Ray Optics: Observation Range Of Optical Systems & The Dimensionless Approach for DOF

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1. Background

This article serves as the supplemental information. It also serves as an introduction for readers wanting to characterize refractive changes in observation range of optical systems (including eyes) from external defocus or image-screen/axial changes. A novel derivation of depth-of-field relations have also been provided for the enthusiastic reader.

The material in this article is of enormous importance towards building an understanding of how lenses ~~corrects~~ compensate for refractive errors of the eye. Math worksheets for µMath+ Android application is supplied along with this text containing a workable version of the formulas used throughout the article.

2. Ray Optics: Thin-lens relation derivation

This section focuses on deriving the relationship between object and image distances for the case of (real) image formation only by an ideal converging thin-lens resulting in the thin-lens relation. The relation applies for the diverging lens as well but will not be attempted here. The 2D illustrations and ray diagrams of this article extend and apply equally well to the 3-Dimensional real world.

An ideal lens itself can be completely defined with only two physical parameters/attributes along with its principal axis – **focal length** and **aperture**. The term focal length or its inverse as lens power always implies the same physical property. These two parameters are sufficient to explain the majority of image formation by actual lenses and their observed behaviour in physical experiments.

In this sense, an ideal lens is essentially a theoretical object and we needn't immediately concern ourselves with the physical implementation details of how it achieves 'image formation' and 'focusing' of rays in reality. It also means that these rays in question can be anything – light, radio waves, sound waves etc. reinforcing the fundamentally theoretical nature of the relation. The concept of ideal lenses is interesting with properties of their own from a purely theoretical perspective.

A lens with curvature only along one axis (meridian) is known as a Cylindrical (CYL) lens while a lens with identical curvature along both independent axes is denoted as Spherical (SPH) lens[1]. It must be noted that words *cylindrical* and *spherical* were meant with regard to appearance of the lens and have little to do with the actual profile (refracting cross-section) of the lens. This profile in the case of most modern lenses is always aspheric. The axis of a cylinder follows the direction of constant curvature with the idea being

1 https://www.clzoptics.com/news/cylindrical-vs-spherical-lenses.html

rotation about its axis should be indistinguishable (due to symmetry). This is required from the standpoint of unique determination of cylindrical axis as shown in Figure 1 below.

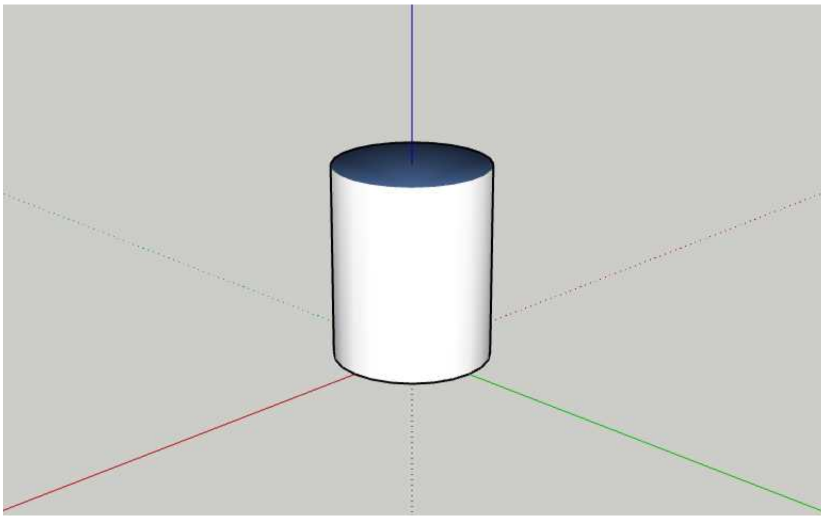

Figure 1: The axis (purple line) of a cylinder is normal to its flat face

2.1 Definition of an Object and its Image in Ray Optics

An object is considered to be a *collection of object points*. From every object point, rays can be assumed to **originate/diverge** in all directions. An image is similarly considered to be a *collection of image points*. At each image point, rays originating from an object point **terminate/converge**.

For an image point, demonstrating convergent intersection of any two rays emerging from an object point is sufficient. A point on an object and its image will always have a one-to-one correspondence. It is very important to note that beyond the image point, away from its object – the rays that converged to form the image point start diverging again. An object can be differentiated from its image in the sense of independent existence. An object can exist independently but an image can not exist independently of an object. It is immensely useful to express Aperture size, object, and image distances in terms of focal length. This dimensionless representation comes at no loss of generality as shown in the later sections.

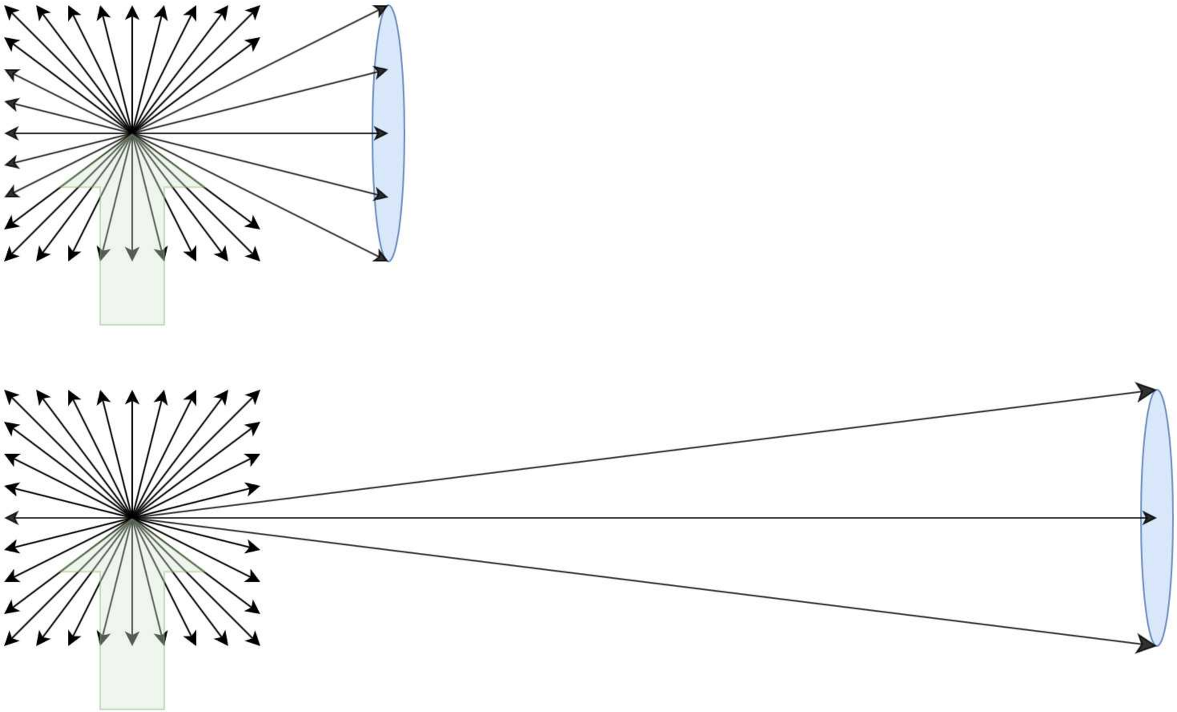

Figure 2: Ray diagram of rays from an object point hitting the lens.

An object point located on an object emits rays in all directions (Figure 2). At distances become larger, the angle between rays emitted by a point object starts decreasing with the rays themselves appearing almost parallel at very large distances from the lens. However, even at infinity, the rays emitted by an object can only appear diverging or parallel and never convergent without help from a lens.

An application of our stricter definition is to consider the commonly mentioned *‘virtual images[2]’* in many textbooks as having properties more in line with objects rather than images because rays seem to originate (diverge) from them. That a ‘virtual image’ lies on the same side of the lens as the object favours consistency of our argument terming ‘virtual images’ as virtual objects. For analysis of systems consisting of multiple optical elements, the virtual object formed by an object and lens combined together can be replaced solely by the virtual object alone. Subsequent imaging of such a virtual object by an optical system (whether eyes or camera) then results in formation of ‘real’ image. Henceforth, the word image in this article refers to a real image.

2.2 Calculating deviation for a ray incident on an ideal lens

Assumption A: The focal length determines the convergence/divergence point (focus) of every ray parallel to the principle axis for a converging/diverging lens respectively shown in Part 1 of Figure 3. This is analogues to a parabolic diverging mirror reflecting all incoming rays parallel to its principal axis on its focus. The Aperture size only determines the extent of admission from principal axis for such parallel rays.

Assumption B: Only the rays passing parallel to the principle axis obey the previous assumption. All rays incident normal to the principal axis pass unaffected through the lens profile. The word ‘thin’ has been used

2 https://en.wikipedia.org/wiki/Virtual_image

to denote this aforementioned property of not affecting rays normal to the principle axis of the lens. There is the implicit assumption that a ray can be first decomposed into its normal and parallel constituents. Ray diagrams given below demonstrate application of the above two assumptions to show deviations for various incident rays by decomposing them into normal and parallel components.

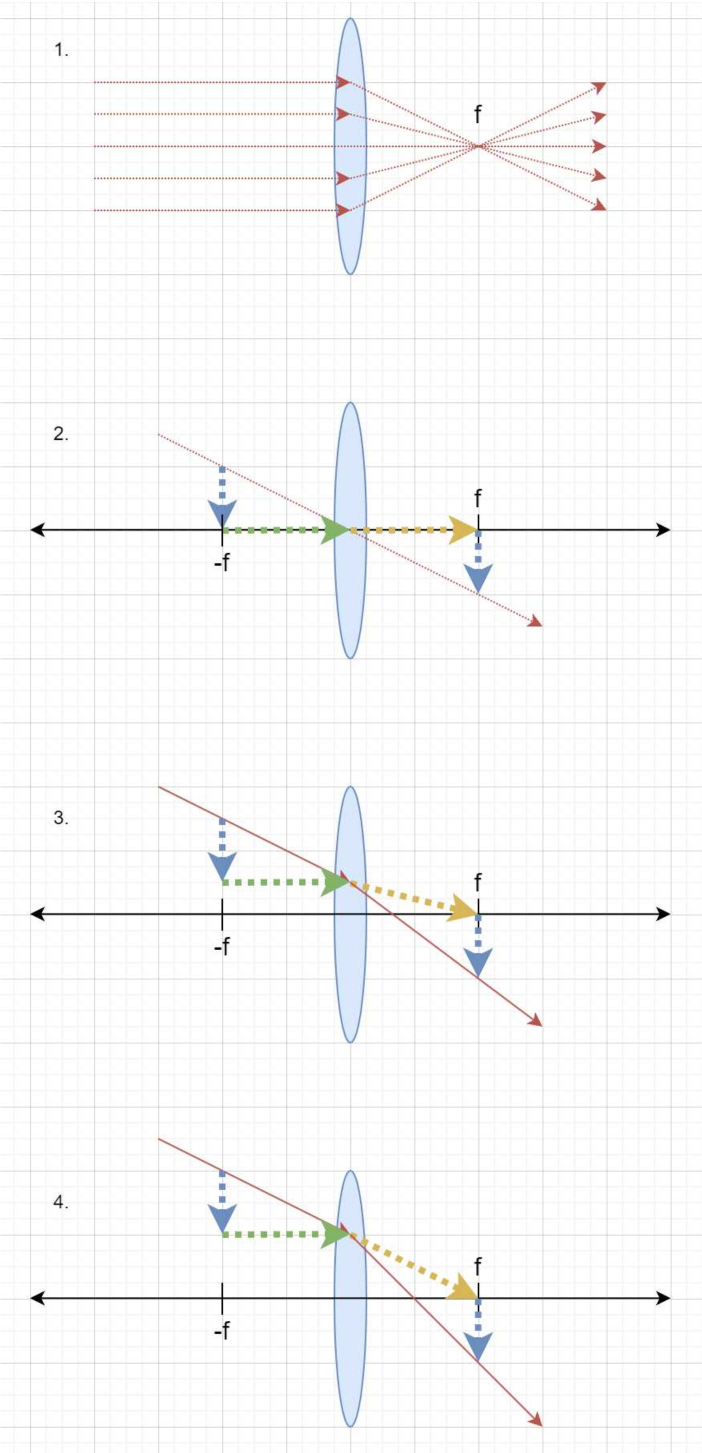

Figure 3: Application of Assumption A (Part 1) and B (Part 2 to 4)

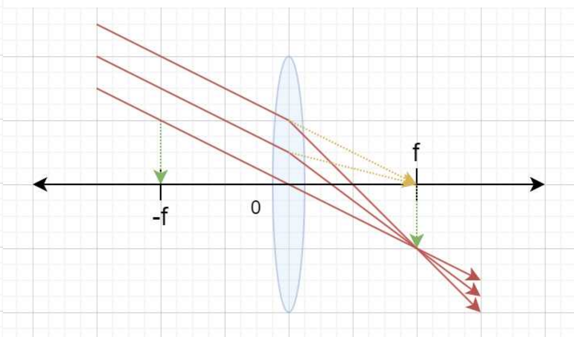

Figure 4: Combined version of 2, 3, and 4 in Figure 3

From Part 2 to 4 shown in Figure 3 and their combined version in Figure 4, the length of the vertical component is the same when determined at the focus. The amount of deviation a ray undergoes depends on both where it is hitting the Lens and the angle it makes.

Consideration of these two assumptions result in simplified rules for all rays incident on an ideal converging (+) lens and is sufficient to explain image formation. Verification for an edge case corresponding to image formation at 2f is provided below in Figure 5. In this particular edge case, we’ve shown convergence of five such rays at a single point when demonstrating that only two meet at the image point would have been enough.

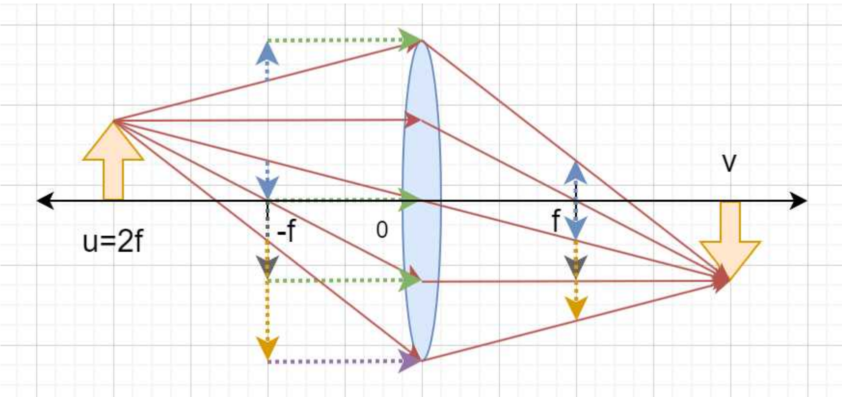

Figure 5: Edge case shown for image formation when u=2f

The rule regarding rays passing unchanged through the optical centre of the lens (as given in many intermediate textbooks[3]) can be explained as a consequence of fundamental assumptions A and B stated above. This rule also determines the magnification ratio because the image point must lie on the unaffected ray passing through the optical centre of the lens.

2.3 The relation between object and image distance

For a converging lens with a given focal length, two distinct cases of rays intersection are possible according to the two assumptions outlined in the previous section 2.2. Both the cases (Part 1 and 2) are shown in Figure 6 below with the same object placed at varying distances with the limiting case (Part 3) of an object placed exactly at the focus.

1. **Image** formation when the object is at a distance farther than the focal length from the lens. Rays from the object point appear to be converging towards the image point. It is important to note beforehand that the image formed in this case is **inverted**.

2. **Virtual *object*** when the object is at a distance closer than the focal length from the lens. Rays appear to be coming from the virtual object.

3 Concepts of Physics - Part 1, For Intermediate Level, Publisher: Bharati Bhawan; Revised Reprint 2015 edition, ISBN-13: 978-8177091878

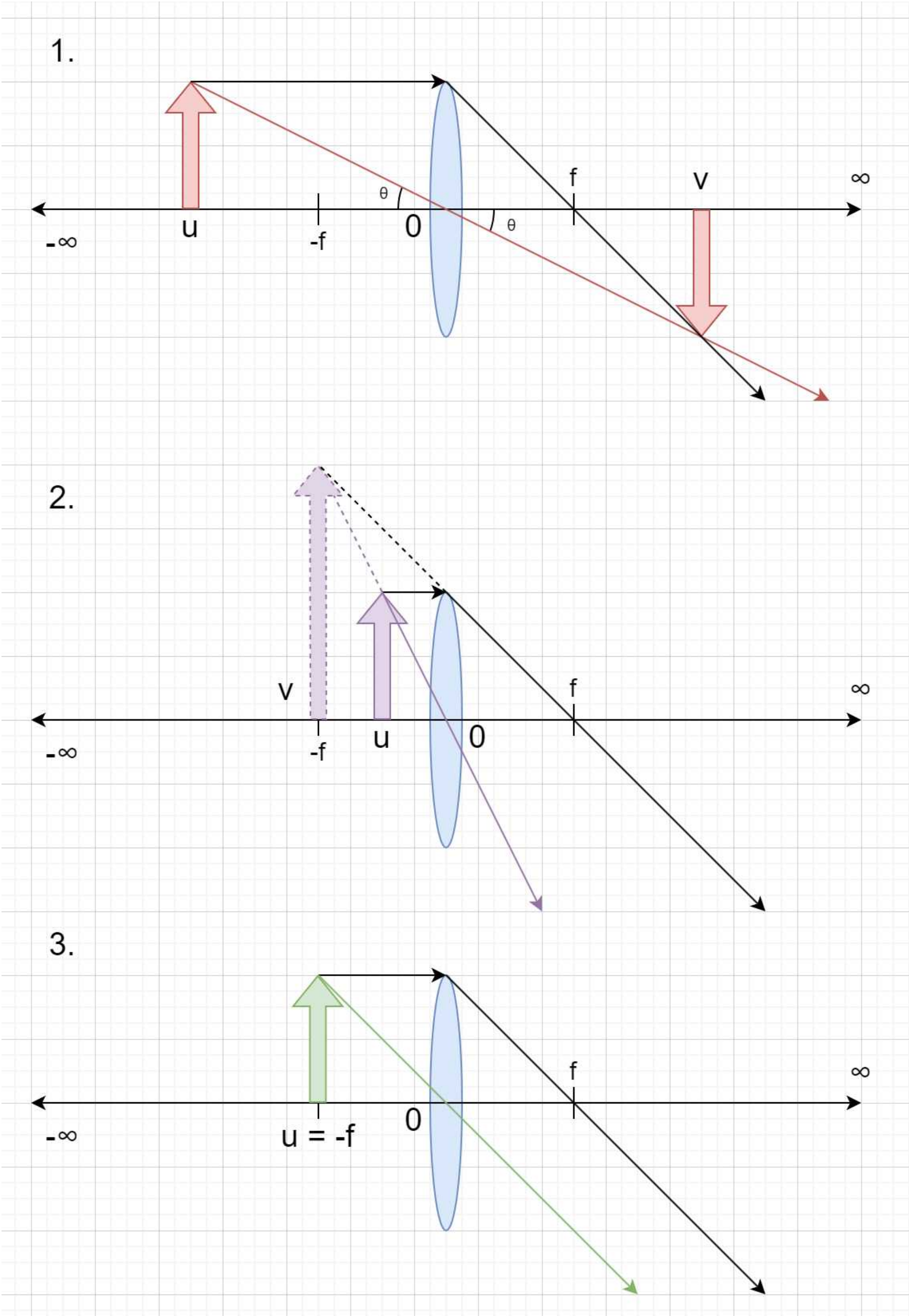

Figure 6: Formation of Image and Virtual Object by a converging lens

The ray passing through focus has the same slope -*h/f* (where *h* is the constant height of object point) while slope of the ray passing through the optical centre depends on the object distance in all three cases. Whether the slope of the latter is greater than or lesser than the former passing through focus determines where these two rays meet and whether an image is formed or not respectively. Precisely, ray from an object point placed farther from the focus and passing through the optical centre will always have a less negative slope.

The derivation of lens relation for an ideal converging lens has been given below (Figure 7) employing dimensionless representation.

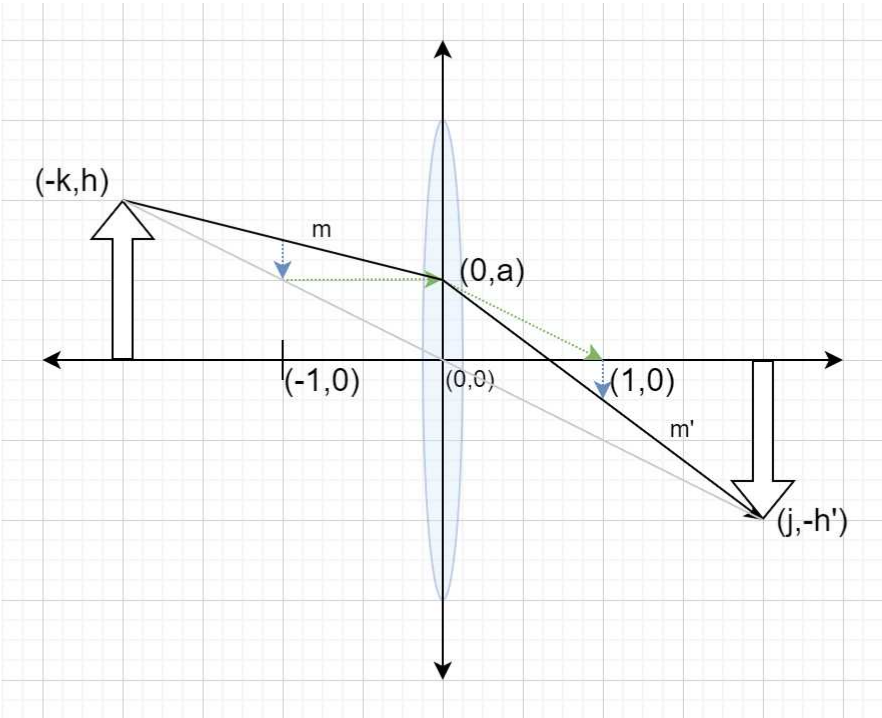

Figure 7: Ray Diagram for dimensionless derivation of relation between image and object distance

For image distance, we assume the origin of the right-handed co-ordinate system centred on the lens. Care must be taken to ensure consistency of units and signs according to conventions while utilizing formulas throughout this article to ensure correctness in calculations. An ideal lens has two foci due to its symmetrical nature. For a converging (+) lens, the focus lies on the opposite side of the lens from object. It depends on the signed power (converging or diverging) of lens that determines the sign of focal length and consequently its position relative to the object. The scope of this article is limited to images formed by a converging lens.

$$magnification(m)\, is\, defined\, as\, \frac{image\, height}{object\, height}=\frac{-h'}{h}=\frac{image\, distance}{object\, distance}=\frac{v}{u}=-\frac{j}{k}$$

$$\textit{We have normalized object and image distances as follows:}$$
$$u = -kf \text{ and } v = jf$$

$$\textit{for image formation, } k > 1 \textit{ which means } -u > f \textit{ or } u < -f$$
$$\textit{slope m when the object point is } (-k, h) \textit{ is given by}$$
$$m = \frac{a-h}{k}$$
$$\textit{for point } (0, a) \textit{ on the lens profile } (y - axis)$$
$$\textit{image formation requires all rays to pass the point } (j, -h') \textit{ after refraction}$$
$$\textit{The slope after refraction } (m') \textit{ is given by,}$$
$$m' = \frac{a-(-h')}{-j} = \frac{a+h'}{-j}$$
$$\textit{The slope after refraction according to Assumption 2 must be}$$
$$m' = m - a \Longrightarrow \frac{a+h'}{-j} = \frac{a-h}{k} - a$$
$$\textit{Now, it's already known from the magnification criteria that}$$
$$\frac{k}{j} = \frac{h}{h'} \Longrightarrow h' = \frac{hj}{k}$$
$$\Longrightarrow \frac{a + \frac{hj}{k}}{-j} = \frac{a-h}{k} - a$$

$$\Longrightarrow \frac{ka+hj}{k} = \frac{hj - aj + kaj}{k}$$
$$\Longrightarrow ka = kaj - aj$$
$$\textit{giving, } k + j = kj$$
$$\textit{which completes the derivation of k and j relation.}$$

$$\textit{To derive the ideal thin lens relation,}$$
$$\textit{we can substitute } k = -\frac{u}{f} \;\&\; j = \frac{v}{f} \textit{ giving}$$
$$-\frac{u}{f} + \frac{v}{f} = -\frac{vu}{f \times f}$$

$$\Longrightarrow -u + v = -\frac{vu}{f}$$
$$\Longrightarrow \frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

2.4 Dimensionless object (k) & image (j) distance relation

The relation between dimensionless object and image distance is $j = \frac{k}{k-1}$

For image formation by a converging lens, k > 1. This simply says that the object needs to be located beyond the focus for image formation. A plot of this equation results in a hyperbola symmetrical around the line x = y (Figure 8). The vertical and horizontal asymptotes given by lines x = 1 & y = 1 correspond to the object at focus and ∞ while also corresponding to its image at ∞ and focus respectively.

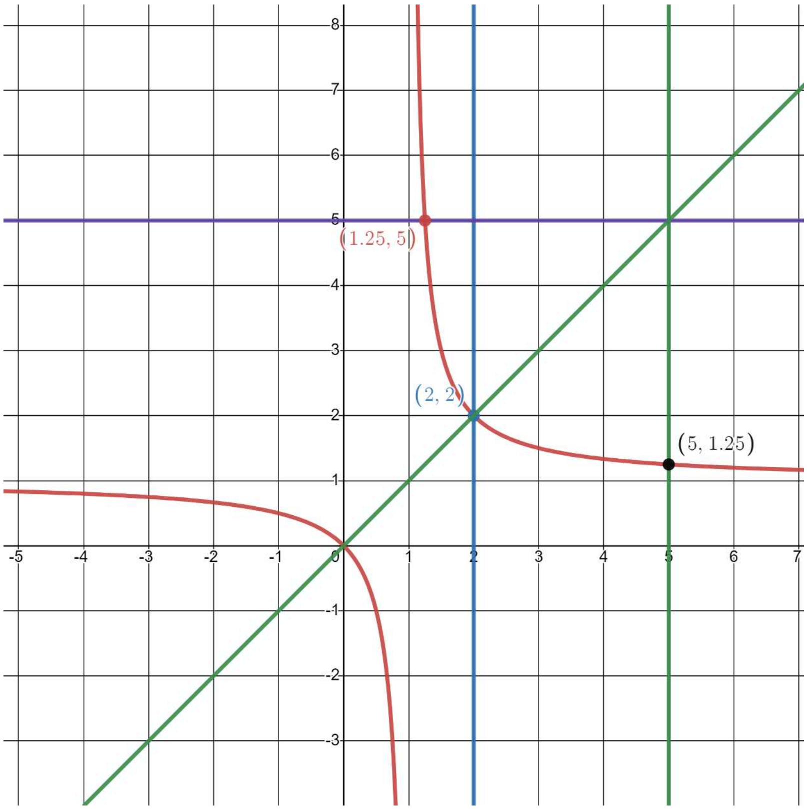

Figure 8: Red Hyperbolic plot representing the dimensionless lens relation

It is evident from the symmetry of the hyperbola in Figure 8 that both an object & its image's dimensionless distances form what can be termed as a mirror pair around the x = y line.

It becomes even more obvious if we solve for k, giving $k=\frac{j}{j-1}$ which can be understood simply as interchanging k with j & j with k in the original relation.

Adding k and j together gives, $k+j=k+\frac{k}{k-1}=\frac{k^2}{k-1}=k\times j$ which has a minimum of 4 at k = j = 2 for positive values as can be seen from the graph below (Figure 9).

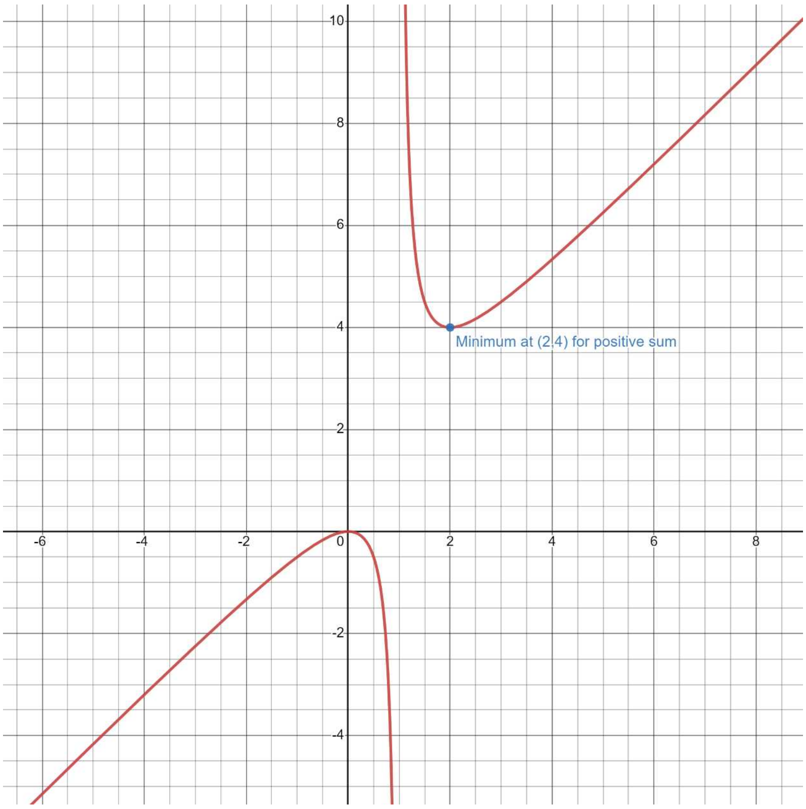

Figure 9: Plot showing variation in sum of dimensionless image and object distance with distance

This shows that for a given distance between an object and its required image, the focal length of the system able to image it can not be greater than the quarter of the given distance. The $(k + j) \geq 4$ rule along with the $k > 1$ rule serve as fundamental constraints for real world implementation of optical systems.

2.5 The power addition rule

The power (inverse of focal length) addition rule can easily be derived from the ideal lens relation by considering two lenses with different focal lengths placed together such that the distance between them can be neglected.

Here, v and u denote the image and object distances respectively with subscripts denoting which lens they are referring to. The power of a lens is the inverse of its focal length. The Dioptre (m^{-1}) is one such derived SI unit of power. Assuming focal length f_1 for Lens L_1 and f_2 for Lens L_2. Then the image distance for Lens L_1 is given by, $v_1 = \frac{u}{1 + \frac{u}{f_1}}$

Because the distance between the two lenses is negligible, this resultant image now gets further refracted as an object by lens L_2. The corresponding final image formation distance is given by (where $u_2 = v_1$).

$$v_2=\frac{v_1}{1+\frac{v_1}{f_2}} \Rightarrow v_2=\frac{\frac{u}{1+\frac{u}{f_1}}}{1+\frac{\frac{u}{f_2}}{1+\frac{u}{f_1}}}=\frac{u}{1+\frac{u}{f_1}+\frac{u}{f_2}}=\frac{u}{1+u(\frac{1}{f_1}+\frac{1}{f_2})}$$

This implies that the combined lenses together act as a lens of focal length $\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f}=P_1+P_2$ which is basically the law of addition of lens powers. For a combination of more than two lenses closely put together, we can proceed by combining two lenses together at a time until only one remains.

3. Implementation of Ideal refracting lenses

What we have described so far is the theoretical description of Ideal lens behaviour. In order to physically implement an ideal lens optically, additional physical considerations are required.

In order to come up with the governing equation for a ideal refractive lens, first we need to get certain concepts out of the way.

The deviation a ray undergoes w.r.t. the normal at the point it encounters another refractive medium is governed by the Snell’s law. This law in turn arises due to the consequence of difference in speed of light in various refractive media. This article assumes prior familiarity with the Snell’s law.

In order to come up with the constraining laws required to realise an ideal refractive lens, We will start with the simple case first and then slowly work our way upwards. All the calculations and formula in this section will use the dimensionless k & j notation introduced in section 2.4.

3.1 Ideal (aspheric) Refracting lens with symmetric profile

Let’s assume an object point at (-2,0) with the optical centre of the lens at origin. The lens itself is oriented with its principal axis as x-axis just like previous examples.

It’s obvious that the image point corresponding to this object point will be at (2, 0). A ray of light diverging from this object point can ‘strike’ the lens anywhere but has to converge at the point (2, 0) for forming an image.

Let’s assume the refractive index of lens medium as n_2 and that of the surrounding as n_1. We will define

$$\eta=\frac{n_2}{n_1}.$$

In order for a ray of light to reach image point from object point, it encounters two medium changes. First from n_1 to n_2 when entering the lens and then from n_2 to n_1 when exiting the lens. If we consider the surroundings to have a refractive index $n_1 = 1$ corresponding to vacuum or roughly approximating air, then the refractive index of lens medium becomes $n_2 = \eta$.

If we assume the lens profile to be symmetric, it means that for this particular case, the light ray must become parallel to the x-axis inside the lens because of the symmetry between object and image point. This is very similar to the prismatic consideration used for deriving the lens maker's relation.
We will derive the relation for exactly this case here as shown in the Figure 10 below.

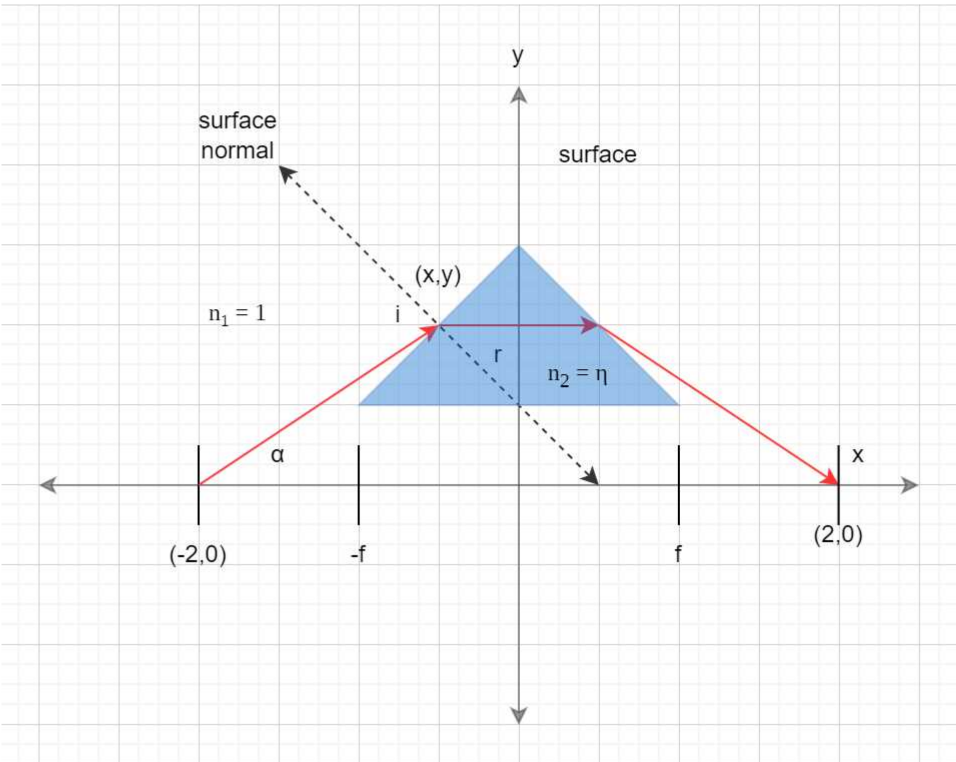

Figure 10: Diagram showing angular relationship using prismatic consideration and parallel symmetry

$$\frac{\sin(i)}{\sin(r)}=\frac{\sin(\alpha+r)}{\sin(r)}=\eta$$

$$\frac{\sin(\alpha)\cos(r)+\cos(\alpha)\sin(r)}{\sin(r)}=\sin(\alpha)\tan(r)+\cos(\alpha)=\eta$$

$$r=\arctan\left(\frac{\eta-\cos(\alpha)}{\sin(\alpha)}\right)$$

Here, α is the angle made by the object ray with the x-axis. r is the angle of refraction from interface normal. r is also the angle made by the interface with the y-axis. The integral for this expression is impossible.

This equation basically implies that the only thing that matters for a lens is a corresponding surface angle from a given angle from point (-2, 0). This alone is sufficient to realise why Fresnel lenses exist and work well for non-imaging applications. The ‘angular gaps’ on the fresnel lens profile that contribute towards its ‘slimness’ also contribute to the unsuitability of Fresnel lenses for imaging applications. The parallel criteria implies that there can not be concavity in the lens in the direction of principle axis.

Plotting these values give us important and interesting insights.

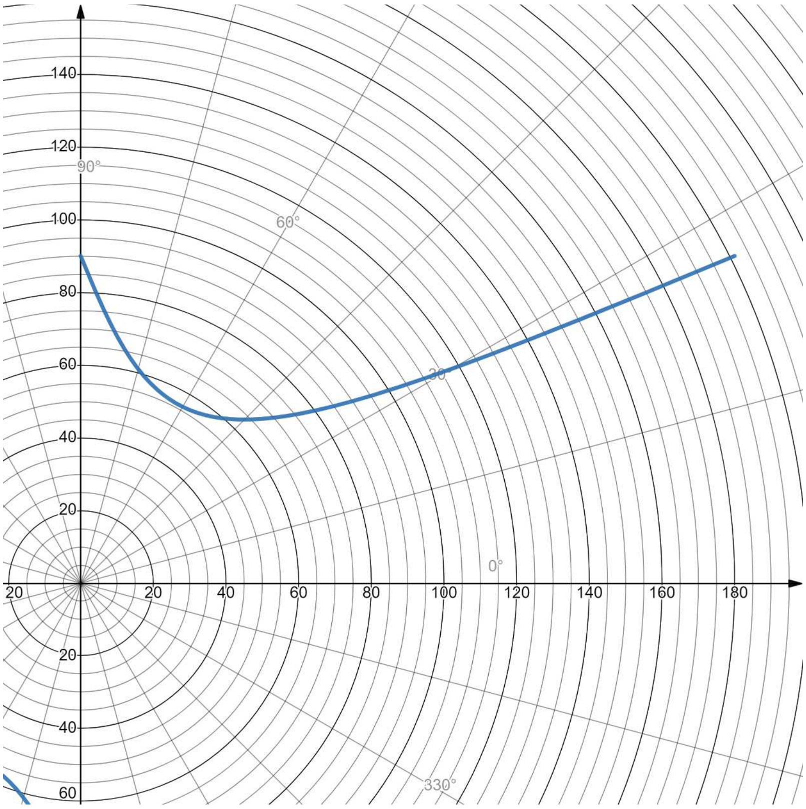

Figure 11: Plot of α vs r for η = sqrt (2)

Careful attention must be paid to the radial axis which denotes angular parameter (α) in this graph. First of which is that the values seem to decrease only till a certain angle and then start increasing again. This becomes even more apparent if we take η = sqrt(2) giving the minima as 45°. The value of η itself must be greater than 1. This corresponds to the point at which total internal reflection (TIR) happens for the lens medium. It is impossible for the lens profile to extend beyond this theoretical limit because of TIR phenomena. This basically means refractive index places a fundamental limit on the aperture size for any lens made of optically refractive media.

For η = sqrt(2), this limit comes out to be 4 × focal length (commonly termed f/0.25). As refractive index increase, this limit tightens even further.

If the desired aperture size is known in advance, a continuous symmetric thick lens profile can be computed numerically by proceeding in a backwards manner. For asymmetric lens profiles, the relation derived above is no longer applicable.
For real world applications, this problem is further complicated due to varying refractive index of the medium for spectral components of light. Any real world design is ultimately the result of multiple such lenses utilised together. The lens profiles themselves are computed based on multiple compromises.

4. The Ideal Lens System (ILS) Setup

Be it Human eye or an optical system like a camera, the purpose is mostly the same - ‘imaging’ objects (which can be located at varying distances) onto the retina/image sensor. For a film camera, the film acts as the image sensor. For a digital camera, an image sensor chip replaces the film. The ray optics involved behind image formation remain unchanged. The term image refers to a ‘real’ image and all mentions of the word ‘lens’ refer to an ideal thin-lens.

The ideal lens system (ILS from now on) consists of an ideal converging lens and a planar image screen (sensor) together with an object completing the optical bench setup shown in Figure 12. The screen is aligned to the principle axis of the lens. For the ILS, tracing the path of rays from an object to its eventual image is very simple. Rays emanating from an object point encounter the lens aperture and hit the image sensor after refraction. The medium in this case can be said to be vacuum/air.

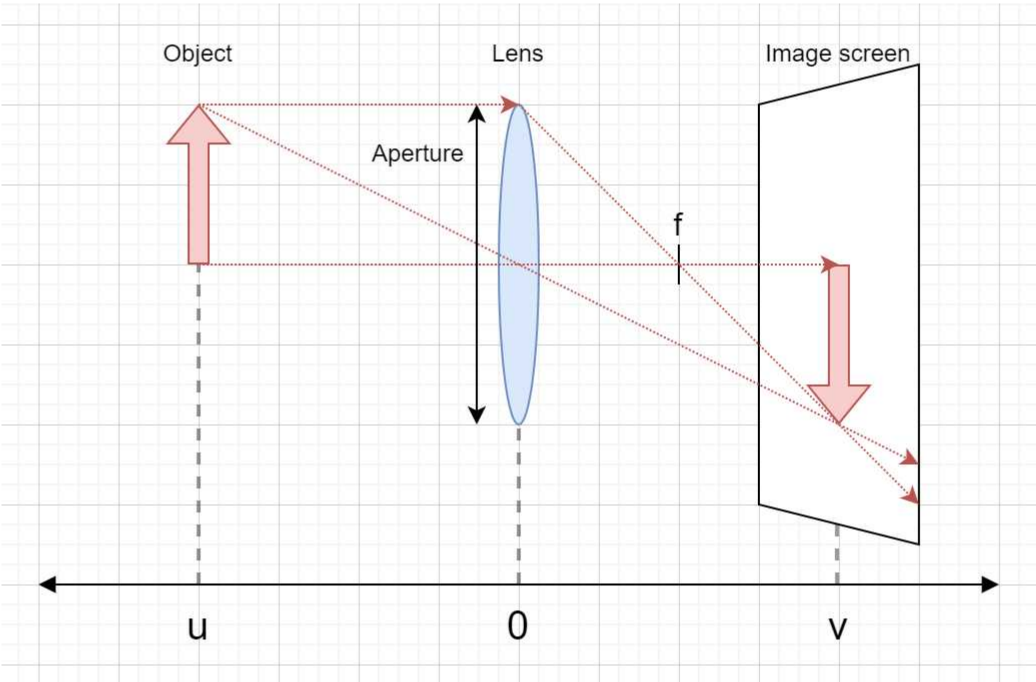

Figure 12: Optical bench diagram of the ILS

Four parameters are necessary to completely describe the setup of an object to be imaged and the ILS, if one also includes Aperture size (having no impact on image distance). The remaining three parameters are object distance (u) with the focal length (f) together determining the image distance (a dependent variable) and the screen distance (s). Distances are measured using Optical Centre as the origin. The **Field of View (FoV)** is defined as the angular-expanse of objects whose images can be formed on the image sensor/screen. It is easily quantified from the ratio of screen height to the screen distance.

4.1 The distinction between image distance and screen distance

For image formation, image distance (v) must coincide with the screen distance (s). Image formation in this context refers to the formation of a focused sharp image for this section and sub-sections. Unless specified otherwise, it will be assumed that depth of field effects are insignificant. Calculation of distances within which screen distance (s) can vary for a given image distance (v) while still forming appreciably sharp images within a CoC (Circle of Confusion) is given in Section 7.

Thus, we can state that the dependence of screen distance (s) on focal length (f) and object distance (u) is already encoded by the ideal lens relation. Knowing any two, the third is easily calculated.

$$s=\frac{fu}{f+u}=\frac{u}{1+\frac{u}{f}}=\frac{1}{\frac{1}{u}+\frac{1}{f}}$$

4.2 The concept of constrained observation range

Fixing screen distance and focal length together also fixes the distance at which an object can be imaged. It is desirable for an optical system to be able to form images of objects located within a distance range instead. Imaging objects can be accomplished by varying either the focal length (f) or the screen distance (s) or some combination of both.

It must be kept in mind that all real-world implementations will always have some physical constraints on the ‘variability’ of independent variables (focal length and screen distance in our case) constraining the values they can take. The range of object distances that can be imaged by a system within constraints will be referred to as its ***observation range*** or focusing range. The farthest and nearest distance extremes of the observation range are commonly designated as the far-point (d_{far}) and near-point (d_{near}) for a system respectively. The Far-Point is defined as the distance of ***far***thest objects a system is able to image while the Near-Point is defined in a similar manner as the distance of ***near***est objects a system is able to image within constraints.

Our approach for studying changes to a system’s observation range results from fixing one of the two independent variables and studying how varying the other affects the range of object distances (d_{near} from d_{far}) that can be imaged. The two cases for our system resulting from fixing focal length (f) and screen distance (s) at a time are discussed in the next two sections.

4.3 Observation range of system when imaging screen distance (s) is fixed (accommodation)

Showing that for such a system,

*the increase in power/decrease in focal length (f) for maintaining **fixed screen distance (s)** solely depends on the object distance (u)*

can be easily done by rearranging the ideal lens law in terms of focal length.

$$By\ the\ thin\ lens\ relation, \frac{1}{f}=\frac{1}{s}-\frac{1}{u}, s\ is\ constant$$

$$For\ an\ object\ at\ \infty, u\rightarrow\infty\Rightarrow\frac{1}{f_\infty}=\frac{1}{s}-\frac{1}{u}=\frac{1}{s}$$

$$For\ an\ object\ closer\ than\ \infty, \frac{1}{f}=\frac{1}{f_\infty}-\frac{1}{u}$$

$$\Rightarrow\frac{1}{f}-\frac{1}{f_\infty}=\Delta\ Power=-\frac{1}{u}$$

$$The\ signed\ term -\frac{1}{u}\ is\ positive\ by\ co-ordinate\ convention.$$

For such a system, the increase in lens power (change in focal length) needed to image objects **closer** than a reference distance depends only on the object distance (u). This lens power increment for imaging closer objects is commonly referred to as accommodation with the reference ‘un-accommodated’ distance taken at infinity. The accommodation required for shifting focus from from far to near point is commonly referred to as a particular system’s accommodation ability. A system is stated to be fully accommodated once lens power reaches constraints corresponding to focusing at its near-point.

Fixing screen distance requires the focal length to vary in accordance with the focused distances. For instance, a system with a fixed screen distance of 25 cm and lens power constrained in the range +4 D to +9 D will have its Far and Near-points at ∞ and 20 cm respectively.

With lens power at +9 D which amounts to +5 D accommodation from the initial +4 D needed for objects at ∞, the system images objects located at 20 cm. Because screen has been kept fixed for this system, the Field-of-View (FOV) remains unchanged.

4.4 Observation range of the system when the focal length (f) is fixed

If the focal length is kept fixed and the screen distance is allowed to vary instead, the dependence of observation range on screen distance is given in a similar manner from the ideal lens relation:

$$\text{By the thin lens relation}, \frac{1}{s}-\frac{1}{u}=\frac{1}{f} \text{ giving } \frac{1}{s}=\frac{1}{f}+\frac{1}{u}$$

$$\text{For an object at } \infty, u \rightarrow \infty$$

$$\text{giving } \frac{1}{s_\infty}=\frac{1}{f}$$

$$\text{For an object closer than } \infty, \frac{1}{s}=\frac{1}{s_\infty}+\frac{1}{u}$$

$$\text{The signed term} \left(\frac{1}{u}\right) \text{is negative here}.$$

Assuming a system like the one described in section 4.3 above but with a fixed focal length (f = 25 cm) instead, we can find out the near and far points of this system in a similar manner.

The k > 1 requirement arrived in section 2.4 mandates that screen distances lesser than the (fixed) focal length of the system can’t result in image formation for all possible object distances. Thus, the screen distance required in this instance must be always greater than 25 cm and can only be increased up to infinity. The far and near point of this system are ∞ and 25 cm respectively.

It is evident that screen distances required for imaging closer objects quickly become very large and even then the system is unable to image objects closer than 25 cm because it is impossible to physically locate the screen beyond infinity. Contrast this to the variable focal length (f) system described earlier where no such physical limit was in place for focal length preventing us from observing closer objects.

Changes to the screen distance implies changes to the overall size of the system and thus the FOV.

5. The Relative Dioptre Scale (RDS) and shifting of Observation ranges

From the description of constrained observation ranges, the need for a tool to visualize optical systems and any subsequent modifications to them whether from defocus or screen distancing was felt. The Relative Dioptre Scale was devised for this exact purpose serving as an intuitive way to visualize change in observation ranges due to induced defocus or changes to the screen distance.

The RDS is simple Cartesian co-ordinate system modified for power (inverse instead of linear distances) onto which observation ranges for optical systems are represented in this article. The x-axis of the RDS represents distances in (D or m^{-1}) with the origin representing infinity (∞). The y-axis was chosen to represent the accommodation ability of the system (also in Dioptre). Thus systems with better accommodation ability are vertically ranked/placed higher-up on the Relative Dioptre Scale.

The idea of representing observation range of optical systems on an inverse length scale stems from the inverse nature of the ideal lens relation itself. This has the important implication of making transformations on the RDS linear with respect to changes in Power.

The ideal lens relation $\frac{1}{s}-\frac{1}{u}=\frac{1}{f}$ itself can then be written as Power of Object distance (in m^{-1}) = Lens Power (in D) – Power of Screen distance (in m^{-1}).

On the RDS, the left end of the system's observation range represents its far-point while the right end its near-point. The observation ranges (d_{near} to d_{far}) of the systems described in sections 4.3 and 4.4 can be represented on the RDS (Figure 13).

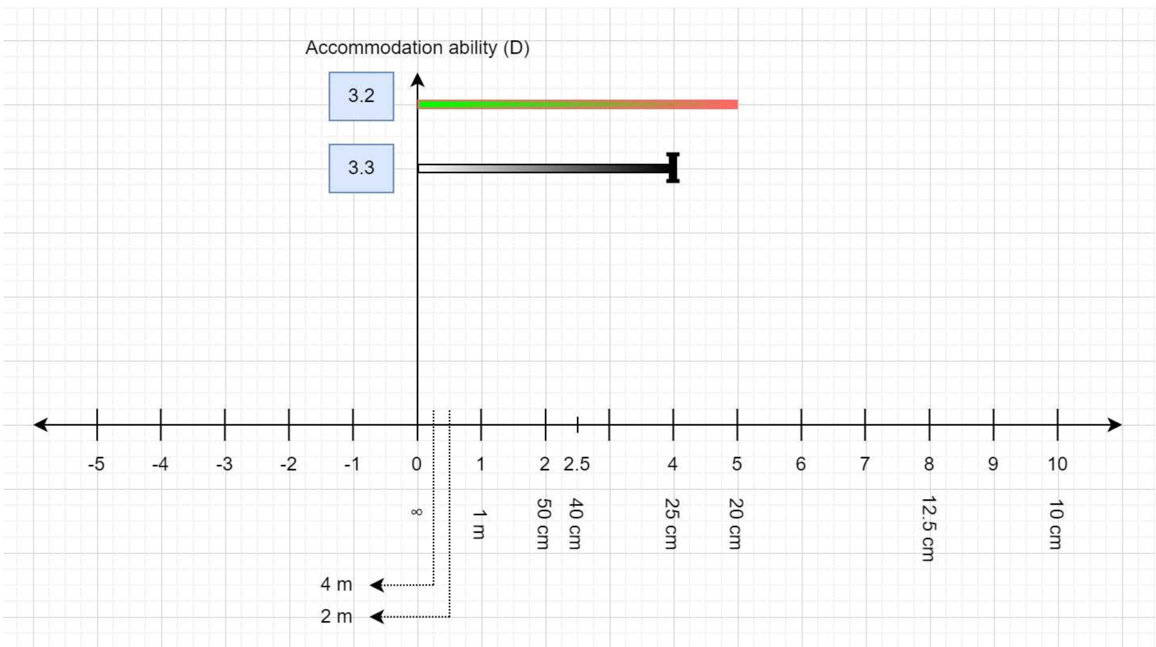

Figure 13 Comparative representation of the observation ranges of the system in section 4.3 and 4.4

The vertical rank/height represents accommodation ability of the systems which is also equal to the length of line segments. Accommodation ability is the power difference between the two extremes (near and far-point) of the observation range.

The gradient from red to green and light to dark was used to indicate reaching the end of the system's observation range due to constraints. The 'I' symbol was used to indicate the impossibility of any physical extension towards the right-hand direction (near-point) for the fixed focal length system described in section 4.4. The sign of object distance on the RDS is inverted from the sign for object distance (u) used in ideal lens relation due to respective co-ordinate conventions and the same must be remembered.

5.1 Shift in observation range after introduction of defocus

What happens to the system in 4.3 when accommodation ability is kept same (+5 D) but the range of values themselves are incremented by +1 D on both ends (+5 D to +10 D from +4 D to +9 D)? Such a change can be easily achieved by introducing another +1 D lens such that the power addition law holds (Section 2.5).

Just like before, it is sufficient to calculate the shifted far and near-points of the system (now 1 m and 16.66 cm respectively). This modified system is represented on the RDS (Figure 14). The original system is shown using slightly thicker grey colour for ease of comparison.

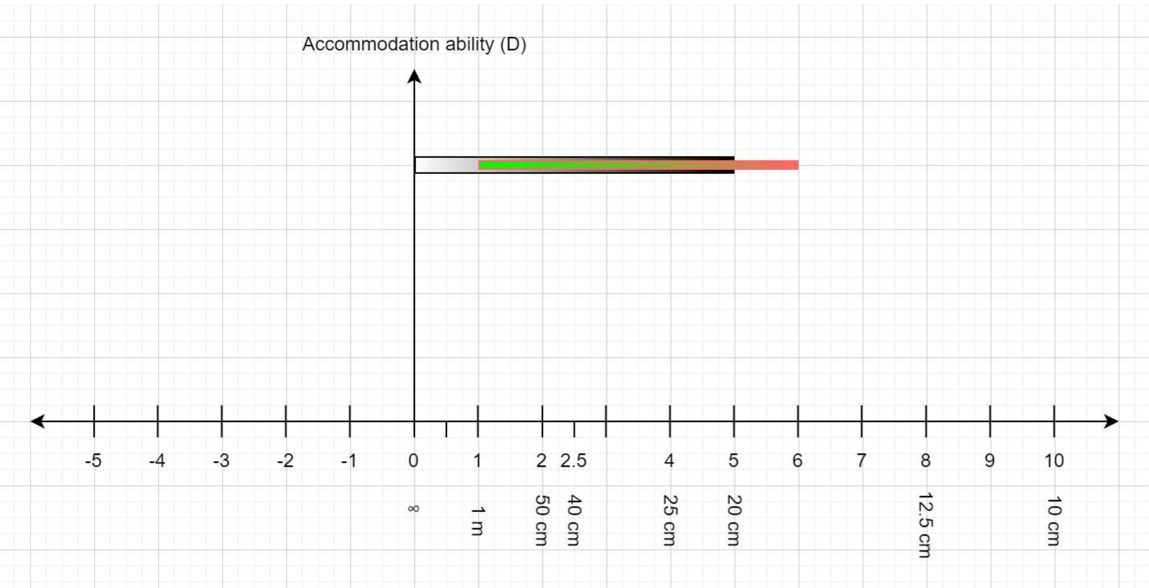

Figure 14: Shift in observation range after introduction of +1 D lens

The modified system's observation range with Lens Power range: +5 D to +10 D gets shifted 1 unit towards the positive/right direction (the sign of introduced lens).

It is evident how an increment in range of lens power values results in right-shifting of the system by the same amount on RDS. Our modifications to the system in 4.3 has rendered the system Myopic (unable to focus on objects farther than 1 m).

The inverse also holds true. It is obvious that introducing a -1 D lens in our newly modified system restores the original accommodation range. This is equivalent to introducing another -1 D lens or removing the previously introduced +1 D lens. Which means that observation range on RDS must also shift towards left by one unit. This is what the word ‘Relative’ means in the RDS. Relative to our modified system, the system in 4.3 can be said to be hyperopic (unable to focus at objects closer than 20 cm).

It should be now easy to understand why *introducing converging (positive) lenses/removing diverging (negative) lenses* to a system is termed as ***Myopic defocus*** – because it renders the system myopic (shifts the observation range towards right).
Similarly, *removing converging (positive) lenses/introducing diverging (negative) lenses* to a system is called ***Hyperopic defocus*** – because it renders the system hyperopic (shifts the observation range toward left).

5.2 Shift in observation range after changes to screen distance (Axial changes)

We’ve previously described shifting of a system’s observation range under induced defocus. Calculating shift in Observation range due to screen distance changes is also as easy as recalculating the far and near-points. Changes to the screen distance can also be termed as Axial changes because it results in changes to the overall size of the system.

For instance, the system in 4.3 has the screen fixed at 25 cm from the lens. If we position the screen 5 cm closer maintaining the same lens power range, the near-point recedes farther from 20 cm to 25 cm (1 D towards left) signifying that the system has become hyperopic. Similarly, if we move the screen 25cm farther from its initial position – the far-point comes closer from ∞ to 50 cm (2 units towards right) rendering the system Myopic. The near-point also shifts closer (by the same unit amount) from 25 cm to 16.66 cm.

6. Experimental verification of shift in observation range from external defocus

The shift in observation range due to screen distance increments act opposite to that of increments in focal lengths. Thus, increase in screen distance results in Myopic shift in observation ranges and decrease in screen distance results in Hyperopic shift respectively. It is easy to experimentally observe the shifting in observation range upon introduction of external defocus as described in section 4.3 using a dedicated camera. Experimental verification of the findings of section 4.4 can also be achieved in a similar manner with the help of an optical bench setup where screen distance can be varied. This has probably been observed and documented several times but we still feel the relative unfamiliarity of phenomena deserves a mention.

Regarding shifting of observation range of a system, reducing (increasing) focal length is analogous to increasing (decreasing) screen distance. This is also intuitive from the way we have defined dimensionless image distance $j=\frac{s}{f}$ where decreasing focal length (f) or increasing screen distance (s) both serve to increase the dimensionless image distance (j) resulting in a decrease in dimensionless object distance (k) signifying Myopic shift and vice-versa. We have also demonstrated how Myopic and hyperopic shift can compensate/cancel each other. It is even possible to compensate observation range shifts from screen distance changes by defocus and vice-versa.

6.1 The Camera Setup

The camera (referring to the combination of camera body paired with a lens) used was a Fujifilm X-S10[4] camera body paired with FUJINON XC35mmF2[5] prime lens.

The light rays from an object first encounter the optical elements inside the camera lens and then go on to meet the image sensor just like the ILS. This can be easily verified by simply positioning a converging lens in front of a bare camera sensor and checking for image formation in the viewfinder if **shoot without lens** mode is present and enabled.

The similarities between image-sensor of a Camera body and the imaging screen for ILS is obvious as shown in Figure 15 below.

Figure 15: The camera body showing the exposed image sensor

Figure 16: The lens showing formation of an inverted image

The camera lens also behaves like an ideal converging lens as shown in Figure 16. The lens comprises of multiple optical elements – 9 elements in 6 groups (including two aspheric elements) to be exact. But for all our experimental purposes, it still behaves close to ideal. The additional optical *elements* are required to adjust focus and minimize aberrations in the image. The larger aperture size of the lens of a dedicated camera permits much shallower Depth-of-Field (DOF) for verification of focus.

The Camera (referring to the combination of camera body paired with a lens) can image close objects roughly 31.5 cm from the lens (if we subtract 35 mm focal length from the specified close focusing distance of 35 cm from the image-sensor plane, all the way up to infinity yielding an accommodation ability of ~3.25 D factoring in some inbuilt headroom for proper focusing at optical infinity (Figure 17).

4 https://fujifilm-x.com/en-in/products/cameras/x-s10/
5 https://fujifilm-x.com/en-in/products/lenses/xc35mmf2/specifications/

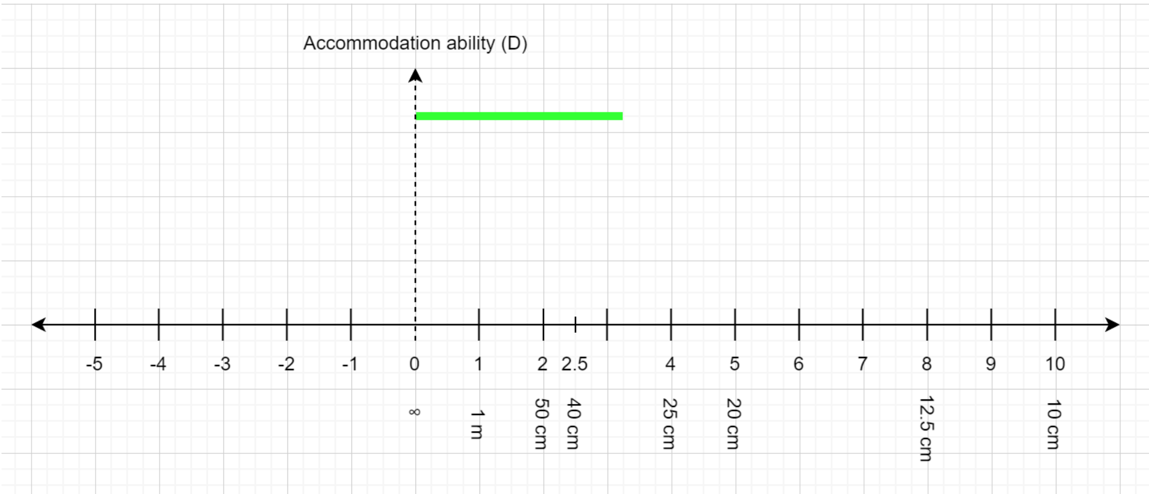

Figure 17: Observation range of XC35mmF2 lens

6.2 Description of the experimental setup

For the experiment, the camera was positioned outside to image a tree at optical infinity across a field (Figure 18).

Figure 18: Image showing the positions of camera and the object

The X-S10 camera body used for this experiment has a handy distance scale option which shows the focusing distance estimated from the position of optical elements as reported by the lens. Position of distance scale[6] with the white focus distance and blue DOF indicator[7] is shown in Figure 19 and 20.

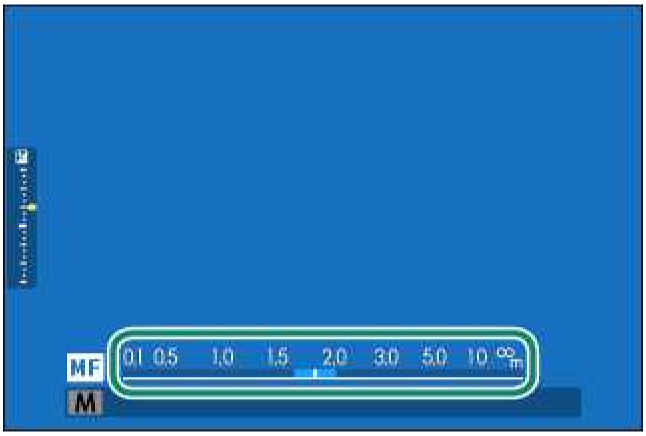

Figure 19: Position of distance scale in the viewfinder

Figure 20: Focus distance and DoF indicators

As can be verified from the white mark near 10m∞ on the Distance scale in the viewfinder (Figure 21), the optical elements inside the camera lens are correctly focused for our subject tree located at optical infinity in the absence of external defocus. Image formation parameters related to image-sensors like exposure or sensitivity is beyond the scope of this article.

6 https://fujifilm-dsc.com/en/manual/x-s10/taking_photo/manual-focus/
7 https://fujifilm-dsc.com/en/manual/x-s10/taking_photo/manual-focus/

Figure 21: Verification of object distance

6.3 Shift in observation range after hyperopic defocus

What happens if one places a diverging (minus) lens (introduce hyperopic defocus) closely in front of camera? In section 5.1, we have already outlined how observation range gets shifted after introduction of external defocus. For instance, introducing a -2.5 D defocus would shift the observation range 2.5 units towards left as shown in Figure 22.

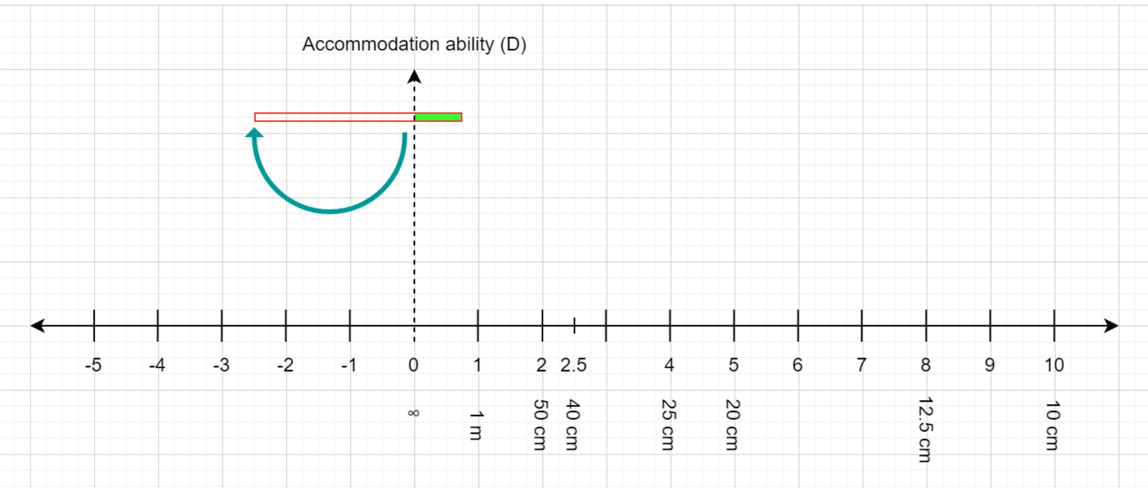

Figure 22: Shift in observation range after introduction of -2.5 D defocus

The hollow red part of the observation range to the left of origin indicates unusable part of the observation range because it is physically impossible to place an object beyond ∞. The remaining useful part of observation range lying closer than infinity is represented with a solid green line. Experimental demonstration and verification for the same using -2.5 D defocus is provided in Figure 23.

Figure 23: Verification of reported focus distance with -2.5 D defocus

The white focus distance indicator is near the 0.5 m mark on distance scale. Coincidentally, this is also the focal length (0.4 m) of the introduced defocus lens. Thus, focusing at an object located at infinity with the camera using hyperopic defocus of -2.5 D results in reported focusing distance of 40 cm at the focus of the introduced lens. The test subject situated at optical infinity is still in acceptable focus and the same can be verified from the blur that results from removing the introduced defocus as shown in Figure 24.

Figure 24: Blur resulting from removing the defocus after focusing for ∞

Due to the nature of the mapping introduced by -2.5 D defocus, we can state that the Camera is *apparently* focused at infinity when the lens optics are *actually* reporting focus around the 40 cm mark. A simple video clip demonstrating the above is supplied with this article.

In a way, we can say that the introduced -2.5 D defocus ‘eats’ away the usable portion of the effective observation range of the lens leaving only the remaining part of the observation range available for image formation. After introduction of defocus, the camera reports its focus distance as equal to the focal length of the diverging lens used even when focusing at infinity like before. This can be understood as the introduced lens ‘mapping’ object distances to their new apparent distances or alternately as the lens apparently ‘bringing’ objects from infinity at its focus (virtual object formation). This also translates to the inability of the camera to lock focus at objects closer than ~1.33 m with the -2.5 D defocus.

We have also repeated the experiment with a -3.0 D defocus so that the similar result can be compared as shown in Figure 25 below.

Figure 25: Verification of reported focus distance with -3.0 D defocus

This time, the image distance (~33 cm) is very case to the close-focusing limit of the lens at ~35 cm.

As one might’ve already observed, placing a lens in front of camera introduces some optical aberrations and slight crop to the image. But for our limited purposes, this has not affected our ability to verify the shift in observation range due to introduced defocus.

External defocus due to an introduced lens can’t alter the accommodation ability because the accommodation ability of a camera happens to be an internal physical constraint decided by how the optical elements are configured to move inside the lens.

6.4 Focusing on objects closer than ∞ under hyperopic defocus

The consequence of introducing a diverging lens in front of a camera for formation of image of objects at infinity has been previously demonstrated in previous sections.

In a way, we can say that the introduced -2.5 D defocus decreases usable effective observation range of the lens leaving only a subset of the observation range available for image formation. This translates as the inability of the camera to achieve focus at objects closer than ~1.33 m with the -2.5 D defocus shown earlier by the green part of observation range shown in Figure 22. This close-focusing distance recedes even further with the -3.0 D defocus rendering the Camera severely hyperopic.

7. Depth-of-field (DOF)

DOF diagrams are provided to scale for ease of visualization (Figure 26). 404

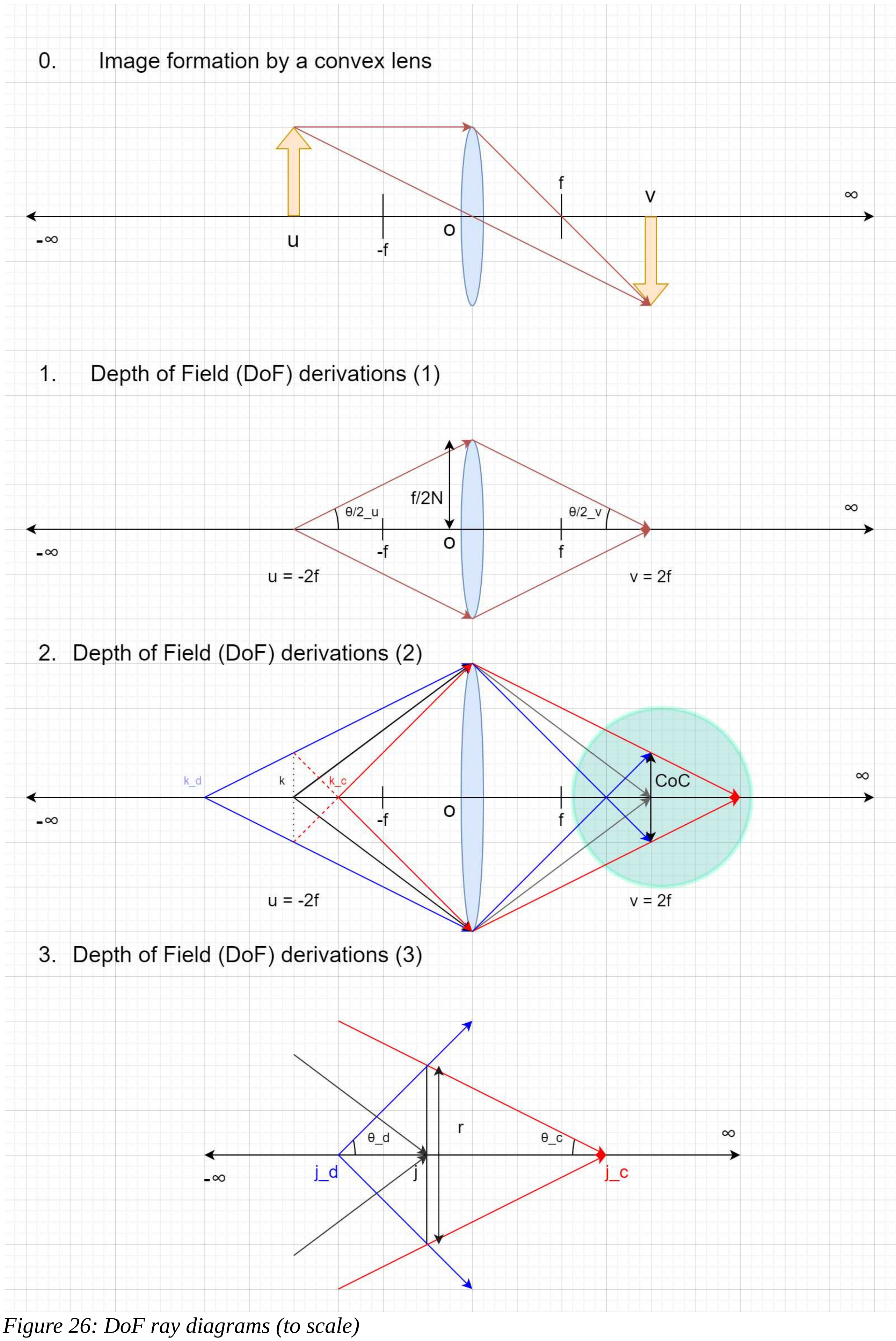

Figure 26: DoF ray diagrams (to scale)

$$\text{An object must be situated at } (-\infty, -f)$$
$$\text{for it to form an image at } (f,\ \infty) \text{ on the other side of the convex lens.}$$

$$\text{In common usage, aperture size } (A) \text{ is usually denoted as F} - \text{number or } \frac{f}{N}.$$
$$A = \frac{f}{N}, \text{ where N is some dimensionless number.}$$
$$\text{For instance, F2 or } \frac{f}{2} \text{ implies N} = 2 \text{ in both the cases}$$
$$\text{and it denotes that the aperture size / opening is half of the focal length.}$$

$$\text{Following this convention,}$$
$$\frac{f}{2N} \text{ denotes the half} - \text{width of aperture size / opening}$$
$$\text{or radius if the aperture is circular}$$

$$\text{we have accordingly}$$

$$\theta_u = 2\tan^{-1}\left(\frac{1}{2Nk}\right)$$
$$\theta_v = 2\tan^{-1}\left(\frac{1}{2Nj}\right)$$

$$\text{To derive D. O. F. we need to find inverse of the problem:}$$

$$\text{for some value of } j \text{ i.e., imaging distance,}$$
$$\text{find the range of } k \text{ given a particular value of N,}$$
$$\text{for which Circle of Confusion (C. o. C.) is lesser than a threshold value.}$$

$$\text{We can make C. O. C. dimensionless by dividing it by } f \text{ to get another quantity } r$$

$$r = \frac{\text{C. O. C.}}{f}$$

$$\text{thus obtained both extreme values of } u \text{ and } v \text{ obey}$$

$$u_d > u > u_c :: v_d < v < v_c$$

$$\text{where the subscript denotes (c)loser and (d)istant object / image respectively}$$

$$\text{This can be understood as a consequence of the hyperbolic relation between}$$
$$\text{object and its image.}$$

$$\text{These same relations must apply to } k \text{ and } j \text{ also}$$

$$k_d > k > k_c :: j_d < j < j_c$$

$$\textit{From consideration of the quadrilateral formed at image junction,}$$
$$\textit{we can see that}$$

$$(j - j_d)\tan\theta_d = (j_c - j)\tan\theta_c$$
$$\implies \frac{j - j_d}{2Nj_d} = \frac{j_c - j}{2Nj_c} = \frac{r}{2}$$

Three quantities equal to each other means that we will get 3C2 = *three equations.*

$$\textit{equating the first two parts, we get}:$$
$$\frac{j - j_d}{j_d} = \frac{j_c - j}{j_c}$$
$$\implies j_c \times j - j_c \times j_d = j_c \times jd - j \times j_d$$

$$\textit{rearrranging we get,}$$
$$j = \frac{2j_c \times j_d}{j_c + j_d}$$

This equation can be checked by putting j_c *and* j_d *equal to* j.

Equating the first and second terms separately with r, we get :

$$\frac{j - j_d}{2Nj_d} = \frac{r}{2}$$
$$\frac{j_c - j}{2Nj_c} = \frac{r}{2}$$

which gives,

$$j_c = \frac{j}{1 - Nr} \; \& \; j_d = \frac{j}{1 + Nr}$$

we can now invert these according to the relation we got earlier

which is $k = \frac{j}{j-1}$, *giving*

$$k_c = \frac{j_c}{j_c - 1} = \frac{\frac{j}{1-Nr}}{\frac{j}{1-Nr} - 1} = \frac{j}{j - 1 + Nr}$$

$$\text{and } k_d = \frac{j_d}{j_d - 1} \Longrightarrow \frac{\frac{j}{1+Nr}}{\frac{j}{1+Nr} - 1} = \frac{j}{j - 1 - Nr}$$

$$k_d - k_c = j \times \left(\frac{1}{j - 1 - Nr} - \frac{1}{j - 1 + Nr} \right)$$

$$\text{here, } k_d - k_f \text{ denotes } \frac{D.o.F.}{f}$$

$$\Longrightarrow D.o.F. = f \times j \times \left(\frac{j - 1 + Nr - (j - 1 - Nr)}{(j - 1 - Nr) \times (j - 1 + Nr)} \right)$$

$$\Longrightarrow D.o.F. = \frac{2f \times j \times Nr}{(j - 1 - Nr) \times (j - 1 + Nr)}$$

$$= \frac{2f \times j \times Nr}{(j-1)^2 - N^2 r^2}$$

How to use this formula for calculations :
firstly calculate j from k.

then putting the respective values of f, N, j & r
in the above formula will give you the D. o. F. in the unit of focal length.

Two special DOF scenarios are described below:

$$\textit{Nearest distance in acceptable focus when focus is at } \infty \left(j = 1,\ k \to \infty\right):$$

$$j_c = \frac{1}{1 - Nr}$$

$$\textit{which gives } k_c = \frac{1}{Nr}$$

$$\textit{the nearest distance of focus while still having objects at } \infty$$
$$\textit{in 'acceptable' focus } (j_d = 1):$$

$$\textit{giving } j = 1 + N \times r$$

$$k = \frac{j}{j-1} = \frac{1 + Nr}{Nr}$$

$$\textit{---FINISHED DOF DERIVATION PART---}$$

$$\textit{we already know that } j = \frac{2j_c \times j_d}{j_c + j_d}$$

$$\textit{substituting } j_x = \frac{k_x}{k_x - 1} \textit{ we get,}$$

$$j = 2 \times \frac{\frac{k_c}{k_c-1} \times \frac{k_d}{k_d-1}}{\frac{k_c}{k_c-1} + \frac{k_d}{k_d-1}} = 2 \times \frac{\frac{k_c}{k_c-1} \times \frac{k_d}{k_d-1}}{\frac{k_c}{k_c-1} + \frac{k_d}{k_d-1}}$$

$$\textit{giving } j = \frac{2k_c \times k_d}{2 \times k_c \times k_d - (k_c + k_d)}$$

$$\textit{finding the same relationship for k results in}$$

$$k = \frac{j}{j-1} = \frac{\frac{2k_c \times k_d}{2 \times k_c \times k_d - (k_c + k_d)}}{\frac{2k_c \times k_d}{2 \times k_c \times k_d - (k_c + k_d)} - 1} = \frac{2k_c \times k_d}{k_c + k_d}$$

$$\textit{This result is due to the interchangable (mirror) nature of the relation.}$$

7.1 Experimental verification for depth-of-field calculations

We will demonstrate use of the above DOF formula for calculating DOF in real world scenarios.

For the X-S10 and XC35mmF2, the screen distance equivalent can be taken approximately 35mm or 3.5 cm equal to the reported focal length of the lens. For a prime lens, the screen distance remains mostly fixed and it is only the focal length of the lens that changes (otherwise there will be FOV breathing). Using this assumption, one can easily utilize the result from section 4.3 to calculate reduction in focal length for distances closer than infinity.

The 6240 x 4160 image sensor of the camera measures 23.5 x 15.6 mm (Width x Height). The pixel pitch comes out to be approximately 23.5 mm ÷ 6240 = ~ 0.0037 mm approximating the CoC. For a manual focus distance of 2.5 m with F-number of 16, the DOF calculations (Figure 28) of ~60cm front-to-back is in line with the value reported in the viewfinder (Figure 27).

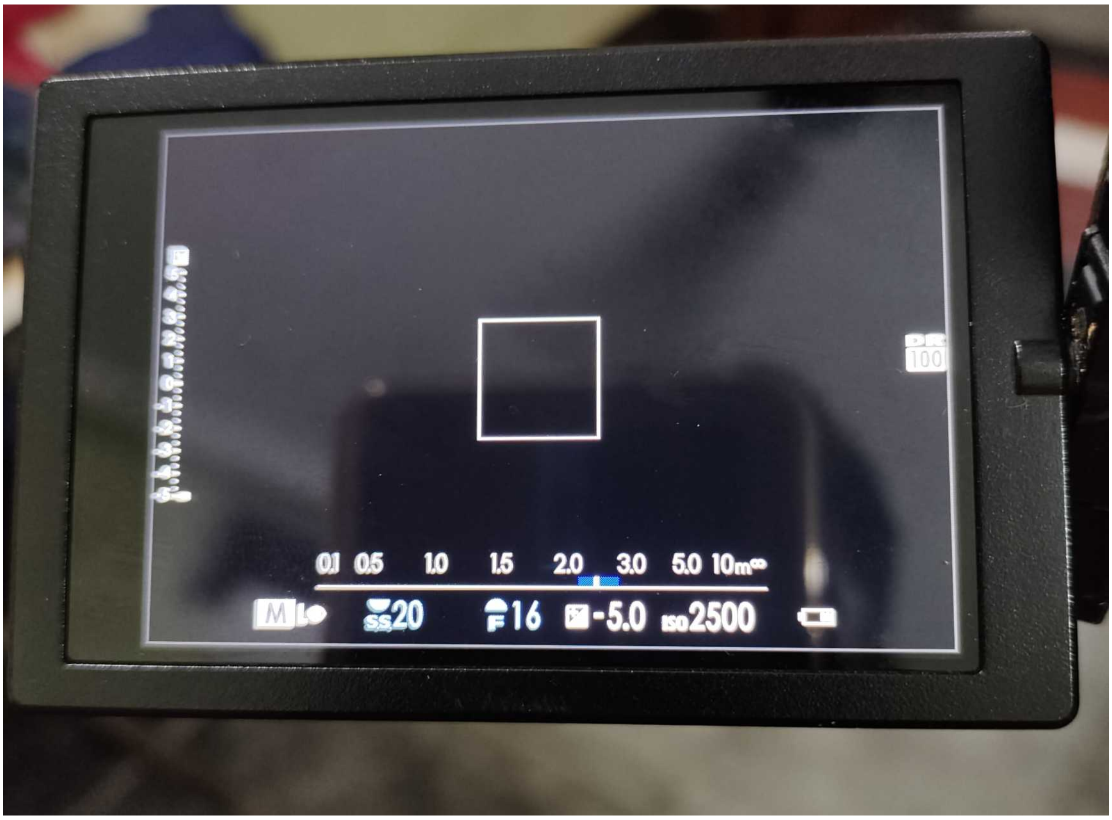

Figure 27: Viewfinder indicating DOF of ~60 cm

$Ac := 0.4 \quad d := \frac{100}{Ac} \quad d = 250.0$ cm

Power in Dioptre and distance in cm

$u := d \quad u := u$ put u for above value

$screen := 3.5$

$$f := \frac{100}{\frac{100}{screen} + Ac} \quad f = 3.452$$

Replace d by object distance you want

$$screen := f \cdot \frac{-u}{f - u} \quad screen = 3.5$$

$$k := \frac{u}{f} \quad k = 72.43$$

$k := k \quad k > 1$, put k for above value

$$j := \frac{k}{k-1} \quad j = 1.014$$

$sum := k + j \quad sum = 73.44$

$$magnification := \frac{-j}{k} \quad magnification = -0.014$$

DOF derivation part

$N := 16 \quad Aperture := \frac{f}{N} \quad Aperture = 0.2157$ cm

Object and image angles subtended

$$theta\_u := \frac{360}{pi} \cdot atan\left(\frac{1}{2 \cdot N \cdot k}\right) \quad theta\_u = 0.04944$$

$$theta\_v := \frac{360}{pi} \cdot atan\left(\frac{1}{2 \cdot N \cdot j}\right) \quad theta\_v = 3.53$$

Radius of CoC

$CoC := \frac{2.35}{6240}$ cm $\quad r := \frac{CoC}{f} \quad r = 1.091E\text{-}4$

k and j close and distant

$$j\_c := \frac{j}{1 - N \cdot r} \quad j\_c = 1.016 \quad j\_c \cdot f = 3.506$$

$$k\_c := \frac{j}{j - 1 + N \cdot r} \quad k\_c = 64.4 \quad k\_c \cdot f = 222.3$$

$$j\_d := \frac{j}{1 + N \cdot r} \quad j\_d = 1.012 \quad j\_d \cdot f = 3.494$$

$$k\_d := \frac{j}{j - 1 - N \cdot r} \quad k\_d = 82.75 \quad k\_d \cdot f = 285.6$$

DOF

$k\_d - k\_c = 18.35$

$DoF := f \cdot (k\_d - k\_c) \quad DoF = 63.33$ cm

$DoF\_alt := \frac{2 \cdot f \cdot j \cdot N \cdot r}{(j-1)^2 - (N \cdot r)^2} \quad DoF\_alt = 63.33$ cm

Nearest distance under DOF keeping focus at ∞ (k_c when j = 1)

$\frac{f}{N \cdot r} = 1977.0$ cm

Nearest focus distance keeping ∞ under DOF (k when j_d = 1)

$\frac{f \cdot (1 + N \cdot r)}{N \cdot r} = 1981.0$ cm

Figure 28: DOF worksheet calculations