Ray Optics For Characterizing Refractive State Of Optical Systems & A Novel Dimensionless Approach For Depth Of Field Calculations

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1. Introduction

This article lays out the fundamental concepts of ray optics and serves as a complete stand-alone reference text for anyone wanting to characterize refractive changes in observation range of optical systems (including eyes) due to defocus (also termed accommodation shift) or imaging-screen changes (also termed axial changes). A novel derivation of depth-of-field relations has also been provided for the enthusiastic reader.

Many eye-specialists/eye-care professionals lack intuitive understanding about what is actually happening due to axial elongation causing an eye to become Myopic. This frequently results in wrong characterization of refractive condition and consequently improper understanding of how prescriptions ~~correct~~ compensate for refractive eye conditions. The material in this article is of enormous importance in that regard also.

The math worksheets for µMath+ Android application is supplied along with this text containing a workable version of the formulas.

2. Ray Optics: Deriving the ideal thin-lens relation

This section focuses on deriving the relationship between object and image distances for the case of (real) image formation only by an ideal converging (also termed convex) thin-lens giving the ideal thin-lens relation. It is trivial to derive the same and the relation remains identical for a concave lens as well but will not be attempted here. The 2D illustrations and ray diagrams of this article extend and apply equally well to the 3-Dimensional real world.

An ideal lens itself can be completely defined with only **two** physical parameters/attributes along with its principal axis – **focal length** and **aperture**. The term focal length or its inverse as lens power always implies the same physical property. These two parameters are sufficient to explain the majority of image formation by actual lenses and their observed behaviour in experiments.

In this sense, an ideal lens is essentially a theoretical object and we needn't immediately concern ourselves with the implementation details of how it achieves 'image formation' and 'focusing' of rays in reality. It also means that these rays in question can be anything – light, radio waves, sound waves etc. reinforcing the purely theoretical nature of the relation. Ideal lenses are interesting objects with properties of their own from a purely theoretical perspective.

A lens with curvature only along one axis is known as a Cylindrical (SPH) lens while a lens with identical curvature along both independent axes is denoted as Spherical (SPH) lens[1]. It must be noted that words *cylindrical* and *spherical* were meant with regard to appearance of the lens and have little to do with the actual refracting profile (cross-section) of the lens. The axis of a cylinder lies along the direction of constant curvature with the idea being rotation about its axis should be indistinguishable (due to symmetry). This is important from the point of unique determination of cylindrical axis as shown in Figure 2.1 below.

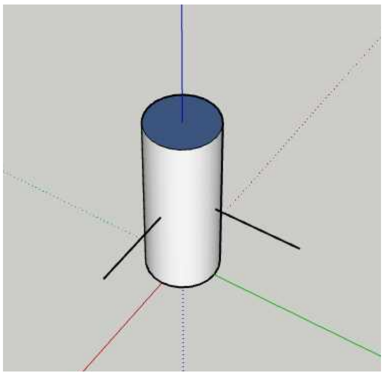

Figure 2.1: The axis (blue line) of a cylinder will always be normal to its flat face

2.1 Definition of an Object and its Image in Ray Optics

An object is considered to be a *collection of object points*. From every object point, rays can be assumed to **originate/diverge** in all directions. An image is similarly considered to be a *collection of image points*. At each image point, rays originating from an object point **terminate/converge**.

For an image point, demonstrating convergent intersection of any two rays emerging from an object point is sufficient. A point on an object and its image will always have a one-to-one correspondence. It is very important to note that beyond the image point, away from its object – the rays that converged to form the image point start diverging again. An object can be differentiated from its image in the sense of independent existence. An object can exist independently but an image can not exist without an object. It is immensely useful to express Aperture size, object, and image distances in terms of focal length. This dimensionless representation comes at no loss of generality as can be seen from the later sections.

1 https://www.clzoptics.com/news/cylindrical-vs-spherical-lenses.html

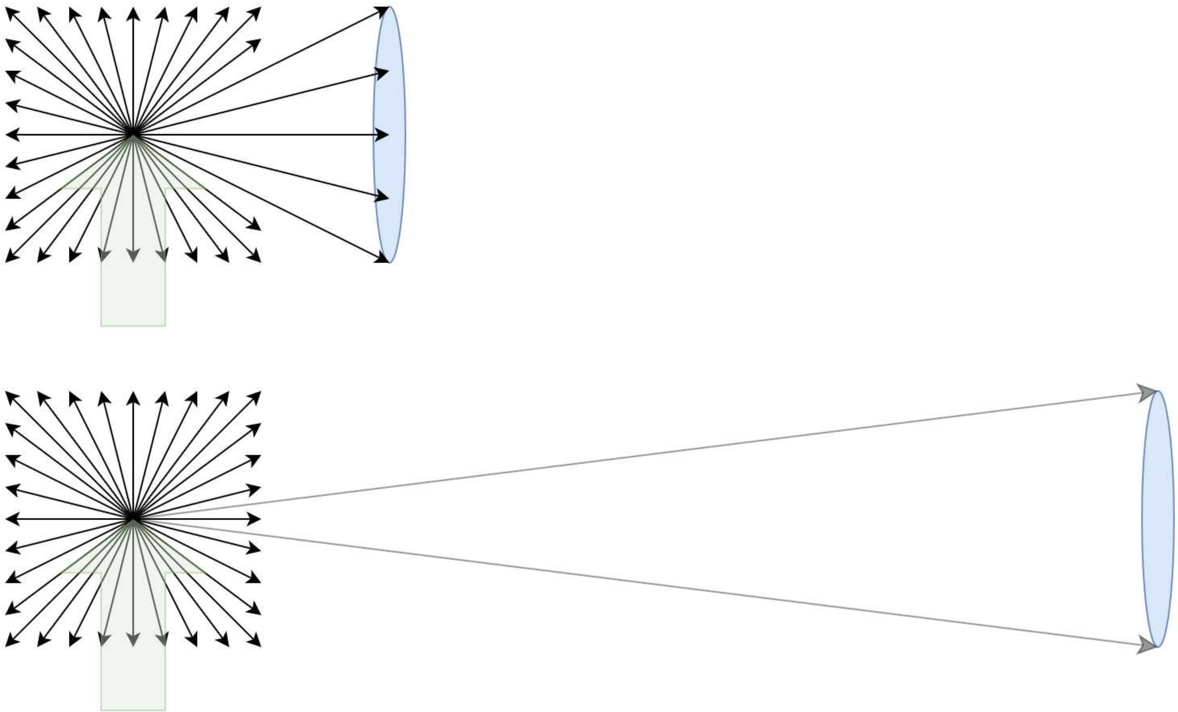

Figure 2.2: Ray diagram of rays hitting the same lens from an object point from far and near.

An object point (on an object) emits rays in all directions (Figure 2.2). At distances become larger, the angle between rays emitted by a point object starts decreasing with the rays themselves appearing almost parallel at very large distances from the lens. However, even at infinity, the rays emitted by an object can only appear diverging or parallel and never converge without help from a lens.

An application of our stricter definition is to consider the commonly mentioned *'virtual images[2]'* in many textbooks as having properties more in line with objects rather than images because rays seem to originate (diverge) from them towards the lens. That a 'virtual' image lies on the same side of the lens as a 'real' object favours consistency of our argument. For analysis of systems consisting of multiple optical elements, the '*virtual image*' formed by an object and lens combined can be replaced solely by the virtual object without affecting the rest of the system. Subsequent imaging of such a virtual object by an optical system (whether our eyes or a camera) then results in an actual real image. Henceforth, the word image in this article should always be taken as referring to a real image.

2.2 Calculating deviation for a ray incident on an ideal lens

Assumption A: The focal length determines the convergence/divergence point (focus) of every ray parallel to the principle axis for a converging/diverging lens respectively shown in Part 1 of Figure 2.3. This is analogues to a parabolic concave mirror reflecting all incoming rays parallel to its principal axis on its focus. The Aperture size only determines the admission extent for such parallel rays on the lens.

Assumption B: Only rays passing parallel to the principle axis obey the previous assumption. All rays incident normal to the principal axis pass unchanged through the lens profile. The word 'thin' can be used to

2 https://en.wikipedia.org/wiki/Virtual_image

denote this aforementioned property of not affecting rays normal to the principle axis of the lens. There is the implicit assumption that a ray can be first decomposed into its normal and parallel constituents. Ray diagrams given below demonstrate application of the above two assumptions to show deviations for various incident rays by decomposing them into normal and parallel components with respect to the principal axis.

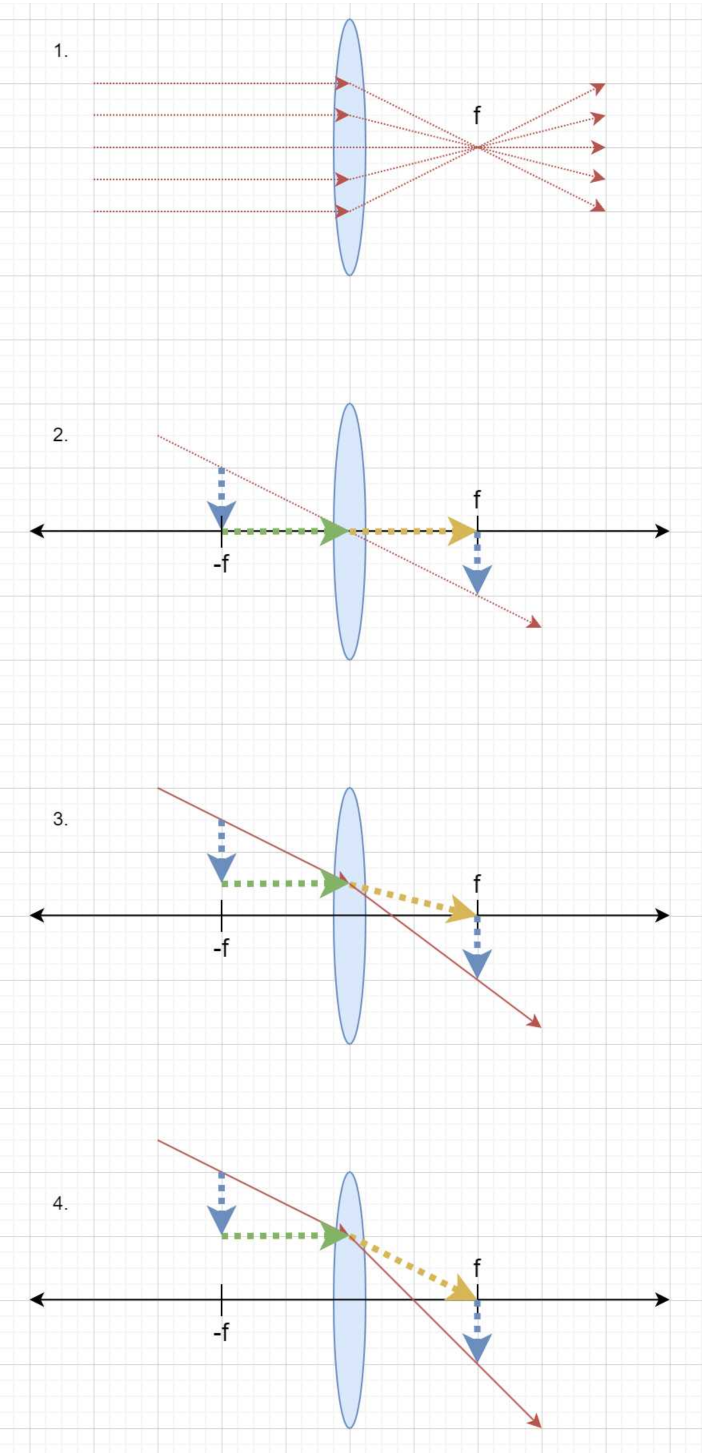

Figure 2.3: Application of Assumption A (Part 1) and B (Part 2 to 4)

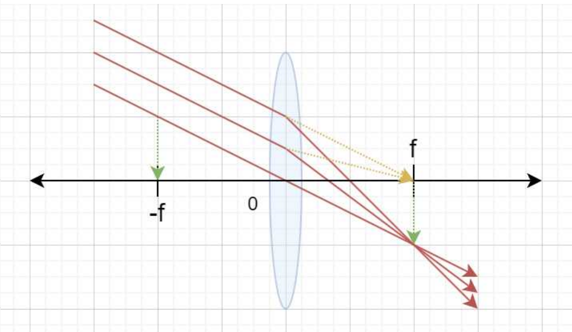

Figure 2.4: Combined version of 2, 3, and 4 in Figure 2.3

From Part 2 to 4 shown in Figure 2.3 and their combined version in Figure 2.4, we can observe that the length of the vertical component is the same when calculated at the focus. The amount of deviation a ray undergoes depends on both where it is hitting the lens Aperture and the angle it makes.

Consideration of these two assumptions results in simplified rules for all rays incident on an ideal converging (+) lens and is sufficient to explain image formation by ideal lenses. A verification for an edge case corresponding to image formation at 2f is provided below in Figure 2.5.

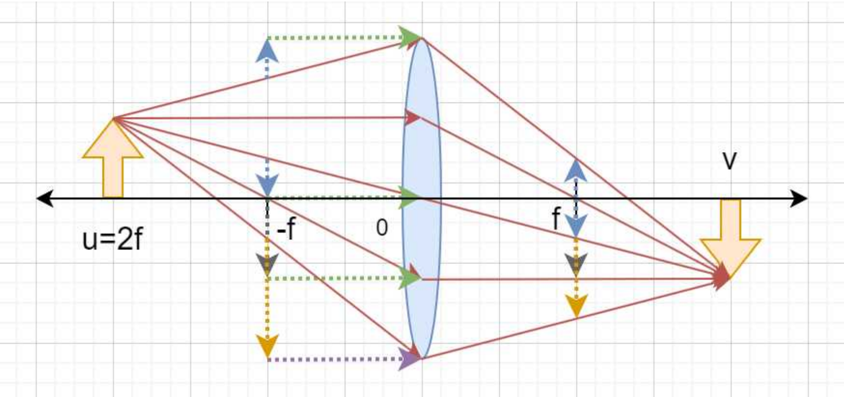

Figure 2.5: Edge case shown for image formation when u=2f

In this particular edge case, we've shown convergence of five such rays at a single point when demonstrating that only two meet at the image point would have been sufficient.

The rule regarding rays passing unchanged through the optical centre of the lens (as given in many intermediate textbooks[3]) can be explained as a consequence of fundamental assumptions A and B stated above. It determines the magnification ratio because the image point must lie on the unaffected ray passing through the optical centre of the lens.

2.3 The relation between object and image distance

For a converging lens with a given focal length, two distinct cases of rays intersection are possible according to the assumptions outlined in the previous section 2.2. Both the cases (Part 1 and 2) are shown in Figure 2.6 below with the same object placed at varying distances with the limiting case (Part 3) of an object placed exactly at the focus.

1. **Image** formation when the object is at a distance farther than the focal length from the lens. Rays from the object point appear to be converging towards the image point. It is important to note beforehand that the image formed in this case is **inverted**.

2. **Virtual** ***object*** when the object is at a distance closer than the focal length from the lens. Rays appear to be coming from the virtual object.

3 Concepts of Physics - Part 1, For Intermediate Level, Publisher: Bharati Bhawan; Revised Reprint 2015 edition, ISBN-13: 978-8177091878

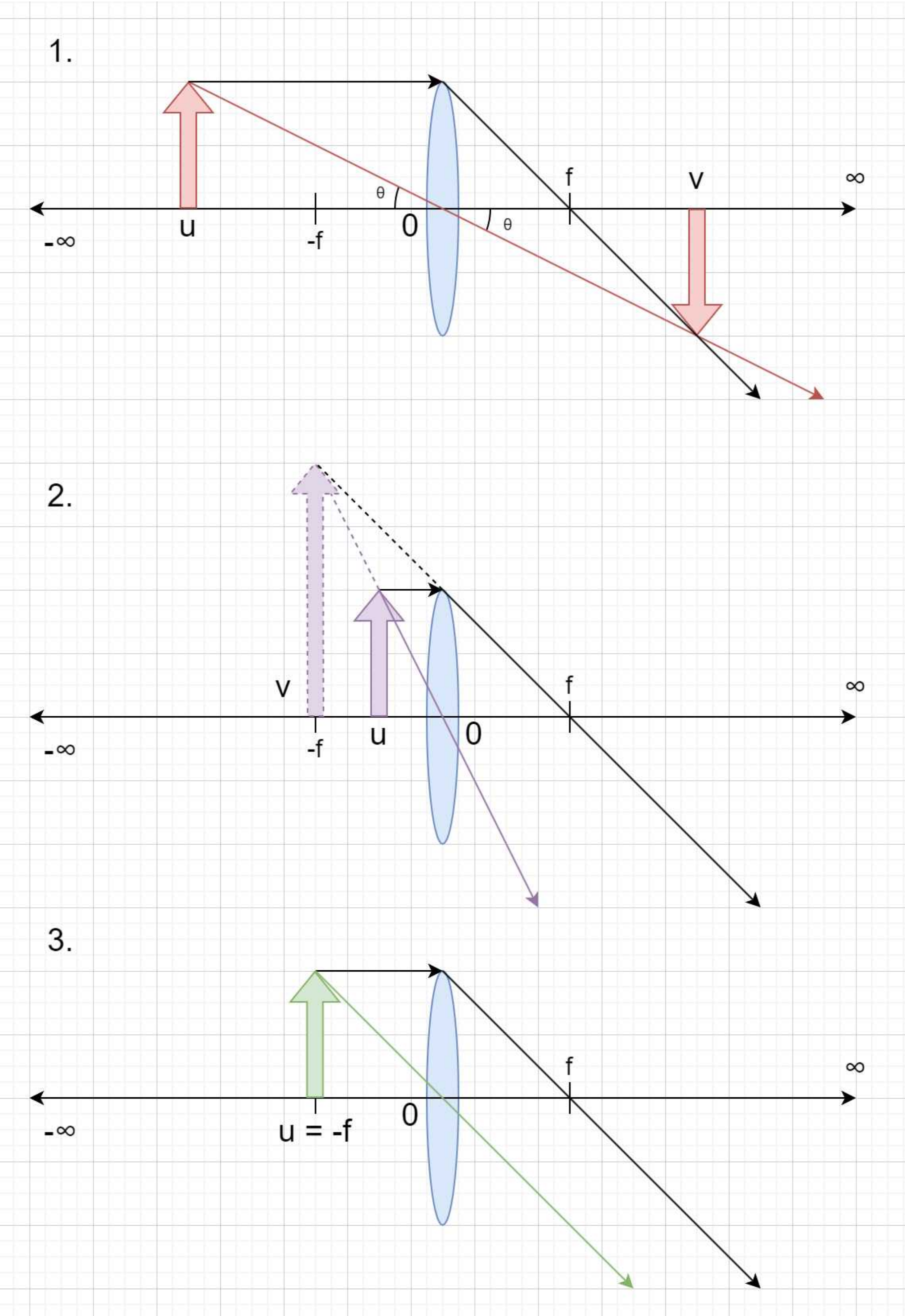

Figure 2.6: Formation of Image and Virtual Object by a converging lens

The ray passing through focus has the same slope *-h/f* (where *h* is the constant height of object point) while slope of the ray passing through the optical centre depends on the object distance in all three cases. Whether the slope of the latter is greater than or lesser than the former passing through focus determines where these two rays meet and whether an image is formed or not. Precisely, ray from an object point placed farther from the focus and passing through the optical centre will always have a less negative slope.

The derivation of ideal lens relation for a converging lens has been given below (Figure 2.7) using dimensionless representation.

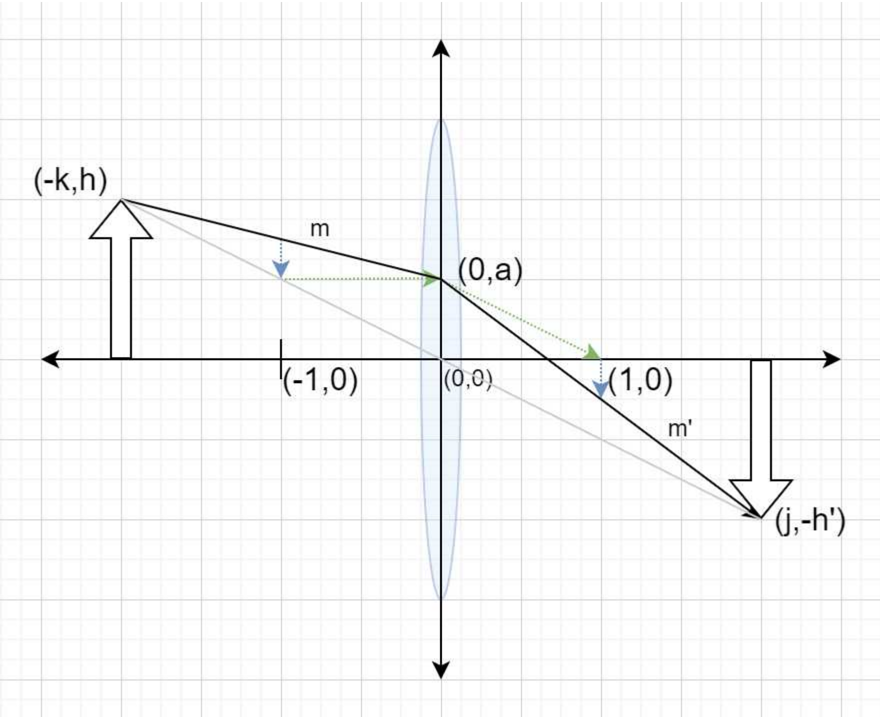

Figure 2.7: Ray Diagram for dimensionless derivation of relation between image and object distance

For calculation of image distance, we assume the origin of the right-handed co-ordinate system centred on the lens. Care must be taken to ensure consistency of units and signs according to conventions while utilizing formulas throughout this article to ensure correctness in calculations. An ideal lens has two foci due to its symmetry. For a converging (+) lens, the focus will always lie on the opposite side of the lens from object. It depends on the signed power (converging or diverging) of lens that determines the sign of focal length and consequently its position relative to the object. For this article, we have only shown images formed by a converging lens.

$$magnification(m)\,is\,defined\,as\,\frac{image\,height}{object\,height}=\frac{-h'}{h}=\frac{image\,distance(v)}{object\,distance(u)}=-\frac{j}{k}$$

$$\text{We have normalized object and image distances as follows:}$$
$$u = -kf \text{ and } v = jf$$

$$\text{for image formation, } k > 1 \text{ which means } -u > f \text{ or } u < -f$$
$$\text{slope } m \text{ when the object point is } (-k, h) \text{ is given by}$$
$$m = \frac{a-h}{k}$$
$$\text{for point } (0, a) \text{ on the lens profile } (y-axis)$$
$$\text{image formation requires all rays to pass the point } (j, -h') \text{ after refraction}$$
$$\text{The slope after refraction } (m') \text{ is given by,}$$
$$m' = \frac{a-(-h')}{-j} = \frac{a+h'}{-j}$$
$$\text{The slope after refraction according to Assumption 2 must be}$$
$$m' = m - a \Longrightarrow \frac{a+h'}{-j} = \frac{a-h}{k} - a$$
$$\text{Now, it's already known from the magnification criteria that}$$
$$\frac{k}{j} = \frac{h}{h'} \Longrightarrow h' = \frac{hj}{k}$$
$$\Longrightarrow \frac{a+\frac{hj}{k}}{-j} = \frac{a-h}{k} - a$$

$$\Longrightarrow \frac{ka+hj}{k} = \frac{hj-aj+kaj}{k}$$
$$\Longrightarrow ka = kaj - aj$$
$$\text{giving, } k + j = kj$$
$$\text{which completes the derivation of } k \text{ and } j \text{ relation.}$$

$$\text{To derive the ideal thin lens relation,}$$
$$\text{we can substitute } k = -\frac{u}{f} \ \& \ j = \frac{v}{f} \text{ giving}$$
$$-\frac{u}{f} + \frac{v}{f} = -\frac{vu}{f \times f}$$

$$\Longrightarrow -u + v = -\frac{vu}{f}$$
$$\Longrightarrow \frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

2.4 Dimensionless object (k) & image (j) distance relation

The relation between dimensionless object and image distance comes out to be $j = \frac{k}{k-1}$

For image formation by a converging lens, k > 1. This simply encodes the result that the object must always be located beyond the focus for image formation. A plot of this equation results in a hyperbola symmetrical around the line x = y (Figure 2.8). The vertical and horizontal asymptotes given by lines x = 1 & y = 1 correspond to the object at focus and ∞ while also corresponding to its image at ∞ and focus respectively.

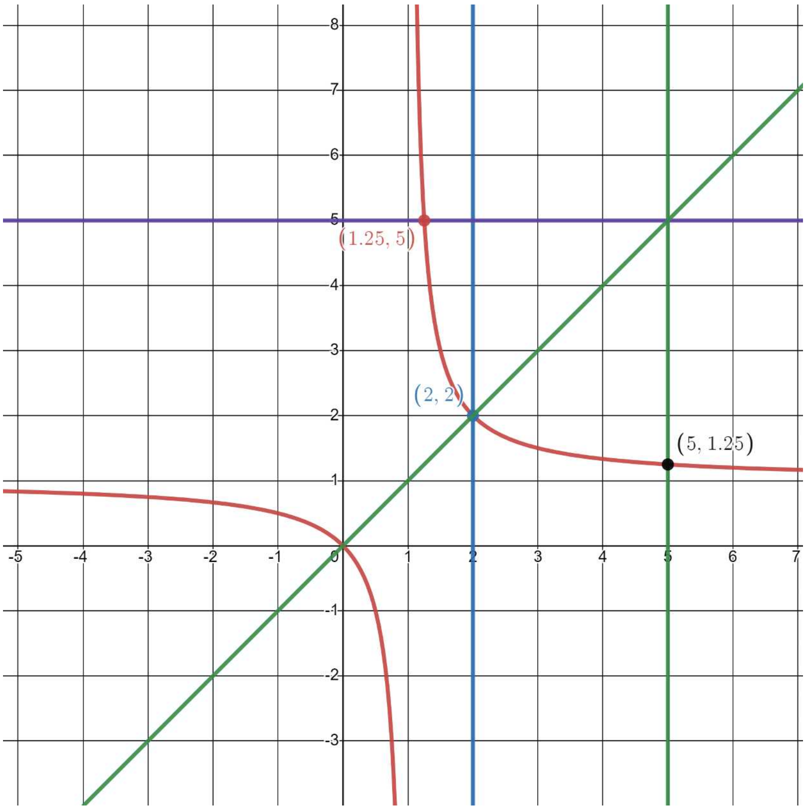

Figure 2.8: Red Hyperbolic plot representing the dimensionless lens relation

It is evident from the symmetry of the hyperbola in Figure 2.8 that both an object & its image's dimensionless distances form what can be termed as a mirror pair around the x = y line.

It becomes even more obvious if we solve for k giving $k = \frac{j}{j-1}$, which can be understood simply as interchanging k with j & j with k in the original relation.

Adding k and j together gives, $k + j = k + \frac{k}{k-1} = \frac{k^2}{k-1} = k \times j$ which has a minimum of 4 at k = j = 2 for positive values as can be seen from the graph below (Figure 2.9).

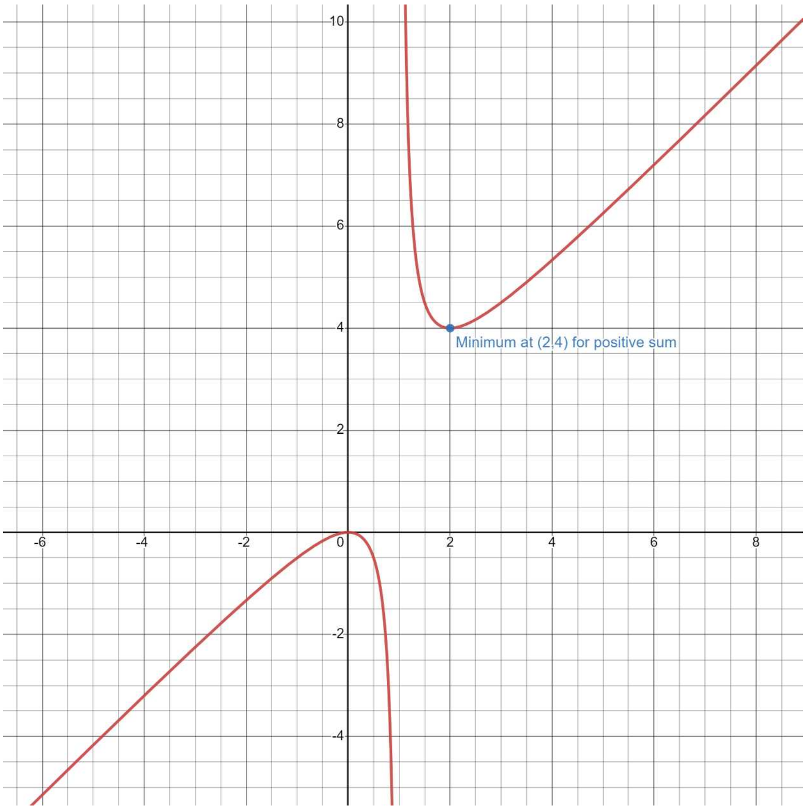

Figure 2.9: Plot showing variation in sum of dimensionless image and object distance with distance

This shows that for a given distance between an object and its required image, the focal length of the system able to image it can not be greater than the quarter of the given distance. The $(k + j) \geq 4$ rule along with the $k > 1$ rule serve as fundamental constraints for real world implementation of optical systems.

2.5 The power addition rule

The power (inverse of focal length) addition rule can easily be derived from the ideal lens relation by considering two lenses with different focal lengths placed together such that the distance between them can be neglected.

Here, v and u denote the image and object distances respectively with subscripts denoting which lens they are referring to. The power of a lens is the inverse of its focal length. The Dioptre (m^{-1}) is one such derived SI unit of power. Assuming focal length f_1 for Lens L_1 and f_2 for Lens L_2. Then the image distance for Lens L_1 is given by, $v_1 = \dfrac{u}{1+\dfrac{u}{f_1}}$

Because the distance between the two lenses is negligible, this resultant image now gets further refracted as an object by lens L_2. The corresponding final image formation distance is given by (where $u_2 = v_1$).

$$v_2=\frac{v_1}{1+\frac{v_1}{f_2}}\Rightarrow v_2=\frac{\frac{u}{1+\frac{u}{f_1}}}{1+\frac{\frac{u}{f_2}}{1+\frac{u}{f_1}}}=\frac{u}{1+\frac{u}{f_1}+\frac{u}{f_2}}=\frac{u}{1+u(\frac{1}{f_1}+\frac{1}{f_2})}$$

This implies that the combined lenses together act as a lens of focal length $\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f}=P_1+P_2$ which is basically the law of addition of lens powers. For a combination of more than two lenses closely put together, we can proceed by combining two lenses together at a time until only one remains.

3. The Simple Lens setup

Be it Human eye or an optical system like a camera, the purpose is mostly the same - 'imaging' objects (which can be located at varying distances) onto the retina/image sensor. For a film camera, the film acts as the image sensor. For a digital camera, an image sensor chip replaces the film. The ray optics involved behind image formation remain unchanged. The term 'image' should be taken to mean real image and all mentions of the word 'lens' refer to an ideal thin-lens.

We will start with an optical bench setup consisting of a converging lens and an image sensor (planar screen) as shown in Figure 3.1. The screen is aligned perpendicular to the principle axis of the lens. For the simple lens system, tracing the path of rays from an object to its eventual image is very simple. Rays emanating from an object point encounter the lens aperture and hit the image sensor after refraction. The medium in this case can be said to be vacuum/air.

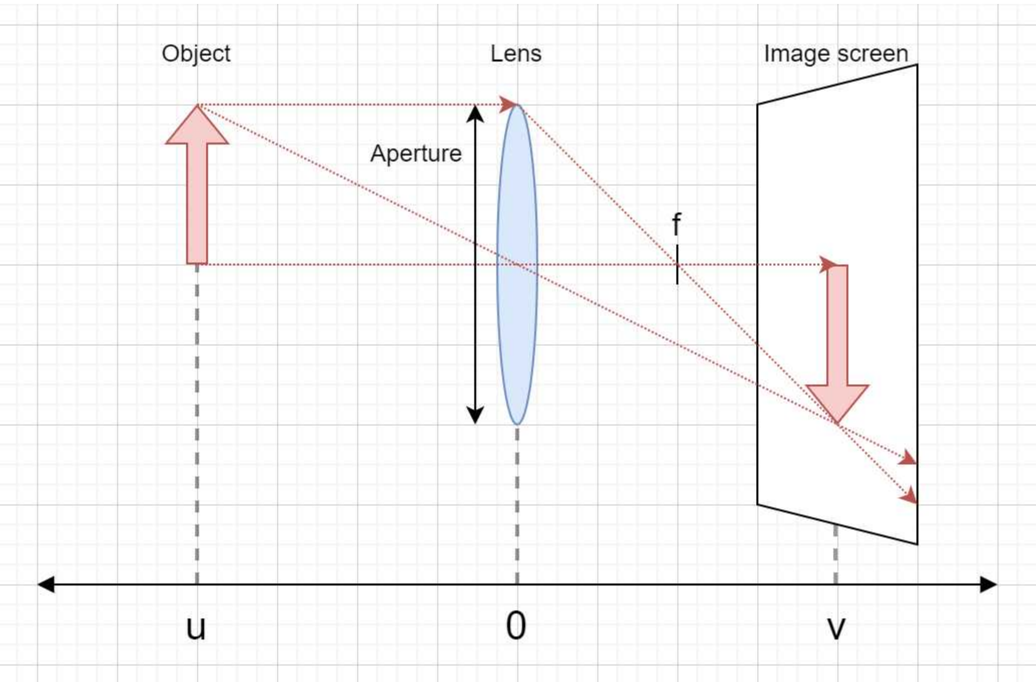

Figure 3.1: Optical bench diagram of the simple lens setup

Four parameters are necessary to completely describe the setup of an object to be imaged and our simple lens system, if one also includes Aperture size (having no impact on image distance). The remaining three parameters are object distance (u) with the focal length (f) {together determining the image distance (v) – a dependent variable} and the screen distance (s). Distances are measured using Optical Centre as the origin.

For image formation, image distance (v) must coincide with screen distance (s). Image formation in this context refers to the formation of a well focused sharp image and will be the cases for this section and sub-sections. The procedure for calculating distances within which screen distance (s) can vary for a given image distance (v) while still forming appreciably sharp images within a CoC (Circle of Confusion) is given in Section 6 of the Appendix accompanying this article.

Assuming the above, we can state that the dependence of screen distance (s) on focal length (f) and object distance (u) is already encoded by the ideal lens relation. Knowing any two, the third is easily calculated.

$$s=\frac{fu}{f+u}=\frac{u}{1+\frac{u}{f}}=\frac{1}{\frac{1}{u}+\frac{1}{f}}$$

The **Field of View (FoV)** is defined as the angular-expanse of objects whose images can be formed on the image sensor/screen. It is easily quantified from the ratio of screen height to the screen distance.

3.1 The concept of constrained observation range

Fixing screen distance and focal length together also fixes the distance at which an object can be imaged. It is desirable for an optical system to be able to form images of objects located within a distance range instead of fixed distance. Imaging objects can be achieved by varying either the focal length (f) or the screen distance (s) or even some combination of both.

It must be kept in mind that all real-world implementations will always have some physical constraints on the 'variability' of independent variables (focal length and screen distance in our case) constraining their allowed values. The range of allowed object distances that can be imaged by such a system within constraints will be referred to as its ***observation range*** throughout this article. The farthest and nearest distance extremes of the observation range are commonly designated as the far-point (d_{far}) and near-point (d_{near}) for a system respectively. The Far-Point is defined as the distance of ***far***thest objects a system is able to image while the Near-Point is defined in a similar manner as the distance of ***near***est objects a system is able to image within its constraints.

Our approach for studying changes to a system's observation range results from fixing one variable and studying how variation of the other affects the range of object distances (d_{near} from d_{far}) that can be imaged. The two cases for our system resulting from fixing focal length (f) and screen distance (s) one at a time are discussed below.

3.2 Observation range of system when imaging screen distance (s) is fixed

Showing that for such a system,

the increase in power/decrease in focal length (f) for maintaining ***fixed screen distance (s)*** *solely depends on the object distance (u)*

can be easily done by rearranging the ideal lens law in terms of focal length.

$$By\,the\,ideal\,lens\,relation\,, \frac{1}{f}=\frac{1}{s}-\frac{1}{u}\,, s\,is\,constant$$

$$For\,an\,object\,at\,\infty\,, u\rightarrow\infty \Rightarrow \frac{1}{f_{\infty}}=\frac{1}{s}-\frac{1}{u}=\frac{1}{s}$$

$$For\,an\,object\,closer\,than\,\infty\,, \frac{1}{f}=\frac{1}{f_{\infty}}-\frac{1}{u}$$

$$\Rightarrow \frac{1}{f}-\frac{1}{f_{\infty}}=\Delta\,Power=-\frac{1}{u}$$

$$The\,signed\,term-\frac{1}{u}\,is\,positive\,because\,object\,distance\,is\,negative\,by\,co-ordinate\,convention\,.$$

For such a system, the increase in lens power (change in focal length) needed to image objects **closer** than a reference distance depends only on the object distance (u). This lens power increment for imaging closer objects is commonly referred to as *accommodation* of the system with the reference ‘un-accommodated’ distance taken at infinity. The accommodation required from far to near point is commonly referred to as a particular system’s accommodation ability. A system is stated to be fully accommodated once lens power reaches constrained extreme corresponding to observing objects at near-point.

Fixing screen distance requires the focal length to vary in accordance with the observed object distances. For instance, a system with a fixed screen distance of 25 cm and lens power constrained in the range +4 D to +9 D will have its Far and Near-points at ∞ and 20 cm respectively.

With lens power at +9 D which amounts to +5 D accommodation from the initial +4 D needed for objects at ∞, the system images objects located at 20 cm. Because screen has been kept fixed for this system, the Field-of-View (FOV) remains unchanged.

3.3 Observation range of the system when the focal length (f) is fixed

If the focal length is kept fixed and the screen distance is allowed to vary instead, the dependence of observation range on screen distance is given in a similar manner from the ideal lens relation:

$$¿\,ideal\,lens\,relation\,, \frac{1}{s}-\frac{1}{u}=\frac{1}{f}\,giving\,\frac{1}{s}=\frac{1}{f}+\frac{1}{u}$$

$$For\,an\,object\,at\,\infty\,, u\rightarrow\infty$$

$$giving\,\frac{1}{s_{\infty}}=\frac{1}{f}$$

$$For\,an\,object\,closer\,than\,\infty\,, \frac{1}{s}=\frac{1}{s_{\infty}}+\frac{1}{u}$$

$$The\,signed\,term\,(\frac{1}{u})\,is\,negative\,here\,.$$

Assuming a system like the one described in section 3.2 above but with a fixed focal length (f = 25 cm) instead, we can find out the near and far points of this system in a similar manner.

The k > 1 requirement arrived in section 2.4 mandates that screen distances lesser than the (fixed) focal length of the system can't result in image formation for any possible object distance. Thus, the screen distance required in this instance must be always greater than 25 cm and can only be increased up to infinity. The far and near point of the system is ∞ and 25 cm respectively.

It is evident that screen distances required for imaging closer objects quickly approach very large values and even then the system is unable to image objects closer than 25 cm because it is impossible to physically locate the screen beyond infinity. Contrast this to the variable focal length (f) system described earlier where no such physical limit was in place for focal length preventing us from observing closer objects.

Changes to the screen distance implies changes to the overall size of the system and thus changes to FOV.

4. Representing observation ranges on the Relative Dioptre Scale (RDS)

From the description of constrained observation ranges, the need for a tool to visualize optical systems and any subsequent modifications to them whether from defocus or screen distancing was felt. The Relative Dioptre Scale was devised for this exact purpose serving as an intuitive way to visualize change in observation ranges due to induced defocus or changes to the screen distance.

The RDS is simple Cartesian co-ordinate system modified for power (inverse instead of linear distances) onto which observation ranges for optical systems are represented in this article. The x-axis of the RDS represents distances in (D or m^{-1}) with the origin representing infinity (∞). The y-axis was chosen to represent the accommodation ability of the system (also in Dioptre). Thus systems with better accommodation ability are vertically ranked/placed higher-up on the Relative Dioptre Scale.

The idea of representing observation range of optical systems on an inverse length scale stems from the inverse nature of the ideal lens relation itself. This has the important implication of making transformations on the RDS linear with respect to changes in Power.

The ideal lens relation $\frac{1}{s} - \frac{1}{u} = \frac{1}{f}$ itself can then be written as Power of Object distance (in m^{-1}) = Lens Power (in D) – Power of Screen distance (in m^{-1}).

On the RDS, the left end of the system's observation range represents its far-point (the farthest a system can focus) while the right end represents its near-point. The observation ranges (d_{near} to d_{far}) of the systems described in sections 3.2 and 3.3 can be represented on the RDS (Figure 4.1).

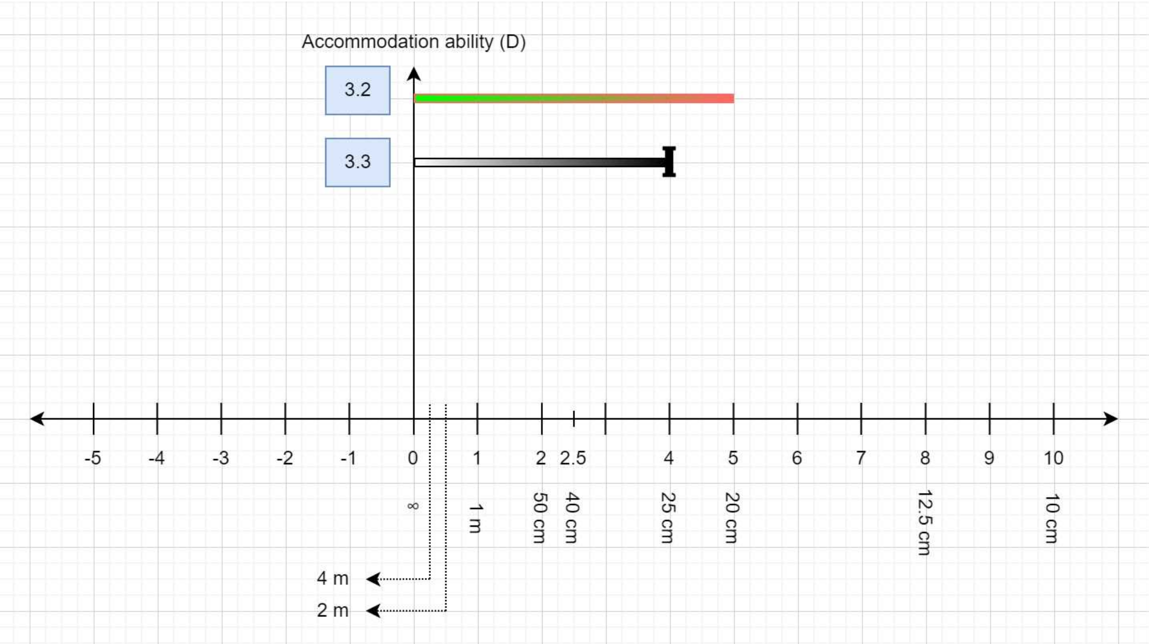

Figure 4.1 Comparative representation of the observation ranges of the system in section 3.2 and 3.3

The vertical rank/height represents accommodation ability of the systems which is also equal to the length of line segments. Accommodation ability of any system is equal to the power difference between the two extremes (near and far-point) of the observation range.

The gradient from red to green and light to dark in this instance was used to indicate reaching the end of the system's observation range due to constraints. The 'I' symbol was used to indicate the impossibility of any physical extension towards the right-hand direction (near-point) for the fixed focal length system described in section 3.3. The sign of object distance (d) on the RDS is inverted from the sign for object distance (u) used in ideal thin lens relation due to co-ordinate conventions and the same must be remembered.

4.1 Shift in observation range after defocus (Accommodation shift)

What happens to the system in 3.2 when accommodation ability is kept same (+5 D) but increased by +1 D on both extremes (+5 D to +10 D)? Such a change can be termed as 'accommodation shift' and the same can be easily achieved by introducing another +1 D defocus such that the power addition law holds (Section 2.5).

Just like before, it is sufficient to calculate the shifted far and near-points of the system (which are now 1 m and 16.66 cm respectively). The modified system is represented on the RDS (Figure 4.2). The original system is shown using slightly thicker grey for ease of comparison.

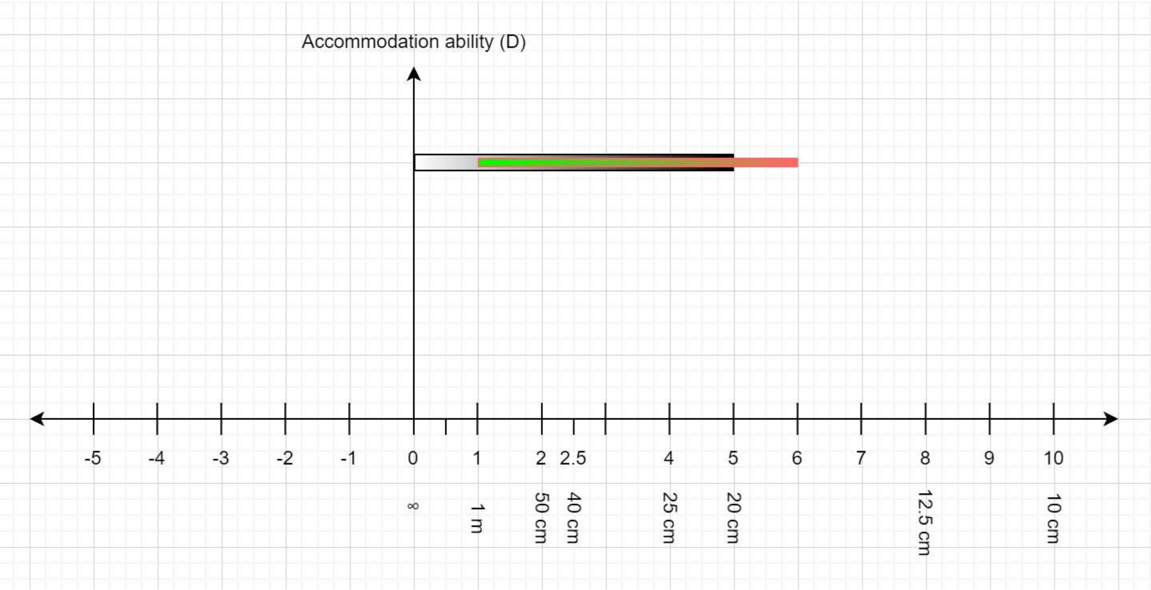

Figure 4.2: Shift in observation range after introduction of +1 D defocus

The modified system's observation range with Lens Power = +5 D to +10 D gets shifted 1 unit towards the positive/right direction (the sign of introduced lens).

It is evident that an increment in both extremes of the lens power range results in right-shifting of the system by the same amount on RDS. Our modifications to system in 3.2 has rendered the system Myopic (unable to image objects farther than 1 m).

The inverse also holds. It is trivial to point out that introducing a -1 D accommodation shift in our newly modified system restores the original accommodation range. This is equivalent to introducing a -1 D lens or removing the previously introduced +1 D lens. Which means that the observation range on RDS must also shift one unit towards left. This is what the word 'Relative' in the RDS stands for. Relative to our modified system, the system in 3.2 can be said to be hyperopic (unable to image objects closer than 20 cm).

It should be now easy to understand why *introducing converging (positive) lenses/removing diverging (negative) lenses* to a system is termed as ***Myopic defocus*** – because it renders the system myopic (shifts the observation range towards right).
Similarly, *removing converging (positive) lenses/introducing diverging (negative) lenses* to a system is called ***Hyperopic defocus*** – because it renders the system hyperopic (shifts the observation range toward left).

4.2 Shift in observation range after changes to screen distance (Axial changes)

We've previously described shifting of a system's observation range under induced defocus. Calculating shift in Observation range due to screen distance changes is also as easy as recalculating the far and near-points. Changes to the screen distance can also be termed as Axial changes because of changes to the overall size of the system.

For instance, the system in 3.2 has the screen fixed at 25 cm from the lens. If we position the screen 5 cm closer maintaining the same lens power range, the near-point recedes farther from 20 cm to 25 cm (1 D towards left) signifying that the system has become hyperopic. Similarly, if we move the screen 25cm farther from its initial position – the far-point comes closer from ∞ to 50 cm (2 units towards right) rendering the system Myopic. The near-point also shifts closer (by the same unit amount) from 25 cm to 16.66 cm.

5. Experimental verification of shift in observation range due to external defocus (Accommodation shift)

The shift in observation range due to screen distance increments act opposite to that of increments in focal lengths. Thus, increase in screen distance results in Myopic shift in observation ranges and decrease in screen distance results in Hyperopic shift respectively. It is easy to experimentally observe the shifting in observation range upon introduction of external defocus as described in section 3.2 using a dedicated camera. Experimental verification of the findings of section 3.3 can also be achieved in a similar manner with the help of an optical bench setup where screen distance can be varied. This has probably been observed and documented several times but we still feel the relative unfamiliarity of phenomena deserves a mention.

Regarding shifting of observation range of a system, reducing (increasing) focal length is analogous to increasing (decreasing) screen distance. This is also intuitive from the way we have defined dimensionless image distance $j=\frac{s}{f}$ where decreasing focal length (f) or increasing screen distance (s) both serve to increase the dimensionless image distance (j) resulting in a decrease in dimensionless object distance (k) signifying Myopic shift and vice-versa. We have also demonstrated how Myopic and hyperopic shift can compensate/cancel each other. It is even possible to compensate Myopic shift from screen distance changes by shift from induced defocus and vice-versa.

5.1 The Camera Setup

The camera (referring to the combination of camera body paired with a lens) used was a Fujifilm X-S10[4] camera body paired with FUJINON XC35mmF2[5] prime lens.

The light rays from an object first encounter the optical elements inside the camera lens and then go on to meet the image sensor just like the simple lens system. This can be easily verified by simply positioning a converging lens in front of a bare camera sensor and checking for image formation in the viewfinder if **shoot without lens** mode is present and enabled.

The similarities between image-sensor of a Camera body and the imaging screen for simple lens model is obvious as shown in Figure 5.1 below.

4 https://fujifilm-x.com/en-in/products/cameras/x-s10/
5 https://fujifilm-x.com/en-in/products/lenses/xc35mmf2/specifications/

Figure 5.1: The camera body showing the exposed image sensor

Figure 5.2: The lens showing formation of an inverted image

The camera lens also behaves like an ideal lens with a variable aperture as shown in Figure 5.2. The lens comprises of multiple optical elements – 9 elements in 6 groups (including two aspheric elements) to be exact. But for all our experimental purposes, it still behaves close to an ideal converging lens evident from the inverted image formation. The additional optical *elements* are required to adjust focus and minimize aberrations in the image. The larger aperture size of the lens of a dedicated camera permits much shallower Depth-of-Field (DOF) for verification of focus.

The Camera (referring to the combination of camera body paired with a lens) can image close objects roughly 31.5 cm from the lens (if we subtract 35 mm focal length from the specified close focusing distance of 35 cm from the image-sensor plane, all the way up to infinity yielding an accommodation ability of ~3.25 D factoring in some inbuilt headroom for proper focusing at optical infinity (Figure 5.3).

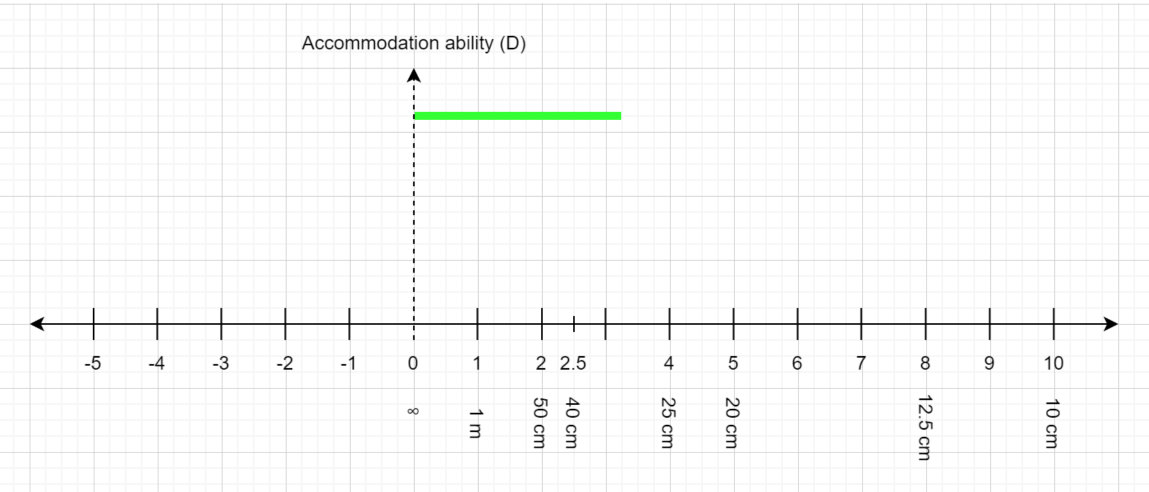

Figure 5.3: Observation range of XC35mmF2 lens

5.2 Description of the experimental setup

For the experiment, the camera was positioned outside to image a tree at optical infinity across a field (Figure 5.4).

Figure 5.4: Image showing the positions of camera and the object

The X-S10 camera body used for this experiment has a handy distance scale option which shows the focusing distance estimated from the position of optical elements as reported by the lens. Position of distance scale[6] with the white focus distance and blue DOF indicator[7] is shown in Figure 5.5 and 5.6.

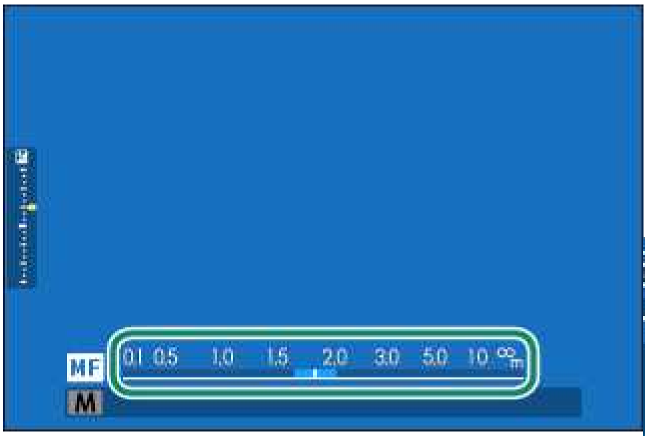

Figure 5.5: Position of distance scale in the viewfinder

Figure 5.6: Focus distance and DoF indicators

As can be verified from the white mark near 10m∞ on the Distance scale in the viewfinder (Figure 5.7), the optical elements inside the camera lens are correctly focused for our subject tree located at optical infinity in

6 https://fujifilm-dsc.com/en/manual/x-s10/taking_photo/manual-focus/
7 https://fujifilm-dsc.com/en/manual/x-s10/taking_photo/manual-focus/

the absence of external defocus. Image formation parameters related to image-sensors like exposure or sensitivity is beyond the scope of this article.

Figure 5.7: Verification of object distance

5.3 Shift in observation range after introducing hyperopic defocus

What happens if one places a diverging (minus) lens (introduce hyperopic defocus) closely in front of camera? In section 4.1, we have already outlined how observation range gets shifted after introduction of external defocus. For instance, introducing a -2.5 D defocus would shift the observation range 2.5 units towards left as shown in Figure 5.8.

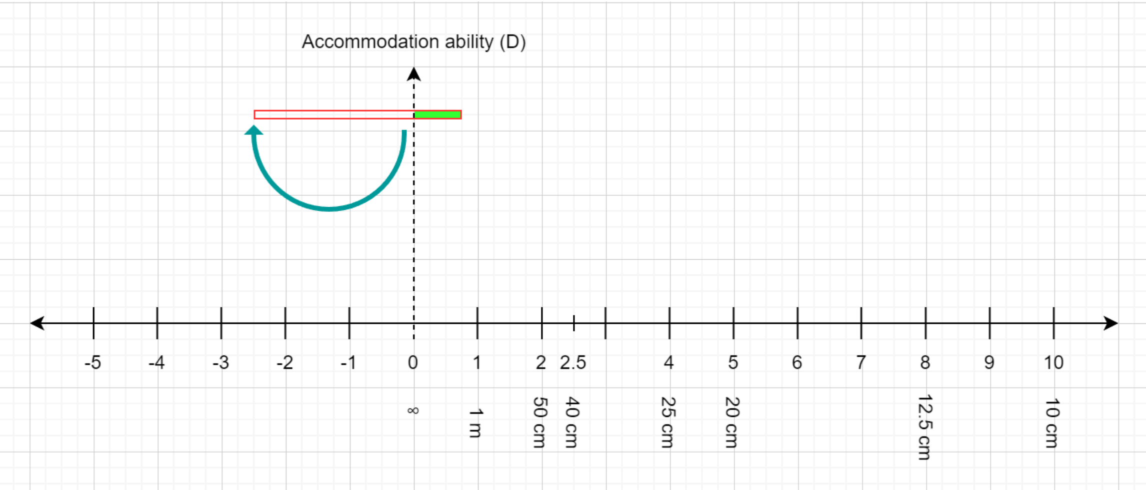

Figure 5.8: Shift in observation range after introduction of -2.5 D defocus

The hollow red part of the observation range to the left of origin indicates unusable part of the observation range because it is physically impossible to place an object beyond ∞. The remaining useful part of observation range lying closer than infinity is represented with a solid green line. Experimental demonstration and verification for the same using -2.5 D defocus is provided in Figure 5.9.

Figure 5.9: Verification of reported focus distance with -2.5 D defocus

The white focus distance indicator is near the 0.5 m mark on distance scale. Coincidentally, this is also the focal length (0.4 m) of the introduced defocus lens. Thus, focusing at an object located at infinity with the camera using hyperopic defocus of -2.5 D results in reported focusing distance of 40 cm at the focus of the introduced lens. The test subject situated at optical infinity is still in acceptable focus and the same can be verified from the blur that results from removing the introduced defocus as shown in Figure 5.10.

Figure 5.10: Blur resulting from removing the defocus after focusing at ∞

Due to the nature of the mapping introduced by -2.5 D defocus, we can state that the Camera is *apparently* focused at infinity when the lens optics are *actually* reporting focus around the 40 cm mark. A simple video clip demonstrating the above is supplied with this article.

In a way, we can say that the introduced -2.5 D defocus 'eats' away the usable portion of the effective observation range of the lens leaving only the remaining part of the observation range available for image formation. After introduction of defocus, the camera reports its focus distance as equal to the focal length of the diverging lens used even when focusing at infinity like before. This can be understood as the introduced lens 'mapping' object distances to their new apparent distances or alternately as the lens apparently 'bringing' objects from infinity at its focus (virtual object formation). This also translates to the inability of the camera to lock focus at objects closer than ~1.33 m with the -2.5 D defocus.

We have also repeated the experiment with a -3.0 D defocus so that the similar result can be compared as shown in Figure 5.11 below.

Figure 5.11: Verification of reported focus distance with -3.0 D defocus

This time, the focus distance (~33 cm) is very case to the close-focusing limit of the lens at ~35 cm.

As one might've already observed, placing a lens in front of camera introduces some optical aberrations and slight crop to the image. But for our limited purposes, this has not affected our ability to verify the shift in observation range under introduced defocus.

External defocus due to an introduced lens is unable to alter the accommodation ability because the accommodation ability of a camera is an internal physical constraint imposed by how its optical elements are configured to move inside the lens.

5.4 Focusing on objects closer than ∞ under hyperopic defocus

The consequence of introducing a diverging lens in front of a camera for formation of image of objects at infinity has been previously demonstrated in previous sections.

In a way, we can say that the introduced -2.5 D defocus decreases usable effective observation range of the lens leaving only a subset of the observation range available for image formation. This translates as the inability of the camera to achieve focus at objects closer than ~1.33 m with the -2.5 D defocus shown earlier by the green part of observation range shown in Figure 5.8. This close-focusing distance recedes even further with the -3.0 D defocus rendering the Camera severely hyperopic.

6. Depth-of-field (DOF)

DOF diagrams are provided to scale for ease of visualization (Figure 6.1).

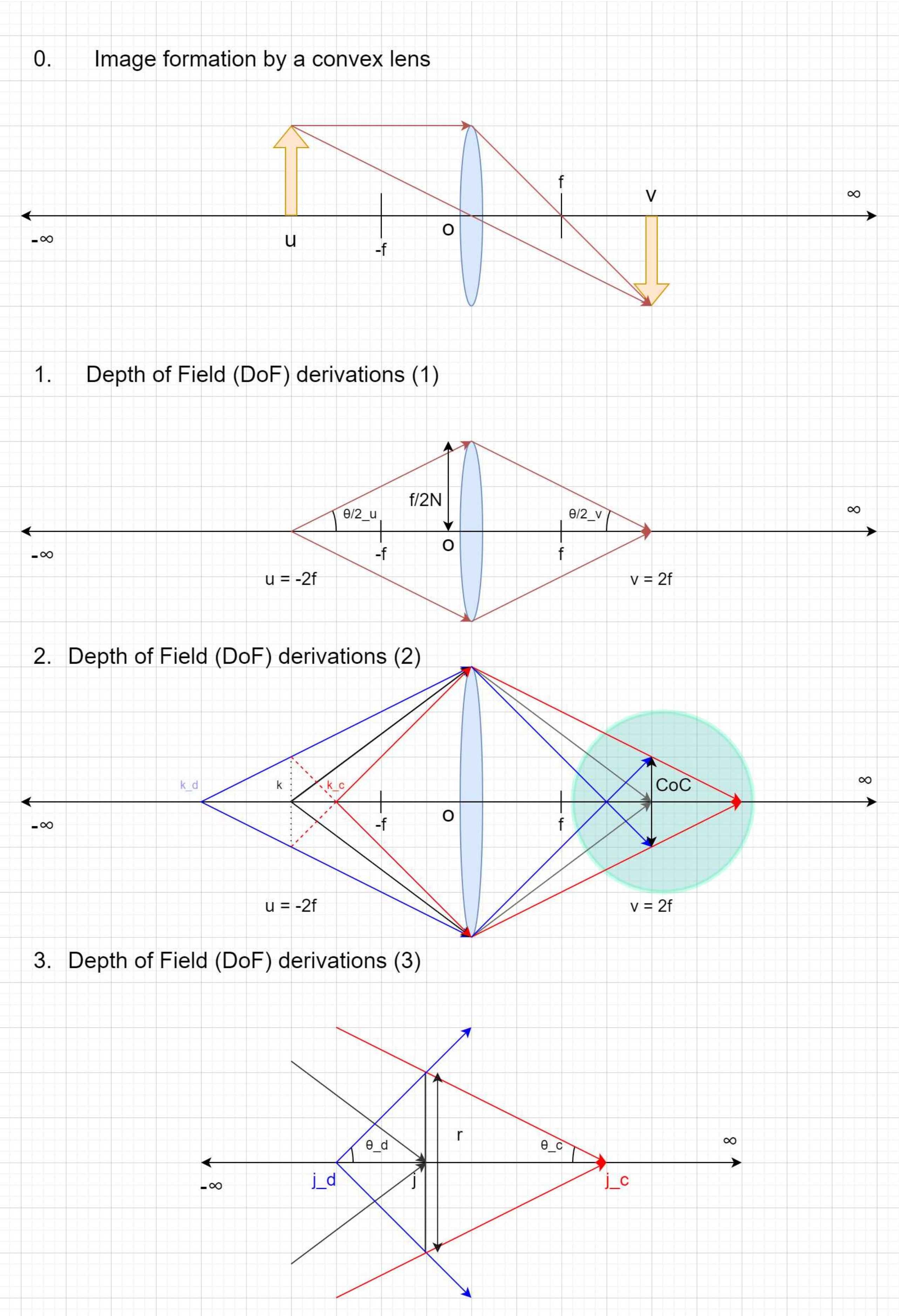

Figure 6.1: DoF ray diagram

$$\textit{An object must be situated at } (-\infty, -f)$$
$$\textit{for it to form an image at } (f,\ \infty) \textit{ on the other side of the convex lens.}$$

$$\textit{In common usage, aperture size } (A) \textit{ is usually denoted as } F-\textit{number or } \frac{f}{N}.$$

$$A = \frac{f}{N}, \textit{ where N is some dimensionless number.}$$

$$\textit{For instance, } F2 \textit{ or } \frac{f}{2} \textit{ implies } N = 2 \textit{ in both the cases}$$
$$\textit{and it denotes that the aperture size / opening is half of the focal length.}$$

$$\textit{Following this convention,}$$
$$\frac{f}{2N} \textit{ denotes the half}-\textit{width of aperture size / opening}$$
$$\textit{or radius if the aperture is circular}$$

$$\textit{we have accordingly}$$

$$\theta_u = 2\tan^{-1}\left(\frac{1}{2Nk}\right)$$
$$\theta_v = 2\tan^{-1}\left(\frac{1}{2Nj}\right)$$

$$\textit{To derive D.O.F. we need to find inverse of the problem:}$$

$$\textit{for some value of } j \textit{ i.e., imaging distance,}$$
$$\textit{find the range of } k \textit{ given a particular value of } N,$$
$$\textit{for which Circle of Confusion (C.o.C.) is lesser than a threshold value.}$$

$$\textit{We can make C.O.C. dimensionless by dividing it by } f \textit{ to get another quantity } r$$

$$r = \frac{C.O.C.}{f}$$

$$\textit{thus obtained both extreme values of } u \textit{ and } v \textit{ obey}$$

$$u_d > u > u_c :: v_d < v < v_c$$

$$\textit{where the subscript denotes (c)loser and (d)istant object / image respectively}$$

$$\textit{This can be understood as a consequence of the hyperbolic relation between}$$
$$\textit{object and its image.}$$

$$\textit{These same relations must apply to } k \textit{ and } j \textit{ also}$$

$$k_d > k > k_c :: j_d < j < j_c$$

$$\textit{From consideration of the quadrilateral formed at image junction, we can see that}$$

$$(j - j_d)\tan\theta_d = (j_c - j)\tan\theta_c$$
$$\implies \frac{j - j_d}{2Nj_d} = \frac{j_c - j}{2Nj_c} = \frac{r}{2}$$

Three quantities equal to each other means that we will get 3C2 = three equations.

$$\textit{equating the first two parts, we get:}$$
$$\frac{j - j_d}{j_d} = \frac{j_c - j}{j_c}$$
$$\implies j_c \times j - j_c \times j_d = j_c \times jd - j \times j_d$$

$$\textit{rearrranging we get,}$$
$$j = \frac{2j_c \times j_d}{j_c + j_d}$$

This equation can be checked by putting j_c *and* j_d *equal to* j.

$$Equating\ the\ first\ and\ second\ terms\ separately\ with\ r,\ we\ get:$$

$$\frac{j - j_d}{2Nj_d} = \frac{r}{2}$$
$$\frac{j_c - j}{2Nj_c} = \frac{r}{2}$$

$$which\ gives,$$
$$j_c = \frac{j}{1 - Nr}\ \&\ j_d = \frac{j}{1 + Nr}$$
$$we\ can\ now\ invert\ these\ according\ to\ the\ relation\ we\ got\ earlier$$
$$which\ is\ k = \frac{j}{j - 1},\ giving$$

$$k_c = \frac{j_c}{j_c - 1} = \frac{\frac{j}{1-Nr}}{\frac{j}{1-Nr} - 1} = \frac{j}{j - 1 + Nr}$$

$$and\ k_d = \frac{j_d}{j_d - 1} \Longrightarrow \frac{\frac{j}{1+Nr}}{\frac{j}{1+Nr} - 1} = \frac{j}{j - 1 - Nr}$$

$$k_d - k_c = j \times \left(\frac{1}{j - 1 - Nr} - \frac{1}{j - 1 + Nr}\right)$$

$$here,\ k_d - k_f\ denotes\ \frac{D.o.F.}{f}$$

$$\Longrightarrow D.o.F. = f \times j \times \left(\frac{j - 1 + Nr - (j - 1 - Nr)}{(j - 1 - Nr) \times (j - 1 + Nr)}\right)$$

$$\Longrightarrow D.o.F. = \frac{2f \times j \times Nr}{(j - 1 - Nr) \times (j - 1 + Nr)}$$

$$= \frac{2f \times j \times Nr}{(j - 1)^2 - N^2 r^2}$$

$$How\ to\ use\ this\ formula\ for\ calculations:$$
$$firstly\ calculate\ j\ from\ k.$$

$$then\ putting\ the\ respective\ values\ of\ f,\ N,\ j\ \&\ r$$
$$in\ the\ above\ formula\ will\ give\ you\ the\ D.o.F.\ in\ the\ unit\ of\ focal\ length.$$

Two DOF scenarios of interest are described below:

$$Nearest\ distance\ in\ acceptable\ focus\ when\ focus\ is\ at\ \infty\ \left(j = 1,\ k\ \rightarrow \infty\right):$$

$$j_c = \frac{1}{1 - Nr}$$

$$which\ gives\ k_c = \frac{1}{Nr}$$

$$the\ nearest\ distance\ of\ focus\ while\ still\ having\ objects\ at\ \infty$$
$$in\ 'acceptable'\ focus\ (j_d = 1):$$

$$giving\ j = 1 + N \times r$$

$$k = \frac{j}{j-1} = \frac{1 + Nr}{Nr}$$

$$---FINISHED\ DOF\ DERIVATION\ PART\ ---$$

$$we\ already\ know\ that\ j = \frac{2j_c \times j_d}{j_c + j_d}$$

$$substituting\ j_x = \frac{k_x}{k_x - 1} we\ get,$$

$$j = 2 \times \frac{\frac{k_c}{k_c-1} \times \frac{k_d}{k_d-1}}{\frac{k_c}{k_c-1} + \frac{k_d}{k_d-1}} = 2 \times \frac{\frac{k_c}{k_c-1} \times \frac{k_d}{k_d-1}}{\frac{k_c}{k_c-1} + \frac{k_d}{k_d-1}}$$

$$giving\ j = \frac{2k_c \times k_d}{2 \times k_c \times k_d - (k_c + k_d)}$$

$$finding\ the\ same\ relationship\ for\ k\ results\ in$$

$$k = \frac{j}{j-1} = \frac{\frac{2k_c \times k_d}{2 \times k_c \times k_d - (k_c + k_d)}}{\frac{2k_c \times k_d}{2 \times k_c \times k_d - (k_c + k_d)} - 1}$$
$$= \frac{2k_c \times k_d}{k_c + k_d}$$

$$This\ result\ is\ due\ to\ the\ interchangable\ (mirror)\ nature\ of\ the\ relation.$$

6.1 Experimental validation for depth-of-field calculations

We will demonstrate use of the above DOF formula for calculating DOF in actual applications.

For the X-S10 and XC35mmF2, the screen distance for this combination of lens can be taken approximately 35mm or 3.5 cm equal to the focal length of the camera used. For a prime lens, the screen distance remains mostly fixed and only the focal length of the lens changes (otherwise there will be FOV breathing). One can easily utilize the result from section 3.2 to calculate reduction in focal length for distances closer than infinity.

The 6240 x 4160 image sensor of the camera measures 23.5 x 15.6 mm (Width x Height).

The pixel pitch comes out to be approximately 23.5 mm ÷ 6240 = ~ 0.0037 mm approximating the CoC.

For a manual mode focus distance of 2.5 m with F-number of 16, the DOF calculations (Figure 6.3) of ~60cm front-to-back is in line with the value reported in the viewfinder (Figure 6.2).

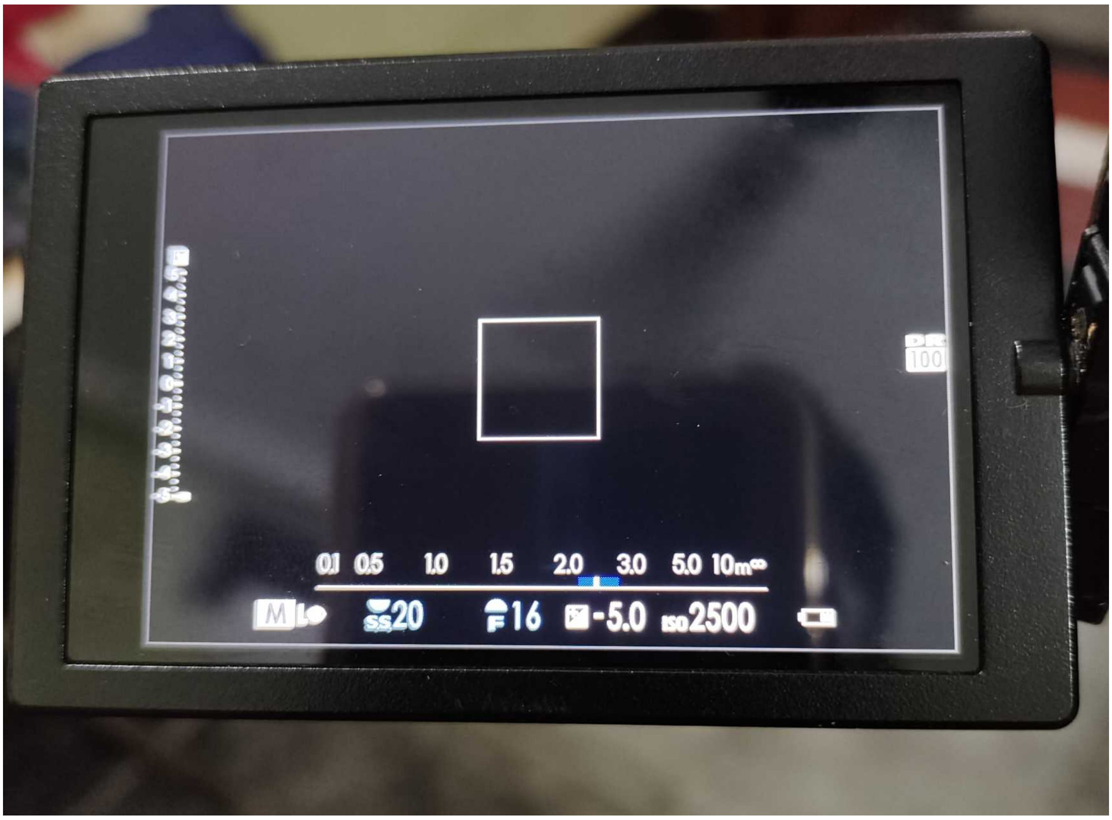

Figure 6.2: Viewfinder indicating DOF of ~60 cm

$Ac := 0.4 \quad d := \frac{100}{Ac} \quad d = 250.0$ cm

Power in Dioptre and distance in cm

$u := d \quad u := u$ put u for above value

$screen := 3.5$

$f := \frac{100}{\frac{100}{screen} + Ac} \quad f = 3.452$

Replace d by object distance you want

$screen := f \cdot \frac{-u}{f - u} \quad screen = 3.5$

$k := \frac{u}{f} \quad k = 72.43$

$k := k \quad k > 1$, put k for above value

$j := \frac{k}{k - 1} \quad j = 1.014$

$sum := k + j \quad sum = 73.44$

$magnification := \frac{-j}{k} \quad magnification = -0.014$

DOF derivation part

$N := 16 \quad Aperture := \frac{f}{N} \quad Aperture = 0.2157$ cm

Object and image angles subtended

$theta\_u := \frac{360}{pi} \cdot atan\left(\frac{1}{2 \cdot N \cdot k}\right) \quad theta\_u = 0.04944$

$theta\_v := \frac{360}{pi} \cdot atan\left(\frac{1}{2 \cdot N \cdot j}\right) \quad theta\_v = 3.53$

Radius of CoC

$CoC := \frac{2.35}{6240}$ cm $\quad r := \frac{CoC}{f} \quad r = 1.091E{-4}$

k and j close and distant

$j\_c := \frac{j}{1 - N \cdot r} \quad j\_c = 1.016 \quad j\_c \cdot f = 3.506$

$k\_c := \frac{j}{j - 1 + N \cdot r} \quad k\_c = 64.4 \quad k\_c \cdot f = 222.3$

$j\_d := \frac{j}{1 + N \cdot r} \quad j\_d = 1.012 \quad j\_d \cdot f = 3.494$

$k\_d := \frac{j}{j - 1 - N \cdot r} \quad k\_d = 82.75 \quad k\_d \cdot f = 285.6$

DOF

$k\_d - k\_c = 18.35$

$DoF := f \cdot (k\_d - k\_c) \quad DoF = 63.33$ cm

$DoF\_alt := \frac{2 \cdot f \cdot j \cdot N \cdot r}{(j-1)^2 - (N \cdot r)^2} \quad DoF\_alt = 63.33$ cm

Nearest distance under DOF keeping focus at ∞ (k_c when j = 1)

$\frac{f}{N \cdot r} = 1977.0$ cm

Nearest focus distance keeping ∞ under DOF (k when j_d = 1)

$\frac{f \cdot (1 + N \cdot r)}{N \cdot r} = 1981.0$ cm

Figure 6.3: DOF worksheet calculations