Advanced Signal Processing Programming Assignment #2:
Matched Filter and Delay Estimation

1 Background

1.1 Delay Estimation and the Matched Filter

The problem considered in class is the following. Assume a given signal s_n, whose values in the range $n \in \{0, .., L-1\}$ are fixed and known, and outside this range are set to zero. We receive a delayed and noisy version of this signal:

$$Y_n = s_{n-W} + Z_n \tag{1}$$

where Z_n is independently identically distributed Gaussian noise with zero mean and variance σ^2. We assume that the delay W is a random variable, and wish to estimate it using $\{Y_n\}$. It was shown in class that the output of the matched filter is a sufficient statistic of $\{Y_n\}$ for W. If we further assume that W is uniform on the set of possible delays $\{1, \ldots, M-1\}$, the optimal MAP estimator for W is the time in which the output of the matched filter is maximized, within $\{1, \ldots, M-1\}$. Sometimes, we assume that the signal may never return, an event that occurs with probability $1-\delta$, and otherwise, the delay is uniform on $\{1, \ldots, M-1\}$. In this case, the MAP estimator is the delay that maximizes the output of the matched filter, as before, provided that the maximum is sufficiently large. Otherwise, if the maximum is too small, the MAP estimator declares that the signal did not return.

In this exercise we shall practice several examples of this problem in a Matlab environment. The signal used in this exercise is a *chirp* signal whose length is 128 samples, depicted in Figure 1. Let us assume that the signal is stored in the Matlab row vector named ”sigvec”. Let us also assume that the channel output is stored in the Matlab row vector named ”chanoutvec”. A numeric example for the usage of this matched filter in low signal-to-noise ratio (SNR) appears in Figure 2. Note that the delayed signal is hardly observed in the middle plot, due to the additive noise. However, it produces a strong peak at the output of the matched filter as can be seen at the bottom plot.

The matched filter can be implemented in the following Matlab code lines:

```
matchedfiltcoeffs = sigvec(end:-1:1);
filtout = filter(matchedfiltcoeffs,1,chanoutvec);
```

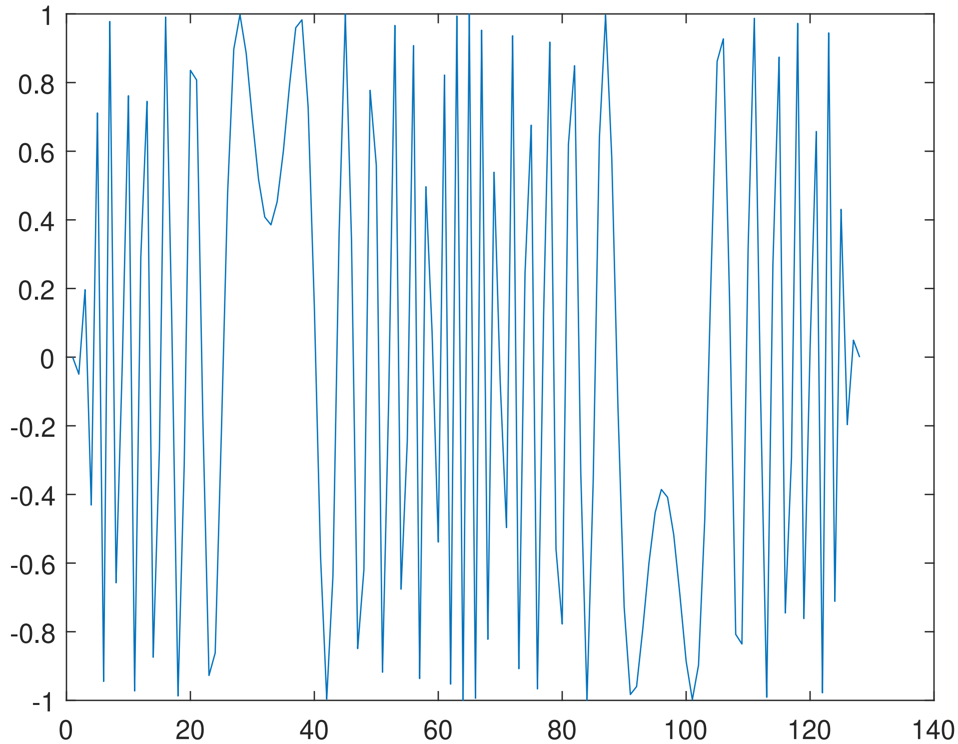

Figure 1: A chirp signal

The first line defines the matched filter as the time reversed version of the signal. The second line performs the filtering. The delay can be estimated according to the location of the maximal value at the matched filter output. Note that the Matlab implementation of the matched filter is a *causal* filter. Namely, its time indexes span from 1 to L and not from $-(L-1)$ to 0, as the theoretical matched filter. Therefore, in order to determine the actual delay, we should subtract L from the location of the maximal matched filter output.

1.2 A Radar Example

A radar (*Radio Detection and Ranging*) is a system using radio waves in order to determine the location of objects. While there are several types of radars, their common concept is sending a designated signal which hits the target, and reflects back to a receiver. The propagation delay between the transmitted and the received signals is then measured, and according to it, the aggregated distance (from the transmitter to the target and back to the receiver) is calculated.

In this exercise we consider a simplified radar system having one transmitting antenna, and two receiving antennas designated to locate objects in a two dimensional plane. We assume that we accurately estimated the propagation delay using a matched filter. The

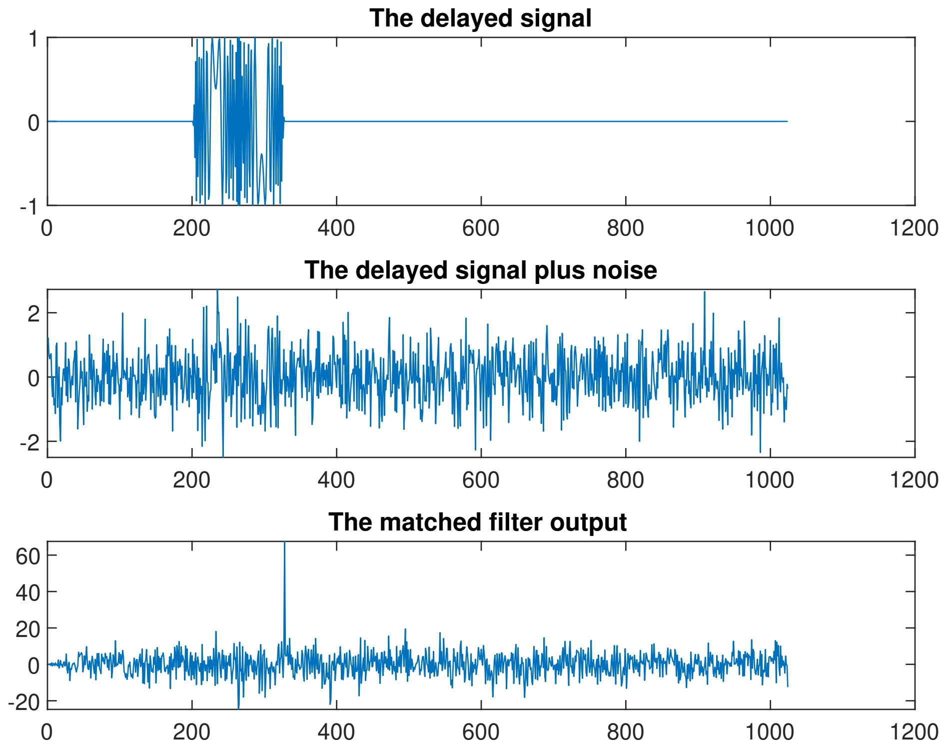

Figure 2: Delay estimation example

relation between the propagation delay w and the aggregated distance d is

$$d = w \cdot c \tag{2}$$

where d is in meters and w is in seconds. The propagation speed of the radio signal is:

$$c \approx 3 \cdot 10^8 [\text{meters per second}] \tag{3}$$

which is the speed of light.

An example for such a radar system is depicted in Figure 3. The transmitting antenna and one of the receiving antennas are located at (x_1, y_1). The second receiving antenna is located at (x_2, y_2) and the target is located at (x, y). The delay measured by the first antenna is w_1, and gives the aggregated distance $d_1 = w_1 \cdot c$, corresponding to the aggregated distance from transmitter at (x_1, y_1) to the target at (x, y) and back to the receiver at (x_1, y_1). The delay measured by the second antenna is w_2, and gives the aggregated distance $d_2 = w_2 \cdot c$, corresponding to the aggregated distance from the transmitting antenna at (x_1, y_1) to the target at (x, y) and back to the receiving antenna (x_2, y_2).

Let us now see how we can calculate the target location (x, y) only according to d_1, d_2, and knowing the locations (x_1, y_1) and (x_2, y_2). The distance from (x_1, y_1) to (x, y) and back to (x_1, y_1), measured as d_1 gives the following equation:

$$2\sqrt{(x - x_1)^2 + (y - y_1)^2} = d_1 \tag{4}$$

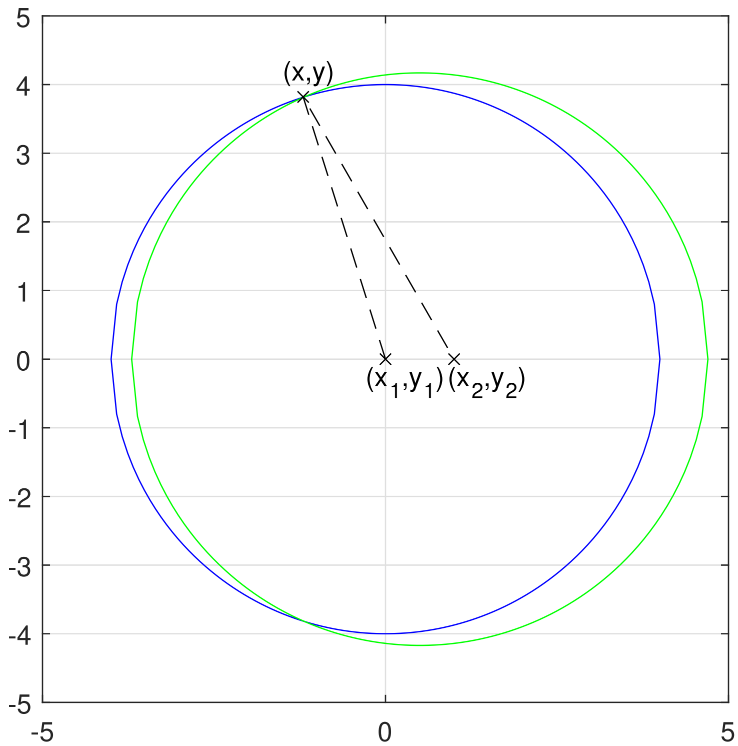

Figure 3: An example for a radar having two antennas

which defines a circle in the XY plane centered in (x_1, y_1). This circle is plotted in blue in Figure 3. The distance from (x_1, y_1) to (x, y) and back to (x_2, y_2), measured as d_2 gives the following equation:

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = d_2. \tag{5}$$

which defines an ellipse whose focal points are (x_1, y_1) and (x_2, y_2). This ellipse is plotted in green in Figure 3. The target (x, y) is exactly at the intersection of the circle and the ellipse. We also note that there is another intersection located at $(x, -y)$. However, we assume we have extra knowledge ensuring that this solution is not feasible. For example, suppose that the radar antennas are located on the front of a self driving car, and the radar is designated to detect objects (like pedestrians or other cars) in front of the car. Since the beams of the antennas are directed forward, it is obvious that the objects cannot be located behind the car.

Let us now see how to calculate (x, y) by solving the equation set (4), (5). Rearranging the equations we can obtain:

$$(x - x_1)^2 + (y - y_1)^2 = \frac{1}{4}d_1^2 \tag{6}$$

$$(x - x_2)^2 + (y - y_2)^2 = \left(d_2 - \frac{1}{2}d_1\right)^2 \tag{7}$$

which is a set of two quadratic equations with two unknowns: x and y. For the sake of simplicity, we assume that the XY plane is shifted and rotated appropriately, so that without loss of generality $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (\Delta, 0)$. In addition, let us denote:

$$A \triangleq \frac{1}{4}d^2, \tag{8}$$

$$B \triangleq \left(d_2 - \frac{1}{2}d_1\right)^2, \tag{9}$$

all in all yielding:

$$x^2 + y^2 = A, \tag{10}$$

$$(x - \Delta)^2 + y^2 = B. \tag{11}$$

Subtracting (11) from (10) we obtain:

$$2x\Delta = A - B + \Delta^2, \tag{12}$$

providing the solution to x:

$$x = \frac{A - B}{2\Delta} + \frac{\Delta}{2}. \tag{13}$$

y can now be obtained by solving (10) with this value of x, obtaining:

$$y = \pm\sqrt{A - x^2}. \tag{14}$$

The negative solution correspond to the negative intersection point observed in Figure 3. As mentioned before, suppose we have extra knowledge ensuring that y is non-negative, rendering the single solution to be:

$$y = \sqrt{A - x^2}. \tag{15}$$

Notes on Practical Implementation

- The radar signal is typically sent in a high radio frequency. For example, a typical radar frequency in the automotive industry is 80GHz. However, the radar signal to be detected is typically in a narrower bandwidth (for example 2GHz) which is *modulated* to the high frequency. In other words, the Fourier transform of the signal is zero outside the range $79 - 81$GHz. In this exercise, we assume for the sake of simplicity that the received signal is already converted to low frequency and sampled to discrete time. In other words, it is assumed that the signals are in discrete time which correspond to low frequencies (which is also called "base-band").

- If the target is moving in high velocity, the frequency of the received signal can differ from that of the transmitted signal. This difference in frequency can cause the matched filter (calculated according to the transmitted signal) not to be perfectly matched to the received signal. This effect is known as the *Doppler Effect* and will be ignored in this exercise.

- Typically, *phased array* radars have more than two antennas (eight antennas or even more).

- Radars operating in high frequencies usually use more involved localization equations as the ones described above. For example, regarding the signal as complex values and calculating the angle of arrival using phase information etc. These methods, are however, beyond the scope of this exercise.

2 Questions

In the following questions you will be asked to use inputs saved in Matlab 'mat' files. The content of the files can be downloaded to Matlab using the 'load' function. The pulse to be used is stored a row vector of length 128, named 'sigvec' and stored in the file **'sigvec.mat'**.

Question 1 - Delay Estimation

Read the radar transmitted pulse signal from **'sigvec.mat'**. The file named **'delayedvecs.mat'** contains two column vectors : 'r1vec' and 'r2vec'of length 2048. Plot the two vectors, and calculate the MAP estimator for their respective delays using the matched filter.

Assume that all the delays have the same probability in the range 1 to 2047-128, and that the signal returns with probability $\delta = 1$.

Question 2 - Delay Estimation with $\delta < 1$

Suppose now that the delay random variable W corresponds to the probabilities:

$$P_W(M) = 1 - \delta \tag{16}$$

$$P_W(w) = \frac{\delta}{M-1}, \text{ for all } w \in [M-1] \tag{17}$$

where $W = M$ corresponds to the event in which the radar signal is not received at all.

a. Write the formula for the MAP estimator of the delay: $\hat{W}_{\mathrm{MAP}}(\mathbf{y})$, including the threshold to which the matched filter output is compared.

b. The file **'delayedvecsQ2.mat'** contains the variables: 'r1', 'r2', 'r3', 'delta1', 'delta2', 'delta3', 'sigma2_1', 'sigma2_2', 'sigma2_3' which correspond to three reception vectors, and their corresponding δ and σ^2. For each one of them, find the threshold value and the delay estimate according to the MAP criterion.

Question 3 - An Airport Radar

In this question we simulate a simplified model of an airport radar. Suppose that the system corresponds to the description in Subsection 1.2. Assume the two antennas (x_1, y_1), (x_2, y_2) are located on two control towers, and the radar is designated to locate a single airplane moving on the ground in front of the two towers. Suppose that the signals in this exercise are sampled at a rate of $F_S = 5 \times 10^6$ samples per second. Namely, the time between two consecutive samples corresponds to 2×10^{-7} seconds. We also ignore the high-radio frequency modulation and de-modulation.

We assume (without loss of generality) that $(x_1, y_1) = (0, 0)$ and $(x_1, y_1) = (\Delta, 0)$ where Δ is 3840 meters. In order to track the moving airplane, a signal (we shall refer to as a *radar pulse*) is sent every $T = 4.096 \times 10^{-4}$ seconds, from the control tower located at $(0, 0)$. The radar pulse used in this question is 'sigvec', the same signal used throughout the exercise.

The localization of the moving airplane is based on two vectors, each corresponding to a different control tower. The vectors represent the signals received at the two control towers antennas, sampled at rate of F_S samples per second. Recalling that the radar pulse is transmitted periodically, every 4.096×10^{-4} seconds , the samples within the first 4.096×10^{-4} seconds, i.e., the first $T \cdot F_S$ samples, correspond to the first transmitted radar pulse. The following $T \cdot F_S$ samples correspond to the second radar pulse and so on.

Write a Matlab function with the following header:

```
function [d1vec,d2vec,xvec,yvec] = radardetect(r1vec,r2vec,sigvec)
```

Note the following points in the implementation of the functions:

- The inputs are three vectors. Assume that the number of samples in each of the vectors r1vec and r2vec vector is $T \cdot F_S \cdot N$, where N is the total number of radar pulses and is an integer. Ignore the shape of the vectors (row or column) and trasnform them to column vectors using vec=vec(:).
- Cut the vectors to N small vectors each corresponding to a different radar pulse. You can use Matlab 'reshape' command.
- For each antenna and each of the N radar pulses, calculate the MAP estimator for the corresponding delay. Store these estimates in the output vectors 'd1vec', 'd2vec', corresponding to the delays to the first and second antennas, respectively.
  Assume all delays are equi-probable, and that the signal returns with probability $\delta = 1$.
- Based on these delay estimates, calculate the estimated location of the airplane, every T seconds. Store these estimates in the vectors 'xvec', 'yvec', where the coordinates of 'xvec' (respectively 'yvec') represent the location of the airplane in the x-axis (respectively, y-axis) every T seconds. 'xvec' and 'yvec' should be in meters.
- Plot the estimated path of the airplane using Matlab plot command (i.e., "plot(xvec,yvec)").

The following should be included in the submission of your assignment:

a. The file **'radarreception.mat'** contains two reception vectors: 'r1vec','r2vec'. Use them as the input to your function and submit the outputs the function. Namely, the vector 'd1vec', 'd2vec', 'xvec', 'yvec' and the figure of the plotted path. Pay attention that the delays should be in units of samples, but the coordinates x and y should be in meters.

b. Submit the code of your function. We will evaluate it on received signals corresponding to the exact scenario described above, but with a different path than the one in the previous item.