Spatial vector algebra cheat sheet

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Spatial vector algebra is a subset of Lie algebra where we follow two conventions that simplify calculations: we use spatial vectors rather than body vectors whenever possible, and Plücker transforms rather than affine transforms to represent members of the Lie group. Like with any other algebra, the more identities we swing, the more proficient we get at it. This cheat sheet lists the ones I have found useful so far. It references both spatial and body vectors. Because when a spatial vector formula resists intuition (not the rareliest occurrence), it can help to explicit all the frames involved.

Notations

We adopt the subscript right-to-left convention for transforms, and superscript notation to indicate the frame of a motion or force vector:

Quantity	Notation
Body angular velocity of frame A in frame B	${}^A\boldsymbol{\omega}_{BA}$
Plücker transform from frame A to frame B	$\boldsymbol{X}_{BA}$
Position of (the origin of) frame B in frame A	${}^A\boldsymbol{p}_B$
Rotation matrix from frame A to frame B	$\boldsymbol{R}_{BA}$
Spatial angular velocity of frame A in frame B	${}^B\boldsymbol{\omega}_{BA}$
Spatial velocity of frame A	$\boldsymbol{v}_A$
World frame (inertial)	W

With these notations frame transforms can be read left to right, for example:

$$\boldsymbol{X}_{CA} = \boldsymbol{X}_{CB}\boldsymbol{X}_{BA} \qquad {}^B\boldsymbol{\omega} = \boldsymbol{R}_{BA}\,{}^A\boldsymbol{\omega} \qquad {}^B\boldsymbol{p}_C = \boldsymbol{R}_{BA}\,{}^A\boldsymbol{p}_C + {}^B\boldsymbol{p}_A$$

We part from Featherstone's notation (${}^B\boldsymbol{X}_A$) to be able to keep track of the original transforms in time derivatives. For example, the angular velocity $\boldsymbol{\omega}_{BA}$ that derivates from the rotation $\boldsymbol{R}_{BA}$ satisfies:

$$\dot{\boldsymbol{R}}_{BA} = \left({}^B\boldsymbol{\omega}_{BA}\times\right)\boldsymbol{R}_{BA} = \boldsymbol{R}_{BA}\left({}^A\boldsymbol{\omega}_{BA}\times\right)$$

Skew-symmetric operator

The operator $\times : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \boldsymbol{a} \mapsto \boldsymbol{a}\times$ turns a 3D vector $\boldsymbol{a}$ into its 3×3 cross-product skew-symmetric matrix:

$$\boldsymbol{a}\times = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

We use the same notation for the operator $\times : \mathbb{R}^6 \to \mathbb{R}^{6\times 6}, \boldsymbol{\xi} \mapsto \boldsymbol{\xi}\times$ on twists:

$$\boldsymbol{\xi}\times = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \times = \begin{bmatrix} \boldsymbol{\omega}\times & \boldsymbol{v}\times \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{\omega}\times \end{bmatrix}$$

Note that the matrix structure is tied to our choice of writing linear coordinates $\boldsymbol{v}$ before angular coordinates $\boldsymbol{\omega}$ in twists $\boldsymbol{\xi}$. (In Featherstone's reference book angular coordinates come first, in which case the linear cross-product block $\boldsymbol{v}\times$ moves to the opposite corner.)

Cross and dot products

Euclidean cross products

ID	Name	Formula
XP1	Vector triple product	$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$
XP2	Rotation of cross product	$\boldsymbol{R}(\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{R}\boldsymbol{a}) \times (\boldsymbol{R}\boldsymbol{b})$
XP3	Cross product by rotated vector	$(\boldsymbol{R}\boldsymbol{v})\times = \boldsymbol{R}(\boldsymbol{v}\times)\boldsymbol{R}^\top$
XP4	Rotation of cross product matrix	$\boldsymbol{R}(\boldsymbol{v}\times) = (\boldsymbol{R}\boldsymbol{v}) \times \boldsymbol{R}$
XP5	Rotation of cross product matrix	$\boldsymbol{R}_{BA} \left({}^A\boldsymbol{v}\times\right) = {}^B\boldsymbol{v} \times \boldsymbol{R}_{BA}$

Spatial cross products

ID	Name	Formula
SC1	Cross product by transformed vector	$(\boldsymbol{X}\boldsymbol{v})\times = \boldsymbol{X}(\boldsymbol{v}\times)\boldsymbol{X}^{-1}$
SC2	Transform of cross product matrix	$\boldsymbol{X}(\boldsymbol{v}\times) = (\boldsymbol{X}\boldsymbol{v}) \times \boldsymbol{X}$
SC3	Transform of cross product matrix	$\boldsymbol{X}_{BA} \left({}^A\boldsymbol{v}\times\right) = {}^B\boldsymbol{v} \times \boldsymbol{X}_{BA}$

Dot products

ID	Property	Formula
IVR1	Invariance by rotation	$(\boldsymbol{R}\boldsymbol{a}) \cdot (\boldsymbol{R}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}$
IVR2	Invariance by dual transforms	$(\boldsymbol{X}\boldsymbol{m}) \cdot (\boldsymbol{X}^*\boldsymbol{f}) = \boldsymbol{m} \cdot \boldsymbol{f}$

Kinematics

Transform matrices

Note that the structure of Plücker transform matrices depends on the order in which angular and linear coordinates are serialized. In this cheat sheet, we serialize linear coordinates first:

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}$$

This results in the following matrices:

ID	Coordinates	Transform	Inverse
TMV	Motion vectors	$\boldsymbol{X}_{BA} = \begin{bmatrix} \boldsymbol{R}_{BA} & \left({}^B\boldsymbol{p}_A\times\right)\boldsymbol{R}_{BA} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{R}_{BA} \end{bmatrix}$	$\boldsymbol{X}_{BA}^{-1} = \begin{bmatrix} \boldsymbol{R}_{BA}^T & -\boldsymbol{R}_{BA}^T\left({}^B\boldsymbol{p}_A\times\right) \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{R}_{BA}^T \end{bmatrix}$
TFV	Force vectors	$\boldsymbol{X}_{BA}^{*} = \begin{bmatrix} \boldsymbol{R}_{BA} & \boldsymbol{0}_{3\times 3} \\ \left({}^B\boldsymbol{p}_A\times\right)\boldsymbol{R}_{BA} & \boldsymbol{R}_{BA} \end{bmatrix}$	$\boldsymbol{X}_{BA}^{-*} = \begin{bmatrix} \boldsymbol{R}_{BA}^T & \boldsymbol{0}_{3\times 3} \\ -\boldsymbol{R}_{BA}^T\left({}^B\boldsymbol{p}_A\times\right) & \boldsymbol{R}_{BA}^T \end{bmatrix}$

Inversions

ID	Inverse of	Formula
INV1	Rotation matrix	$\boldsymbol{R}_{AB} = \boldsymbol{R}_{BA}^{-1} = \boldsymbol{R}_{BA}^{\top}$
INV2	Angular velocity	${}^A\boldsymbol{\omega}_{AB} = -{}^A\boldsymbol{\omega}_{BA}$

Time derivatives

ID	Quantity	Spatial derivative	Body derivative
TD1	Rotation matrix	$\dot{\boldsymbol{R}}_{BA} = {}^B\boldsymbol{\omega}_{BA} \times \boldsymbol{R}_{BA}$	$\dot{\boldsymbol{R}}_{BA} = \boldsymbol{R}_{BA}\left({}^A\boldsymbol{\omega}_{BA}\times\right)$

ID	Quantity	Spatial vector algebra	Screw algebra
TD2	Rotation matrix	$\dot{\boldsymbol{R}}_{BA} = {}^B\left(\boldsymbol{\omega}_A - \boldsymbol{\omega}_B\right) \times \boldsymbol{R}_{BA}$	$\dot{\boldsymbol{R}}_{BA} = {}^B\left(\boldsymbol{\omega}_{WA} - \boldsymbol{\omega}_{WB}\right) \times \boldsymbol{R}_{BA}$
TD3	Plücker transform	$\dot{\boldsymbol{X}}_{BA} = {}^B\left(\boldsymbol{v}_A - \boldsymbol{v}_B\right) \times \boldsymbol{X}_{BA}$	$\dot{\boldsymbol{X}}_{BA} = {}^B\left(\boldsymbol{v}_{WA} - \boldsymbol{v}_{WB}\right) \times \boldsymbol{X}_{BA}$
TD4	Plücker to world	$\dot{\boldsymbol{X}}_{WB} = \left(\boldsymbol{v}_B\times\right)\boldsymbol{X}_{WB}$	$\dot{\boldsymbol{X}}_{WB} = {}^W\boldsymbol{v}_{WB} \times \boldsymbol{X}_{WB}$
TD5	Plücker to world	(not a spatial identity)	$\dot{\boldsymbol{X}}_{WB} = \boldsymbol{X}_{WB}\left({}^B\boldsymbol{v}_{WB}\times\right)$

Proof of some identities

INV2

We start from the time derivative of the identity:

$$\boldsymbol{R}_{AB}\boldsymbol{R}_{BA} = \boldsymbol{I}_3$$

Using the identities $\dot{\boldsymbol{R}}_{AB} = \left({}^A\boldsymbol{\omega}_{AB}\times\right)\boldsymbol{R}_{AB}$ and $\dot{\boldsymbol{R}}_{BA} = \boldsymbol{R}_{BA}\left({}^A\boldsymbol{\omega}_{BA}\times\right)$, we get:

$$\left({}^A\boldsymbol{\omega}_{AB}\times\right)\boldsymbol{R}_{AB}\boldsymbol{R}_{BA} + \boldsymbol{R}_{AB}\boldsymbol{R}_{BA}\left({}^A\boldsymbol{\omega}_{BA}\times\right) = \boldsymbol{0}_{3\times 3}$$

And thus ${}^A\boldsymbol{\omega}_{AB} = -{}^A\boldsymbol{\omega}_{BA}$.

TD1 $\Leftrightarrow$ TD2

We can check that TD1 and TD2 are equivalent, that is, ${}^B\left(\boldsymbol{\omega}_{WA} - \boldsymbol{\omega}_{WB}\right) = {}^B\boldsymbol{\omega}_{BA}$. For that, let's go back to the time derivative of the rotation matrix:

$$
\begin{aligned}
\dot{\boldsymbol{R}}_{BA} = {}^B\boldsymbol{\omega}_{BA} \times \boldsymbol{R}_{BA} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{R}_{BW}\boldsymbol{R}_{WA}\right) & \\
&= \dot{\boldsymbol{R}}_{BW}\boldsymbol{R}_{WA} + \boldsymbol{R}_{BW}\dot{\boldsymbol{R}}_{WA} \\
&= {}^B\boldsymbol{\omega}_{BW} \times \boldsymbol{R}_{BW}\boldsymbol{R}_{WA} + \boldsymbol{R}_{BW}\boldsymbol{R}_{WA}\left({}^A\boldsymbol{\omega}_{WA}\times\right) \\
&= {}^B\boldsymbol{\omega}_{BW} \times \boldsymbol{R}_{BA} + \boldsymbol{R}_{BA}\left({}^A\boldsymbol{\omega}_{WA}\times\right) \\
&= \left(-\,{}^B\boldsymbol{\omega}_{WB}\right) \times \boldsymbol{R}_{BA} + {}^B\boldsymbol{\omega}_{WA} \times \boldsymbol{R}_{BA} \\
&= \left({}^B\boldsymbol{\omega}_{WA} - {}^B\boldsymbol{\omega}_{WB}\right) \times \boldsymbol{R}_{BA}
\end{aligned}
$$

To go further

Textbooks

A reference book on spatial vector algebra is Roy Featherstone's Rigid Body Dynamics Algorithms. Its tables are cheat sheets of their own. The book itself is better as an implementation reference than for learning things, as it often assumes the reader is already familiar with screw theory. For first-time learners, Modern Robotics might be a better place to start, or A Mathematical Introduction to Robotic Manipulation for those who like their math fresh.

Other cheat sheets

- Kinematics and dynamics cheat sheet of the Kindr library
- Representing attitude: Euler angles, unit quaternions, and rotation vectors for rotations only
- SE(3) algebra cheat sheet of the Pinocchio library

See also

- From spatial to body acceleration for a practical use case
- Research Notes on Spatial Velocities by Jan Carius
- Screw axes with more details on central and noncentral axes
- Some notes on Lie groups by Wilson Jallet