Tapl notes

suspen

2026 - 01 - 14

Contents

1 untyped system

1.1 untyped Lambad-Calculus

Exercise I:

5.2.2 :

$$\begin{aligned} \text{plus} &:= \lambda m.\lambda n.\lambda s.\lambda z.\ m\ s\ (n\ s\ z) \\ \text{suc} &:= \lambda n.\lambda s.\lambda z.\ s\ (n\ s\ z) \\ \text{suc}' &:= \lambda n.\lambda s.\lambda z.\ \text{plus}\ n\ (\lambda s.\lambda z.\ s\ z) \end{aligned}$$

5.2.3 : Is it possible to define multiplication on Church numerals without using plus?

$$\text{mul} := \lambda m.\lambda n.\lambda s.\lambda z.\ m\ (n\ s)\ z$$

5.2.4 : Define a term for raising one number to the power of another.

$$\text{pow} := \lambda m.\lambda n.\lambda s.\lambda z.\ n\ (\text{mul}\ m)\ (\lambda s.\lambda z.\ s\ z)$$

5.2.5 : Use prd to define a subtraction function.

$$\begin{aligned} \text{pair} &:= \lambda f.\lambda s.\lambda b.bfs \\ \text{fst} &:= \lambda p.p\ \text{tru} \\ \text{snd} &:= \lambda p.p\ \text{fls} \\ \text{zz} &:= \text{pair}\ c_0 c_0 \\ \text{ss} &:= \lambda p.\ \text{pair}\ (\text{snd}\ p)\ (\text{plus}\ c_1\ (\text{snd}\ p)) \\ \text{prd} &:= \lambda m.\ \text{fst}\ (m\ \text{ss}\ \text{zz}) \\ \text{sub} &:= \lambda m.\lambda n.\ n\ (\text{prd}\ m)\ z \end{aligned}$$

5.2.7 : Write a function equal that tests two numbers for equality and returns a Church boolean. For example,

1.2 de Bruijn index

Exercise II:

6.1.1

$$c_0 := \lambda s.\lambda z.\ z \overset{\text{nameless}}{\Longrightarrow} \lambda.\lambda.\ 0$$

$$c_2 := \lambda s.\lambda z.\ s\ (s\ z) \overset{\text{nameless}}{\Longrightarrow} \lambda.\lambda.\ 1\ (1\ 0)$$

$$\text{plus} := \lambda m.\lambda n.\lambda s.\lambda z.\ m\ s\ (n\ z\ s) \overset{\text{nameless}}{\Longrightarrow} \lambda.\lambda.\lambda.\lambda.\ 3\ 1\ (2\ 0\ 1)$$

$$\text{fix} := \lambda f.(\lambda x.\ f\ (\lambda y.\ (x\ x)\ y))\ (\lambda x.\ f(\lambda y.\ (x\ x)\ y)) \overset{\text{nameless}}{\Longrightarrow} \lambda.(\lambda.1\ (\lambda.\ (1\ 1)\ 0))\ (\lambda.\ 1\ (\lambda.\ (1\ 1)\ 0))$$

6.14

$$
\begin{aligned}
&\texttt{remove}_\Gamma(\texttt{t}) := \\
&\quad \mid \texttt{x} \quad\quad\ \Rightarrow \texttt{index}_\Gamma(\texttt{x}) \\
&\quad \mid \lambda\ \texttt{x . t} \Rightarrow \lambda.\texttt{remove}_{\texttt{x} :: \Gamma}(\texttt{t}) \\
&\quad \mid \texttt{t1 t2} \ \Rightarrow \texttt{remove}_\Gamma(\texttt{t1})\ \texttt{remove}_\Gamma(\texttt{t2})
\end{aligned}
$$

newname := choose the first variable name that is not already in dom(Γ).

$$
\begin{aligned}
&\texttt{restore}_\Gamma(\texttt{t}) := \\
&\quad \mid \texttt{i} \quad\quad \Rightarrow \Gamma\texttt{[i]} \\
&\quad \mid \lambda.\ \texttt{t} \quad \Rightarrow \lambda\texttt{newname}_{\mathrm{dom}(\Gamma)}.\texttt{restore}_\Gamma(\texttt{t}) \\
&\quad \mid \texttt{t1 t2} \Rightarrow \texttt{restore}_\Gamma(\texttt{t1})\ \texttt{restore}_\Gamma(\texttt{t2})
\end{aligned}
$$

6.2.2

$$\uparrow^2 (\lambda.\lambda.\ 1\ (0\ 2)) = \lambda.\lambda\ 1\ (0\ 4)$$

$$\uparrow^2 (\lambda.\ 0\ 1\ (\lambda.\ 0\ 1\ 2)) = \lambda.\ 0\ 3\ (\lambda.\ 0\ 1\ 4)$$

Definition 1.1 shifting :

The d-palce shift of a term t above cutoff c, written $\uparrow_c^d (t)$, is defined as follows:

$$\uparrow_c^d (k) = \begin{cases} k, k < c \\ k + d, k \geq c \end{cases}$$

$$\uparrow_c^d (\lambda.t) = \lambda.\uparrow_{c+1}^d (t)$$

$$\uparrow_c^d (t_1\ t_2) = \uparrow_c^d (t_1)\ \uparrow_c^d (t_2)$$

We write $\uparrow^d (t)$ for $\uparrow_0^d (t)$.

Definition 1.2 substitution : The substitution of a term s for variable number j in a term t, written $[j \mapsto s]t$, is defined as follows:

$$[j \mapsto s]k = \begin{cases} s, k = j \\ k, \text{otherwise} \end{cases}$$

$$[j \mapsto s](\lambda.\, t_1) = \lambda.\, [j + 1 \mapsto \uparrow^1(s)]t_1$$

$$[j \mapsto s](t_1\ t_2) = ([j \mapsto s]t_1\ [j \mapsto s]t_2)$$

when evaluate the application $(\lambda.M)\ v$, the substitution has little different, is defined as follows:

$$(\lambda.M)\ v = \uparrow^{-1} ([0 \mapsto \uparrow^1 (v)]M)$$