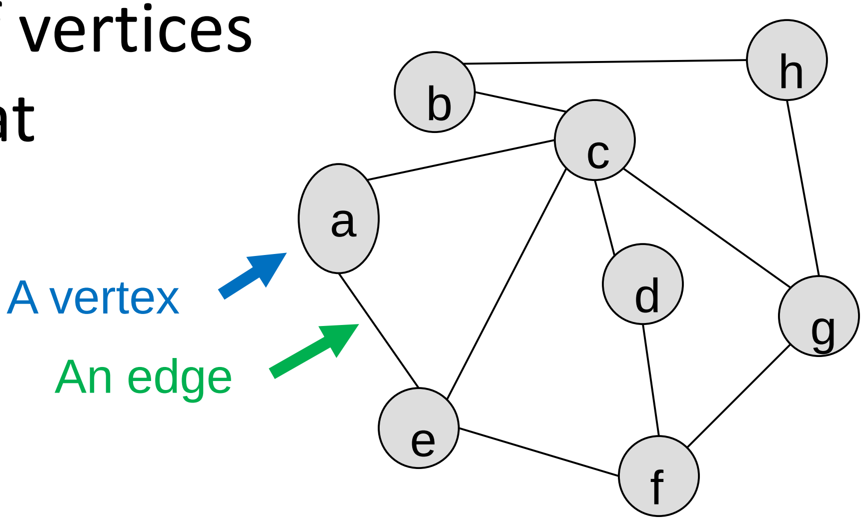
CSCI 104
Graph Representation and Traversals

Mark Redekopp
David Kempe
Sandra Batista

GRAPH REPRESENTATIONS

Graph Notation

- A **graph** is a collection of vertices (or nodes) and edges that connect vertices



V	E
a	(a,c)
b	(a,e)
c	(b,h)
d	(b,c)
e	(c,e)
f	(c,d)
g	(c,g)
h	(d,f)
	(e,f)
	(f,g)
	(g,h)

|V|=n=8

|E|=m=11

- Let **V** be the set of vertices
- Let **E** be the set of edges
- Let **|V| or n** refer to the number of vertices
- Let **|E| or m** refer to the number of edges

Graphs in the Real World

- Social networks
- Computer networks / Internet
- Path planning
- Interaction diagrams
- Bioinformatics

Basic Graph Representation

- Can simply store edges in a list
  - Unsorted
  - Sorted

V

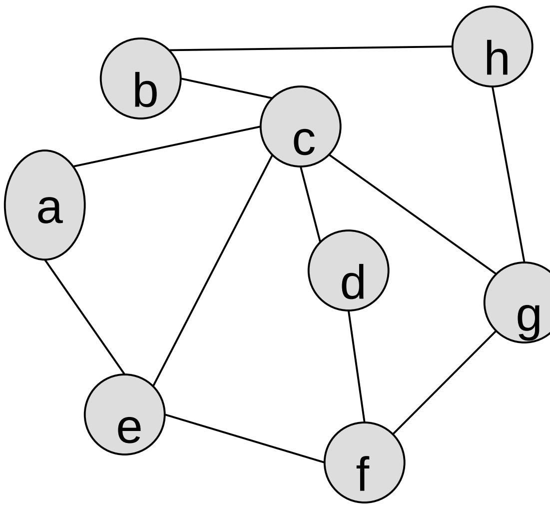
a
b
c
d
e
f
g
h

|V|=n=8

E

(a,c)
(a,e)
(b,h)
(b,c)
(c,e)
(c,d)
(c,g)
(d,f)
(e,f)
(f,g)
(g,h)

|E|=m=11



Graph ADT

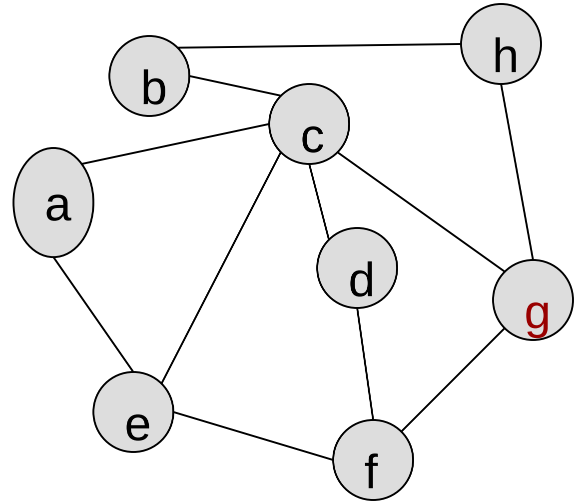
- What operations would you want to perform on a graph?
- `addVertex() : Vertex`
- `addEdge(v1, v2)`
- `getAdjacencies(v1) : List<Vertices>`
  - `Returns any vertex with an edge from v1 to itself`
- `removeVertex(v)`
- `removeEdge(v1, v2)`
- `edgeExists(v1, v2) : bool`

```
#include<iostream>
using namespace std;

template <typename V, typename E>
class Graph{

};
```

Perfect for templating the data associated with a vertex and edge as V and E

More Common Graph Representations

- Graphs are really just a list of lists
  - List of vertices each having their own list of adjacent vertices
- Alternatively, sometimes graphs are also represented with an adjacency matrix
  - Entry at (i,j) = 1 if there is an edge between vertex i and j, 0 otherwise



List of Vertices | Adjacency Lists

List of Vertices	Adjacency Lists
a	c,e
b	c,h
c	a,b,d,e,g
d	c,f
e	a,c,f
f	d,e,g
g	c,f,h
h	b,g

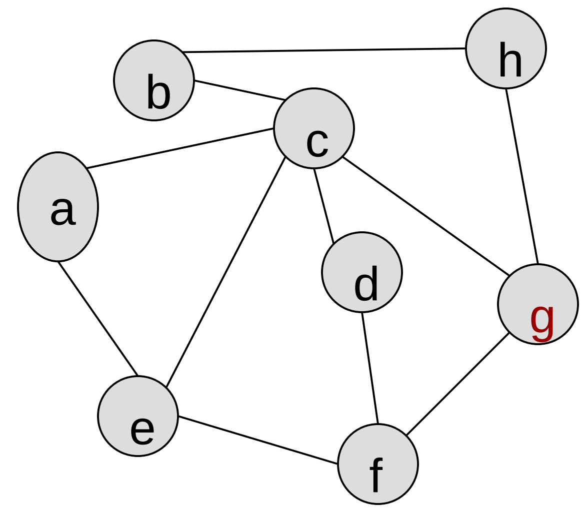
How would you express this using the ADTs you've learned?

	a	b	c	d	e	f	g	h
a	0	0	1	0	1	0	0	0
b	0	0	1	0	0	0	0	1
c	1	1	0	1	1	0	1	0
d	0	0	1	0	0	1	0	0
e	1	0	1	0	0	1	0	0
f	0	0	0	1	1	0	1	0
g	0	0	1	0	0	1	0	1
h	0	1	0	0	0	0	1	0

Adjacency Matrix Representation

Graph Representations

- Let |V| = n = # of vertices and |E| = m = # of edges
- Adjacency List Representation
  - O(________________) memory storage
  - Existence of an edge requires O(______________) time
- Adjacency Matrix Representation
  - O(________________) storage
  - Existence of an edge requires O(__________) lookup



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e	a,c,f
f	d,e,g
g	c,f,h
h	b,g

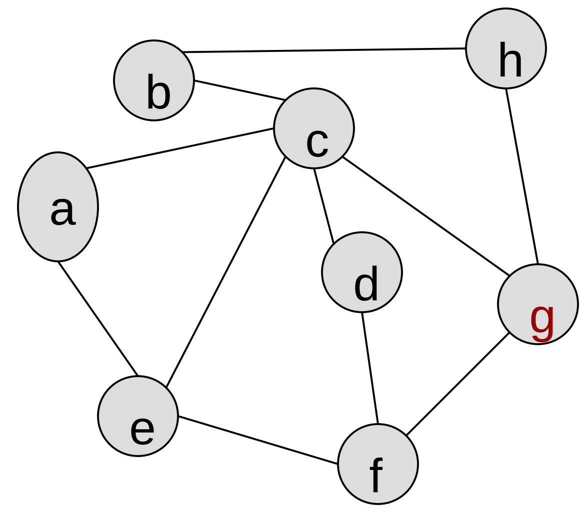
How would you express this using the ADTs you've learned?

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**
a	0	0	1	0	1	0	0	0
b	0	0	1	0	0	0	0	1
c	1	1	0	1	1	0	1	0
d	0	0	1	0	0	1	0	0
e	1	0	1	0	0	1	0	0
f	0	0	0	1	1	0	1	0
g	0	0	1	0	0	1	0	1
h	0	1	0	0	0	0	1	0

Adjacency Matrix Representation

Graph Representations

- Let |V| = n = # of vertices and |E| = m = # of edges
- Adjacency List Representation
  - O(|V| + |E|) memory storage
  - Define **degree** to be the number of edges incident on a vertex (deg(a) = 2, deg(c) = 5, etc.
  - Existence of an edge requires searching the adjacency list in O(deg(v))
- Adjacency Matrix Representation
  - O($|V|^2$) storage
  - Existence of an edge requires O(1) lookup (e.g. matrix[i][j] == 1)



List of Vertices	Adjacency Lists
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b	c,h
c	a,b,d,e,g
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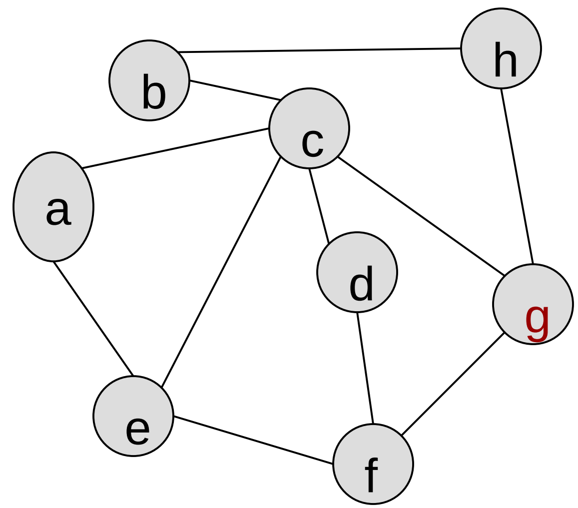
e	a,c,f
f	d,e,g
g	c,f,h
h	b,g

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**
a	0	0	1	0	1	0	0	0
b	0	0	1	0	0	0	0	1
c	1	1	0	1	1	0	1	0
d	0	0	1	0	0	1	0	0
e	1	0	1	0	0	1	0	0
f	0	0	0	1	1	0	1	0
g	0	0	1	0	0	1	0	1
h	0	1	0	0	0	0	1	0

Adjacency Matrix Representation

Graph Representations

- Can 'a' get to 'b' in two hops?
- Adjacency List
  - For each neighbor of a…
  - Search that neighbor's list for b
- Adjacency Matrix
  - Take the dot product of row a & column b



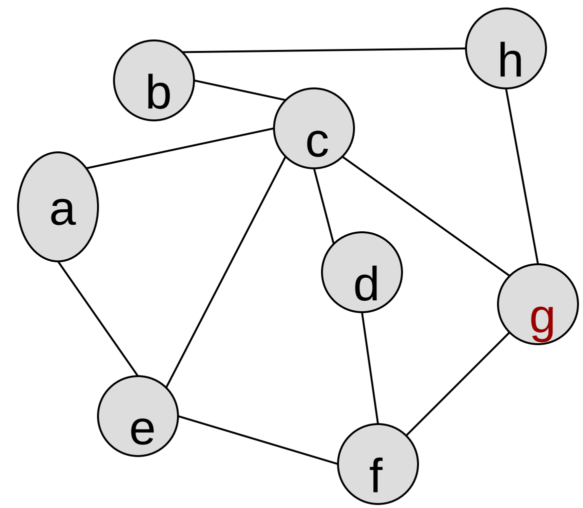
a	c,e
b	c,h
c	a,b,d,e,g
d	c,f
e	a,c,f
f	d,e,g
g	c,f,h
h	b,g

	a	b	c	d	e	f	g	h
a	0	0	1	0	1	0	0	0
b	0	0	1	0	0	0	0	1
c	1	1	0	1	1	0	1	0
d	0	0	1	0	0	1	0	0
e	1	0	1	0	0	1	0	0
f	0	0	0	1	1	0	1	0
g	0	0	1	0	0	1	0	1
h	0	1	0	0	0	0	1	0

Adjacency Matrix Representation

Graph Representations

- Can 'a' get to 'b' in two hops?
- Adjacency List
  - For each neighbor of a…
  - Search that neighbor's list for b
- Adjacency Matrix
  - Take the dot product of row a & column b



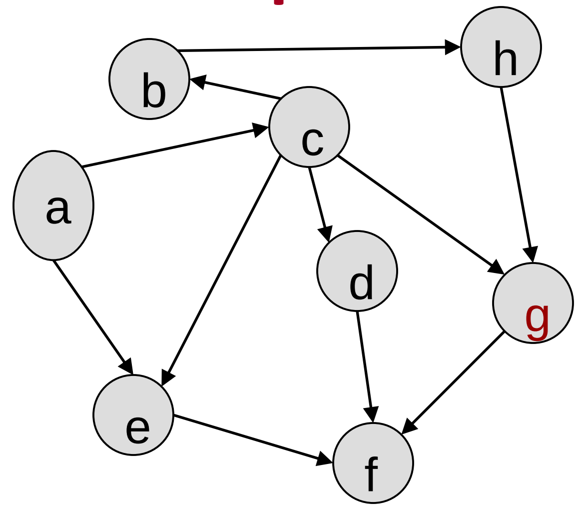
a	c,e
b	c,h
c	a,b,d,e,g
d	c,f
e	a,c,f
f	d,e,g
g	c,f,h
h	b,g

	a	b	c	d	e	f	g	h
a	0	0	1	0	1	0	0	0
b	0	0	1	0	0	0	0	1
c	1	1	0	1	1	0	1	0
d	0	0	1	0	0	1	0	0
e	1	0	1	0	0	1	0	0
f	0	0	0	1	1	0	1	0
g	0	0	1	0	0	1	0	1
h	0	1	0	0	0	0	1	0

Adjacency Matrix Representation

Directed vs. Undirected Graphs

- In the previous graphs, edges were **undirected** (meaning edges are 'bidirectional' or 'reflexive')
  - An edge (u,v) implies (v,u)
- In **directed** graphs, links are unidirectional
  - An edge (u,v) does not imply (v,u)
  - For Edge (u,v): the **source** is u, **target** is v
- For adjacency list form, you may need 2 lists per vertex for both predecessors and successors



List of Vertices / Adjacency Lists

List of Vertices	Adjacency Lists
a	c,e
b	h
c	b,d,e,g
d	f
e	f
f	
g	f
h	g

Target

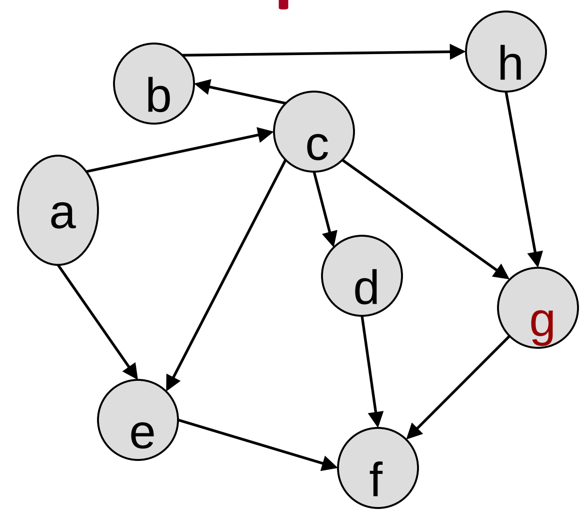
Source

	a	b	c	d	e	f	g	h
a	0	0	1	0	1	0	0	0
b	0	0	0	0	0	0	0	1
c	0	1	0	1	1	0	1	0
d	0	0	0	0	0	1	0	0
e	0	0	0	0	0	1	0	0
f	0	0	0	0	0	0	0	0
g	0	0	0	0	0	1	0	0
h	0	0	0	0	0	0	1	0

Adjacency Matrix Representation

Directed vs. Undirected Graphs

- In directed graph with edge (src,tgt) we define
  - Successor(src) = tgt
  - Predecessor(tgt) = src
- Using an adjacency list representation *may* warrant two lists predecessors and successors



List of Vertices

Vertex	Succs (Outgoing)	Preds (Incoming)
a	c,e	
b	h	c
c	b,d,e,g	a
d	f	c
e	f	a,c
f		d, e, g
g	f	c,h
h	g	b

Target / Source

	a	b	c	d	e	f	g	h
a	0	0	1	0	1	0	0	0
b	0	0	0	0	0	0	0	1
c	0	1	0	1	1	0	1	0
d	0	0	0	0	0	1	0	0
e	0	0	0	0	0	1	0	0
f	0	0	0	0	0	0	0	0
g	0	0	0	0	0	1	0	0
h	0	0	0	0	0	0	1	0

Adjacency Matrix Representation

Graph Runtime, |V| = n, |E| =m

Operation vs Implementation for Edges	Add edge	Delete Edge	Test Edge	Enumerate edges for single vertex
Unsorted array or Linked List				
Sorted array				
Adjacency List				
Adjacency Matrix				

Graph Runtime, |V| = n, |E| =m

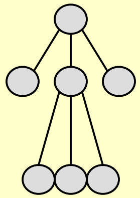
Operation vs Implementation for Edges	**Add edge**	**Delete Edge**	**Test Edge**	**Enumerate edges for single vertex**
Unsorted array or Linked List	$\Theta(1)$	$\Theta(m)$	$\Theta(m)$	$\Theta(m)$
Sorted array	$\Theta(m)$	$\Theta(m)$	$\Theta(\log m)$ [if binary search used]	$\Theta(\log m)+\Theta(deg(v))$ [if binary search used]
Adjacency List	Time to find List for a given vertex + $\Theta(1)$	Time to find List for a given vertex + $\Theta(deg(v))$	Time to find List for a given vertex + $\Theta(deg(v))$	Time to find List for a given vertex + $\Theta(deg(v))$
Adjacency Matrix	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(v)$

A graph with restrictions

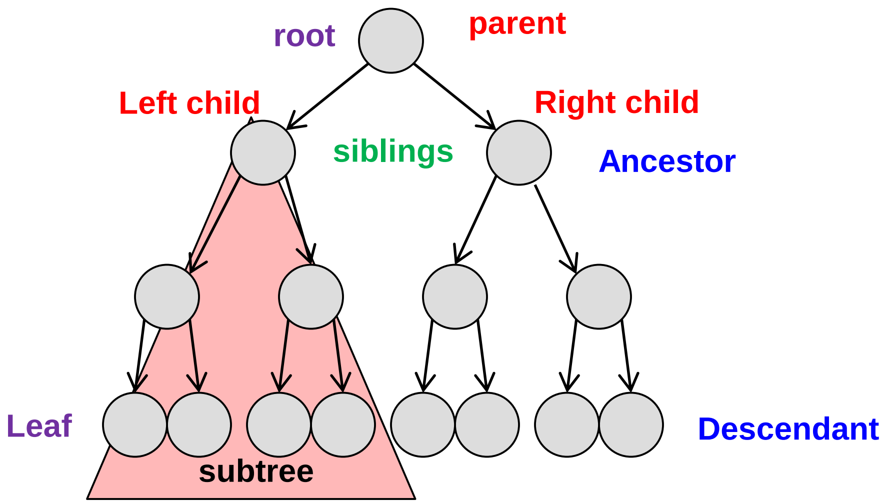
TREES

Tree Definitions – Part 1

- **Definition**: A connected, acyclic (no cycles) graph with:
  - A root node, r, that has 0 or more subtrees
  - Exactly one path between any two nodes
- In general:
  - Nodes have exactly one parent (except for the root which has none) and 0 or more children
- d-ary tree
  - Tree where each node has at most d children
  - Binary tree = d-ary Tree with d=2



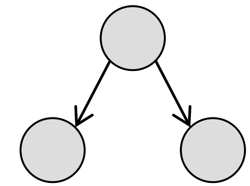
A 3-ary (trinary) tree



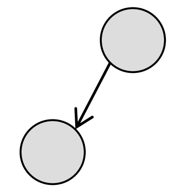
Terms:
- **Parent(i)**: Node directly above node i
- **Child(i)**: Node directly below node i
- **Siblings**: Children of the same parent
- **Root:** Only node with no parent
- **Leaf:** Node with 0 children
- **Height:** Number of nodes on longest path from root to any leaf
- **Subtree(n)**: Tree rooted at node n
- **Ancestor(n)**: Any node on the path from n to the root
- **Descendant(n):** Any node in the subtree rooted at n

Tree Definitions – Part 2

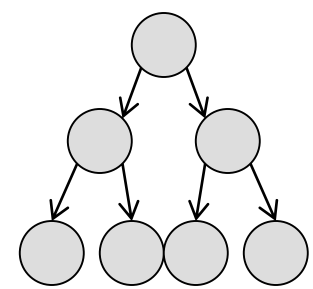
- Tree height: maximum # of nodes on a path from root to any leaf
- **Full** d-ary tree, T, where
  - Every vertex has 0 or d children and all leaf nodes are at the same level (i.e. adding 1 more node requires increasing the height of the tree)
- **Complete** d-ary tree
  - **Top h-1 levels are full** AND **bottom level is filled left-to-right**
  - Each level is filled left-to-right and a new level is not started until the previous one is complete
- **Balanced** d-ary tree
  - Tree where, for EVERY node, the subtrees for each child differ in height by at most 1



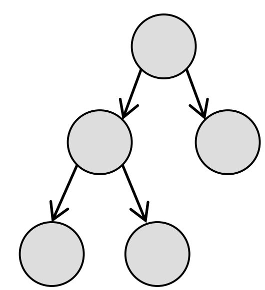
Full



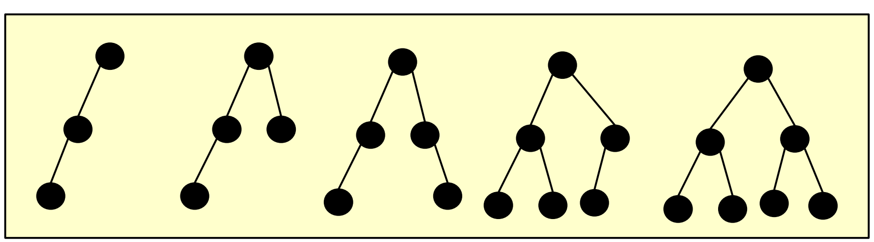
Complete, but not full



Full



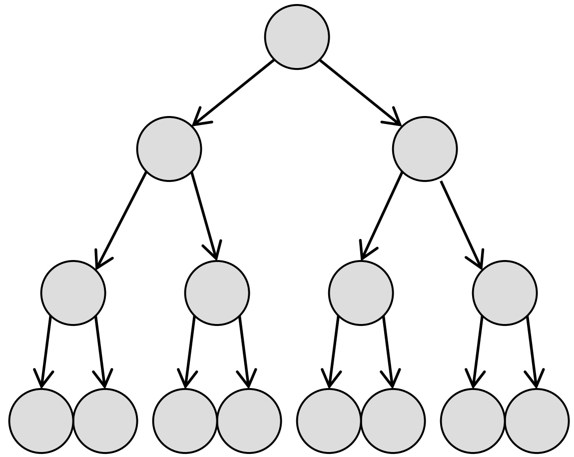
Complete, but not full



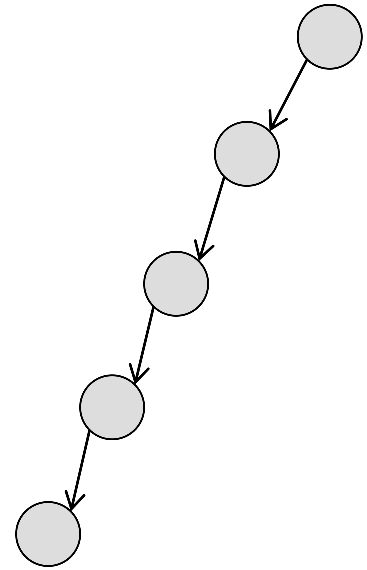
DAPS, 6th Ed. Figure 15-8

Tree Height

- A full binary tree of n nodes has height, h=$\lceil log_2(n+1) \rceil$
  - This implies the minimum height of any tree with n nodes is $\lceil log_2(n+1) \rceil$
- The maximum height of a tree with n nodes is, ___



15 nodes => height $log_2(16)$ = 4



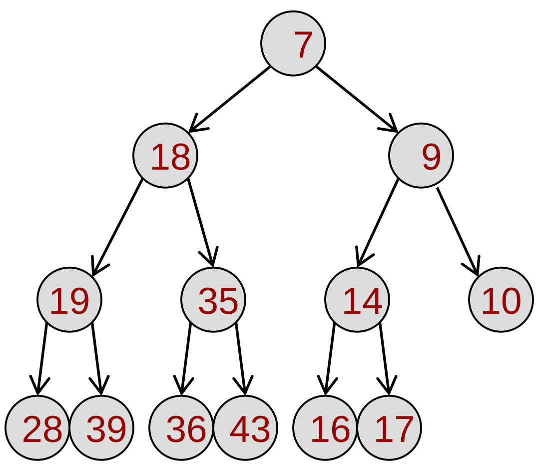
5 nodes => height = __

Array-based and Link-based

TREE IMPLEMENTATIONS

Array-Based Complete Binary Tree

- Binary tree that is complete (i.e. only the lowest-level contains empty locations and items added left to right) can be stored nicely in an array (let's say it starts at index 1 and index 0 is empty)
- Can you find the mathematical relation for finding the index of node i's parent, left, and right child?
  - Parent(i) = ___________
  - Left_child(i) = ____________
  - Right_child(i) = ____________



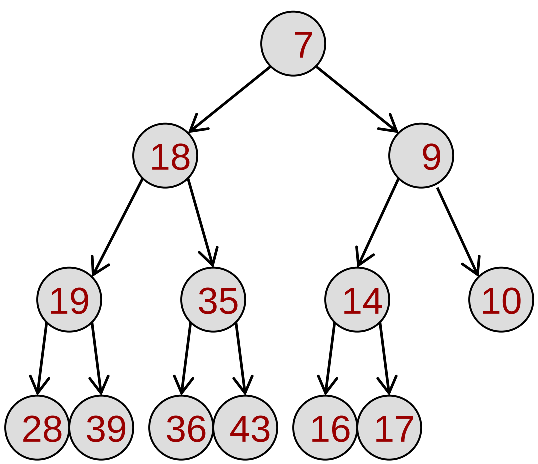
0	1	2	3	4	5	6	7	8	9	10	11	12	13
em	7	18	9	19	35	14	10	28	39	36	43	16	17

parent(5) = ________
Left_child(5) = _________
Right_child(5) = __________

Array-Based Complete Binary Tree

- Binary tree that is complete (i.e. only the lowest-level contains empty locations and items added left to right) can be stored nicely in an array (let’s say it starts at index 1 and index 0 is empty)
- Can you find the mathematical relation for finding node i's parent, left, and right child?
  - Parent(i) = i/2
  - Left_child(i) = 2*i
  - Right_child(i) = 2*i + 1

0	1	2	3	4	5	6	7	8	9	10	11	12	13
em	7	18	9	19	35	14	10	28	39	36	43	16	17

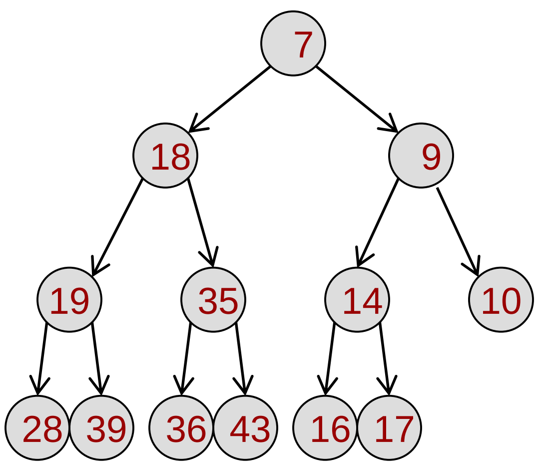


parent(5) = 5/2 = 2
Left_child(5) = 2*5 = 10
Right_child(5) = 2*5+1 = 11

Non-complete binary trees require much more bookeeping to store in arrays…usually link-based approaches are preferred

0-Based Indexing

- Now let's assume we start the root at index 0 of the array
- Can you find the mathematical relation for finding the index of node i's parent, left, and right child?
  - Parent(i) = __________
  - Left_child(i) = ___________
  - Right_child(i) = ___________



0	1	2	3	4	5	6	7	8	9	10	11	12
7	18	9	19	35	14	10	28	39	36	43	16	17

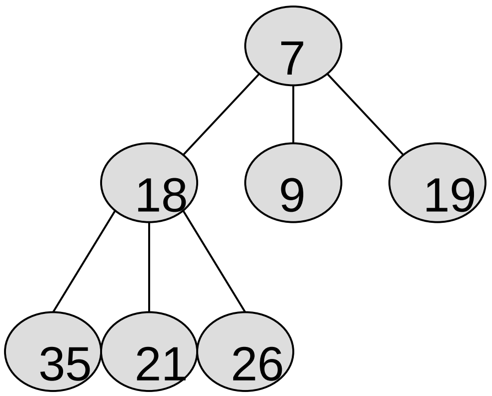
parent(5) = ________

Left_child(5) = _________

Right_child(5) = __________

D-ary Array-based Implementations

- Arrays can be used to store d-ary **complete** trees
  - Adjust the formulas derived for binary trees in previous slides in terms of **d**



A 3-ary (trinary) tree

0	1	2	3	4	5	6
7	18	9	19	35	21	26

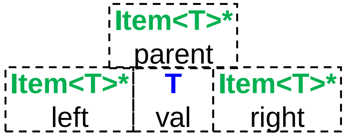
Link-Based Approaches

- For an arbitrary (**non-complete**) d-ary tree we need to use pointer-based structures
  - Much like a linked list but now with two pointers per Item
- Use NULL pointers to indicate no child
- Dynamically allocate and free items when you add/remove them

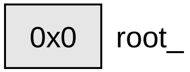
```
#include<iostream>
using namespace std;

template <typename T>
struct Item {
  T val;
  Item<T>* left,right;
  Item<T>* parent;
};
// Bin. Search Tree
template <typename T>
class BinTree
{
 public:
 BinTree();
 ~BinTree();
 void add(const T& v);
 ...
 private:
 Item<T>* root_;
};
```

Item<T> blueprint:



class BinTree<T>:



Link-Based Approaches

- Add(5)
- Add(6)
- Add(7)