Applications of Opinion Dynamics

A Credit Seminar Report (EE694)

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by

Shubham Daheriya
(23M1089)

Under the Supervision of

Prof. Dwaipayan Mukherjee

Department of Electrical Engineering

INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

Mumbai - 400076, India

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Shubham Daheriya
(23M1089)

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Chapter 1

Introduction

1.1 Overview

In recent years, the intersection of social network analysis (SNA) and control theory has witnessed a remarkable evolution, propelled by advancements in mathematical modeling, complex networks theory, and the tools for analyzing vast datasets. The 20th century marked a pivotal shift in social and behavioral sciences, transitioning from individualistic approaches to dynamic problems of changing group life.

This is essentially a literature survey of the research paper named "A tutorial on modeling and analysis of dynamic social networks. Part I" by Proskurnikov and Tempo Proskurnikov and Tempo (2017). It primarily deals with models of opinion dynamics. First, we will discuss some graph theory concepts which will help us understand these opinion dynamics models. Then in later sections we will understand some agent-based models with real-valued scalar and vector opinions, whereas other models are either skipped or mentioned briefly. All the models, considered in this paper, deal with an idealistic closed community, which is neither left by the agents nor can acquire new members. Hence the size of the group, denoted by $n \geq 2$, remains unchanged.

1.2 Problem Statement

Despite the well-developed theory of SNA, the realm of social systems remained relatively unexplored by modern control theory. The gap persisted due to a lack of mathematical models capable of describing the dynamics of social groups and tools for the quantitative analysis of large-scale social systems. The spontaneous order observed in many natural and engineered networks contrasted with the irregular and sophisticated dynamics characterizing social communities, where opinions often fail to reach consensus, leading to persistent disagreement and clustering.

Recent years have witnessed a concerted effort to bridge the gap between SNA and dynamical systems, giving rise to Dynamical Social Networks Analysis (DSNA) and temporal or evolutionary networks. This convergence has opened avenues for the application of multi-agent and networked control to social groups, paving the way for the emerging science of dynamic social networks.

Chapter 2

Prerequisite

2.1 Approaches to opinion dynamics modeling

Individuals' opinions represent their cognitive orientations towards objects such as specific issues, events, or other individuals—manifested as attitudes or subjective certainties of belief. Mathematically, opinions are scalar or vector quantities associated with social actors.

Models of social dynamics can be classified into two major classes:

Macroscopic Models: Macroscopic models of opinion dynamics are akin to models of continuum mechanics, based on Euler's formalism. This approach to opinion modeling is also known as Eulerian or statistical. Macroscopic models describe how the distribution of opinions (e.g., vote preferences in an election or referendum) evolves over time. The statistical approach is commonly employed in "sociodynamics" and evolutionary game theory, where the "opinions" of players denote their strategies. Some macroscopic models date back to the 1930s–40s.

Microscopic Models: Microscopic, or agent-based, models of opinion formation describe how opinions of individual social actors, hereafter called agents, evolve. There is an analogy between the microscopic approach, also called aggregative, and the Lagrangian formalism in mechanics. Unlike statistical models, which are suitable for very large groups (mathematically, as the number of agents tends to infinity), agent-based models can describe both small-size and large-scale communities.

2.2 Basic notions from graph theory

1) **Directed Graph:** We define a graph by $G(V, E)$, where V is the set of nodes of graph G and E is the set of edges of graph G. A graph is considered directed if and only if $(v_1, v_2) \neq (v_2, v_1)$ where $v_1, v_2 \in V$ and $(v_1, v_2), (v_2, v_1) \in E$.

2) **Undirected Graph:** A graph $G(V, E)$ is termed undirected if and only if $(v_1, v_2) = (v_2, v_1)$, where $v_1, v_2 \in V$ and $(v_1, v_2), (v_2, v_1) \in E$.

3) **Walk of Length L:** For any graph given by $G = (V, E)$, a walk of length L is defined as the set of nodes $v_1, v_2, v_3, \ldots, v_L$ such that there exists an edge between v_i and v_{i+1} for $1 \leq i < L$.

4) **Degree Matrix:** For an undirected graph, the degree matrix is defined as $D = [D_{ij}]$, where D_{ij} is given by $D_{ij} = \deg(v_i)$ if $i = j$ and 0 otherwise. Here, $\deg(v_i)$ is the degree of node v_i.

5) **Adjacency Matrix:** For an undirected graph, the adjacency matrix is defined as $A = [A_{ij}]$, where $A_{ij} = 1$ if v_i and v_j are connected, and 0 otherwise.

6) **Laplacian Matrix:** The Laplacian matrix for an undirected graph $G(V, E)$ is given by $L = D - A$.

7) **Algebraic Connectivity:** Algebraic connectivity measures the connectivity of the graph $G(V, E)$. Let the Laplacian matrix have eigenvalues such that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_{\max}$. The second eigenvalue λ_2 is called algebraic connectivity. If $\lambda_2 = 0$, the graph is connected; otherwise, it is disconnected.

8) **Fiedler Vector:** The eigenvector corresponding to algebraic connectivity is called a Fiedler vector.

9) **Perron-Frobenius Theorem:** For $A \in \mathbb{R}^{n \times n}$, where A is the adjacency matrix for strongly connected graphs, there exists a unique eigenvalue $\lambda_{\max}$ such that the eigenvector U corresponding to $\lambda_{\max}$ has all positive entries.

10) **Geodesics:** Any path from vertex v_i to v_j is called a geodesic if it is the minimum length between v_i and v_j, and this path length is termed the geodesic length.

11) **Stochastic Matrix:** A stochastic matrix is formed by transition probabilities for each node. When forming a stochastic matrix, we assume a self-loop exists and then form the matrix corresponding to the group.

Chapter 3

Opinion Dynamics Models

3.1 French-DeGroot Model

One of the foundational agent-based models of opinion formation was proposed by the social psychologist French in his influential paperFrench (1956), integrating social network analysis (SNA) and systems theory. French's model, later generalized by DeGroot as "iterative opinion pooling,"Degroot (1974) outlines a straightforward procedure enabling rational agents to converge on a consensus.

Originally, French aimed to create a mathematical model for social power, indicating an individual's ability to control group behavior and revealing a deep connection between opinion formation and centrality measures.

3.1.1 The French-DeGroot Model of Opinion Formation

The French-DeGroot model describes a discrete-time process of opinion formation in a group of n agents, denoted by $x_1, \ldots, x_n$. Initially considering scalar opinions $x_i \in \mathbb{R}$, the model's key parameter is a stochastic $n \times n$ matrix of influence weights $W = (w_{ij})$, where $w_{ij} \geq 0$. The influence weight $w_{ij} > 0$ indicates that agent j influences agent i at each step of the opinion iteration.

Mathematically, the vector of opinions $x(k) = (x_1(k), \ldots, x_n(k))$ follows the equation

$$x(k+1) = Wx(k), \quad k = 0, 1, \ldots.$$

This is equivalent to the system of equations

$$x_i(k+1) = \sum_{j=1}^{n} w_{ij} x_j(k), \quad \forall i, k = 0, 1, \ldots.$$

The self-influence weight $w_{ii} \geq 0$ indicates an agent's openness to assimilating others' opinions. For example, $w_{ii} = 0$ implies open-mindedness, while $w_{ii} = 1$ and $w_{ij} = 0$ for $j \neq i$ imply stubbornness.

More generally, if agent opinions are vectors of dimension m, represented as rows $x_i = (x_{i1}, \dots, x_{im})$, forming an opinion matrix $X = (x_{ij}) \in \mathbb{R}^{n \times m}$, the equation becomes

$$X(k+1) = WX(k), \quad k = 0, 1, \dots.$$

This generalization is referred to as the French–DeGroot model.

3.1.2 History of the French-DeGroot Model

French introduced a special case of the model in his seminal paper, associating it with a graph G where nodes correspond to agents, each with a self-loop. The French model updates an agent's opinion to the mean value of opinions displayed to it, as determined by the weighted graph.

DeGroot's general model, proposed in 1974, originated in applied statistics and was suggested as a heuristic procedure to find a "consensus of subjective probabilities." DeGroot's iterative opinion pooling replaced complex convex optimization with a simple weighted averaging algorithm, allowing decentralized opinion updates.

3.1.3 Algebraic Convergence Criteria

The convergence and consensus of the French-DeGroot model depend on algebraic criteria. A key result states that the model is convergent (regular) if and only if $\lambda = 1$ is the only eigenvalue of W on the unit circle, and it reaches consensus (fully regular) if this eigenvalue is simple.

For large-scale networks, graph-theoretic conditions are more practical. The model is convergent if all closed strong components are aperiodic, and it reaches consensus if the graph is quasi-strongly connected with a single aperiodic closed strong component.

3.1.4 Dual Markov Chain and Social Power

Viewing W as transition probabilities of a Markov chain, the model's convergence implies the chain's convergence to a stationary distribution. This distribution represents a consensus where the Markov chain forgets its history, and all essential classes (closed strong components) are aperiodic.

The final opinion in the French-DeGroot model is determined by the stubborn agents' opinions, indicating their social power. The centrality measure, akin to eigenvector centrality, identifies influential nodes in the network.

3.1.5 Stubborn Agents in the French-DeGroot Model

The presence of stubborn agents, whose opinions remain unchanged, can lead to non-consensus situations. A stubborn agent corresponds to a source node in the graph $G[W]$, and consensus depends on the configuration of source nodes. The model converges if all source nodes are connected by walks to all other nodes in $G[W]$.

3.1.6 Example

3.1.6.1 Example 1: Influence in a Group

Consider the French model with $n = 3$ agents **?**, corresponding to the graph in Fig. 3.1. One can expect that the "central" node 2 corresponds to the most influential agent in the group. This is confirmed by a straightforward computation: solving the system of equations $p_\infty = p_\infty W$ and $p_\infty^1 = 1$, one obtains the vector of social powers $p_\infty = \left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$.

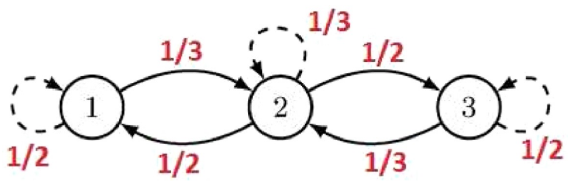

FIGURE 3.1: Graph corresponding to Example 1.

3.1.6.2 Example 2: Weighted French-DeGroot Model

Consider the French-DeGroot model, corresponding to the weighted graph in Fig. 3.2.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

It can be shown that the steady opinion vector of this model is $x(\infty) = (x_1(0), \frac{x_1(0)}{2} + \frac{x_3(0)}{2}, x_3(0))$.

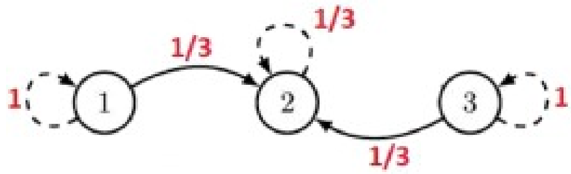

FIGURE 3.2: Weighted graph corresponding to Example 2.

3.2 Abelson's Model

In his influential work(Abelson (1964)), Abelson proposed a continuous-time counterpart of the French-DeGroot model. Besides this model and its nonlinear extensions, he formulated a key problem in opinion formation modeling, referred to as the community cleavage problem or Abelson's diversity puzzle.

3.2.1 Abelson's Models of Opinion Dynamics

To introduce Abelson's model, we first consider an alternative interpretation of the French-DeGroot model. Recalling that $1 - w_{ii} = \sum_{j \neq i} w_{ij}$, one has

$$\frac{dx_i(k+1)}{dt} - \frac{dx_i(k)}{dt} = \sum_{j \neq i} w_{ij}[x_j(k) - x_i(k)] \quad \forall i. \tag{3.1}$$

The experiments with dyadic interactions ($n = 2$) show that "the attitude positions of two discussants... move toward each other(Abelson (1967))". Equation (3.1) stipulates that this argument holds for simultaneous interactions of multiple agents: adjusting its opinion $x_i(k)$ by $\frac{dx_i(k)}{dt}$, agent i shifts it towards $x_j(k)$ as

$$x_i' = x_i + \frac{dx_i}{dt} \Rightarrow |x_j - x_i'| = (1 - w_{ij})|x_j - x_i|.$$

The increment in the ith agent's opinion $\frac{dx_i(k)}{dt}$ is the "resultant" of these simultaneous adjustments. Supposing that the time elapsed between two steps of the opinion iteration is very small, the model (3.1) can be replaced by the continuous-time dynamics:

$$\frac{dx_i(t)}{dt} = \sum_{j \neq i} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \ldots, n.$$

Here $A = (a_{ij})$ is a non-negative (but not necessarily stochastic) matrix of infinitesimal influence weights (or "contact rates"). The infinitesimal shift of the ith opinion

$dx_i(t) = \frac{dx_i(t)}{dt}dt$ is the superposition of the infinitesimal shifts $a_{ij}(x_j(t) - x_i(t))dt$ of agent i towards the influencers. A more general nonlinear mechanism of opinion evolution is

$$\frac{dx_i(t)}{dt} = \sum_{j \neq i} a_{ij} g(x_i, x_j)(x_j(t) - x_i(t)) \quad \forall i.$$

Here $g : \mathbb{R} \times \mathbb{R} \to (0, 1]$ is a coupling function, describing the complex mechanism of opinion assimilation.

In this section, we mainly deal with the linear Abelson model (3.2.1), whose equivalent matrix form is

$$\frac{dx(t)}{dt} = -L[A]x(t),$$

where $L[A]$ is the Laplacian matrix.

3.2.2 Convergence and Consensus Conditions

Note that Corollary 6, applied to the M-matrix $L[A]$ and $\lambda_0 = 0$, implies that all Jordan blocks, corresponding to the eigenvalue $\lambda_0 = 0$, are trivial and for any other eigenvalue λ of the Laplacian $L[A]$, one has $\mathrm{Re}(\lambda) > 0$. Thus, the model (3.2.1) is Lyapunov stable (yet not asymptotically stable) and, unlike the French-DeGroot model, is always convergent.

Corollary 3.1. *For any nonnegative matrix A, the limit $P_\infty = \lim_{t\to\infty} e^{-L[A]t}$ exists, and thus the vector of opinions in (3.2.1) converges:*

$$x(t) \underset{t\to\infty}{\longrightarrow} x_\infty = P_\infty x(0).$$

The matrix P_∞ is a projection operator onto the Laplacian's null space and is closely related to the graph's structure.

Similar to the discrete-time model (**??**), the system (3.2.1) reaches a consensus if the final opinions coincide $x_\infty^1 = \ldots = x_\infty^n$ for any initial condition $x(0)$. Obviously, consensus means that the null space is spanned by the vector 1_n, i.e., $P_\infty = \frac{1}{n} 1_n 1_n^\top$, where $1_n \in \mathbb{R}^n$ is a vector of ones. By noticing that $x = 1_n$ is an equilibrium point, one has $P 1_n = 1_n$ and thus $P_\infty 1_n = 1$. Since P commutes with $L[A]$, it can be easily shown that $P_\infty L[A] = 0$. Recalling that $L[A]$ has a nonnegative left eigenvector p such that $p^\top L[A] = 0$ due to Corollary 6 and $\dim \ker L[A] = 1$, one has $p_\infty = cp$, where $c > 0$. Combining this with Lemma 8, one obtains the following consensus criterion.

Theorem 3.2. *The linear Abelson model (3.2.1) reaches consensus if and only if the underlying graph is quasi-strongly connected (i.e., has a directed spanning tree).*

In this case, the opinions converge to the limit:

$$\lim_{t\to\infty} x^1(t) = \ldots = \lim_{t\to\infty} x^n(t) = p_\infty x(0),$$

where $p_\infty \in \mathbb{R}^n$ *is the nonnegative vector, uniquely defined by the equations* $p_\infty L[A] = 0$ *and* $p_\infty 1_n = 1$.

Remark: The vector p_∞ *may be treated as a vector of the agents' social powers or a centrality measure on the nodes of the graph.*

3.2.3 The Community Cleavage Problem

Admitting that in general, the outcome of consensus is "too strong to be realistic," Abelson formulated a fundamental problem, called the community cleavage problem or Abelson's diversity puzzle. The informal formulation, stated in Abelson, was: "Since universal ultimate agreement is an ubiquitous outcome of a very broad class of mathematical models, we are naturally led to inquire what on earth one must assume in order to generate the bimodal outcome of community cleavage studies." In other words, the reasons for social cleavage, that is, persistent disagreement among the agents' opinions (e.g., clustering), are to be identified. This requires finding mathematical models of opinion formation that are able to capture the complex behavior of real social groups, yet simple enough to be rigorously examined.

As discussed in Section 3.6, one of the reasons for opinion clustering is the presence of stubborn agents, whose opinions are invariant. In the models (3.2.1) and (3.2.1), agent i is stubborn if and only if $a_{ij} = 0$ for all j, corresponding thus to a source node of the graph.

In the next sections, we consider more general models with "partially" stubborn or prejudiced agents.

Chapter 4

Conclusions

In this literature survey, we aimed to comprehend the gap between Social Network Analysis and Control, exploring how the introduction of new mathematical models describing the dynamics of social groups has facilitated advancements in complex networks theory and multi-agent systems. A prior understanding of graph theory is necessary to grasp these mathematical models.

The French-DeGroot model provided a foundational understanding of opinion convergence, with algebraic criteria shedding light on convergence and consensus conditions. Abelson's model, a continuous-time counterpart, raised intriguing questions about community cleavage and diversity.

While the report covers two significant models, it's essential to acknowledge that the research paper encompasses more models awaiting exploration.

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