PO Box 5042,
Woodbridge,
NJ 07095-5042.

Dear New Jersey Turnpike Authority,

We are a group of students having conducted a project focusing on solving the problem of merging after tollbooths. And we are sending this letter to let you know the problems that may exist in current toll plaza and suggesting some possible ways to improve it.

First of all, by analyzing the design of existing toll plazas on the highways, we find out the following weaknesses of them. At most toll plazas, the vehicles go disorderedly after toll for following reasons. For some toll plazas, there are no leading lines after toll, so drivers just go at their own discretion, thus their routes become unpredictable. And even there are some orientation lines, if two vehicles arrive at the merging point at the same time, only one of them can go through at one time and it will also cause some problems. Disorder after toll leads to both harm to the efficiency of the highway entrance and more risk for traffic accidents.

Therefore, to solve the above problem, we introduced a new model called "Control Time Model" (CTM). In our model, the releases of cars are controlled at the booth, ensuring that the time interval from any two releases of cars are more that the "safe time". Therefore, the merging becomes more orderly and safer.

We run some simulations with our models to test and compare the efficiency between the CTM and current existing models. The result is that, the efficiency of CTM is only slightly smaller than existing model (about 0.6 throughput less per lane per hour), but we haven't considered the time wasted when accidents happen, which is a lot more likely to take place in existing models theoretically.

And also according to the simulation, the risk of car crashes is also limited to a lower level, which is 13.66% less than current existing models.

In terms of expenditure of construction and maintenance, as our model is much more orderly, the total length of the merging area can be reduced.

Therefore, a decrease in the total expense is expected, let alone the fact that maintenance of plaza could be a lot easier if accident rate decreases distinctively.

All in all, our conclusion is that the CTM performs better than existing models in all aspects including but not limited to efficiency, risk elimination as well as total cost. Therefore, it is worthwhile to consider the future construction with our model design.

Thank you for considering our model and wish you a bright future.

Sincerely,

Team#70545
MCM2017

Control Time Model:
Wait One Second and Start Your Journey

Contents

1 Introduction

1.1 Background

Traveling on highways in recent years are quite convenient ways for people to go to another city. With a compared high speed limit on highway, people are able to travel a lot faster. However, building highways are quite expensive and it also takes a lot for the maintenance. Therefore, building of toll plazas along the highway is a common practice all around the world. Meanwhile, there are many factors to be considered during the design and the construction of the tollbooths. Firstly, the lands and the construction fee for the booths and roads are very expensive, so we need to minimize the area. And as the vehicles on the highway always have a high speed, it is also quite dangerous when they want to merge into a lane. Finally, the design of the booths is also supposed to guarantee the efficiency, which is described by the quantity throughout the highway for a certain time interval.

1.2 Restatement of the problem

As required by the question, we are supposed to determine the shape, size and merging pattern at the toll plaza depended on number of lanes on the highway in one direction (L) and total number of tollbooths in one direction (B). Meanwhile, we will also justify our approach by comparing it with the existing models in terms of risk, cost and efficiency. What's more, the implement of our plan in various situations (e.g. mixture of the self-driving vehicle and different proportion distribution of types of tollbooths) will also be shown.

1.3 Literature review

The optimal number of tollbooths needed to minimize the average waiting time is well-studied and simulated based on different real situations (Corwen et al, 2005). Though the definition of "optimal" varies, similar suggestions have been given. Tollbooths should be implemented conforming to encouraged behaviors, e.g. faster booths should be put on the left (Spann et al, 2005), which is incorporated in our model. Some literature suggested that tollbooths employ no barrier to ensure a relatively smooth flow (Kane, 2005), but we contend that the uplift of barrier takes negligible amount of time, the benefit of which cannot be compared with the chaos and potential risk if some vehicles go through toll plaza directly. Therefore, barriers are included in our model.

2 Assumptions

To simplify the real life situation, we will make the following assumptions as a start of construction of our models.

- **Most of the drivers are rational.** They will choose the path as suggested by signals at entrance, and act as risk averters. It is reasonable to make this assumption, because without it, it's meaningless to make any rules as people won't obey them.

- **The arrival time lapse between the first vehicle and the second vehicle in the heavy traffic follows uniform distribution.** By the heavy traffic assumption, the second vehicle comes to tollbooth before the first vehicle leaves.
- **All the paths of the vehicles follow either constant speed motion or constant acceleration motion.** As the average speed for the vehicles around the tollbooths are relatively low.
- **Vehicles are queuing at the entrance of the tollbooth one by one with low speed in the heavy traffic situation** . And we further assume them to be stationary in our models.
- **The service by human-staffs at the tollbooth is quite time-consuming, and the approximate mean service time is around 15 seconds, which follows a normal distribution.** This assumption is appropriate, because driver need to bring the fee to the staff and the staff may need to prepare changes for it. And the time varies from people and different situations.
- **The length of vehicles is omitted during the calculation for the size and shape for the merging area.** It is valid because the magnitude of the length of toll plaza is a lot larger than the length of vehicles.

3 Notations

Notation	Definition	Unit
L	Number of outgoing lanes in highway	N/A
B	Number of tollbooths	N/A
D	Distance between the tollbooth and the end of the plaza	m
d_l	Width of the lane in highway	m
d_b	Width of the lane in tollbooth area	m
r	Radius for vehicles' turning	m
T_h	Time of toll service per vehicle human-staffed tollbooths	s
t_0	Time of control (to be defined in Section 4.2)	s
v_m	Speed limit in merging area	m/s
a_s	Starting acceleration of vehicles after toll payment	m/s^2
a_b	Deceleration of braking vehicles	m/s^2

4 Models

In this section, we first introduce and analyze the existing model, and then two new models invented by us. And the focus for the new models will only be the establishment of the first new model (control time model) and second new model (waiting area model) only deviates from the first one slightly.

4.1 Existing Model(EM)

Under our investigation, the existing solution for the merging after toll can be roughly divided into two types.

The first solution is in short called "no solution", which doesn't give the drivers any instruction after toll and let them go at their own discretion. Then the second and more commonly used solution is to instruct the vehicles to merge at some pre-determined merging point.

For the first solution, the efficiency of it is mainly based on the drivers' own discretions, which can vary a lot among different people. And in the light traffic situations, this solution may be optimal, for the reason that drivers can go whatever path they what, and in other words their own efficiency is maximized. Meanwhile, with the light traffic assumption, there are little probability to crash. However, it will be fairly chaotic in heavy traffic, because it is very likely that driver's own optimal path can cross each other. And then they will merge at any point so it is also hard to predict what drivers will do. Therefore, in heavy traffic, it is neither efficient nor safe.

And the second solution provides a relatively orderly merging pattern. Vehicles merge at some certain merging points and at least the driving direction is predictable as long as the drivers are rational and always follow the instruction. However, some problems also occur at the merging point. As for most time, the number of tollbooths are larger than the number of lanes on the highway, there always exist the situation that vehicles from several different booths need to merge into one lane. So different vehicles from different booths can arrive at the merging point simultaneously, and then only one of them can get through at a time and others have to wait. This situation can be quite dangerous because drivers always want to go first, and it also increase the deficiency.

4.2 Control Time Model (CTM)

Given aforementioned deficiencies of existing merging pattern, we propose a new model, partially based on the current one. Instead of having all the vehicles moving and merging at their own discretion, control time model will control the departure time of vehicles to ensure a smooth and safe emerging process.

Specifically, for situations where two booths merge into one lane, the second vehicle will only be allowed to proceed t_0 seconds after the first vehicle moves forward. The time t_0 is defined as the control time. Similarly, for situations where three booths merge into one lane, the third vehicle will be allowed to proceed t_0 seconds after the second vehicle moves forward, whilst the second vehicle t_0 seconds after the first vehicle, as shown in Figure 1 and Figure 2.

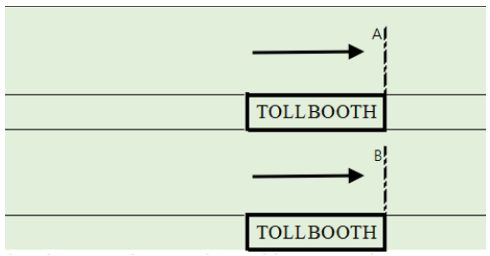

Figure 1: Either A or B is open

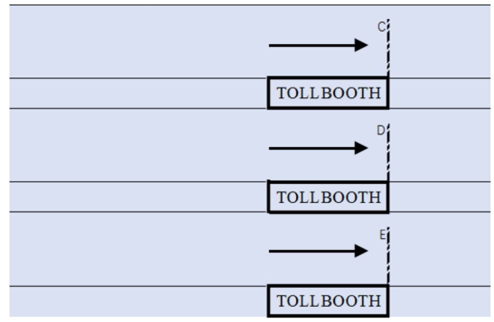

Figure 2: Either C, D or E is open

In this way, the regulated merging of the vehicles into another lane would be more efficient than the situation where vehicles are proceeding without regulation, for drivers should take time to make decision when multiple booths merge into one lane simultaneously, let alone the risk for doing so.

We model that vehicles start with constant acceleration a_s until reach the maximum speed v_m in the straight path. They then immediately starts merging into their prescribed road in two consecutive tangent circle arcs.

Figure 3: Two or three tollbooth egress lanes merge into one lane

Now we evaluate the "appropriate" control time. When emergency happens, one vehicle take severe brake action with acceleration a_b, after response time t_{res}, the posterior vehicle take severe brake action with the same acceleration. Consider the distance of the two vehicles ($t = 0$ is the time when emergencies happen):

$$d = v_m t_{res} + \int_0^t v_{posterior}(t)dt - \int_0^t v_{previous}(t)dt$$

where

$$v_{previous} = \begin{cases} v_m, & t \leq t_{res} \\ v_m - a_b(t - t_{res}), & t_{res} < t \leq t_{res} + \frac{v_m}{a_b} \\ 0, & t_{res} + \frac{v_m}{a_b} < t \end{cases} \tag{1}$$

Solve the equation $d \geq 0$, we get $t_0 \geq t_{res}$. By saying "appropriate" control time, we take $t_0 = 1s > t_{res} = 0.8s$ (Lee et al, 2002).

After describing the control time model, we first calculate the throughput of the toll plaza.

Let i_k be the number of booths corresponding with the k^{th} lane. For example, if the first three booths merge into the first lane, we say $i_1 = 3$.

By computer simulation (details are included in Appendix), we have t_{i_k} close to $\frac{T}{i_k}$ when $i_k \ll \frac{T}{t_0}$ (especially when the variance of service time is small), and t_{i_k} increases as i_k increases, and the increase speed is quite large when $i_k > \frac{T}{t_0}$. where t_{i_k} is the averaged time for a car to come into k^{th} lane.

Therefore, to maximize the efficiency, we'll keep i_k as small as possible, whereas the throughput comes suboptimal if $i_k > \frac{T}{t_0}$ for some k. The optimal throughput is closed to $\frac{B}{T}$ car(s) per second.

By substituting B by 8, the expectation of human service time T by 15. According to our optimization method above, when we keep the i_k as small as possible, which is in other words, let them distributed evenly(e.g. {2,3,3}). We will get the optimal throughput closed to $\frac{B}{T} \approx 0.533$ car per second.

Then we calculate the size of the plaza, by doing so we discuss the value of D.

Define $f : \{1, 2, \ldots, B\} \to \{1, 2, \ldots, L\}$ which is non-decreasing, and $f(i) = j$ if and only if vehicles from i^{th} booth merge into j^{th} lane. It obviously follows that $i_k = \|f^{-1}(\{k\})\|$. We call the mapping f a "merging pattern".

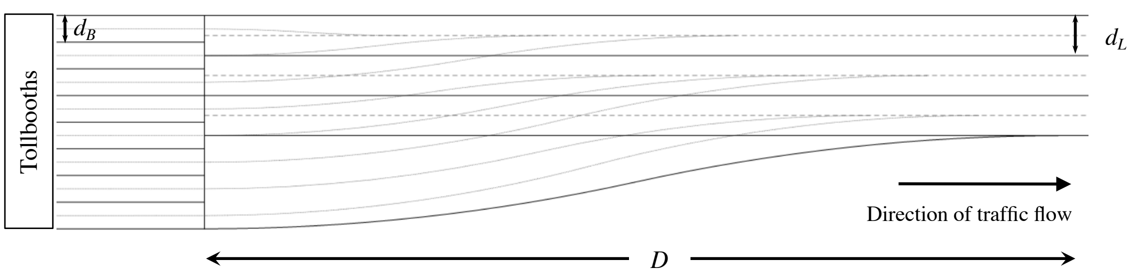

Figure 4: $B = 8$ tollbooth egress lanes merge into $L = 3$ lanes

By geometric relations, we have

$$\left(\frac{D - \frac{v_m^2}{2a}}{2}\right)^2 + \left(r - \frac{(i-\frac{1}{2})d_B - (j-\frac{1}{2})d_L)}{2}\right)^2 = r^2$$

Observed from above relation, a certain (i, j) prescribes a lower bound of D, and for a certain merging pattern, the distance between the tollbooth and the end of the plaza that we eventually employ is the greatest lower bound of D. The less the greatest lower bound is, the smaller the size will be, and so will the cost of plaza.

By property of quadratic function, for a certain j, the greater i is, the greater D is. We have

$$\sum_{k=1}^{j} i_k d_B - j d_L \geq \left(\sum_{k=1}^{j} i_k + 1\right) d_B - (k+1) d_L$$

since $d_L > d_B$. That implies if j is changed into $j+1$ whilst i is changed into $i+1$, D will get smaller. Therefore, the global maximum must be obtained at one of the local maximum points, and then we let x be

$$x = \max_{j \in \{1,2,\ldots,L\}} \left\{ \frac{(\sum_{k=1}^{j} i_k - \frac{1}{2})d_B - (j - \frac{1}{2})d_L}{2} \right\}$$

And for given B and L, the optimal choice of merging pattern f would be the case if we can minimize x to be

$$x_{min} = \min_{\{i_k\} \in \{\|f^{-1}(\{k\})\| : \forall f\}} x$$

After trying out some pairs of small (B, L), we induce that for any given small number B, L (which can be put in practical use) satisfying $B > L$, there exists a merging pattern f_m such that

$$x = \frac{(B - \frac{1}{2})d_B - (L - \frac{1}{2})d_L)}{2} := x_0$$

Since for any i_k we have the relation $f(B) = L$, which implies above expression will appear in every calculation of x for all i_k, hence $x_{min} \geq x_0$, hence the existence of x_0 guarantees "=", providing us with the optimal merging pattern to minimize D.

By substituting $B = 8, L = 3, d_B = 2.5m, d_L = 3.75m, r = 115m, v_m = 15m/s, a = 2.78m/s^2$, the minimum of D is $105.5m$, when $(i_1, i_2, i_3) \in \{(3,3,2), (3,2,3), (2,3,3)\}$.

4.3 Waiting Area Model(WAM)

After completing the above model, something seems to be deficient also appears in our model. In the control time model, vehicles may need to wait at the booths even after the service finished, while there are a lot of vehicles queuing after it. So we wonder that whether it is better to build a waiting area at a plaza (as shown in Figure 5), so that the vehicles whose service has finished can go to the waiting area, then the consecutive vehicle can be serviced. So in this model, the staff spending more time serving instead of doing nothing while the vehicles are just waiting at the booths.

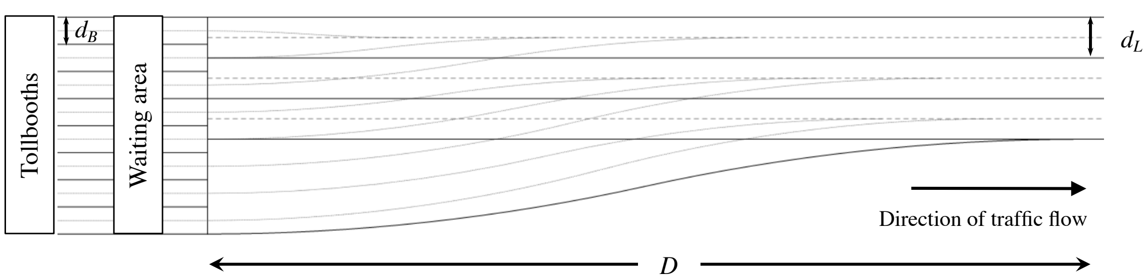

Figure 5: $B = 8$ tollbooth egress lanes merge into $L = 3$ lanes

We will test and compare this model with control time model in the following sections, to see whether it is efficient or not.

5 Analysis and Results

In this section we will analyze throughput, accident prevention and cost of each models introduced in former section.

5.1 Throughput

Throughput (θ) here is defined to be the number of vehicles per hour passing the point where the end of the plaza joins one of the the L outgoing traffic lanes. Here we discuss only one outgoing lane merged from two lanes in human-staffed tollbooth area in heavy traffic where vehicles come continuously during heavy traffic so that the cashier work continually, as shown in Figure 6 .

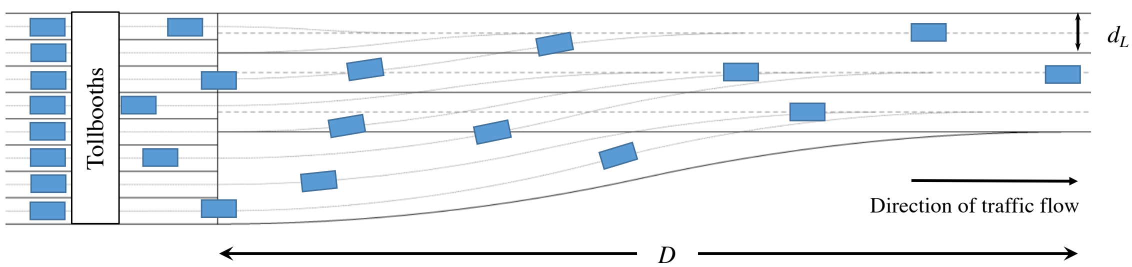

Figure 6: $B = 8$ tollbooth egress lanes merge into$L = 3$ lanes in heavy traffic

5.1.1 Throughput of existing model

The toll collection service time is a random variable which can be seen as a sum of random variables including time needed by driver to search money or credit vehicled, time needed by cashier to take related record, etc. According to the central limit theorem, we can assume that the service time of i^{th} vehicle ($t_{h,i}$) of the each lane follow normal distribution with $\mu = 15$ and $\sigma^2 = 4$ independently and identically.

$$T_{h,i} \sim N(15, 4)$$

Suppose the first vehicle coming into the tollbooth area at $t = 0$, then the time of the n^{th} vehicles coming out from a lane t_n should equal to the sum of the service time of itself and former vehicles.

$$t_n = \sum_{i=1}^{n} T_{h,i}$$

As can be seen that t_n is the sum of normally distributed random variables, thus

$$t_n \sim N(15n, 4n)$$

Besides we introduce a concept Post-Encroachment Time (PET) which is defined to be the time lapse between end of encroachment of turning vehicle and the time that the through vehicle actually arrives at the potential point of collision (Gettman & Head, 2003). When PET is less than reaction time, rational drivers would decelerate in order to maintain a safety distance, corresponding to reaction time $t_r = 1s$, with former vehicle. Consequently, more time is needed to pass the merging area which lower the efficiency and throughput. Denote PET of i^{th} vehicle as PET_i, then total delayed time T_d is

$$T_d = \sum_{PET_i < t_r} (t_r - PET_i)$$

Considering the total delayed time T_d, the mean time needed of a vehicle coming out from merging area is

$$\bar{t} = \frac{t_n + T_d}{2n}$$

Since there are two lanes working simultaneously and assume the motion in the merging area of each vehicle is the same, the throughput of the outgoing lane should be

$$\theta = \frac{3600}{\bar{t}}$$

After computer simulation with $N = 1000$ times, the results are shown in Figure. 7

5.1.2 Throughput of control time model

Now we consider the model with control time (t_0). Suppose the n^{th} vehicle coming out from lane a at $t_{a,n}$ with service time $T_{a,n}$ the m^{th} vehicle coming out from lane b at $t_{b,m}$ with service time $T_{b,m}$ where

$$T_{a,n} \sim N(15, 4) \quad \text{and} \quad T_{b,m} \sim N(15, 4)$$

Assume the 1^{st} vehicle comes to lane a at $t = 0$ and after ΔT the 2^{nd} vehicles comes to lane b where $\Delta T \sim U(0, 15)$. If we have

$$t_{a,n} + T_{a,n+1} < t_{b,m} + T_{b,m+1}$$

which means the $(n+1)^{th}$ vehicle in lane a completes toll collection service before the $(m+1)^{th}$ vehicle in lane b, then

$$t_{a,n+1} = max\{t_{a,n} + T_{a,n+1}, t_{a,n} + t_0, t_{b,m} + t_0\}$$

otherwise

$$t_{b,m+1} = max\{t_{b,m} + T_{b,m+1}, t_{a,n} + t_0, t_{b,m} + t_0\}$$

Set $max\{t_{a,n}, t_{b,m}\} < 3600$ in order to simulate time period of one hour. Once if

$$t_{a,n^*} > 3600 \quad \text{or} \quad t_{b,m^*} > 3600$$

we can obtain the throughput as

$$\theta = n^* + m^* - 1$$

After computer simulation with $N = 1000$ times, the results are shown in Figure 7

5.1.3 Throughput of waiting area model

Denote $t_{as,n}$ as the time when n^{th} vehicle completes toll collection and starts to wait in lane b, $t_{bs,m}$ as the time when m^{th} vehicle completes toll collection and starts to wait in lane b. The iteration equations in former section can be adapted to the follows. If

$$t_{as,n} + T_{a,n+1} < t_{bs,m} + T_{b,m+1}$$

which means the $(n+1)^{th}$ vehicle in lane a completes toll collection service before the $(m+1)^{th}$ vehicle in lane b, then

$$t_{a,n+1} = max\{t_{as,n} + T_{a,n+1}, t_{a,n} + t_0, t_{b,m} + t_0\}$$

otherwise

$$t_{b,m+1} = max\{t_{bs,m} + T_{b,m+1}, t_{a,n} + t_0, t_{b,m} + t_0\}$$

Set $max\{t_{a,n}, t_{b,m}\} < 3600$ in order to simulate time period of one hour. Once if

$$t_{a,n^*} > 3600 \quad \text{or} \quad t_{b,m^*} > 3600$$

we can obtain the throughput as

$$\theta = n^* + m^* - 1$$

After computer simulation with $N = 1000$ times, the results are shown in Figure 7

5.1.4 Comparison and hypothesis tests of throughputs

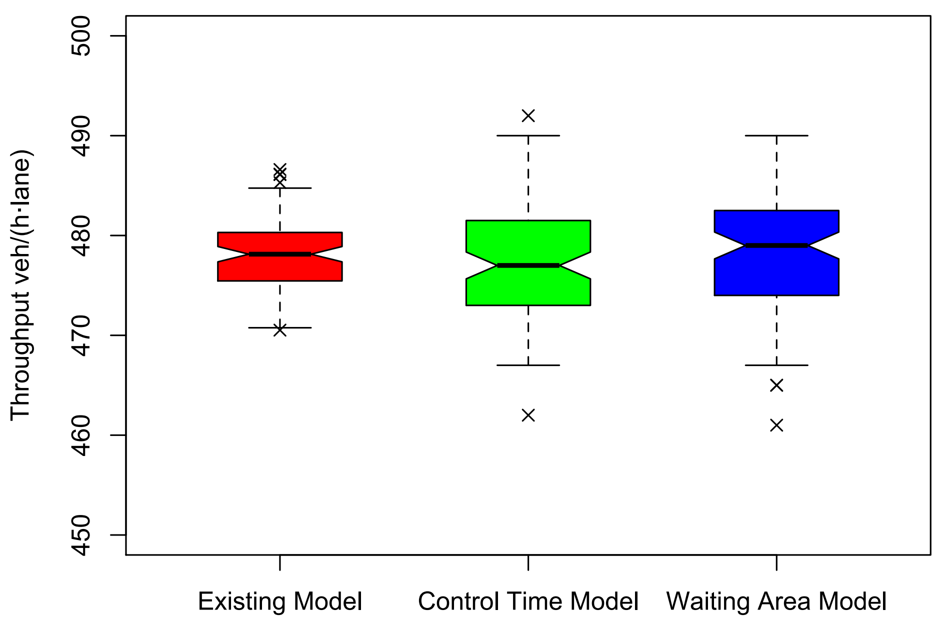

Figure 7: Comparison of throughput of models

It can be seen from Figure 7 that the throughputs differ slightly among models. Therefore, we conduct hypothesis test for means at 5% level of significance.

$$
\begin{aligned}
H_0 :& \text{There is no difference in average throughputs} \\
& \text{in existing modeland control time model, i.e. } \mu 1 = \mu 2 \\
H_1 :& H_0 \text{ is wrong, i.e.} \mu 1 \neq \mu 2
\end{aligned}
$$

Table 1: Statistics of two models

Model	Size	Mean	S.d.	SEM
Existing Model	1000	477.830	4.054	0.128
Control Time Model	1000	477.422	5.843	0.185
Waiting Area Model	1000	479.081	6.039	0.191

$$
\begin{aligned}
\text{difference in means} &= 477.830 - 477.422 = 0.408 \\
\text{overall standard error} &= \sqrt{0.128^2 + 0.185^2} = 0.225 \\
\text{test statistic} &= \frac{0.408}{0.225} = 1.813 \\
\text{p-value} &= 0.0348 > 0.025
\end{aligned}
$$

Therefore, H_0 is not rejected at 5% level of significance. We conclude that there is no significant difference of means of throughputs in existing model and control time model.

However, by similar method, when we make hypothesis test for means of the later two models, the conclusion is that there is significant difference of means of throughputs in existing model and control time model.

The analysis of throughputs above lead to the following results:

- Control time mechanism would not lower the efficiency and throughput. It just transfer delayed time T_d in merging area under existing model to control time t_0 in tollbooth area under control time model, as shown in
- Control time mechanism can improve the orderliness of the pattern of vehicles coming out from tollbooth into merging area which thereby improves the safety level of the merging pattern, to be discussed in section 5.2.
- Even though waiting area design could improve throughput, the increment $1.6veh/h$ is slight compared to the magnitude of throughput. Besides, the cost of land as well as other expense with regard to waiting area can be relatively high. Therefore, the disadvantageous of waiting area outweigh the benefit it brings so that we should disvehicled this model.

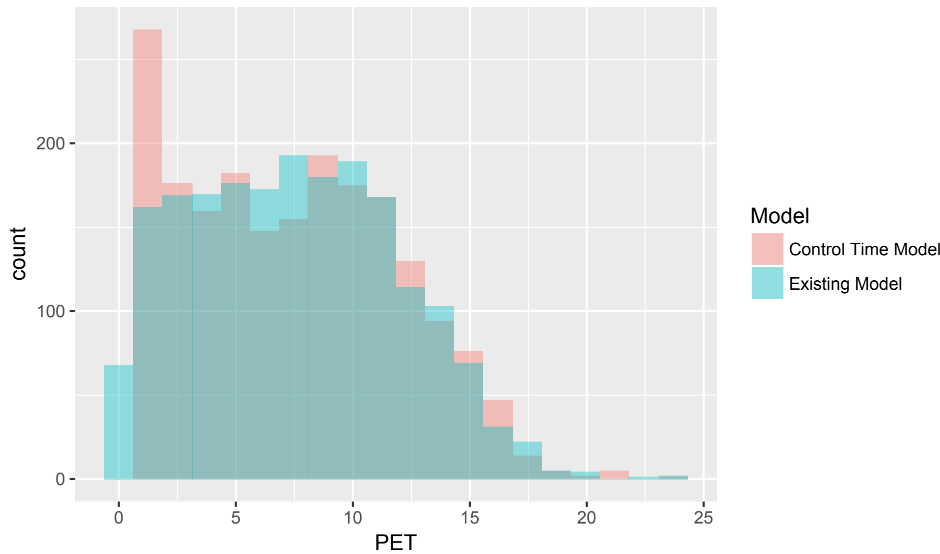

Figure 8: PET of existing model and PET of control time model

5.2 Accident prevention

This section will analyze accident rate in merging area in terms of PET. When PET is small, it becomes difficult for drivers to make decision promptly to avoid collision, and thus accident is more likely to happen. Therefore, we introduce following function to associate the probability of accident with PET.

$$p = \frac{e^{-PET}}{100} \tag{2}$$

When $PET = 0$, two vehicles arrive the potential point of collision theoretically while drivers could manage speed along the way in order to avoid collision, and hence the probability of collision under this circumstance can be set 0.01. After computer simulation based on the iteration equations with $N = 4000$ times, the PET of existing model and PET of control time model are shown in Figure 8.

It can be seen that under existing model, a significant part of PET lies in the interval $[0, 1]$ where accident is more likely to happen. While under control time model, due to the control mechanism, the minimum PET is control time $t_0 = 1s$ which guarantee low accident risk. The expectations of accident rate by simulation on function are shown in Table 2. The control time model can reduce the accident rate by 13.66%. What's more, the low rate of accident would make vehicles pass merging area without hindrance, which improve efficiency and throughput. Therefore, The control time model has better performance in accident prevention than existing model.

Table 2: Accident rate of each model

Model	Accident per 100,000 vehicles
Existing Model	61.416
Control Time Model	53.024

5.3 Cost

This section analyzes cost of merging area and tollbooth area, including fixed cost and variable cost per year, i.e. annual operation cost. Fixed cost mainly depends on cost of land and tollbooth construction. We can estimate area of land (A) by supposing the marginal curve of the out-most lane is symmetric, as shown in Figure 9.

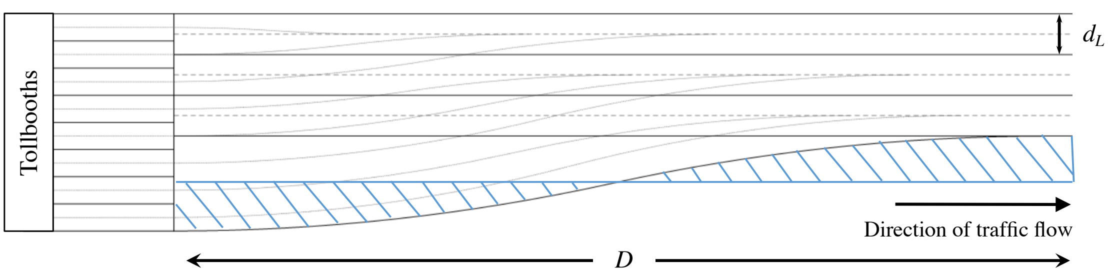

Figure 9: Estimation of A by symmetry of shaded area

Tollbooth construction cost can be divided to three kinds including human-staffed, electronic and automated tollbooths.

$$\text{Fixed cost} = \frac{1}{2}c_1 A + (c_h p_h + c_e p_e + c_a p_a)B$$

where

$$A = \frac{v_m^2}{a}Bd_b + (Bd_b + Ld_l)(D - \frac{v_m^2}{2a})$$
$$c_l = \text{construction cost of land per } m^2$$
$$c_h = \text{construction cost per automated tollbooth}$$
$$c_e = \text{construction cost per electronic tollbooth}$$
$$c_a = \text{construction cost per human-staffed tollbooth}$$

Substitute $B = 8, L = 3, d_B = 2.5, d_L = 3.75, c_l = 172.83, c_h = 58500, c_e = 15400, c_a = 58000$ (Pietrzyk,1992), $p_h = 4/8, p_e = 1/8, p_a = 3/8$ and get fixed cost = $\$739,320$. Similarly, annual operation cost of tollbooth area can be estimated by

$$\text{Annual operation cost} = (c'_a p_a + c'_h p_h + c'_e p_e)B$$

$$c'_a = \text{operation cost of automated tollbooth per year}$$
$$c'_h = \text{operation cost of salary per cashier per year}$$
$$c'_e = \text{operation cost of each electronic device per year}$$

Substitute $c'_h = 141900, c'_e = 4200, c'_a = 43300$ (Pietrzyk,1992)$, p_h = 4/8, p_e = 1/8, p_a = 3/8$ and get annual operation cost = $\$701,700$.

6 Possible Modification

As required in the problem, our report will then proceed to analyze the influences of some deviation of the conditions on our model. Some possible modifications may be made to improve the utility of our model in those situations.

6.1 Mixture of autonomous vehicles and human-driving vehicles

With the improvements in artificial intelligence and vehicle industry recent years, autonomous vehicles have become much more reliable and popular. However, some dangerous occasions may occur as the auto-driving system is not completely stable for all time. Therefore, even in some countries people are allowed to take the autonomous vehicle to the street, there are always required to be a driving sitting on the driver's seat and prepared to take over at any time.

If autonomous vehicles are allowed on the street, the situation of merging may become much more complicated. Firstly, sometimes the autonomous vehicles fail to detect the marking on the road and therefore go a wrong lane (Sam, 2016). It will be very dangerous if the autonomous vehicle went wrong when merging. Because there are a lot of complicated lanes at the merging area, and some may even cross each other, which are difficult for an autonomous vehicle to recognize and are hard to determine the turning angle accurately.

And one possible solution is to instruct the autonomous vehicles to go to the most left lane. Reason to constraint them to this lane is that the vehicles go through it will just need to go straight till the very end of merging area, which is also the optimal path that the computer will generate even that it fails to detect the signal on the road.

6.2 Change in proportion of different types of tollbooths

Nowadays, there are roughly 3 different types of tollbooths, they are conventional (human-staffed), automated and electronic toll collection booths. The main difference among these three different types is that they have different service times. The conventional booths have the longest service time, which is around 15 seconds in our assumption. And the automated ones have the second longest service time, and we assume it to be around 8 seconds. Meanwhile, vehicles equipped with ETC transponder are able to pass the ETC booths even without stop (with a maximal passing speed of $15mph$).

The model we introduced before works out quite good for both conventional and automated booths as explained before. However, it may not work well with ETC since they don't need to stop at the booth. First of all, one thing that we would like to classify is that the proportion of ETC booths will be at either a minority or majority position. It is because that the estimated capacity of a single ETC booth is $2000/h$, and this number is much larger than the need now considering the proportion of the drivers who possess a ETC transponder. So if the ETC booths are to be included in a toll plaza, we will place it at the most left lane, as we only need one to satisfy all demands for it. And we will arrange the other booths and lanes with the algorithm we introduced before. Of course, if the users of ETC are going to increase significantly in the future, the proportion of ETC booths can also be increased. However, one thing to be noticed is that always one booth merges to one lane on highway, provided that the efficiency of ETC booths are high and we do not need to increase the number of booths, which will only make it riskier when merging.

7 Sensitivity test

This section conducts sensitivity tests of throughput and cost. We define the sensitivity of a function $A(X)$ in terms of X to be $\frac{\frac{dA}{A}}{\frac{dX}{X}}$. If A is a discrete function, we change dA into $\Delta A = A(X+1) - A(X)$.

7.1 Sensitivity test of throughput

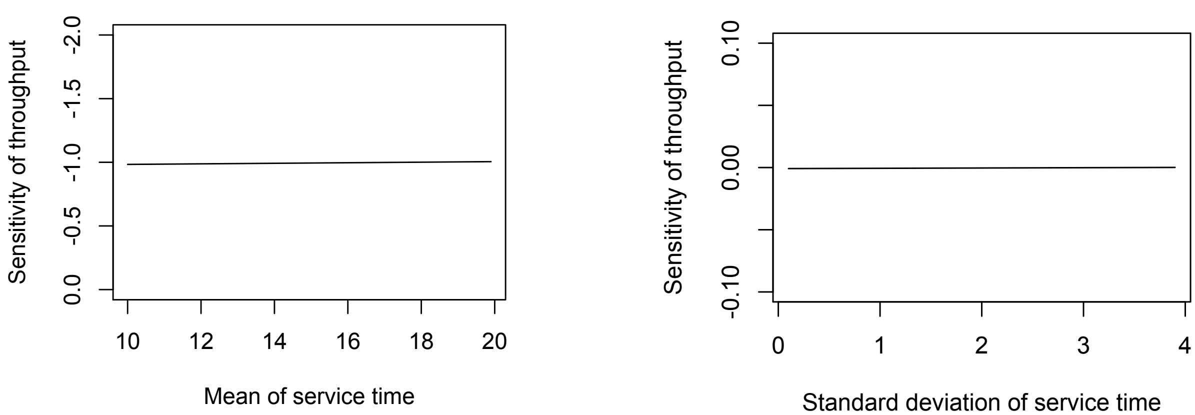

Figure 10: Sensitivity of throughput in terms of mean and standard deviation of service time.

As shown in 10, the sensitivity of throughput in terms of mean of service time T is a constant -1. Recall that in section 4.2, the optimal throughput is

$$\theta = \frac{B}{T}$$

The inverse ratio relationship of θ and T implies unit constant sensitivity of θ with respect to T, which is consistent with sensitivity test result. Besides, due to the law of large number, the volatility of service time finally performs slight effects on throughput and thus the sensitivity is zero.

7.2 Sensitivity test of fixed cost

By our assumption stated in Section 4.2 that the greatest lower bound is provided by the most outside lane and booth, combined with the formula to calculate the fixed cost, we get

$$\text{Fixed cost} = \frac{1}{2}c_1 A + (c_h p_h + c_e p_e + c_a p_a)B$$

The initial value of

$$\begin{aligned}&(v_m, a, B, L, d_B, d_L, p_a, p_e, p_h, c_l, c_e, c_h, c_a)\\=&(15, 2.78, 8, 3, 2.5, 3.75, 3/8, 1/8, 4/8, 172.84, 15400, 58500, 58000)\end{aligned}$$

We calculate the sensitivity of the fixed cost with respect to the number of booths B, we get the graph as following And it shows that the the sensitivity of B around 8 is about 1.15 and

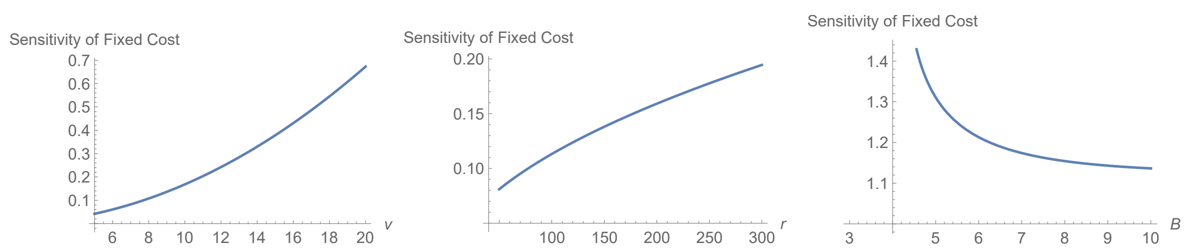

Figure 11: Sensitivity of fixed cost in terms of v_m, r and B

always above 1, and the decreasing speed is very low, which means that D is always sensitive in terms of B. So increasing B can have significant influence on D, and we may need to evaluate more whether to build more booths considering the balance of cost and efficiency.

Similarly, we can calculate the sensitivity of fixed cost with respect to v_m and r(as shown in the Figure 11).

The sensitivity of v_m around 0.38, and it remains below 1 as it vary from $10m/s$ to $20m/s$, which shows that the cost is not sensitive to the speed limit. So our cost estimation should be roughly correct with slight deviation with v_m. Finally, with similar analysis, sensitivity in terms of r is always around 0.1 and 0.2 from $50m$ to $200m$. So the radius only have little influence on our final result.

8 Conclusion

In Section 4 we describe three models, namely existing models, CTM, WAM. By simulation in Section 5, we discard WAM and point out the deficiencies of EM model and employ CTM. Therefore, the following conclusion is based on CTM.

The shape of the plaza should enable cars from every booth to merge into its prescribed lane with two consecutive arcs. By the assumption stated in Section 4.2, the shape of the plaza is determined by the most outside booth merging into the most outside lane. The boundary should make sure that the most outside vehicles have sufficient space on its right side. Take the example of $B = 8, L = 3$, the shape and merging pattern are as follows.

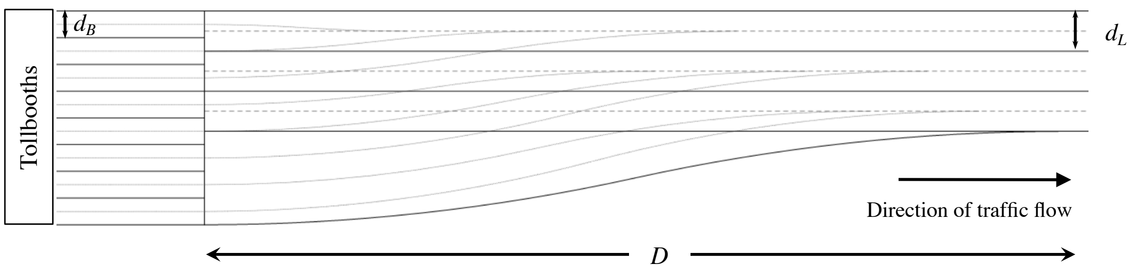

Figure 12: Shape and merging pattern of merging area

Note that in no case will cars from different booths merge into one lane simultaneously as is stated in our control time model, so it does not matter whether the distance between two adjacent trajectories merging into one lane becomes smaller or not. Additionally, leading lines should be drawn on the ground to ensure that vehicles take intended paths.

The size of merging area is given by formula

$$A = \frac{v_m^2}{a} Bd_b + (Bd_b + Ld_l)(D - \frac{v_m^2}{2a})$$

for one direction. This includes the acceleration paths (the left term) and merging (turning) paths (the right term). Multiply it by a factor 2 if we are to calculate for two directions. For the above example $B = 8, L = 3$, the area for one direction is $1824.93m^2$

In heavy traffic (defined as the situation where there is at least one car at every service booths), the throughput of plaza is approximately $3600B/T$ vehicles per hour (if all the tollbooths are human-staffed) , computed by induction of computer simulation. By substituting $B = 8, T_h = 15$, we get 1920 vehicles per hour from human-staffed lanes. In light traffic, the throughput is less than 3600B/T. Overall, there is no statistical significant of difference between throughput of CTM and existing model. This comparison does not include the consideration of accident. If we take into account the fact that CTM reduces the accident rate by 13.66% and that accident is likely to cause traffic jam and reduce toll plaza's throughput, the superiority of CTM would be more distinct. The cost of construction (fixed cost) and operation (variable cost) of the merging area and tollbooth area are calculated respectively. The fixed cost is given by the formula

$$\text{Fixed cost} = \frac{1}{2} c_1 A + (c_h p_h + c_e p_e + c_a p_a)B$$

the denotation is given in Section 5.3. In the B=8, L=3 example, the fixed cost is \$739320. The annual cost is given by similar formula

$$\text{Annual operation cost} = (c'_a p_a + c'_h p_h + c'_e p_e)B$$

which, in our example, turns to be \$701, 700 per year.

9 Strengths and Weaknesses

9.1 Strengths

By computer simulation, as the expression of throughput θ only contains the number of booths B and service time T, the employment of regulation in control time model does not reduce the throughput of the plaza (control time t_0 is not in the expression). However, the risk of accident decreases to a great extent, which ensures a safe and smooth flow of vehicles.

Moreover, we consider service time T following normal distribution instead of assuming a constant, which can better approximate real situation. Additionally, the control time mechanism could decrease the size of merging area compared with existing model where vehicles merge at their discretion. Last but not least, we use multiple programming platforms to run computer simulations (Mathematica, MATLAB and R), which provides us with accurate and convincing data to support our models.

9.2 Weaknesses

We assume the vehicles adopt two tangent uniform circular motion to merge into its prescribed lane, which may not be the real trajectory of vehicles. However, we can draw some leading lines on the ground to guide vehicles. Additionally, we do not consider the situation where vehicles break down and block certain booths and/or lanes.

The arrival of vehicles should follow Poisson distribution, but as we did not have the real-world data to estimate λ, we instead assume that the difference of two consecutive vehicles' arrival follows a uniform distribution, which may lead to some inaccuracies. This problem, together with the difficulties to determine the expectation and variance of service time T, could be addressed if data collected in real-world situation could be provided.

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Appendix

Some useful codes in R

1. Efficiency simulations

1.1 Efficiency simulation of existing model

```
x=numeric(1000)
y=numeric(1000)
mean.ex=numeric(1000)
PET.ex=numeric(1999)
for (j in 1:1000) {
  t1=rnorm(1000,15,4)
  t2=rnorm(1000,15,4)
  for (i in 1:1000){
    x[i]=sum(t1[1:i])
    y[i]=sum(t2[1:i])
    time.x.y=sort(c(x,y))
  }
  k=1:1999
  PET.ex[k]=time.x.y[k+1]-time.x.y[k]
  T.dl=sum(1-PET.ex[PET.ex<1])
  T.dl
  mean.ex[j]=(x[1000]+T.dl)/2000
}
Th.ex=3600/mean.ex
Th.ex
mean(Th.ex)
```

1.2 Efficiency simulation of CTM

```
k=1
stat.2=matrix(0,2,1000)
m=1:1000
for (m in 1:1000) {
  i=0
  j=0
  x=0
  y=runif(1,0,15)
  T1=rnorm(1000,15,2)
  T2=rnorm(1000,15,2)
  while (max(x,y)<=3600){
    if (x+T1[i+1]<y+T2[j+1]) {
```

```
      x=max(x+T1[i+1],x+k,y+k)
      i=i+1
    }
    else {
      y=max(y+T2[j+1],x+k,y+k)
      j=j+1
    }
  }
  stat.2[1,m]=i
  stat.2[2,m]=j
}
stat.2
Th.ct=stat.2[1,]+stat.2[2,]-1
Th.ct
mean(Th.ct)
```

1.3 Efficiency simulation of WAM

```
k=1
stat.3=matrix(0,2,1000)
m=1:1000
for (m in 1:1000) {
  i=0
  j=0
  x=0
  y=runif(1,0,)
  X=0
  Y=0
  T1=rnorm(1000,15,2)
  T2=rnorm(1000,15,2)
  while (max(X,Y)<=3600) {
    if (x+T1[i+1]<y+T2[j+1]) {
      X=max(x+T1[i+1],X+k,Y+k)
      x=x+T1[i+1]
      i=i+1
    }
    else {
      Y=max(y+T2[j+1],X+k,Y+k)
      y=y+T2[j+1]
      j=j+1
    }
  }
  stat.3[1,m]=i
  stat.3[2,m]=j
```

```
}
stat.3
Th.wa=stat.3[1,]+stat.3[2,]-1
Th.wa
mean(Th.wa)
```

2 Risk evaluation by simulations

2.1 Simulation of existing model

```
x=numeric(1000)
y=numeric(1000)
mean.ex=numeric(1000)
PET.ex=numeric(1999)
t1=rnorm(1000,15,2)
t2=rnorm(1000,15,2)
for (i in 1:1000){
  x[i]=sum(t1[1:i])
  y[i]=sum(t2[1:i])
  time.x.y=sort(c(x,y))
}
k=1:1999
PET.ex[k]=time.x.y[k+1]-time.x.y[k]
PET.ex
```

2.2 Simulation of CTM

```
k=1
i=0
j=0
x=0
y=runif(1,0,15)
T1=rnorm(1100,15,2)
T2=rnorm(1100,15,2)
X=numeric(1100)
Y=numeric(1100)
while (i+j<2000){
  if (x+T1[i+1]<y+T2[j+1]){
    x=max(x+T1[i+1],x+k,y+k)
    i=i+1
  }
  else{
    y=max(y+T2[j+1],x+k,y+k)
    j=j+1
  }
  X[i]=x
```

```
  Y[j]=y
}
X=X[which(X>0)]
Y=Y[which(Y>0)]
time.x.y=sort(c(X,Y))
PET.ct=numeric(i+j-1)
n=1:(i+j-1)
PET.ct[n]=time.x.y[n+1]-time.x.y[n]
PET.ct
```