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Global Stabilization of the Unstable Reaction-wheel Pendulum

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Abstract—The paper deals with the problem of the Reaction Wheel Pendulum stabilization about unstable (inverted) position for arbitrary initial conditions. Considered mechanical system consists of a physical pendulum with a symmetric disk attached to the end of the pendulum, which is free to spin about an axis parallel to the axis of rotation of the pendulum. The disk is actuated by a DC-motor. The coupling torque generated by the angular acceleration of the disk is used to control of the pendulum. The switching control law is proposed to swinging up the pendulum and balancing it about the inverted position. The nonlinear swinging up control law is proposed ensuring global stabilization of the pendulum about inverted position. The Energy-based Speed-gradient (EBSG) control scheme is used to designing the swinging-up controller. The modification of the EBSG method is proposed to ensure attainability of the inverted position of the pendulum for all initial states of the system. The balance controller is designed on the basis of the Variable Structure Control with forced sliding mode. Numerical simulation results are presented showing achievement of the posed control goal by means of the control action of small magnitude.

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1. INTRODUCTION

The problems of control for oscillatory mechanical systems have significant theoretical value and increasing interest for practice. Different kind of laboratory equipment, learning tools, the algorithmic and software support are developing worldwide for research and educational purposes in the fields of mechatronics and automatic control. Among others, laboratory pendulum set-ups are widespread for the aims of research and education. The pendulum systems are characterized by highly nonlinear dynamics, which allows explore and demonstrate the rather complex nonlinear phenomena such as instability, chaos, synchronization, the Sommerfeld effect, reveals the need for application of modern control design methods, such as Lyapunov and energy-based control, neural networks and fuzzy systems etc., see [2, 15, 16, 19, 21, 22, 25].

The Reaction-wheel Pendulum is a physical pendulum with a symmetric disk attached to its end. The disc is free to spin about an axis parallel to the axis of rotation of the pendulum. The disk is actuated by a DC-motor and the torque generated by the angular acceleration of the disk, can be used to actively control of the system. Motion control of the pendulum is fulfilled by means of changes in the direction and rotation speed of the inertia flywheel. In its turn, the flywheel is swinging by changing the rotation of the drive motor under the control voltage, applying to the armature circuit. This voltage is controlled in accordance with commands from the controller, being bounded in absolute value. The present paper is devoted to the following problem of the Reaction-wheel pendulum control: lifting the pendulum up to the certain neighborhood of the raised (unstable) position and stabilization the pendulum in the vicinity of this position for arbitrary initial conditions and the bounded control action. Typically, initially the pendulum is at the lower equilibrium point and the flywheel is not spinning.

Fig. 1. Photo of the Reaction-wheel Pendulum.

Such a mechanical system is described in several papers, e.g., see [6, 7, 21, 24, 25], and is a part of a laboratory equipment of some Russian and foreign universities. As an example, Fig. 1 shows a photograph of a pendulum with a flywheel, manufactured by the *Mechatronics Systems Inc.* This set-up is installed at the Teaching and Research Complex "Manipulators and Pendulum Systems" of the Saint Petersburg Interuniversity Academic shared center "Mechatronic and Mobile Complexes." The other example is the pendulum laboratory set-up of the Research Institute of Mechanics of Moscow State University, described in [6, 7].

In [24] two design methods for control (swinging up and stabilization) of the Reaction-wheel pendulum are considered: the *feedback linearization* and the *passification-based* methods.[1] It is shown, that the considered system is locally feedback linearizable by means of a certain local diffeomorphism in the state space and a nonlinear feedback. To stabilize unstable state of the pendulum, a modal control technique is applied to the linearized model. For swinging the pendulum up, the *energy-based approach* is used. This approach is combined in [24] with the partial feedback linearization technique, ensuring passivity of the zero-dynamics for the closed-loop system. For switching between the pendulum swinging up and stabilization, the appropriate switching algorithm is used.

The similar pendulum system is used in [7] as a device for testing the stabilization algorithms for unstable plants in the case of controls, bounded in the absolute value. In [7], the algorithms for stabilization of the unstable (inverted) equilibrium position of the pendulum are designed and experimentally tested for "small" (about $20^{\rm o}$) initial deviations from the desired position. The method of [7] is applicable to the linear plant models with one positive pole and the negative real parts of the other ones. This method ensures the maximum possible attraction domain of the equilibrium under the given control constraints. The control law of [7] has a form of a saturated linear feedback. It is shown in [7], that this type of control ensures stabilization of the equilibrium for all initial conditions, under which it is possible in principle (in this sense, the authors of [7] are pointing out to the "optimality" of the proposed control), but the computation costs for control of [7] are significantly less than the computation costs for the time-optimal control. The method of [7] can be applied for various problems of unstable plants stabilization, for example, for control of multi-stage launch vehicles at the initial phase of the each stage operation, when the angular perturbations are especially large.

[1] Description of these methods can be found in [8].

In the present paper the problem of the pendulum stabilization in the upper position for *arbitrary* initial conditions is considered. As well as in [24], a switching algorithm is used. The algorithm solves the following particular subtasks: putting the pendulum to the neighborhood of the desired (inverted) equilibrium point and balancing around this position. For achieving the first goal, the Speed-gradient algorithm is employed [12, 13], where the energy-based objective function [2, 4, 11] is chosen. To achieve the second objective, the sliding mode control is used. Additionally, as opposed to cite Grishin02, the posed problem is solved for *arbitrary* initial conditions. The control method of this paper was partially presented in [1].

The paper is organized as follows. Design and the mathematical model of the Reaction-wheel pendulum are presented in Section 2. Section 3 is devoted to solving the subproblem–1: swinging up the pendulum to the prescribed neighborhood of the desired position. The subproblem–2, balancing the pendulum in the upright position, is considered in Section 4. In Section 5 the switching algorithm is described and the numerical simulation results for pendulum control by means of this algorithm are presented. In this Section, the control problem under a priori uncertainty of the plant model parameters is also considered. To solve this problem as applied to system of nonlinear differential equations which are not resolved with respect the highest derivative, the identification algorithm with the Implicit Tunable Model is proposed and numerically studied.

2. DESIGN AND THE MATHEMATICAL MODEL OF THE REACTION-WHEEL PENDULUM

Following [7], let us briefly describe the design of the Reaction-wheel Pendulum of the Research Institute of Mechanics of the MSU and present its mathematical model.

The single-pendulum with the mounted flywheel is considered, see Figs. 1 and 2. The pendulum is free to rotate in the vertical plane. Its pivot pin lies in a horizontal plane and is mounted on a stationary base. The pivot pin of the flywheel is attached to the pendulum. The axes of rotation of the pendulum and the flywheel are parallel each other. The flywheel is driven by the DC electric motor which is mounted coaxially with the flywheel on the pendulum. Control of the motor is performed through the interface device from the PC. Measuring rotation angles of the pendulum and the flywheel is provided by the optical encoder.

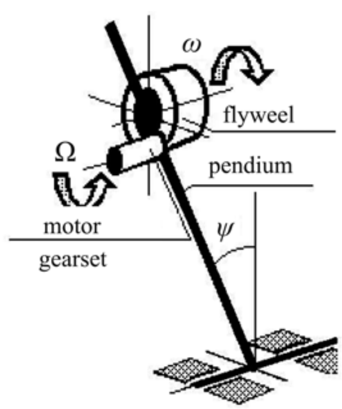

Fig. 2. Schematics of the pendulum setup.

Model of the pendulum with the flywheel dynamics is given in [7] as follows:

$$\begin{cases} J\chi\ddot{\psi} + (J_r + \chi J_m)\dot{\omega} = (M\,b + m\,h)g\chi \sin\psi \\ (J_r + \chi J_m)\chi\ddot{\psi} + (J_r + \chi^2 J_m)\dot{\omega} = \chi T, \end{cases} \tag{1}$$

where ψ is the pendulum rotation angle; ω denotes the angular velocity of the flywheel with respect to the pendulum; J_m is the moment of inertia of the flywheel about its main axis (the axis of rotation); J_r stands for the principal moment of inertia of the motor's rotor; $J = J_v + J_r + J_m + mh^2$ denotes the total moment of inertia of the system pendulum-flywheel-rotor, where J_v stands for the moment of inertia of the pendulum about its axis of rotation; g is the acceleration of gravity; M, m denote masses of the pendulum and the motor (respectively); b, h are the distance from the axes of rotation to the centers of mass of the pendulum and the flywheel; T is the electromagnetic torque, applied to the rotor; χ is the gearbox transfer ratio, $\omega = \chi\Omega$, where Ω is the rotational velocity of the rotor. Taking into account the back-emf reaction of the armature, the torque T can be approximated (neglecting the electromagnetic time constant) in the following form

$$T = c_1 u - c_2 \omega \chi^{-1}, \tag{2}$$

where u is the control voltage, applied to the motor armature; c_1 and c_2 denote the motor parameters.

Equations (1) and (2) describe the nonlinear dynamical system of the third-order with the state variables $\{\psi, \dot{\psi}, \omega\}$ and the control action $u(t)$. The vertical (unstable) equilibrium position corresponds to the value of $\psi^* = 0$ (or, more accurate, $\psi^* = \pm 2\pi n$, $n = 0, 1, \dots$).

It is easy to see that the linearization of (1), (2) with respect to the upper equilibrium point leads to an unstable third-order linear model with one positive real eigenvalue. It is obvious that linearized system can not be stabilized by means of the bounded control action for arbitrary initial conditions.

Specificity of the nonlinear dynamics of the pendulum allows, however, ensuring such a global stabilization. For this aim let us use the described below algorithms for solving particular subproblems, which are running under the supervision of the switching control law.

3. ALGORITHM FOR BRINGING THE PENDULUM INTO AN UPRIGHT POSITION

Consider the problem of bringing the pendulum into the neighborhood of the desired equilibrium position for arbitrary initial conditions, demanding simultaneously, that the pendulum angular velocity at the upper point should be close to zero. For the design the algorithm, let us use the *Speed-gradient method* taking the energy-based objective function. This method is briefly described below.

3.1. Speed-gradient Method with Energy-based Objective Function

Most of the problems arising in the nonlinear control systems design relate to the aims of stabilization or tracking. In such tasks the control objective function can be expressed in terms of deviation between current object state $x(t)$ and some given (e.g., by means of the reference model) trajectory $x_*(t)$. In present, the growing interest is observed to other control problems such as control of chaotic and periodic oscillations that are not leads directly to the habitual stabilization or tracking problems. A typical example is the problem of swinging the pendulum [14, 15, 20, 26]. For solving the problems of a such kind, the *energy approach* has been applied [20]. In the row of papers [2, 4, 11, 17, 18] this approach has been further extended and generalized by virtue of combination with the *Speed-gradient* method [12, 13]. Let us briefly recall its main principles.

Let the plant be modeled in the following state-space form

$$\dot{x} = F(x, u, t), \quad t \geq 0, \tag{3}$$

where $x(t) \in \mathbb{R}^n$ stands for the plant state vector, $u(t) \in \mathbb{R}^m$ is the vector of input (control) variables, $F(\cdot) : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is the continuously differentiable in x vector-function. The problem is to find the control law $u(t) = U\{x(s), u(s) : 0 \le s \le t\}$ ensuring fulfillment of the following control goal:

$$Q_t \to 0 \quad \text{as } t \to \infty, \tag{4}$$

where Q_t is the given goal functional $Q_t = Q(x(s), u(s) : 0 \le s \le t)$.

For designing the Speed-gradient (SG) algorithm for the local cost functional $Q_t = Q(x(t), t)$, which is described by means of a smooth scalar objective function $Q(x, t) \ge 0$, the function $\omega(x, u, t)$, representing the speed of change of Q_t along the system (3) trajectories is used. It is easy to see that the following expression $\omega(x, u, t) = (\nabla_x Q)^{\mathrm{T}} F(x, u, t)$ is valid. In accordance with the SG method [12, 13], the control signal must be changed along the antigradient on u of the function $\omega(x, u, t)$. The *combined* SG-algorithm, merging the integral and signal components has the following form [5, 13]:

$$\frac{d}{dt}(u + \psi(x, u, t)) = -\Gamma \nabla_u \omega(x, u, t), \tag{5}$$

where $\psi(\cdot)$ is a certain function, satisfying the following *pseudogradient condition*: $\psi^{\mathrm{T}} \nabla_u \omega \ge 0$, Γ is the symmetric nonnegative definite $(m \times m)$-matrix (the *matrix gain*, $\Gamma = \Gamma^{\mathrm{T}} \ge 0$). The main particular forms of the algorithm (5) are the SG-algorithm in the *differential form* [12, 13]:

$$\dot{u}(t) = -\Gamma \nabla_u \omega(x, u, t), \quad \Gamma = \Gamma^{\mathrm{T}} > 0, \tag{6}$$

and the SG-algorithm in the *finite form*

$$u = \psi(x, u, t), \tag{7}$$

which, in turn, yields to the following “proportional” and relay algorithms:

$$u = -\Gamma \nabla_u \omega(x, u, t), \quad \Gamma = \Gamma^{\mathrm{T}} > 0, \tag{8}$$

$$u = -\Gamma_1 \operatorname{sgn}(\nabla_u \omega(x, u, t)), \quad \Gamma_1 = \operatorname{diag}\{\gamma_i\}, \ \gamma_i > 0 \tag{9}$$

(the function $\operatorname{sgn}(z)$ of vector z is understood elementwise).

The basic idea, underlying in the algorithms (5)–(9), lies in decreasing the function $\dot{Q}$ along the closed-loop system trajectories. As a result, at sufficiently large t and under a number of additional conditions, the inequality $\dot{Q} < 0$ is valid and the objective function Q_t decreases.

Let us combine the SG-method with the energy approach, taking the deviation of the controlled process energy from the specified value as the objective function $Q(x, t)$. Following [4, 11, 17], let us demonstrate the application of this method for control problems energy oscillations of conservative systems.

Conservative models arise in many technical applications when the viscous or Coulomb friction in system are negligible. An ideal rotor (flywheel), physical and conical pendulums, an extra-atmospheric spacecraft may serve as examples of such kind of systems. The conservative model can be represented in the following canonical Hamiltonian form:

$$\dot{p} = -\left(\frac{\partial H}{\partial q}\right)^{\mathrm{T}} + Bu, \quad \dot{q} = \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}}, \tag{10}$$

where p, $q \in \mathbb{R}^n$ are the vectors of generalized coordinates and momenta, n is a number of degrees of freedom; $H = H(p, q)$ is the Hamiltonian (the total system energy), which is assumed to be a continuously differentiable function on its arguments; $u = u(t)$ is the input control action (the generalized force); $B(p, q)$ is $m \times n$ matrix-function, $B \in \mathbb{R}^{m \times n}$, $m \le n$.

Let us define the control objective as an asymptotic tendency of the total energy $H(p,q)$ to the given value H_*: $\lim\limits_{t\to\infty} H\big(p(t),q(t)\big) = H_*$. This objective can be represented as (4) if one takes $x = \mathrm{col}\{p,q\}$ and define the objective function as

$$Q(p,q) = \frac{1}{2}\big(H(p,q) - H_*\big)^2. \tag{11}$$

According to the described above SG-scheme, let us find the function $\dot{Q}$—the time derivative of the function (11) along the system (10) trajectories. We obtain $\dot{Q} = (H - H_*)\left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} Bu$. Then the SG-algorithms (8), (9) take the form:

$$u = -\gamma(H - H_*)B^{\mathrm{T}}\left(\frac{\partial H}{\partial p}\right), \tag{12}$$

$$u = -\gamma \operatorname{sgn}\left((H - H_*)B^{\mathrm{T}}\left(\frac{\partial H}{\partial p}\right)\right). \tag{13}$$

In what follows, the energy-based Speed-gradient design method is employed for swinging the pendulum up to the neighborhood of the inverted (unstable) balance position for arbitrary initial conditions by means of the bounded feedback control.

3.2. Design and Examination of the Speed-gradient Control Law Based on the Energy Objective Function

Firstly, let us consider the aim of swinging the pendulum up to the neighborhood of the desired balance position. To this end, let us derive the control law ensuring achievement of the total system energy value, which conforms to the inverted (unstable) balance position for arbitrary initial conditions. Let us apply the oscillations energy control and employ the foregoing Speed-gradient method. To derive the SG algorithms, the objective function (4) in the form of a square of the deviation between the total mechanical energy $H(x)$ and its desired value H^* is chosen [4, 17, 23].

The kinetic energy E of the considered mechanical system has a form [7]:

$$E = 0.5\left(J_v\dot{\psi}^2 + J_r(\dot{\psi} + \Omega)^2 + J_m(\dot{\psi} + \omega)^2 + mh^2\dot{\psi}^2\right). \tag{14}$$

The potential system energy $\Pi(\psi)$ is as follows

$$\Pi(\psi) = (M\,b + m\,h)g\cos\psi. \tag{15}$$

Therefore, the total mechanical energy of the system $H(\psi,\dot{\psi},\omega) = E(\psi,\dot{\psi},\omega) + \Pi(\psi)$ has the following form:

$$\begin{aligned} H(\psi,\dot{\psi},\omega) &= (M\,b + m\,h)\,g\cos\psi + 0.5\left(J + J_m + mh^2\right)\dot{\psi}^2 \\ &+ \left(J_r\chi^{-1} + 2J_m\right)\omega\dot{\psi} + \left(J_m + 0.5J_r\chi^{-2}\right)\omega^2. \end{aligned} \tag{16}$$

Let us choose the objective function Q in the form of a square of the deviation between the total mechanical energy H and its desired value H^*:

$$Q(\psi,\dot{\psi},\omega) = 0.5\left(H(\psi,\dot{\psi},\omega) - H^*\right)^2. \tag{17}$$

Complying with the general Speed-gradient design scheme, let us find the time derivative of the function $Q(\psi, \dot{\psi}, \omega)$ along the system (1), (2) trajectories as

$$\dot{Q}(\psi, \dot{\psi}, \omega) = -(H(\psi, \dot{\psi}, \omega) - H^*)\Big((M\,b + m\,h)g\dot{\psi}\sin\psi + \Big(J + J_m + mh^2\Big)\ddot{\psi}\dot{\psi} + (J_r\chi^{-1} + 2J_m)(\ddot{\psi}\omega + \dot{\psi}\dot{\omega}) + \Big(2J_m + J_r\chi^{-2}\Big)\dot{\omega}\Big). \tag{18}$$

The Speed-gradient algorithms in the proportional and the relay forms are as follows:

$$u = \gamma(H^* - H)\,\sigma(\dot{\psi}, \omega), \quad \text{the proportional form,} \tag{19}$$

$$u = \gamma\,\mathrm{sgn}\big((H^* - H)\,\sigma(\dot{\psi}, \omega)\big), \quad \text{the relay form,} \tag{20}$$

where the function $\sigma(\dot{\psi}, \omega)$ stands for the partial derivative $\partial\dot{H}(\psi, \dot{\psi}, \omega)/\partial u$. To find function $\sigma(\dot{\psi}, \omega)$, Eqs. (1) and (2) should be solved with respect to higher derivative terms, see the next subsection for the details. For practice, the following piecewise-linear form (the proportional algorithm with a saturation) may be useful:

$$u = \gamma\mathrm{sat}\big(K(H^* - H)\,\sigma(\dot{\psi}, \omega)\big), \tag{21}$$

where $\mathrm{sat}(\cdot)$ denotes the saturation function, K is a certain (sufficiently large) gain. If $K \to \infty$, algorithm (21) turns to relay algorithm (20). It should be noticed that for algorithms (20), (21), the restrictions, imposed on the control signal are expressed explicitly by the parameter γ: namely, the inequality $|u(t)| \le \gamma$ is fulfilled. In the case of proportional algorithm (19) usage, parameter γ should be chosen based on the possible values of the system state variables.

In the sequel, the following numerical values of the system parameters are used [7]:

$$M = 1\text{ kg}, \quad m = 3\text{ kg}, \quad b = 0.1\text{ m}, \quad h = 0.13\text{ m}, \quad J = 0.12\text{ kg} \times \text{m}^2,$$
$$J_m = 0.03\text{ kg} \cdot \text{m}^2, \quad J_r = 10^{-4}\text{ kg} \times \text{m}^2, \quad J_v = 3.92 \times 10^{-2}\text{ kg} \times \text{m}^2,$$
$$\chi = 0.1, \quad c_1 = 8 \times 10^{-2}\text{ N} \times \text{m/V}, \quad c_2 = 7.6 \times 10^{-3}\text{ N} \times \text{m} \times \text{s}.$$

Substituting these values into (16), one obtains the following expression:

$$\Pi(\psi) = 4.81\cos\psi,$$
$$H(\psi, \dot{\psi}, \omega) = 4.81\cos\psi + 0.1\dot{\psi}^2 + 0.06\,\omega\,\dot{\psi} + 0.035\,\omega^2. \tag{22}$$

Let us rewrite (1), (2) in the form of explicit differential equations. This leads to the following model of the system (for the sake of compactness, the system parameters are presented numerically in SI units):

$$\begin{cases} \ddot{\psi} - 6.14\,\omega - 50.1\sin\psi = -6.46\,u \\ \dot{\omega} + 23.8\,\omega + 38.8\sin\psi = 25\,u. \end{cases} \tag{23}$$

Linearizing the model (23) in the vicinity of $\psi^* = 0$, one obtains the linear model spectrum (the set of eigenvalues) λ as $\lambda = \{-24.2,\ -6.0,\ 6.5\}$.

Substituting the expressions for $\ddot{\psi}$, $\dot{\omega}$ and taking into account the numerical values of the system parameters, one obtains from (23) the following expression:

$$\dot{Q}(\psi, \dot{\psi}, \omega) = (H(\psi, \dot{\psi}, \omega) - H^*)\big((0.23\dot{\psi} + 1.36\omega)\,u - 1.29\,\omega^2 + 0.34\omega\sin\psi + 2.88\dot{\psi}\sin\psi - 0.217\omega\dot{\psi}\big).$$

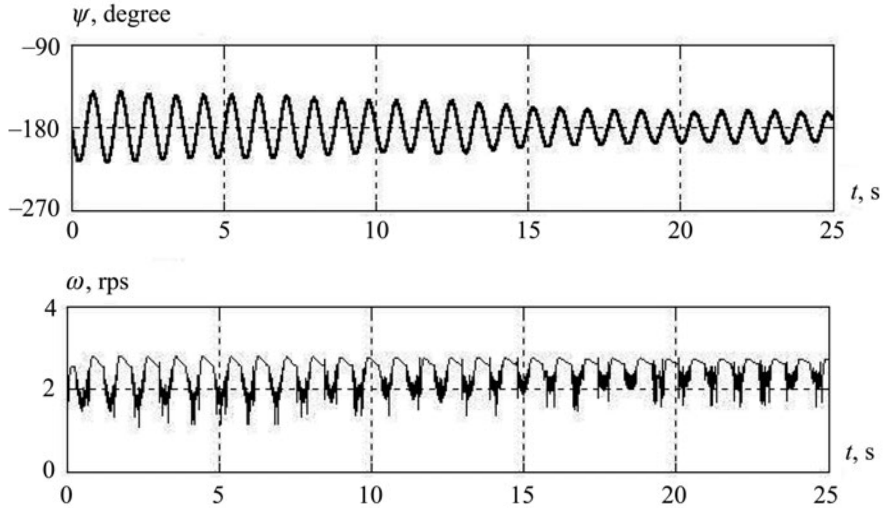

Fig. 3. Swinging the pendulum up via algorithm (16), (20), $\gamma = 16$ V, $H^* = 4.81$ J.

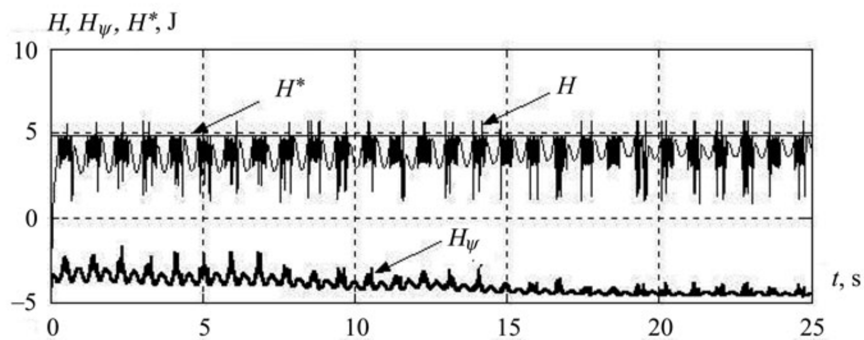

Fig. 4. Oscillations energy time histories. Algorithm (16), (20) is used, $\gamma = 16$ V, $H^* = 4.81$ J.

It follows that

$$\frac{\partial \dot{Q}}{\partial u} = \left(H(\psi, \dot{\psi}, \omega) - H^*\right) \times \left(0.23\,\dot{\psi} + 1.36\,\omega\right). \tag{24}$$

Therefore, the signal $\sigma(\dot{\psi}, \omega)$ in control laws (19)–(21) has a form $\sigma(\dot{\psi}, \omega) = 0.23\,\dot{\psi} + 1.36\,\omega$. The common factor in the expression for σ may be taken out of the brackets and, hereinafter, may be omitted.

Let us simulate the oscillations excitation via algorithm (16), (20) for a lower initial position of the pendulum ($\psi(0) = \pi$) and zero initial rotating speeds of the pendulum and the flywheel. From (22) one obtains that if the pendulum is in upper (unstable) position and no rotation of the flywheel and the pendulum exists, the total mechanical energy of the system $H = 4.81$ J. Let us use the value obtained as a reference one, assuming that $H^* = 4.81$ J. Choose the control signal magnitude γ as $\gamma = 16$ V. The simulation results are plotted in Figs. 3 and 4. As is seen from the plots, the total system energy tends rapidly to the neighborhood of the reference value, but, nevertheless, the desired excitation of oscillations does not occur because the main part of the energy falls to the flywheel rotation with an average speed about 15 $\mathrm{s}^{-1} \approx 2.4$ rps. Figure 4 also demonstrates the time history of the component $H_\psi(\psi, \dot{\psi}) = (M\,b + m\,h)\,g\cos\psi + 0.5(J + J_m + mh^2)\,\dot{\psi}^2$. This term denotes the pendulum energy for the case if the flywheel is stopped. Notice, that lesser values of γ do not make possible ensuring the prescribed energy of the system due the energy dissipation.

The results obtained show that the SG-algorithm (for the case if the control magnitude is sufficiently large) ensures achievement of the demanded total energy of oscillations, but this energy does not distributed properly between the parts of the mechanical system and is transmitted mainly to the flywheel rotation. In the case when $\psi(0) \approx 0$, the complex oscillations appear and the flywheel rotates to and fro, but, nevertheless, the upper (unstable) equilibrium is not an attracting point of the system trajectories.

Let us modify the control law, using the partial energy H_ψ instead of the total one $H(\psi, \dot{\psi}, \omega)$.

3.3. The Modified Control Strategy: Control Based on the Partial Energy

As above, let us obtain the energy-based SG control laws, using the partial component

$$H_\psi(\psi, \dot{\psi}) = (Mb + mh)g\cos(\psi) + 0.5(J + J_m + mh^2)\dot{\psi}^2 \tag{25}$$

of the system energy instead of the total mechanical energy $H(\psi, \dot{\psi}, \omega)$. The component $H_\psi(\psi, \dot{\psi})$ represents the energy of the pendulum with stopped flywheel. Introduce the modified control goal

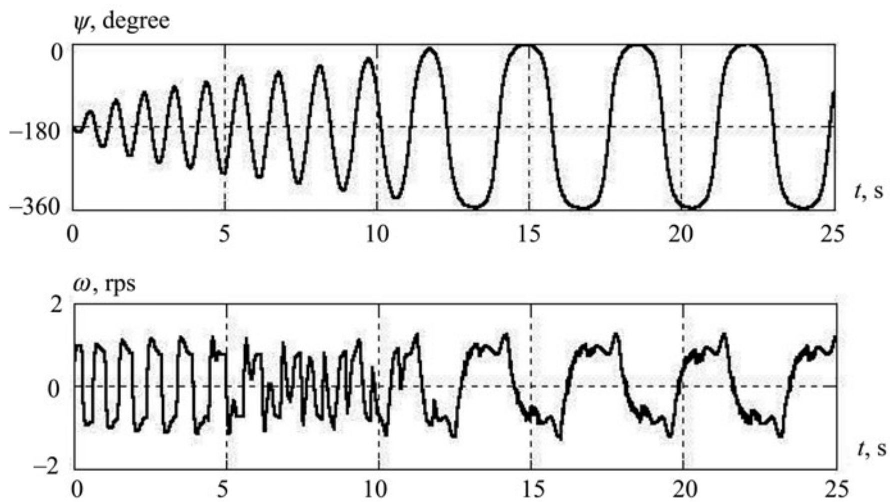

Fig. 5. Swinging up the pendulum. Algorithm (25) is used, (29), $\gamma = 6$ V, $H^* = 4.81$ J.

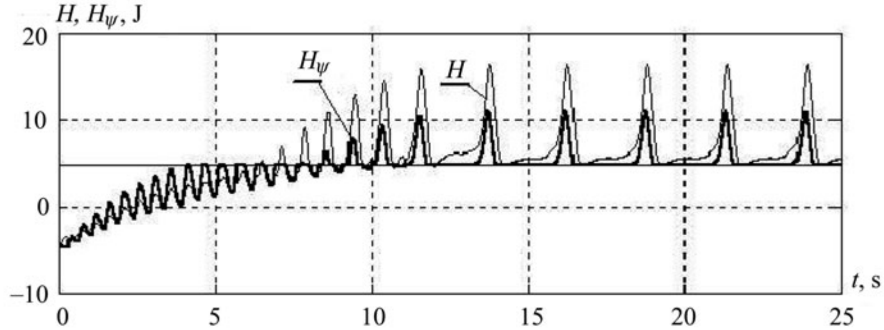

Fig. 6. Oscillations energy time histories. Algorithm (25), (29) is used, $\gamma = 6$ V, $H^* = 4.81$ J.

as follows:

$$Q(\psi, \dot{\psi}) = 0.5 \left(H_\psi(\psi, \dot{\psi}) - H^*\right)^2. \tag{26}$$

Calculating, as above, the time derivative of the function $Q(\psi, \dot{\psi})$ along the system (1), (2) trajectories, one obtains the following expression:

$$\dot{Q} = \left(H_\psi(\psi, \dot{\psi}) - H^*\right)\left(\alpha_1 \sin\psi \times \dot{\psi} + (\alpha_2 u + \alpha_3 \omega + \alpha_4 \sin\psi)\dot{\psi}\right), \tag{27}$$

where α_i, $i = 1, \ldots, 4$ are certain coefficients, depending on the system parameters. Calculating the partial derivative of the function $\dot{Q}$ with respect to the control action u and taking into account that $\alpha_2 < 0$, one obtains the following control laws for putting the pendulum to the required state:

$$u = \gamma(H_\psi - H^*)\,\dot{\psi}, \quad \text{the proportional form,} \tag{28}$$

$$u = \gamma \,\mathrm{sgn}\big((H_\psi - H^*)\,\dot{\psi}\big), \quad \text{the relay form.} \tag{29}$$

The simulation results for system (1), (2) with the control law (25), (29) and the control signal magnitude $\gamma = 6$ V, are depicted in Figs. 5 and 6. It is seen from the plots that the modified control law ensures fulfillment of the control aim: putting the pendulum to the lifted-up position with a slow angular velocity in its vicinity. The simulations show that the proposed algorithm (25), (29) requires half as large control action magnitude than that of [7].

4. BALANCING CONTROL LAW

For stabilizing the pendulum in the upper position simultaneously with stopping the wheel rotation, let us apply the final form (7) of the SG-algorithm. To design the control law, ensuring appearance of the stable sliding motion in the certain neighborhood of the desired balance state, let us use the linearized plant model. To this end, let us apply the commonly known control design technique of [10] (see also [3]).

Let us linearize (23) in the vicinity of the point $\psi^* = 0$ (note, that (23) is linear on ω and u, therefore the values of ω^* and u^* may not be taken into account at the linearisation). With respect to the vector $x = \mathrm{col}\{\Delta\psi, \Delta\dot{\psi}, \Delta\omega\}$, denoting the deviations from the prescribed balance state, the state-space equation $\dot{x} = Ax + Bu$ has the following matrices A and B:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 50.1 & 0 & 6.14 \\ -38.8 & 0 & -23.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -6.46 \\ 25 \end{bmatrix}. \tag{30}$$

Assuming that the whole state vector x in (30) is measurable, let us find the vector $g \in \mathbb{R}^3$ so that the numerator $B(s)$ of the transfer function $W(s) = g^{\mathrm{T}}(p\mathrm{I} - A)^{-1}B$ be a given Hurwitz polynomial. Let us employ the Butterworth polynomial [9] as a "standard" one. Setting $g_1 = 1$, one obtains the polynomial $B(s)$ in the form $B(s) = (-6.46g_2 + 25g_3)s^2 - (6.46 + 0.25g_2)s - 0.25 - 10^3 g_3$. Equating the coefficients of $B(s)$ to those of the Butterworth polynomial $D(s) = s^2 + 1.4\Omega_0 s + \Omega_0^2$ and taking $\Omega_0 = 5$ s^{-1}, one obtains the values of g_2, g_3 as $g_2 = 0.23$ s, $g_3 = 0.023$ s. Finally, the following control law for pendulum stabilization around the inverted position simultaneously with stopping the flywheel rotation is obtained:

$$u = -\gamma \,\mathrm{sgn}\,\sigma, \quad \sigma = g_1(\psi \mod 2\pi) + g_2\dot{\psi} + g_3\omega, \tag{31}$$

where $g_1 = 1$, $g_2 = 0.23$, $g_3 = 0.023$.

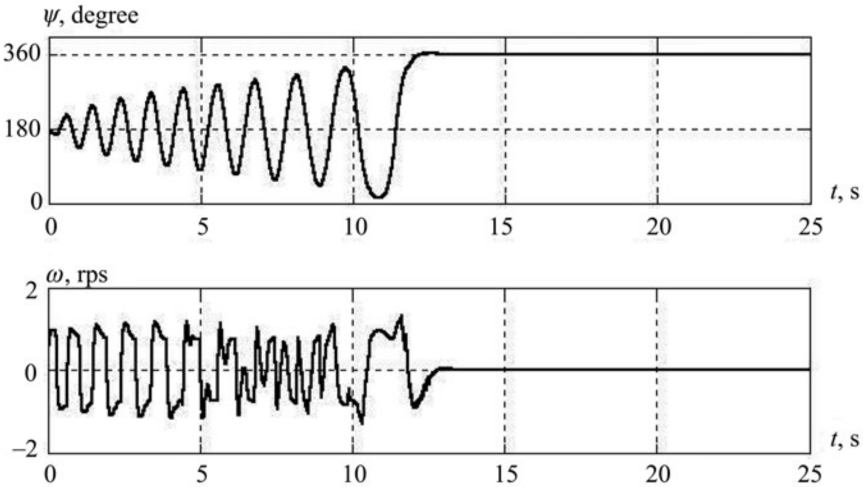

Fig. 7. Swinging up and balancing the pendulum in accordance with hybrid control algorithm (25), (29), (31) for $\gamma = 6$ V.

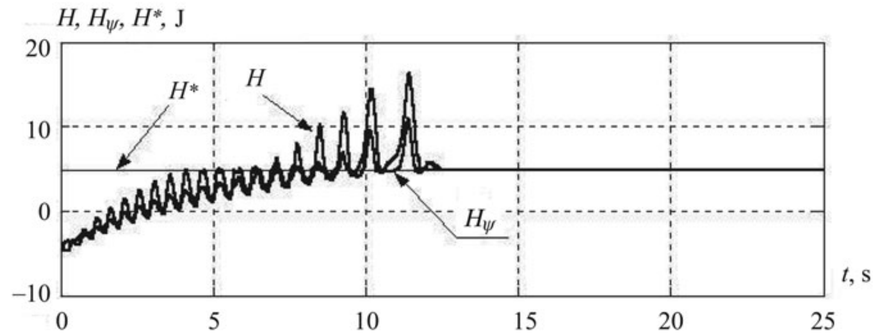

Fig. 8. Oscillation energy time histories. Algorithm (25), (29), (31) is used.

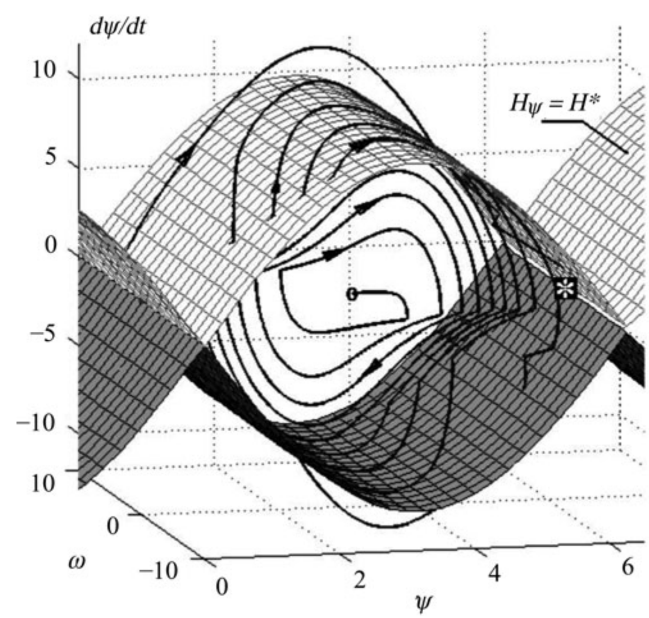

Fig. 9. Phase plot. Initial condition—**o** (the stable equilibrium point), final state—$*$ (the unstable equilibrium point).

5. HYBRID CONTROL STRATEGY

Let us use the condition $\mathrm{abs}(\psi \mod 2\pi) < \Delta_\psi$ for switching the control action from (25), (29) to (31).

Consider the joint performance of the control algorithms (25), (29) and (31). Pick up the threshold Δ_ψ as $\Delta_\psi = 40$ deg. The simulation results are presented in Figs. 7 and 8. As is seen from the plots, the oscillations are excited, their energy and magnitude increase until the prescribed region is achieved. Then the controller is switched over, ensuring balancing around the inverted position. The corresponding phase plot in the space $\{\psi, \omega, \dot{\psi}\}$ is depicted in Fig. 9 along with the equipotential surface of the system partial energy H_ψ.

6. CONCLUSIONS

In the paper the hybrid control strategy for global stabilization of the inverted Reaction Wheel Pendulum is presented and examined. The control aim is achieved by solution of the following two particular control problems: swinging the pendulum up to the certain neighborhood of the inverted position and balancing it about this position. For the design of the swing-up controller, the Speed-gradient control strategy with a partial energy goal function is employed. The second aim is achieved by means of the sliding-mode controller. In the contrary of [7], the proposed method makes possible to stabilize the Reaction Wheel Pendulum for arbitrary initial conditions and, in addition, requires half as large control action magnitude than that of [7].

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