Math 357
Expositional homework 04

Assigned: 2024–02–14 (W)
Due: 2024–02–26 (M)

The goal of this homework is to understand modules—including their connection to and generalization of vector spaces—via a specific and important example: $F[t]$-modules. The exercises are adapted from Dummit & Foote, 3e, Sections 10.1 and 10.3.

Let F be a field, let t be an indeterminate, let V be an F-module (aka F-vector space), and let $T : V \to V$ be an F-module homomorphism (aka F-linear transformation).

(a) Use the linear transformation T to give V the structure of an $F[t]$-module. In particular, verify that the ring action of $F[t]$ on V that you define satisfies the module axioms. *Hint:* See p 340.

(b) Let $U \leqslant V$ be an F-submodule (aka F-subspace). U is T**-stable** (aka T**-invariant**) if for all $u \in U$, $T(u) \in U$.[1] Show that $W \subseteq V$ is an $F[t]$-submodule if and only if W is a T-stable F-subspace of V. *Hint:* See p 341.

(c) View V as an F-vector space. Recall that the **endomorphism ring of** V is the set $\mathrm{End}_F(V)$ of all F-linear transformations from V to itself, equipped with operations $+$ and $\times$ (see pp 346–7). There is a natural map

$$\begin{aligned} F &\to \mathrm{End}_F(V) \\ \alpha &\mapsto \alpha I \end{aligned}$$

where $I : V \to V$ is the identity map, and αI is the map

$$\begin{aligned} \alpha I : V &\to V \\ v &\mapsto \alpha \cdot I(v) \end{aligned}$$

The image of F in $\mathrm{End}_F(V)$ is called the **subring of scalar transformations**.[2] Prove or disprove the following statement: Let T be a scalar transformation. If V is a nonzero cyclic $F[t]$-module, then $\dim_F V = 1$. *Hint:* See p 352.

(d) Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let $T \in \mathrm{End}_F(V)$ be rotation by π radians around the origin. Show that every F-subspace of V is an $F[t]$-submodule for T. *Hint:* Classify all F-subspaces of V.

[1] One nice property of a T-stable submodule U is that the module homomorphism $T : V \to V$, restricted to U, again gives a module homomorphism $T|_U : U \to U$. Under favorable conditions, we can decompose $V = U \oplus W$ and understand the operation of T on V by studying the operation of its restrictions to the "smaller" spaces U and W.

[2] In the general setting of R-modules M, the analogous map $R \to \mathrm{End}_R(M)$ may not be injective, as it is when $R = F$ a field.

Solutions

Exercises (a)–(c) are discussed in the pages referenced in the hints.

Exercise (d)

Using elementary linear algebra, we can show that, up to isomorphism, vector (sub)spaces are completely characterized by their dimension. In particular, because $\dim_{\mathbf{R}} \mathbf{R}^2 = 2$, any subspace $U \leqslant \mathbf{R}^2$ is one of the following three types:

1. $\dim U = 0$, corresponding to the zero subspace, $\{0\}$.
2. $\dim U = 1$, corresponding to a line through the origin in $\mathbf{R}^2$.
3. $\dim U = 2$, corresponding to the entire vector space $\mathbf{R}^2$.

Let U be an $\mathbf{R}$-subspace of $\mathbf{R}^2$. Using the construction in exercise (a), we have that U is an $F[t]$-submodule if and only if U is T-stable. Geometrically, it is clear that T fixes each of the three types of subspaces listed above. (Algebraically, only the case $\dim U = 1$ requires work, and that work is easy.) Thus, for the given T, every $\mathbf{R}$-subspace of $\mathbf{R}^2$ is an $F[t]$-submodule for T, as desired.