Supplemental Information: Modified Linear Failure Rate Distribution for Bathtub Hazard Data

1 Introduction

The Modified Linear Failure Rate Odds Ratio Distribution Generator used the Odds Ratio:

$$F_{\text{RT–EOR-H}}(x;\alpha,\beta) = Q\left[\alpha R^{\beta}(x)\right] \quad \text{for } R(x) = \frac{G(x;\Theta)}{\overline{G}(x;\Theta)} \tag{1}$$

With the Linear Failure Rate function as $LFR(t) = 1 - \exp\left(-at - bt^2\right)$

2 The Linear Failure Rate Odds Ratio-G Distribution

The M-LFRD-G is expressed in term of $R(x;\Theta)$ and $\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}$ as follows:

$$\begin{aligned} F_{M-LFRD-G}(x) &= 1 - \exp\left\{-kR^{\beta}(x) - \lambda R^{2\beta}(x)\right\} \\ &= 1 - \exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\} \end{aligned} \tag{2}$$

$$f_{M-LFRD-G}(x) = \beta r(x) R^{\beta-1}(k + 2\lambda R^{\beta}(x)) \exp\left\{-kR^{\beta}(x) - \lambda R^{2\beta}(x)\right\} \tag{3}$$

$$\begin{aligned} &= \beta g(x;\Theta)\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}}\left[k + 2\lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta}\right] \\ &\times \exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\} \end{aligned} \tag{4}$$

2.1 Expansion of the PDF

Remark 1. The PDF of M-LFRD-G can be denoted in linear combination as follows:

$$f_{\text{M-LFRD-G}}(x) = \sum_{i,j,m=0}^{\infty} v_R\, u_R(x;\Theta) = \sum_{i,j,m,n=0}^{\infty} c_G\, u_G(x;\Theta) \tag{5}$$

where the coefficients and component functions are defined as:

$$v_R = \binom{i}{j}\binom{1}{m} \cdot \frac{\beta(-1)^{i} k^{1-m+i-j} 2^{m} \lambda^{m+j}}{\beta(i+j+m)}$$

$$c_G = \binom{i}{j}\binom{1}{m}\binom{-\beta(i+j+m+1)-1}{n} \cdot \frac{\beta(-1)^{i+n} k^{i-j-m+1} 2^{m} \lambda^{m+j}}{i!(n+\beta(i+j+m+1))} \tag{6}$$

$$u_R(x;\Theta) = \beta(i+j+m) \cdot r(x) \cdot R^{\beta(i+j+m)-1}(x)$$

$$u_G(x;\Theta) = (n+\beta(i+j+m+1)) \cdot g(x;\Theta) \cdot G^{n+\beta(i+j+m+1)-1}(x;\Theta) \tag{7}$$

Letting $r(x) = \frac{g(x;\Theta)}{(1-G(x;\Theta))^2}$, the exponential expansion is given by:

$$\begin{aligned}
\exp\{-kR^{\beta}(x) - \lambda R^{2\beta}(x)\} &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(kR^{\beta}(x) + \lambda R^{2\beta}(x)\right)^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{j=0}^{i} \binom{i}{j} k^{i-j} \lambda^j R^{\beta(i+j)}(x) \\
&= \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \binom{i}{j} k^{i-j} \lambda^j R^{\beta(i+j)}(x)
\end{aligned}$$

Additionally, we expand:

$$(k + 2\lambda R^{\beta}(x)) = \sum_{m=0}^{\infty} \binom{1}{m} k^{1-m} 2^m \lambda^m R^{\beta m}(x) \tag{8}$$

Substituting these into the PDF:

$$\begin{aligned}
f_{\text{M-LFRD-G}}(x) &= \beta r(x) R^{\beta-1}(x) \sum_{m=0}^{\infty} \binom{1}{m} k^{1-m} 2^m \lambda^m R^{\beta m}(x) \\
&\quad \times \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \binom{i}{j} k^{i-j} \lambda^j R^{\beta(i+j)}(x) \\
&= \sum_{i,j,m=0}^{\infty} \binom{i}{j} \binom{1}{m} \frac{\beta(-1)^i k^{1-m+i-j} 2^m \lambda^{m+j}}{i!} r(x) R^{\beta(i+j+m+1)-1}(x) \\
&= \sum_{i,j,m=0}^{\infty} v_{\scriptscriptstyle R}\, u_{\scriptscriptstyle R}(x;\Theta)
\end{aligned}$$

Since $r(x) = R'(x) = \frac{g(x;\Theta)}{(1-G(x;\Theta))^2}$, we substitute:

$$\begin{aligned}
f_{\text{M-LFRD-G}}(x) &= \beta g(x;\Theta)(1 - G(x;\Theta))^{-2} \sum_{i,j,m=0}^{\infty} \binom{i}{j} \binom{1}{m} \frac{(-1)^i k^{1-m+i-j} 2^m \lambda^{m+j}}{i!} \\
&\quad \times \left(\frac{G(x;\Theta)}{1 - G(x;\Theta)} \right)^{\beta(i+j+m+1)-1}
\end{aligned}$$

Now expand the negative power term:

$$(1 - G(x;\Theta))^{-1-\beta(i+j+m+1)} = \sum_{n=0}^{\infty} \binom{-1 - \beta(i+j+m+1)}{n} (-1)^n G^n(x;\Theta)$$

Substituting this in, we get:

$$\begin{aligned}
f_{\text{M-LFRD-G}}(x) &= \sum_{i,j,m=0}^{\infty} \binom{i}{j}\binom{1}{m} \frac{\beta(-1)^i k^{1-m+i-j} 2^m \lambda^{m+j}}{i!} g(x;\Theta) \\
&\times \sum_{n=0}^{\infty} \binom{-1-\beta(i+j+m+1)}{n} (-1)^n G^n(x;\Theta) \\
&= \sum_{i,j,m,n=0}^{\infty} \binom{i}{j}\binom{1}{m}\binom{-1-\beta(i+j+m+1)}{n} \frac{\beta(-1)^{i+n} k^{1-m+i-j} 2^m \lambda^{m+j}}{i!} g(x;\Theta) \\
&\times G^{\beta(i+j+m+1)+n-1}(x;\Theta)
\end{aligned}$$

Rewriting the expression using coefficient $c_{i,j,m,n}$ and function u_G, we obtain:

$$f_{\text{M-LFRD-G}}(x) = \sum_{i,j,m,n=0}^{\infty} c_{i,j,m,n} \cdot u_G(x;\Theta)$$

with:

$$\begin{aligned}
c_G &= \binom{i}{j}\binom{1}{m}\binom{-\beta(i+j+m+1)-1}{n} \cdot \frac{\beta(-1)^{i+n} k^{1-m+i-j} 2^m \lambda^{m+j}}{i!(\beta(i+j+m+1)+n)} \\
u_G(x;\Theta) &= (\beta(i+j+m+1)+n) \cdot g(x;\Theta) \cdot G^{\beta(i+j+m+1)+n-1}(x;\Theta)
\end{aligned}$$

2.2 Hazard Rate

Remark 2. The hazard rate function of the Modified Linear Failure Rate is defined as follows:

$$\begin{aligned}
hrf_{M-LFRD-G}(x) &= \frac{f_{M-LFRD-G}(x)}{1-F_{M-LFRD-G}(x)} \\
&= \beta r(x) R^{\beta-1}(x)\left(k+2\lambda R^{\beta}(x)\right) \\
&= \beta g(x;\Theta) \frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}} \left(k+2\lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta}\right) \qquad (9)
\end{aligned}$$

Meanwhile, the reverse hazard rate function is denoted as follows:

$$\begin{aligned}
\tau_{M-LFRD-G}(x) &= \frac{f_{M-LFRD-G}}{F_{M-LFRD-G}} \\
&= \frac{\beta r(x) R^{\beta-1}(x)(k+2\lambda R^{\beta}(x)) \times \exp\left\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\right\}}{1-\exp\left\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\right\}}
\end{aligned}$$

$$= \frac{\beta g(x;\Theta)\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}}\left(k+2\lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta}\right)\exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\}} \tag{10}$$

2.3 Quantile Function Derivation

Remark 3. The quantile function is derived by denoting $F_{M-LFRD-G} = p$ then solving $G(x;\Theta)$ by p:

$$\begin{aligned}
F_{M-LFRD-G}(x) = 1-\exp\left\{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}-\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right\} &= p\\
\exp\left\{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}-\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right\} &= 1-p\\
-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}-\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta} &= \log(1-p)\\
\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}+k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}+\log(1-p) &= 0\\
\lambda\left(\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}\right)^{2}+k\left(\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}\right)+\log(1-p) &= 0 \qquad (11)
\end{aligned}$$

By quadratic formula:

$$\begin{aligned}
\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta} &= \frac{-k+\sqrt{k^2-4\lambda\log(1-p)}}{2\lambda} = Z\\
\frac{G(x;\Theta)}{1-G(x;\Theta)} &= \frac{-k+\sqrt{k^2-4\lambda\log(1-p)}}{2\lambda} = Z^{1/\beta}\\
G(x;\Theta) &= \frac{Z^{1/\beta}}{1+Z^{1/\beta}} = \frac{1}{\frac{1}{Z^{1/\beta}}+1} = \frac{1}{Z^{-1/\beta}+1}\\
G(x;\Theta) &= \frac{1}{\left(\frac{-k+\sqrt{k^2-4\lambda\log(1-p)}}{2\lambda}\right)^{-1/\beta}+1}\\
G(x;\Theta) &= \frac{1}{\left(\frac{2\lambda}{-k+\sqrt{k^2-4\lambda\log(1-p)}}\right)^{1/\beta}+1}
\end{aligned}$$

2.4 Moments

Remark 4. The p^{th} ordinary/raw moment is given as:

$$\mu_p = E(Y^p) = \int_{-\infty}^{\infty} x^p f_{M-LFRD-G}(x;\Theta)dx = \sum_{i,j,m,n=0}^{\infty} c_G \int_{-\infty}^{\infty} x^p u_{G_{\beta^*-1}}(x;\Theta)dx \tag{12}$$

Here, $E(Y^p)$ denotes the expected value of Y^p, where Y is a random variable following the expanded PDF above.

Remark 5. The s^{th} incomplete moments, denoted I_s, is given below:

$$I_s(z) = \int_0^z x^s f_{M\text{-}LFRD\text{-}G}(x)\,dx = \sum_{i,j,m,n=0}^{\infty} c_G \int_0^z x^s u_{G_{\beta^*-1}}(x;\Theta)\,dx \tag{13}$$

Remark 6. The moment generating function (MGF) is derived as:

$$MGF_Y(t) = E\left(e^{tY}\right) = \sum_{i,j,m,n=0}^{\infty} c_G E\left(e^{tZ_{i,j,m,n}}\right) = \sum_{i,j,m,n=0}^{\infty} c_G M_{Z_{i,j,m,n}}(t) \tag{14}$$

Where $E(e^{tY})$ is the expected value of the exponentiated generalized distribution $Z_{i,j,m,n}$ with the parameter $\beta^* = n + \beta(i+j+m+1)$.

2.5 Renyi Entropy

Theorem 2. The Renyi Entropy is derived by the formula:

$$\begin{aligned}
I_R(\varphi) &= (1-\varphi)^{-1}\log\left[\int_{-\infty}^{\infty} f^{\varphi}(x)dx\right] \\
&= (1-\varphi)^{-1}\Bigg[\varphi\log\beta + \log\int_{-\infty}^{\infty} r^{\varphi}(x)R^{\varphi(\beta-1)}(x)(k+2\lambda R^{\beta}(x))^{\varphi} \\
&\times \exp\{-k\varphi R^{\beta}(x) - 2\lambda\varphi R^{2\beta}(x)\}dx\Bigg] \qquad (15)\\
&= (1-\varphi)^{-1}\log\left[\int_{-\infty}^{\infty}\left(\beta g(x;\Theta)\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}}\left[k+2\lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta}\right]\right.\right. \\
&\left.\left.\times\exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\}\right)^{\varphi} dx\right] \\
&= (1-\varphi)^{-1}\left\{\varphi\log\beta + \log\int_{-\infty}^{\infty} g^{\varphi}(x;\Theta)\frac{[G(x;\Theta)]^{\varphi(\beta-1)}}{[\overline{G}(x;\Theta)]^{\varphi(\beta+1)}}\right.
\end{aligned}$$

$$\times \left(k + 2\lambda \left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)} \right]^{\beta} \right)^{\varphi} \exp\left\{ -k\varphi \left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)} \right]^{\beta} - \lambda\varphi \left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)} \right]^{2\beta} \right\} dx \Bigg\} \tag{16}$$

where $\varphi > 0$ and $\varphi \neq 1$. Apply the same expansion technique for the pdf, i.e. exponential series and binomial series,

$$\exp\left\{ -k\varphi \left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)} \right]^{\beta} - \lambda\varphi \left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)} \right]^{2\beta} \right\} \tag{17}$$

We can generalize the Rényi Entropy as below

$$\begin{aligned}
I_R(\varphi) &= (1-\varphi)^{-1} \Bigg\{ \varphi \log\beta + \log \Bigg[\sum_{a,b,l,u=0}^{\infty} \frac{(-1)^{a+u}\varphi^a}{a!} 2^l k^{\varphi+a-b-l} \lambda^{b+l} \binom{a}{b} \\
&\times \binom{-\varphi(\beta+1)-\beta(a+b+l)}{u} \left(\frac{\varphi}{\beta(\varphi+a+b+l)+u} \right)^{\varphi} \\
&\times \int_{-\infty}^{\infty} \left(\frac{\beta(\varphi+a+b+l)+u}{\varphi} \right)^{\varphi} g^{\varphi}(x;\Theta) \left[G(x;\Theta)\right]^{\varphi(\beta+1)\beta(a+b+l)} dx \Bigg] \Bigg\} \\
&= (1-\varphi)^{-1} \Bigg\{ \varphi \log\beta + \log \Bigg[\sum_{a,b,l,u}^{\infty} \frac{(-1)^{a+u}\varphi^a}{a!} 2^l k^{\varphi+a-b-l} \lambda^{b+l} \binom{a}{b} \\
&\times \binom{-\varphi(\beta+1)-\beta(a+b+l)}{u} \left(\frac{\varphi}{\beta(\varphi+a+b+l)+u} \right)^{\varphi} \\
&\times \int_{-\infty}^{\infty} \left[\left(\frac{\beta(\varphi+a+b+l)+u}{\varphi} \right) g(x;\Theta) \left[G(x;\Theta)\right]^{\frac{\varphi(\beta+1)-\beta(a+b+l)}{\varphi}} \right]^{\varphi} dx \Bigg] \Bigg\} \\
&= (1-\varphi)^{-1} \Bigg\{ \varphi \log\beta + \log \Bigg[\sum_{a,b,l,u}^{\infty} \frac{(-1)^{a+u}\varphi^a}{a!} 2^l k^{\varphi+a-b-l} \lambda^{b+l} \binom{a}{b} \\
&\times \binom{-\varphi(\beta+1)-\beta(a+b+l)}{u} \left(\frac{\varphi}{\beta(\varphi+a+b+l)+u} \right)^{\varphi} exp^{(1-\varphi)I_{REG}} \Bigg] \Bigg\}
\end{aligned} \tag{18}$$

where $\beta(\varphi+a+b+l)+u = \varphi(\beta-1)+\beta(a+b+l)+u+\varphi$

where I_{REG}, as indicated below, is the Rényi Entropy of the exponentiated generalized distribution of the parameter $\beta^* = \frac{\beta(\varphi+a+b+l)+u}{\varphi}$

$$I_{REG} = \frac{1}{1-\varphi}\int_{-\infty}^{\infty}\left(\frac{\beta(\varphi+a+b+l)+u}{\varphi}g(x;\Theta)\left[G(x;\Theta)\right]^{\frac{\varphi(\beta-1)+\beta(a+b+l)+u}{\varphi}}\right)^{\varphi}dx$$
$$= \frac{1}{1-\varphi}\int_{-\infty}^{\infty}\left(\beta^{*}g(x;\Theta)\left[G(x;\Theta)\right]^{\beta^{*}-1}\right)^{\varphi}dx \tag{19}$$

2.6 Order Statistics

Consider $X_1, X_2, \ldots, X_n$ as n independent and identically distributed (i.i.d.) random variables.

Theorem 3. The PDF of the i-th order statistic, denoted as $f_{i:n}(x)$, is given by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f_{M-LFRD-G}(x)\left[F_{M-LFRD-G}(x)\right]^{i-1}\left[\overline{F}_{M-LFRD-G}(x)\right]^{n-i}$$
$$= \frac{n!f_{M-LFRD-G}(x)}{(i-1)!(n-i)!}\left[1-\overline{F}_{M-LFRD-G}(x)\right]^{i-1}\left[\overline{F}_{M-LFRD-G}(x)\right]^{n-i}$$
$$= \frac{n!f_{M-LFRD-G}(x)}{(i-1)!(n-i)!}\sum_{s=0}^{i-1}\binom{i-1}{s}(-1)^{s}\overline{F}^{s}_{M-LFRD-G}(x)\overline{F}^{n-i}_{M-LFRD-G}(x)$$
$$= \frac{n!f_{M-LFRD-G}(x)}{(i-1)!(n-i)!}\sum_{s=0}^{i-1}\binom{i-1}{s}(-1)^{s}\overline{F}^{s+n-i}_{M-LFRD-G}(x)$$
$$= \frac{n!f_{M-LFRD-G}(x)}{(i-1)!(n-i)!}\sum_{s=0}^{i-1}\binom{i-1}{s}(-1)^{s}\left(\exp\left\{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}-\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right\}\right)^{s+n-i}$$
$$= \frac{n!}{(i-1)!(n-i)!}\beta g(x;\Theta)\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}}\left[k+2\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}\right]$$
$$\times\exp\left\{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}-\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right\}$$
$$\times\sum_{s=0}^{i-1}\binom{i-1}{s}(-1)^{s}\left(\exp\left\{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}-\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right\}\right)^{s+n-i}$$
$$= \frac{n!}{(i-1)!(n-i)!}\sum_{s=0}^{i-1}\binom{i-1}{s}(-1)^{s}\beta g(x;\Theta)\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}}\left[k+2\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}\right]$$
$$\times\left(\exp\left\{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}-\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right\}\right)^{s+n-i+1}$$
$$= \frac{n!}{(i-1)!(n-i)!}\sum_{s=0}^{i-1}\binom{i-1}{s}(-1)^{s}\beta g(x;\Theta)\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}}\frac{s+n-i+1}{s+n-i+1}$$

$$\times \left[k + 2\lambda \left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}\right] \left(\exp\left\{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta} - \lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right\}\right)^{s+n-i+1}$$

$$= \frac{n!}{(i-1)!(n-i)!}\sum_{s=0}^{i-1}\binom{i-1}{s}\frac{(-1)^s}{s+n-i+1} f_{M-LFRD-G}(x;k^*,\lambda^*,\beta) \quad (20)$$

2.7 Mean Residual Life

Remark 7. The survival function, mean residual life, and mean time to failure, are derived as follows:

$$\begin{aligned}
S(x;\Theta) &= 1 - F_{M-LFRD-G}(x;\Theta) \\
&= \exp\{-kR^{\beta}(x) - 2\lambda R^{2\beta}(x) && (21)\\
&= \exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\} && (22)\\
MRL(x;\Theta) &= E[T-t|T>t] \\
&= \frac{1}{S(t,\Theta)}\int_t^{\infty} S(x)dx && (23)\\
&= \frac{1}{S(t,\Theta)}\int_0^{\infty} S(t+x)dx \\
&= \frac{1}{S(t,\Theta)}\int_t^{\infty} e^{-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta} - \lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}} dx \\
&= \frac{1}{S(t,\Theta)}\int_t^{\infty}\sum_p^{\infty}\frac{\left(-k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta} - \lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right)^p}{p!}dx \\
&= \frac{1}{S(t,\Theta)}\sum_p^{\infty}\frac{(-1)^p}{p!}\int_t^{\infty}\sum_i^{p}\frac{p!}{i!(p-i)!}\left(k\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{\beta}\right)^i\left(\lambda\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta}\right)^{p-i} dx \\
&= \frac{1}{S(t,\Theta)}\sum_p^{\infty}\sum_i^{p}\frac{(-1)^p k^i\lambda^{p-i}}{i!(p-i)!}\int_t^{\infty}\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta p-\beta i} dx && (24)\\
MTTF(x;\Theta) &= \int_0^{\infty} S(x)dx \\
&= \sum_p^{\infty}\sum_i^{p}\frac{(-1)^p k^i\lambda^{p-i}}{i!(p-i)!}\int_0^{\infty} R^{2\beta p-\beta i}(x) && (25)\\
&= \sum_p^{\infty}\sum_i^{p}\frac{(-1)^p k^i\lambda^{p-i}}{i!(p-i)!}\int_0^{\infty}\left(\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right)^{2\beta p-\beta i} dx && (26)
\end{aligned}$$

2.8 Probability Weighted Function

Remark 8. For p, q, and $\tau \geq 0$

$$\begin{aligned} PW(x;p,q,\tau) &= E\left[x^{\tau}(F_{M-LFRD-G}(x))^{p}(1-F_{M-LFRD-G}(x))^{q}\right] \\ &= \int_{-\infty}^{\infty} x^{\tau}(F_{M-LFRD-G}(x))^{p}(1-F_{M-LFRD-G}(x))^{q} f_{M-LFRD-G}(x)dx \end{aligned} \tag{27}$$

Let $\tau = 0$

$$PW(x;p,q) = \int_{-\infty}^{\infty} (F_{M-LFRD-G}(x))^{p}(1-F_{M-LFRD-G}(x))^{q} f_{M-LFRD-G}(x)dx \tag{28}$$

Set $p = 1$ and $q = 0$ for lower tail sensitivity to measure the behavior near the beginning of the distribution

$$\begin{aligned} PW(x) &= \int_{-\infty}^{\infty} (F_{M-LFRD-G}(x)) f_{M-LFRD-G}(x)dx \\ &= \int_{-\infty}^{\infty} (1-\exp\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\})\beta r(x)R^{\beta-1}(x) \\ &\times (k+2\lambda R^{\beta}(x))\exp\{-kR^{\beta}(x)-\lambda R^{2\beta}(x)\}dx \qquad (29) \\ &= \int_{-\infty}^{\infty} \left[1-\exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\}\right] \beta g(x;\Theta)\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}} \\ &\times \left[k+2\lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta}\right]\exp\left\{-k\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^{2\beta}\right\}dx \qquad (30) \end{aligned}$$

3 Methods of Parameter Estimation

3.1 MLE for Uncensored Data

Let $\Delta = (k,\lambda,\beta,\Theta)^{T}$, then the likelihood function is as follow,

$$\begin{aligned} \Delta &= \prod_{i=1}^{n} f_{M-LFRD-G}(x_i;k,\lambda,\beta,\Theta) \\ &= \prod_{i=1}^{n} \beta r_i(x)R^{\beta-1}(x_i)(k+2\lambda R^{\beta}(x_i))\exp\left\{-kR^{\beta}(x_i)-\lambda R^{2\beta}(x_i)\right\} \qquad (31) \\ &= \prod_{i=1}^{n} \beta g(x_i;\Theta)\frac{G(x_i;\Theta)^{\beta-1}}{\overline{G}(x_i;\Theta)^{\beta+1}}\left[k+2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right] \\ &\times \exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\} \qquad (32) \end{aligned}$$

Denote $Z(x_i;\Theta) = \frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}$ ln-likelihood function of Δ is as follow,

$$\ell(\Delta) = n\ln\beta + \sum_{i=1}^{n}\log g(x_i;\Theta) + (\beta-1)\sum_{i=1}^{n}\log G(x_i;\Theta) - (\beta+1)\sum_{i=1}^{n}\log(1-G(x_i;\Theta))$$
$$+\sum_{i=1}^{n}\log\left(k+2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right) - \sum_{i=1}^{n}k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \sum_{i=1}^{n}\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta} \tag{33}$$

The first derivative of the ln-likelihood function with respect to Δ is as follows:

$$\frac{\partial\ell}{\partial k} = \sum_{i=1}^{n}\frac{1}{\left(k+2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right)} - \sum_{i=1}^{n}\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} \tag{34}$$

$$\frac{\partial\ell}{\partial \lambda} = \sum_{i=1}^{n}\frac{2\left[\frac{G(x_i;\Theta)}{1-G(x_i;\Theta)}\right]^{\beta}}{\left(k+2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right)} - \sum_{i=1}^{n}\left[\frac{G(x_i;\Theta)}{1-G(x_i;\Theta)}\right]^{2\beta} \tag{35}$$

$$\frac{\partial\ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)} + \sum_{i=1}^{n}\frac{2\lambda\left[\frac{G(x_i;\Theta)}{1-G(x_i;\Theta)}\right]^{\beta}}{\left(k+2\lambda\left[\frac{G(x_i;\Theta)}{1-G(x_i;\Theta)}\right]^{\beta}\right)}\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}$$
$$-k\sum_{i=1}^{n}\left[\frac{G(x_i;\Theta)}{1-G(x_i;\Theta)}\right]^{\beta}\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)} - 2\lambda\sum_{i=1}^{n}\left[\frac{G(x_i;\Theta)}{1-G(x_i;\Theta)}\right]^{2\beta}\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)} \tag{36}$$

$$\frac{\partial\ell}{\partial \Theta_t} = \sum_{i=1}^{n}\frac{1}{g(x_i;\Theta)}\frac{\partial g(x_i;\Theta)}{\partial\Theta_t} + (\beta-1)\sum_{i=1}^{n}\frac{1}{G(x_i;\Theta)}\frac{\partial G(x_i;\Theta)}{\partial\Theta_k}$$
$$+(\beta+1)\sum_{i=1}^{n}\frac{1}{1-G(x_i;\Theta)}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}$$
$$+\sum_{i=1}^{n}\frac{2\lambda\beta}{\left(k+2\lambda\left[\frac{G(x_i;\Theta)}{1-G(x_i;\Theta)}\right]^{\beta}\right)}\frac{G(x_i;\Theta)^{\beta-1}}{[1-G(x_i;\Theta)]^{\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}$$
$$-k\beta\sum_{i=1}^{n}\frac{G(x_i;\Theta)^{\beta-1}}{[1-G(x_i;\Theta)]^{\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t} - 2\lambda\beta\sum_{i=1}^{n}\frac{G(x_i;\Theta)^{2\beta-1}}{[1-G(x_i;\Theta)]^{2\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t} \tag{37}$$

3.1.1 Asymptotic Confidence Intervals

We base our calculation on the asymptotic normality distributions for $k, \lambda, \beta, \Theta$, as the sample size approaches infinity. Specifically,

$$\sqrt{n}(\Delta - \hat{\Delta}) \sim N_4(0, I^{-1})$$

where

$$I(\Delta) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial k^2} & \frac{\partial^2 \ell}{\partial k \partial \lambda} & \frac{\partial^2 \ell}{\partial k \partial \beta} & \frac{\partial^2 \ell}{\partial k \partial \Theta} \\ . & \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} & \frac{\partial^2 \ell}{\partial \lambda \partial \Theta} \\ . & . & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \Theta} \\ . & . & . & \frac{\partial^2 \ell}{\partial \Theta^2} \end{bmatrix}$$

Based on this matrix, we can have the approximate variance-covariance matrix evaluated at $\hat{\Delta} = (\hat{k}, \hat{\lambda}, \hat{\beta}, \hat{\Theta})'$ the MLE of $(k, \lambda, \beta, \Theta)'$ as

$$I^{-1}(\hat{\Delta}) = \begin{bmatrix} var(\hat{k}) & cov(\hat{k}, \hat{\lambda}) & cov(\hat{k}, \hat{\beta}) & cov(\hat{k}, \hat{\Theta}) \\ . & var(\hat{\lambda}) & cov(\hat{\lambda}, \hat{\beta}) & cov(\hat{\lambda}, \hat{\Theta}) \\ . & . & var(\hat{\beta}) & cov(\hat{\beta}, \hat{\Theta}) \\ . & . & . & var(\hat{\Theta}) \end{bmatrix} \tag{38}$$

Hence, the $100(1-\delta)\%$ asymptotic confidence intervals (ACIs) for each Δ_t is given by

$$ACI_t = \left[\hat{\Delta}_t - Z_{\frac{\delta}{2}} \sqrt{\hat{I}_{tt}}, \hat{\Delta}_t + Z_{\frac{\delta}{2}} \sqrt{\hat{I}_{tt}}\right], \tag{39}$$

where $\hat{I}_{tt}$ is the (t,t) diagonal elements of $I_n(\hat{\Delta})^{-1}$ for t = 1,2,3,4 and $Z_{\frac{\delta}{2}}$ is the upper δ^{th} percentile of the standard normal distribution.

3.2 MLE for Right-Censored Data

Remark 6. The log-likelihood function of the M-LFRD-G model can be expressed in censored random sample with the PDF and survival function as follows:

$$\begin{aligned} \ell_C(\Delta) &= \sum_{i=1}^{n} \delta_i \log f(y_i) + \sum_{i=1}^{n} (1 - \delta_i) \log S(y_i) \qquad (40) \\ &= \sum_{i=1}^{n} \delta_i \log \beta + \sum_{i=1}^{n} \delta_i \log g(y_i, \Theta) + (\beta - 1) \sum_{i=1}^{n} \delta_i G(y_i, \Theta) \\ &+ (\beta + 1) \sum_{i=1}^{n} \delta_i \overline{G}(y_i, \Theta) + \sum_{i=1}^{n} \delta_i \log \left[k + 2\lambda \left(\frac{G(y_i, \Theta)}{\overline{G}(y_i, \Theta)}\right)^{\beta}\right] \end{aligned}$$

$$
- k \sum_{i=1}^{n} \left(\frac{G(y_i, \Theta)}{\overline{G}(y_i, \Theta)} \right)^{\beta} - \lambda \sum_{i=1}^{n} \left(\frac{G(y_i, \Theta)}{\overline{G}(y_i, \Theta)} \right)^{2\beta} \tag{41}
$$

3.3 Least Square and Weighted Least Square Estimation

We start with the Least Square function as follows:

$$
\begin{aligned}
LS(\boldsymbol{\Delta}) &= \sum_{i=1}^{n} \left(F(x_i, \boldsymbol{\Delta}) - \frac{i}{n+1} \right)^2 \\
&= \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right)^2
\end{aligned} \tag{42}
$$

Take derivative of $LS(\boldsymbol{\Delta})$ with respect to $\boldsymbol{\Delta}$,

$$
\begin{aligned}
\frac{\partial LS}{\partial k} &= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\quad \times \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\}
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\frac{\partial LS}{\partial \lambda} &= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\quad \times \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\}
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
\frac{\partial LS}{\partial \beta} &= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\quad \times \left[k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} \log\left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right) + 2\lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \log\left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right) \right] \\
&\quad \times \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\} \\
&= 2 \sum_{i=1}^{n} \left(1 - \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\} - \frac{i}{n+1} \right) \\
&\quad \times \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} \log\left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right) \left(k + 2\lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} \right) \\
&\quad \times \exp\left\{ -k \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{\beta} - \lambda \left[\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right]^{2\beta} \right\}
\end{aligned} \tag{45}
$$

$$\begin{aligned}
\frac{\partial LS}{\partial \Theta_t} &= 2\sum_{i=1}^{n}\left(1-\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}-\frac{i}{n+1}\right)\\
&\times\left[k\beta\frac{(G(x_i;\Theta))^{\beta-1}}{(\overline{G}(x_i;\Theta))^{\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}+2\lambda\beta\frac{(G(x_i;\Theta))^{2\beta-1}}{(\overline{G}(x_i;\Theta))^{2\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}\right]\\
&\times\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\\
&= 2\beta\sum_{i=1}^{n}\left(1-\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}-\frac{i}{n+1}\right)\\
&\times\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}\frac{(G(x_i;\Theta))^{\beta-1}}{(\overline{G}(x_i;\Theta))^{\beta+1}}\left[k+2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right]\\
&\times\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\} \qquad (46)
\end{aligned}$$

$$WLS(\Delta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F_{P-LRFD-G}(x_i,\Delta)-\frac{i}{n+1}\right)^2. \qquad (47)$$

3.4 Maximum Product Spacing Approach of Estimation

For $\Delta=(k,\lambda,\beta,\Theta)^T$, the MSP is as follows:

$$MPS(\mathbf{\Delta})=\left\{\prod_{i=1}^{n+1}D_i(x_i,\mathbf{\Delta})\right\}^{\frac{1}{n+1}} \text{ where } D_i=\begin{cases}F(x_1,\mathbf{\Delta}), i=1\\ F(x_i,\mathbf{\Delta})-F(x_{i-1},\mathbf{\Delta}), i=2,3,...,n\\ 1-F(x_n,\mathbf{\Delta}), i=n+1\end{cases}$$

Then the $MPS(\mathbf{\Delta})$ can be rewritten as follows:

$$\begin{aligned}
MPS(\mathbf{\Delta}) &= \left\{F(x_1,\mathbf{\Delta})\left(1-F(x_n,\mathbf{\Delta})\right)\prod_{i=2}^{n}\left(F(x_i,\mathbf{\Delta})-F(x_{i-1},\mathbf{\Delta})\right)\right\}^{\frac{1}{n+1}}\\
&= \left\{\left[1-\exp\left\{-k\left[\frac{G(x_1;\Theta)}{\overline{G}(x_1;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_1;\Theta)}{\overline{G}(x_1;\Theta)}\right]^{2\beta}\right\}\right]\right.\\
&\times\left(\exp\left\{-k\left[\frac{G(x_n;\Theta)}{\overline{G}(x_n;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_n;\Theta)}{\overline{G}(x_n;\Theta)}\right]^{2\beta}\right\}\right)\\
&\times\prod_{i=2}^{n}\left[\exp\left\{-k\left[\frac{G(x_{i-1};\Theta)}{\overline{G}(x_{i-1};\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_{i-1};\Theta)}{\overline{G}(x_{i-1};\Theta)}\right]^{2\beta}\right\}\right.
\end{aligned}$$

$$- \exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\Bigg]\Bigg\}^{\frac{1}{n+1}} \tag{48}$$

Let $\left(-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right) = X_i$ and $\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)} = A_i$ will make the following functions format better. Then,

$$\begin{aligned}
\frac{\partial X_i}{\partial k} &= -\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} = -A_i^{\beta} \\
\frac{\partial X_i}{\partial \lambda} &= -\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta} = -A_i^{2\beta} \\
\frac{\partial X_i}{\partial \beta} &= -k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)} - 2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)} \\
&= -\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\left(k + 2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right)\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)} \\
&= -A_i^{\beta}\left(k + 2\lambda A_i^{\beta}\right)\log A_i \\
\frac{\partial X_i}{\partial \Theta_t} &= -k\beta\frac{[G(x_i;\Theta)]^{\beta-1}}{\left[\overline{G}(x_i;\Theta)\right]^{\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial \Theta_t} - 2\lambda\beta\frac{(G(x_i;\Theta))^{2\beta-1}}{(\overline{G}(x_i;\Theta))^{2\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial \Theta_t} \\
&= -\beta\frac{[G(x_i;\Theta)]^{\beta-1}}{\left[\overline{G}(x_i;\Theta)\right]^{\beta+1}}\left(k + 2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right)\frac{\partial G(x_i;\Theta)}{\partial \Theta_t} \\
&= -\frac{\beta A_i^{\beta-1}}{\overline{G}(x;\Theta)^2}(k + 2\lambda A_i^{\beta})\frac{\partial G(x_i;\Theta)}{\partial \Theta_t}
\end{aligned}$$

Using X_i and A_i, the $MPS(\mathbf{\Delta})$ becomes:

$$MPS(\mathbf{\Delta}) = \left\{\left[1 - e^{X_1}\right]\left(e^{X_n}\right)\prod_{i=2}^{n}\left[e^{X_{i-1}} - e^{X_i}\right]\right\}^{\frac{1}{n+1}} \tag{49}$$

We can maximize $H = \log(MPS)$, then take the derivative of H with respect to $\mathbf{\Delta}$:

$$\begin{aligned}
H(\mathbf{\Delta}) &= \frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(x_i, \mathbf{\Delta}) \\
&= \frac{1}{n+1}\left[\log(1 - e^{X_1}) + \log e^{X_n} + \sum_{i=2}^{n}\log(e^{X_{i-1}} - e^{X_i})\right]
\end{aligned}$$

$$= \frac{1}{n+1}\left[\log(1-e^{X_1}) + X_n + \sum_{i=2}^{n}\log(e^{X_{i-1}} - e^{X_i})\right] \tag{50}$$

$$\begin{aligned}
\frac{\partial H}{\partial \Theta_t} &= \frac{\partial H}{\partial X}\frac{\partial X}{\partial \Theta_t} \\
&= \frac{1}{n+1}\left[\frac{e^{X_1}\frac{\beta A_1^{\beta-1}}{\overline{G}(x_1;\Theta)^2}(k+2\lambda A_1^{\beta})\frac{\partial G(x_1;\Theta)}{\partial \Theta_t}}{1-e^{X_1}} - \frac{\beta A_n^{\beta-1}}{\overline{G}(x_n;\Theta)^2}(k+2\lambda A_n^{\beta})\frac{\partial G(x_n;\Theta)}{\partial \Theta_t}\right. \\
&\left. + \sum_{i=2}^{n}\frac{-\frac{\beta A_{i-1}^{\beta-1}e^{X_{i-1}}}{\overline{G}(x_{i-1};\Theta)^2}(k+2\lambda A_{i-1}^{\beta})\frac{\partial G(x_{i-1};\Theta)}{\partial \Theta_t} + \frac{\beta A_{i-1}^{\beta-1}e^{X_i}}{\overline{G}(x_{i-1};\Theta)^2}(k+2\lambda A_{i-1}^{\beta})\frac{\partial G(x_{i-1};\Theta)}{\partial \Theta_t}}{e^{X_{i-1}} - e^{X_i}}\right]
\end{aligned} \tag{51}$$

$$\frac{\partial H}{\partial X_i} = \frac{1}{n+1}\left[\frac{-e^{X_1}X_1'}{1-e^{X_1}} + X_n' + \sum_{i=2}^{n}\frac{e^{X_{i-1}}X_{i-1}' - e^{X_i}X_i'}{e^{X_{i-1}} - e^{X_i}}\right] \tag{52}$$

$$\frac{\partial H}{\partial k} = \frac{1}{n+1}\left[\frac{e^{X_1}\left[\frac{G(x_1;\Theta)}{\overline{G}(x_1;\Theta)}\right]^{\beta}}{1-e^{X_1}} - \left[\frac{G(x_n;\Theta)}{\overline{G}(x_n;\Theta)}\right]^{\beta} + \sum_{i=2}^{n}\frac{-\left[\frac{G(x_{i-1};\Theta)}{\overline{G}(x_{i-1};\Theta)}\right]^{\beta}e^{X_{i-1}} + \left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}e^{X_i}}{e^{X_{i-1}} - e^{X_i}}\right] \tag{53}$$

$$\frac{\partial H}{\partial \lambda} = \frac{1}{n+1}\left[\frac{e^{X_1}\left[\frac{G(x_1;\Theta)}{\overline{G}(x_1;\Theta)}\right]^{2\beta}}{1-e^{X_1}} - \left[\frac{G(x_n;\Theta)}{\overline{G}(x_n;\Theta)}\right]^{2\beta} + \sum_{i=2}^{n}\frac{-\left[\frac{G(x_{i-1};\Theta)}{\overline{G}(x_{i-1};\Theta)}\right]^{2\beta}e^{X_{i-1}} + \left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}e^{X_i}}{e^{X_{i-1}} - e^{X_i}}\right] \tag{54}$$

$$\begin{aligned}
\frac{\partial H}{\partial \beta} &= \frac{\partial H}{\partial X_i}\frac{\partial X_i}{\partial \beta} \\
&= \frac{1}{n+1}\left[\frac{e^{X_1}A_1^{\beta}\left(k+2\lambda A_1^{\beta}\right)\log A_1}{1-e^{X_1}} - A_n^{\beta}\left(k+2\lambda A_n^{\beta}\right)\log A_n\right. \\
&\left. + \sum_{i=2}^{n}\frac{-e^{X_{i-1}}A_{i-1}^{\beta}\left(k+2\lambda A_{i-1}^{\beta}\right)\log A_{i-1} + e^{X_i}A_i^{\beta}\left(k+2\lambda A_i^{\beta}\right)\log A_i}{e^{X_{i-1}} - e^{X_i}}\right] \\
&- \frac{\beta A_i^{\beta-1}}{\overline{G}(x;\Theta)^2}(k+2\lambda A_i^{\beta})\frac{\partial G(x_i;\Theta)}{\partial \Theta_t}
\end{aligned} \tag{55}$$

$$\frac{\partial H}{\partial \Theta_t} = \frac{\partial H}{\partial X}\frac{\partial X}{\partial \Theta_t}$$

$$= \frac{1}{n+1}\left[\frac{e^{X_1}\frac{\beta A_1^{\beta-1}}{\overline{G}(x_1;\Theta)^2}(k+2\lambda A_1^{\beta})\frac{\partial G(x_1;\Theta)}{\partial\Theta_t}}{1-e^{X_1}} - \frac{\beta A_n^{\beta-1}}{\overline{G}(x_n;\Theta)^2}(k+2\lambda A_n^{\beta})\frac{\partial G(x_n;\Theta)}{\partial\Theta_t}\right.$$
$$\left. + \sum_{i=2}^{n}\frac{-\frac{\beta A_{i-1}^{\beta-1}e^{X_{i-1}}}{\overline{G}(x_{i-1};\Theta)^2}(k+2\lambda A_{i-1}^{\beta})\frac{\partial G(x_{i-1};\Theta)}{\partial\Theta_t} + \frac{\beta A_{i-1}^{\beta-1}e^{X_i}}{\overline{G}(x_{i-1};\Theta)^2}(k+2\lambda A_{i-1}^{\beta})\frac{\partial G(x_{i-1};\Theta)}{\partial\Theta_t}}{e^{X_{i-1}}-e^{X_i}}\right] \tag{56}$$

3.5 Cramér-von Mises Approach of Estimation

We differentiate $CM(x;\mathbf{\Delta})$ with respect to $\mathbf{\Delta}=(k,\lambda,\beta,\Theta)^T$

$$\begin{aligned}
CM(x;\mathbf{\Delta}) &= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}\left(F(x_i;\mathbf{\Delta}) - \frac{2i-1}{2n}\right)^2 \\
&= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}\left(1-\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\} - \frac{2i-1}{2n}\right)^2 \\
&= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2(n-i)+1}{2n} - \exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\right)^2 \\
&= \frac{1}{12n^2} + \frac{1}{n}\sum_{i=1}^{n}A_i^2
\end{aligned} \tag{57}$$

Take the partial derivative of $CM(x;\mathbf{\Delta})$

$$\frac{\partial CM}{\partial k} = \frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\right) \tag{58}$$

$$\frac{\partial CM}{\partial \lambda} = \frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\right) \tag{59}$$

$$\begin{aligned}
\frac{\partial CM}{\partial \beta} &= \frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\log\left(\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right) + 2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\log\left(\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right)\right]\right. \\
&\quad \left.\times\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\right)
\end{aligned} \tag{60}$$

$$\begin{aligned}
\frac{\partial CM}{\partial \Theta_t} &= \frac{2}{n}\sum_{i=1}^{n}\left(A_i\left[k\beta\frac{G(x_i;\Theta)^{\beta-1}}{\overline{G}(x_i;\Theta)^{\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t} + 2\lambda\beta\frac{G(x_i;\Theta)^{2\beta-1}}{\overline{G}(x_i;\Theta)^{2\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}\right]\right. \\
&\quad \left.\times\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\right)
\end{aligned} \tag{61}$$

3.6 Anderson and Darling Approach of Estimation

Take the differentiate of AD equation with respect to $\boldsymbol{\Delta}$.
$\Delta = (k, \lambda, \beta, \Theta)^T$

$$
\begin{aligned}
AD(\boldsymbol{\Delta}) &= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_i, \boldsymbol{\Delta}) + \log \overline{F}(x_{n+1-i}, \boldsymbol{\Delta})] \\
&= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1 - \exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\right)\right. \\
&\left.+\log\left(\exp\left\{-k\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{2\beta}\right\}\right)\right] \\
&= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1 - \exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}\right)\right. \\
&\left.-k\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{2\beta}\right] \qquad (62)
\end{aligned}
$$

Now take the derivative of $AD(\boldsymbol{\Delta})$ by

$$
\begin{aligned}
\frac{\partial AD(\boldsymbol{\Delta})}{\partial k} &= -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}\right. \\
&\left.+\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{\beta}\right] \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial AD(\boldsymbol{\Delta})}{\partial \lambda} &= -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right. \\
&\left.+\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{2\beta}\right] \qquad (64)
\end{aligned}
$$

$$
\frac{\partial AD(\boldsymbol{\Delta})}{\partial \beta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta} - \lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}\right.
$$

$$\times\left(k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}+2\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right)\log\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}$$

$$+k\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{\beta}\log\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}+2\lambda\left[\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\right]^{2\beta}\log\frac{G(x_{n+1-i};\Theta)}{\overline{G}(x_{n+1-i};\Theta)}\Bigg] \tag{65}$$

$$\frac{\partial AD(\mathbf{\Delta})}{\partial\Theta_t}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}{1-\exp\left\{-k\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{\beta}-\lambda\left[\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right]^{2\beta}\right\}}\right.$$
$$\times\left(k\beta\frac{G_(x_i;\Theta)^{\beta-1}}{\overline{G}(x_i;\Theta)^{\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}+2\lambda\beta\frac{G_(x_i;\Theta)^{2\beta-1}}{\overline{G}(x_i;\Theta)^{2\beta+1}}\frac{\partial G(x_i;\Theta)}{\partial\Theta_t}\right)$$
$$\left.+k\beta\frac{G_(x_{n+1-i};\Theta)^{\beta-1}}{\overline{G}(x_{n+1-i};\Theta)^{\beta+1}}\frac{\partial G(x_{n+1-i};\Theta)}{\partial\Theta_t}+2\lambda\beta\frac{G_(x_{n+1-i};\Theta)^{2\beta-1}}{\overline{G}(x_{n+1-i};\Theta)^{2\beta+1}}\frac{\partial G(x_{n+1-i};\Theta)}{\partial\Theta_t}\right] \tag{66}$$

3.7 Bayesian Estimation

The Bayesian model is constructed by the product of the likelihood function and prior distribution $\pi(\Delta)$ for $\Delta=(k,\lambda,\beta,\Theta)'$, which derived the posterior distribution of Δ represented by $\phi(\Delta|\mathcal{D})$. $\pi(\Delta)$ denoted the distribution of Δ before observing the data $D: t_1,t_2,...,t_n$. Based on the Bayes theorem, we can obtain the distribution of the posterior as $\Delta|\mathcal{D}$ defined:

$$\pi(\Delta|\mathcal{D})=\frac{\mathcal{L}(\mathcal{D}|\Delta)\pi(\Delta)}{\int_{\Theta}\mathcal{L}(\mathcal{D}|\Delta)\pi(\Delta)d\Delta} \tag{67}$$
$$\propto\mathcal{L}(\mathcal{D}|\Delta)\pi(\Delta) \tag{68}$$

where $\int_{\Theta}\mathcal{L}(\mathcal{D}|\Delta)\pi(\Delta)d\Delta$ is the normalizing constant of the posterior distribution of Δ, Θ is the parameter space, and $\mathcal{L}(\mathcal{D}|\Delta)$ represents the likelihood function of M-LFRD-G distribution, and is given by

$$\mathcal{L}(\mathcal{D}|\Delta)=\beta^n\prod_{i=1}^{n}g(x_i;\Theta)\frac{G(x_i;\Theta)^{\beta-1}}{\overline{G}(x_i;\Theta)^{\beta+1}}\left[k+2\lambda\left(\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right)^{\beta}\right]$$
$$\times\exp\left[-k\sum_{i=1}^{n}\left(\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right)^{\beta}-\lambda\sum_{i=1}^{n}\left(\frac{G(x_i;\Theta)}{\overline{G}(x_i;\Theta)}\right)^{2\beta}\right] \tag{69}$$

In this study, the parameters $k, \lambda, \beta, \Theta$ are supposed to have gamma priors following the works of Kundu Howlader [48], Soliman et al. [49], and Abba and Wang [29]. Let $\pi(\Delta_t)$ denote the gamma prior having (ν_t, γ_t) as the hyper-parameters, with PDF given by

$$\pi(\Delta_t) = \frac{\nu_t^{\gamma_t}}{\Gamma(\gamma_t)} \Delta_t^{\gamma_t - 1} e^{-\nu_t \Delta_t}, \tag{70}$$

where $\nu_t > 0, \gamma_t > 0, t = 1, 2, 3, 4$

Thus, $k \sim \pi(k|\nu_1, \gamma_1)$,$\lambda \sim \pi(\lambda|\nu_2, \gamma_2)$, $\beta \sim \pi(\beta|\nu_3, \gamma_3)$, $\Theta \sim \pi(\Theta|\nu_4, \gamma_4)$. The joint posterior distribution of $\Delta|\, \mathcal{D}$ is therefore obtained after substituting Eq.[69]-[70] as

$$\begin{aligned}
\pi(\Delta|\, \mathcal{D}) \propto\ & \frac{k^{\gamma_1 - 1} \lambda^{\gamma_2 - 1} \beta^{n + \gamma_3 - 1}}{\Theta^{-\gamma_4 + 1}} \prod_{i=1}^{n} g(x_i; \Theta) \frac{G(x_i; \Theta)^{\beta - 1}}{\overline{G}(x_i; \Theta)^{\beta + 1}} \\
& \times \left[k + 2\lambda \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] \\
& \times \exp\left(- \left[\alpha_1 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] k - \left[\alpha_2 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{2\beta} \right] \lambda - \alpha_3 \beta - \alpha_4 \Theta \right)
\end{aligned} \tag{71}$$

We can determine the marginal posterior densities from 71 for the $k, \lambda, \beta, \Theta$ as

$$\begin{aligned}
\pi(k|\, \mathcal{D}) &\propto k^{\gamma_1 - 1} \prod_{i=1}^{n} \left[k + 2\lambda \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] \exp\left(- \left[\alpha_1 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] k \right) \\
\pi(\lambda|\, \mathcal{D}) &\propto \lambda^{\gamma_2 - 1} \prod_{i=1}^{n} \left[k + 2\lambda \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] \exp\left(- \left[\alpha_2 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{2\beta} \right] \lambda \right) \\
\pi(\beta|\, \mathcal{D}) &\propto \beta^{n + \gamma_3 - 1} \prod_{i=1}^{n} \frac{G(x_i; \Theta)^{\beta - 1}}{\overline{G}(x_i; \Theta)^{\beta + 1}} \left[k + 2\lambda \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] \\
&\quad \times \exp\left(- \left[\alpha_1 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] k - \left[\alpha_2 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{2\beta} \right] \lambda - \alpha_3 \beta \right) \\
\pi(\Theta|\, \mathcal{D}) &\propto \Theta^{\gamma_4 - 1} \prod_{i=1}^{n} g(x_i; \Theta) \frac{G(x_i; \Theta)^{\beta - 1}}{\overline{G}(x_i; \Theta)^{\beta + 1}} \left[k + 2\lambda \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] \\
&\quad \times \exp\left(- \left[\alpha_1 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{\beta} \right] k - \left[\alpha_2 + \sum_{i=1}^{n} \left(\frac{G(x_i; \Theta)}{\overline{G}(x_i; \Theta)} \right)^{2\beta} \right] \lambda - \alpha_4 \Theta \right)
\end{aligned} \tag{72}$$

Applying the square error loss function, the Bayes estimators of $k, \lambda, \beta, \Theta$, survival function S(t), and HRF h(t), are derived as follows:

$$\begin{aligned}
\hat{k}_* &= \mathbb{E}(k|\,\mathcal{D}) = \int_\Theta k\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{\lambda}_* &= \mathbb{E}(\lambda|\,\mathcal{D}) = \int_\Theta \lambda\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{\beta}_* &= \mathbb{E}(\beta|\,\mathcal{D}) = \int_\Theta \beta\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{\Theta}_* &= \mathbb{E}(\Theta|\,\mathcal{D}) = \int_\Theta \Theta\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{S}_* &= \mathbb{E}(S(t;\Delta)|\,\mathcal{D}) = \int_\Theta S(t;\Delta)\pi(\Delta|\,\mathcal{D})d\Delta \\
\hat{h}_* &= \mathbb{E}(h(t;\Delta)|\,\mathcal{D}) = \int_\Theta h(t;\Delta)\pi(\Delta|\,\mathcal{D})d\Delta
\end{aligned} \tag{73}$$

Calculating the Bayes estimates via the posterior means may be infeasible with no closed-form solutions of the marginal posterior densities $\pi(k|\,\mathcal{D}), \pi(\lambda|\,\mathcal{D}), \pi(\beta|\,\mathcal{D})$, and $\pi(\Theta|\,\mathcal{D})$.
Hence, for a posterior sample $\Delta_s, s = 1, 2, ..., N$ generated from $\pi(\Delta|\,\mathcal{D})$, the approximate Bayes estimates of M-LFRD-G parameters $k, \lambda, \beta, \Theta$, as well as the reliability function S(t) and HRF h(t) are calculated as

$$\begin{aligned}
\hat{k}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} k_s \\
\hat{\lambda}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} \lambda_s \\
\hat{\beta}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} \beta_s \\
\hat{\Theta}_* &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} \Theta_s \\
\hat{S}_*(t) &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} S(t;\Delta_s) \\
\hat{h}_*(t) &\approx \frac{1}{N-\wp}\sum_{s=\wp+1}^{N} h(t;\Delta_s),
\end{aligned} \tag{74}$$

where $\wp$ represents the number of burn-in observations/ iterations prior to stationarity of the samples. It is highly advisable to run the simulation for m parallel chains (m = 3, 4 or 5) for better assessment of the sampler convergence. Therefore, we can proceed to compute the posterior means for m parallel chains as follows:

$$\hat{k}_* \approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} k_{(s,b)}$$

$$\hat{\lambda}_* \approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} \lambda_{(s,b)}$$

$$\hat{\beta}_* \approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} \beta_{(s,b)}$$

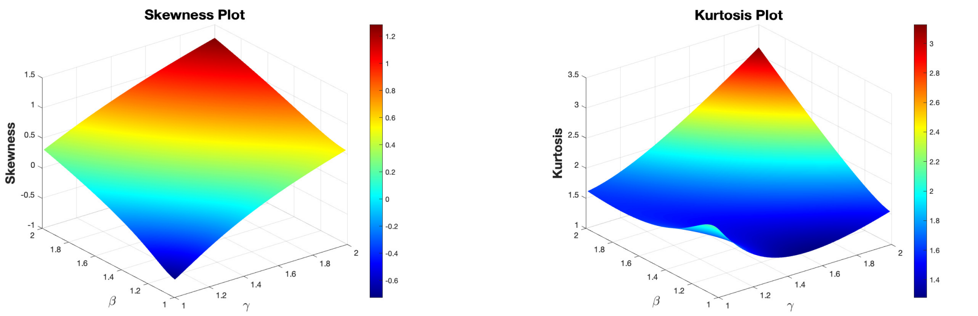
$$\hat{\Theta}_* \approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} \Theta_{(s,b)}$$

$$\hat{S}_*(t) \approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} S(t;\Delta_{(s,b)})$$

$$\hat{h}_*(t) \approx \frac{1}{N-\wp}\sum_{b=1}^{m}\sum_{s=\wp+1}^{N} h(t;\Delta_{(s,b)}), \tag{75}$$

4 Special Cases

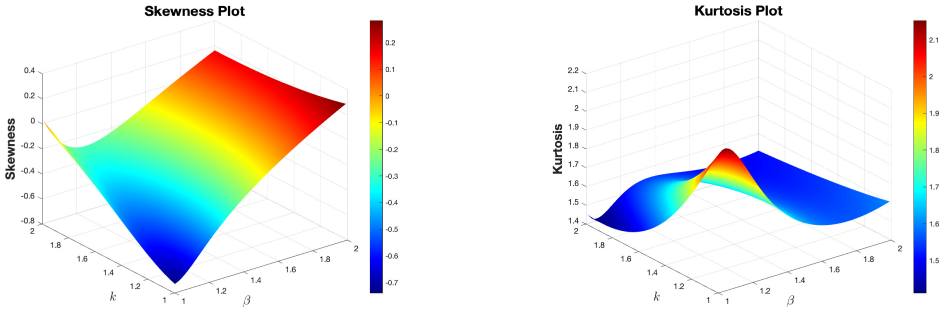
Below are the additional illustrations of the various skewness and kurtosis in pairs of parameters for M-LFRD-G.



(a) Skewness: β vs. γ

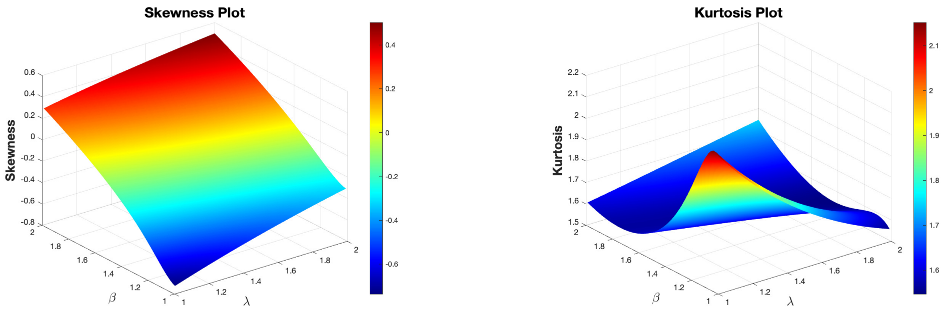
(b) Kurtosis: β vs. γ

Fig. 1: **LFRE: Comparison of Skewness and Kurtosis for β and γ parameters.**



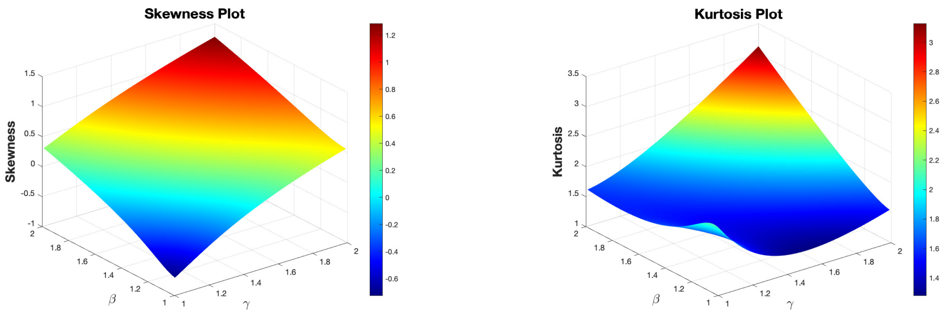
(a) Skewness: β vs. k (b) Kurtosis: β vs. k

Fig. 2: **LFRE: Comparison of Skewness and Kurtosis for β and k parameters.**



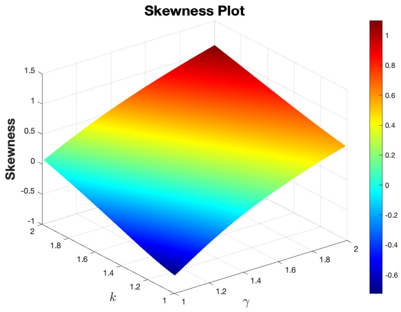
(a) Skewness: β vs. λ (b) Kurtosis: β vs. λ

Fig. 3: **LFRE: Comparison of Skewness and Kurtosis for β and λ parameters.**

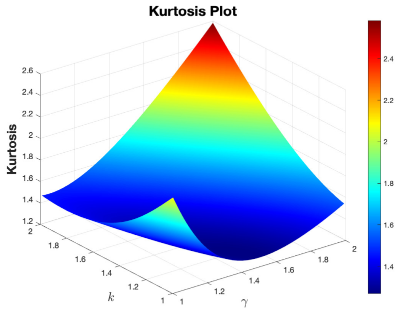


(a) Skewness: γ vs. β (b) Kurtosis: γ vs. β

Fig. 4: **LFRE: Comparison of Skewness and Kurtosis for γ and β parameters.**

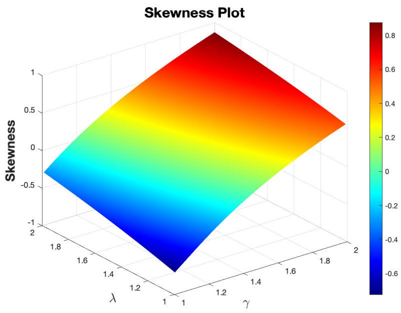


(a) Skewness: γ vs. k

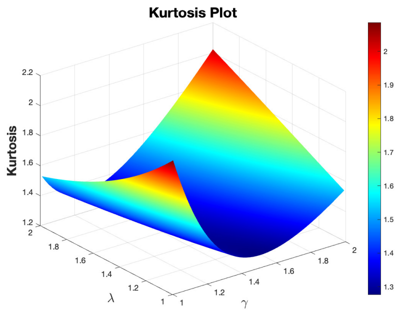


(b) Kurtosis: γ vs. k

Fig. 5: **LFRE: Comparison of Skewness and Kurtosis for γ and k parameters.**

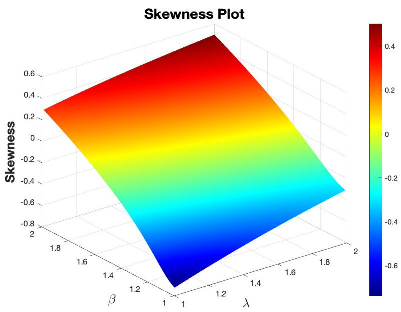


(a) Skewness: γ vs. λ

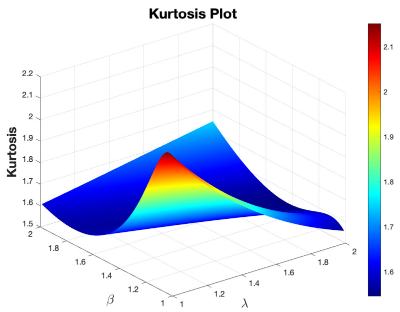


(b) Kurtosis: γ vs. λ

Fig. 6: **LFRE: Comparison of Skewness and Kurtosis for γ and λ parameters.**

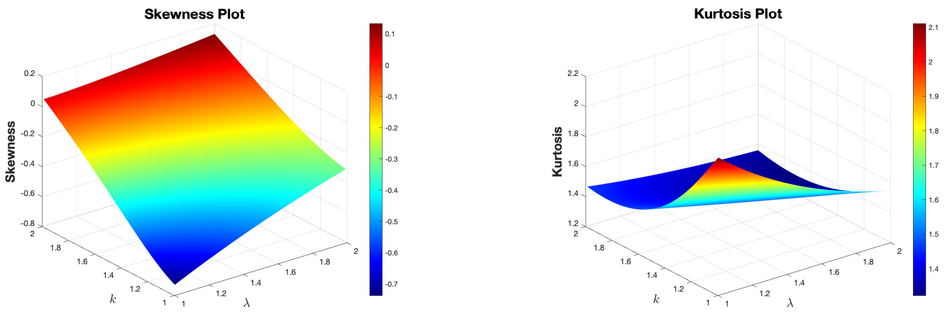


(a) Skewness: λ vs. β



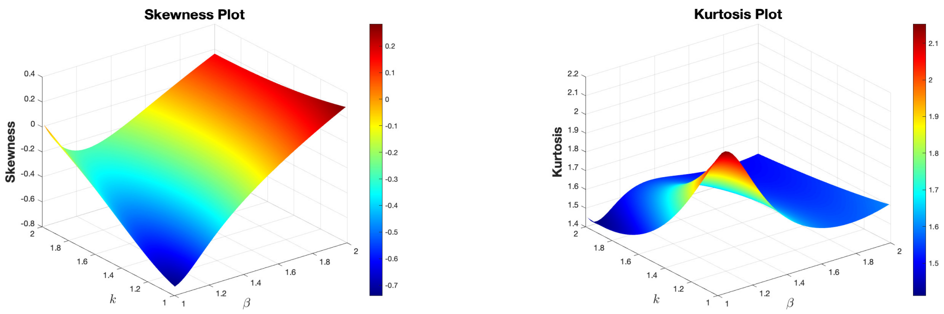
(b) Kurtosis: λ vs. β

Fig. 7: **LFRE: Comparison of Skewness and Kurtosis for λ and β parameters.**



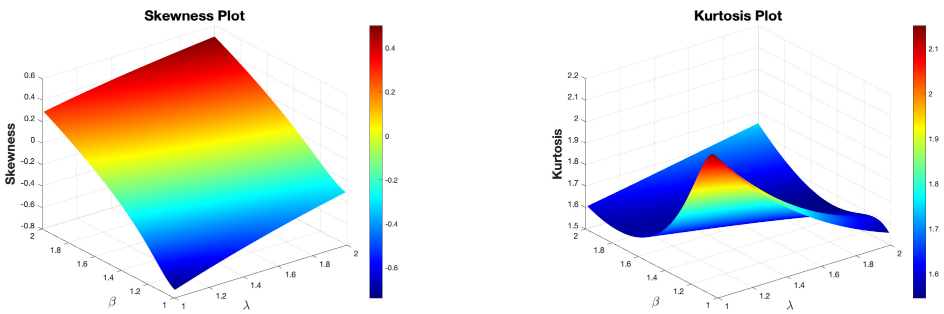
(a) Skewness: λ vs. k (b) Kurtosis: λ vs. k

Fig. 8: **LFRE: Comparison of Skewness and Kurtosis for λ and k parameters.**



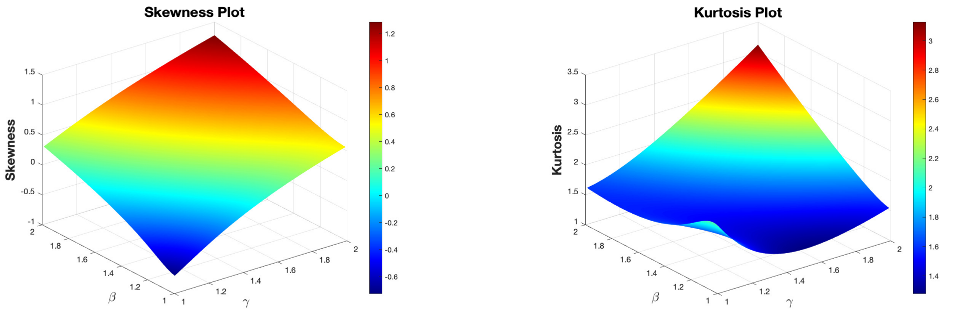
(a) Skewness: β vs. k (b) Kurtosis: β vs. k

Fig. 9: **LFRU: Comparison of Skewness and Kurtosis for β and k parameters.**



(a) Skewness: β vs. λ (b) Kurtosis: β vs. λ

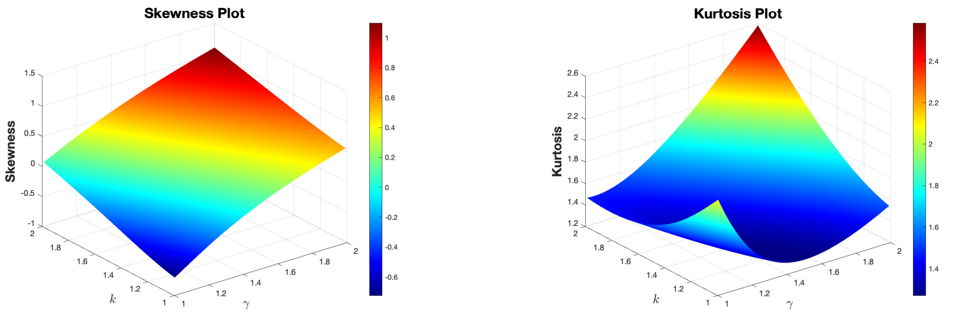
Fig. 10: **LFRU: Comparison of Skewness and Kurtosis for β and λ parameters.**



(a) Skewness: γ vs. β

(b) Kurtosis: γ vs. β

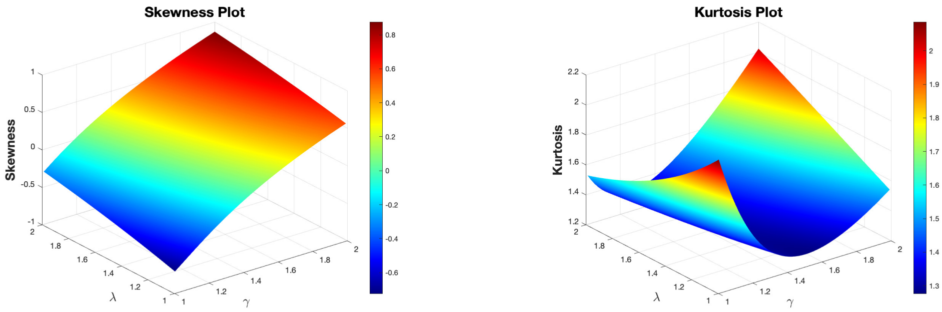
Fig. 11: **LFRU: Comparison of Skewness and Kurtosis for γ and β parameters.**



(a) Skewness: γ vs. k

(b) Kurtosis: γ vs. k

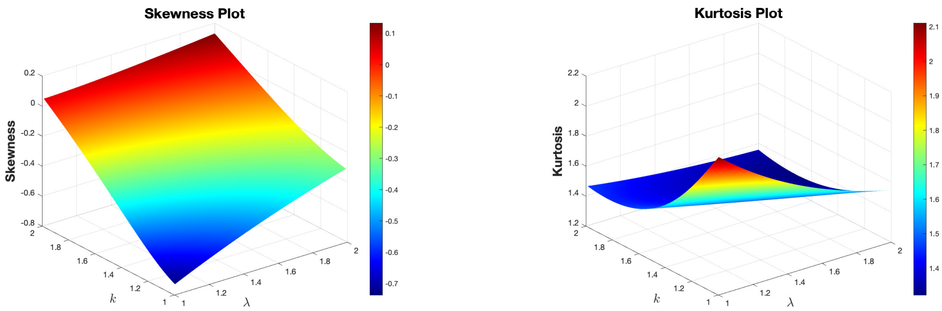
Fig. 12: **LFRU: Comparison of Skewness and Kurtosis for γ and k parameters.**



(a) Skewness: γ vs. λ

(b) Kurtosis: γ vs. λ

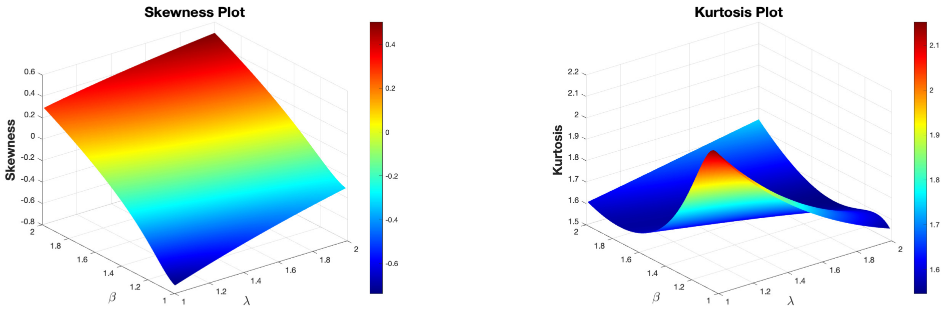
Fig. 13: **LFRU: Comparison of Skewness and Kurtosis for γ and λ parameters.**



(a) Skewness: k vs. λ

(b) Kurtosis: k vs. λ

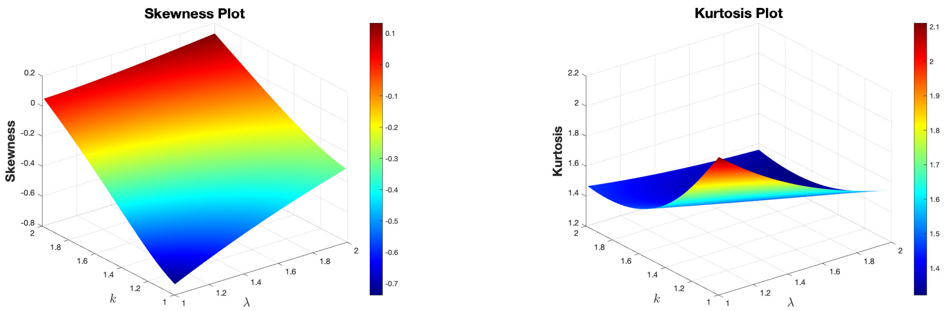
Fig. 14: **LFRU: Comparison of Skewness and Kurtosis for k and λ parameters.**



(a) Skewness: λ vs. β

(b) Kurtosis: λ vs. β

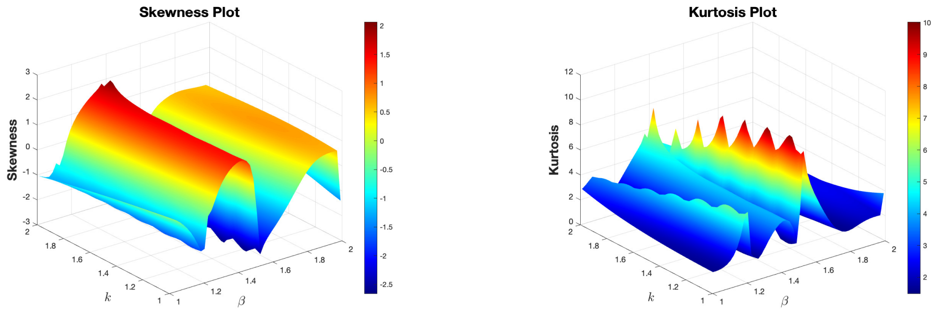
Fig. 15: **LFRU: Comparison of Skewness and Kurtosis for λ and β parameters.**



(a) Skewness: λ vs. k

(b) Kurtosis: λ vs. k

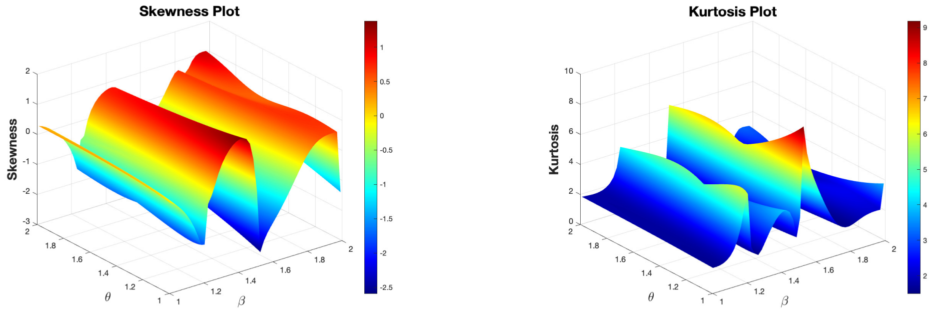
Fig. 16: **LFRU: Comparison of Skewness and Kurtosis for λ and k parameters.**



(a) Skewness: β vs. k

(b) Kurtosis: β vs. k

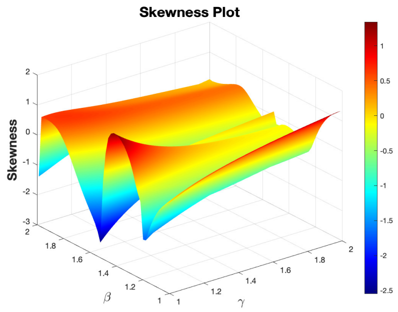
Fig. 17: **LFRP: Comparison of Skewness and Kurtosis for β and k parameters.**



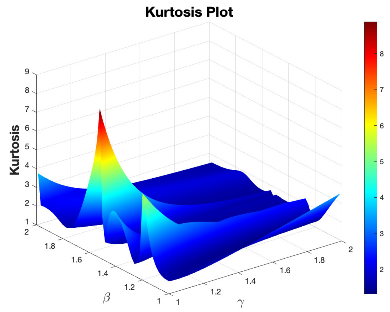
(a) Skewness: β vs. θ

(b) Kurtosis: β vs. θ

Fig. 18: **LFRP: Comparison of Skewness and Kurtosis for β and θ parameters.**

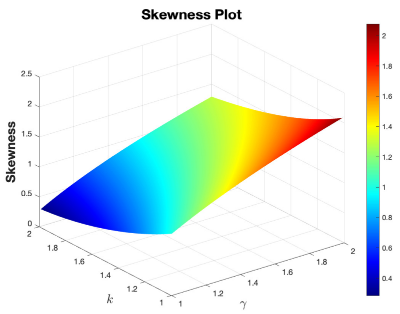


(a) Skewness: γ vs. β

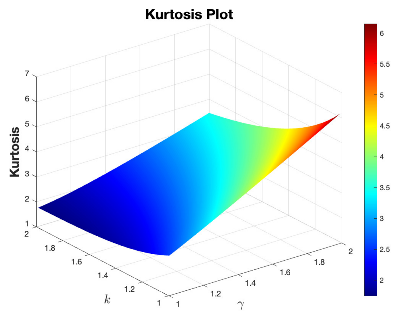


(b) Kurtosis: γ vs. β

Fig. 19: **LFRP: Comparison of Skewness and Kurtosis for γ and β parameters.**

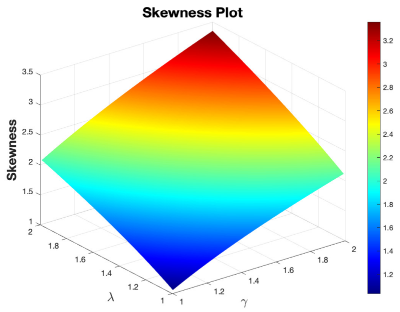


(a) Skewness: γ vs. k

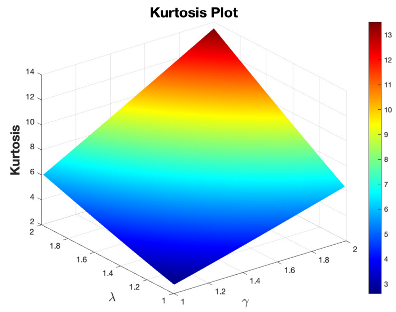


(b) Kurtosis: γ vs. k

Fig. 20: **LFRP: Comparison of Skewness and Kurtosis for γ and k parameters.**

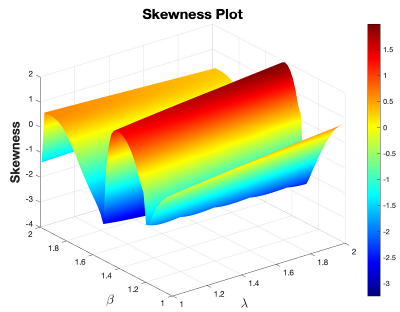


(a) Skewness: γ vs. λ

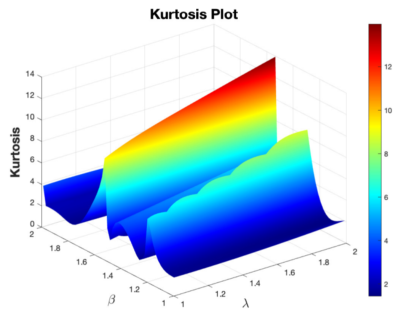


(b) Kurtosis: γ vs. λ

Fig. 21: **LFRP: Comparison of Skewness and Kurtosis for γ and λ parameters.**

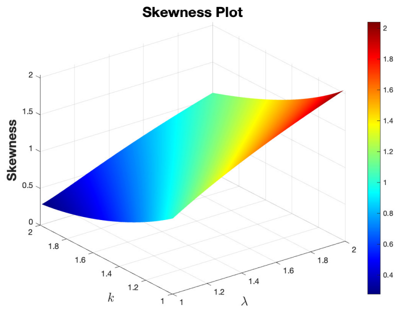


(a) Skewness: λ vs. β

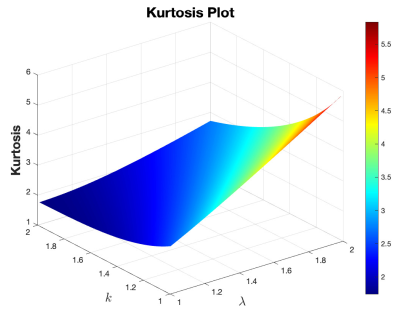


(b) Kurtosis: λ vs. β

Fig. 22: **LFRP: Comparison of Skewness and Kurtosis for λ and β parameters.**

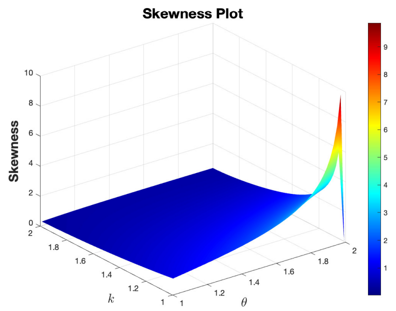


(a) Skewness: λ vs. k

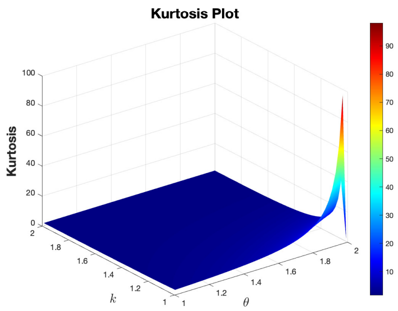


(b) Kurtosis: λ vs. k

Fig. 23: **LFRP: Comparison of Skewness and Kurtosis for λ and k parameters.**



(a) Skewness: θ vs. k



(b) Kurtosis: θ vs. k

Fig. 24: **LFRP: Comparison of Skewness and Kurtosis for θ and k parameters.**