SUPPLEMENTARY INFORMATION

ADVANCING RELIABILITY AND MEDICAL DATA ANALYSIS THROUGH NOVEL STATISTICAL DISTRIBUTION EXPLORATION

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1. Abbreviations

The following abbreviations are used in this paper:

RB-TL-EHL-G	Ristić-Balakrishnan-Topp-Leone-Exponentiated half Logistic-G
cdf	cumulative distribution function
pdf	probability density function
hrf	hazard rate function
Exp-G	exponentiated-G
PWMs	probability weighted moments
RB-TL-EHL-LLoG	RB-TL-EHL-Log-Logistic
RB-TL-EHL-W	RB-TL-EHL-Weibull
RB-TL-EHL-Lx	RB-TL-EHL-Lomax
MLEs	maximum likelihood estimates
Mean	mean estimate
ABias	average bias
RMSE	root mean square error
OEHL-BXII	odd exponentiated half-logistic-Burr XII
APExLLD	alpha power extended log-logistic
APTLW	power Topp-Leone Weibull
AIC	Akaike Information Criterion
CAIC	Consistent Akaike Information Criterion
BIC	Bayesian Information Criterion
W^*	Cramér-von Mises statistic
A^*	Anderson-Darling statistic
K-S	Kolmogorov-Smirnov statistic
ECDF	empirical cumulative distribution function
TTT	total time on test
K-M	Kaplan-Meier

2. Elements of Score Vector

The elements of the score vector U are given as follows:

$$\frac{\partial \ell_n}{\partial \sigma} = \frac{n(\Gamma'(\sigma))}{\Gamma(\sigma)} + \sum_{i=1}^{n} \ln\left(-\log\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b}\right),$$

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial b} &= \frac{n}{b} - (\sigma-1)\sum_{i=1}^{n} \frac{\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b} \ln\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]}{\left(-\log\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b}\right)\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b}} \\
&+ \sum_{i=1}^{n} \ln\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial a} &= \frac{n}{a} - 2b(\sigma-1)\sum_{i=1}^{n} \frac{\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b-1}\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)}{\left(-\log\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b}\right)\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b}} \\
&\times \left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi}\right)^{a} \ln\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right) - \sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a} \ln\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)}{\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)} \\
&+ 2(b-1)\sum_{i=1}^{n} \frac{\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}_i(x;\Psi)}\right)^{a}\right)\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a} \ln\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi}\right)}{\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]} + \ln\left[G(x_i;\Psi)\right] \\
&- \ln\left[1+\overline{G}(x_i;\Psi)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial \Psi_k} &= -4ba(\sigma-1)\sum_{i=1}^{n} \frac{\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b-1}\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)}{\left(-\log\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b}\right)\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]^{b}} \\
&\times \frac{\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a-1}\frac{\partial G(x_i;\Psi)}{\partial \Psi_k}}{\left[1+\overline{G}(x_i;\Psi)\right]^{2}} + 4a(b-1)\sum_{i=1}^{n} \frac{\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a-1}\frac{\partial G(x_i;\Psi)}{\partial \Psi_k}}{\left[1-\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)^{2}\right]\left[1+\overline{G}(x_i;\Psi)\right]^{2}} \\
&+ a\sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a-1}\frac{\partial G(x_i;\Psi)}{\partial \Psi_k}}{\left(1-\left(\frac{G(x_i;\Psi)}{1+\overline{G}(x_i;\Psi)}\right)^{a}\right)\left[1+\overline{G}(x_i;\Psi)\right]^{2}} + \sum_{i=1}^{n} \frac{\frac{\partial g(x_i;\Psi)}{\partial \Psi_k}}{[g(x_i;\Psi)]} + (a-1)\frac{\frac{\partial G(x_i;\Psi)}{\partial \Psi_k}}{[G(x_i;\Psi)]} \\
&- (a+1)\frac{\frac{\partial \overline{G}(x_i;\Psi)}{\partial \Psi_k}}{\left[1+\overline{G}(x_i;\Psi)\right]}.
\end{aligned}
$$

3. Some Probability Distributions

3.1. **The gamma exponentiated Lindley log-logistic (GELLLoG) Distribution.** The pdf of the gamma exponentiated Lindley log-logistic (GELLLoG) distribution is given by

$$
\begin{aligned}
f_{GELLLoG}(x;\lambda,c,\alpha,\delta) &= \frac{1}{\Gamma(\delta)}\left[-\log\left(1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha}\right)\right]^{\delta-1}\\
&\times \quad \alpha\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha-1}\\
&\times \quad \frac{(1+x^c)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right],
\end{aligned}
$$

for $\lambda, c, \alpha, \delta > 0$.

3.2. **The exponentiated half-logistic odd Weibull-Topp-Leone-log logistic (EHLOW-TL-LLoG) Distribution.** The pdf of the exponentiated half-logistic odd Weibull-Topp-Leone-log logistic (EHLOW-TL-LLoG) distribution is given by

$$
\begin{aligned}
f_{EHLOW-TL-BXII}(x;\alpha,\beta,\delta,\lambda,\gamma) &= \frac{4\alpha\beta\delta\lambda\gamma x^{\lambda-1}(1+x^{\lambda})^{-2\gamma-1}[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha\beta-1}}{(1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha})^{\beta+1}}\\
&\times \quad \exp(-t)\big(1+\exp(-t)\big)^{-2}\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta-1},
\end{aligned}
$$

where $t=\left[\frac{[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}{1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}\right]^{\beta}$, for $\alpha,\beta,\delta,\lambda,\gamma>0$ (We obtain the EHLOW-TL-LLoG distribution from the EHLOW-TL-BXII distribution by setting $\gamma = 1$).

3.3. **The odd exponentiated half-logistic- Burr XII (OEHL-BXII) Distribution.** The pdf of the odd exponentiated half-logistic- Burr XII (OEHL-BXII) distribution is given by

$$
f_{OEHLBXII}(x;\alpha,\lambda,a,b)=\frac{2\alpha\lambda abx^{a-1}\exp(\lambda[1-(1+x^a)^b])(1-\exp(\lambda[1-(1+x^a)^b]))^{\alpha-1}}{(1+x^a)^{-b-1}(1+\exp(\lambda[1-(1+x^a)^b]))^{\alpha+1}},
$$

for $\alpha,\lambda,a,b>0$,

3.4. **The alpha power extended log-logistic (APExLLD) Distribution.** The pdf of the alpha power extended log-logistic (APExLLD) distribution is given by

$$
f_{APExLLD}(x;\alpha,a,b,c) = \frac{ac\log(\alpha)\left(\frac{x}{b}\right)^{-a-1}}{b(\alpha-1)}\left[\left(\frac{x}{b}\right)^{-a}+1\right]^{-c-1}\alpha^{\left[\left(\frac{x}{b}\right)^{-a}+1\right]^{-c}},
$$

for $\alpha,a,b,c>0$ and $x>0$.

3.5. **The alpha power Topp-Leone Weibull (APTLW) Distribution.** The pdf of the alpha power Topp-Leone Weibull (APTLW) distribution is given by

$$
\begin{aligned}
f_{APTLW}(x;\theta,\alpha,\beta,\lambda) &= \frac{2\beta\theta\lambda\log(\alpha)x^{\beta-1}\exp\left(-2\lambda x^{\beta}\right)}{\alpha-1}\left(1-\exp\left(-2\lambda x^{\beta}\right)\right)^{\theta-1}\\
&\times \quad \alpha^{\left(1-\exp\left(-2\lambda x^{\beta}\right)\right)^{\theta}},
\end{aligned}
$$

for $\theta,\alpha,\beta,\lambda>0$.

4. Data Used in the paper

4.1. **Eruptions Data.** This data set was reported by Professor Jim Irish, and can be accessed at `http://www.statsci.org/data/oz/kiama.html`. Regarding the Kiama Blowhole eruptions, the following data is provided:

83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

4.2. **Remission Times Data.** This data is from [1], and it is about the remission times (in months) of 128 patients suffering from bladder cancer. The dataset values are as follows:
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

4.3. **Active Repair Times Data.** The data set was reported by [2], and it represents active repair times for airborne communication transceivers. The observations are
0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

References

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[2] Bent Jorgensen. *Statistical properties of the generalized inverse Gaussian distribution*, volume 9. Springer Science & Business Media, 2012.

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