Linear Shooting Method:

Used for linear differential equations with boundary values and the following form

$$y'' = f(x, y, y')$$

Consider a linear boundary value problem of the form:

$$y''(x) = p(x)y'(x) + q(x)y(x) + r(x) \qquad \dots (1)$$

with the following conditions:

- The boundary conditions are $y(a) = \alpha$ and $y(b) = \beta$, $a \leq x \leq b$
- $p(x)$, $q(x)$ and $r(x)$ are continuous on $[a, b]$
- $q(x) > 0$ on $[a, b]$

We can find a solution for the ODE (1) as:

$$y(x) = u(x) + \frac{\beta - u(b)}{v(b)} v(x)$$

such that $u(x)$ is the solution of the following differential equation with initial-value (referred as Equation 1 in the code)

$$u''(x) = p(x)u'(x) + q(x)u(x) + r(x), \;\; u(a) = \alpha, \;\; u'(a) = 0 \qquad \dots (2)$$

and $v(x)$ is the solution of the following differential equation with initial value (referred as Equation 2 in the code)

$$v''(x) = p(x)v'(x) + q(x)v(x), \;\; v(a) = 0, \;\; v'(a) = 1 \qquad \dots (3)$$

The ODE (2) can be decomposed into a system of linear first order differential equations considering $u_1 = u$ and $u_2 = u_1'$.

$$u_1'(x) = u_2(x)$$

$$u_2'(x) = p(x)\, u_2(x) + q(x)u_1(x) + r(x)$$

Taking an iterative approach,

$$u_{1,i}' = u_{2,i}$$

$$u_{2,i}' = p(x_i)u_{2,i} + q(x_i)u_{1,i} + r(x_i)$$

The step-size is taken as $h = \frac{b-a}{N}$ where N is the number of iterations of the calculation and subsequently $x_i = a + ih$. Here, $u_{1,0} = u_1(x_0) = u(a) = \alpha$ and $u_{2,0} = u_2(x_0) = u'(a) = 0$. The numerical values of $u_{1,i}$ and $u_{2,i}$ are calculated using fourth order Runge-Kutta method as such:

$$u_{1,i+1} = u_{1,i} + \frac{k_{11} + 2k_{21} + 2k_{31} + k_{41}}{6}$$

$$u_{2,i+1} = u_{2,i} + \frac{k_{12} + 2k_{22} + 2k_{32} + k_{42}}{6}$$

where

$$
\begin{array}{ll}
k_{11} = hu_{2,i} & k_{12} = h\left[p(x_i)u_{2,i} + q(x_i)u_{1,i} + r(x_i)\right] \\
k_{21} = h\left(u_{2,i} + \frac{1}{2}k_{12}\right) & k_{22} = h\left[p\left(x_i + \frac{h}{2}\right)\left(u_{2,i} + \frac{1}{2}k_{12}\right) + q\left(x_i + \frac{h}{2}\right)\left(u_{1,i} + \frac{1}{2}k_{11}\right) + r\left(x_i + \frac{h}{2}\right)\right] \\
k_{31} = h\left(u_{2,i} + \frac{1}{2}k_{22}\right) & k_{32} = h\left[p\left(x_i + \frac{h}{2}\right)\left(u_{2,i} + \frac{1}{2}k_{22}\right) + q\left(x_i + \frac{h}{2}\right)\left(u_{1,i} + \frac{1}{2}k_{21}\right) + r\left(x_i + \frac{h}{2}\right)\right] \\
k_{41} = h\left(u_{2,i} + \frac{1}{2}k_{32}\right) & k_{42} = h\left[p(x_i + h)\left(u_{2,i} + k_{32}\right) + q(x_i + h)\left(u_{1,i} + k_{31}\right) + r(x_i + h)\right]
\end{array}
$$

Similarly, for $v(x)$, we have,

$$v_1'(x) = v_2(x)$$

$$v_2'(x) = p(x)\, v_2(x) + q(x)v_1(x)$$

Taking an iterative approach,

$$v_{1,i}' = v_{2,i}$$

$$v_{2,i}' = p(x_i)v_{2,i} + q(x_i)v_{1,i}$$

Here, $v_{1,0} = v_1(x_0) = v(a) = 0$ and $v_{2,0} = v_2(x_0) = v'(a) = 1$. The numerical values of $v_{1,i}$ and $v_{2,i}$ are calculated using fourth order Runge-Kutta method as such:

$$v_{1,i+1} = v_{1,i} + \frac{k_{11} + 2k_{21} + 2k_{31} + k_{41}}{6}$$

$$v_{2,i+1} = v_{2,i} + \frac{k_{12} + 2k_{22} + 2k_{32} + k_{42}}{6}$$

where

$$
\begin{array}{ll}
k_{11} = hv_{2,i} & k_{12} = h\left[p(x_i)v_{2,i} + q(x_i)v_{1,i}\right] \\
k_{21} = h\left(v_{2,i} + \frac{1}{2}k_{12}\right) & k_{22} = h\left[p\left(x_i + \frac{h}{2}\right)\left(v_{2,i} + \frac{1}{2}k_{12}\right) + q\left(x_i + \frac{h}{2}\right)\left(v_{1,i} + \frac{1}{2}k_{11}\right)\right] \\
k_{31} = h\left(v_{2,i} + \frac{1}{2}k_{22}\right) & k_{32} = h\left[p\left(x_i + \frac{h}{2}\right)\left(v_{2,i} + \frac{1}{2}k_{22}\right) + q\left(x_i + \frac{h}{2}\right)\left(v_{1,i} + \frac{1}{2}k_{21}\right)\right] \\
k_{41} = h\left(v_{2,i} + \frac{1}{2}k_{32}\right) & k_{42} = h\left[p(x_i + h)\left(v_{2,i} + k_{32}\right) + q(x_i + h)\left(v_{1,i} + k_{31}\right)\right]
\end{array}
$$