Derivations

$(x_k - \theta)^2$

$2(x_k - \theta) \quad (0-1)$

For Unknown mean, θ (μ)

$$P(x_k|\theta) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x_k-\theta)^t \Sigma^{-1} (x_k-\theta)\right]$$

Now, it's easier to maximize if we add ln:

$$\ln P(x_k|\theta) = -\frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}(x_k-\theta)^t \Sigma^{-1}(x_k-\theta)$$

Now do the partial derivation along with θ,

$$\nabla_\theta P(x_k|\theta) = 0$$

$$\Rightarrow \frac{\partial}{\partial\theta}\left[-\frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}(x_k-\theta)^t \Sigma^{-t}(x_k-\theta)\right] = 0.$$

$$\Rightarrow -\frac{1}{2}\Sigma^{-1} 2(x_k-\theta)(0-1) = 0.$$

$$\Rightarrow \Sigma^{-1}(x_k-\theta) = 0.$$

$$\Rightarrow \sum_{k=1}^{n}(x_k-\hat{\theta}) = 0.$$

$$\Rightarrow n\hat{\theta} = \sum_{k=1}^{n} x_k$$

$$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

$$\frac{\partial}{\partial\theta}\left(-\frac{1}{2}(x_k-\theta)^2 \Sigma^{-1}\right)$$

$$= -\frac{1}{2} 2(x_k-\theta)(0-1)\Sigma^{-1}$$

$$= (-1)(x_k-\theta)(-1)\Sigma^{-1}$$

$$= \Sigma^{-1}(x_k-\theta)$$

For unknown mean and variance,

Let $\theta = [\mu, \sigma^2]$ $\quad \theta_1 = \mu \quad \theta_2 = \sigma^2$

$$\ln P(x_k|\theta) = -\frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}(x_k - \theta_1)^T \Sigma^{-1}(x_k - \theta_1)$$

$$= -\frac{d}{2}\ln 2\pi - \frac{1}{2}\ln \theta_2 - \frac{1}{2}(x_k - \theta_1)^T \theta_2^{-1}(x_k - \theta_1)$$

$\nabla_\theta (\ln P(x_k|\theta)) = 0$

with respect to $\theta_1 = \frac{1}{\theta_2}(x_k - \theta_1)$ —— (i)

with respect to $\theta_2 = -\frac{1}{2}\theta_2 + \frac{1}{2\theta_2^2}(x_k - \theta_1)^2$ —— (ii)

From (i)

$$\sum_{k=1}^{n} \frac{1}{\bar{\theta}_2}(x_k - \bar{\theta}_1) = 0$$

$$\Rightarrow \sum_{k=1}^{n} (x_k - \bar{\theta}_1) = 0$$

$$\Rightarrow n\bar{\theta}_1 = \sum_{k=1}^{n} x_k$$

$$\boxed{\bar{\theta}_1 = \mu = \frac{1}{n}\sum_{k=1}^{n} x_k}$$

From (ii)

$$\sum_{k=1}^{n} -\frac{1}{2\bar{\theta}_2} + \frac{(x_k - \bar{\theta}_1)^2}{2\bar{\theta}_2^2} = 0$$

$$\Rightarrow \sum_{k=1}^{n} \frac{(x_k - \bar{\theta}_1)^2}{2\bar{\theta}_2^2} = \sum_{k=1}^{n} \frac{1}{2\bar{\theta}_2}$$

$$\Rightarrow \sum_{k=1}^{n} (x_k - \bar{\theta}_1)^2 = \sum_{k=1}^{n} \frac{2\bar{\theta}_2^2}{2\bar{\theta}_2} = \sum_{k=1}^{n} \bar{\theta}_2$$

$$\sum_{k=1}^{n} \bar{\theta}_2 = \sum_{k=1}^{n} (x_k - \bar{\theta}_1)^2$$

$$\Rightarrow n\bar{\theta}_2 = \sum_{k=1}^{n} (x_k - \mu)^2$$

$$\Rightarrow \bar{\theta}_2 = \frac{1}{n}\sum_{k=1}^{n} (x_k - \mu)^2$$

$$\boxed{\sigma^2 = \frac{1}{n}\sum_{k=1}^{n} (x_k - \mu)^2}$$

Proved