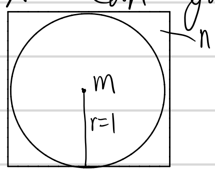
github shuyueliu
lsy1992l016
CoCalc freya_lsy@163.com
lsy12345

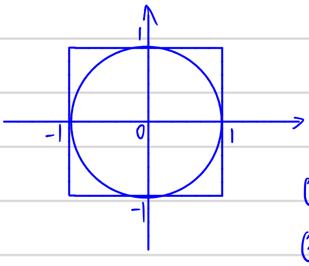
Basic Monte carlo L03S01

1. What is def of MC ?

2. Ex: Can you approximate π ?



suppose: ① I make n-shots to the rectangular, and m-shots fall in circle.
② shots are uniformly distributed, then $\pi \approx \frac{m}{n} \cdot 4$



① Draw N random points uniformly in rectangular $(-1,1)\times(-1,1)$ i.e.
$\{z_i = (x_i, y_i) : x_i \sim U(-1,1)$
$y_i \sim U(-1,1)$ iid. $1 \le i \le N\}$.

② Identify the number N_1 of points inside the circle.

③ $\hat{\pi} = 4 \cdot \frac{N_1}{N}$

RK: π is deterministic value. But its approximation $\hat{\pi}$ is random.

Def: A random estimator (approximation) $\hat{\alpha}$ to a deterministic value α is called MC (Monte carlo)

① $\text{Bias} = \mathbb{E}[\hat{\alpha}] - \alpha$

② $MSE(\hat{\alpha}) = \mathbb{E}[(\hat{\alpha}-\alpha)^2] = |\text{Bias}|^2 + Var(\hat{\alpha})$

proof: $MSE(\hat{\alpha}) = \mathbb{E}[(\hat{\alpha}-\alpha)^2] = \mathbb{E}(\beta^2)$, value $\beta = \hat{\alpha} - \alpha$

$= Var(\beta) + (\mathbb{E}(\beta))^2 = Var(\hat{\alpha}-\alpha) + [\mathbb{E}(\hat{\alpha}-\alpha)]^2 = Var(\hat{\alpha}) + (\mathbb{E}(\hat{\alpha}) - \mathbb{E}(\alpha))^2$

$= Var(\hat{\alpha}) + (\mathbb{E}(\hat{\alpha}) - \alpha)^2 = Var(\hat{\alpha}) + |\text{Bias}|^2$

draft: $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

$Var(X) = Var(X + C)$ C is a constant.

$\mathbb{E}(X+C) = \mathbb{E}(X) + C$

The smaller MSE, the better MC.

MC is called unbiased if $\text{Bias} = 0$. i.e. $\mathbb{E}(\hat{\alpha}) = \alpha$

ex:

In approximation π by $\hat{\pi}$ ① Find 1^{st}, 2^{nd} moments of $\hat{\pi}$ ② Find Bias & MSE of $\hat{\pi}$

(Def m^{th} moment is $\mathbb{E}(X^m)$)

soln: ① $\mathbb{E}(\hat{\pi}) = \mathbb{E}(4 \cdot \frac{N_1}{N}) = \frac{4}{N}\mathbb{E}(N_1) = \frac{4}{N}\mathbb{E}\left(\sum_{i=1}^{N} I_{\{|z_i|<1\}}\right) = \frac{4}{N}\sum_{i=1}^{N}\mathbb{E}(I_{\{|z_i|<1\}}) = \frac{4}{N}\left(\mathbb{E}[I_{\{|z_1|<1\}}] + \mathbb{E}[I_{\{|z_2|<1\}}] + \cdots + \mathbb{E}[I_{\{|z_N|<1\}}]\right)$

$= \frac{4}{N} \cdot N \cdot \mathbb{E}(I_{\{|z_1|<1\}}) = 4 \cdot P(|z_1|<1) = 4 \cdot \frac{\text{Circle Area}}{\text{Rectangular Area}} = 4 \cdot \frac{\pi \cdot 1^2}{2\times 2} = \pi$

Recall: $I_A = \begin{cases} 1, & \text{if } A \text{ occur} \\ 0, & \text{if } A \text{ not occur} \end{cases}$ $\mathbb{E}(X) + \mathbb{E}(Y) = \mathbb{E}(X+Y)$ $Var(X \cdot C) = C^2 \cdot Var(X)$

$\mathbb{E}(I_A) = \mathbb{P}(A)$ $Var(X+Y) = Var(X) + Var(Y)$ if $X \perp Y$

$\mathbb{E}(\hat{\pi}^2) = [\mathbb{E}(\hat{\pi})]^2 + Var(\hat{\pi}) = \pi^2 + Var(\hat{\pi})$

$Var(\hat{\pi}) = Var(4 \cdot \frac{N_1}{N}) = \frac{16}{N^2}Var(N_1) = \frac{16}{N^2}Var\left(\sum_{i=1}^{N} I_{\{|z_i|<1\}}\right) = \frac{16}{N^2}\sum_{i=1}^{N} Var[I_{\{|z_i|<1\}}] = \frac{16}{N^2}\cdot N \cdot Var(I_{\{|z_1|<1\}}) = \frac{16}{N}\cdot \mathbb{P}\{|z_1|<1\}(1 - P\{|z_1|<1\})$

$= \frac{16}{N} \cdot \frac{\pi}{4} \cdot (1 - \frac{\pi}{4}) = \frac{\pi(4-\pi)}{N}$

?

$\Rightarrow$

$\mathbb{E}(\hat{\pi}^2) = \pi^2 + \frac{\pi(4-\pi)}{N} = \frac{(N\pi - \pi + 4)\pi}{N}$

② $\text{Bias} = \mathbb{E}(\hat{\pi}) - \pi = 0$

$MSE(\hat{\pi}) = |\text{Bias}|^2 + Var(\hat{\pi}) = 0 + \frac{\pi(4-\pi)}{N} = \frac{\pi(4-\pi)}{N}$

Def: Given a series of estimator of α, say $\{\alpha_n, n\in\mathbb{N}\}$, we say (α_n) is consistent if $\boxed{p-\lim_{n\to\infty}\alpha_n = \alpha}$

prop: If $MSE(\hat{\alpha}_n)\to 0$, then $\hat{\alpha}_n$ is consistent.

ex: If N is fixed, then $Bias(\hat{\pi})=0$

prop: Suppose n ppl. do the some unbiased approximation, their out comes are $\{\alpha^{(i)} : 1\le i\le n\}$
$\sum_{i=1}^{n}\hat{\alpha}^{(i)}\to\alpha$ always surely.

Pf: Law of large number (LLN)

ex: Let $\hat{\pi}_N$ be estimator with N total points. Then $\{\hat{\pi}_N : N\in\mathbb{N}\}$ is a consistent estimator of π, because
$$MSE(\hat{\pi}_N) = \frac{\pi(4-\pi)}{N}\xrightarrow[\text{as } N\to\infty]{} 0$$

Def: ① "$p-\lim_{n\to\infty}\alpha_n=\alpha$" or "$\alpha_n\xrightarrow[n\to\infty]{}\alpha$" in prob" if $\lim_{n\to\infty}\mathbb{P}(|\alpha_n-\alpha|>\varepsilon)=0$, $\forall\varepsilon>0$.
② "$\lim_{n\to\infty}\alpha_n=\alpha$" or "$\alpha_n\xrightarrow[n\to\infty]{}\alpha$" a.s if $\mathbb{P}(\lim_{n\to\infty}\alpha_n=\alpha)=1$
③ "$L_2-\lim_{n\to\infty}\alpha_n=\alpha$" or "$\alpha_n\xrightarrow[n\to\infty]{}\alpha$" in L^2" if $\lim_{n\to\infty}\mathbb{E}|\alpha_n-\alpha|^2=0$

ex: Justify the statement "$P-\lim_{n\to\infty}\alpha_n=\alpha$" implies "$L_2-\lim_{n\to\infty}\alpha_n=\alpha$"

Ans: No. because following counter-example:

Let $\mathbb{P}=U([0,1])\longrightarrow$ probability

$\Omega=[0,1]\longrightarrow$ sample space

$\alpha_n : [0,1]\to\mathbb{R}$ s.t.
$$\alpha_n(\omega)=\begin{cases} n, & \text{if } 0<\omega<\frac{1}{n^2}\\ 0, & \text{otherwise}\end{cases} \Rightarrow \alpha_2(\omega)=\begin{cases}\alpha, & \text{if } 0<\omega<\frac{1}{4}\\ 0, & \text{otherwise.}\end{cases}$$

$\alpha=[0,1]\longrightarrow\mathbb{R}$. s.t.

$\alpha(\omega)=0$

$\mathbb{R}^2=\mathbb{R}\times\mathbb{R}=\{(x,y): x\in\mathbb{R}, y\in\mathbb{R}\}$. $z\in\mathbb{R}^2$ means a point on $\mathbb{R}^2$-plane $|z|=\sqrt{|z_x|^2+|z_y|^2}$
$$I_{\{|z|<1\}}=\begin{cases}1 & \text{if } |z|<1\\ 0 & \text{if } |z|\ge 1\end{cases}$$

if $x=\begin{cases}a_1 & P_1\\ a_2 & P_2\\ \vdots & \\ a_n & P_n\end{cases}$ Then $\mathbb{E}(X)=\sum_{i=1}^{n}a_iP_i$ if $X=I_A=\begin{cases}1 & \mathbb{P}(A)\\ 0 & 1-\mathbb{P}(A)\end{cases}$

$Var(X)=E(X^2)-(E(X))^2=\sum_{i=1}^{n}a_i^2P_i-\left(\sum_{i=1}^{n}a_iP_i\right)^2$

$\mathbb{E}(I_A)=\mathbb{E}(X)=1\cdot\mathbb{P}(A)+0\cdot[1-\mathbb{P}(A)]=\mathbb{P}(A)$

$\mathbb{E}(I_A^2)=1^2\cdot\mathbb{P}(A)+0^2\cdot(1-\mathbb{P}(A))=\mathbb{P}(A)$

$Var(I_A)=\mathbb{E}(I_A^2)-[\mathbb{E}(I_A)]^2=\mathbb{P}(A)-\mathbb{P}^2(A)=\mathbb{P}(A)\cdot[1-\mathbb{P}(A)]$

Sep 20. L03 S02

Recall prop1: If $MSE(\hat{\alpha}_n) \to 0$, then $\hat{\alpha}_n$ is consistent. (prop 1 is the consequence of the following fact.)

prop 2: $L_2-\lim_{n\to\infty} \alpha_n = \alpha$ implies $p-\lim_n \alpha_n = \alpha$

pf: $\forall \varepsilon > 0$, $\mathbb{P}(|\alpha_n - \alpha| > \varepsilon) = \mathbb{P}\left(\frac{\alpha_n-\alpha}{\varepsilon} > 1\right) \leq \mathbb{E}\frac{(\alpha_n-\alpha)^2}{\varepsilon^2}$ $(\longrightarrow 0)$ by Chebyshev inequality, b/c $L_2-\lim_{n\to\infty} \alpha_n = \alpha$.

RK Chebyshev inequality is used above i.e. $\mathbb{P}(|X|>1) \leq \mathbb{E}(|X|)$

pf RHS $= \int_{\mathbb{R}} |x| p(x) dx$, when $p \sim X$ $\geq \int_{-\infty}^{-1} p(x)\,dx + \int_1^{\infty} p(x)\,dx$ = LHS

prop If F is a strictly increasing CDF, $U \sim U(0,1)$. uniform on r.v on $[0,1]$

then $X = F^{-1}(U)$ has its CDF F.

pf $\mathbb{P}(X<a) \stackrel{?}{=} F(a)$

$= \mathbb{P}(F^{-1}(U) < a)$

$= \mathbb{P}(U < F(a))$

$= F(a)$

$U \sim U(0,1)$

$\mathbb{P}(U<\frac{1}{2}) = \frac{1}{2}$

$\mathbb{P}(U<\tilde{a}) = \tilde{a}$ if $\tilde{a} \in (0,1)$

A company sells option.

if ICC crashes within 10 yrs, Buyer will obtain \$100, otherwise \$0

Suppose $\mathbb{P}$(ICC crashes within 10 yr) = 1%, then what is fair price of the option?

L04 S01 Black Sholes formula

BS model assumes the distribution of stock as lognormal. In particular, it writes:

$$\ln \frac{S(T)}{S(0)} \sim N\left((r-\tfrac{1}{2}\sigma^2)T,\ \sigma^2 T\right)$$

The call & put price with maturity T & K will be known as C_0 & P_0 given as below:

$$\begin{cases} C_0 = \mathbb{E}\left[e^{-rT}\cdot(S(T)-K)^+\right] = S_0\,\phi(d_1) - Ke^{-rT}\phi(d_2) \\ P_0 = \mathbb{E}\left[e^{-rT}\cdot(S(T)-K)^-\right] = Ke^{-rT}\phi(-d_2) - S_0\,\phi(-d_1) \end{cases}$$

$$\begin{cases} d_1 = \dfrac{(r+\frac{1}{2}\sigma^2)T - \ln\frac{K}{S_0}}{\sigma\sqrt{T}} \\ d_2 = \dfrac{(r-\frac{1}{2}\sigma^2)T - \ln\frac{K}{S_0}}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{cases}$$

Verify put-call parity: $C_0 - P_0 = S(0) - e^{-rT}K$

$\Rightarrow$ B-S-valuation (rewrite code)

L04 S02 Calibration of BSM Volatility

In simple terms, problem of calibration is to find parameters for the proposed model such that observed market quotes of liquidly traded plain vanilla options are replicated as closely as possible. The general approach is that, one defines an error function that is to be minimized.

In this below, we will use Sep-30-2014 market quote on call options underlying Euro Stock 50 - BSM · model - RMSE as an error function.

Calibration by minimizing RMSE

Suppose there are N options available in the market quote with various strike and maturities. We denote the market quote of N options by $\{C_n^*: n=1,2,\cdots,N\}$

Correspondingly, there are N BSM theoretical prices available as a function of the volatility. We denote it by: $\{C_n(\sigma): n=1,2,\cdots,N\}$

The error function RMSE is defined by: $$RMSE(\sigma) = \sqrt{\frac{1}{N}\sum_{n=1}^{N} |C_n^* - C_n(\sigma)|^2}$$

Our goal is to find out the calibrated volatility $\hat{\sigma}$, which is given by: $\hat{\sigma} = \arg\min_{\sigma} RMSE(\sigma)$

Sep 27

Zero Coupon Bond $(T) \to$ maturity $\quad$ payoff $(T) = 1$

suppose r is rate, today's price is $p(0,T)$ (0 for today, T for maturity)

then $p(0,T) \cdot e^{rT} = 1$

$$p(0,T) = e^{-rT}$$

Stock $\sim$ BSM(σ) $\quad$ suppose S_t is price at t

W_t is Brownian Motion

$$S_t = S_0 \cdot \exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma W_t\} \longrightarrow \text{EMM } \mathbb{Q}.$$

$$W_t = (W_{t_1} - W_{t_0}) + (W_{t_2} - W_{t_1}) + \cdots + (W_{t_n} - W_{t_{n-1}}) = \Delta W_{t_1} + \Delta W_{t_2} + \cdots + \Delta W_{t_n}$$

proposition ① $\{\Delta W_{t_1}, \Delta W_{t_2}, \cdots, \Delta W_{t_n}\}$ are independent.

② $\Delta W_{t_n} = W_{t_n} - W_{t_{n-1}} \sim \mathcal{N}(0, \Delta t_n)$ $\quad (\Delta t_n = t_n - t_{n-1})$

Asian Option (Geometric)

Payoff for GAC (T, K, n) $\quad \Pi_T^c = (A_T - K)^+$ $\quad$ where $\Delta T = \frac{T}{n}$

(GAC: Geom Asian call)

$$A_T = (S_{t_1} \cdot S_{t_2} \cdots S_{t_n})^{\frac{1}{n}}$$

payoff for GAP (T, k, n) $\quad \Pi_T^p = (A_T - k)^-$

Goal what's the price of GAC & GAP

Ans $\Pi_0^c = \mathbb{E}^{\mathbb{Q}}[e^{-rT}\Pi_T^c]$ $\qquad \Pi_0^p = \mathbb{E}^{\mathbb{Q}}[e^{-rT}\Pi_T^p]$

$0 \xrightarrow{\Delta t} t_1 \xrightarrow{\Delta t} t_2 \quad \cdots \quad t_{n-1} \xrightarrow{\Delta t} t_n = T$

$$S(t_i) = S_0 \cdot \exp\{i\mu\Delta t + \sigma\sqrt{\Delta t}\sum_{j=1}^{i} z_j\}$$

$$\prod_{i=1}^{n} S(t_i) = S_0^n \cdot \exp\{\sum_{i=1}^{n}(i\mu\Delta t + \sigma\sqrt{\Delta t}\sum_{j=1}^{i} z_j)\} = S_0^n \cdot \exp\{\frac{\mu T(n+1)}{2} + \sigma\sqrt{\Delta t}\sum_{i=1}^{n}\sum_{j=1}^{i} z_j\}$$

$$= S_0^n \cdot \exp\{\frac{\mu T(n+1)}{2} + \sigma\sqrt{\Delta t} \cdot \underbrace{\sum_{j=1}^{n}(n+1-i)z_j}_{a}\}$$

a is sum of normal random variables, thus is another normal random variable. its mean is.

$$\mathbb{E}(a) = \mathbb{E}\left(\sum_{j=1}^{n}(n+1-i)z_i\right) = \sum_{j=1}^{n}\mathbb{E}(n+1-i)z_i = 0$$

$$\mathrm{Var}(a) = \mathrm{Var}(\cdots) = \sum_{j=1}^{n}\mathrm{Var}(n+1-i)z_i = \sum_{j=1}^{n}(n+1-i)^2 \cdot \underbrace{\mathrm{Var}(z_i)}_{=1}$$

$$= n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$\therefore a \sim \mathcal{N}(0, \frac{n(n+1)(2n+1)}{6}) = \sqrt{\frac{n(n+1)(2n+1)}{6}}\,\hat{z}$, where $\hat{z} \sim \mathcal{N}(0,1)$

$$A_T = (S_{t_1} \cdot S_{t_2} \cdots S_{t_n})^{\frac{1}{n}} = \left[\prod_{i=1}^{n}(S_{t_i})\right]^{\frac{1}{n}} = S_0 \cdot \exp\{\frac{1}{n} \cdot (\frac{\mu T(n+1)}{2} + \sigma\sqrt{\Delta t} \cdot \sum_{i=1}^{n}(n+1-i)\hat{z})\} \quad (n\Delta t = T)$$

$$= S_0 \cdot \exp\{\frac{\mu(n+1)}{2n}T + \frac{\sigma\sqrt{T}}{n} \cdot \sqrt{\frac{n(n+1)(2n+1)}{6}}\,\hat{z}\}$$

Recall BSM-Call $(S_0, k, 0, T, r, \sigma)$

If $S_t = S_0 \cdot \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z\}$. then $\mathbb{E}[e^{-rT}(S_T - k)^+] = \text{BSM-Call}(S_0, k, 0, T, r, \sigma)$

Now I want $\Pi_0^C = \mathbb{E}\left[e^{-rT}(A_T - K)^+\right]$ where $\begin{cases} A_T = S_0 \cdot \exp\{(\hat{r} - \frac{1}{2}\hat{\sigma}^2)T + \hat{\sigma}\sqrt{T}\hat{Z}\} \\ \hat{\sigma} = \frac{\sigma}{n} \cdot \sqrt{\frac{(n+1)(2n+1)}{6}} \\ \hat{r} - \frac{1}{2}\hat{\sigma}^2 = \frac{(n+1)\mu}{2n} \end{cases}$

$$\therefore \quad \Pi_0^C = e^{\hat{r}T - rT}\,\mathbb{E}\left[e^{-\hat{r}T}(A_T - K)^+\right] = e^{(\hat{r}-r)T} \cdot \text{BSM-Call}(S_0, K, 0, T, \hat{r}, \hat{\sigma})$$

L05 Approximation to BSM Option Valuation by CRR

We denote CRR Model by $\mathcal{M}^{CRR} = \{(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), T, (S, B)\}$

the time horizon is $t \in \{0, \Delta t, \dots, T\} := \mathbb{T}$

Bond price follows binomial tree: $B_t = e^{-r(T-t)}$

Stock price follows $S_{t+\Delta t} = S_t \cdot m_t$

where $m_t \in \{u, d\}$ is a random variable with 2 possible values with:

$u = e^{\sigma\sqrt{\Delta t}}$ $\quad d = e^{-\sigma\sqrt{\Delta t}}$

[A1] $\sigma > r$, under [A1], there exists unique risk-neutral probability Q given by:

$$q = Q(m_t = u) = \frac{e^{r\Delta t} - d}{u - d}$$

$$1 - q = Q(m_t = d) = \frac{u - e^{r\Delta t}}{u - d}$$

[Q] if [A1] is violated, then does it still make the model arbitrage free?

CRR call value can be evaluted backwardly from $C_0 = e^{-rT} \mathbb{E}_0^Q (C_T)$

L06

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i}\; Z_{i+1}$$

$$\Rightarrow \hat{W}(t_{i+1}) = \hat{W}(t_i) + \frac{1}{\sqrt{n}} Z_{i+1}, \quad \text{for } i = 0, 1, \dots, n-1$$

Generating GBM path.

GBM is given by

$$X(t) = x_0 \cdot \exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma W(t)\}$$

$$\Rightarrow \hat{X}(t) = x_0 \cdot \exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma \hat{W}(t)\}$$

Application to Arithmetic asian option price.

call option price:

$$C(T) = (A(T) - K)^+ \qquad A(T) = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$$

$$\Rightarrow C_0 = \mathbb{E}\left[e^{-rT}(A(T) - K)^+\right]$$

Oct 04 § Binomial option pricing

Recall.

BSM. $S_0 = \text{Init}$, $\text{vol} = \sigma$, $\text{rate} = r$

Then it implies stock under EMM $\mathbb{Q}$ $\frac{dS_t}{S_t} = r\,dt + \sigma\, dW_t$

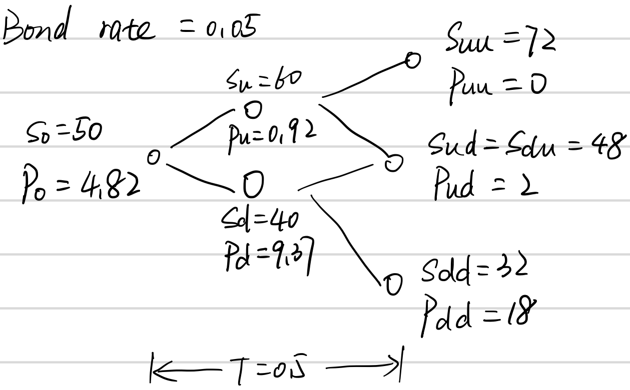
or $S_t = S_0 \cdot \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W_t\}$.

We consider Binomial Tree model, denoted by

★ $\text{BinTree}(S_0 = 50, N = 2, T = 0.5, u = 1.2, d = 0.8, r = 0.05, K = 50)$

① Bond rate $= 0.05$

②



☆ option types:

① Eu Call $(T = 0.5, K = 50)$ $\quad \text{Payoff}|_T = (S_T - K)^+$

② Eu Put (T, K) $\quad \text{Payoff}|_T = (S_T - K)^-$

③ Am Call (T, K) $\quad \text{Payoff}|_t = (S_t - K)^+, \quad t \le T$

④ Am Put (T, K) $\quad \text{Payoff}|_t = (S_t - K)^-, \quad t \le T$

Q. $S_t \xrightarrow{q} uS_t$, $S_t \xrightarrow{1-q} dS_t$ what's the EMM?

$S_t \cdot e^{r\Delta t} = q \cdot uS_t + (1-q)\, dS_t$

$\Rightarrow \boxed{q = \dfrac{e^{r\Delta t} - d}{u-d}}$

Recall Martingale

$\{X_t : t \in T\}$ is a mtgl process, if $\mathbb{E}[X_{t+h} | X_t] = X_t$

Recall mtgl (discrete)

$\{X_t : t \in \mathbb{N}\}$ is a mtgl process if. $\mathbb{E}[X_{t+1} | X_t] = X_t$

ex1

$X_0 = 100$, $q = \frac{1}{2}$: $1000 = X_1(u)$; $1-q = \frac{1}{2}$: $99 = X_1(d)$

① If $Q = (q, 1-q) = (\frac{1}{2}, \frac{1}{2})$, then its not mtgl. b/c $\mathbb{E}[X_1] = 1000 \times \frac{1}{2} + 99 \times \frac{1}{2} = \frac{1099}{2} \neq X_0$

② Is there mtgl measure? (i.e. Find Q s.t. $\mathbb{E}^Q(X_1) = X_0$)

$1000\, q + 99(1-q) = 100 \Rightarrow \underline{q = \frac{1}{901}}$

ex2

$X_0 = 100$, q: $1000 = X_1(u)$; $1-q$: $101 = X_1(d)$

Find mtgl measure?

solu: $\mathbb{E}(X_1) = 1000q + 101(1-q) = 100 \Rightarrow$ $\underline{q \text{ does not exist.}}$

ex3

$X_0 = 100$, q_1: $101 = X_1(u)$; q_2: $100 = X_1(e)$; $1-q_1-q_2$: $99 = X_1(d)$

Find mtgl measure?

solu: $101 q_1 + 100 q_2 + 99(1-q_1-q_2) = 100 \Rightarrow 2q_1 + q_2 = 1$

$Q = (0, 1, 0)$ or $(\frac{1}{2}, 0, \frac{1}{2})$ or (q_1, q_2, q_1), $q_1 \in [0, \frac{1}{2}]$

Def

$T = \{0, \Delta t, 2\Delta t, \cdots, (n-1)\Delta t, n\Delta t = T\} = \{n\Delta t : n = 0, 1, \cdots, n\}$

Stock price: $S_{n\Delta t}$ Bond rate: r

Q is EMM if $\mathbb{E}[S_{(n+1)\Delta t} | S_{n\Delta t}] = e^{r\Delta t} S_{n\Delta t}$ or $S_{n\Delta t} = \mathbb{E}[\underbrace{e^{-r\Delta t} S_{(n+1)\Delta t}}_{\text{discounted Stock Price}} | S_{n\Delta t}]$

RK.

Discounted stock price is a mtgl

$S_t \xrightarrow{q} uS_t$, $S_t \xrightarrow{1-q} dS_t$ mtgl: $\boxed{q = \dfrac{e^{r\Delta t} - d}{u-d} \qquad \Delta t = \dfrac{T}{n}}$

constructing BinEuOpt

set-up-parameters: $M = N + 1$ # of terminal nodes.

$u = 1 + p_u$, $d = 1 - p_d$ } # up/down factor

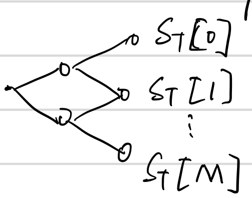
$q_u = (e^{r\Delta t} - d)/(u-d)$

$q_d = 1 - q_u$

$\Delta t = \frac{T}{N}$ # Time length of one period

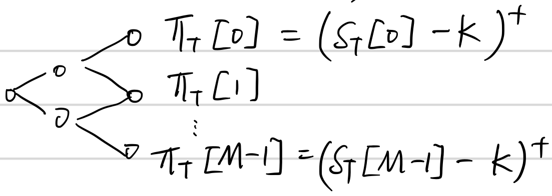
$df = e^{-r\Delta t}$ # discount factor

Init - stock - tree: compute: $\{S_T[i],\ i = 0, 1, \dots, M-1\}$



Init - payoff - tree:

ex: EuCall $(T = 0.5,\ K = 50)$



Traverse - tree (Backward)

$$\Pi_N[\ ] \to \Pi_{N-1}[\ ] \to \Pi_{N-2}[\ ] \cdots \to \Pi_0[0]$$

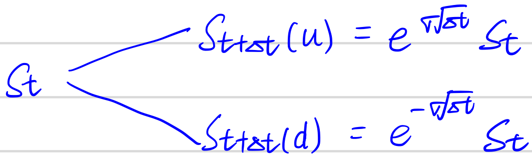
$$\Pi_{i-1}[j] = e^{-r\Delta t}\left(q\,\Pi_i[j] + (1-q)\,\Pi_i[j+1]\right)$$

※. Given BSM(σ), rate $= r$, we have BSM formula for pricing EuCall(T, K), EuPut(T, K)

Arithmetic Asian option
Barrier option
$\}$ has no formula

☆ CRR$(S_0 = 50,\ N = 2,\ T = 0.5,\ \sigma = 0.3,\ r = 0.05)$ = Bin Tree$(S_0 = 50,\ N = 2,\ T = 0.5,\ u = e^{\sigma\sqrt{\Delta t}},\ d = \frac{1}{u},\ r = 0.05)$

$\xrightarrow{N \to \infty}$ BSM$(S_0 = 50,\ T = 0.5,\ \sigma = 0.3,\ r = 0.05)$ where $\Delta t = T/N$



ex: what is price of Put $(T = 0.5,\ K = 50)$

Q1: for BSM$(S_0 = 50,\ r = 0.05,\ \sigma = 0.3)$ = ?

Q2: for CRR$(S_0 = 50,\ N = 200,\ T = 0.5,\ \sigma = 0.3,\ r = 0.05)$ = ?