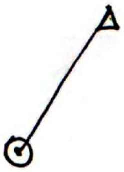


①

忽略动量轮产生的扭矩 τ，有

$$Mgl\sin\theta = J\ddot{\theta}$$

又 $\tau = J_r\ddot{\theta}_r$，则

$$Mgl\sin\theta - J_r\ddot{\theta}_r = J\ddot{\theta}$$

动量轮的扭矩由电机提供，即

$$kI = J_r\ddot{\theta}_r$$

（k为扭矩系数，I为电流大小）

故综上：

$$Mgl\sin\theta - kI = J\ddot{\theta}$$

②

1) K_E

$$\begin{cases} KE_x = \frac{1}{2}m\left(l\cdot\frac{d}{dt}\sin\theta\right)^2 = \frac{1}{2}m\,l^2(\dot{\theta}\cos\theta)^2 \\ KE_y = \frac{1}{2}m\left(l\cdot\frac{d}{dt}\cos\theta\right)^2 = \frac{1}{2}m\,l^2(\dot{\theta}\sin\theta)^2 \\ KE = KE_x + KE_y = \frac{1}{2}ml^2\dot{\theta}^2 \end{cases}$$

$$\underbrace{KE_{translational}}_{\text{平动}} = \frac{1}{2}\underbrace{m_1(l_1\dot{\theta}_1)^2}_{\text{摆杆}} + \frac{1}{2}\underbrace{m_2(l_2\dot{\theta}_1)^2}_{\text{动量轮}}$$

$$\underbrace{KE_{inertial}}_{\text{惯性(转动)}} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2(\dot{\theta}_1+\dot{\theta}_2)^2$$

2) $PE = m_1g(-l_1\cos\theta_1) + m_2g(-l_2\cos\theta_1)$

3) 拉格朗日量 $L = KE - PE$，令 $q_1=\theta_1, q_2=\theta_2$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} -m_1gl_1\sin\theta_1 - m_2gl_2\sin\theta_1 \\ 0 \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} I_1\dot{\theta}_1 + I_2(\dot{\theta}_1+\dot{\theta}_2) + m_1l_1^2\dot{\theta}_1 + m_2l_2^2\dot{\theta}_1 \\ I_2(\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} I_2\ddot{\theta}_2 + \ddot{\theta}_1(I_1+I_2+m_1l_1^2+m_2l_2^2) \\ I_2\ddot{\theta}_1 + I_2\ddot{\theta}_2 \end{bmatrix}$$

又 $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \sum$ 外τ（拉格朗日方程

（简化）$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \Rightarrow \begin{cases} I_2\ddot{\theta}_2 + \ddot{\theta}_1(I_1+I_2+m_1l_1^2+m_2l_2^2) = -m_1gl_1\sin\theta_1 - m_2gl_2\sin\theta_1 \\ I_2(\ddot{\theta}_1+\ddot{\theta}_2) = 0 \quad (\tau) \; ? \end{cases}$

这里暂时忽略输入力矩

$$\Rightarrow \begin{cases} \ddot{\theta}_1 = -g\dfrac{(m_1l_1+m_2l_2)\sin\theta_1}{I_1+m_1l_1^2+m_2l_2^2} \\ \ddot{\theta}_2 = g\dfrac{(m_1l_1+m_2l_2)\sin\theta_1}{I_1+m_1l_1^2+m_2l_2^2} \end{cases}$$

加入输入力矩 τ

$$\begin{bmatrix} m_1 l_1^2 + m_2 l_2^2 + I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + 0 = \begin{bmatrix} -(m_1 l_1 + m_2 l_2) g \sin\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau$$

在倒立平衡点做近似化　$\sin\theta \approx \pi - \theta$　$(\theta \approx \pi)$，有

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(m_1 l_1 + m_2 l_2) g}{(m_1 l_1^2 + m_2 l_2^2 + I_1)} & 0 & 0 & 0 \\ -\frac{(m_1 l_1 + m_2 l_2) g}{m_1 l_1^2 + m_2 l_2^2 + I_1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 - 180 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{(m_1 l_1^2 + m_2 l_2^2 + I_1)} \\ \frac{1}{I_2} + \frac{1}{(m_1 l_1^2 + m_2 l_2^2 + I_1)} \end{bmatrix} [\tau]$$

▷ 测量杆质量 m_1，动量轮质量 m_2（带电机），设若 $m_2 >> m_1$，则 $l_1 = l_2 = l_{stick}$，再求得 I_1、I_2

两个矩阵便可以列出

摆杆质量m，质心位置l

反应轮 M， L

摆杆相对于正法线夹角θ_1

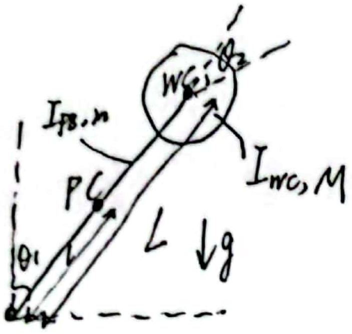
反应轮相对于摆杆夹角θ_2

(3)

应用拉格朗日法：

势能 $V_P = mlg\cos\theta_1$

$V_W = MLg\cos\theta_1$ $\Rightarrow V = mlg\cos\theta_1 + MLg\cos\theta_1$



动能

$$\begin{cases} T_{PT}\ (\text{平动}) = \frac{1}{2}(ml^2)\dot\theta_1^2 \\ T_{PR}\ (\text{转动}) = \frac{1}{2}I_{PC}\dot\theta_1^2 \end{cases} \qquad \begin{cases} T_{WT} = \frac{1}{2}(ML^2)\dot\theta_1^2 \\ T_{WR} = \frac{1}{2}I_{WC}(\dot\theta_1+\dot\theta_2)^2 \end{cases}$$

$I_{PB} = I_{PC} + ml^2$

$$\Rightarrow T = \frac{1}{2}I_{WC}(\dot\theta_1+\dot\theta_2)^2 + \frac{1}{2}(ML^2)\dot\theta_1^2 + \frac{1}{2}I_{PB}\dot\theta_1^2$$

拉格朗日量 $L = T - V = \frac{1}{2}I_{WC}(\dot\theta_1+\dot\theta_2)^2 + \frac{1}{2}(ML^2)\dot\theta_1^2 + \frac{1}{2}I_{PB}\dot\theta_1^2 - mlg\cos\theta_1 - MLg\cos\theta_1$

$$= \frac{1}{2}I_{WC}(\dot\theta_1+\dot\theta_2)^2 + \frac{1}{2}a\dot\theta_1^2 - bg\cos\theta_1 \qquad \begin{cases} a = ML^2 + I_{PB} \\ b = \cancel{(m+M)L}\ ml + ML \end{cases}$$

$$= \frac{1}{2}I_{WC}\dot\theta_1^2 + \frac{1}{2}I_{WC}\dot\theta_2^2 + I_{WC}(\dot\theta_1\dot\theta_2) + \frac{1}{2}a\dot\theta_1^2 - bg\cos\theta_1$$

有 $\frac{\partial L}{\partial \theta_1} = bg\sin\theta_1$ ， $\frac{\partial L}{\partial \dot\theta_1} = I_{WC}\dot\theta_1 + I_{WC}\dot\theta_2 + a\dot\theta_1$

$\frac{\partial L}{\partial \theta_2} = 0$ ， $\frac{\partial L}{\partial \dot\theta_2} = I_{WC}\dot\theta_2 + I_{WC}\dot\theta_1$

计算扭矩值：
(拉格朗日力学方程)

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \Rightarrow I_{WC}\ddot\theta_1 + I_{WC}\ddot\theta_2 + a\ddot\theta_1 - bg\sin\theta_1 = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = \tau \Rightarrow I_{WC}\ddot\theta_2 + I_{WC}\ddot\theta_1 = \tau \end{cases}$$

线性化 $\Rightarrow$

$$\begin{cases} a\ddot\theta_1 + I_{WC}\ddot\theta_1 + I_{WC}\ddot\theta_2 - bg\theta_1 = 0 \\ I_{WC}\ddot\theta_1 + I_{WC}\ddot\theta_2 - 0 = \tau \end{cases} \Rightarrow \begin{cases} \ddot\theta_1 = \dfrac{(ml+ML)g\theta_1 - \tau}{ML^2 + I_{PB}} \\ \ddot\theta_2 = -\dfrac{(ml+ML)g\theta_1 - \tau}{ML^2 + I_{PB}} + \dfrac{\tau}{I_{WC}} \end{cases}$$