Lectures on

Fourier Analysis

Delivered By

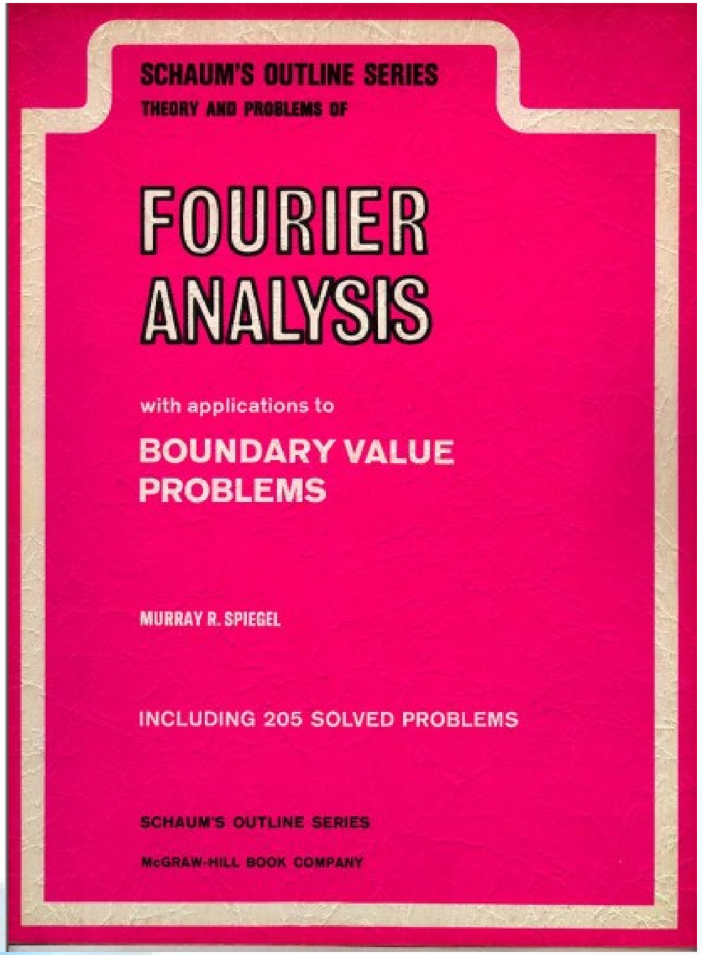
Md. Nuruzzaman
Assistant Professor
Department of Mathematics
Mawlana Bhashani Science and Technology
Santosh,Tangail-1902
Mobile: 01740066434
Email: mnmath24@gmail.com

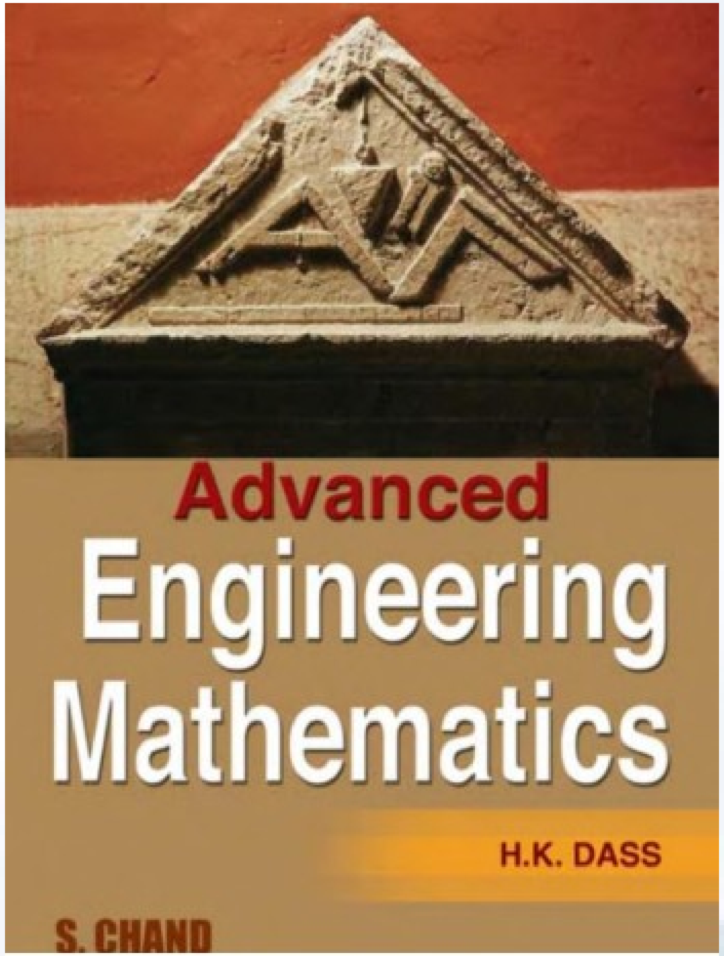
Outlines

- **Fourier series**
  - Periodic function
  - Piecewise continuous function
  - Fourier series
  - Half range Fourier series
  - Application of Fourier series to solve Boundary Value problems

- **Fourier Integral and Fourier Transformation**
  - Fourier Transformation and inverse Fourier Transformation
  - Application of Fourier Transform

Reference books





PERIODIC FUNCTIONS

A function $f(x)$ is said to have a *period* P or to be *periodic* with period P if for all x, $f(x+P) = f(x)$, where P is a positive constant. The least value of $P > 0$ is called the *least period* or simply *the period* of $f(x)$.

Example 1.

The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$, since $\sin(x+2\pi)$, $\sin(x+4\pi)$, $\sin(x+6\pi)$, ... all equal $\sin x$. However, 2π is the *least period* or *the period* of $\sin x$.

Example 2.

The period of $\sin nx$ or $\cos nx$, where n is a positive integer, is $2\pi/n$.

Example 3.

The period of $\tan x$ is π.

Example 4.

A constant has any positive number as a period.

Other examples of periodic functions are shown in the graphs of Fig. 2-1.

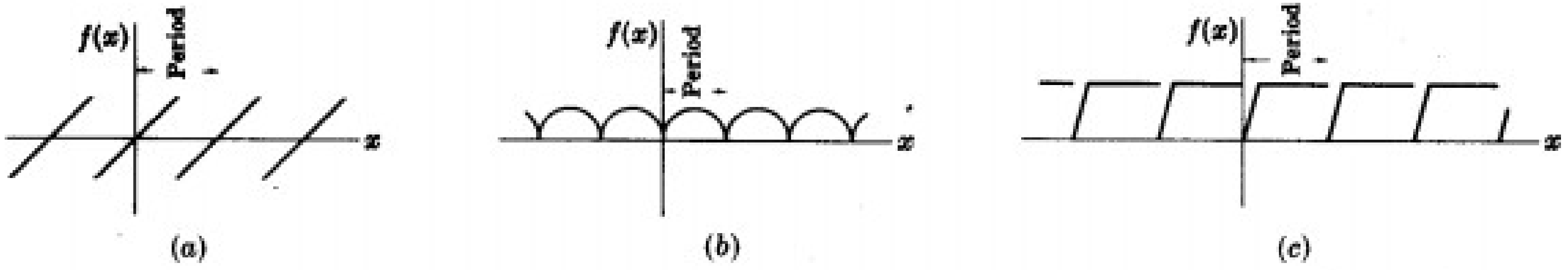


Fig. 2-1

Examples

Graph each of the following functions.

$$(a)\quad f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases} \qquad \text{Period} = 10$$

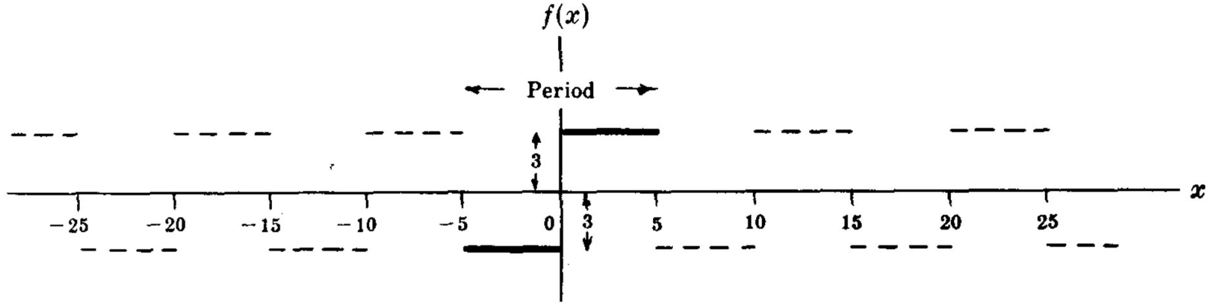


Fig. 2-3

Since the period is 10, that portion of the graph in $-5 < x < 5$ (indicated heavy in Fig. 2-3 above) is extended periodically outside this range (indicated dashed). Note that $f(x)$ is not defined at $x = 0, 5, -5, 10, -10, 15, -15$, etc. These values are the *discontinuities* of $f(x)$.

$$(b)\quad f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad \text{Period} = 2\pi$$

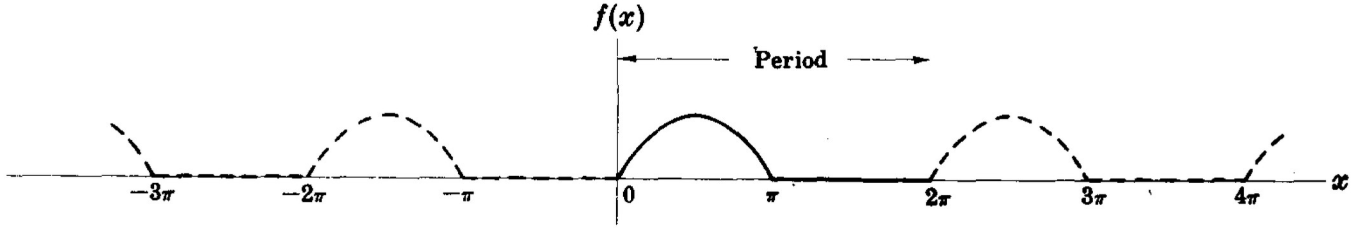


Fig. 2-4

Refer to Fig. 2-4 above. Note that $f(x)$ is defined for all x and is continuous everywhere.

(c) $$f(x) = \begin{cases} 0 & 0 \leqq x < 2 \\ 1 & 2 \leqq x < 4 \\ 0 & 4 \leqq x < 6 \end{cases} \qquad \text{Period} = 6$$

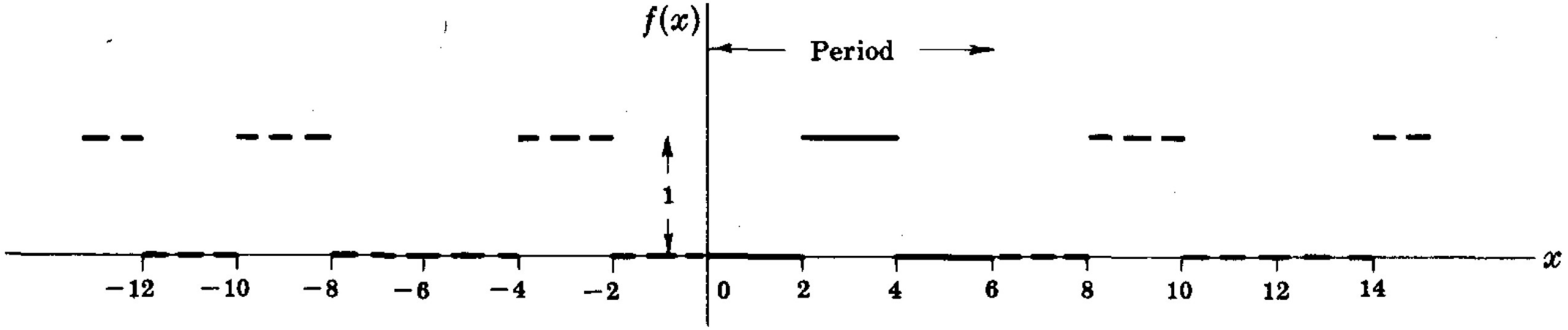


Fig. 2-5

Refer to Fig. 2-5 above. Note that $f(x)$ is defined for all x and is discontinuous at $x = \pm 2, \pm 4, \pm 8, \pm 10, \pm 14, \ldots$.

Fourier Series

DEFINITION OF FOURIER SERIES

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

DIRICHLET CONDITIONS

Theorem 2-1: Suppose that

(i) $f(x)$ is defined and single-valued except possibly at a finite number of points in $(-L, L)$

(ii) $f(x)$ is periodic with period $2L$

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$

Then the series *(1)* with coefficients *(2)* or *(3)* converges to

(a) $f(x)$ if x is a point of continuity

(b) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

For a proof see Problems 2.18–2.23.

According to this result we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{4}$$

at any point of continuity x. However, if x is a point of discontinuity, then the left side is replaced by $\frac{1}{2}[f(x+0)+f(x-0)]$, so that the series converges to the mean value of $f(x+0)$ and $f(x-0)$.

PIECEWISE CONTINUOUS FUNCTIONS

A function $f(x)$ is said to be *piecewise continuous* in an interval if (i) the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and (ii) the limits of $f(x)$ as x approaches the endpoints of each subinterval are finite. Another way of stating this is to say that a piecewise continuous function is one that has at most a finite number of finite discontinuities. An example of a piecewise continuous function is shown in Fig. 2-2. The functions of Fig. 2-1(a) and (c) are piecewise continuous. The function of Fig. 2-1(b) is continuous.

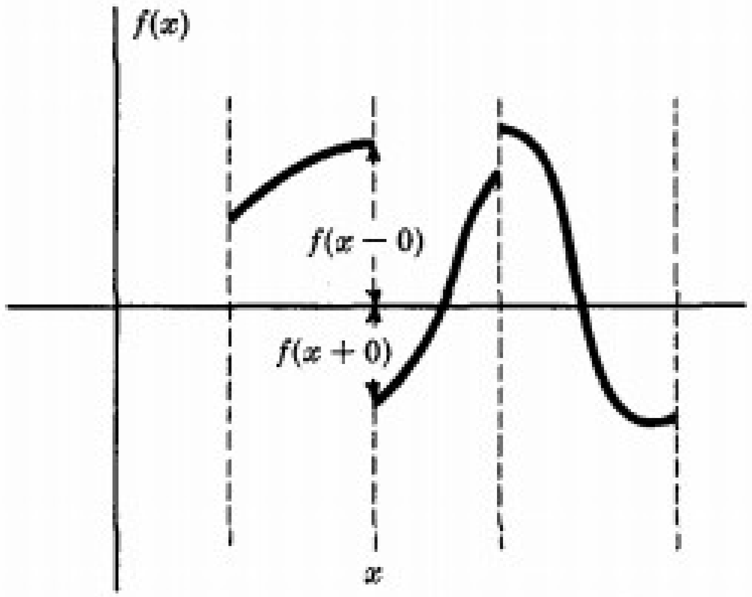


Fig. 2-2

The *limit of* $f(x)$ *from the right* or the *right-hand limit* of $f(x)$ is often denoted by $\lim_{\epsilon \to 0} f(x+\epsilon) = f(x+0)$, where $\epsilon > 0$. Similarly, the *limit of* $f(x)$ *from the left* or the *left-hand limit* of $f(x)$ is denoted by $\lim_{\epsilon \to 0} f(x-\epsilon) = f(x-0)$, where $\epsilon > 0$. The values $f(x+0)$ and $f(x-0)$ at the point x in Fig. 2-2 are as indicated. The fact that $\epsilon \to 0$ and $\epsilon > 0$ is sometimes indicated briefly by $\epsilon \to 0+$. Thus, for example, $\lim_{\epsilon \to 0+} f(x+\epsilon) = f(x+0)$, $\lim_{\epsilon \to 0+} f(x-\epsilon) = f(x-0)$.

2.2. Prove $\displaystyle\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = \int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = 0$ if $k = 1, 2, 3, \ldots$.

2.3. Prove (a) $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$

(b) $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$

where m and n can assume any of the values $1, 2, 3, \ldots$.

See page 26(spiegel)

2.4. If the series $A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots$,

(a) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, (c) $A = \frac{a_0}{2}$.

(a) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \quad \text{if } m \neq 0 \end{aligned} \tag{2}$$

Thus

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(b) Multiplying (1) by $\sin\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= b_m L\end{aligned}$$

Thus

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(c) Integration of (1) from $-L$ to L, using Problem 2.2, gives

$$\int_{-L}^{L} f(x)\,dx = 2AL \qquad \text{or} \qquad A = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (a), we find $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ and so $A = \frac{a_0}{2}$.

2.5. (*a*) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \quad \text{Period} = 10$$

(*b*) Write the corresponding Fourier series.

(*c*) How should $f(x)$ be defined at $x = -5$, $x = 0$ and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

The graph of $f(x)$ is shown in Fig. 2-6 below.

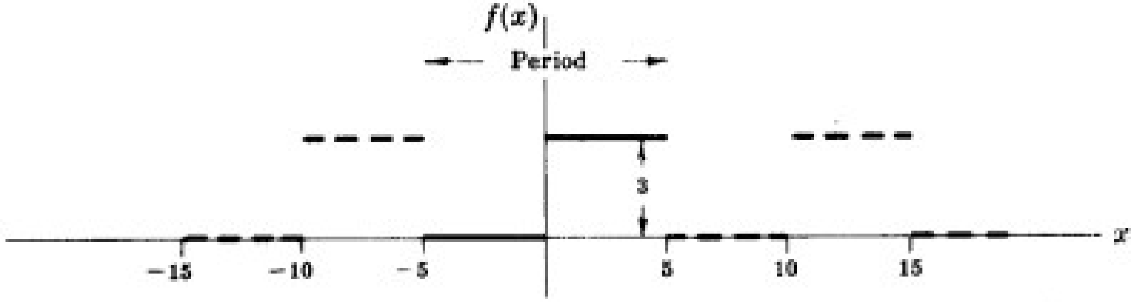


Fig. 2-6

(*a*) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. Then

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0$$

If $n = 0$, $\quad a_n = a_0 = \dfrac{3}{5}\displaystyle\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

(*b*) The corresponding Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right)$$

(*c*) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x = -5$, 0 and 5, which are points of discontinuity, the series converges to $(3+0)/2 = 3/2$, as seen from the graph. The series will converge to $f(x)$ for $-5 \leqq x \leqq 5$ if we redefine $f(x)$ as follows:

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases} \qquad \text{Period} = 10$$

2.6. Expand $f(x) = x^2$, $0 < x < 2\pi$, in a Fourier series if the period is 2π.

The graph of $f(x)$ with period 2π is shown in Fig. 2-7.

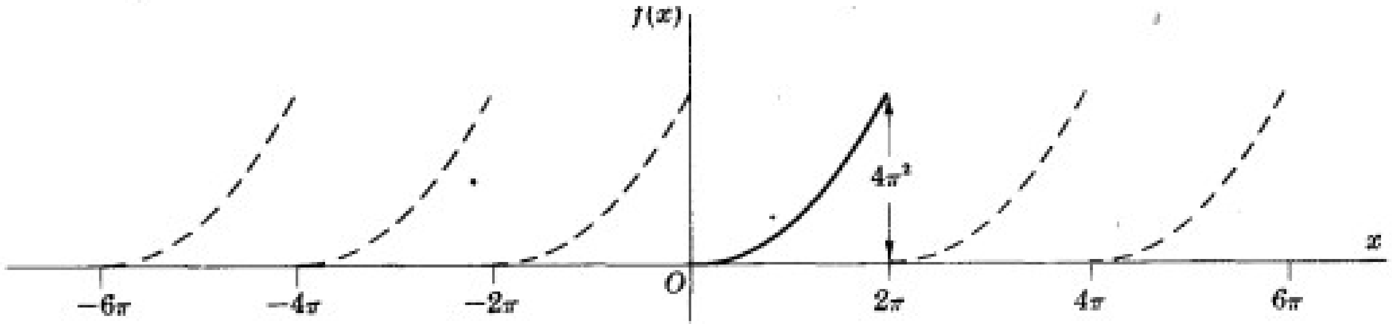


Fig. 2-7

Period $= 2L = 2\pi$ and $L = \pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2}, \quad n \neq 0$$

If $n = 0$, $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{8\pi^2}{3}$.

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

Then $f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$ for $0 < x < 2\pi$.

Example 1. *Find the Fourier series representing*

$$f(x) = x, \quad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Example 2. *Given that* $f(x) = x + x^2$ *for* $-\pi < x < \pi$, *find the Fourier expression of* $f(x)$.
Deduce that $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots$ (*U.P. II Semester, Summer 2003*)

Page 852, HK.DAS

Example 3. *Find the Fourier series of the function*

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < -\dfrac{\pi}{2} \\ 0 & \text{for} \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ +1 & \text{for} \quad \dfrac{\pi}{2} < x < \pi. \end{cases}$$

2.34. Graph each of the following functions and find its corresponding Fourier series, using properties of even and odd functions wherever applicable.

(a) $f(x) = \begin{cases} 8 & 0 < x < 2 \\ -8 & 2 < x < 4 \end{cases}$ Period 4

(b) $f(x) = \begin{cases} -x & -4 \leqq x \leqq 0 \\ x & 0 \leqq x \leqq 4 \end{cases}$ Period 8

(c) $f(x) = 4x, \; 0 < x < 10,$ Period 10

(d) $f(x) = \begin{cases} 2x & 0 \leqq x \leqq 3 \\ 0 & -3 < x < 0 \end{cases}$ Period 6

2.35. In each part of Problem 2.34, tell where the discontinuities of $f(x)$ are located and to what value the series converges at these discontinuities.

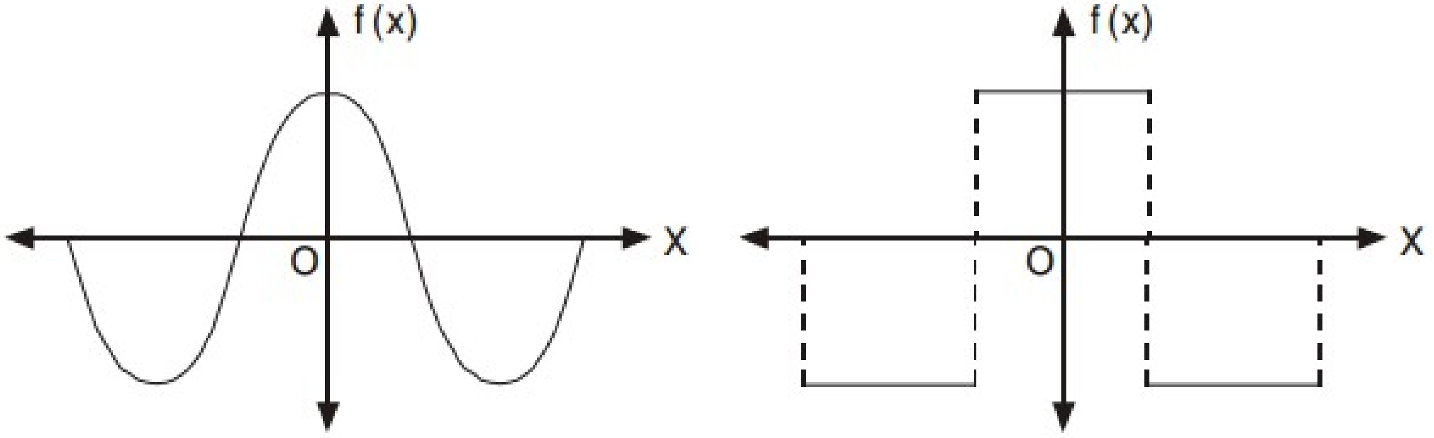
2.36. Expand $f(x) = \begin{cases} 2 - x & 0 < x < 4 \\ x - 6 & 4 < x < 8 \end{cases}$ in a Fourier series of period 8.

Spiegel

12.8 (a) EVEN FUNCTION

A function $f(x)$ is said to be even (or symmetric) function if, $f(-x) = f(x)$

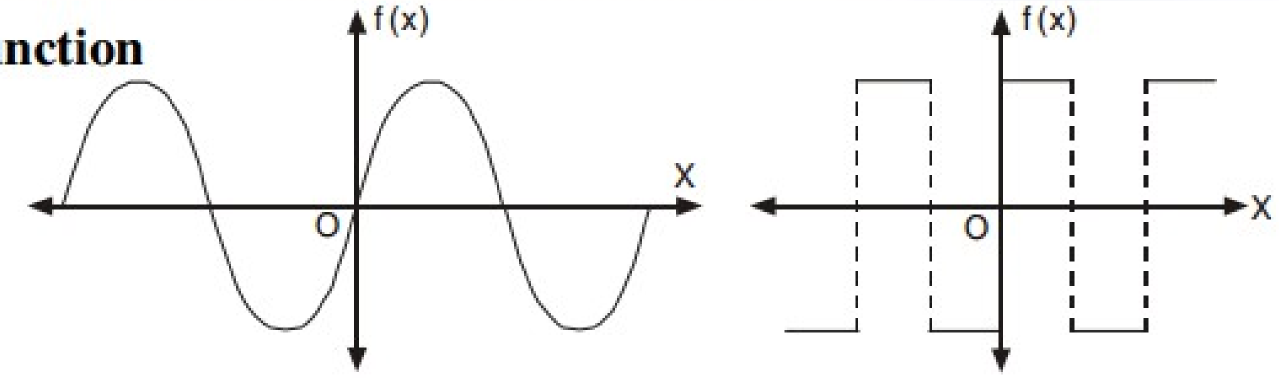
The graph of such a function is symmetrical with respect to y-axis [$f(x)$ axis]. Here y-axis is a mirror for the reflection of the curve.



The area under such a curve from $-\pi$ to π is double the area from 0 to π.

$$\therefore \qquad \int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

(b) Odd Function



A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

Here the area under the curve from $-\pi$ to π is zero.

$$\int_{-\pi}^{\pi} f(x)\, dx = 0$$

Expansion of an even function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

As $f(x)$ and $\cos nx$ are both even functions .

$\therefore$ The product of $f(x).\cos nx$ is also an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

As $\sin nx$ is an odd function so $f(x).\sin nx$ is also an odd function. We need not to calculate b_n. It saves our labour a lot.

The series of the even function will contain only cosine terms.

Expansion of an odd function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad [f(x).\cos nx \text{ is odd function.}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$[f(x).\sin nx$ is even function.$]$

The series of the odd function will contain only sine terms.

The function shown below is neither odd nor even so it contains both sine and cosine terms

2.10. If $f(x)$ is even, show that (a) $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $b_n = 0$.

(a)
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Letting $x = -u$,

$$\frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_0^L f(-u)\cos\left(\frac{-n\pi u}{L}\right)du = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du$$

since by definition of an even function $f(-u) = f(u)$. Then

$$a_n = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

2.12. Expand $f(x) = x$, $0 < x < 2$, in a half-range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 2-12 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4$, $L = 2$.

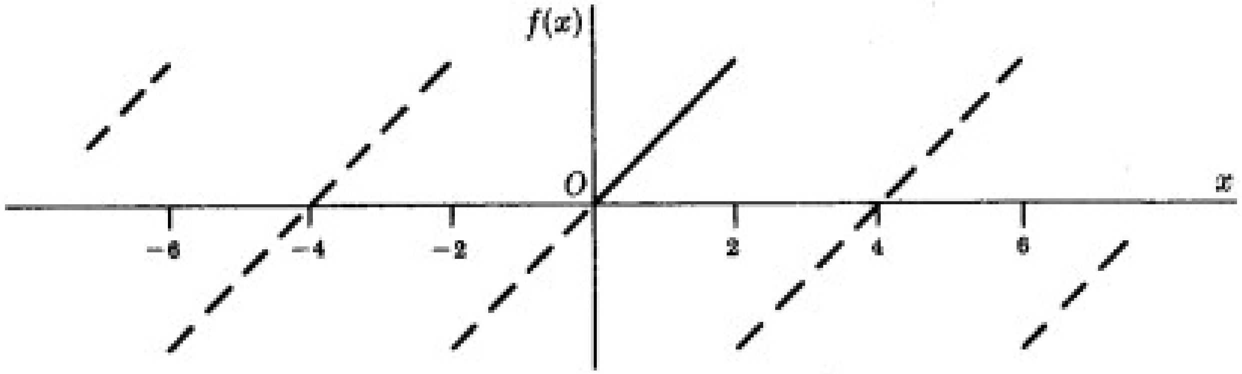


Fig. 2-12

Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then
$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(b) Extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 2-13 below. This is the *even extension* of $f(x)$. Then $2L = 4$, $L = 2$.

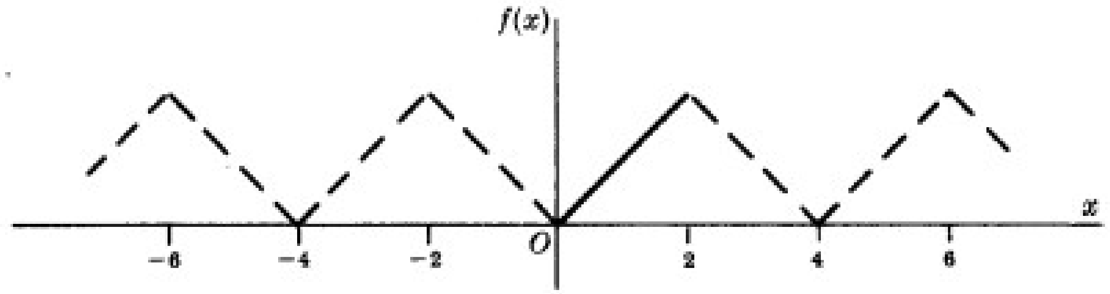


Fig. 2-13

Thus $b_n = 0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2$$

$$= \frac{-4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{if } n \neq 0$$

If $n = 0$, $\quad a_0 = \int_0^2 x\,dx = 2.$

Then $$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

It should be noted that although both series of (a) and (b) represent $f(x)$ in the interval $0 < x < 2$, the second series converges more rapidly.

2.11. Expand $f(x) = \sin x,\ 0 < x < \pi,$ in a Fourier cosine series.

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence we extend the definition of $f(x)$ so that it becomes even (dashed part of Fig. 2-11). With this extension, $f(x)$ is defined in an interval of length 2π. Taking the period as 2π, we have $2L = 2\pi$, so that $L = \pi$.

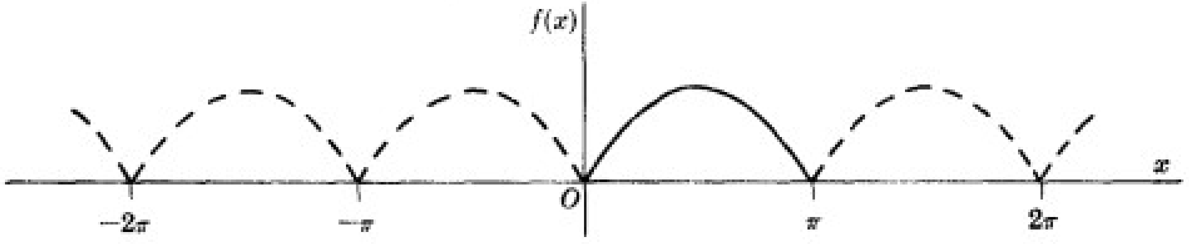


Fig. 2-11

By Problem 2.10, $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^\pi \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi \{\sin(x+nx) + \sin(x-nx)\}\,dx = \frac{1}{\pi}\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\Bigg|_0^\pi$$

$$= \frac{1}{\pi}\left\{\frac{1-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi - 1}{n-1}\right\} = \frac{1}{\pi}\left\{-\frac{1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right\}$$

$$= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} \quad \text{if } n \neq 1$$

For $n = 1$, $\quad a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{2}{\pi}\frac{\sin^2 x}{2}\Bigg|_0^\pi = 0.$

Then $$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos n\pi)}{n^2-1}\cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right)$$

Example 8. *Find the Fourier series expansion of the periodic function of period* 2π

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Page 862, HK.DAS

Example 9. *Obtain a Fourier expression for*

$$f(x) = x^3 \quad \text{for} \quad -\pi < x < \pi.$$

Page 863, HK.DAS

2.12. Expand $f(x) = x$, $0 < x < 2$, in a half-range (*a*) sine series, (*b*) cosine series.

Page 32, Spiegel

Example 10. *Represent the following function by a Fourier sine series :*

$$f(t) = \begin{cases} t, & 0 < t \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} < t \le \pi \end{cases}$$

Page 864, HK.DAS

2.37. (a) Expand $f(x) = \cos x,\ 0 < x < \pi$, in a Fourier sine series.

(b) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leqq x \leqq \pi$?

2.38. (a) Expand in a Fourier series $f(x) = \cos x,\ 0 < x < \pi$, if the period is π; and (b) compare with the result of Problem 2.37, explaining the similarities and differences if any.

2.39. Expand $f(x) = \begin{cases} x & 0 < x < 4 \\ 8 - x & 4 < x < 8 \end{cases}$ in a series of (a) sines, (b) cosines.

2.40. Prove that for $0 \leqq x \leqq \pi$,

(a) $$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right)$$

(b) $$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)$$

Obtain a Fourier series to represent the function

$$f(x) = |x| \quad \text{for} -\pi < x < \pi$$

and hence deduce $\quad \dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

$$\textbf{Ans.} \quad \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$

Example 12. *A periodic function of period 4 is defined as*

$$f(x) = |x|, \quad -2 < x < 2.$$

Find its Fourier series expansion.

PARSEVAL'S IDENTITY

2.13. Assuming that the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-L, L)$, prove Parseval's identity

$$\frac{1}{L}\int_{-L}^{L} \{f(x)\}^2\,dx \quad = \quad \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where the integral is assumed to exist.

If $f(x) \;=\; \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$, then multiplying by $f(x)$ and integrating term by term from $-L$ to L (which is justified since the series is uniformly convergent), we obtain

$$\begin{aligned}\int_{-L}^{L}\{f(x)\}^2\,dx &= \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right\} \\ &= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad (1)\end{aligned}$$

where we have used the results

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \;=\; La_n, \qquad \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \;=\; Lb_n, \qquad \int_{-L}^{L} f(x)\,dx \;=\; La_0 \qquad (2)$$

obtained from the Fourier coefficients.

The required result follows on dividing both sides of (*1*) by L. Parseval's identity is valid under less restrictive conditions than imposed here. In Chapter 3 we shall discuss the significance of Parseval's identity in connection with generalizations of Fourier series known as *orthonormal series*.

Example 20. *By using the sine series for f(x) = 1 in $0 < x < \pi$ show that*

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots.$$

Solution. sine series is $f(x) = \sum b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{-2}{n\pi}[\cos n\pi - 1] = \frac{-2}{n\pi}[(-1)^n - 1]$$

$= \dfrac{2}{n\pi}$ if n is odd.

$= 0$ if n is even

Then, the sine series is

$$1 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \dots..$$

$$\int_0^c [f(x)]^2\,dx = \frac{c}{2}\left[b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + ...\right]$$

$$\int_0^\pi (1)^2\,dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \left(\frac{4}{7\pi}\right)^2 + ...\right]$$

$$[x]_0^\pi = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right]$$

$$\pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$

Proved.

Example 21. *If* $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2 - x), & 1 < x < 2 \end{cases}$

using half range cosine series, show that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\pi^4}{96}$