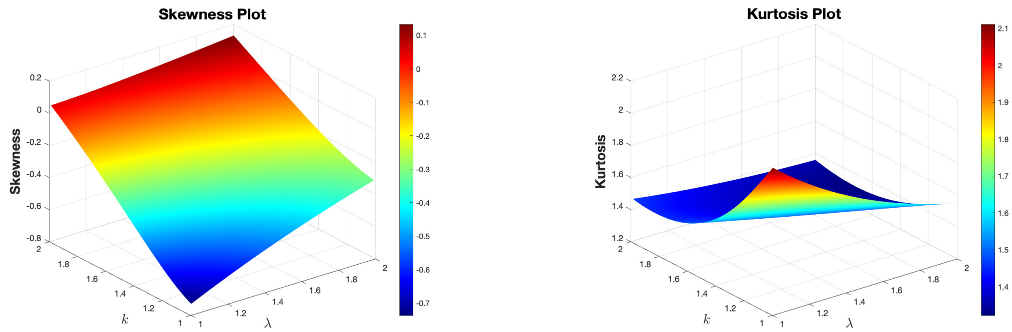
Figure 14: **LFRU: Comparison of Skewness and Kurtosis for k and λ parameters.**



(a) Skewness: λ vs. β

(b) Kurtosis: λ vs. β

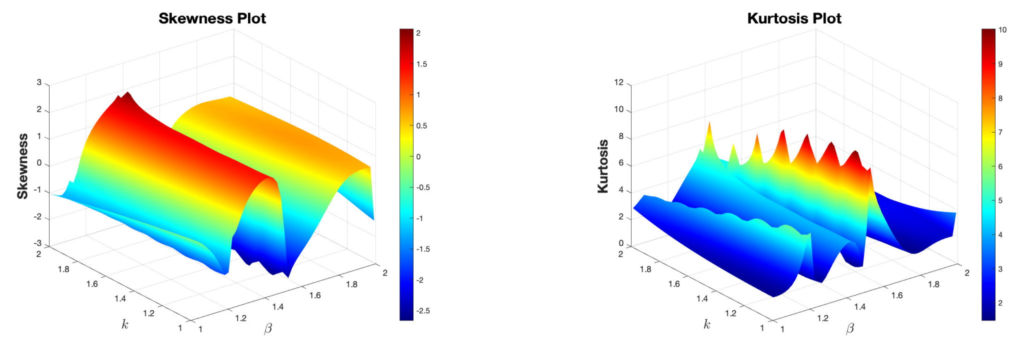
Figure 15: **LFRU: Comparison of Skewness and Kurtosis for λ and β parameters.**



(a) Skewness: λ vs. k

(b) Kurtosis: λ vs. k

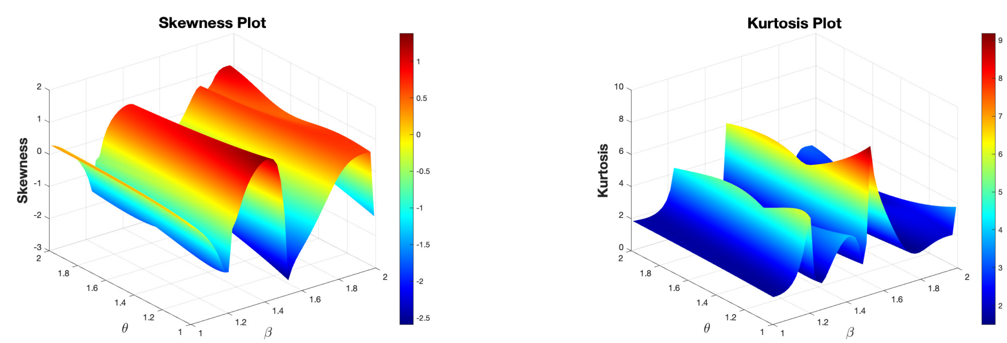
Figure 16: **LFRU: Comparison of Skewness and Kurtosis for λ and k parameters.**



(a) Skewness: β vs. k

(b) Kurtosis: β vs. k

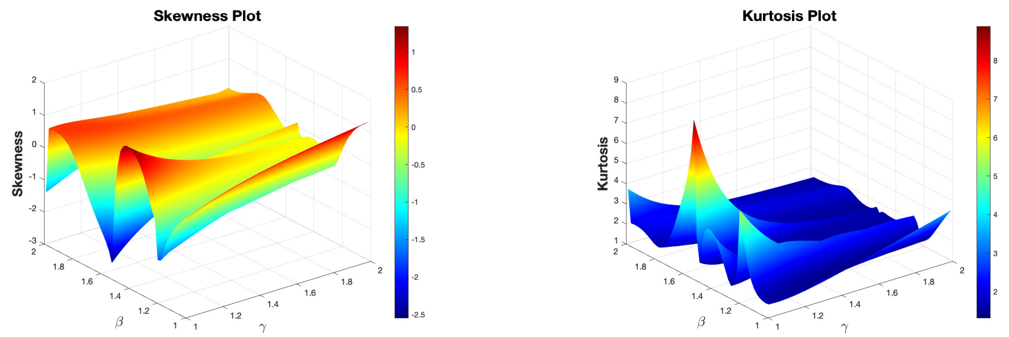
Figure 17: **LFRP: Comparison of Skewness and Kurtosis for β and k parameters.**



(a) Skewness: β vs. θ

(b) Kurtosis: β vs. θ

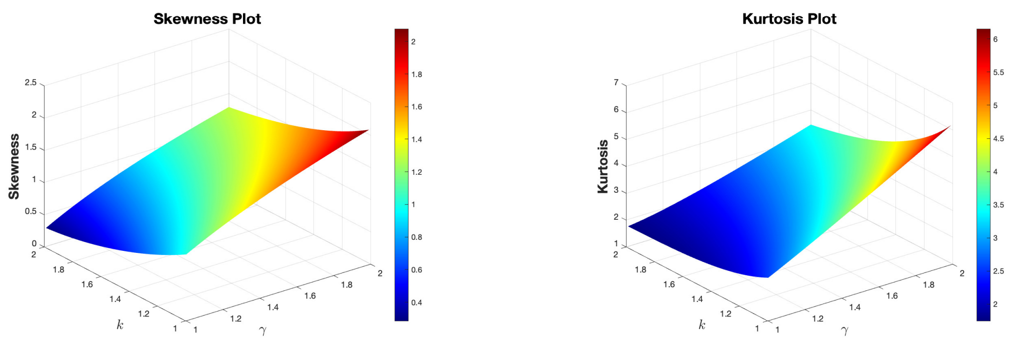
Figure 18: **LFRP: Comparison of Skewness and Kurtosis for β and θ parameters.**



(a) Skewness: γ vs. β

(b) Kurtosis: γ vs. β

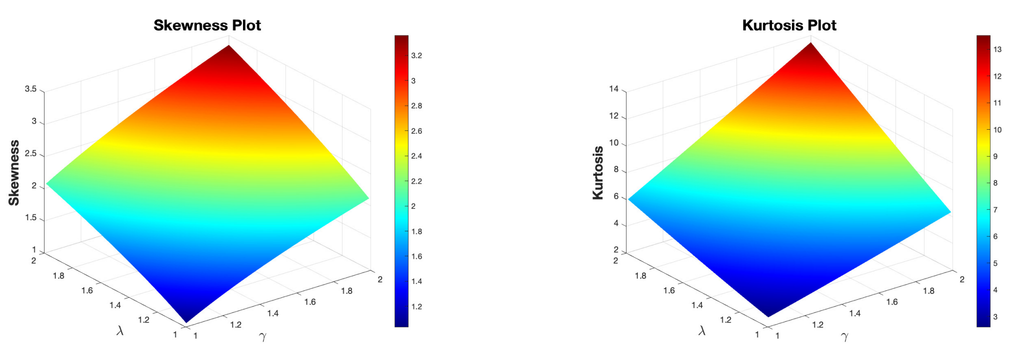
Figure 19: **LFRP: Comparison of Skewness and Kurtosis for γ and β parameters.**



(a) Skewness: γ vs. k

(b) Kurtosis: γ vs. k

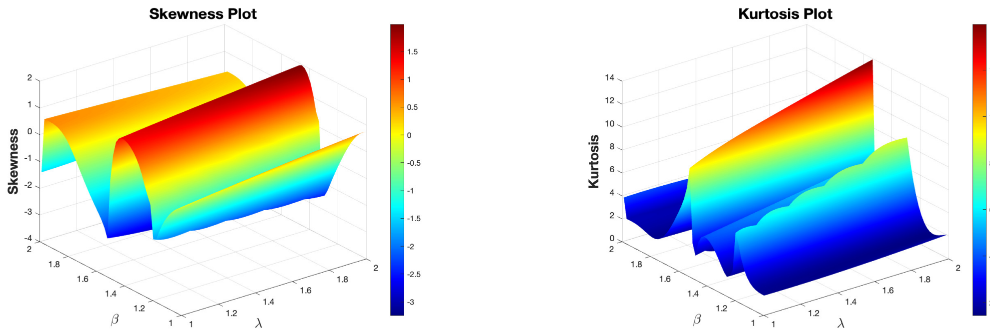
Figure 20: **LFRP: Comparison of Skewness and Kurtosis for γ and k parameters.**



(a) Skewness: γ vs. λ

(b) Kurtosis: γ vs. λ

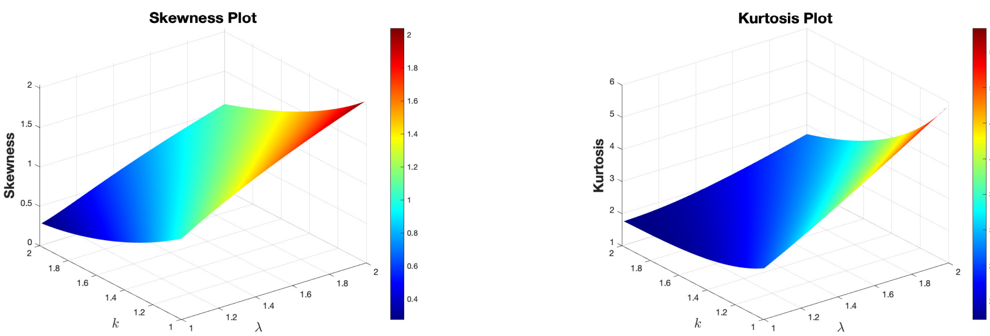
Figure 21: **LFRP: Comparison of Skewness and Kurtosis for γ and λ parameters.**



(a) Skewness: λ vs. β

(b) Kurtosis: λ vs. β

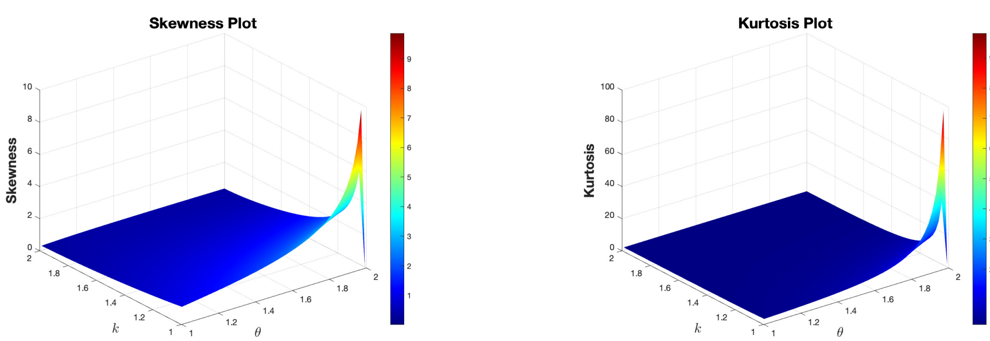
Figure 22: **LFRP: Comparison of Skewness and Kurtosis for λ and β parameters.**



(a) Skewness: λ vs. k

(b) Kurtosis: λ vs. k

Figure 23: **LFRP: Comparison of Skewness and Kurtosis for λ and k parameters.**



(a) Skewness: θ vs. k

(b) Kurtosis: θ vs. k

Figure 24: **LFRP: Comparison of Skewness and Kurtosis for θ and k parameters.**