CONTROL OF AN INVERTED PENDULUM WITH A REACTION WHEEL

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CONTROL OF AN INVERTED PENDULUM WITH A REACTION WHEEL

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ABSTRACT

This report describes the control of unstable systems; inverted pendulum with a cart and inverted pendulum with a reaction wheel. We discuss the dynamics of the inverted pendulum and control theories which are appropriate for controlling the inverted pendulum with a cart and reaction wheel separately. We derived mathematical model of both systems. These mathematical models are either nonlinear and linear format. We also discuss proportional-integral-derivative (PID) and linear quadratic regulator (LQR) control theories, and SIMULINK model. This project can be evolved to control a single axis reaction wheel for satellite in future applications, since the control sense is similar to the inverted pendulum.

1 INTRODUCTION

In space industry, satellites have built during last years for different missions in terms of Earth observation, communication etc. and to either make stable the structure or pointing accurate, the attitude control systems are used. Attitude control is orientation of the satellite body with respect to inertial frame or near objects' frame. Attitude determination and control system uses sensors and actuators. As actuator, reaction wheel which is called also as flywheel system is one may be used to control the body in applied axis. Three axis attitude control and stability of the satellite can be provided via three or more flywheels. Working principle of the reaction wheel for attitude control is leaned to Newton's third law of the motion; hence the attitude of the satellite will be in opposite direction of the motion of the reaction wheel. Thus, reaction wheel is a wise actuator for attitude control for the satellites, and this project is aimed to understand controlling a structure with a reaction wheel.

In this report, in order to understand the reaction wheel dynamics; inverted pendulum with a cart is modified to inverted pendulum with a reaction wheel, yet the project has begun with the inverted pendulum on a cart model. The pendulum is weight suspended at a fixed point, thereby it can swing freely. In a situation out of equilibrium point, the pendulum needs to be controlled for stabilizing and backing towards the equilibrium point, and it is provided via a cart or a reaction wheel.

During controlling process of the inverted pendulum on a cart and with reaction wheel, firstly mathematical models of the systems have been derived, and then state space forms as well as corresponding state functions have been built. Control of the inverted pendulum on a cart and inverted pendulum with reaction wheel were realized in MATLAB and SIMULINK and as control method proportional-integral-derivative (PID) and linear quadratic regulator (LQR) used, and their control theories have been described also. Finally, SIMULINK model of these systems have written detailed and consist of plant model (inverted pendulum on a cart, inverted pendulum with reaction wheel, and actuator) and control model with PID and LQR.

2 PROJECT DESCRIPTION

The reaction wheel is a good actuator to control a spacecraft in desired axis which is used in CubeSat control systems. This project is aimed to understand the control unstable systems and learn the control theories. There are several methods to make stable the unstable systems, and in this project literature researching has been made for control theories. During the project two unstable systems have been investigated and controlled which are inverted pendulum with a cart and inverted pendulum with a reaction wheel.

To control an unstable system one need to define the mathematical modeling of the complete system. After mathematical modeling, state space equations are required and one control method has to be applied. To do that, LQR and PID have been investigated and learned. All in all, purpose of the controlling the inverted pendulum with a cart is understanding the process of the controlling unstable system in terms of required steps such as mathematical model derivation, control theory etc.

After understanding the control process of the inverted pendulum with a cart, next step of the project is control of the inverted pendulum with a reaction wheel. Aim of the reaction wheel inverted pendulum control is performing interest with dynamic model of the system and derivation of the mathematical model of the reaction wheel inverted pendulum to create control design and simulations in MATLAB and SIMULINK. Consequently, purpose of the using reaction wheel is stabilizing the inverted pendulum about its vertical equilibrium (0°) via controlling the angular velocity of the pendulum.

Briefly, this project is aimed to understanding the working principle of the reaction wheel and process of the controlling an unstable system such as inverted pendulum with cart and flywheel. The project topics that have been studied and described in this report are; mathematical modeling for inverted pendulum with a car and inverted pendulum with a reaction wheel, motor dynamics, state space representations, state functions, control theory which consists of PID control and LQR control, and corresponding SIMULINK model.

3 MODEL PARAMETERS

Model of the inverted pendulum on a cart and with reaction wheel were implemented with parameters. These parameters were taken from MATLAB Tutorial[1] for inverted pendulum on a cart and Two-Axis Reaction Wheel Inverted Pendulum[2] article which is written by Petter Brevik. Identification of these parameters are significant before create a model on SIMULINK. Model parameters are different for inverted pendulum and pendulum with reaction wheel models and these were given in following sections.

3.1 Model Parameters of Inverted Pendulum on a Cart

As shown in Figure 1, inverted pendulum model consist of two main parts which are pendulum and cart. Model parameter values of inverted pendulum were given in Table 1. These parameters are length of the pendulum L, mass of the cart and pendulum M, m respectively, coefficient of friction k and moment of inertia of the pendulum I.

Table 1:Model Parameters of Inverted Pendulum

Parameters	**Values**
L	0.3 m
M	0.5 kg
m	0.2 kg
k	0.1
I	0.006 kgm^2

3.2 Model Parameters of Inverted Pendulum with a Reaction Wheel

Model of the pendulum with reaction wheel has been implemented with parameters as shown in table below. System of the reaction wheel pendulum consists of a pendulum and a reaction wheel, and pendulum has fixed from a point which is located at ground. In mathematical modeling of the reaction wheel pendulum there are some parameters and these parameters have described in this section.

The distance between center of the mass of the pendulum and fixing point is referred as “l” and the distance between center of the mass of the reaction wheel and fixing point is referred as “L”. Masses of the pendulum and reaction wheels are “m” and “M”, respectively. Also, pendulum moment of inertia is referred as “I_{PB}”, and reaction wheel moment of inertia is referred as “I_{WC}”. Numerical values of all these parameters have been written in the figure, and these values have obtained from "Two-Axis Reaction Wheel Inverted Pendulum" technical paper.

Table 2:Model Parameters of Pendulum with Reaction Wheel

Parameters	**Values**
L	0.330 m
l	0.185 m
M	0.58 kg
m	0.033 kg
$\boldsymbol{I_{WC}}$	2.262×10^{-4} kgm^2
$\boldsymbol{I_{PB}}$	3.765×10^{-4} kgm^2

4 MATHEMATICAL MODELLING

4.1 Mathematical Modeling of Inverted Pendulum on a Cart:

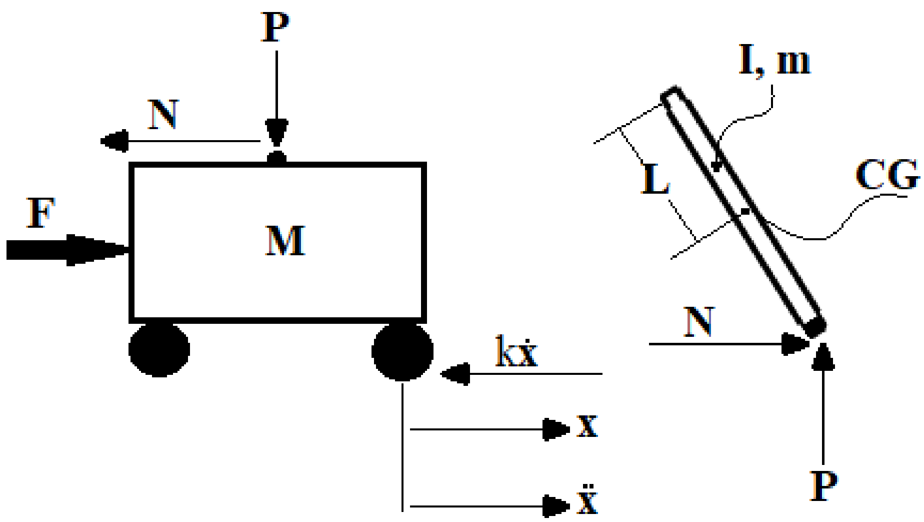

Figure 1:Free-Body Diagrams of Cart and Inverted Pendulum[1]

Controlling of an inverted pendulum can be start with the derivation of nonlinear equations. Before doing it, free-body diagram of inverted pendulum on a cart system must be understood very well. Figure 1 shows the free-body diagram of the inverted pendulum on a cart. The system consists of pendulum and the cart. This pendulum is mounted on the center of the top of the cart. In Figure 1, x and $\dot{x}$ represent the position and velocity of the cart, respectively. In addition, θ and $\dot{\theta}$ represent the angle and angular velocity of the pendulum, respectively. Pendulum is unstable without control signal. So, one has to apply a force to the cart that creates reaction forces on the pendulum body. To describe this system, non-linear equations of motion of the system need to be derived according to free-body diagram. Cart and pendulum individually have one degree of freedom, which are x and θ, so torque-balance and force-balance equations are calculated as follows [3].

$$\frac{d^2x}{dt^2} = \frac{1}{M}\sum\nolimits_{cart} F_x = \frac{1}{M}\left(F - N - k\frac{dx}{dt}\right) \tag{1}$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{I}\sum\nolimits_{pend} \tau = \frac{1}{I}(NLcos(\theta) - PLsin(\theta)) \tag{2}$$

where P and N are interaction forces between cart and pendulum, k$\dot{x}$ is friction force and L is the distance between center of the mass of the pendulum and top of the pendulum, M and m is the mass of cart and pendulum respectively and I is the moment of inertia of the pendulum which is the inertia about a fixed axis can be described as the body's resistance to change to its rotation[2]. In addition to that, θ is the angle of pendulum.

Then x and y equations for pendulum were written as

$$m\frac{d^2x_p}{dt^2} = \frac{1}{M}\sum\nolimits_{pend} F_x = N \tag{3}$$

$$m\frac{d^2y_p}{dt^2} = \sum\nolimits_{pend} F_y = P - mg \tag{4}$$

where x_p and y_p are functions of θ. With using these formulas interaction forces N and P were written as

$$N = m\frac{d^2x_p}{dt^2} \tag{5}$$

$$P = m(\frac{d^2y_p}{dt^2} + g) \tag{6}$$

Time derivative of x_p can be represented in terms of θ and it was written below

$$x_p = x - Lsin(\theta) \tag{7}$$

$$\dot{x}_p = \dot{x} - Lcos(\theta)\dot{\theta} \tag{8}$$

$$\ddot{x}_p = \ddot{x} + Lsin(\theta)\dot{\theta}^2 - Lcos(\theta)\ddot{\theta} \tag{9}$$

Also, time derivative of y_p can be represented in terms of θ and it was given below

$$y_p = Lcos(\theta) \tag{10}$$

$$\dot{y}_p = -Lsin(\theta)\dot{\theta} \tag{11}$$

$$\ddot{y}_p = -Lcos(\theta)\dot{\theta}^2 - Lsin(\theta)\ddot{\theta} \tag{12}$$

To find the $\ddot{\theta}$ and $\ddot{x}$, Eq. 5 and 6 were substituted into the Eq.1 and 2.

$$\frac{d^2x}{dt^2} = \frac{1}{M}\left(F - m\frac{d^2x_p}{dt^2} - k\frac{dx}{dt}\right) \tag{13}$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{I}\left(m\frac{d^2x_p}{dt^2}Lcos(\theta) - m(\frac{d^2y_p}{dt^2} + g)Lsin(\theta)\right) \tag{14}$$

Then, Eq. 9 and 12 were substituted into the Eq.13 and 14 and $\ddot{x}$ and $\ddot{\theta}$ were gathered to left side of the equations.

$$\frac{d^2x}{dt^2} = \frac{F + \frac{Lm^2gcos(\theta)\sin(\theta)}{I + mL^2} + \frac{Lmsin(\theta)(d\theta)}{dt} - k\left(\frac{dx}{dt}\right)}{M + m + (L^2m^2\cos^2(\theta))/(I + L^2m)} \tag{15}$$

$$\frac{d^2\theta}{dt^2} = \frac{mL}{I + mL^2}\left(\frac{d^2x}{dt^2}cos(\theta) - gsin(\theta)\right) \tag{16}$$

Finally, Eq.15 and 16 were put the inside of the each other, then $\ddot{x}$ and $\ddot{\theta}$ were found.

$$\ddot{x} = \frac{I + mL^2}{(I + mL^2)(M + m) - m^2L^2\cos(\theta)}(F - mLsin(\theta)(\dot{\theta})^2 + \frac{m^2L^2gsin(\theta)\cos(\theta)}{I + mL^2} - k\dot{x}) \tag{17}$$

$$\ddot{\theta} = \frac{(m + M)(mL)}{(I + mL^2)(M + m) - m^2L^2\cos(\theta)}\left(\frac{\cos(\theta)}{M + m}(F - mLsin(\theta)\dot{\theta}^2 - k\dot{x}) + gsin(\theta)\right) \tag{18}$$

4.1.1 Linearization of Inverted Pendulum on a Cart Model:

Equations were linearized about vertically upward equilibrium position, namely θ=0. In this position the deviation of the pendulum's angle is small enough to control for this nonlinear system. Therefore, cos(θ)=1, sin(θ)= θ and $\dot{\theta}^2$=0 were approximate. As a result, substituting into nonlinear equations, we have the two linearized equations of motion [4].

$$\ddot{x} = \frac{I + mL^2}{(I + mL^2)(M + m) - m^2L^2}(F + \frac{m^2L^2g\theta}{I + mL^2} - k\dot{x}) \tag{19}$$

$$\ddot{\theta} = \frac{(m + M)(mL)}{(I + mL^2)(M + m) - m^2L^2}\left(\frac{1}{M + m}(F - k\dot{x}) + g\theta\right) \tag{20}$$

4.2 Mathematical Modelling of Inverted Pendulum with a Reaction Wheel:

As the first step of the controlling the inverted pendulum with a reaction wheel, mathematical model of the system has been derived. In this section derivation of the mathematical modeling of the inverted pendulum with reaction wheel is described. Firstly, nonlinear model is derived with help of the Lagrangian approach, and then this nonlinear model will be linearized.

In order to control the inverted pendulum with reaction wheel, angles are needed to be understood because the main issue is controlling the pendulum angle that fixes it to desired location via rotation of the reaction wheel. That is to say, denominations of the angles are quite important and during the derivation of the equations of motion they must be written correctly. So, reaction wheel pendulum has two degrees of freedom and, θ_1 is an angle which is measured between pendulum and fixing point, and θ_2 is an angle which is measured between line of pendulum and arbitrary axis on reaction wheel. These angles have been shown in Figure 2. In addition, $\dot{\theta}_1$ and $\dot{\theta}_2$ are angular velocities of the pendulum and reaction wheel, respectively. To control of the system, θ_1, $\dot{\theta}_1$ and $\dot{\theta}_2$ must equal to 0 for vertical upward equilibrium, but θ_2 can be left uncontrolled because we do not need to control angle of the reaction wheel rather than its velocity [5].

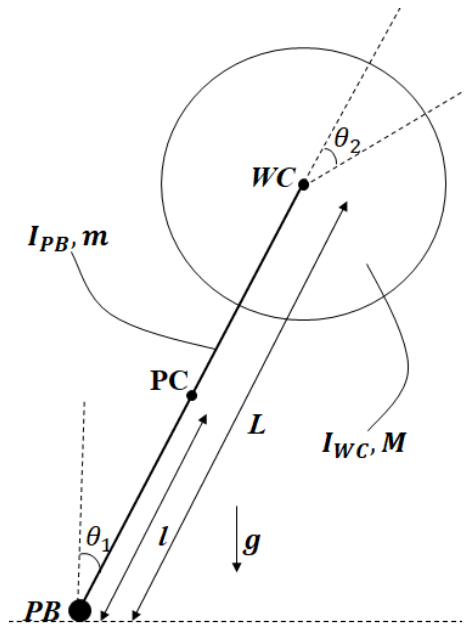

Figure 2: Free Body Diagram of the Reaction Wheel Inverted Pendulum[2]

To derive the mathematical modeling of the reaction wheel inverted pendulum, equations of motion need to be derived, and to do that most elegant method is Lagrangian method. This method is simpler than vector forces and accelerations because it works with scalar energy functions. Before Lagrangian equation, potential and kinetic energy equations need to be obtained. Potential energy equations of the pendulum and reaction wheel have written down as follows.

$$V_P = mlgcos(\theta_1) \tag{21}$$

$$V_W = MLgcos(\theta_1) \tag{22}$$

Eq.21 is potential energy of pendulum with respect to PB, and Eq.22 is potential energy of the reaction wheel with respect to PB. Here, m and M are masses of the pendulum and wheel, respectively. In addition, l and L are distance between center of masses of the pendulum and reaction wheel according to fixing point PB, respectively. Also, kinetic energy equations of the pendulum and reaction wheel have written down in terms of rotational and translational kinetic energies as follows.

$$T_{PT} = \frac{1}{2}(ml^2)\dot{\theta}_1^2 \tag{23}$$

$$T_{PR} = \frac{1}{2}I_{PC}\dot{\theta}_1^2 \tag{24}$$

Where T_{PT} and T_{PR} are translational and rotational kinetic energy of the pendulum, respectively. Also, I_{PC} is moment of inertia of the pendulum, and $\dot{\theta}_1$ is angular velocity for pendulum.

$$T_{WT} = \frac{1}{2}(ML^2)\dot{\theta}_1^2 \tag{25}$$

$$T_{WR} = \frac{1}{2}I_{WC}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \tag{26}$$

In Eq.26, $\dot{\theta}_1 + \dot{\theta}_2$ is total angular velocity. Eventually, kinetic energy of the pendulum can be simplified as $\frac{1}{2}I_{PB}\dot{\theta}_1^2$ according to parallel axis theorem $(I_{PC} + ml^2)$. Besides, sum of the kinetic energy of the pendulum and reaction wheel can be called as T;

$$T = \frac{1}{2}I_{WC}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2}(ML^2)\dot{\theta}_1^2 + \frac{1}{2}I_{PB}\dot{\theta}_1^2 \tag{27}$$

And potential energy of the system is;

$$V = mlgcos(\theta_1) + MLgcos(\theta_1) \tag{28}$$

Up to now, to derive the equation of motion with Lagrangian equation; kinetic and potential energies have been obtained. To use the Lagrangian equation, one need to remember the details as described below.

"The Lagrangian method begins by defining a set of generalized coordinates, $q_1, \ldots, q_n$, to represent an n-degree-of-freedom system. These generalized coordinates are typically position coordinates (distances or angles). In a multi-body system, the kinetic and potential energies can be computed for each body independently and then added together to form the energies of the complete system. This is an important advantage of the Lagrangian method and works because energy is a scalar valued, as opposed to vector valued, function. Once the kinetic and potential energies are determined, the Lagrangian, $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$, is then defined as the difference between the kinetic and potential energies. The Lagrangian is, therefore, a function of the generalized coordinates and their derivatives." [2]

The form to express the reaction wheel inverted pendulum equations of motions is given below in terms of Lagrangian;

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_k}\right) - \frac{\delta L}{\delta_k} = \tau_k, \qquad k = 1, \ldots\ldots, n \tag{29}$$

Here, the τ_k is a variable which represent the force or torque. So, equations which will be derived via Lagrangian method are equivalent with the equations derived by Newton's second law. The Lagrangian function is;

$$L = T - V = \frac{1}{2} I_{WC}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2}(ML^2)\dot{\theta}_1^2 + \frac{1}{2} I_{PB}\dot{\theta}_1^2 - mlgcos(\theta_1) + MLgcos(\theta_1)$$

$$L = \frac{1}{2} I_{WC}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2}(ML^2 + I_{PB})\theta_1^2 - mlgcos(\theta_1\dot{)} - MLgcos(\theta_1)$$

$$L = \frac{1}{2} I_{WC}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2} a\dot{\theta}_1^2 - bgcos(\theta_1)$$

Where a is $ML^2 + I_{PB}$ and b is $ml + ML$. Finally, Lagrangian function can be written as given below;

$$L = \frac{1}{2} I_{WC}\dot{\theta_1}^2 + \frac{1}{2} I_{WC}\dot{\theta_2}^2 + \frac{1}{2} I_{WC}\left(\dot{\theta}_1\dot{\theta}_2\right)^2 + \frac{1}{2} a\dot{\theta}_1^2 - bgcos(\theta_1) \tag{30}$$

Partial derivatives of the Lagrangian function are;

$$\frac{\delta L}{\delta \dot{\theta}_1} = a\dot{\theta}_1 + I_{WC}\dot{\theta}_1 + I_{WC}\dot{\theta}_2 \tag{31}$$

$$\frac{\delta L}{\delta \theta_1} = bgsin(\theta_1) \tag{32}$$

$$\frac{\delta L}{\delta \dot{\theta}_2} = I_{WC}\dot{\theta}_1 + I_{WC}\dot{\theta}_2 \tag{33}$$

$$\frac{\delta L}{\delta \theta_2} = 0 \tag{34}$$

Then, Lagrangian equations to Lagrange mechanics for applied torque value.

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\theta_1}}\right) - \frac{\delta L}{\delta \theta_1} = 0 \tag{35}$$

$$a\ddot{\theta_1} + I_{WC}\ddot{\theta_1} + I_{WC}\ddot{\theta_2} - bgsin(\theta_1) = 0 \tag{36}$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\theta_2}}\right) - \frac{\delta L}{\delta \theta_2} = \tau \tag{37}$$

$$I_{WC}\ddot{\theta_1} + I_{WC}\ddot{\theta_2} - 0 = \tau \tag{38}$$

4.2.1 Linearization of Inverted Pendulum with a Reaction Wheel:

In order to linearize this system one may consider to equilibrium position of vertically upward situation. Since, there will be of course some deviations, these are really small angles for control this nonlinear system. As a result of this, $\sin(\theta_1) = \theta_1$ approximation has been done, and linearized kinematic equations yields as follows;

$$a\ddot{\theta_1} + I_{WC}\ddot{\theta_1} + I_{WC}\ddot{\theta_2} - bg(\theta_1) = 0$$

$$I_{WC}\ddot{\theta_1} + I_{WC}\ddot{\theta_2} - 0 = \tau$$

$$\ddot{\theta_1} = \frac{(ml + ML)g\theta_1 - \tau}{(ML^2 + I_{PB})} \tag{39}$$

$$\ddot{\theta_2} = -\frac{(ml + ML)g\theta_1}{ML^2 + I_{PB}} + \frac{(ML^2 + I_{PB} + I_{WC})\tau}{(ML^2 + I_{PB})I_{WC}} \tag{40}$$

5 MOTOR DYNAMICS [6]

One of the most widely known actuator in control theory is of course the DC motor. It can provide some torque for the system as well as translational motion. One may require this common actuator for both pendulum on a cart model, and with a reaction wheel version of it, yet it is not necessary to model motor dynamics for cart system, since it is already actuated by some input force on the cart body. Moreover, one may require additional dynamics ,which are related to DC motor, for pendulum with a reaction wheel in order to control the model more precisely and realistic. Calculations are based on the University of Michigan control tutorials. In the following figure, the free-body diagram and circuit model of an arbitrary DC motor is illustrated by University of Michigan control tutorial website.

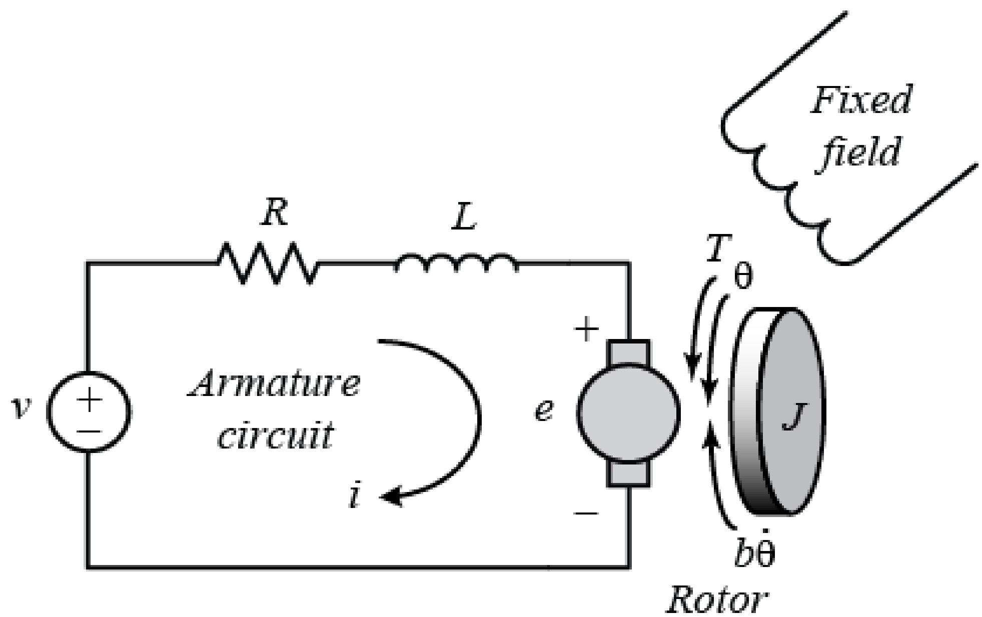

Figure 3: Circuit and Free-body diagram of an electric motor[3]

Generated torque by a DC motor is generally proportioned an armature current, and it depends on its magnetic field strength. In the following calculations, it is assumed that the magnetic field is constant, resulting to relation between motor torque (Kt) and armature current (i). So, the following relation mentions this idea as;

$$T = K_t i \tag{41}$$

Moreover, According to University of Michigan tutorial, the back emf (e) is proportional to the angular velocity of the DC motor shaft by a constant factor of Ke. This relation is given as follows;

$$e = K_e \, \dot{\theta} \tag{42}$$

In fact, the motor torque and back emf constant are same in SI units, i.e Kt = Ke, therefore, one can use K to represent both of them.

To complete the relation of torque and motor current, one can define the differential equations based on Newton's 2nd Law and Kirchhoff's voltage law, according to the Michigan Tutorials. So, this relation can be given as follows;

$$I_{wc}\ddot{\theta} + b\dot{\theta} = Ki \tag{43}$$

$$L \, \frac{di}{dt} + Ri = V - K\dot{\theta} \tag{44}$$

So, for the sake of clarity, V is for voltage, θ is for angular velocity of the rotor, R is for resistance on circuit, L is for inductance. Those manuscripts are not related to other free body diagrams that are given in the previous chapters.

For this analysis, the motor parameters are just made up, yet it is ensured that made up variables are acceptable for real motor parameters, since we made a cross comparison between real motors, and used motor for real application of referenced projects. So, the following table shows the physical parameters of the motor for inverted pendulum with a reaction wheel pendulum.

Table 3: Selected Motor parameters for Analysis

Moment of Inertia of the Rotor (kg.m^2)	2.262e-4
Motor Friction Constant (N.m.s)	0.2
Back EMF Constant(V/rad/s)	1e-3
Motor Torque Constant (N.m/A)	1e-3
Electric Resistance (Ohm)	1
Electric Inductance (H)	0.5

At the end of this chapter, one can re-mentioned that we assumed that the input of this system is voltage, and the corresponding output is going to be the resulting torque due to the motion of the rotor. The rotor and shaft are assumed to be rigid, and there is a friction between them, which is proportional to shaft angular velocity. At last, notice that we have not given any DC Motor or another actuator in hand. As a result of this, one has to made up the motor parameters, but it must be ensured to be within the possible ranges of a real DC motor as much as possible. This assurance is done by cross comparison between real world applications, and DC motors.

6 STATE SPACE REPRESENTATION

In this chapter, the state space representation of actuator and both of the mathematical models are mentioned. But before this, one may introduce the linear state space concept.

Let us consider a control system;

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \tag{45}$$

$$y = \boldsymbol{C}\boldsymbol{x} + Du \tag{46}$$

where;

y = Output signal   u = Control signal   **x =** State vector (n-vector)

A= n × n constant matrix   **B**= n × 1 constant matrix   **C=**1 × n constant matrix   D= constant

From now, one can choose the control signal to be;

$$u = -\boldsymbol{K_G}\boldsymbol{x} \tag{47}$$

So, as indicated in Eq.47, the control signal is determined by an instantaneous state, which means that it is state feedback. The 1 × n matrix $\boldsymbol{K_G}$ is called state feedback gain matrix. We assume that all state variables are available for the feedback, in other words, all the state variables are assumed as measurable.

6.1 State-Space Representation of Linear Inverted Pendulum on a Cart Model

To control the system, state-space representation of a system was obtained. Standard form of state-space representation as

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = Ax + Bu \ \ , y = Cx + Du$$

where the u is the force of linear inverted pendulum model. A,B,C and D were found with using the linear equations of $\ddot{\theta}$ and $\ddot{x}$, as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -ak & \frac{a(m^2L^2g)}{I+mL^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{bk}{M+m} & bg & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ a \\ 0 \\ \frac{b}{M+m} \end{bmatrix},\ \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}, a = \frac{I + mL^2}{(M+m)(I+mL^2) - (m^2L^2)}, b = \frac{(M+m)(mL)}{(I+mL^2)(m+M) - (m^2L^2)}$$

For the inverted pendulum system, outputs are the cart's position and the pendulum angle. The main reason to represent the state-space vector is to use these for control the pendulum. Finally, state-space representation of linear inverted pendulum model is

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -ak & \frac{a(m^2L^2g)}{I+mL^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{bk}{M+m} & bg & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ a \\ 0 \\ \frac{b}{M+m} \end{bmatrix} F \tag{48}$$

6.2 State Space Representation for Inverted Pendulum with a Reaction Wheel:

State space is defined as n dimensional space which coordinate axes include x_1 to x_n axis. For state space analysis there are three types of variables which are input variables, output variables, and state variables. In this section, state space representation of the reaction wheel pendulum is described as follow in general form.

During making state space, equations which derived previously for angular acceleration of the pendulum and reaction wheel inverted pendulum are written in matrix form. To obtain the state space of the reaction wheel inverted pendulum the forms are used given below;

$$\ddot{x} = Ax + Bu \tag{49}$$

$$y = Cx + Du \tag{50}$$

Then, as the result obtaining state space is;

$$\ddot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ bg/a & 0 & 0 \\ -bg/a & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/a \\ \dfrac{a + I_{WC}}{aI_{WC}} \end{bmatrix} \tau \tag{51}$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{52}$$

Where, $a = ML^2 + I_{PB}$ and $b = ml + ML$.

6.3 Actuator State Space Representation:

The state space representation of DC motor can be determined according to the given representation in Eq.45. As a result of that, the DC motor dynamics equations that are derived in Motor Dynamics chapter, can be written as follows;

$$\frac{d^2\theta}{dt^2} = -\frac{b}{I_{wc}}\dot{\theta} + \frac{K}{I_{wc}} + 0V \tag{53}$$

$$\frac{di}{dt} = \frac{V}{L} - \frac{K}{L}\dot{\theta} - \frac{R}{L}i \tag{54}$$

Since we want to control our torque and it is related to angular velocity and current, one can define the state space as follows;

$$\frac{d}{dt}\begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} = \begin{bmatrix} -\dfrac{b}{I_{wc}} & \dfrac{K}{I_{wc}} \\ -\dfrac{K}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} V \tag{55}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} \tag{56}$$

7 TRANSFER FUNCTIONS

In this section, the transfer function realization of both linearized mathematical models obtained firstly for inverted pendulum on a cart, following with reaction wheel model.

7.1 Transfer Function of Inverted Pendulum on a Cart:

According to the linearized equation of motion of inverted pendulum on a cart, one can define the transfer function as follows;

Firstly, one need to take Laplace transform of $\ddot{\theta}$ and $\ddot{x}$ as follows;

$$\boldsymbol{\Theta}(\boldsymbol{s})s^2 = \frac{(M+m)(mL)}{(I+mL^2)(M+m)-m^2L^2}\left(\frac{1}{M+m}(\boldsymbol{U}(\boldsymbol{s}) - k\boldsymbol{X}(\boldsymbol{s})s) + g\boldsymbol{\Theta}(\boldsymbol{s})\right) \tag{57}$$

$$\boldsymbol{X}(\boldsymbol{s})s^2 = \frac{I+mL^2}{(I+mL^2)(M+m)-m^2L^2}\left(\boldsymbol{U}(\boldsymbol{s}) + \frac{m^2L^2g\boldsymbol{\Theta}(\boldsymbol{s})}{I+mL^2} - k\boldsymbol{X}(\boldsymbol{s})s\right) \tag{58}$$

So, calling;

$$a = \frac{(M+m)(mL)}{(I+mL^2)(M+m)-m^2L^2} \qquad b = \frac{I+mL^2}{(I+mL^2)(M+m)-m^2L^2}$$

$$c = \frac{1}{M+m} \qquad d = \frac{m^2L^2g}{I+mL^2}$$

It yields the following form;

$$\boldsymbol{\Theta}(\boldsymbol{s})s^2 = a(c(\boldsymbol{U}(\boldsymbol{s}) - k\boldsymbol{X}(\boldsymbol{s})s) + g\boldsymbol{\Theta}(\boldsymbol{s})) \tag{59}$$

$$\boldsymbol{X}(\boldsymbol{s})s^2 = b(\boldsymbol{U}(\boldsymbol{s}) + d\boldsymbol{\Theta}(\boldsymbol{s}) - k\boldsymbol{X}(\boldsymbol{s})s) \tag{60}$$

After some mathematical manipulations, one can get the following relationship;

$$\boldsymbol{\Theta}(\boldsymbol{s}) = \frac{ac}{s^2 - ag}(\boldsymbol{U}(\boldsymbol{s}) - k\boldsymbol{X}(\boldsymbol{s})s) \tag{61}$$

Putting this into **X(s)** equation and we get;

$$\frac{\boldsymbol{X}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})}=\frac{(b)s^2+(bdac)s-bag}{s^4+(bk)s^3-(ag)s^2-(agbk)s+bdack} \quad (62)$$

Now, we get the transfer function for cart. From this point one can find the transfer function for pendulum as follows;

$$\frac{\boldsymbol{\Theta}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})}=\frac{ac}{s^2-ag}-\frac{ack}{(s^2-ag)}\frac{((b)s^2+(bdac)s-bag)}{(s^4+(bk)s^3-(ag)s^2-(agbk)s+bdack)} \quad (63)$$

Since it is hard to simplify, it kept as the above expression.

7.2 Transfer Function of Inverted Pendulum with a Reaction Wheel:

According to the linearized equation of motion of inverted pendulum with reaction wheel, one can define the transfer function as follows;

Firstly, one need to take Laplace transform of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ as follows;

$$\boldsymbol{\Theta_1}(\boldsymbol{s})s^2=\frac{(ml+ML)g\boldsymbol{\Theta_1}(\boldsymbol{s})-\boldsymbol{U}(\boldsymbol{s})}{(ML^2+I_{PB})} \quad (64)$$

$$\boldsymbol{\Theta_2}(\boldsymbol{s})s^2=\frac{\boldsymbol{U}(\boldsymbol{s})}{I_{WC}}+\frac{\boldsymbol{U}(\boldsymbol{s})-(ml+ML)g\boldsymbol{\Theta_1}(\boldsymbol{s})}{(ML^2+I_{PB})} \quad (65)$$

So, calling;

$$e=\frac{(ml+ML)g}{(ML^2+I_{PB})} \qquad f=\frac{1}{(ML^2+I_{PB})}$$

It yields to following form;

$$\boldsymbol{\Theta_1}(\boldsymbol{s})s^2=e\boldsymbol{\Theta_1}(\boldsymbol{s})-f\boldsymbol{U}(\boldsymbol{s}) \quad (66)$$

$$\boldsymbol{\Theta_2}(\boldsymbol{s})s^2=I_{WC}^{-1}\boldsymbol{U}(\boldsymbol{s})+f\boldsymbol{U}(\boldsymbol{s})-e\boldsymbol{\Theta_1}(\boldsymbol{s}) \quad (67)$$

After some mathematical manipulations, one can get the transfer function for the pendulum as follows;

$$\frac{\boldsymbol{\Theta_1}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})} = \frac{f}{e - s^2} \tag{68}$$

From now on, one can put this relation to $\boldsymbol{\Theta_2}(\boldsymbol{s})$ and can get the following result;

$$\frac{\boldsymbol{\Theta_2}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})} = \frac{s^2(I_{WC}^{-1} + f) - I_{WC}^{-1}e}{s^4 - es^2} \tag{69}$$

7.3 Transfer Function of Actuator:

Applying the Laplace transform to the dynamic equations of actuator (DC motor), and it can be yields as follows;

$$s(Js + b)\boldsymbol{\Theta}(\boldsymbol{s}) = K\boldsymbol{I}(\boldsymbol{s}) \tag{70}$$

$$(Ls + R)\boldsymbol{I}(\boldsymbol{s}) = \boldsymbol{V}(\boldsymbol{s}) - sK\boldsymbol{\Theta}(\boldsymbol{s}) \tag{71}$$

So, by eliminating **I(s),** one can get transfer function for rotational speed as follows;

$$\frac{\boldsymbol{\Theta}(\mathbf{s})}{\boldsymbol{V}(\boldsymbol{s})} = \frac{K}{(Js + b)(Ls + R) + K^2} \tag{72}$$

8 CONTROL THEORY

In this chapter control theory for both inverted pendulum on a cart and inverted pendulum with a reaction wheel is discussed. Two different controllers are planned to design. Firstly, PID controller is designed for given state space representation of both mathematical models, including motor, after that the LQR one with the same state space representations.

8.1 PID Control

PID is the most commonly known and conventional feedback controller among the other controllers. It measures the output, and compares it with the desired one, following to adjusting a gain value for the corresponding control signal depending on the error value. PID stands for Proportional, Integral and Derivative. So, basically, it alters the control signal with respect to given present error, which is actually under the responsibility of proportional part, also the accumulated error in the past, which is carried by integral part, and the predicted future error, namely input for derivative part. The following figure illustrates the simple block diagram for PID controller.

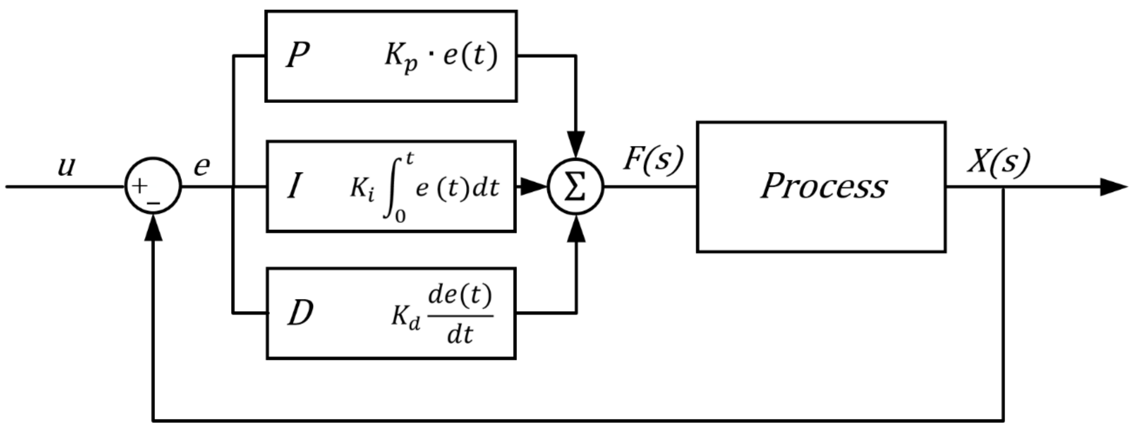

Figure 4: PID feedback block diagram.[4]

8.1.1 PID For Inverted Pendulum on a Cart

In this subsection, PID control approach for inverted pendulum on a cart is discussed. In order to design some controller, one has to define the design constraints, or in other words, the required angle and cart positions. Since we are trying to balance a rod that is on a cart which is driven by the wheels, the initial condition of the rod is crucial, and yet the linearized model dynamics necessitates us to give initial pendulum angle as much as close to the 0 degrees, which is upward equilibrium position for the pendulum. Moreover, we wanted to limit our pendulum angle oscillation region, such that it never trends on being larger than $\pm 10^{o}$. In addition, we are assumed that the cart has placed into infinitely long road, so no need to define a constraint for it, but the velocity of the cart can be finite, such that it can be maximum 1 m/s. This constraint is arbitrary, since we do not have any DC motor to actuate the cart. One more constraint can defined by making an assumption about settling time for the upward equilibrium condition, namely the settling time for pendulum angle. So, for an arbitrariness, we are required to control the system no more than 5 seconds. At the end, one can tabulate the design requirements as following table;

Table 4: Design requirements for inverted pendulum on a cart.

Pendulum Angle	Cannot be more than $\mathbf{10^{o}}$ while controlling.
Cart Velocity	Cannot reach more than 1 m/s.
Pendulum Angle Settling Time	Must be controlled within 5 seconds.

8.1.1.1 System Structure:

The system structure is defined inside SIMULINK. Firstly, the plant model is created from the derived state-space representation, rather than using transfer functions. We assumed that there is a disturbance force acting on the pendulum. Since we want to control pendulum position, which is desired to be balanced at vertical upward position, one need to scheme feedback loop, and a PID controller. As a result of that, the following feedback scheme is defined inside SIMULINK.

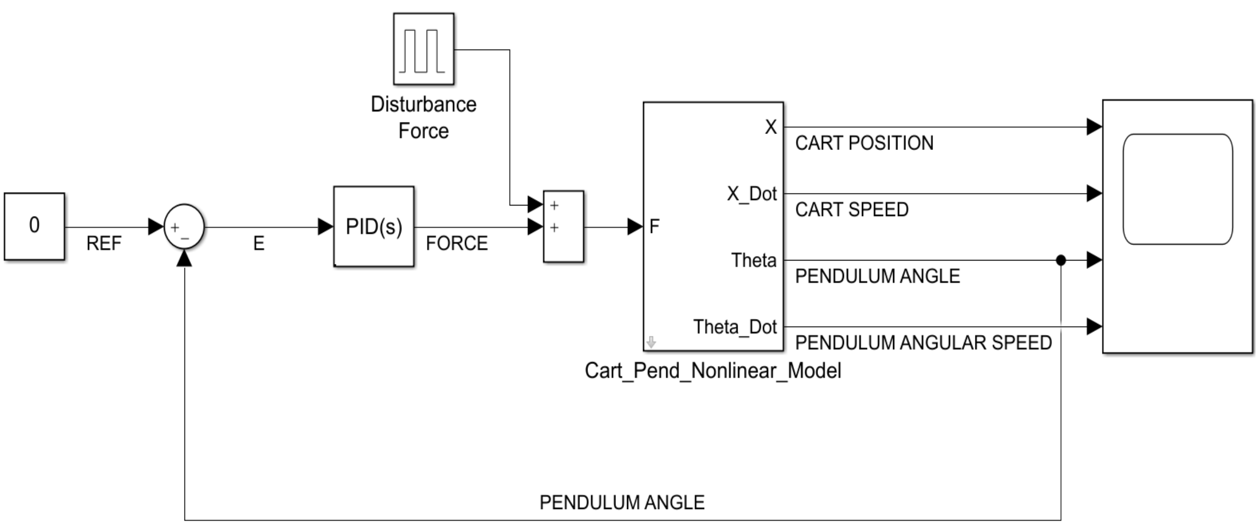

Figure 5: PID Control Structure for Inverted Pendulum on a Cart.

8.1.1.2 Results:

The system is created in Section 8.1.1.1, and from now on, one need to determine the gain for this structure following with the control results. In order to determine the gain, one may use the tune property of the PID function in SIMULINK, which automatically generates P, I and D gains for given plant. Therefore, one can manipulate the tuned gains to ensure that the resultant gain is sufficient enough for given design requirements. The pendulum initial condition is -0.05 rad, and cart position is 0m initially.

Firstly, one can check the results for tuned gains, and the results are yielding as follows;

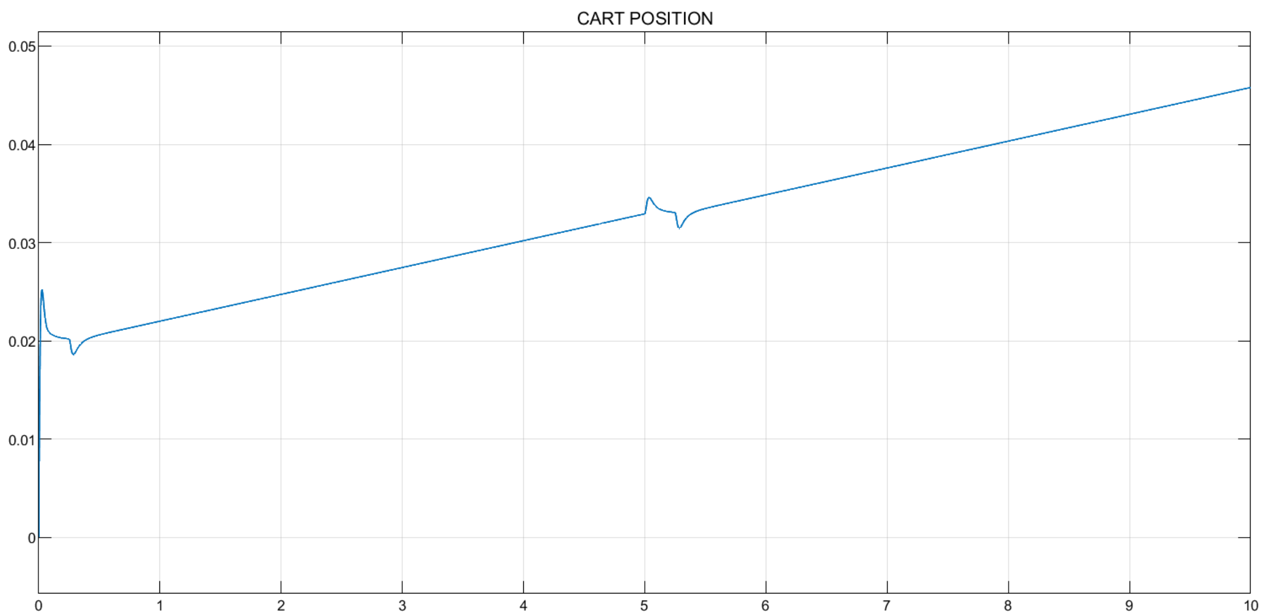

Figure 6:Cart Position(m) vs. Time(s) Graph(Inverted Pendulum PID Tuned Gain)

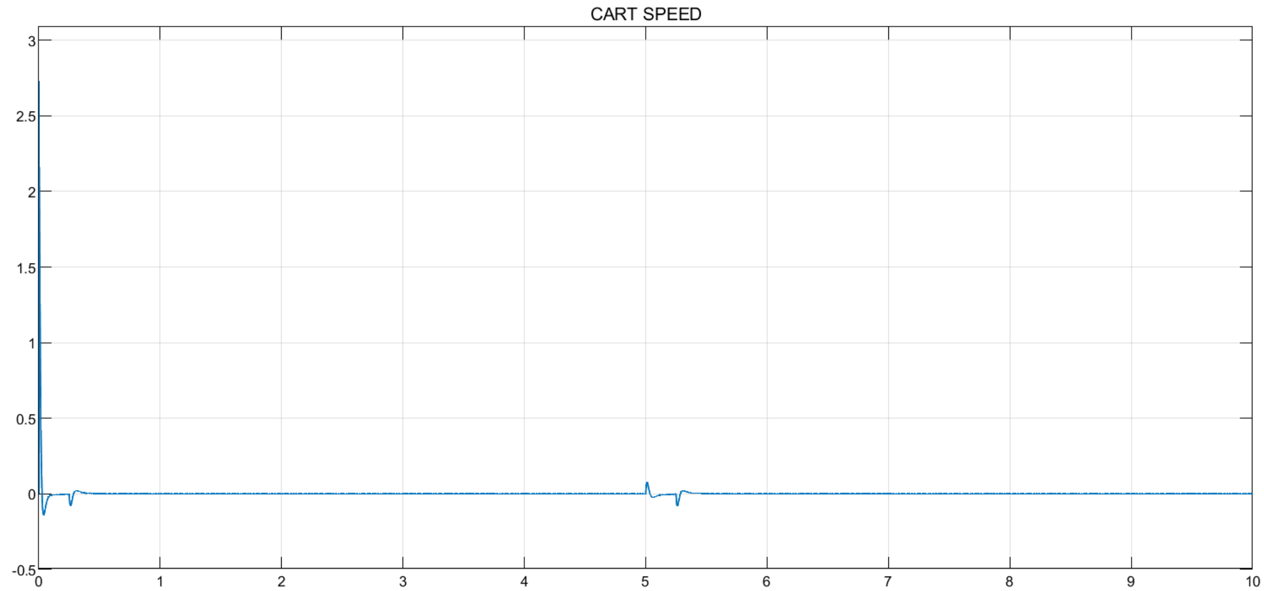

Figure 7:Velocity(m/s) vs. Time(s) Graph(Inverted Pendulum PID Tuned Gain)

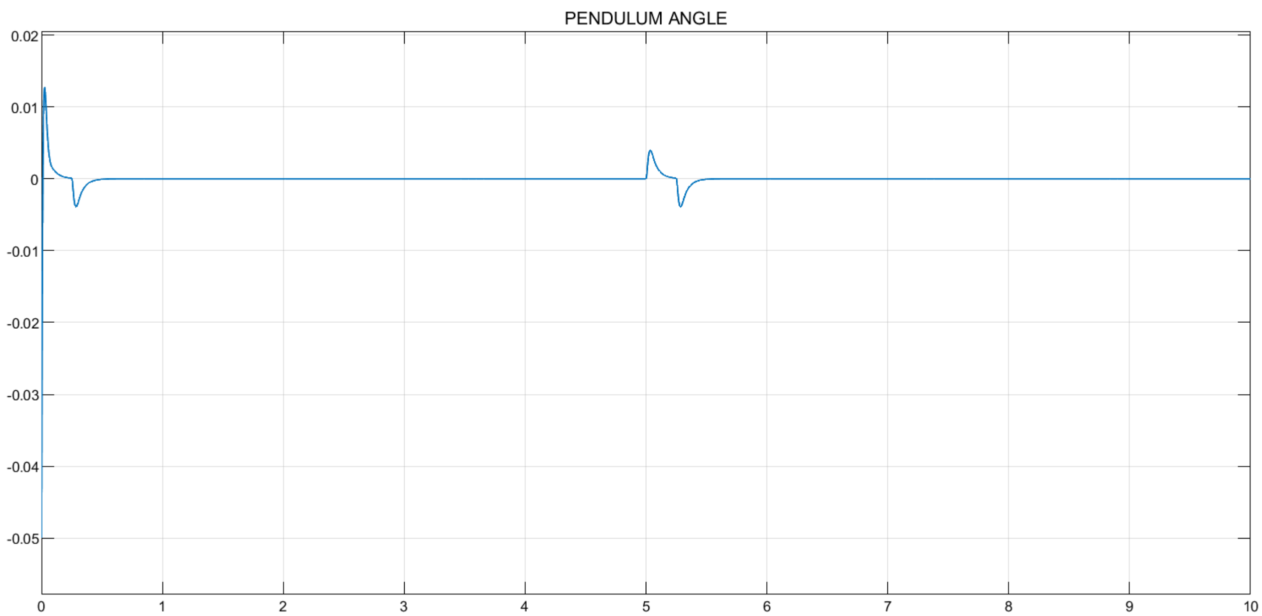

Figure 8:Pendulum Angle(rad) vs. Time(s)(Inverted Pendulum PID Tuned Gain)

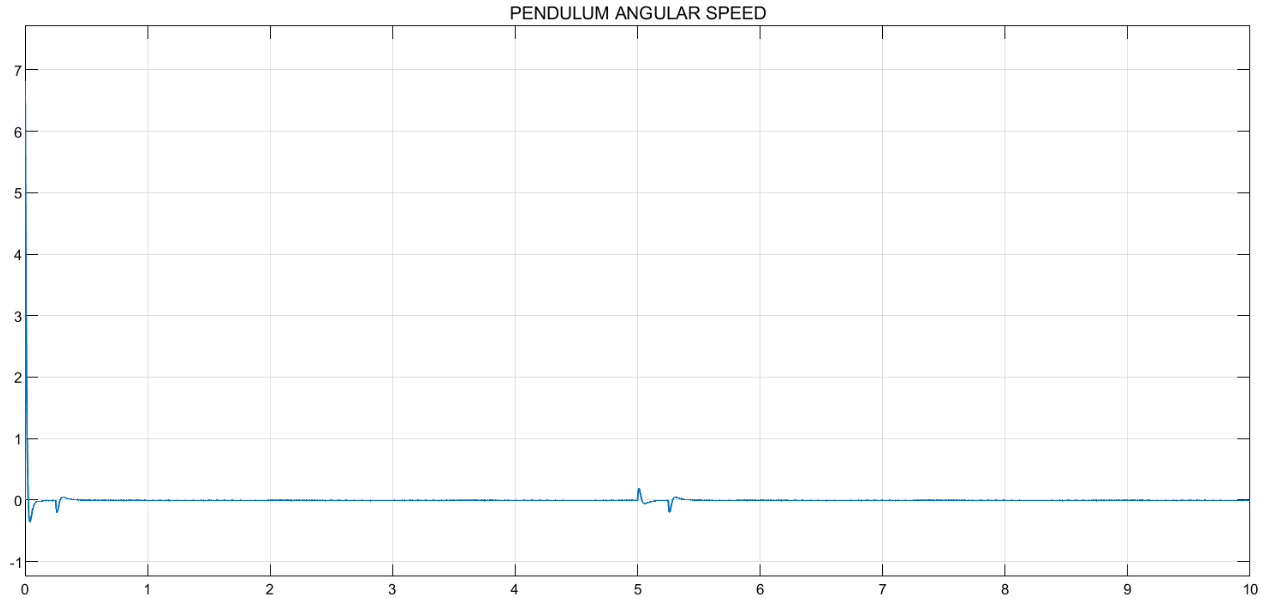

Figure 9:Pendulum Angular Speed(rad/s) vs. Time(s) (Inverted Pendulum PID Tuned Gain)

As seen from these results, the impulse force acting every 5 seconds, and the system responds it with fine control. We want to control the pendulum angle, and it seems it is controlling it, yet the settling time is less than 1 second, and cart velocity is greater than the design constraint. So, one need to alter the gain values and need to obtain the optimum result that satisfies the design requirements. Also, alternation of gain value will be make the design more adaptable to the real application. So, the tuned gain values are given as follows;

$$P = 2223.7429, I = 30940.2995, D = 30.4140 \quad (73)$$

So, one need to alter this gain value. Let us introduce the altered one and check the results whether it satisfies the design requirements or not. Keeping the same initial conditions for inverted pendulum.

$$P = 100, I = 2000, D = 10\ , Filter\ Coefficient = 800 \quad (74)$$

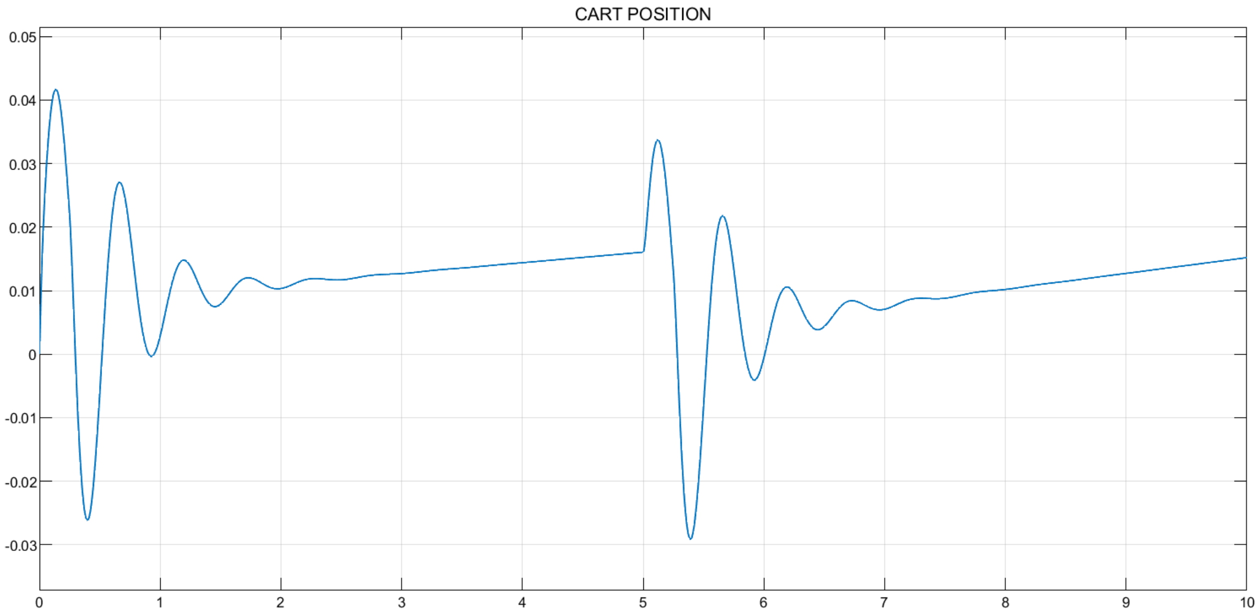

Figure 10:Cart Position(m) vs. Time(s) Graph(Inverted Pendulum PID Altered Gain)

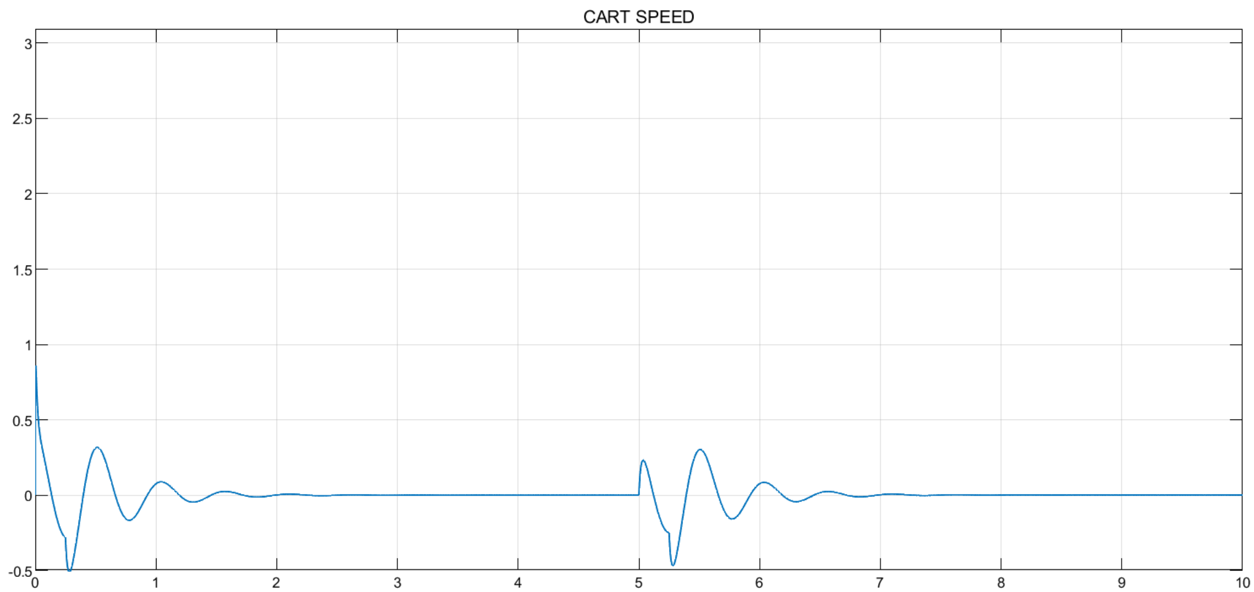

Figure 11:Velocity(m/s) vs. Time(s) Graph(Inverted Pendulum PID Altered Gain)

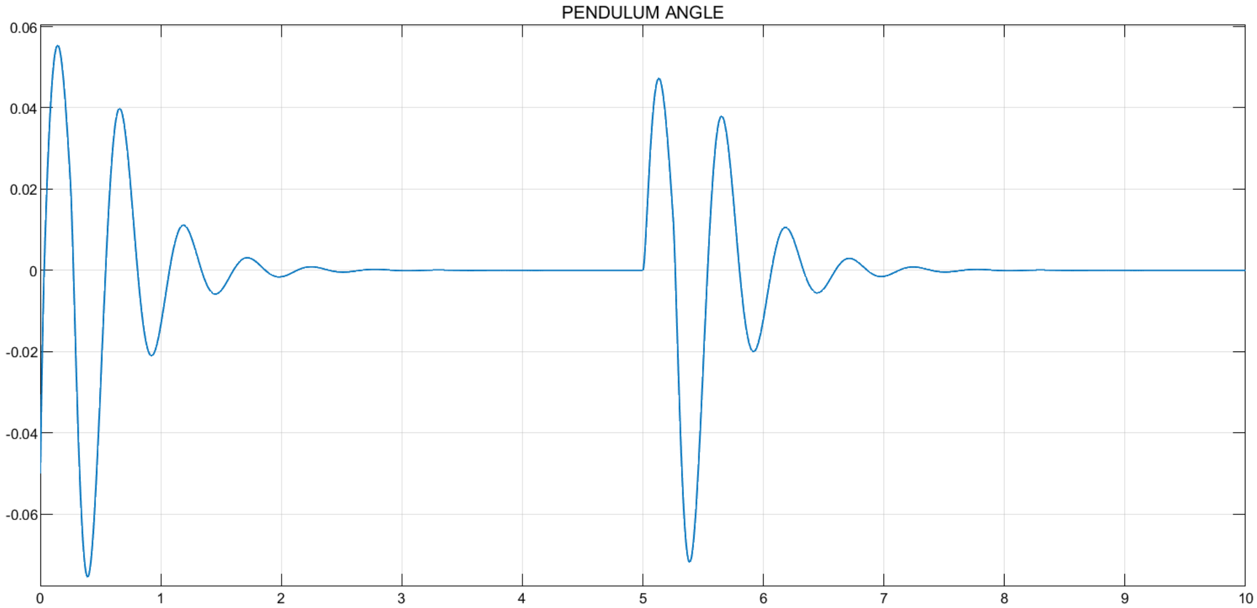

Figure 12:Pendulum Angle(rad) vs. Time(s) (Inverted Pendulum PID Altered Gain)

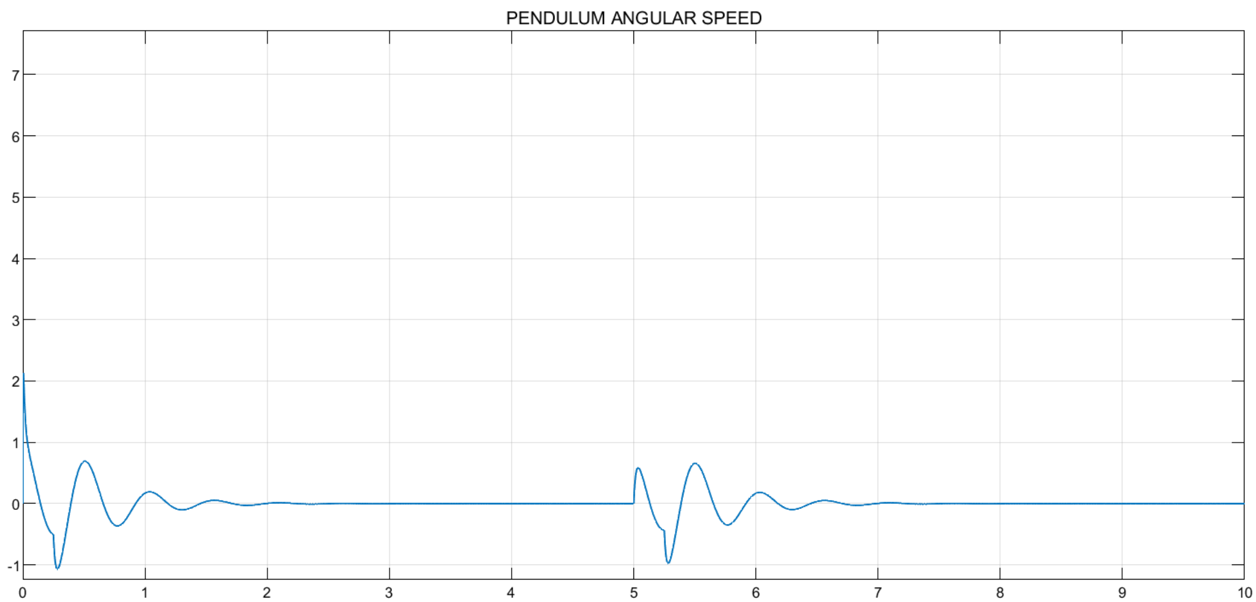

Figure 13:Pendulum Angular Speed(rad/s) vs. Time(s) (Inverted Pendulum PID Altered Gain)

As seen from these results, we satisfied the design requirements in terms of cart velocity and pendulum angle constrains. The cart speed is less than 1m/s and the maximum pendulum angle reaches to 4^o. Also, it settles the pendulum angle from 0.05 rad to 0 about 3 seconds, which means that it also satisfies the settling time. Notice that the disturbance force is acting every 5 second, and the system again controls itself. Even in those conditions, the design requirements are satisfied also.

8.1.2 PID for Inverted Pendulum with a Reaction Wheel

In this subsection, the PID control is applied for inverted pendulum with reaction wheel. Same as the previous step, one has to define the design constraints. From now on, we are trying to balance a rod that is pined to the ground, with a fly wheel. So, if the fly wheel rotates around its rotation axis, the system tends to rotate in the opposite direction of the fly wheel rotation due to the conservation of angular momentum. The most crucial condition of the rod is its initial angle, and one need to use the linearized model of the system, because it allows us the control the system around upward balance condition which is $\pm 10^o$ around of it. More to the constraints, we wanted to limit our pendulum angle oscillation region, such that it never trends on being larger than $\pm 10^o$ such as in the case of inverted pendulum on a cart. In addition, since there is no cart, and we only want to control pendulum angle, the "reaction wheel position" which is actually its rotational position, is redundant, and no need to calculate at all, yet the velocity of the pendulum must be finite, such that it can be maximum 250 rpm. This constraint is arbitrary, since we do not have any DC motor to actuate the reaction wheel, but during the SIMULINK modelling, we introduced the motor dynamics into the reaction wheel. Also, one can defined a time constraint by assuming a settling time for the upward equilibrium condition, namely the settling time for pendulum angle. So, for an arbitrariness, we are required to control the system no more than 5 seconds. At the end, one can tabulate the design requirements as following table;

Table 5: Design requirements for inverted pendulum with reaction wheel.

Pendulum Angle	Cannot be more than $\mathbf{10^o}$ while controlling.
Reaction Wheel Angular Speed	Cannot reach more than 250 rpm.
Pendulum Angle Settling Time	Must be controlled within 5 seconds.

8.1.2.1 System Structure:

System structure is created inside SIMULINK. The plant model is created from the derived state-space representation, rather than using transfer functions for reaction wheel pendulum. We assumed that there is not any disturbance force acting on the pendulum. In this system, we introduced the motor as an actuator for the reaction wheel. Since we want to control pendulum angle, which is desired to be balanced at vertical upward position, one has to make following feedback loop, and a PID controller. The design system structure is given in the following figure.

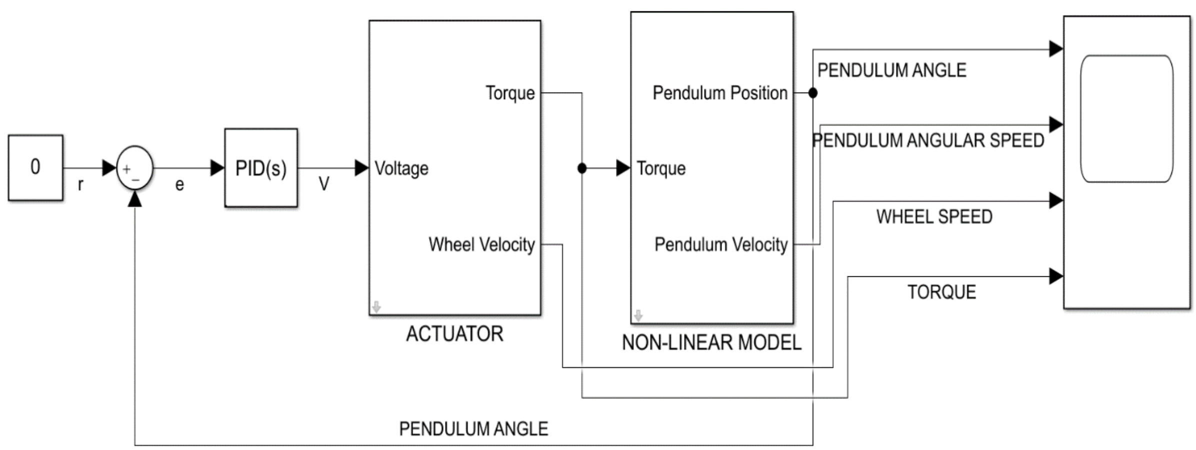

Figure 14:PID Control Structure for Inverted Pendulum with Reaction Wheel

8.1.2.2 Results:

The system is generated in the previous section is now going to be analyzed. From this point, one need to determine the gain for the corresponding structure following with the control results. In order to determine the gain, one may use the tune property of the PID function in SIMULINK, which automatically generates P, I and D gains for given plant. Therefore, one can manipulate the tuned gains to ensure that the resultant gain is sufficient enough for given design requirements. The pendulum Initial condition is set equal to 0.01 rad initially.

So, let us check the results for tuned gains as follows;

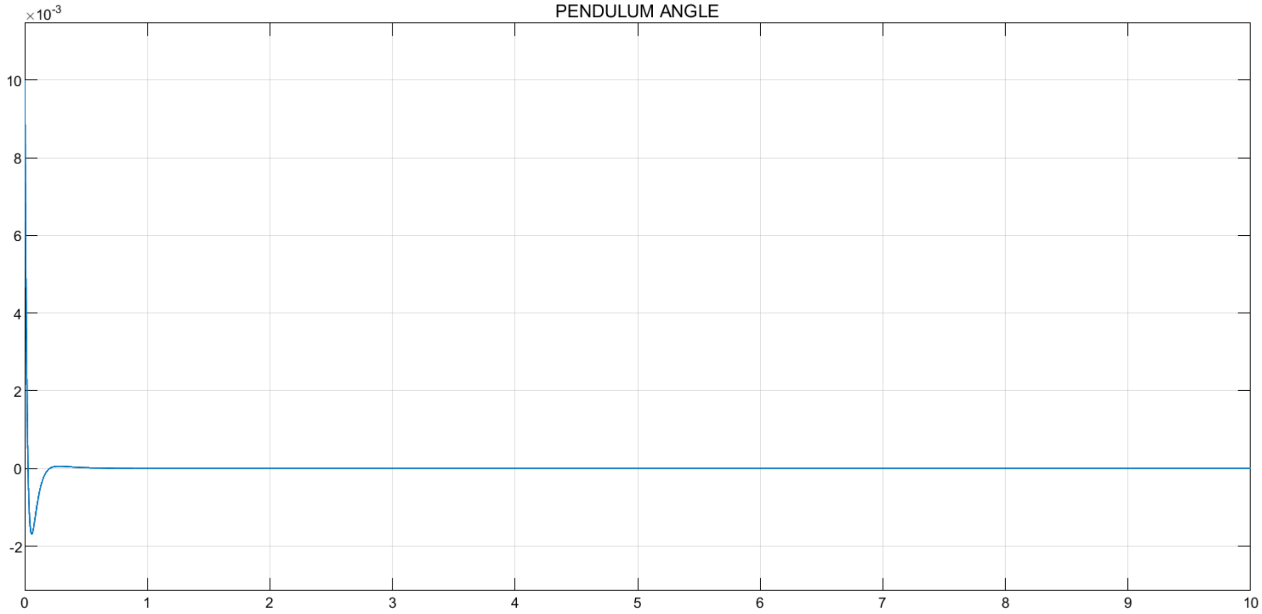

Figure 15:Pendulum Angle(rad) vs. Time(s) (Reaction Wheel Tuned Gain)

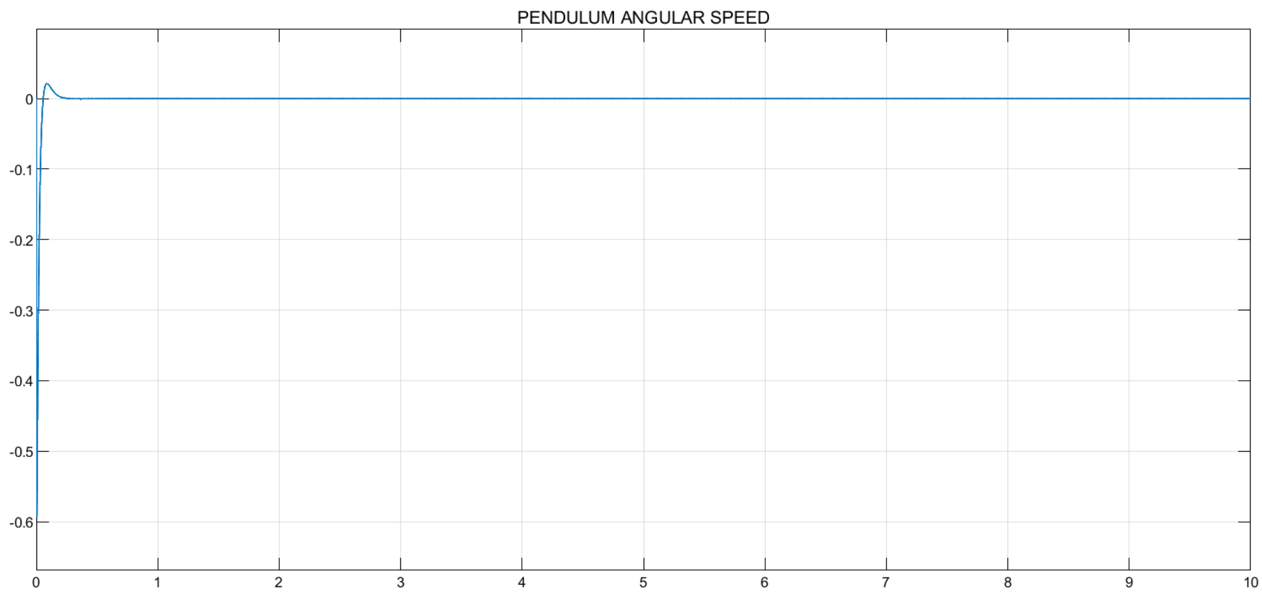

Figure 16:Pendulum Angular Speed (rad/sec) vs. Time(s) (Reaction Wheel Tuned Gain)

Figure 17:Wheel Angular Speed(rad/sec) vs. Time(s) (Reaction Wheel Tuned Gain)

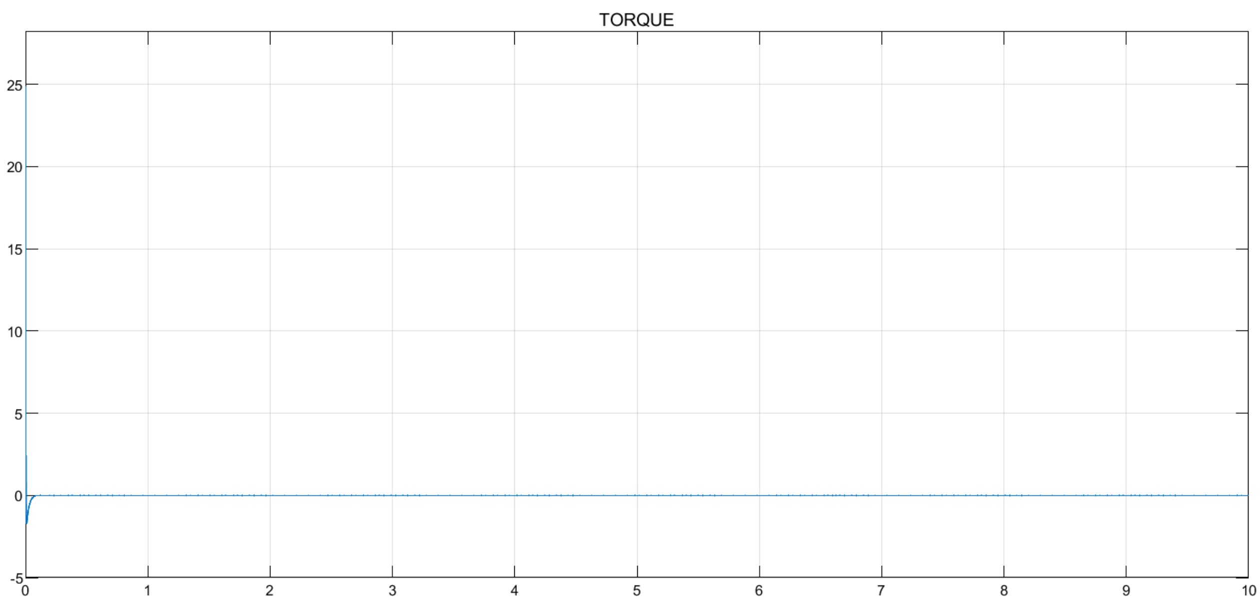

Figure 18:Torque (N.m) vs. Time(s) (Reaction Wheel Tuned Gain)

As seen from these results, the system is controlled with the tuned gain, yet it does not satisfies the design requirements. Since the wheel speed is 140 rad/sec (1336 rpm) at the beginning, and we are requiring something near to 26 rad/sec (150 rpm), it cannot be accepted. Moreover, the torque value is too high in initial state, and the settling time for the pendulum angle is too low, namely 0.2 sec or so. As a result of this, the tuned gain values are not good for our design, so one again need to alter this PID gain, but before this let us show the tuned gain;

$$P = -32515200.8518, I = -142636633.1604, D = -1600441.3951 \tag{75}$$

So, the altering the gain values, and keeping the initial conditions same, it will give as the results as follows;

$$P = -3000000, I = -100000000, D = -100000 \tag{76}$$

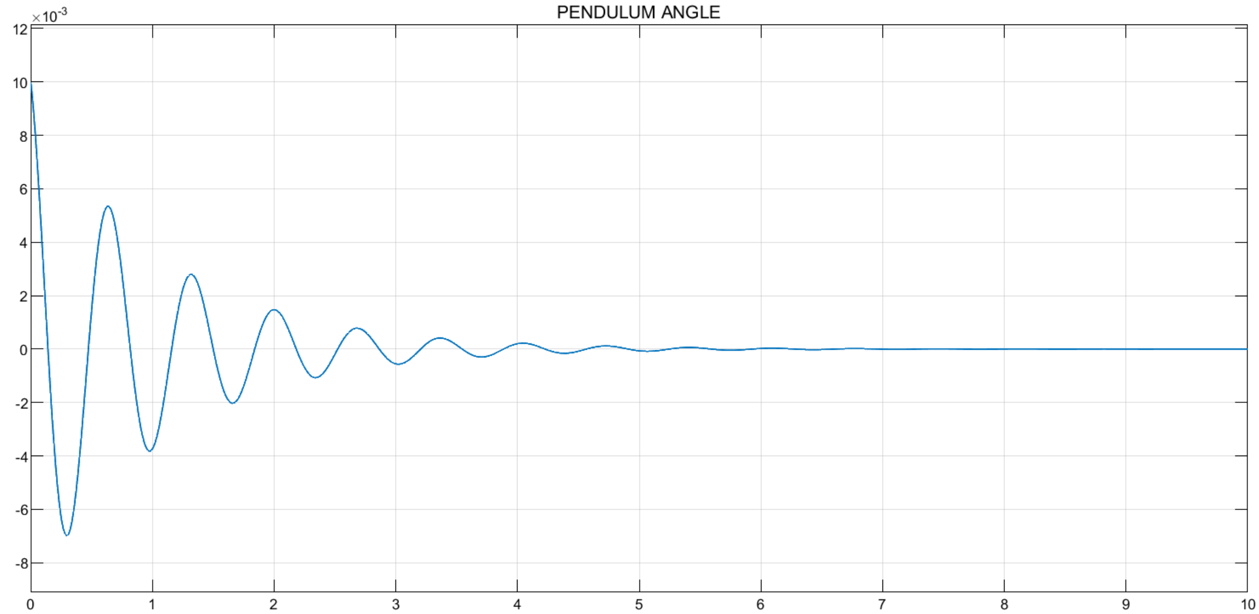

Figure 19: Pendulum Angle(rad) vs. Time(s) (Reaction Wheel Altered Gain)

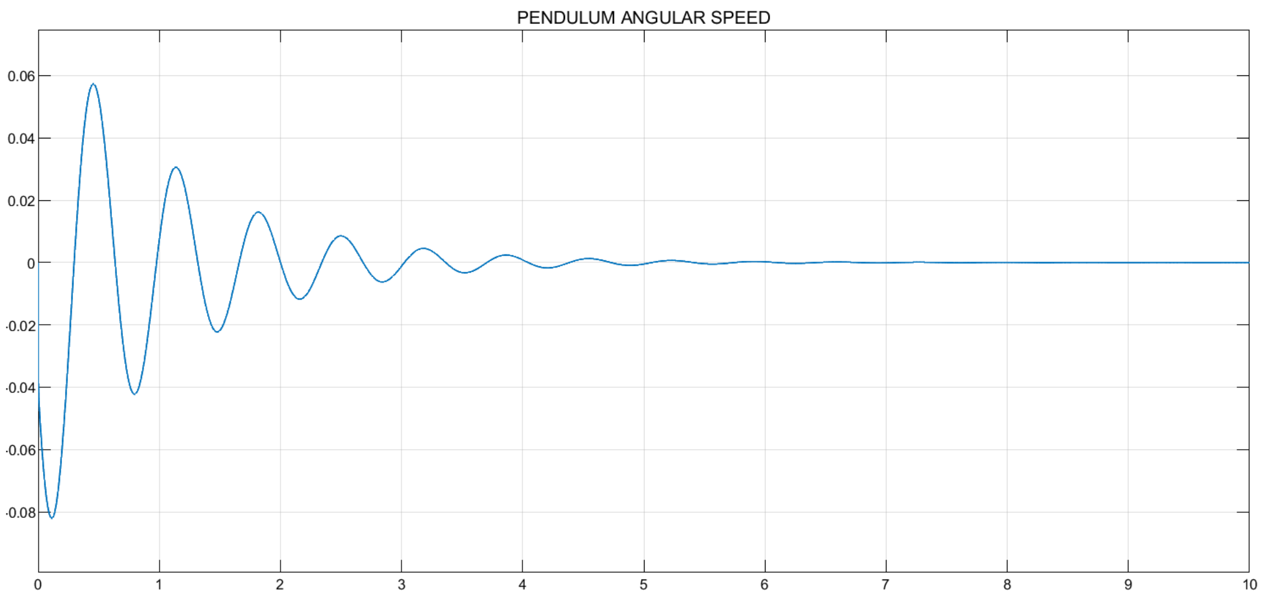

Figure 20: Pendulum Angular Speed(rad/s) vs. Time(s) (Reaction Wheel Altered Gain)

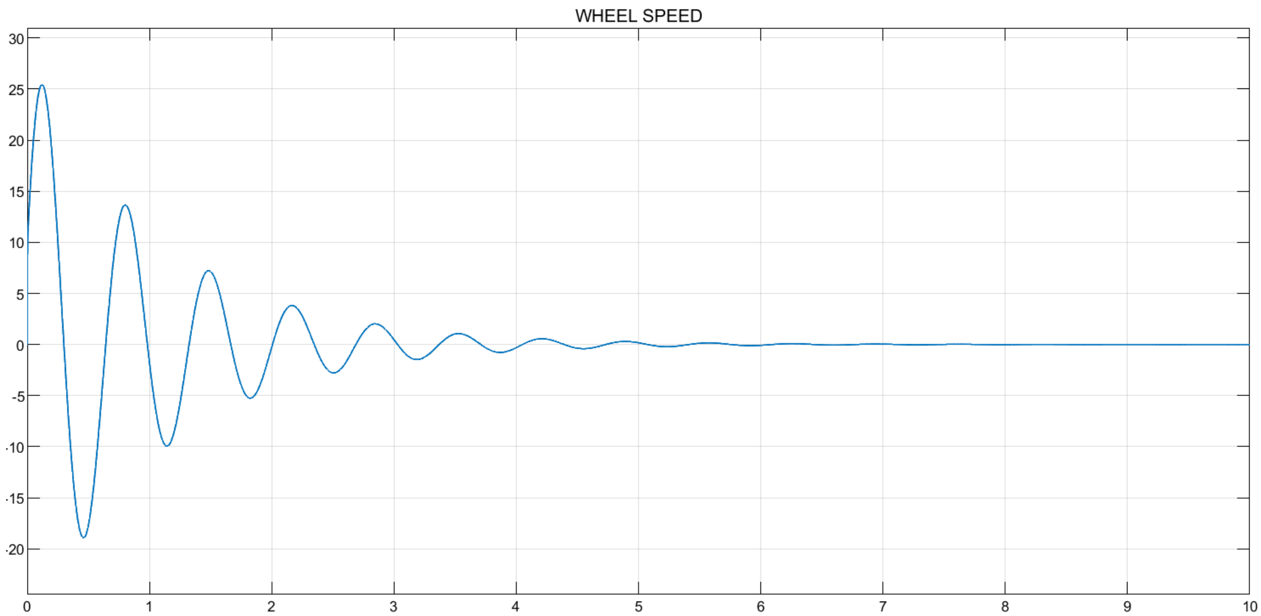

Figure 21: Wheel Speed(rad/s) vs. Time(s) (Reaction Wheel Altered Gain)

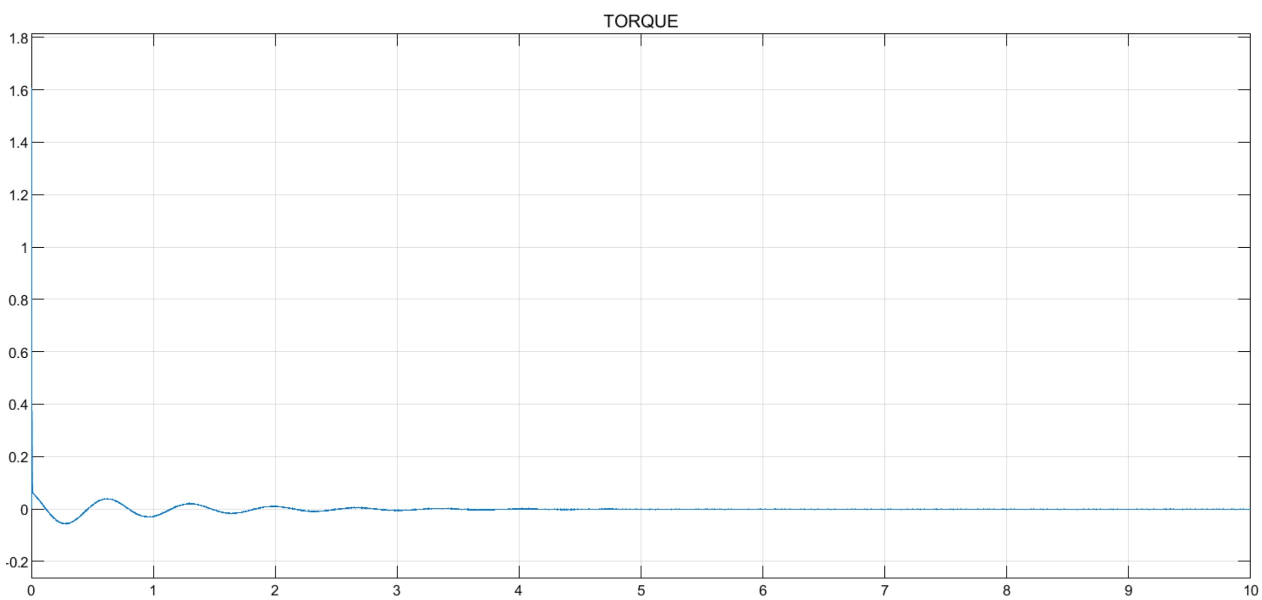

Figure 22: Torque(N.m) vs. Time(s) (Reaction Wheel Altered Gain)

As seen from these results, we have satisfied our design requirements. The wheel velocity is slightly higher than 25 rad/s (238 rpm), and the settling time of pendulum angle is 4 sec or so. Also, the pendulum angle is 0.01 rad initially as expected and it yields to be the maximum value. The second maximum is around 0.007 rad, so which means that it is around 4^o. Since it is less than 10^o, it is already satisfied. Notice that we did not consider any disturbance torque for reaction wheel. There is only one thing, that the torque value is initially 1.5 Nm, yet it is still better than tuned gain torque.

8.2 LQR Control

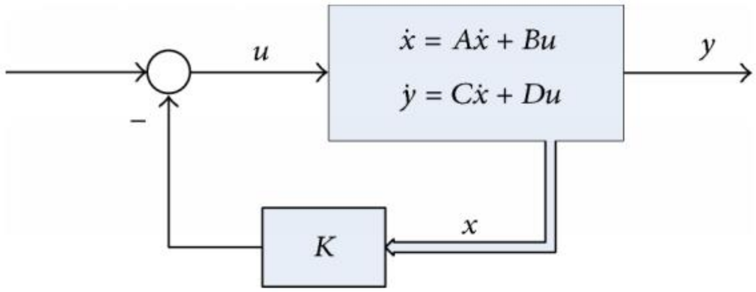

Figure 23:Structure of LQR[5]

The LQR is a well-known method that provides optimally controlled feedback gains to enable the closed-loop stable and high performance design of systems.[7] The LQ controller takes all the model states as input and uses weight matrices Q and R to minimize the cost function .LQR control method helps to find optimal feedback gain vector K by choosing closed-loop characteristic. Pole placement and LQR method appear similar but their technic to find K are different. K matrix is found with using Quadratic cost function and it was given below equation

$$J = \int_0^t (x^T Q x + u^T R u) dt \tag{77}$$

where Q and R cost weights. Q>0 and R≥0 are constant, positive-definite matrices and Q matrix penalizes the transient state deviation, and the R matrix penalizes control effect.[2] This cost function says the how bad if the state is really slow to converge to the stable position. With the changing of cost weight, pendulum can be converge slowly or fast. Cost weight Q has same dimension as states.

Cost function finds the K for the state feedback control system which is added to the state space representation of the system.

$$u = -Kx \tag{78}$$

After finding the gain matrix K, state space representation is obtain as

$$\dot{x} = (A - BK)x \tag{79}$$

8.2.1 LQR For Inverted Pendulum on A Cart

In this subsection, LQR control approach is applied to the inverted pendulum on a cart. In this project, it is used to alternative controller for PID. The purpose is kept same, we are still trying to balance a rod on a cart. The only change is control approach and corresponding block diagram. One need to calculate the gain for the control signal by using derived state space matrices. This step is done by using MATLAB LQR function. Basically, it gives us an optimum gain for given state space matrices. Same as PID approach, the most crucial part to control the pendulum angle is of course the rod's initial angle, and one need to use the linearized model of the system, because it allows us the control the system around upward balance condition which is $\pm 10^o$ around of it. Same design requirements are mandatory for this approach also. So, recalling the requirement table for inverted pendulum on a cart as follows;

Table 6: Design requirements for inverted pendulum on a cart.

Pendulum Angle	Cannot be more than $\mathbf{10^o}$ while controlling.
Cart Velocity	Cannot reach more than 1 m/s.
Pendulum Angle Settling Time	Must be controlled within 5 seconds.

Also, LQR gain calculation requires some cost matrices, which are namely Q and R. Q has same dimension as number of states in the state space form and R is just a constant. Since the diagonal parts of Q matrix determines the cost for every individual state, we need to manipulate them. For inverted pendulum on a cart, we want to control pendulum angle not very aggressively, and our cart velocity is limited, resulting to following cost matrices, which are calculated by trial and error method to ensure design requirements;

$$\boldsymbol{Q} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{79}$$

$$R = 0.008 \tag{80}$$

$$K = [-111.8034 \quad -158.8952 \quad 564.7350 \quad 112.0341] \tag{81}$$

8.2.1.1 System Structure:

In this section, the system structure is discussed for inverted pendulum on a cart model with LQR controller. The most significant difference between PID and LQR controllers is their position in block diagram. PID is placed after the plant model, while ,since LQR only generates a gain, that is placed to the feedback loop as a gain constant.

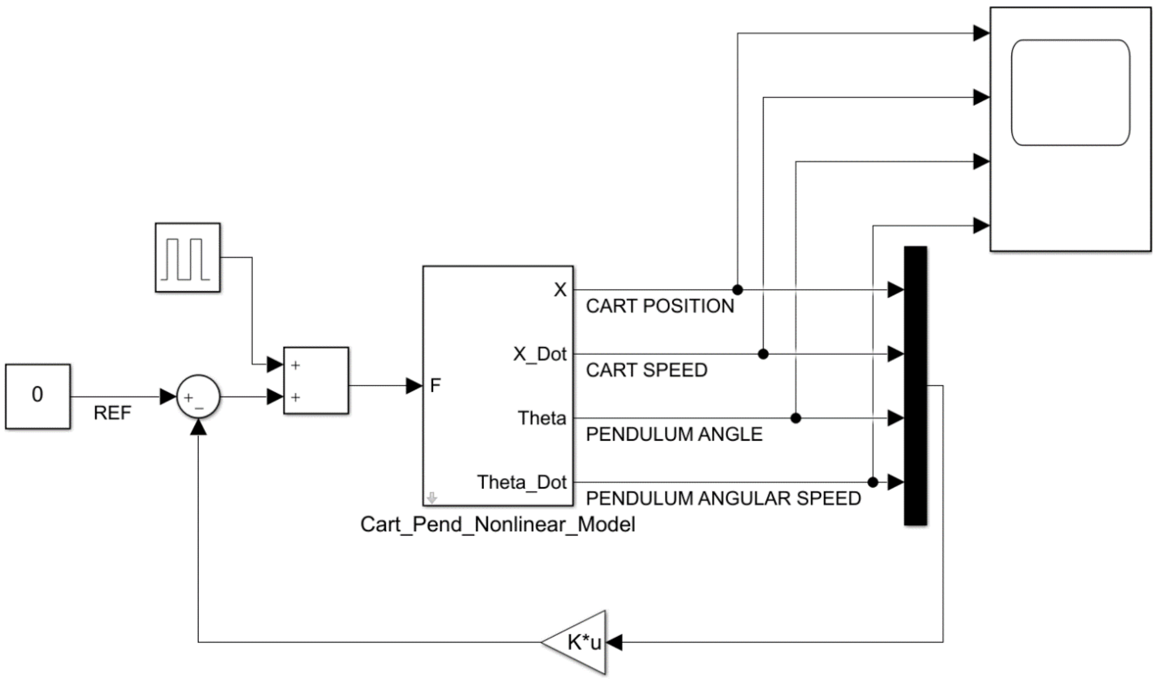

Figure 24:LQR Control Structure for Inverted Pendulum on a Cart.

8.2.1.2 Results:

In this section, results of inverted pendulum on a cart after controlling it with LQR is mentioned. Initial condition of the pendulum angle is kept same as PID case, which means that it is -0.05 rad. Initial cart position is also kept same, namely it is at 0m with respect to the defined reference frame.

Same disturbance force is assumed acting on the body (every 5 second with an amplitude of 10), and the motor dynamics are not considered for inverted pendulum on a cart. State space model of cart-pendulum is used for this analysis, rather than transfer function.

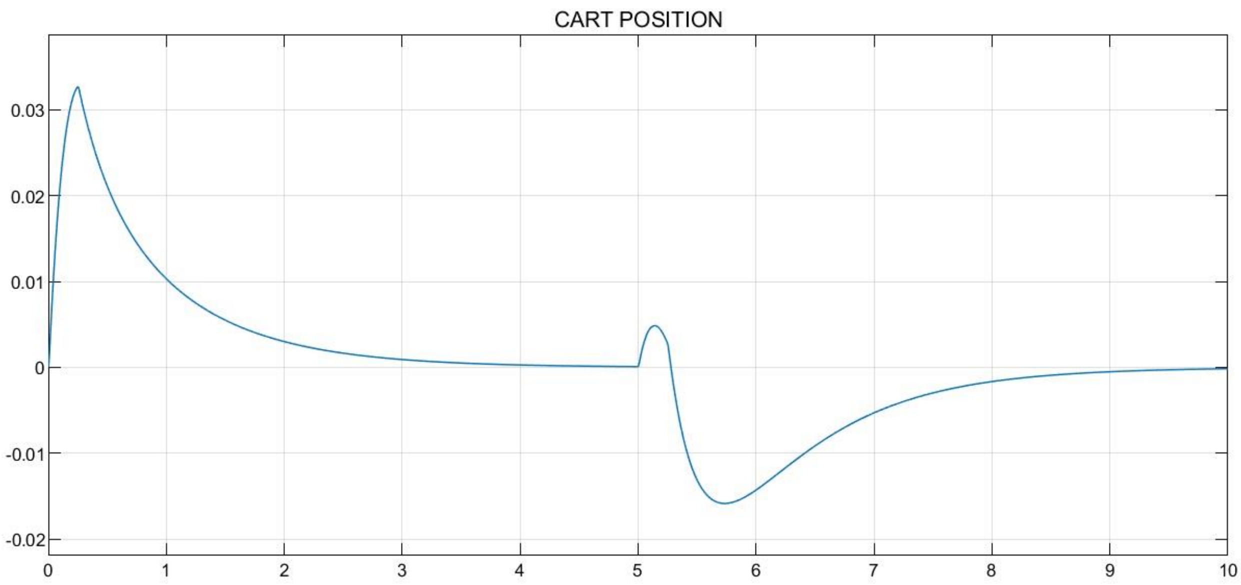

Figure 25: Cart Position(m) vs. Time(s) (Inverted Pendulum on a Cart LQR)

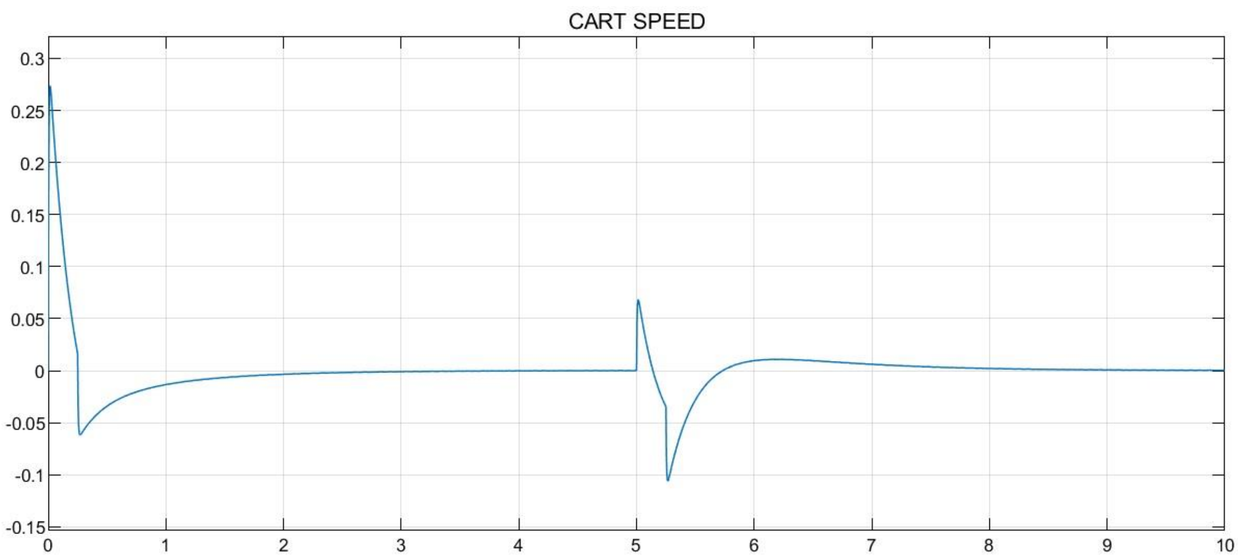

Figure 26: Cart Speed (m/s) vs. Time(s) (Inverted Pendulum on a Cart LQR)

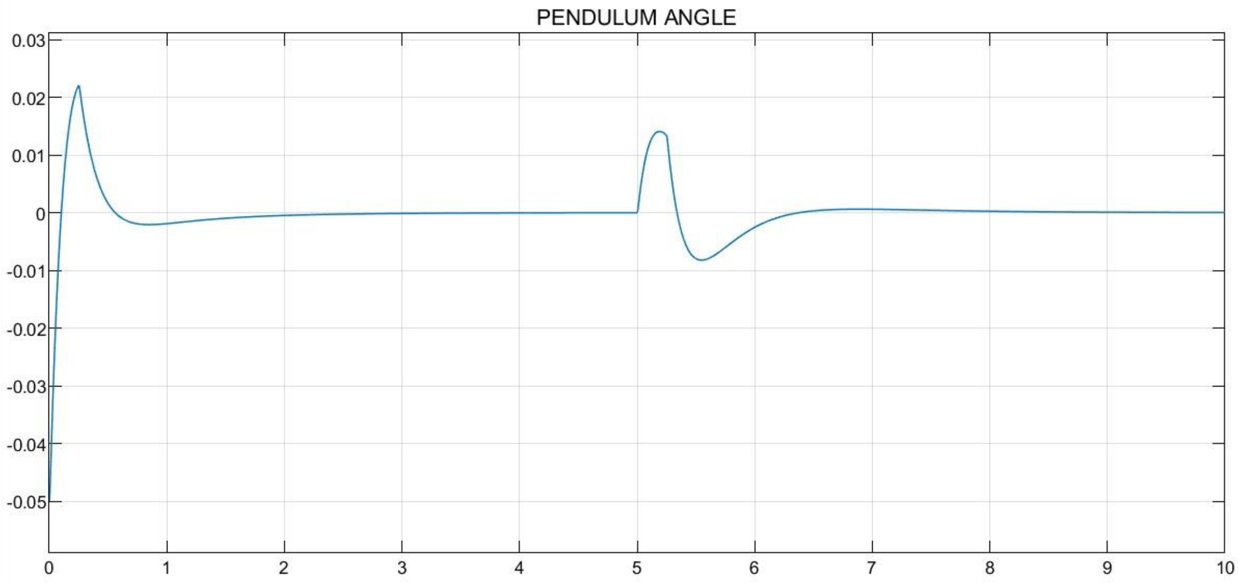

Figure 27: Pendulum Angle(rad) vs. Time(s) (Inverted Pendulum on a Cart LQR)

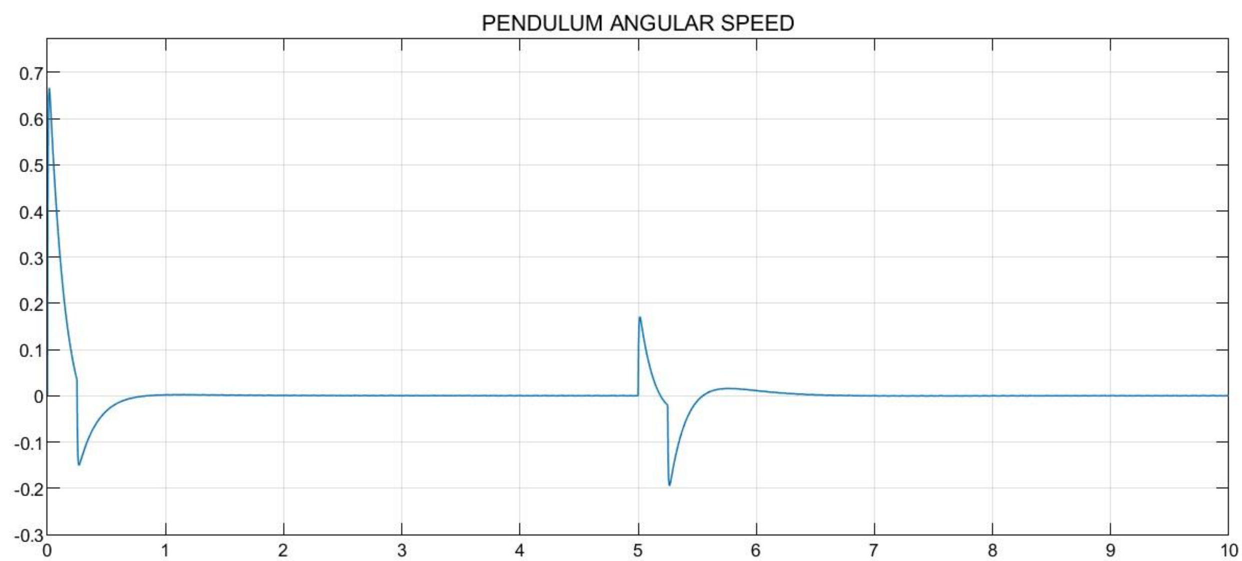

Figure 28: Pendulum Angular Speed(rad/s) vs. Time(s) (Inverted Pendulum on a Cart LQR)

As seen from these results, we have satisfied the design requirements, namely pendulum angle constraint, cart velocity constraint and required settling time for pendulum angle. In first look, it seems controlling very aggressively, but it can be acceptable. Cart speed is 0.25 m/s at maximum, and compared to the PID, it predominates PID in terms of cart speed. About pendulum angle, it got maximum value of around 0.02 rad, which is again satisfies the requirements, and less than PID.

8.2.2 LQR For Inverted Pendulum with Reaction Wheel

In this section, LQR control method is used to control an inverted pendulum with reaction wheel. It is again used to alternative controller for PID. The purpose is kept same, we are still trying to balance a rod with a principle of conservation of angular momentum. Only change is control approach and corresponding block diagram as mentioned in the previous section. One need to calculate the gain for the control signal by using derived state space matrices. Also, we introduced actuator to this model, which creates torque as in the PID case, in order to make a cross comparison. The gain is again calculated in MATLAB, with the LQR build in function. Basically, it gives us an optimum gain for given state space matrices. Same as PID approach, the most crucial part to control the pendulum angle is of course the rod's initial angle, and one need to use the linearized model of the system, because it allows us the control the system around upward balance condition which is $\pm 10^{o}$ around of it, same conditions as the cart case. Same design requirements are mandatory for this approach also. So, recalling the requirement table for inverted pendulum with reaction wheel as follows;

Table 7: Design requirements for inverted pendulum with reaction wheel.

Pendulum Angle	Cannot be more than $\mathbf{10^{o}}$ while controlling.
Reaction Wheel Angular Speed	Cannot reach more than 250 rpm.
Pendulum Angle Settling Time	Must be controlled within 5 seconds.

Also, LQR gain calculation requires some cost matrices, which are namely Q and R. Q has same dimension as number of states in the state space form and R is just a constant. Since the diagonal parts of Q matrix determines the cost for every individual state, we need to manipulate them. For inverted pendulum with reaction wheel, we want to control pendulum angle not very aggressively, and our cart velocity is limited, resulting to following cost matrices, which are calculated by trial and error method to ensure design requirements;

$$\boldsymbol{Q} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} \tag{82}$$

$$R = 0.0008 \tag{83}$$

$$K = [-1110346.1115 \quad -201460.1003 \quad -351.4723] \tag{84}$$

8.2.2.1 System Structure:

In this section, the system structure is discussed for inverted pendulum with reaction wheel model with LQR controller. The most significant difference between PID and LQR controllers is their position in block diagram. PID is placed after the plant and actuator model, while ,since LQR only generates a gain, that is placed to the feedback loop as a gain constant. Again, note that, we are not considering any disturbance torque for our model as in PID case.

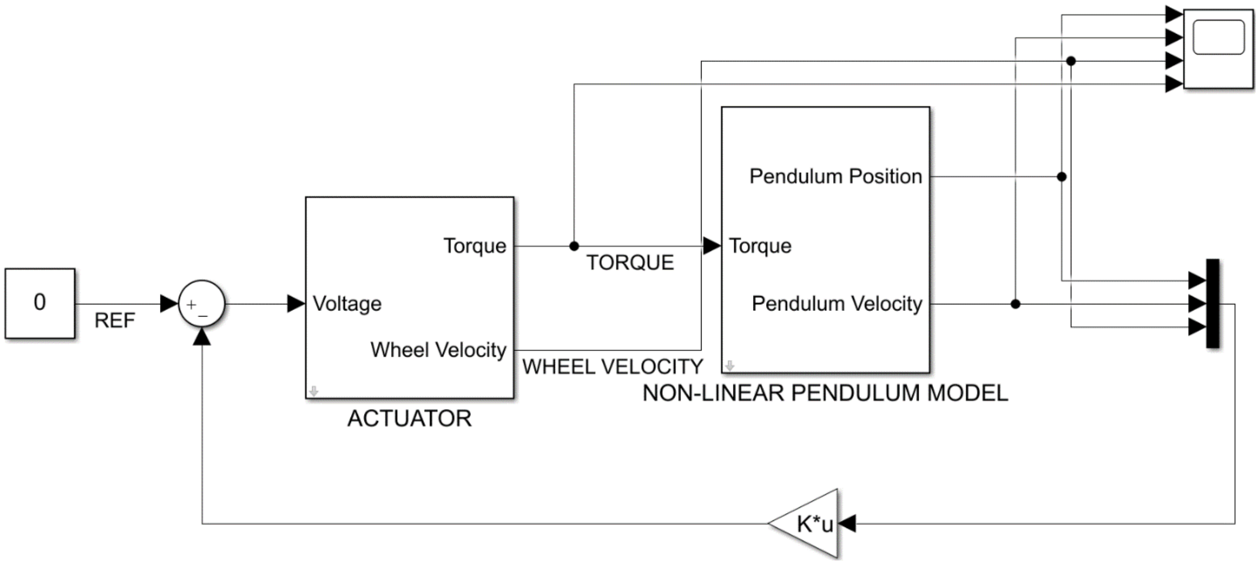

Figure 29: LQR Control Structure for Inverted Pendulum with Reaction Wheel.

8.2.2.2 Results:

In this section, results of inverted pendulum with a reaction wheel that is controlled by LQR is discussed. Initial condition of the pendulum angle is kept same as PID case, which means that it is 0.01 rad. Assumed no disturbance force is acting on the body, and the motor dynamics are included for inverted pendulum with reaction wheel. State space model of reaction is used for this analysis, rather than transfer function.

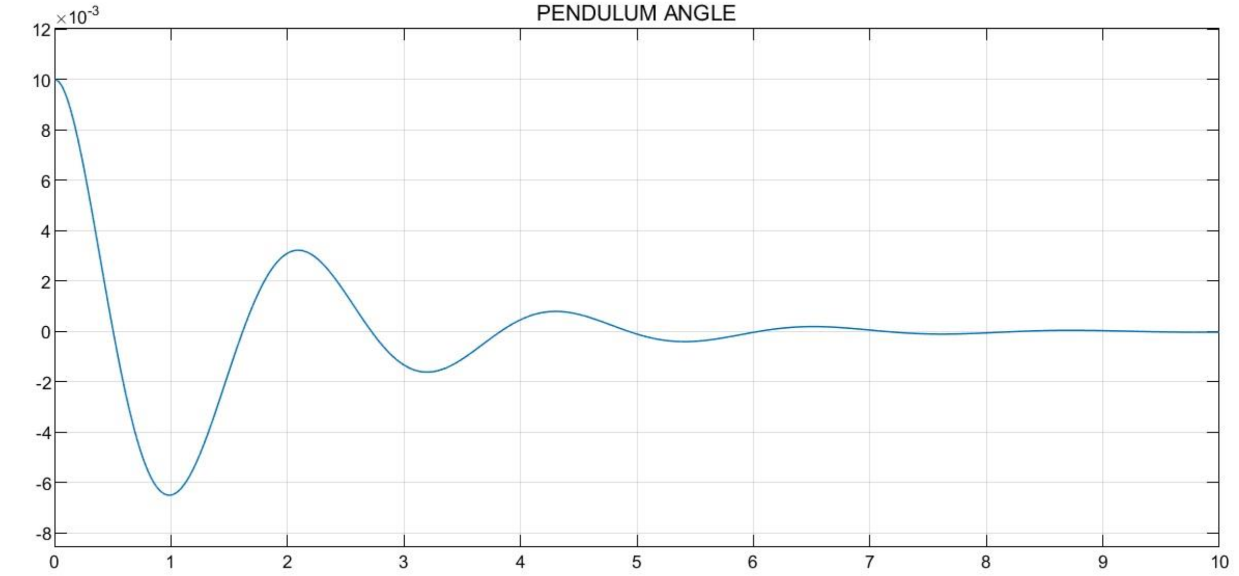

Figure 30: Pendulum Angle(rad) vs. Time(s) (Reaction Wheel LQR)

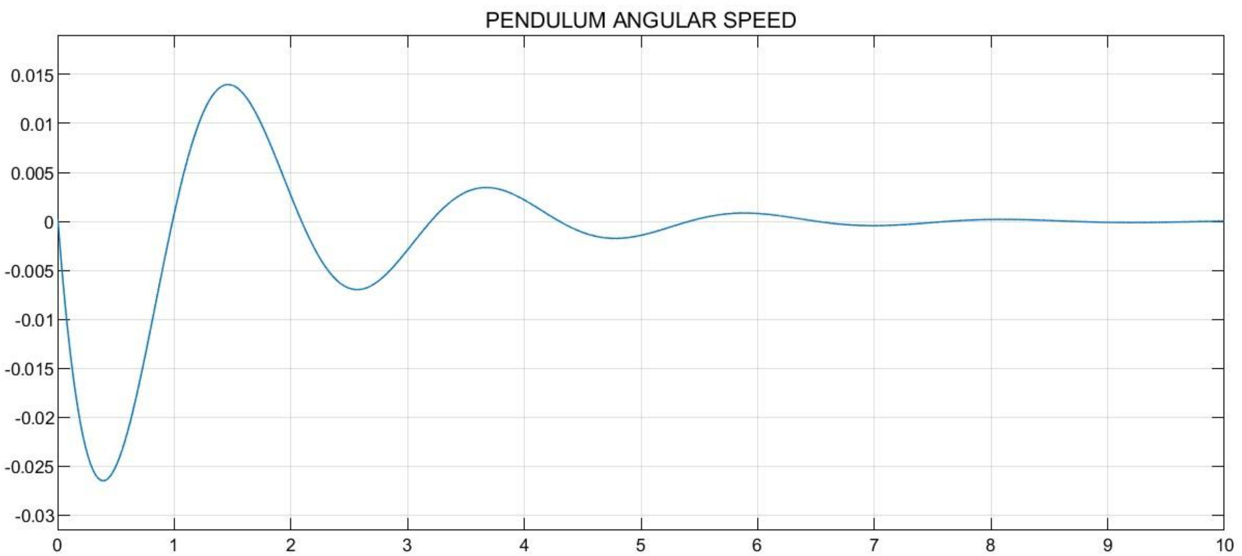

Figure 31: Pendulum Angular Speed(rad/s) vs. Time(s) (Reaction Wheel LQR)

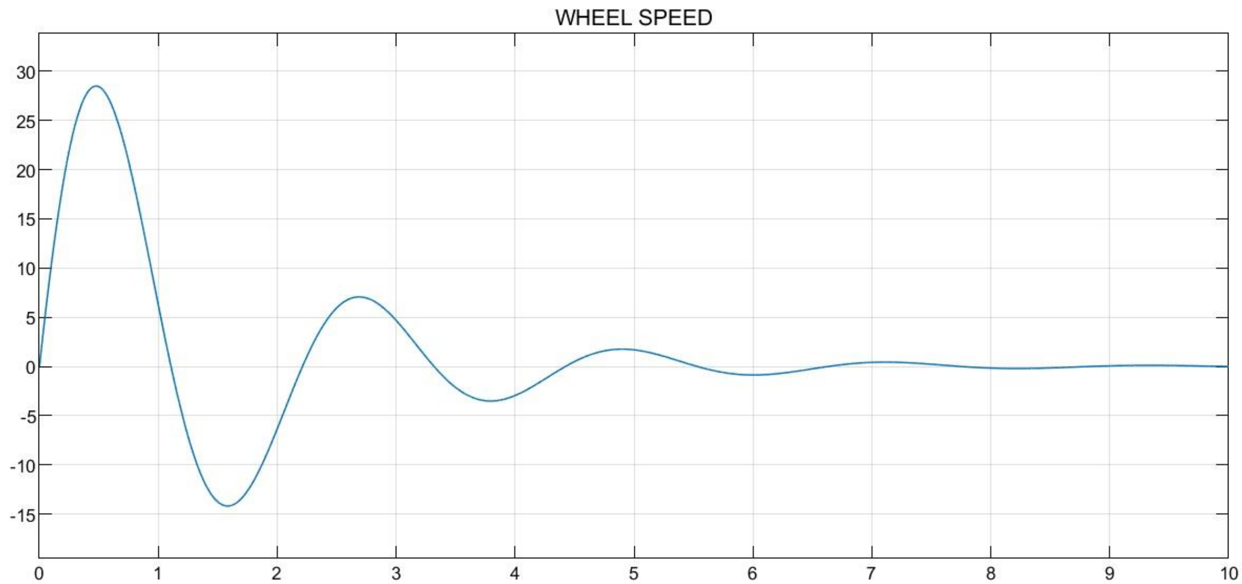

Figure 32: Wheel Speed (rad/s) vs. Time(s) (Reaction Wheel LQR)

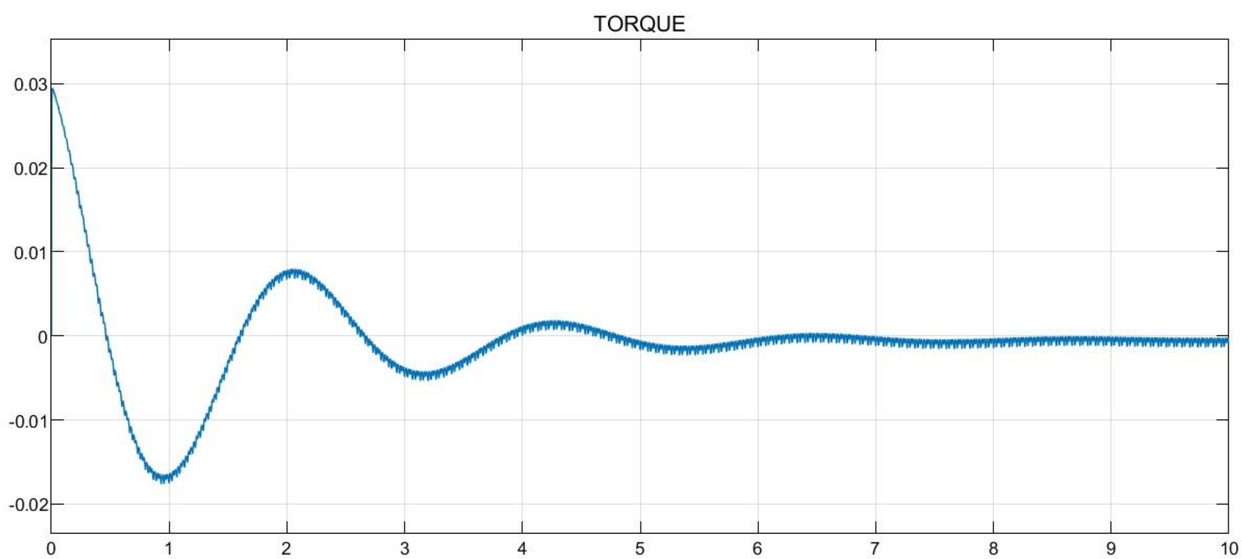

Figure 33: Torque (N.m) vs. Time(s) (Reaction Wheel LQR)

As seen from these results, we have achieved the desired control requirements. In first look, the wheel velocity is slightly higher than 26 rad/s (257 rpm) which is not good, since our constraint is 250 rpm. The pendulum angle is nearly same as the PID results, but the pendulum angular speed is predominated by the LQR against PID. Also, torque value is smaller in this controller (0.03N.m initially) compared to PID, which is also good.

9 CONCLUSION

In this graduation project, controlling of the inverted pendulum with a cart and inverted pendulum with a reaction wheel have been done, and understood the process of the making stable the unstable inverted pendulum systems. The attitude control system is a quite important subsystem of a spacecraft and reaction wheel is an actuator to control the spacecraft in space with respect to the inertial fixed frame or other body's frame. For this reason, this project aimed to understand the control theories for the reaction wheel.

During this project, firstly the subject has been selected as inverted pendulum and it is made stable with a cart, and to do that firstly mathematical modeling of the inverted pendulum has been derived and then state-space equations have been written. Then, to control the system; LQR and PID have been used. At the end of the controlling the inverted pendulum with a cart, the new subject (inverted pendulum with reaction wheel) studied. Same with cart inverted pendulum the mathematical modeling has been obtained and state-space equations have been written. After these steps, to make stable the unstable reaction wheel inverted pendulum PID and LQR have used.

In the overall aspect, students tried to control both inverted pendulum on a cart and with reaction wheel by two different controllers, namely LQR and PID. Both controllers have their own advantages and disadvantages. Beside of that, PID is predominated by LQR in some cases and vice versa.

For inverted pendulum on a cart, PID controller is used with tuned gain and altered gain. These two cases are required because of initially defined design requirements are not satisfied by the tuned gain. By trial and error method, the tuned gain is altered, and the most optimum gain is obtained in order to satisfy the requirements, because settling time of the tuned gain PID was very aggressive (it can control the system within 0.1 seconds which is too swift.). Also, the cart velocity seems too much, namely 2.7 m/s , while the design requirement limits it as 1m/s. As a result of that, students decided to alternate the tuned gain by a trial and error method to find a gain value that satisfies the requirements. At the end of PID controller section, we got 0.05 rad maximum pendulum angle, maximum cart speed of 0.8m/s and about 3 seconds of settling time, which completely satisfies the design requirements. On the consideration of LQR, one need a gain value that satisfies the design requirement also. So, in this aspect, an arbitrary cost matrix is created and with the help of LQR function in MATLAB, optimum gain value is obtained. As a result of that, students got 0.02 rad maximum pendulum angle, maximum cart speed of 0.26 m/s and settling time of 2 seconds. Those results are comparably better than PID results for inverted pendulum on cart case.

For inverted pendulum with reaction wheel, same approach is followed. Firstly, PID controller is design then it is compared with the LQR. For PID case, tune option is used, and once again very aggressive outputs are obtained, namely 140 rad/sec for maximum wheel speed and 25N.m for maximum torque. Those values are too high for design requirements, including the settling time for pendulum angle, 0.1 sec or so. Therefore, one need to alternate this gain value, and the corresponding out puts are; 25 rad/sec for maximum wheel speed, 1.6N.m for maximum torque and nearly 4 second of settling time for pendulum angle. These results are good for design requirements, but one can still make a comparison between LQR and PID. So, in LQR design, the cost function is assumed that it gives the optimum value for the requirements. At the end, outputs are obtained and compared with the PID results. Hence, LQR results are basically gives us; 26 rad/sec for maximum wheel speed, and 0.03 N.m maximum torque. The wheel speed result is nearly at the edge of our constraint, so it is not good for the design.

As a conclusion, one can say that for inverted pendulum on a cart is controlled much better in LQR approach than PID. On the contrast of that, control of inverted pendulum with reaction wheel is carried out by PID is more preferable than LQR.

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FIGURE REFERENCES

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