Differential Equations & Linear Algebra

Stephen Wolff[1]

September 6, 2025

[1]Contact Stephen.Wolff@rice.edu with corrections and comments.

Contents

Course Description

Resources

Computer software is helpful for visualizing differential equations. Java versions of dfield (direction field) and pplane (phase plane) software are available here for free download. The authors of Farlow *et al.* (2018) promote the software Interactive Differential Equations, integrated into their book.

Slope fields in these notes were created using the desmos slope field generator.

Remarks

The authors encourage "keeping a scholarly journal" (see page xvii in Farlow *et al.* (2018)). This is a fancy way to say, write your thoughts on a regular (daily) basis.

To Do (Priority List)

- *Add discussion of generalized eigenvectors, jordan normal form to chapter(s) on linear algebra (in the section about eigendinge, or in a separate (optional?) section).*
- *Give cohesive, unified treatment of higher-order linear ODEs and linear systems.*

Chapter 1

First-Order Differential Equations

1.1 Dynamical Systems: Modeling

Key Ideas

- Differential equation; ODE, PDE; order

1.1.1 Models and Differential Equations

A key idea in modeling (indeed, life!) is to simplify — just enough, and no more.[1]

The mathematical models we will discuss can be classified into two types:

1. Continuous-time models. Modeled using differential equations.

2. Discrete-time models. Modeled using iterative equations.

Some people distinguish scalar models (i.e. models with only one dependent variable) from vector models (i.e. models with one or more dependent variables). Scalar models can be viewed as special cases of vector models.

Dynamical systems are systems that change over time.

- A **differential equation (DE)** is an equation that contains derivatives.

- An **ordinary differential equation (ODE)** is a DE that contains only ordinary derivatives (no partial derivatives).

- A **partial differential equation (PDE)** is a DE that contains partial derivatives.

[1] Or, as Einstein put it, "It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience."

The **order** of a DE is the highest-order derivative in the equation.

Loosely speaking, a DE is **linear** if its dependent variables and their derivatives are not multiplied by any others (including themselves). Otherwise it is **nonlinear**.

Example 1.1. This terminology is only important insofar as it allows us to communicate ideas. To this end, let's try our hand classifying two differential equations (order, ordinary versus partial, linear versus nonlinear, independent and dependent variables).

(a) (Hooke's law) 2nd-order linear ODE; IV t; DV x

$$m\frac{d^2x}{dt^2} + kx = 0$$

(b) 5th-order nonlinear PDE; IVs y, z; DVs s, t

$$z\frac{\partial^3 z}{\partial t^3} + \frac{\partial^5 y}{\partial^2 s \partial t \partial^2 s} = y^2 z$$

Rates of change of a quantity are often related to the quantity in a specified way. Some common ways are modeled as follows (k is a constant of proportionality):

- The rate of change of y is **(directly) proportional** to y:

$$\frac{dy}{dt} = ky$$

- The rate of change of y is **inversely proportional** to t:

$$\frac{dy}{dt} = \frac{k}{t}$$

One often sees these mixed and matched, e.g.

- The rate of change of y is directly proportional to the product of s and y^2 and inversely proportional to $\sqrt{t}$:

$$\frac{dy}{dt} = \frac{ksy^2}{\sqrt{t}}$$

Example 1.2. Let's practice. *(better for students to practice building the models ex nihilo; skip this exercise)*

(a) The rate of change of temperature T of an object is directly proportional to the different between the temperature M of its surroundings and T:

$$\frac{dT}{dt} = k(M - T), \quad k \in \mathbf{R}_{>0}$$

(b) The transmission rate of a disease in a population of fixed size L is directly proportional to the number of infected individuals N and the number of individuals not yet infected:

$$\frac{dN}{dt} = kN(L - N), \quad k \in \mathbf{R}_{>0}$$

1.2 Qualitative Analysis

Key Ideas

- General solution, particular solution
- Initial value problem (IVP)
- Equilibrium solution; stable, unstable equilibria
- Isocline

Recall that a differential equation is an equation that contains derivatives. The order of a differential equation is the highest-order derivative in the equation.

In this chapter we'll study first-order ordinary differential equations (ODEs). These equations have the general form

$$\frac{dy}{dt} = f(t, y) \tag{1.2.1}$$

We can view solutions to a differential equation analytically and graphically.

1. Analytical. A solution to (1.2.1) is a function $y(t)$ that, when substituted into (1.2.1) produces an identity (i.e. both sides of the equation are the same) on a suitable domain for t.

2. Graphical. A solution to (1.2.1) is a function whose slope at each point is specified by the derivative.

1.2.1 Analytical Solutions

Analytically, in general, finding solutions is (relatively) hard, and checking them is (relatively) easy. To check whether a given function solves a differential equation, plug the function into the differential equation, and see if the equation is satisfied (on a suitable domain for the independent variable(s)).

Example 1.3 (Example 1.2.1)**.** Show that $y(t) = \sqrt{1-t^2}$ solves the differential equation

$$\frac{dy}{dt} = -\frac{t}{y}.$$

What is the largest interval in $\mathbf{R}$ for which this solution is valid?

Solution. We check whether a proposed function is a solution by plugging it into the DE and seeing whether the equation is satisfied. For the given $y(t)$, the left side equals

$$\frac{dy}{dt} = \frac{1}{2}\left(1-t^2\right)^{-\frac{1}{2}}(-2t) = -\frac{t}{\sqrt{1-t^2}} = -\frac{t}{y},$$

which equals the right side. Hence the given $y(t)$ is a solution.

The given differential equation is not defined when $y = 0$. For our solution $y(t) = \sqrt{1-t^2}$, this corresponds to the t values

$$\sqrt{1-t^2} = y(t) \underset{\text{set}}{=} 0 \qquad \Leftrightarrow \qquad t = \pm 1.$$

Also, our solution $y(t)$ is not defined for

$$1 - t^2 < 0 \qquad \Leftrightarrow \qquad t < -1 \text{ or } t > 1.$$

We conclude that the given solution $y(t)$ is valid on the open interval $(-1, 1)$.

(reword the following) A **general solution** is a specification of all possible solutions to a differential equation, usually using parameters. A **particular solution** is one member of that family of general solutions. An **initial value problem (IVP)** is a differential equation with specified initial conditions, e.g.,

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

where t_0, y_0 are constants. The initial condition says that the point (t_0, y_0) is a point on the graph of $y(t)$.

Example 1.4 (Example 1.2.3)**.** Consider the first-order ODE

$$\frac{dy}{dt} = 3t^2.$$

Show that the general solution is $y(t) = t^3 + c$. Show that the initial condition $y(0) = 1$ corresponds to the particular solution with $c = 1$.

Solution. We check whether a proposed function is a solution by plugging it into the DE and seeing whether the DE is satisfied. This is the case. The initial condition $y(0) = 1$ corresponds to the point $(t, y) = (0, 1)$. Algebraically, the equation defining the general solution must hold at this point, so

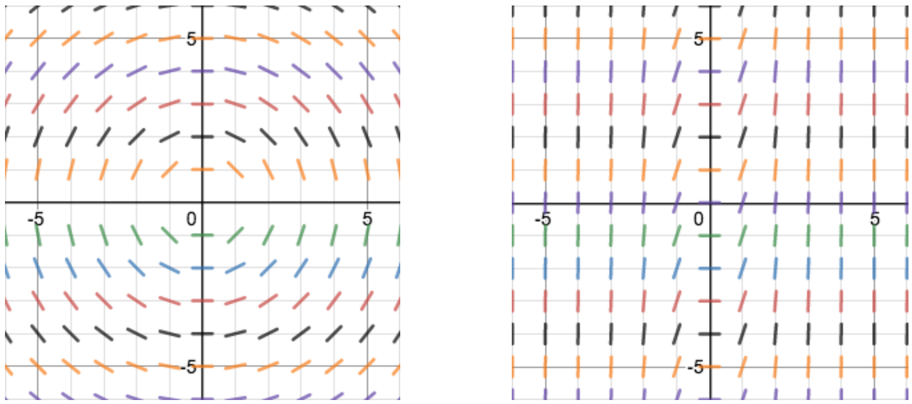
$$1 = y(0) = 0^3 + c = c.$$

Graphically, the point $(t, y) = (0, 1)$ must be on the graph of $y(t)$. Different values of c correspond to (vertical) translations of the graph *(add graphic like Figure 1.2.2)*. The graph containing the point $(0, 1)$ corresponds to $c = 1$.

Warning! Not all differential equations have closed-form solutions, for example (see page vii in Farlow *et al.* (2018)),

$$\frac{dy}{dt} = y^2 - t, \qquad \qquad \frac{dy}{dt} = e^{ty^2}.$$

So we develop other, non-analytical tools.



(a) (b)

Figure 1.2.1: Slope fields in the (t, y) plane for selected 1st-order ODEs. (a) $y' = -\frac{t}{y}$ in Example 1.3. (b) $y' = 3t^2$ in Example 1.4.

1.2.2 Graphical Solutions

We can view the first-order ODE (1.2.1) as telling us the slope of y at every point (t, y). A **direction field** (or **slope field**) is a plot of the slopes $f(t, y)$ at the corresponding points (t, y). Slope fields for the 1st-order ODEs in Examples 1.3 and 1.4 are shown in Figure 1.2.1. Loosely speaking, if we follow the slope field, we can plot solutions (even if we don't know the equations for them!).

1.2.3 Concavity

We can address concavity of solutions to the general 1st-order ODE (1.2.1) without solving it. Recall that the concavity of the graph of $y(t)$ is given by its second derivative, $y''(t)$:

- $y''(t) > 0$ means the graph of y opens up at $(t, y(t))$.
- $y''(t) = 0$ gives no information about concavity.
- $y''(t) < 0$ means the graph of y opens down at $(t, y(t))$.

Differentiating (1.2.1) using the chain rule, we find

$$y'' = \frac{d}{dt} f(t, y) = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) f(t, y).$$

The function $f(t, y)$ is the right side of (1.2.1); we can compute its partial derivatives.

Example 1.5. Analyze concavity for the 1st-order ODE $y' = 3t^2$ from Example 1.4.

Solution. For this ODE,

$$f(t, y) = 3t^2.$$

we compute

$$\frac{\partial f}{\partial t} = 6t, \qquad\qquad \frac{\partial f}{\partial y} = 0.$$

Hence

$$y'' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f = 6t.$$

We conclude that any solution curve $y(t)$ is

- concave up when $6t = y'' > 0$, i.e. $t > 0$.
- concave down when $6t = y'' < 0$, i.e. $t < 0$.

In this case, we know the general form of a solution curve, namely $y(t) = t^3 + c$. Note that the concavity of these graphs is exactly as we concluded above.

Example 1.6. Analyze concavity for the 1st-order ODE $\frac{dy}{dt} = -\frac{t}{y}$ from Example 1.3.

Solution. Here,

$$f(t, y) = -\frac{t}{y},$$

so

$$\frac{\partial f}{\partial t} = -\frac{1}{y}, \qquad\qquad \frac{\partial f}{\partial y} = \frac{t}{y^2}.$$

Hence

$$y'' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f = -\frac{1}{y} + \frac{t}{y^2}\left(-\frac{t}{y}\right) = -\frac{1}{y} - \frac{t^2}{y^3} = -\frac{y^2 + t^2}{y^3}.$$

With some thought, we find it is helpful to consider three separate cases, based on the value of y:

- If $y = 0$, we have noted that the DE is not defined.
- If $y < 0$, then the denominator $y^3 < 0$, so y'' has the same sign as $y^2 + t^2$, which is always positive (in this case, by hypothesis, $y \neq 0$). Hence in the lower half-plane, solutions are concave up.
- If $y > 0$, then the denominator $y^3 > 0$, so y'' has the opposite sign as $y^2 + t^2$, which is always negative (again, in this case, $y \neq 0$). Hnece in the upper half-plane, solutions are concave down.

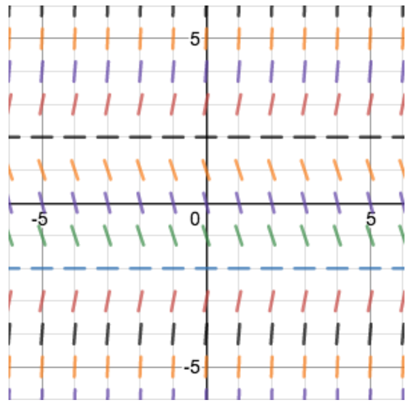


Figure 1.2.2: Slope field for the 1st-order ODE in Example 1.7.

1.2.4 Equilibria and Stability

An **equilibrium solution** is a solution that does not change over time. What does this mean for first-order ODEs? Well, to say that the solution $y(t)$ does not change over times means

$$y'(t) = 0.$$

Integrating this, we see that equilibrium solutions are constant functions $y(t) = c$.

Graphically (and loosely speaking), an equilibrium solution is **stable** if nearby solutions approach it as $t \uparrow +\infty$, and **unstable** if nearby solutions move away from it as $t \uparrow +\infty$.

Example 1.7 (Example 1.2.8)**.** Analyze the following first-order ODE for concavity, equilibrium solutions, and their stability.

$$y' - y^2 + 4 = 0$$

Solution. Equilibrium calculation:

$$f(t,y) = y^2 - 4 = 0 \qquad \Leftrightarrow \qquad y = \pm 2.$$

Concavity calculation:

$$y'' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f(t,y) = 0 + 2y(y^2-4) = 2y(y+2)(y-2).$$

We find (e.g., by using sign lines, by calculating y'' at interior points, etc.)

- For $y \in (-\infty, -2) \cup (0, 2)$, y is concave down.
- For $y \in (-2, 0) \cup (2, +\infty)$, y is concave up.

We conclude that $y = -2$ is an unstable equilibrium; $y = 2$ is a stable equilibrium.

A slope field for this ODE is graphed in Figure 1.2.2. Notice how the equilibria and concavity we computed algebraically appear graphically in the slope field.

1.2.5 Isoclines

An **isocline**[2] of the first-order ODE (1.2.1) is a curve in the (t, y) plane along which the slope (as given by the differential equation) is constant. Thus, an isocline is a graph $f(t, y) = c$.

Example 1.8. Describe the isoclines for the 1st-order ODE $y' = -\frac{t}{y}$ in Example 1.4.

Solution. The isocline with slope c is the set of points (t, y) satisfying

$$-\frac{t}{y} = y' = c.$$

In this case, we can solve for y as a function of t:

$$y = -\frac{1}{c}t.$$

Thus the isocline with slope c is the line $y = -\frac{1}{c}t$, through the origin in the (t, y) plane. Notice how this algebraic result appears graphically in the slope field in Figure 1.2.1(a).

[2] In Greek, the root "isos" means "equal", and "klinein" means "slope".

1.3 Quantitative Analysis

Key Ideas

- Separable first-order ODE; how to identify and solve them

A first-order ODE is **separable** if the independent and dependent variables can be separated as follows:

$$y' = f(t)g(y),$$

for some functions f and g.

Recall that equilibrium solutions (solutions $y(t)$ that do not change with time) are constants, i.e. have the form $y(t) \equiv c$. Thus for equilibrium solutions,

$$f(t)g(y) = y' = 0,$$

so either (i) $f(t) = 0$ or (ii) $g(y) = 0$, *for all* t *in the relevant domain*. Hence (i) says that f is the zero function; in this case, the original ODE must be $y' = 0$. If our original ODE is not $y' = 0$, then $f \not\equiv 0$, and we conclude that equilibrium solutions can be found by solving $g(y) = 0$.

Example 1.9 (Exercises 1.3.2,4,8,10)**.** Which of the following first-order ODEs are separable?

(a) $y' = y - y^3$

(b) $y' = \ln(ty)$

(c) $y' = t\ln(y^{2t}) + t^2$

(d) $ty' = 1 + y^2$

Solution. All but (b) are separable.

(proof of separation of variables — just the chain rule)

1.3.1 Separation of variables: Practical guide

1. Determine if the DE is separable. If yes, rewrite it in the form $y' = f(t)g(y)$.

2. Set $g(y) = 0$ and solve for any equilibrium solutions.

3. Assume $g(y) \not\equiv 0$, and rewrite the DE as

$$\frac{dy}{g(y)} = f(t)dt.$$

4. If possible, integrate each side separately. This will give a one-parameter family of solutions (i.e. the general solution).

5. If possible, solve the result for y as an explicit function of t.

6. If the DE is an IVP, use the initial condition to find the particular solution.

Example 1.10 (Example 1.3.3)**.** Solve the IVP

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y}, \quad y(0) = 1.$$

Solution. Step 1: This DE is separable. It is already in the desired form $y' = f(t)g(y)$, with

$$f(t) = 3t^2 + 1, \qquad g(y) = \frac{1}{1+2y}.$$

Step 2: $g(y) \neq 0$ for all y, so there are no equilibrium solutions. Also, from the given DE, we must impose $y \neq \frac{1}{2}$. Step 3:

$$(2y+1)dy = (3t^2+1)dt.$$

Step 4: We can integrate each side explicity:

$$y^2 + y = t^3 + t + c.$$

Step 6: Applying the initial condition $y(0) = 1$, we find $c = 2$. Step 5: To solve for y as an explicit function of t, we can complete the square or use the quadratic equation:

$$y = -\frac{1}{2} \pm \frac{1}{2}\sqrt{4t^3+4t+9}.$$

This is two solutions. We want the one containing the point $(0,1)$ from the initial condition. This corresponds to the $+$ solution (graphically, the upper branch of the graph *(show graph)*). Note that this solution is defined for t such that $y > -\frac{1}{2}$ (recall our requirement from Step 2).

1.4 Numerical Analysis

Key Ideas

- Euler method; Runge–Kutta methods
- Types of error in numerical approximation; tradeoffs

This section presents methods for approximating solutions to DEs. We'll apply each method to a concrete example at the end. The power of these methods lies in their computer implementations.

Focus on the geometry of each method; the algebra follows.

Throughout this section we consider the first-order ODE IVP

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{1.4.1}$$

1.4.1 Euler Method

The Euler method is an iterative numerical method to approximate solutions to first-order ODEs.[3]

The Euler method produces a piecewise-linear approximation to the solution, as follows:

- Start at the point (t_0, y_0) given in the initial condition.
- The DE (1.4.1) gives the slope of the tangent line to a solution curve here, namely $y' = f(t_0, y_0)$.
- Move a little bit, say h units in the positive x-direction, along this tangent line. This puts us at the point

  $$(t_1, y_1) = (t_0 + h, y_0 + hf(t_0, y_0)) .$$

  Connect the points (t_0, y_0) and (t_1, y_1) with a straight line segment (part of the tangent line to the solution curve at (t_0, y_0)).
- Iterate this process.

(clean up the following) Here's a web application that illustrates Euler's method. (With the default 1st-order linear ODE $\frac{dy}{dx} = 2xy$, try step sizes $h = 0.3, 0.1, 0.01$ — and also $h = 0.5$, to warn about the Singularity!)

[3] It's also a minor character in the film "Hidden Figures".

1.4.2 Error

Numerical approximation has two types of error:

(i) **Roundoff error**: Error arising from computer truncation of numbers.

(ii) **Discretization error**: Error arising from using a discrete process to estimate a continuous one.

Taylor series expansions show that the global discretization error for Euler's method is order 1 in the step size h, written $\mathcal{O}(h)$. This is a shorthand way to say that for each step n, the difference in y-values between the actual solution $y(t_n)$ and the approximation (t_n, y_n) from Euler's method satisfies

$$|y_n - y(t_n)| \leqslant Ch$$

for some nonnegative constant C.

Note that decreasing the step size h decreases the discretization error. Essentially, more steps means more chances to update the slope, which allows us to follow the actual solution curve more closely. However, more steps means more times the computer needs to truncate the approximations, so decreasing h increases the roundoff error.

(insert graphic illustrating tradeoff)

1.4.3 Runge–Kutta Methods

The Runge–Kutta methods are a family of iterative numerical methods to approximate solutions to ODEs. We'll look at two.

Second-order Runge–Kutta (RK2) Similar to the Euler method, but at each step n, use the slope at the midpoint of the tangent segment. Essentially, at each step $n+1$, we use the tangent line at the current point (t_n, y_n) to "peek" ahead to the slope of the tangent halfway along the segment, and follow this slope from the current point (t_n, y_n) to the next point (t_{n+1}, y_{n+1}). Explicitly,

$$\begin{aligned} t_{n+1} &= t_n + h, \\ y_{n+1} &= y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right). \end{aligned}$$

Fourth-order Runge–Kutta (RK4) Uses a weighted average of the slopes at four points in the t-interval $(t_n, t_n + h)$:

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + \frac{h}{6}\left(k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}\right), \end{aligned}$$

where

$$\begin{aligned}
k_{n,1} &= f(t_n, y_n) \\
k_{n,2} &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n,1}\right) \\
k_{n,3} &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n,2}\right) \\
k_{n,4} &= f(t_n + h, y_n + hk_{n,3}).
\end{aligned}$$

(explain each $k_{n,i}$ geometrically — what slope is being used for each?)

Here's a web application that illustrates the fourth-order Runge–Kutta method. WolframAlpha can also perform numerical approximation, e.g., using the syntax "Runge-Kutta method, dy/dt = yourFunction,y(t0) = y0, from tStart to tEnd,h = stepSize".

Exercise 1.1. Consider the first-order ODE IVP

$$y' = t + y, \quad y(0) = 0.$$

Approximate the value $y(0.5)$ using the Euler method, RK2, and RK4, each with step size $h = 0.5$.

Solution. Note that this DE is nonseparable. Let's do each method in turn. For each method, our starting point is $(t_0, y_0) = (0, 0)$.

Euler method (EM).

$$y_1 = y_0 + hf(t_0, y_0) = 0 + (0.5)f(0, 0) = 0$$

Second-order Runge–Kutta (RK2).

$$y_1 = 0 + (0.5)f(0 + 0.25, 0 + (0.25)f(0, 0)) = (0.5)f(0.25, 0) = 0.125$$

Fourth-order Runge–Kutta (RK4).

$$\begin{aligned}
k_{1,1} &= f(0, 0) = 0 \\
k_{1,2} &= f(0 + 0.25, 0 + (0.25)0) = f(0.25, 0) = 0.25 \\
k_{1,3} &= f(0 + 0.25, 0 + (0.25)(0.25)) = f\left(\frac{1}{4}, \frac{1}{16}\right) = \frac{5}{16} = 0.3125 \\
k_{1,4} &= f\left(0 + 0.5, 0 + (0.5)\frac{5}{16}\right) = f\left(\frac{1}{2}, \frac{5}{32}\right) = \frac{21}{32} = 0.65625
\end{aligned}$$

so

$$y_{n+1} = 0 + \frac{0.5}{6}(0 + 2(0.25) + 2(0.3125) + 0.65625) = 0.1484375$$

Later we'll see methods (e.g., variation of parameters, integrating factors) that will allow us to solve this IVP exactly as *(correct the following — sign is off)*

$$y(t) = e^t - t - 1.$$

Hence the exact solution is $y(0.5) = e^{0.5} - 0.5 - 1 \approx -0.14872$.

Table 1.4.1 compares the errors from the three methods.

Method	Approximation y_1	Actual error	Bound on global error
EM	0.0	0.1487	$h = 0.5$
RK2	0.125	0.0237	$h^2 = 0.25$
RK4	0.1484	0.0003	$h^4 = 0.0625$

Table 1.4.1: Estimate, actual error, and bound on global error for Exercise 1.1.

1.5 Theoretical Analysis

Key Ideas

- Existence; uniqueness
- Picard's theorem

When looking for solutions, we are interested in whether there are any solutions (existence), and if so, how many (uniqueness).

1.5.1 Motivating Examples

Example 1.5.1 in Farlow *et al.* (2018) illustrates existence and uniqueness of a zero of a function using results from calculus.

Example 1.11 (1.5.3)**.** Consider the first-order (nonlinear) ODE

$$y' = \sqrt{y}, \quad y(0) = y_0. \tag{1.5.1}$$

Analyze existence and uniqueness for all values $y_0 \in \mathbf{R}$.

Solution. $y_0 < 0$. Note that for $y < 0$, the DE is not defined (in $\mathbf{R}$, we cannot take the square root of a negative number). Thus the slope field for this DE is not defined for $y < 0$. (The slope field is plotted in Figure 1.5.1.) It follows that when $y_0 < 0$, no solution exists for (1.5.1).

$y_0 = 0$. Observe that the constant function $y(t) \equiv 0$ solves (1.5.1). So in this case, a solution exists. Is it unique?

Claim: When $y_0 = 0$, there are infinitely many solutions to (1.5.1). To show this, note that (1.5.1) is separable. Separating variables and solving gives

$$y(t) = \frac{1}{4}(t + c)^2.$$

Applying the initial condition gives $c = 0$. Thus

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{4}t^2 & \text{if } t \geqslant 0 \end{cases} \tag{1.5.2}$$

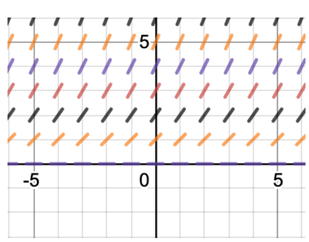


Figure 1.5.1: Slope field for the 1st-order nonlinear ODE in Example 1.11.

is also a solution to (1.5.1). In fact, one can check[4] that, for any $c \in \mathbf{R}_{\geqslant 0}$,

$$y(t) = \begin{cases} 0 & \text{if } t < c \\ \frac{1}{4}(t-c)^2 & \text{if } t \geqslant c \end{cases}$$

is also a solution to (1.5.1). (The graphs of these solutions in the (t, y) plane are horizontal translations of the graph of (1.5.2), c units to the right.) So (1.5.1) has infinitely many solutions when $y_0 = 0$, as claimed.

(comment or add analysis of $y_0 > 0$?)

1.5.2 Picard's Theorem

Theorem 1.12 (Picard's theorem for first-order ODEs with IV)**.** *Consider the IVP*

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{1.5.3}$$

Let $f(t, y)$ be continuous on some open rectangle

$$R = \left\{(t, y) \in \mathbf{R}^2 \,|\, t \in (a, b), y \in (c, d)\right\},$$

and let $(t_0, y_0) \in R$. Then on some interval $(t_0 - h, t_0 + h)$ ($h \in \mathbf{R}_{>0}$), there exists a solution to (1.5.3). Moreover, if $\frac{\partial f}{\partial y}$ is continuous on R, then this solution is unique.

Remark 1.13. Three remarks.

1. Picard's theorem is a local result: When the hypotheses are satisfied, it guarantees a solution exists on some (possibly very small) interval around t_0.

2. Picard's theorem gives sufficient conditions for a solution to exist and be unique. If the conditions are not satisfied, this does not mean a solution does not exist (or is not unique)! For example, consider the first-order nonlinear ODE[5]

$$y' = \frac{1}{t^2 + y^2}.$$

The slope field is plotted in Figure 1.5.2. One can check that, for any initial condition $(t_0, y_0) \neq (0, 0)$, the hypotheses of Picard's theorem for both existence and uniqueness are satisfied. At $(0, 0)$, neither f nor $\frac{\partial f}{\partial y}$ is defined, let alone continuous. However, one can check that a solution curve does pass through this point *(verify)*, with infinite slope at the origin.

3. Note that in Example 1.11, $f(t, y) = \sqrt{y}$ is not defined, and hence not continuous, on any open rectangle containing $(t_0, y_0) = (0, 0)$. Hence the hypotheses in Picard's theorem are not satisfied.

[4]E.g., by substituting the function into (1.5.1).
[5]See pages 87–89 of Farlow *et al.* (2018).

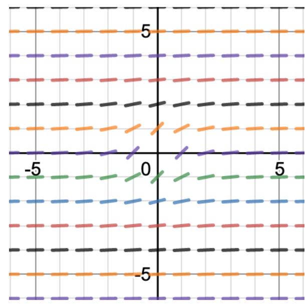


Figure 1.5.2: Slope field for the 1st-order nonlinear ODE $\frac{dy}{dt} = \frac{1}{t^2+y^2}$.

(see the following Wikipedia articles ; in light of which, adjust and enhance the above?)

- *Picard–Lindelöf theorem*
- *Lipschitz continuity*
- *Fixed-point iteration*
- *Banach fixed-point theorem*
- *Grönwall's inequality*

1.5.3 Another Example

Example 1.14 (1.5.5)**.** Consider the first-order ODE IVP

$$y' = t^2 + ty^2, \quad y(-4) = -2.$$

Analyze existence and uniqueness of solutions, using Picard's theorem.

Solution. For this IVP,

$$f(t, y) = t^2 + ty^2.$$

This function is continuous everywhere, in particular, on any open rectangle containing $(-4, -2)$. Hence Picard's theorem guarantees a solution exists on some interval around $t = -4$. Moreover,

$$\frac{\partial f}{\partial y} = 2ty$$

is continuous everywhere, in particular, on any open rectangle we chose to apply the first part of Picard's theorem. Thus Picard's theorem guarantees the solution whose existence we concluded above is unique in that interval.

Chapter 2

Linearity and Nonlinearity

2.1 Linear Equations

Key Ideas

- Vector space
- Linear operator
- Homogeneous, nonhomogeneous equations
- Differential operator

We encountered the linearity in the setting of ODEs briefly in Section 1.1. We may have seen linear systems of equations, like

$$\begin{aligned} 2x + 3y &= -1 \\ 3x + 2y &= 1. \end{aligned}$$

Few real-world systems are linear. Two reasons to study linear systems:

1. Linear systems are relatively simple, thus relatively easy and well understood.
2. Nonlinear systems can be approximated by linear systems.

This second point is key, and suggests an approach to studying general (hence probably nonlinear) systems:

- Approximate the system by a linear system.
- Use the tools and theory of linear systems to analyze the approximation.

2.1.1 Vector Spaces: A First Look

Linear systems are naturally (and most powerfully) viewed in the framework of vector spaces. We have a first look at vector spaces here. Our goal will be to get a feeling for how things work in a vector space. We'll worry about formal definitions and results later (see Chapter 3).

In the world of vector spaces, there are two basic objects:

1. vectors, which we'll denote $\mathbf{v}, \mathbf{v}_i$, etc.; and
2. scalars, which we'll denote a, a_i, etc.

We are allowed to combine these objects in the following two ways:

1. Add two vectors: $\mathbf{v}_1 + \mathbf{v}_2$.
2. Multiply a vector by a scalar: $a\mathbf{v}$.

Iterating these two allowed combination techniques, we get general expressions of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n.$$

(Formally, such an expression is called a **linear combination**.) The form of this expression resembles the examples of linearity we saw previously. In particular, vectors $\mathbf{v}_i$ always appear "to the power 1": they can be multiplied by scalars, but never by other vectors; they can't appear inside functions; etc.

(give concrete examples ; relate explicitly (!) to our study of DEs here)

(reconcile this vague section with much more precise subsection following ? try to give a "feeling for how things work" approach for linear maps)

2.1.2 Linear Maps

Let k be a field (e.g., $k = \mathbf{R}$ or $\mathbf{C}$), and let V, W be vector spaces over the same field k. A **linear map** on V is a function $L : V \to W$ such that

(i) for all vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2).$$

(ii) for all vectors $\mathbf{v} \in V$, for all scalars $a \in k$,

$$L(a\mathbf{v}) = aL(\mathbf{v}).$$

(give examples)

An equation $F(x_1, \ldots, x_n) = C$ is **linear** (with respect to the variables x_i) if it has the form

$$a_1x_1 + \ldots + a_nx_n = C,$$

where the a_j are not functions of the x_i. If $C = 0$, then the equation is **homogeneous**.

Remark 2.1. In a linear equation, variables are to the first power only, not multiplied by any other variable.

An ODE $F(y, y', \dots, y^{(n)}) = f(t)$ is **linear** if it has the form

$$a_n(t)y^{(n)} + \dots + a_1(t)y' + a_0(t)y = f(t),$$

for some functions $a_i(t)$, all defined over some common interval I (so the ODE is defined). If $f(t) \equiv 0$ on this interval, then the DE is **homogeneous**.

A DE that cannot be written in linear form is called **nonlinear**.

Remark 2.2. Note that a linear ODE requires only that the dependent variable y and its derivatives appear linearly in the equation. In particular, the coefficients $a_i(t)$ can be nonlinear. Similarly, one can say an equation is linear with respect to some variables and not others. For example,

$$t^2 x_2 + \frac{1}{\sqrt{t}} x_1 = 0$$

is a homogeneous linear equation with respect to the variables x_i, but nonlinear with respect to t.

Example 2.3 (2.1.1)**.** For each of the following ODEs, give its order, linearity, homogeneity, and type of coefficients (variable or constant).

(a) $y' + ty = 1$

(b) $y'' + yy' + y = t$

(c) $y'' + t^2 y' + y^2 = 0$

(d) $y'' + 3y' + 2y = 0$

(e) $y'' + y = \sin y$

(f) $y^{(4)} + 3y = \sin t$

2.1.3 Notation

The following notation is convenient, but whenever we use it, we should be sure to specify what we mean the first time we use it.

- A boldface variable denotes a vector of variables, e.g.,

$$\mathbf{x} = [x_1, \dots, x_n].$$

- Similarly, for a function $y(t)$, we'll often denote its vector of derivatives (for specified n)

$$\mathbf{y} = [y^{(n)}, y^{(n-1)}, \dots, y', y]$$

- We're used to representing functions by letters, e.g.,

$$f(x) = x^2 - 4x + 3.$$

  In the function viewpoint, the symbol f stands for "take the input (call it x) and return the result of the expression defining f (here, $x^2 - 4x + 3$)". For linear equations and differential equations, we'll often use similar notation, with the letter L (for "linear"). We call such a function L a **linear operator**.

- Sometimes we'll denote the derivative with respect to a given independent variable (e.g., t) by D, called the **differential operator**. E.g., $D(y) = \frac{dy}{dt}$, $D^2(y) = \frac{d^2y}{dt^2}$, etc. With this notation, we can write

$$y'' + 3y' + 2y = 0 \quad \text{as} \quad L(y) = 0, \quad \text{where} \quad L(y) = (D^2 + 3D + 2)y.$$

Example 2.4 (2.1.5)**.** For the linear ODEs in Example 2.3, what is the corresponding linear operator L?

2.1.4 The Superposition and Nonhomogeneous Principles

Let k be a field, let V, W be vector spaces over k, and let $L : V \to W$ be a linear map.

Proposition 2.5 (Superposition Principle)**.** *Let $\mathbf{v}_1, \mathbf{v}_2$ be solutions to the homogeneous linear equation*

$$L(\mathbf{v}) = \mathbf{0}_W. \tag{2.1.1}$$

Then any linear combination $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ is also a solution to (2.1.1).

Proposition 2.6 (Nonhomogeneous Principle)**.** *Let $\mathbf{w}_0 \in W$, let $\mathbf{v}_p$ be a (p for "particular") solution to the inhomogeneous equation*

$$L(\mathbf{v}) = \mathbf{w}_0, \tag{2.1.2}$$

and let $\mathbf{v}_h$ be a (h for "homogeneous") solution to the corresponding homogeneous equation

$$L(\mathbf{v}) = \mathbf{0}_W. \tag{2.1.3}$$

Then $\mathbf{v}_p + \mathbf{v}_h$ also solves (2.1.2). Moreover, every solution to (2.1.2) has this form.

Proof. Let $\mathbf{v}_p \in V$ solve (2.1.2), and let $\mathbf{v}_h \in V$ solve (2.1.3). Then

$$L(\mathbf{v}_p + \mathbf{v}_h) = L(\mathbf{v}_p) + L(\mathbf{v}_h) = \mathbf{w}_0 + \mathbf{0}_W = \mathbf{w}_0.$$

Thus $\mathbf{v}_p + \mathbf{v}_h$ also solves (2.1.2).

For the second result, let $\mathbf{v} \in V$ solve (2.1.2). In particular, $L(\mathbf{v}) = \mathbf{w}_0$. Using the particular solution $\mathbf{v}_p$ above, write

$$\begin{aligned} \mathbf{v} &= \mathbf{v} + \mathbf{0}_V \\ &= \mathbf{v} + (\mathbf{v}_p - \mathbf{v}_p) \\ &= \mathbf{v}_p + (\mathbf{v} - \mathbf{v}_p). \end{aligned}$$

Claim: This last expression has the desired form, $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_h$. To verify this, we must check that $\mathbf{v} - \mathbf{v}_p$ is a solution to the homogeneous equation (2.1.3). We can do this by feeding it to L:

$$L(\mathbf{v} - \mathbf{v}_p) = L(\mathbf{v}) - L(\mathbf{v}_p) = \mathbf{w}_0 - \mathbf{w}_0 = \mathbf{0}_W.$$

□

2.1.5 Solving Nonhomogeneous Linear Equations

The practical upshot from all this is that it gives us a method to solve nonhomogeneous linear equations $L(\mathbf{v}) = \mathbf{w}_0$.

1. Find all solutions $\mathbf{v}_h$ to the corresponding homogeneous equation.
2. Find one particular solution $\mathbf{v}_p$ to the nonhomogeneous equation.
3. By the nonhomogeneous principle, all solutions to the nonhomogeneous linear equation have the form $\mathbf{v}_p + \mathbf{v}_h$.

Example 2.7 (2.1.8)**.** Let $a, b \in \mathbf{R}$. Find all solutions to the first-order ODE

$$y' + ay = b. \tag{2.1.4}$$

Solution. We follow the method outlined above.

- Step 1: Find all y_h. The corresponding homogeneous equation is

  $$y' + ay = 0.$$

  This DE is separable. Solving, we find

  $$y(t) = ce^{-at}$$

  for $c \in \mathbf{R}$.

- Step 2: Find one y_p. By inspection, the constant function $y(t) = \frac{b}{a}$ solves (2.1.4).

- Step 3: Add. By the nonhomogeneous principle, all solutions to (2.1.4) have the form

  $$y(t) = y_p(t) + y_h(t) = \frac{b}{a} + ce^{-at},$$

  for some $c \in \mathbf{R}$.

2.2 Solving First-Order Linear ODEs

Key Ideas

- Variation of parameters (how to derive and use)
- Integrating factors (how to derive and use)

This section introduces two methods to solve first-order linear ODEs: variation of parameters, and integrating factors. Variation of parameters is applicable to a limited class of differential equations (linear ODEs). Integrating factors work only on differentials that can be made exact *(verify and clarify)*. Being able to recognize when these methods apply, and being able to follow how they are derived and applied, is more important than memorizing the formulas.

2.2.1 Variation of Parameters

The method of variation of parameters (VoP) can be applied to any linear ODE or system of ODEs.

Consider the general first-order linear ODE

$$y' + p(t)y = f(t). \tag{2.2.1}$$

In Section 2.2 we saw that all solutions to this ODE have the form $y = y_p + y_h$, where y_p is a particular solution to (2.2.1), and y_h is a solution to the corresponding homogeneous equation

$$y' + p(t)y = 0. \tag{2.2.2}$$

Step 1: Find all y_h. (2.2.2) is separable. Solving,

$$y_h(t) = e^{-\int p(t)\,dt + C} = ce^{-\int p(t)\,dt}.$$

Step 2: Find one y_p. The key idea is to take the homogeneous solution, but treat the constant c as a function of t. That is, we guess that a particular solution has the form

$$y_p(t) = v(t)y_h(t) = v(t)e^{-\int p(t)\,dt}. \tag{2.2.3}$$

(Note that we can absorb the parameter c from $y_h(t)$ into the unknown function $v(t)$.) Substituting this into the original ODE, we have

$$\begin{aligned}
f &\underset{\text{set}}{=} y_p' + py_p \\
&= v'y_h + vy_h' + py_p \\
&= v'y_h + v(-p)y_h + py_p \\
&= v'y_h - p(vy_h) + py_p \\
&= v'y_h,
\end{aligned}$$

where to get the third line, we use the homogeneous equation (2.2.2), solved for y'; and to get the last line, we use that $y_p = vy_h$ (where for y_h, we set the parameter $c = 1$). Solving for v, we find

$$v(t) = \int (y_h)^{-1} f \, dt = \int e^{\int p(t)\,dt} f(t)\, dt$$

Substituting this result into (2.2.3) gives a particular solution to (2.2.1):

$$y_p(t) = v(t) y_h(t) = e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt} f(t)\, dt.$$

Step 3: Add. By the nonhomogeneous principle, any solution y to (2.2.1) is given by

$$y = y_p + y_h = e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt} f(t)\, dt + ce^{-\int p(t)\,dt}. \tag{2.2.4}$$

VoP is clarified by example.

Example 2.8 (2.1.7)**.** Find all solutions to the first-order ODE

$$y' - y = t. \tag{2.2.5}$$

Solution. We follow the steps in Section 2.1 and use VoP to find y_p.

- Step 1: Find all y_h. The corresponding homogeneous equation is

  $$y' - y = 0.$$

  This DE is separable. Solving, we find

  $$y(t) = c_1 e^t$$

  for $c_1 \in \mathbf{R}$.

- Step 2: Find one y_p. Using VoP, we guess a particular solution of the form

  $$y_p(t) = v(t) y_h(t) = v(t) e^t.$$

  Substituting this into (2.2.5),

  $$t \underset{\text{set}}{=} v' e^t + v e^t - v e^t = v' e^t.$$

  Solving for $v(t)$ (using integration by parts),

  $$v(t) = \int e^{-t} t \, dt = -te^{-t} + \int e^{-t} dt = -te^{-t} - e^{-t} + c_2,$$

  for some $c_2 \in \mathbf{R}$. (Because we need only one particular solution, we may take $c_2 = 0$, if desired.) Thus

  $$y_p(t) = v(t) y_h(t) = \left(-te^{-t} - e^{-t} + C\right) e^t = -t - 1 + c_2 e^t.$$

  One might also be able to obtain y_p "by inspection", i.e. by observing that $y_p(t) = -t - 1$ solves (2.2.5). There is nothing wrong with guess-and-check!

- Step 3: Add. By the nonhomogeneous principle, all solutions to (2.2.5) have the form

$$y(t) = y_p(t) + y_h(t) = -t - 1 + ce^t,$$

for some $c \in \mathbf{R}$.

One can verify that $y(t)$ indeed solves (2.2.5), by plugging it in. *(include direction fields, explain how their geometry relates to nonhomogeneous principle)*

2.2.2 Integrating Factors

The method of integrating factors has a severely limited scope: It applies to exact ODEs only *(explicate, link reference)*.

Motivation for the method of integrating factors comes by viewing the left side of (2.2.1) as the result of the product rule. That is, we'd like to find a function $\mu(t)$ such that

$$\mu\left(y' + py\right) \underset{\text{set}}{=} \frac{d}{dt}(\mu y). \tag{2.2.6}$$

Writing out both sides,

$$\mu y' + \mu p y = \mu' y + \mu y',$$

which if $y \not\equiv 0$ simplifies to

$$\mu p = \mu'.$$

This first-order linear ODE is separable. Solving,

$$\ln|\mu(t)| = \int p(t)\,dt \qquad \Rightarrow \qquad \mu(t) = ce^{\int p(t)\,dt},$$

for any $c \in \mathbf{R}$. (Recall our goal: To find some function $\mu(t)$ satisfying (2.2.6). Thus we may choose c to make our lives easy, e.g., $c = 1$. (Taking $c = 0$ is valid, but it doesn't help us. Why not?))

Multiplying both sides of the original DE (2.2.1) by our integrating factor $\mu(t)$ and using (2.2.6), we have

$$\mu f = \mu\left(y' + py\right) = \frac{d}{dt}(\mu y).$$

Integrating both sides of this and solving for y gives

$$y(t) = (\mu)^{-1}\left(\int \mu f\,dt + c\right) = e^{-\int p(t)\,dt}\int e^{\int p(t)\,dt} f(t)\,dt + ce^{-\int p(t)\,dt}.$$

Note that this is the same final result we obtained via the method of variation of parameters (see (2.2.4)).

Example 2.9 (2.2.2)**.** Find all solutions to the first-order linear ODE (2.2.5), using integrating factors.

Solution. Here $p(t) = -1$, so we can take (setting $c = 1$)

$$\mu(t) = e^{-t}.$$

Then

$$e^{-t}t = \mu f = \frac{d}{dt}(\mu y) = \frac{d}{dt}\left(e^{-t}y\right),$$

so (using integration by parts to evaluate the integral)

$$y(t) = (\mu)^{-1}\int e^{-t}t\,dt = e^{t}\left(-e^{-t}t - e^{-t} + c\right) = -t - 1 + ce^{t},$$

for some $c \in \mathbf{R}$. This agrees with the general solution we found using the method of variation of parameters.

(add brief discussion of how to test for exactness ; implications for applying integrating factors)

2.3 Growth and Decay

Key Ideas

- Growth (or decay) equation

2.3.1 Growth Equation

Let $k \in \mathbf{R}$. The first-order linear ODE

$$y' = ky \tag{2.3.1}$$

is called the **growth equation** if $k > 0$, and the **decay equation** if $k < 0$. The constant k is called the **rate of growth** or **rate of decay**, respectively.

Solving (2.3.1),[1]

$$\begin{aligned} \frac{dy}{y} &= k\,dt \\ \ln|y| &= kt + C \\ |y| &= e^{kt+C} = e^C e^{kt}. \end{aligned}$$

A priori, y can be positive or negative; we can capture this in the last equation by replacing e^C (which is positive for any $C \in \mathbf{R}$) with $A \in \mathbf{R}$:

$$y(t) = Ae^{kt}.$$

If the situation we are modeling involves adding or removing material via other processes, we must include terms in our ODE to reflect this.

Example 2.10 (Exercise 2.3.12)**.** The half-life of E104 is 0.15 seconds, and it can be produced at a rate of 2×10^{-5} micrograms per second. If we start with no E104, how much is present after t seconds?

Solution. Let $y(t)$ denote the amount (in micrograms) of E104 present at time t (in seconds). Let $p = 2 \times 10^{-5}$ denote the rate of production of E104 (in micrograms per second), and let k denote the rate of decay (in micrograms per second).

First let's specify how y changes with t, and the initial condition:

$$\frac{dy}{dt} = p + ky, \qquad\qquad y(0) = 0.$$

(Note that in this example, $k < 0$. We could write $-k$ in place of $+k$ in our ODE; if we do this, we need to use the same convention when we analyze the growth

[1] One can solve (2.3.1) via multiple methods: e.g., separable, variation of parameters, integrating factors.

equation, below, to determine the half-life.) The ODE is separable (view the entire right side as a function of y). Separating variables and solving, we get

$$\begin{aligned}(p+ky)^{-1}dy &= dt\\ \frac{1}{k}\ln|p+ky| &= t+c_1\\ |p+ky| &= e^{k(t+c_1)} = e^{kt}e^{kc_1},\end{aligned}$$

which we can write as

$$p+ky = c_2e^{kt},$$

where $c_2 = \pm e^{kc_1}$. Thus

$$y(t) = -\frac{1}{k}\left(p - c_2e^{kt}\right). \tag{2.3.2}$$

We're told the value of p, but what is the value of k? Note that k is not 0.15; the half-life has units of seconds, whereas the rate k has units micrograms per second. We can determine k from the given information about the half-life. Ignoring the production process, the decay of E104 follows the decay equation,

$$z(t) = Ae^{kt}.$$

Evaluating this at the half-life $t = 0.15$, when (by definition) exactly half the original amount of E104 remains, we find

$$\frac{1}{2}A = \frac{1}{2}z(0) = z(0.15) = Ae^{0.15k}.$$

Solving for k, we obtain

$$\ln\frac{1}{2} = 0.15k \qquad \Leftrightarrow \qquad k = -\frac{1}{0.15}\ln 2 = -\frac{20}{3}\ln 2 \approx -4.6210.$$

Note that $k < 0$, indicating decay, as expected.

Now that we know both p and k, we can apply the initial condition $y(0) = 0$ to (2.3.2) to determine c_2:

$$0 = y(0) = -\frac{1}{k}(p - c_2) \qquad \Leftrightarrow \qquad c_2 = p.$$

(N.B. In this case, given the form of the general solution (2.3.2) and the y value of the initial condition, we could have used the initial condition to solve for c_2, even without knowing k.)

We conclude that the amount of E104 present after t seconds is

$$y(t) = -\frac{p}{k}\left(1 - e^{kt}\right) \approx 4.328 \times 10^{-6}\left(1 - e^{-4.6210t}\right).$$

2.3.2 Compound Interest

An annual interest rate r compounded n times per year means in each compounding period the capital grows $\frac{r}{n}$. Let $A(t)$ denote capital in year t, and A_0 capital at time $t = 0$. Then

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

Continuous compounding corresponds to $n \to \infty$. Using the fact that

$$\lim_{n \uparrow +\infty} \left(1 + \frac{x}{n}\right)^n = e^x,$$

and function–limit interchange, we can model continuous compounding by

$$A(t) = A_0 e^{rt},$$

i.e. the growth equation!

If there are regular contributions or withdraws of amount a from an account with interest rate r compounded continuously, then

$$\frac{dA}{dt} = rA + a.$$

One can check that the solution to this ODE, with initial condition $A(0) = A_0$, is

$$A(t) = A_0 e^{rt} + \frac{a}{r}\left(e^{rt} - 1\right).$$

2.4 Linear Models

Key Ideas

- Mixing problems: quantity = concentration × volume
- Heat change: rate proportional to the difference in temperature

2.4.1 Mixing Problems

Let $x(t)$ denote the amount (not concentration!) of substance dissolved at time t.

$$\frac{dx}{dt} = \text{rate in} - \text{rate out},$$

where each rate is given by

$$\text{rate} = \text{concentration} \times \text{flow rate}.$$

This last equation is just the time derivative of the related rates equation[2]

$$\text{quantity} = \text{concentration} \times \text{volume},$$

which comes from the definition of concentration:

$$\text{concentration} = \frac{\text{quantity}}{\text{volume}}$$

Example 2.11 (Example 2.4.2)**.** Saltwater containing 1 kg/L of salt is poured at 1 L/min into a tank that initially contains 100 L of fresh water. The stirred mixture is drained off at 2 L/min.

(a) Give the IVP the amount of salt in the tank satisfies until the tank is empty.

(b) Find the amount of salt in the tank at time t.

Solution. First let's set some notation (i.e. name things). We're interested in the amount of salt, so let $S(t)$ denote the amount of salt (in kg) in the tank at time t. We will quickly see that the total volume of liquid in the tank is also of interest, so let $V(t)$ denote the amount of liquid (in L) in the tank at time t.

The rate of change of volume of liquid in the tank is

$$\frac{dV}{dt} = \text{flow rate in} - \text{flow rate out} = 1\text{L/min} - 2\text{L/min} = -1\text{L/min}.$$

[2] As explained above, this "related rates equation" is just a rearrangement of the definition of concentration. We call it a related rates equation because in our applications, quantity and volume will often be functions of time.

Let $t = 0$ correspond to the time we start pouring in saltwater and draining off mixture. The problem specifications tell us that

$$V(0) = 100.$$

Thus we have a 1st-order linear ODE describing the volume:

$$\frac{dV}{dt} = -1, \qquad\qquad V(0) = 100.$$

This is an IVP for the volume, not the amount of salt, but (as we'll quickly see) we'll need $V(t)$ to define the IVP for the amount of salt. Solving this IVP, we find that the volume of liquid in the tank at time t is

$$V(t) = 100 - t.$$

Note that $V(t) < 0$ when $t > 100$. In our setting, negative volume doesn't make sense, so our model will only apply to $t \in [0, 100]$.

(a) The problem specification tells us about the rate of change of salt. To clarify our analysis, we'll break this change into two parts: what we're pouring in, and what we're draining out. For each part, the amount of salt is found by multiplying the concentration (in kg/L) by the flow (in L/min). That is,

$$\text{Rate of change of Quantity (kg/min)} = \text{Concentration (kg/L)} \times \text{Flow (L/min)}.$$

Notice how the units cancel in this equation:

$$\frac{\text{kg}}{\text{L}} \times \frac{\text{L}}{\text{min}} = \frac{\text{kg}}{\text{min}}.$$

We compute

$$\begin{aligned} \text{rate in} &= (1\text{kg/L})(1\text{L/min}) = 1\text{kg/min} \\ \text{rate out} &= \left(\frac{S}{100-t}\right)(2\text{L/min}) = \frac{2S}{100-t}. \end{aligned}$$

The rate of change of salt in the tank is found by combining these two:

$$\frac{dS}{dt} = (\text{rate in}) - (\text{rate out}) = 1 - \frac{2S}{100-t}.$$

This is the ODE part of the IVP. We still need an initial condition. Reading the problem specification, we see a natural initial condition to apply: When the process starts, i.e. the time we've designated $t = 0$, the tank contains pure fresh water, i.e. no salt. In our model, this translates to the initial condition

$$S(0) = 0.$$

Thus the IVP describing the amount of salt in the tank is

$$\frac{dS}{dt} = 1 - \frac{2S}{100-t}, \qquad S(0) = 0, \tag{2.4.1}$$

where this ODE is defined only on the t-interval $[0, 100]$.

(b) The amount of salt in the tank at time t is the solution to this IVP. We can solve using the methods of variation of parameters or of integrating factors.

Variation of parameters. The corresponding homogeneous ODE is

$$\frac{dS}{dt} + \frac{2}{100-t}S = 0 \qquad \Leftrightarrow \qquad \frac{dS}{dt} = -\frac{2S}{100-t}.$$

This ODE is separable:

$$-\frac{1}{2}S^{-1}\,dS = (100-t)^{-1}\,dt.$$

Integrating both sides, we get

$$\begin{aligned} -\frac{1}{2}\ln|S| &= -\ln|100-t| + c_1 \\ \ln|S| &= 2\ln|100-t| - c_2 \\ |S| &= e^{-c^2}\,|100-t|^2. \end{aligned}$$

In our model, $t \in [0, 100]$, so for all t we consider, $100 - t \geqslant 0$. Thus we can remove the absolute value on the right. Moreover, if we replace e^{-c_2} (which is always positive) with c_3 (which can be any real number), then we can remove the absolute value on the left. This leaves us with the general solution to the corresponding homogeneous ODE:

$$S_h(t) = c_3(100-t)^2.$$

Variation of parameters says to look for a particular solution S_p to the nonhomogeneous ODE by replacing the constant parameter c_3 with a function $v(t)$ of t:

$$S_p = v(t)(100-t)^2. \tag{2.4.2}$$

If this solves the nonhomogeneous equation, then plugging it in yields an equality, which we can solve for v:

$$\begin{aligned} S_p' &= 1 - \frac{2S}{100-t} = \frac{100-t-2S}{100-t} \\ v'(100-t)^2 - 2v(100-t) &= \frac{100-t-2(v(100-t)^2)}{100-t} = 1 - 2v(100-t) \\ v'(100-t)^2 &= 1. \end{aligned}$$

This is a separable ODE, which we can solve for v:

$$\begin{aligned} dv &= (100-t)^{-2}\,dt \\ v &= (100-t)^{-1} + c. \end{aligned}$$

Plugging this into (2.4.2) yields the (oxymoronic) general particular solution

$$S_p(t) = \left((100-t)^{-1} + c\right)(100-t)^2 = (100-t) + c(100-t)^2.$$

Our work says that for any value for the constant $c \in \mathbf{R}$ of integration, this is a particular solution. Because we're looking for just one particular solution, we can choose c to be whatever value we like. To make our lives simple, let's choose $c = 0$, corresponding to the particular solution

$$S_p(t) = 100 - t.$$

Finally, invoking the nonhomogeneous principle, we conclude that any solution $S(t)$ to the ODE in (2.4.1) has the form

$$S(t) = S_p + S_h = (100-t) + c_3(100-t)^2.$$

Integrating factors. Rewriting our model's ODE in the form

$$S' + \frac{2}{100-t}S = 1,$$

we see that, in the notation we used for our general analysis of integrating factors (see Section 2.2.2),

$$p(t) = \frac{2}{100-t}, \qquad\qquad f(t) = 1.$$

Thus

$$\mu(t) = c_1 e^{\int p(t)\,dt} = c_1 e^{\int \frac{2}{100-t}\,dt} = c_2 e^{-2\ln|100-t|} = c_2(100-t)^{-2}.$$

The value of the scalar c_2 multiplying this integrating factor doesn't matter *(make sure this is mentioned in the theoretical discussion)*, so let's make our lives easy and set $c_2 = 1$:

$$\mu(t) = (100-t)^{-2}.$$

Then

$$\begin{aligned} S(t) &= (\mu)^{-1}\left(\int \mu f\,dt + c\right) \\ &= (100-t)^2\left(\int (100-t)^{-2}(1)\,dt + c\right) \\ &= (100-t)^2\left((100-t)^{-1} + c\right) \\ &= (100-t) + c(100-t)^2. \end{aligned}$$

(Note that this agrees with our result from variation of parameters.)

Once we have the general solution to the nonhomogeneous equation, we apply the initial condition to pin down the value of the parameter c:

$$0 = S(0) = (100-0) + c(100-0)^2 \qquad \Leftrightarrow \qquad c = -\frac{1}{100}.$$

We conclude that, for any $t \in [0, 100]$, the amount of salt (in kg) in the tank at time t is given by

$$S(t) = -\frac{1}{100}(100-t)^2 + (100-t).$$

2.4.2 Newton's Law of Cooling

Let T denote the temperature of an object, M denote the (uniform) temperature of its surroundings. Then

$$\frac{dT}{t} = k(M-T),$$

where $k \in \mathbf{R}_{>0}$ denotes a constant of proportionality. This coincides with our experience: If the surroundings are hotter than an object ($M - T > 0$), then the object will heat up ($T' > 0$). If the surroundings are colder than an object, then the object will cool down.

2.5 Nonlinear Models

Key Ideas

- Autonomous ODE
- Phase line
- Classification of equilibrium points
- Logistic equation
- Bifurcation; bifurcation point

2.5.1 Autonomous First-Order ODEs

Definition 2.12. A first-order ODE is **autonomous** (also called **time-invariant** or **stationary**) if it has the form

$$\frac{dy}{dt} = f(y). \tag{2.5.1}$$

The slopes depend only on y, not t. Thus the direction field in the (t, y) plane looks the same along any vertical line (i.e. at any value of t). What does this imply about the following features?

- **Isoclines.** By definition, isoclines are curves with constant slope, $\frac{dy}{dt} = c$. Here, $\frac{dy}{dt} = f(y)$ does not depend on t, so $\frac{dy}{dt} = 0$. Thus, if we have an isocline, then its constant slope must be $c = 0$, i.e. it must be a horizontal line. This agrees with the form of the ODE (2.5.1) : $\frac{dy}{dt}$ depends only on y, not t, so changing t (i.e. moving horizontally in the (t, y) plane) does not change the slope.

- **Solution curves.** Let $y(t)$ be a solution curve. Any horizontal translation y_1 of y can be written as $y_1(t) = y(t - t_0)$ for some fixed t_0. By the chain rule,
$$\frac{dy_1}{dt} = \frac{d}{dt}\left(y(t - t_0)\right) = y'(t - t_0) \cdot 1 = f(t - t_0, y) = f(y),$$
because by hypothesis y' does not depend on t. Thus $y_1(t)$ is also a solution. This says that new solution curves can be obtained from old ones by horizontal translation.

Phase Line

For autonomous first-order ODEs, the slope $\frac{dy}{dt}$ depends only on the single variable y, so we can capture all slope information in one dimension, i.e. along the y-line. Plotting the sign of the slope at each point on the y-axis, we obtain what is called the **phase line**. The phase line enables easy classification of equilibrium points P:

- **stable** (or **sink**): arrows on either side of P point toward P
- **unstable** (or **source**): arrows on either side of P point away from P
- **semistable** (or **node**): arrows on one side point toward P, on the other, away from P

Example 2.13 (Example 2.5.1)**.** Analyze the solutions to the first-order nonlinear ODE

$$y' = (y-1)(y-3)(y-5)^2.$$

Solution. *(plot phase line (y-axis), then add t-axis and sketch solution curves in (t, y) plane ; see Figure 2.5.6(b) (p 91))* Three equilibrium solutions: $y \equiv 1, 3, 5$. Analyzing the sign of y' in each region of the y-axis, we see these equilibria are stable, unstable, and semistable, respectively.

2.5.2 Logistic Equation

The growth equation

$$\frac{dy}{dt} = ky$$

models a growth rate that is always directly proportional to y, and unbounded growth. This is not realistic in most settings. Replacing the constant k by a function $k(y)$ allows for the growth rate to vary with the population size y. Specifically, setting

$$k(y) = r - ay,$$

with $a, r \in \mathbf{R}_{>0}$, gives the **logistic equation**. Let $L = \frac{r}{a}$. Then the logistic equation writes as

$$\frac{dy}{dt} = r\left(1 - \frac{y}{L}\right) y.$$

The parameter r is called the **intrinsic growth rate**. The parameter L is called the **carrying capacity**.

(sketch phase line (two equilibrium solutions), solution curves ; see Figure 2.5.7 (p 92))

The logistic equation, like all autonomous DEs, is separable. The logistic equation can be solved using partial fraction decomposition:

$$\begin{aligned}
\frac{dy}{r(1-\frac{y}{L})y} &= r\,dt \\
\left(\frac{1}{y} + \frac{\frac{1}{L}}{1-\frac{y}{L}}\right) dy &= r\,dt \\
\ln\left|\frac{y}{1-\frac{y}{L}}\right| = \ln|y| + \ln\left|1 - \frac{y}{L}\right| &= rt + c. \qquad (2.5.2)
\end{aligned}$$

If an initial condition $y(t_0) = y_0$ is given, then we can determine the constant c of integration. Moreover, using our qualitative analysis of the logistic equation above, we see that

- if $y_0 > L$, then $y(t) > L > 0$ for all t, so
$$|y| = y, \qquad \left|1 - \frac{y}{L}\right| = \frac{y}{L} - 1.$$
- if $0 < y_0 < L$, then $0 < y(t) < L$ for all t, so
$$|y| = y, \qquad \left|1 - \frac{y}{L}\right| = 1 - \frac{y}{L}.$$
- if $y_0 < 0$, we can analyze as above, but for models of population, this region of the (t, y) plane is not relevant.

In either case when $y_0 > 0$ *(for own edification, look at $y_0 < 0$, too, see what holds)*, using the above to rewrite (2.5.2) without absolute values, exponentiating, letting $C = e^c$, and applying the initial condition $y(0) = y_0$ (note here we assume $t_0 = 0$), we obtain
$$y(t) = \frac{L}{1 + \left(\frac{L}{y_0} - 1\right) e^{-rt}}.$$

Threshold Equation

For some populations, if the population level falls below a **threshold level** T, then the population will end up extinct. *(sketch the types of solution curves we want — see Figure 2.5.8)* Note that this phase line is the negative of the phase line for the logistic equation, in the sense that all signs are flipped. So insert a minus sign into the original ODE, and change L to T:
$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right) y.$$
Alternatively (more unifying view), just take $r < 0$ in the logistic equation. *(rewrite this subsection under this unifying view? Yes! Do it!)* The solutions to the threshold equation have the same form as those for the logistic equation, but with e^{rt} in the denominator.

2.5.3 Bifurcation

As we saw with the logistic equation, changing the parameter values (e.g., in the logistic equation, changing r from positive to negative) can change the nature of solutions: stability, even number. A **bifurcation point** is a parameter value at which the number or type of equilibria changes.

Example 2.14 (Example 2.5.3)**.** Consider the first-order ODE

$$y' = y(a - y). \tag{2.5.3}$$

Discuss bifurcation.

Solution. Equation (2.5.3) describes a 1-parameter family of ODEs. Equilibria at $y = 0$ and $y = a$. Classify based on sign of a. Bifurcation point at $a = 0$. *(include Figures 2.5.10 and 2.5.11)*

(include more examples from Blanchard Devaney Hall?)

Chapter 3

Linear Algebra

3.1 Matrices

Key Ideas

- How to add and multiply matrices; transpose
- Special matrices: identity, zero, diagonal, symmetric
- The standard inner product of vectors; orthogonal; norm
- How to differentiate matrices; product rule

This section introduces concepts and notation and will be dry.

(illustrate all the following with examples)

A **matrix** is a rectangular array. element or entry, row, column. Elements in a matrix are typically indexed with subscripts denoting row, then column: $a_{i,j}$ denotes the element in row i and column j.

The **order** of a matrix are its dimensions, written row $\times$ columns.

The **diagonal** of a matrix are the elements $a_{i,i}$.

The special cases when a matrix has only one column (i.e. an $m \times 1$ matrix) or one row (i.e. a $1 \times n$ matrix) are sometimes called a **column vector** and a **row vector**, respectively. *(add warning that while not inaccurate, this terminology can be misleading?)*

Two matrices are equal if they have the equal order and all corresponding elements are equal.

A **zero matrix** is a matrix with all elements 0.

A **diagonal matrix** is a matrix whose only nonzero elements (if any) appear on the diagonal.

An **identity matrix** is a square diagonal matrix with all diagonal elements equal to 1.

Matrices are added (and hence subtracted) element-wise.

Matrices can be multiplied by a scalar, by multiplying all elements of the matrix by the scalar.

For example,

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} - 2\begin{bmatrix}2 & -1 & 1\\-1 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} + \begin{bmatrix}-4 & 2 & -2\\2 & -4 & -6\end{bmatrix} = \begin{bmatrix}-3 & 4 & 1\\6 & 1 & 0\end{bmatrix}.$$

The set of all $m \times n$ matrices with elements in a field k (e.g., $\mathbf{R}$ or $\mathbf{C}$) is sometimes denoted $M_{m\times n}(k)$. If we equip this set with the operations of matrix addition and scalar multiplication by elements of that field, then we get a vector space. (We'll learn more about vector spaces in Section 3.5.)[1]

(provide discussion or reference to more robust discussion of inner product)

An **inner product** or **scalar product** is a function that eats two vectors (from the same vector space) and spits out a scalar (from the underlying field). We might see these properly defined later. For now, we'll consider a special case to help define matrix multiplication: Given two vectors

$$\mathbf{u} = (u_1, \ldots, u_n) \qquad \text{and} \qquad \mathbf{v} = (v_1, \ldots, v_n)$$

of the same length n, form their standard inner product by multiplying corresponding elements, then adding everything up:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n.$$

Two vectors are **orthogonal** if their inner product is 0.

The (standard) **norm** or **length** of a vector $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

3.1.1 Matrix Multiplication

For matrix multiplication, order matters! That is, $AB \neq BA$, in general!

We can multiply two matrices when the number of columns of the first equals the number of rows of the second. In this case, we multiply each row i of the first matrix by each row j of the second matrix to get the (i, j) element of the product. Note that row A_i is a row vector with n elements, and column B_j is a column vector with n elements, so each computation is the standard inner product defined above.

(example of matrix multiplication)

(mention view of matrix multiplication as first matrix acting on the rows or columns of the second matrix. i.e. block multiplication. this view is helpful in general, and in particular in subsequent discussions of Gauss–Jordan reduction algorithm)

Matrix multiplication is associative and distributive on left and right, but not commutative in general. Multiplication by identity matrices and zero matrices behave like multiplication by 1 and 0 in a field.

[1] In fact, when $m = n$, we can equip the set $M_{n\times n}(k)$ with another operation — matrix multiplication — in addition to matrix addition and scalar multiplication, and we get a structure called a (noncommutative) algebra over k.

The **transpose** of an $m \times n$ matrix A, denoted A^T, is the matrix obtained by reflecting the elements of A across the diagonal:

$$A^T_{i,j} = A_{j,i}.$$

We can think of this as "reflecting the matrix A through the main diagonal".

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

If we think about transpose as reflection, note that the elements on the main diagonal (i.e. $a_{1,1} = 1$ and $a_{2,2} = 5$) don't move.

Properties of the matrix transpose:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $kA^T = kA^T$
- $AB^T = B^T A^T$ (note the order!)

A square matrix A is **symmetric** if it's symmetric, i.e. if $A^T = A$.

For matrices with entries that are (differentiable) functions, we define the derivative of a matrix to be the matrix of corresponding derivatives. Matrix differentiation follows rules we are used to from single-variable calculus; just be careful to remember that matrix multiplication is not commutative, so we must write rules like the product rule "correctly".

Let $\mathbf{A}, \mathbf{B}$ be compatible matrices with entries differentiable functions of t, and let c be a scalar.

- $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
- $(c\mathbf{A})' = c\mathbf{A}'$
- $(\mathbf{AB})' = \mathbf{A}'\mathbf{B} + \mathbf{AB}'$

Example 3.1 (Example 3.1.14)**.** Let

$$\mathbf{A} = \begin{bmatrix} \sin t & \cos t \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} t^2 & t \\ 2t & 3 \end{bmatrix}.$$

Compute $(\mathbf{AB})'$ directly and via the product rule.

Solution. Direct computation. Matrix multiplication gives

$$\mathbf{AB} = \begin{bmatrix} t^2 \sin t + 2t \cos t & t \sin t + 3 \cos t \\ t^2 & t \end{bmatrix}.$$

The derivative of a matrix is computed by differentiating each entry, so, using the product rule where appropriate, we obtain

$$\begin{aligned}(\mathbf{AB})' &= \begin{bmatrix} 2t\sin t + t^2\cos t + 2\cos t - 2t\sin t & \sin t + t\cos t - 3\sin t \\ 2t & 1 \end{bmatrix} \\ &= \begin{bmatrix} (t^2+2)\cos t & t\cos t - 2\sin t \\ 2t & 1 \end{bmatrix}.\end{aligned}$$

Product rule. Computing the derivative of $\mathbf{A}$ and $\mathbf{B}$ (entry-wise), we find

$$\mathbf{A}' = \begin{bmatrix} \cos t & -\sin t \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B}' = \begin{bmatrix} 2t & 1 \\ 2 & 0 \end{bmatrix}.$$

The matrix version of the product rule gives[2]

$$\begin{aligned}(\mathbf{AB})' &= \mathbf{A}'\mathbf{B} + \mathbf{A}\mathbf{B}' \\ &= \begin{bmatrix} \cos t & -\sin t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} t^2 & t \\ 2t & 3 \end{bmatrix} + \begin{bmatrix} \sin t & \cos t \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2t & 1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} t^2\cos t - 2t\sin t & t\cos t - 3\sin t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2t\sin t + 2\cos t & \sin t \\ 2t & 1 \end{bmatrix} \\ &= \begin{bmatrix} (t^2+2)\cos t & t\cos t - 2\sin t \\ 2t & 1 \end{bmatrix}.\end{aligned}$$

[2]Remember to mind the order of multiplication! Does the order of addition matter?

3.2 Systems of Linear Equations

Key Ideas

- Questions of existence, uniqueness, and effectiveness
- System of linear equations; matrix representation
- Elementary row operations
- Reduced row echelon form (RREF)
- Gauss–Jordan reduction algorithm for solving linear systems
- Rank of a matrix

A **system of linear equations** is a set of (one or more) linear equations, e.g.,

$$\begin{aligned} 3x - 3y + 2z &= 6 \\ 3x + 6y - 2z &= 18. \end{aligned}$$

that are to be solved *simultaneously*. That is, a solution to a system of linear equations must solve all the equations in the system.

The general form for a system of linear equations is

$$\begin{aligned} a_{1,1}x_1 + \ldots + a_{1,n}x_n &= b_1 \\ &\vdots \\ a_{m,1}x_1 + \ldots + a_{m,n}x_n &= b_m. \end{aligned}$$

Using matrix multiplication, we can rewrite this system, on the left separating the coefficients $a_{i,j}$ from the variables x_k:

$$\begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \ldots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Denoting these matrices $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$, respectively, we can write the original system of linear equations in the compact matrix form

$$\mathbf{Ax} = \mathbf{b}.$$

A system of equations is **homogeneous** if $\mathbf{b} = \mathbf{0}$.

With linear systems, and (mathematical) life more generally, mathematicians and others are interested in questions of existence, uniqueness, and effectiveness:

- Existence : Does a solution exist?
- Uniqueness : Is the solution unique? (How many solutions are there?)
- Effectiveness : How do we find solutions?

3.2.1 Elementary Row Operations

Let $\mathbf{A}$ be an $m \times n$ matrix, and denote row i of $\mathbf{A}$ by $\mathbf{A}_{i,\bullet}$. There are three important operations we can perform on the rows of $\mathbf{A}$, called **elementary row operations**.

1. Interchange row i and row j.
2. Multiply row i by a nonzero (!) constant c.
3. Add a constant multiple c times row j to another row i.[3]

Note that these elementary row operations correspond to left-multiplying our matrix $\mathbf{A}$ by the following matrices, respectively (blank entries denote 0):

$$\begin{bmatrix} 1 & & & \\ & \ddots & 1 & \\ & 1 & \ddots & \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & c & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & c & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

When dealing with systems of linear equations, these elementary row operations correspond to the following operations on the equations:

1. Interchanging the order of the equations.
2. Scaling an equation by a nonzero constant.
3. Adding a constant multiple of one equation to another.

We will see that these operations produce an **equivalent system**, i.e. another system of linear equations with exactly the same solutions as the original system.[4]

There is an analogous notion of elementary column operations, corresponding to right-multiplication of $\mathbf{A}$ by analogous matrices.

3.2.2 Row Reduced Echelon Form

One convenient method for solving a system $\mathbf{Ax} = \mathbf{b}$ of linear equations involves applying elementary row operations to the **augmented matrix**, i.e. the matrix obtained by appending the $m \times 1$ vector $\mathbf{b}$ to the end of the $m \times n$ matrix of coefficients $\mathbf{A}$:

$$\left[\begin{array}{c:c} \mathbf{A} & \mathbf{b} \end{array}\right] = \left[\begin{array}{ccc:c} a_{1,1} & \dots & a_{1,n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & \dots & a_{m,n} & b_m \end{array}\right].$$

We continue applying elementary row operations to the augmented matrix until we obtain a matrix in **reduced row echelon form (RREF)**, i.e. a matrix satisfying the following properties:

[3] Note that here we allow $c = 0$, and $i = j$. What happens in each case?

[4] Briefly, elementary row operations produce equivalent systems because they correspond to left-multiplication of both the coefficient matrix $\mathbf{A}$ and the constant vector $\mathbf{b}$ by invertible (!) matrices.

1. All rows of entirely zeros are at the bottom.
2. The leftmost nonzero entry of each row, called a **pivot**, is 1.
3. Each pivot lies strictly to the right of the pivot in the row above it.
4. Each pivot is the only nonzero entry in its column.

A column containing a pivot is called a **pivot column**.[5]

One can show that the RREF of a matrix is unique. That is, any sequence of elementary row operations that yields a matrix in RREF yields the same matrix in RREF *(reword for clarity)*.

A matrix satisfying conditions 1–3 is said to be in **row echelon form (REF)**. In particular, a matrix in REF can have nonzero entries lying above a pivot. The REF is sometimes preferred to the RREF for computational reasons. *(elaborate)*

The **row reduction algorithm** (also known as **Gauss–Jordan reduction algorithm**, or **Gauss–Jordan elimination**) is an effective method for solving a system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ ("effective" in the sense that it implements a procedure).

1. **Write augmented matrix.** Form the augmented matrix $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{b} \end{array}\right]$.
2. **Row reduce.** Use elementary row operations to put the augmented matrix in RREF.
3. **Read solutions.** Read off solutions from the RREF matrix. Because elementary row operations produce an equivalent system, these solutions are exactly the solutions to our original system of linear equations.

(maybe relocate this next bit? but keep in mind the second solution to the subsequent example uses these ideas)

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in $\mathbf{R}$. One can check that

$$\begin{aligned} L : \mathbf{R}^n &\to \mathbf{R}^m \\ \mathbf{x} &\mapsto \mathbf{A}\mathbf{x} \end{aligned}$$

is a linear map. Therefore the superposition and nonhomogeneous principles apply to L. In particular, as we have seen, any solution $\mathbf{x}$ to the nonhomogeneous linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.2.1}$$

can be written in the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h,$$

[5] Note that elementary row operations never interchange columns, so a pivot column in the RREF corresponds to the same column in the original matrix.

where $\mathbf{x}_p$ is a particular solution to (3.2.1), and $\mathbf{x}_h$ is a general solution to the associated homogeneous equation (i.e. (3.2.1) with $\mathbf{b} = \mathbf{0}$).

The RREF of the augmented matrix for the corresponding homogeneous equation is the same as the RREF of the augmented matrix for the nonhomogeneous equation, with the final column (the one corresponding to $\mathbf{b}$) replaced by $\mathbf{0}$. This is because the RREF of $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{b} \end{array}\right]$ is obtained by applying the same elementary row operations to the coefficient matrix $\mathbf{A}$ and the constant vector $\mathbf{b}$ — $\mathbf{b}$ can be any vector! When $\mathbf{b} = \mathbf{0}$, the elementary row operations on $\mathbf{b} = \mathbf{0}$ always return $\mathbf{0}$.

Let's learn by example.

Example 3.2 (Example 3.2.6)**.** Use the row reduction algorithm to solve the system of linear equations

$$\begin{aligned} 3x - 3y + 2z &= 6 \\ 3x + 6y - 2z &= 18. \end{aligned}$$

Solution. The corresponding augmented matrix is

$$\left[\begin{array}{ccc:c} 3 & -3 & 2 & 6 \\ 3 & 6 & -2 & 18 \end{array}\right].$$

Applying elementary row operations, we obtain the (unique) RREF

$$\begin{aligned} \left[\begin{array}{ccc:c} 3 & -3 & 2 & 6 \\ 3 & 6 & -2 & 18 \end{array}\right] &\xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc:c} 3 & -3 & 2 & 6 \\ 0 & 9 & -4 & 12 \end{array}\right] \\ &\xrightarrow{R_1 = \frac{1}{3}R_1, R_2 = \frac{1}{9}R_2} \left[\begin{array}{ccc:c} 1 & -1 & \frac{2}{3} & 2 \\ 0 & 1 & -\frac{4}{9} & \frac{4}{3} \end{array}\right] \\ &\xrightarrow{R_1 = R_1 + R_2} \left[\begin{array}{ccc:c} 1 & 0 & \frac{2}{9} & \frac{10}{3} \\ 0 & 1 & -\frac{4}{9} & \frac{4}{3} \end{array}\right]. \end{aligned} \tag{3.2.2}$$

Checking the rows of the RREF, we see the system is consistent (no rows with all zeros except a nonzero number after the dashed line). Columns 1 and 2 are pivot columns, so the corresponding variables x and y are basic variables. The remaining variable, z, is a free variable. Using $z = t$ as our parameter, we conclude that the solutions to our original system are

$$\mathbf{x} = \begin{bmatrix} \frac{10}{3} - \frac{2}{9}t \\ \frac{4}{3} + \frac{4}{9}t \\ t \end{bmatrix},$$

for any $t \in \mathbf{R}$.

Alternatively, we can apply the superposition and nonhomogeneous principles. Solving the nonhomogeneous system gives the RREF matrix in (3.2.2); setting $t = 0$ gives a (i.e. one) particular solution

$$\mathbf{x}_p = \begin{bmatrix} \frac{10}{3} \\ \frac{4}{3} \\ 0 \end{bmatrix}.$$

As discussed above, applying the row reduction algorithm to the corresponding homogeneous equation yields the same RREF matrix to the left of the dashed line, and all zeros to the right:

$$\left[\begin{array}{ccc:c} 1 & 0 & \frac{2}{9} & 0 \\ 0 & 1 & -\frac{4}{9} & 0 \end{array}\right].$$

This homogeneous system has the general solution

$$\mathbf{x}_h = \begin{bmatrix} -\frac{2}{9}t \\ \frac{4}{9}t \\ t \end{bmatrix},$$

for any $t \in \mathbf{R}$. Hence by the nonhomogeneous principle, any solution has the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} \frac{10}{3} \\ \frac{4}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{2}{9} \\ \frac{4}{9} \\ 1 \end{bmatrix},$$

for any $t \in \mathbf{R}$.

A system of equations is **consistent** if it has at least one solution. If it has no solutions, then it is **inconsistent**.

Proposition 3.3. *Consider the system of linear equations with corresponding matrix equation* $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{b} \end{array}\right]$.

(a) The system is inconsistent if and only if its RREF has a row of the form $\left[\begin{array}{ccc:c} 0 & \dots & 0 & c \end{array}\right]$ *for some nonzero constant* $c \neq 0$.[6]

(b) The system has a unique solution if and only if every column in its RREF corresponding to $\mathbf{A}$ *is a pivot column.*

(c) The system is ***underdetermined*** *(and hence has more than one solution) if and only if it is consistent and one or more columns in its RREF corresponding to* $\mathbf{A}$ *is not a pivot column.*

If column j is a pivot column, then the corresponding variable x_j is called a **basic variable** or **leading variable**. All other variables are called **free variables**.

Definition 3.4. The **rank** of a matrix is the number of pivot columns in its RREF.

Because the number of pivot columns equals the number of nonzero rows (think about this!), the rank of a matrix is also the number of nonzero rows in its RREF.

Let $\mathbf{A}$ be an $m \times n$ matrix, and consider the system of m equations in n variables corresponding to

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{3.2.3}$$

[6]Note that such a row corresponds to the equation $0 = 0x_1 + \ldots + 0x_n = c \neq 0$, a contradiction or "inconsistency".

- If $\operatorname{rank} \mathbf{A} = n$, then (3.2.3) has a unique solution.
- If $\operatorname{rank} \mathbf{A} < n$, then either (3.2.3) is inconsistent, or solutions are not unique. (Which case holds depends on $\mathbf{b}$.)

3.3 The Inverse of a Matrix

Key Ideas

- Inverse matrix
- Determining existence, computing inverse using RREF

Definition 3.5. Let $\mathbf{A}$ be an $n \times n$ matrix. An **inverse matrix** of $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I}_n \qquad \text{and} \qquad \mathbf{BA} = \mathbf{I}_n,$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. If an inverse matrix exists, then $\mathbf{A}$ is **invertible**, and we typically denote its inverse by $\mathbf{A}^{-1}$.

Remark 3.6. Remarks.

1. From the definition, one can show that if an inverse matrix exists, then it is unique.
2. Another term for invertible is **nonsingular**. A matrix that is not invertible is **singular**.
3. For a nonsquare $m \times n$ matrix $\mathbf{A}$, we can define the notion of a **generalized inverse** or **pseudoinverse**. See, for example, the Moore–Penrose inverse.

Proposition 3.7 (Properties of the Matrix Inverse)**.** *Let* $\mathbf{A}, \mathbf{B}$ *be invertible* $n \times n$ *matrices over the same field* k*. Then the following are true.*

(a) $\mathbf{A}^{-1}$ *is invertible, and*

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}.$$

(b) $\mathbf{AB}$ *is invertible, and (note the order!)*

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

(c) $\mathbf{A}^{\mathrm{T}}$ *is invertible, and*

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}.$$

3.3.1 Computing the Inverse Matrix

Let $\mathbf{A}$ be an $n \times n$ matrix. We can use the reduced row echelon form (RREF) and the row reduction algorithm introduced in Section 3.2 to determine whether $\mathbf{A}$ has an inverse, and, if so, to compute $\mathbf{A}^{-1}$.

Here's the procedure:

1. Form the $n \times (2n)$ augmented matrix $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{I}_n \end{array}\right]$.
2. Apply the row reduction algorithm.
3. If the left $n \times n$ block is now $\mathbf{I}_n$, then the right $n \times n$ block is $\mathbf{A}^{-1}$. Otherwise, $\mathbf{A}$ is not invertible.

Pictorially, if $\mathbf{A}$ is invertible, then this procedure transforms

$$\left[\begin{array}{c:c} \mathbf{A} & \mathbf{I}_n \end{array}\right] \qquad \text{into} \qquad \left[\begin{array}{c:c} \mathbf{I}_n & \mathbf{A}^{-1} \end{array}\right].$$

Why does this work? View this problem as follows. We are looking for an $n \times n$ matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n. \tag{3.3.1}$$

The matrix $\mathbf{A}^{-1}$ is unknown; denote its columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$, which we view as $n \times 1$ vectors. Viewing (3.3.1) as block multiplication, with each column of $\mathbf{A}^{-1}$ and $\mathbf{I}_n$ separate, we see that (3.3.1) is equivalent to the n matrix equations

$$\mathbf{A}\mathbf{x}_i = \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the $n \times 1$ vector with 1 in entry i and 0s everywhere else. To solve for each $\mathbf{x}_i$, we can apply the row reduction algorithm to the augmented matrix $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{e}_i \end{array}\right]$. Because the row reduction algorithm corresponds to left-multiplication, we can append as many $n \times 1$ vectors $\mathbf{b}_i$ to $\mathbf{A}$ as we like, run the algorithm, and simultaneously produce solutions to all the equations $\mathbf{A}\mathbf{x} = \mathbf{b}_i$.

Example 3.8. *(can skip, given next example)* Determine whether the following matrices are invertible and, if so, find their inverse.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -1 & -1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

Solution. Should find

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ -1 & 1 & 1 \\ -3 & 2 & 1 \end{bmatrix},$$

and $\mathbf{B}$ is not invertible.

Example 3.9 (Example 3.3.4)**.** Consider the following third-order homogeneous ODE:

$$y^{(3)} - 2y^{(2)} - y^{(1)} + 2y = 0,$$

with initial conditions

$$y(0) = b_0, \qquad y'(0) = b_1, \qquad y''(0) = b_2,$$

for constants $b_i \in \mathbf{R}$. We will see *(link ref)* that the general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}, \tag{3.3.2}$$

for $c_i \in \mathbf{R}$. (Though we haven't yet seen how to find this solution, one can check it by substituting it into the original IVP.)

Suppose we wanted to (or had to) solve this IVP for various initial conditions, i.e. for various vectors $\mathbf{b} = (b_0, b_1, b_2)$. We can simplify our work by leaving the b_i general, solving for the c_j in terms of the b_i, then adjusting the b_i as needed. And we can use matrix algebra to do this.

The initial conditions are on the 0th, first, and second derivatives of the solution y. So, given (3.3.2), we compute

$$\begin{aligned} y'(t) &= 2c_1 e^{2t} + c_2 e^{t} - c_3 e^{-t} \\ y''(t) &= 4c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}. \end{aligned}$$

Applying the initial conditions, we get

$$\begin{aligned} y(0) &= c_1 + c_2 + c_3 \underset{\text{set}}{=} b_0 \\ y'(0) &= 2c_1 + c_2 - c_3 \underset{\text{set}}{=} b_1 \\ y''(0) &= 4c_1 + c_2 + c_3 \underset{\text{set}}{=} b_2, \end{aligned}$$

which we can write in matrix form as (remember that the c_i are unknown, i.e. our variables)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}.$$

If we could find the inverse (if it exists) of the matrix $\mathbf{A}$ of coefficients, then we could multiply both sides of this equation by $\mathbf{A}^{-1}$ and have a solution for the c_i in terms of the b_j. We know how to do this! Applying the row reduction algorithm to $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{I}_n \end{array}\right]$, we find

$$\mathbf{A}^{-1} = \frac{1}{6} \begin{bmatrix} -2 & 0 & 2 \\ 6 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix}.$$

Thus $\mathbf{c} = \mathbf{A}^{-1}\mathbf{b}$ gives the c_i, for any initial condition vector $\mathbf{b} = (b_0, b_1, b_2)$.

3.4 Determinants and Cramer's Rule

Key Ideas

- Determinant; characterizing properties
- How to compute the determinant recursively

Once we know what all the words mean, the following definition of determinant is powerful.

Definition 3.10. Let $n \in \mathbf{Z}_{>0}$, and let A be a field (more generally, a commutative ring with multiplicative identity) of scalars. The **determinant** is the unique function

$$\det : M_{n\times n}(A) \to A$$

that is n-linear and alternating in the columns, and normalized so that $\det \mathbf{I} = 1$.

Here $M_{n\times n}(A)$ denotes the set of $n \times n$ matrices with entries in A. To say that det is n-linear means it is linear in each of its n columns. To say that det is alternating means that if one column of $\mathbf{A}$ is a linear combination of the others, then $\det \mathbf{A} = 0$. We'll learn about linear combinations in Section **??**. *(add ref)*

In the short-term, we'll use determinants to compute things. The following recursive procedure will serve us well.

Definition 3.11. Base case. The determinant of a 1×1 matrix is given by its entry:

$$\det \begin{bmatrix} a \end{bmatrix} = a.$$

Inductive step. The determinant of an $n \times n$ matrix $\mathbf{A}$ is given by

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{i,j} (-1)^{i+j} \det \mathbf{M}_{i,j}, \tag{3.4.1}$$

where $\mathbf{M}_{i,j}$ denotes the (i, j) **minor** of $\mathbf{A}$, i.e. the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from $\mathbf{A}$.

Remark 3.12. Remarks.

1. The quantity $(-1)^{i+j} \det \mathbf{M}_{i,j}$ is called the **cofactor** of the entry $a_{i,j}$ of $\mathbf{A}$.

2. Notice that at each stage of the recursion, we feed det an $n \times n$ matrix, and it returns an expression involving the determinant of (finitely many) $(n-1) \times (n-1)$ matrices. So at each stage, the size of the problem is reduced by exactly 1. Eventually this size will be 1×1, and we can apply the known formula for the base case.

3. Definition (3.4.1) is obtained by summing the cofactors of $a_{i,j}$, scaled by $a_{i,j}$, along row i. This is called **expansion along row** i. One can do the same procedure along column j, i.e. **expansion along column** j, giving

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{i,j}(-1)^{i+j} \det \mathbf{M}_{i,j}.$$

One can show that both procedures, for any row i or column j, give the same result.

For example, using this iterative definition along row 1, we can compute the determinant of a 2×2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a(-1)^{1+1} \det \begin{bmatrix} d \end{bmatrix} + b(-1)^{1+2} \det \begin{bmatrix} c \end{bmatrix} = ad - bc.$$

Example 3.13. Compute the determinant of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 0 & 5 & 2 \\ 0 & 0 & 4 & 0 \\ 2 & 0 & 2 & 3 \end{bmatrix}.$$

Solution. At each stage of the recursive process, we are free to choose any row or column to expand along. Let's choose what makes our lives easy!

To start, expand along column 2:

$$\det \mathbf{A} = 2(-1)^{1+2} \det \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

To compute the determinant on the right, expand along row 2:

$$\det \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 0 \\ 2 & 2 & 3 \end{bmatrix} = 4(-1)^{2+2} \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Finally, expand along any row or column, or use the "alternative" base-case formula for the determinant of 2×2 matrices, to get

$$\det \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 0 \\ 2 & 2 & 3 \end{bmatrix} = 4(3 - 4) = -4.$$

Using this, we find

$$\det \mathbf{A} = -2(-4) = 8.$$

3.4.1 Elementary Row Operations and Determinants

The elementary row operations have the following effects on the determinant:

1. Interchanging two distinct (!) rows (or columns) changes the sign.
2. Multiplying a row (or column) by a constant c multiplies the determinant by c.
3. Adding a constant multiple of one row (or column) to another does not change the determinant.

All of these are consequences of the fact that elementary row (or column) operations correspond to left- (or right-) multiplication, and the following fact:

Proposition 3.14. *Let* $\mathbf{A}, \mathbf{B}$ *be* $n \times n$ *matrices. Then*

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}).$$

(figure out how you want to present equivalent characterizations of invertibility. see pp 151, 159. do not (!!!) have students memorize)

Other properties of determinants.

Proposition 3.15. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix.*

(a) $\det \mathbf{A}^T = \det \mathbf{A}$.

(b) If $\det \mathbf{A} \neq 0$*, then* $\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1}$.

(c) If $\mathbf{A}$ *is an upper-triangular or lower-triangular matrix (i.e. all nonzero entries lie on and above or on and below the main diagonal), then* $\det \mathbf{A}$ *is the product of (just) the diagonal entries.*

(d) If $\mathbf{A}$ *has a row or column of all zeros, then* $\det \mathbf{A} = 0$.

(e) If one row (or column) of $\mathbf{A}$ *is a linear combination of the others, then* $\det \mathbf{A} = 0$*. In particular, if* $\mathbf{A}$ *has two rows or two columns that are equal, then* $\det \mathbf{A} = 0$.

3.4.2 Cramer's Rule

Cramer's rule is a method to solve an $n \times n$ system of linear equations that has a unique solution. As such, its scope is limited. Moreover, its computational complexity doesn't beat the row reduction algorithm. A naive implementation of Cramer's method has complexity $O(n!n)$; methods using LU decomposition can reduce this complexity to $O(n^4)$; even more clever implementations can reduce this complexity to $O(n^3)$. The row reduction algorithm has complexity $O(n^3)$. For more details, see [1], [2].

Proposition 3.16 (Cramer's Rule)**.** *Let* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *be the matrix representation of an* $n \times n$ *system of linear equations, and let* $\det \mathbf{A} \neq 0$ *(so the system has a unique solution). Let* $\mathbf{A}_i$ *denote the* $n \times n$ *matrix obtained by replacing column* i *of* $\mathbf{A}$ *with the column vector* $\mathbf{b}$*. Then for each* i*,*

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}}.$$

3.5 Vector Spaces and Subspaces

Key Ideas

- Vector space: definition and examples
- Subspace

Vector spaces have linear structure: One can "add" vectors, and "multiply" a vector by a scalar, but one cannot "multiply" vectors.[7] Thus objects in a vector space look like $a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$, where the a_i are scalars and the $\mathbf{v}_i$ are vectors — a linear expression in the vectors $\mathbf{v}_i$.

Nonlinear phenomena in the real world can be approximated by linear structures. This makes vector spaces a powerful tool.

3.5.1 Definition

Definition 3.17. Let k be a field (e.g., $\mathbf{R}$ or $\mathbf{C}$). A **vector space** over k is a set V with two operations

- vector addition: $+ : V \times V \to V$
- scalar multiplication: $\cdot : k \times V \to V$

that satisfy the following axioms: For all $\mathbf{v}, \mathbf{v}_i \in V$, for all $a, a_i \in k$,

VA Vector addition

1. associative: $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$
2. commutative: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$
3. identity: There exists a **zero vector**, denoted $\mathbf{0} \in V$, such that for all $\mathbf{v} \in V$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$
4. inverse: For all $\mathbf{v} \in V$, there exists its **negative** $-\mathbf{v} \in V$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

SM Scalar multiplication

1. identity: For all $\mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$
2. compatibility: $a_1(a_2\mathbf{v}) = (a_1a_2)\mathbf{v}$

DL Distributive laws

1. $(a_1 + a_2)\mathbf{v} = a_1\mathbf{v} + a_2\mathbf{v}$

[7] If the vector space (secretly or not-so-secretly) admits the structure of an algebra, then we can define a multiplication of vectors.

2. $a(\mathbf{v}_1 + \mathbf{v}_2) = a\mathbf{v}_1 + a\mathbf{v}_2$

Remark 3.18. Three remarks.

1. The elements of V are called **vectors**. The elements of k are called **scalars**.

2. As functions, the operations of vector addition and scalar multiplication are **closed**. That is, adding two vectors in V must always give a vector in V, and scalar multiplication must always give a vector in V.

3. Don't memorize or worry about this definition, at first. Instead, let's focus on examples. The abstract definition above pulls out the "important" features of all these examples.

Example 3.19. Some examples of vector spaces. *(give very concrete illustrations!)*

1. Let k be a field. k^n, the set of n-vectors with components in k, equipped with the usual operations of component-wise addition of vectors and scalar multiplication, is an (n-dimensional) vector space over k.

2. $M_{m\times n}(k)$, the set of $m \times n$ matrices with entries in the field k, equipped with the usual matrix addition and scalar multiplication, is a vector space over k (of dimension mn).

3. Let X be a nonempty set, and let k be a field. The set of functions $f : X \to k$, equipped with pointwise addition and scalar multiplication of functions, i.e.

$$(f + g)(x) = f(x) + g(x), \qquad (af)(x) = af(x),$$

forms a vector space over k.

4. The set of solutions to a linear homogeneous (!) ODE, equipped with the addition and scalar multiplication of functions above, form a vector space over k. This is exactly what the superposition principle, discussed in Section *(add ref)*, is saying — we just didn't yet have the language of vector spaces when we first met it. The dimension of this vector space equals the degree of the ODE. (The adjective "homogeneous" is critical in this example. Why?)

5. Let k be a field. The set $\mathcal{P}_n(k)$ of polynomials in an indeterminate (i.e. variable) t with coefficients in k of degree less than or equal to n, equipped with the usual addition and scalar multiplication, forms a vector space over k (of dimension $n + 1$).

6. Let $I \subseteq \mathbf{R}$ be an interval. The set $\mathscr{C}^n(I)$ of all functions $f : I \to \mathbf{R}$ that have at least n continuous derivatives, equipped with the addition and scalar multiplication of functions above, forms a vector space over $\mathbf{R}$.

Definition 3.20. Let k be a field, and let V be a vector space over k. A subset $W \subseteq V$ is a **subspace** of V if W is nonempty and closed under the vector space operations (vector addition and scalar multiplication) induced by V.

One can check both closure properties for W at once by checking simple linear combinations: That is, for all vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$, and for all scalars $a_1, a_2 \in k$, check that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \in W.$$

For any vector space V, $\{\mathbf{0}\}$ and V are always subspaces. These are often called the **trivial subspaces**. All other subspaces are called **nontrivial**.

Example 3.21 (Exampl 3.5.9)**.** Let $I = [0, 1] \subseteq \mathbf{R}$, and let $V = \mathscr{C}^0(I)$ be the set of continuous functions $f : I \to \mathbf{R}$. Show that the subset W of functions $f \in V$ such that $f(0) = 0$ is a subspace of V.

Solution. We must verify nonempty and closure.

Nonempty. Observe that the zero function $z : I \to \mathbf{R}$ given by $x \mapsto 0$ is continuous on I and satisfies $z(0) = 0$. Hence $z \in W$, so W is nonempty.

Closure. We check linear combinations. For all $f_1, f_2 \in V$, and for all $a_1, a_2 \in \mathbf{R}$, the linear combination $a_1f_1 + a_2f_2$ is continuous on I, and

$$\begin{aligned}(a_1f_1 + a_2f_2)(0) &= (a_1f_1)(0) + (a_2f_2)(0)\\ &= a_1f_1(0) + a_2f_2(0)\\ &= a_1 0 + a_2 0\\ &= 0.\end{aligned}$$

Hence $a_1f_1 + a_2f_2 \in W$, so W is closed under the two vector space operations.

3.6 Bases and Dimension

Key Ideas

- Linear combination, span
- Linear (in)dependence
- How to determine linear (in)dependence of functions
- Wronskian (a function!)
- Basis of a vector space; standard basis
- Rank of a matrix

Let k be a field, and let V be a vector space over k.

Definition 3.22. Let $\mathbf{v}_1, \dots, \mathbf{v}_m \in V$. A **linear combination** of these vectors is a vector sum of the form

$$a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m,$$

for scalars $a_1, \dots, a_m \in k$.

Definition 3.23. Let $\mathbf{v}_1, \dots, v_m \in V$. The **span** of these vectors is the set of all linear combinations:

$$\mathrm{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_m) = \{a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m \mid a_1, \dots, a_m \in k\}.$$

By construction, the span of a set of vectors is a subspace of V. By convention, the span of the empty set is the zero vector space: $\mathrm{Span}(\varnothing) = \{0\}$.

Let $W \subseteq V$. If $\mathrm{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_m\}) = V$, then we say that these vectors **span** W.

Also from the definition of span, it follows that $\mathbf{v} \in \mathrm{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_m\})$ if and only if the matrix equation

$$\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_m \\ | & & | \end{bmatrix} \mathbf{x} = \mathbf{v}$$

has at least one solution $\mathbf{x}$.

Definition 3.24. Let $\mathbf{A}$ be an $n \times m$ matrix with entries in k. The **column space of $\mathbf{A}$** is the span of the columns of $\mathbf{A}$, viewed as n-vectors.

Definition 3.25. The vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ are **linearly independent** if their only linear combination that produces the zero vector is the linear combination with all coefficients equal to zero: That is, these vectors are linearly independent if

$$a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m = \mathbf{0} \qquad \Rightarrow \qquad a_1, \dots, a_m = 0.$$

If these vectors are not linearly independent, then they are **linearly dependent**.

This definition equivalent to saying that these vectors are linearly dependent if one of them can be expressed as a linear combination of the others. Another equivalent definition is that these vectors are linearly independent if and only if the solution $\mathbf{x} = \mathbf{0}$ to

$$\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_m \end{bmatrix} \mathbf{x} = \mathbf{0}$$

is unique.

In the case $V = k^n$, and the vectors $\mathbf{v}_i$ are given in terms of coordinate vectors (with respect to some basis), then we can write $[\mathbf{v}_1, \dots, \mathbf{v}_m]$ as an $n \times m$ matrix A with entries in k. We can apply the Gauss–Jordan reduction algorithm to put A in reduced row echelon form (RREF). Our earlier study of RREF shows that the original vectors are linearly independent if and only if every column is a pivot column. (Why?)

Example 3.26 (Example 3.6.9)**.** Let $k = \mathbf{R}$, and let $V = \mathscr{C}^\infty(\mathbf{R})$. Consider

$$\mathbf{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} \qquad \mathbf{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} \qquad \mathbf{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}.$$

Show that these vectors are linearly independent.

Solution. We apply the definition of linear (in)dependence. By definition, the vectors $\mathbf{v}_i$ are linearly independent if and only if the equation

$$a_1\mathbf{v}_1(t) + a_2\mathbf{v}_2(t) + a_3\mathbf{v}_3(t) = \mathbf{0} \tag{3.6.1}$$

has only the trivial solution, i.e. all $a_i = 0$. If this equation holds for all $t \in \mathbf{R}$, then in particular, it must hold for any $t \in \mathbf{R}$ we want. Consider $t = 0$. Evaluating the vectors $\mathbf{v}_i$ at $t = 0$, (3.6.1) becomes

$$\mathbf{0} = a_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

This equation has a unique solution if and only if the 3×3 matrix is invertible, if and only if its determinant is nonzero. We compute its determinant to be -1 (e.g., using expansion by minors down the first column). Therefore, the unique solution to (3.6.1) at $t = 0$ is the trivial one, and therefore the vectors $\mathbf{v}_i$ are linearly independent.

3.6.1 Linear Independence of Functions

We present two methods to check linear independence of functions. This will be of great importance to us later *(add ref)*.

Evaluate m functions at m points If we're asked to evaluate linear (in)dependence of m functions, we can evaluate each of them at m distinct points, then analyze the resulting system of m equations in the m variables a_i (the coefficients in the linear (in)dependence relation).

Example 3.27 (Example 3.6.10)**.** Consider $\mathscr{C}^\infty(\mathbf{R})$ as a vector space over $\mathbf{R}$. Show that the following functions are linearly independent:

$$f_1(t) = e^t \qquad f_2(t) = 5e^{-t}, \qquad f_3(t) = e^{3t}.$$

Solution. Let z denote the zero function on $\mathbf{R}$. The functions f_1, f_2, f_3 are linearly independent if and only if the only solution to

$$a_1 e^t + 5a_2 e^{-t} + a_3 e^{3t} = a_1 f_1(t) + a_2 f_2(t) + a_3 f_3(t) = z(t) = 0, \tag{3.6.2}$$

as functions (!), is the trivial one. Consider the points $t = -1, 0, 1$. Evaluating (3.6.2) at these values of t gives

$$\begin{aligned} e^{-1}a_1 + 5ea_2 + e^{-3}a_3 &= 0 \\ a_1 + 5a_2 + a_3 &= 0 \\ ea_1 + 5e^{-1}a_2 + e^3 a_3 &= 0. \end{aligned}$$

Rewrite this system of equations as a matrix equation:

$$\begin{bmatrix} e^{-1} & 5e & e^{-3} \\ 1 & 5 & 1 \\ e & 5e^{-1} & e^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Computing the determinant of the 3×3 matrix (e.g., via expansion by minors along row 2), we get

$$\begin{aligned} \det \begin{bmatrix} e^{-1} & 5e & e^{-3} \\ 1 & 5 & 1 \\ e & 5e^{-1} & e^3 \end{bmatrix} &= 5 \det \begin{bmatrix} e^{-1} & e & e^{-3} \\ 1 & 1 & 1 \\ e & e^{-1} & e^3 \end{bmatrix} \\ &= 5\left(-(e^4 - e^{-4}) + (e^2 - e^{-2}) - (e^{-2} - e^2)\right) \\ &= 5\left(-e^4 + e^{-4} + 2e^2 - 2e^{-2}\right) \neq 0. \end{aligned}$$

Therefore the vectors (functions) f_1, f_2, f_3 are linearly independent in V.

The benefit of this method is that the functions do not have to be differentiable. The (severe) limitation of this method is that, although a nonzero determinant certifies linear independence, a zero determinant does not guarantee linear dependence! We could choose a "bad" set of m points. (For example, consider the functions $f_1(t) = 1$ and $f_2(t) = t^2$, and the points $t = \pm 1$.)

Compute the wronskian *(holds in a more general setting?)*

Definition 3.28. Let $I \subseteq \mathbf{R}$, and let $f_1, \dots, f_m \in \mathscr{C}^{m-1}(I)$. The **wronskian** W of these functions the function $W : I \to \mathbf{R}$ given by the determinant of the matrix whose rows are successive derivatives of each function:

$$W = W[f_1, \dots, f_m] = \det \begin{bmatrix} f_1 & \dots & f_m \\ f_1' & \dots & f_m' \\ \vdots & \ddots & \vdots \\ f_1^{(m-1)} & \dots & f_m^{(m-1)} \end{bmatrix}.$$

The following theorem says that the wronskian detects linear independence. Note that, as stated, the implication is only one way! Example 3.6.12 gives an example where the converse fails, i.e. where two linearly independent functions have a zero (function) wronskian. *(add to notes)*

Theorem 3.29. *If $W[f_1, \dots, f_m]$ does not equal the zero function on I, then the f_i are linearly independent.*

Example 3.30 (Example 3.6.11)**.** Let $V = \mathscr{C}^\infty(\mathbf{R})$, considered as a vector space over $\mathbf{R}$, and let $f_1, f_2, f_3 \in V$ be given by

$$\begin{aligned} f_1(t) &= t^2 + 1 \\ f_2(t) &= t^2 - 1 \\ f_3(t) &= 2t + 5. \end{aligned}$$

Show that $\{f_1, f_2, f_3\}$ is linearly independent.

Solution. We have three functions, all of which are twice (in fact, infinitely) continuously differentiable. Thus the wronskian test applies. We compute the wronskian (using expansion along row 3):

$$\begin{aligned} W &= \det \begin{bmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{bmatrix} \\ &= 4 \det \begin{bmatrix} t^2+1 & t^2-1 & 2t+5 \\ t & t & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= 4\left((t^2 - 1 - (2t^2 + 5t)) - (t^2 + 1 - (2t^2 + 5t)) + 0\right) \\ &= 4\left(-t^2 - 5t - 1 - (-t^2 - 5t + 1)\right) \\ &= 4(-2) = -8 \neq 0. \end{aligned}$$

Their wronskian is not the zero function (!), so by the wronskian test, f_1, f_2, f_3 are linearly independent.

3.6.2 Bases

Let k be a field, and let V be a vector space over k. The (ordered) set $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of vectors $\mathbf{v}_i \in V$ is a **basis** of V if it satisfies the following two properties:

(i) span: $\mathrm{Span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}) = V$.

(ii) linearly independent: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

The definition above works for finite dimensional vector spaces. (We'll define the dimension of a vector space soon.) There is a more general definition that applies to both finite and infinite dimensional vector spaces.

Definition 3.31. Let k be a field, and let V be a vector space over k. A **basis** for V is a subset $B \subseteq V$ that satisfies the following two properties

(i) span: for all $\mathbf{v} \in V$, there exist finitely many vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in B$ such that $\mathbf{v} \in \mathrm{Span}(B)$, i.e. there exist scalars $a_1, \ldots, a_n \in k$ such that

$$\mathbf{v} = a_1\mathbf{b}_1 + \ldots + a_n\mathbf{v}_n.$$

(ii) linear independence: every finite subset $\{b_1, \ldots, b_n\} \subseteq B$ is linearly independent in V.

Definition 3.32. Let k be a field, and let $V = k^n$ be a vector space over k. The **standard basis** of k^n is the ordered set $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$, where $\mathbf{e}_i$ is the vector with 1 in entry i and 0 everywhere else.

By definition, every field k has elements $0, 1$ with $0 \neq 1$. As column vectors, the standard basis looks like

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Definition 3.33. The **dimension** of a (finite dimensional) vector space is the number of vectors in a basis.

One can show that a given vector space can have different bases, but these bases always have the same number of vectors.

Definition 3.34. A vector space V is **infinite dimensional** if no finite set of vectors spans V.

Example 3.35 (Example 3.6.18)**.** The space $\mathcal{P}(k)$ of all polynomials in a single indeterminate (say, t) with coefficients in k, considered as a vector space over k, is infinite dimensional. To see this, we argue by contradiction: Suppose $\mathcal{P}(k)$ were finite dimensional. Then by definition, there exists some finite basis $(p_1, \ldots, p_n)$ of polynomials. Let d be the maximum degree of these polynomials $p_1, \ldots, p_n$. One can check (e.g., using the methods described above) that the monomial t^{d+1} is not a linear combination of $p_1, \ldots, p_n$. Hence $(p_1, \ldots, p_n)$ is not a basis, a contradiction.

3.6.3 Column Space

(remind definition of column space, defined earlier this section)

The pivot columns of $\mathbf{A}$ form a basis for the column space of $\mathbf{A}$.

Definition 3.36. The **rank** of $\mathbf{A}$ is the dimension of the column space:

$$\operatorname{rank} \mathbf{A} = \dim \operatorname{Col}(\mathbf{A}).$$

3.7 Linear Transformations

Key Ideas

- Linear transformation, endomorphism
- Relationship between linear transformations and matrices

Let k be a field, and let V, W be vector spaces over k.

Definition 3.37. A **linear transformation** from V to W is a function $T : V \to W$ that preserves the vector space structure, i.e. vector addition and scalar multiplication. That is, for all $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, and for all $a \in k$,

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2), \qquad T(a\mathbf{v}) = aT(\mathbf{v}).$$

As with any function, the space of inputs (here, V) is called the **domain** (or **source**), and the space of outputs (here, W) is called the **codomain** (or **target**).

Remark 3.38. When the domain and codomain are the same vector space V, one calls T an **enomorphism** of V, or sometimes a **linear operator** on V. (Warning: Different people use the word *operator* to mean different things.)

Exercise 3.1. Show that any linear transformation $T : V \to W$ maps the zero vector in V to the zero vector in W. (Hint: Use the fact that, by definition, T preserves scalar multiplication.)

Definition 3.39. The **image** of T is the set of vectors in W that T outputs, for all possible inputs in V:

$$\operatorname{im} T = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V : T(\mathbf{v}) = \mathbf{w}\}.$$

In fact, $\operatorname{im} T$ is a subspace of W.

Theorem 3.40 (Image is a vector subspace)**.** *Let k be a field, let V, W be vector spaces over k, and let $T : V \to W$ be a linear transformation. The image $\operatorname{im} T$ is a subspace of W.*

Proof. We check that the axioms of a subspace hold for $\operatorname{im} T$.

Nonempty. By definition, V has a zero vector $\mathbf{0}_V$. Because T is a linear transformation, $T(\mathbf{0}_V) = \mathbf{0}_W$. Hence $\mathbf{0}_W \in \operatorname{im} T$, so $\operatorname{im} T \neq \varnothing$.

Closed under linear combinations. Let $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{im} T$, and let $a_1, a_2 \in k$. By definition of the image, there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$. Using this and the hypothesis that T is linear, we have

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) = T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2).$$

Because V is a vector space, it is closed under scalar multiplication and vector addition; in particular,

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \in V.$$

Hence

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 \in \operatorname{im} T,$$

as desired. □

Let $I = [a, b] \subseteq \mathbf{R}$ be a finite interval. Recall that $\mathscr{C}^1(I)$ is the set of functions $f : I \to \mathbf{R}$ that are continuously differentiable.

Definition 3.41. The **derivative operator** is the function

$$\begin{aligned} D : \mathscr{C}^1(I) &\to \mathscr{C}(I) \\ f &\mapsto f' \end{aligned}$$

The **integration operator** is the function

$$\begin{aligned} I : \mathscr{C}(I) &\to \mathbf{R} \\ f &\mapsto \int_a^b f(t)\,dt. \end{aligned}$$

One can check that both operators D and I are linear.
(include Table 5.1.3 of common linear transformations?)

Example 3.42 (Example 5.1.6)**.** Consider $\mathbf{R}^2$ as a vector space over $\mathbf{R}$, and fix $\theta \in \mathbf{R}$. Show that the linear transformation $R_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ with the associated matrix

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

corresponds to counterclockwise rotation by the angle θ around the origin.

Solution. Relate how R_θ acts on the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ to the columns of $\mathbf{R}_\theta$. Confirm R_θ acts on each as claimed. Next, by definition of basis, any $\mathbf{v} \in \mathbf{R}^2$ can be written (uniquely!) as

$$\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2.$$

Now use linearity.

3.7.1 Linear Transformations and Matrices

Linear transformations and matrices have a beautiful relationship. Let k be a field, let $m, n \in \mathbf{Z}_{>0}$, and let V, W be vector spaces over k with $\dim V = m$ and $\dim W = n$. Then every $n \times m$ matrix $\mathbf{A}$ with entries in k corresponds to a linear transformation $T : V \to W$, *once we have chosen a basis for* V *and a basis for* W. More precisely, let $\mathbf{b}$ be the coordinates of $\mathbf{v}$, viewed as an $m \times 1$ vector, with respect to the chosen basis for V. Then $T(\mathbf{v})$ is the vector in W whose coordinates with respect to the chosen basis for W is the matrix product

$$T(\mathbf{v}) = \mathbf{Ab}.$$

Note that the order of the matrix product is $n \times 1$, as required by $\dim W = n$.

Conversely, given a linear transformation $T : V \to W$ and bases $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ for V and $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ for W, T corresponds to an $n \times m$ matrix whose columns are the coordinate vectors of the images $T(\mathbf{v}_i)$ of the basis vectors of V, with respect to the basis for W:

$$\begin{bmatrix} | & & | \\ T(\mathbf{v}_1)_{\mathscr{B}_W} & \ldots & T(\mathbf{v}_m)_{\mathscr{B}_W} \\ | & & | \end{bmatrix}.$$

In the case when $V = W$, i.e. our linear transformation is an endomorphism $T : V \to V$, the relationship gets even more interesting. The set of all endomorphisms on V, denoted $\mathrm{End}(V)$, equipped with addition, composition (the "multiplication" operation), and scalar multiplication of functions, forms a structure called an associative algebra with (multiplicative) identity (loosely, almost a field, but elements may not commute under multiplication, and not all elements have multiplicative inverses; think of all $m \times m$ matrices (and read on!)). Endomorphisms that are both injective and surjective are called **automorphisms** of V. The set of all automorphisms of V, equipped with function composition, forms a structure called a group (loosely, we can "multiply" and "undo" multiplication, but again elements may not commute). When $\dim V = m$ is finite, then $\mathrm{End}(V)$ corresponds to the set of $m \times m$ matrices $M_{m \times m}(k)$, and $\mathrm{Aut}(V)$ corresponds to the invertible $m \times m$ matrices, denoted $\mathrm{GL}_n(k)$.[8]

One upshot of this is that T has an inverse if and only if its associated matrix $\mathbf{A}$ (as defined above) is invertible, with respect to any (and hence every) basis on V.

Remark 3.43. The way this is presented in Farlow *et al.* (2018) is simplified but misleading. What are referred to as vectors on page 291 are actually coordinates with respect to a basis.

Example 3.44 (Example 5.1.8(a))**.** Consider $\mathbf{R}^2$ and $\mathbf{R}^3$ as vector spaces over $\mathbf{R}$. Find the matrix $\mathbf{A}$ describing the linear transformation

$$\begin{aligned} T : \mathbf{R}^2 &\to \mathbf{R}^3 \\ (x, y) &\mapsto (x - y, x + y, 2x) \end{aligned}$$

[8] See Wikipedia, Linear map: Endomorphisms and automorphisms.

with respect to the standard bases on $\mathbf{R}^2$ and $\mathbf{R}^3$.

Solution. *(be more precise, with explicit reference to bases?)* Looking at the dimensions of the domain and codomain of T, we see that our matrix $\mathbf{A}$ has order 3×2. From our discussion above, $\mathbf{A}$ satifies

$$\mathbf{A} = \begin{bmatrix} | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ T((1,0)) & T((0,1)) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Example 3.45 (Example 5.1.8(b))**.** Let $D^2 : \mathcal{P}^3(\mathbf{R}) \to \mathcal{P}^1(\mathbf{R})$ be the second-derivative operator. Find the matrix $\mathbf{A}$ describing D^2 with respect to the standard bases $(1, t, \ldots, t^n)$ on $\mathcal{P}^n(k)$.

Solution. We compute

$$D^2(1) = 0, \qquad D^2(t) = 0, \qquad D^2(t^2) = 2, \qquad D^2(t^3) = 6t,$$

so

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ D^2(1) & D^2(t) & D^2(t^2) & D^2(t^3) \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

3.8 Properties of Linear Transformations

Key Ideas

- Injective, surjective, bijective
- Rank of a linear transformation
- Kernel
- Rank–nullity theorem

Definition 3.46. Let X, Y be sets, and let $f : X \to Y$ be a function.

- f is **injective** (or **one-to-one**) if for all $x_1, x_2 \in X$,
$$f(x_1) = f(x_2) \qquad \Rightarrow \qquad x_1 = x_2.$$
That is, different inputs yield different outputs.
- f is **surjective** (or **onto**) if for all $y \in Y$, there exists some (possibly more than one) $x \in X$ such that
$$f(x) = y.$$
- f is **bijective** if it is both injective and surjective.

Definition 3.47. The **rank** of a linear transformation is the dimension of its image:
$$\operatorname{rank} T = \dim(\operatorname{im} T).$$

(add explicit remark relating this definition to column rank from earlier)
(careful with following result, vectors versus coordinates)

Theorem 3.48. *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with corresponding matrix $\mathbf{A} \in M_{m \times n}(\mathbf{R})$. The image of T is the column space of $\mathbf{A}$:*
$$\operatorname{im} T = \operatorname{Col} \mathbf{A}.$$
The pivot columns of $\mathbf{A}$ form a basis for $\operatorname{im} T$. Hence
$$\operatorname{rank} T = \dim(\operatorname{im} T) = \dim(\operatorname{Col} \mathbf{A}) = \#\textit{ of pivot columns in } \mathbf{A}$$

Definition 3.49. Let k be a field, let V, W be vector spaces over k, and let $T : V \to W$ be a linear transformation. The **kernel** (or **nullspace**) of T is the set of vectors that are mapped to $\mathbf{0}_W$ by T:
$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}.$$

Recall (see Theorem 3.40) that the image $\operatorname{im} T$ is a subspace of W.

Theorem 3.50 (Kernel is a vector subspace)**.** *The kernel* $\ker T$ *is a subspace of* V.

Proof. We check that the axioms of a subspace hold for $\ker T$.

Nonempty. By definition, V has a zero vector $\mathbf{0}_V$. Because T is a linear transformation, $T(\mathbf{0}_V) = \mathbf{0}_W$. Hence $\mathbf{0}_V \in \ker T$, so $\ker T \neq \varnothing$.

Closed under linear combinations. Let $\mathbf{v}_1, \mathbf{v}_2 \in \ker T$, and let $a_1, a_2 \in k$. Using linearity of T, we compute

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) = a_1 T(\mathbf{v}_1) + a_2 T(\mathbf{v}_2) = a_1\mathbf{0}_W + a_2\mathbf{0}_W = \mathbf{0}_W.$$

Hence

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \in \ker T,$$

as desired. □

The dimension of the kernel $\dim(\ker T)$ is sometimes called the **nullity** of T.

Theorem 3.51 (Rank–nullity theorem)**.** *Let* k *be a field, let* V, W *be vector spaces over* k *with* $\dim V < \infty$*, and let* $T : V \to W$ *be a linear transformation. Then*

$$\dim(\ker T) + \dim(\operatorname{im} T) = \dim V.$$

Note that the first term on the left is the nullity, and the second term on the left is the rank (both of T).

Example 3.52 (Examples 5.2.3,7,10(b))**.** For $n = 2, 4$, consider $\mathbf{R}^n$ as a vector space over $\mathbf{R}$. Let $T : \mathbf{R}^4 \to \mathbf{R}^2$ be the linear transformation associated to the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix}$$

with respect to the standard bases on $\mathbf{R}^4$ and $\mathbf{R}^2$.

(a) Find $\operatorname{im} T$ and its dimension. Give a basis for $\operatorname{im} T$.

(b) Find $\ker T$ and its dimension. Give a basis for $\ker T$.

(c) Verify that the rank–nullity theorem holds for T.

Solution. Each part in turn.

Part (a). Recall that

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \\ | & | & | & | \end{bmatrix}$$

(where the column vectors $\mathsf{T}(\mathbf{e}_i)$ shown are the coordinates of the image vectors with respect to the standard basis on $\mathbf{R}^2$). Any vector $\mathbf{v}$ in the domain $\mathbf{R}^4$ is a linear combination of these basis vectors, i.e.,

$$\mathbf{v} = a_1\mathbf{e}_1 + \ldots + a_4\mathbf{e}_4$$

for unique scalars $a_1, \ldots, a_4 \in \mathbf{R}$. By linearity of T,

$$\mathsf{T}(\mathbf{v}) = \mathsf{T}(a_1\mathbf{e}_1 + \ldots + a_4\mathbf{e}_4) = a_1\mathsf{T}(\mathbf{e}_1) + \ldots + a_4\mathsf{T}(\mathbf{e}_4).$$

This shows that any vector in the image $\operatorname{im}\mathsf{T}$ is a linear combination of the images of the standard basis vectors. Hence

$$\operatorname{im}\mathsf{T} = \operatorname{Span}\{\mathsf{T}(\mathbf{e}_1), \ldots, \mathsf{T}(\mathbf{e}_4)\} = \operatorname{Span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}\right\}.$$

This set of four vectors is certainly redundant in specifying the span (why?). Let's use the fact that the pivot columns of $\mathbf{A}$ form a basis for $\operatorname{im}\mathsf{T}$ to winnow down our set. We find the pivot columns by transforming $\mathbf{A}$ into RREF.

$$\begin{aligned}\mathbf{A} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} &\rightsquigarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ -1 & 2 & -2 & -3 \end{bmatrix} \\ &\rightsquigarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ &\rightsquigarrow \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}\end{aligned}$$

Thus columns 1 and 3 of the original matrix $\mathbf{A}$ form a basis for $\operatorname{im}\mathsf{T}$:

$$\operatorname{im}\mathsf{T} = \operatorname{Span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}\right\}.$$

(In fact, any two linearly independent vectors in $\mathbf{R}^2$ form a basis for $\operatorname{im}\mathsf{T}$. Why?) This basis has 2 vectors, so $\dim(\operatorname{im}\mathsf{T}) = 2$.

Part (b). Reduce, reuse, recycle! (Sorry, corniness must be contagious...) By definition, vectors $\mathbf{v}$ in the kernel $\ker\mathsf{T}$ are precisely the vectors that solve the homogeneous linear system $\mathbf{A}\mathbf{v} = \mathbf{0}$. We have seen that vectors that solve this system are precisely those that solve the equivalent system $\mathbf{B}\mathbf{v} = \mathbf{0}$ where $\mathbf{B}$ is the RREF of $\mathbf{A}$. (Why? Hint: What do the elementary row operations that we use to transform $\mathbf{A}$ into $\mathbf{B}$ correspond to in matrix operations? What effect do these have on the homogeneous equation? Is this reversible?) So $\mathbf{v} \in \ker\mathsf{T}$ if and only if $\mathbf{v}$ solves

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \mathbf{0}.$$

Only columns 1 and 3 are pivot columns, so variables v_2 and v_4 are free variables. Hence every $\mathbf{v} \in \ker \mathsf{T}$ has the form

$$\mathbf{v} = \begin{bmatrix} 2v_2 - 3v_4 \\ v_2 \\ 0 \\ v_4 \end{bmatrix} = v_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for $v_2, v_4 \in \mathbf{R}$. That is,

$$\ker \mathsf{T} = \operatorname{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

It is straightforward to check that these two vectors are linearly independent in $\mathbf{R}^4$, so they form a basis of $\ker \mathsf{T}$. Hence $\dim(\ker \mathsf{T}) = 2$.

Part (c). Rank plus nullity is

$$\dim(\operatorname{im} \mathsf{T}) + \dim(\ker \mathsf{T}) = 2 + 2 = 4 = \dim \mathbf{R}^4,$$

as promised by the rank–nullity theorem.

3.8.1 Solutions to Nonhomogeneous Systems

(do a better job setting up the link) In our study of DEs, we encountered the nonhomogeneous principle, which states that the general solution to a nonhomogeneous linear ODE can be expressed as the sum of a particular solution and the general solution to the corresponding homogeneous equation. *(add ref)* We also saw that we could write a linear ODE using a linear operator. This is a special case (in the context of ODEs) of a result that holds for linear transformations in general.

Theorem 3.53 (Nonhomogeneous principle for linear transformations)**.** *Let k be a field, let V, W be vector spaces over k, let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation, let $\mathbf{b} \in \mathsf{W}$, and let $\mathbf{v}_p$ be any particular solution to*

$$\mathsf{T}(\mathbf{v}) = \mathbf{b}.$$

Then the set S of all solutions to this nonhomogeneous equation is

$$\mathsf{S} = \{\mathbf{v}_p + \mathbf{v}_h \,|\, \mathbf{v}_h \in \ker \mathsf{T}\}.$$

Note that $\mathbf{v}_h \in \ker \mathsf{T}$ is equivalent to the statement that $\mathbf{v}_h$ solves $\mathsf{T}(\mathbf{v}) = \mathbf{0}$ — in our earlier language, the corresponding homogeneous equation. *(use this common language — "corresponding homogeneous equation", as opposed to "corresponding homogeneous ODE", etc. — throughout)*

3.9 Eigenvalues and Eigenvectors

Key Ideas

- Eigenvalue, eigenvector, eigenspace
- Characteristic equation and polynomial
- How to find all eigenvalues and eigenvectors of a (square) matrix
- Algebraic and geometric multiplicity of eigenvalues
- Linear independence of eigenvectors with distinct eigenvalues

Let k be a field, let V be a vector space over k, and let $T : V \to V$ be a linear transformation from V to itself (!).

Definition 3.54. An **eigenvalue** (also called **characteristic value**) of T is a scalar $\lambda \in k$ such that there exists some nonzero (!) vector $\mathbf{v} \in V$ satisfying

$$T(\mathbf{v}) = \lambda \mathbf{v}.$$

Such a vector $\mathbf{v}$ is called an **eigenvector** of T corresponding to λ.

Remark 3.55. A few remarks.

1. Eigenvectors are vectors that are scaled (not rotated) by the linear transformation. They are scaled by their corresponding eigenvalue.
2. Note that the notion of eigenvector (and hence eigenvalue) only makes sense for linear transformations from a vector space *to itself*.
3. Eigen*vectors* must be nonzero, but eigen*values* can be zero.
4. The eigenvalue $\lambda = 0$ corresponds to (nonzero) eigenvectors satisfying $T(\mathbf{v}) = 0\mathbf{v} = \mathbf{0}$, i.e. to nonzero vectors in the kernel $\ker T$.

(be more precise with this next bit? the eigenvalues and eigenvectors still belong to the associated T and V, the matrix formulation just identifies the vectors' coordinates, which correspond bijectively with the vectors in V) We have seen how to associate a linear transformation to a matrix, and vice versa. If our vector space V is finite dimensional, say $\dim V = n$, then after choosing a basis, we can associate T to an $n \times n$ matrix $\mathbf{A} \in M_{n\times n}(k)$. This association allows us to define eigenvalues and eigenvectors for (square) matrices:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}.$$

In our work, we'll deal mostly with matrices, so in what follows we present all things eigen in the matrix context.

Exercise 3.2. Let $\mathbf{v}$ be an eigenvector for the linear transformation $T : V \to V$ with corresponding eigenvalue λ. Show that for any nonzero $a \in k$, the vector $a\mathbf{v}$ is also an eigenvector for T with associated eigenvalue λ. Hence eigenvectors are not unique (unless the field k has only one nonzero element).

3.9.1 Computing Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ matrix with coefficients in k. By definition, an eigenvalue λ for $\mathbf{A}$ (more precisely, the linear transformation T corresponding to $\mathbf{A}$) satisfies the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} = \lambda\mathbf{I}_n\mathbf{v}$$

for some $\mathbf{v} \neq \mathbf{0}$. We can rewrite this equation as

$$(\mathbf{A} - \lambda\mathbf{I}_n)\,\mathbf{v} = \mathbf{0}.$$

We have seen *(add ref)* that this equation has a nonzero solution $\mathbf{v}$ if and only if the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}_n$ is singular, i.e.

$$\det(\mathbf{A} - \lambda\mathbf{I}_n) = 0. \tag{3.9.1}$$

Recall that the determinant of a matrix is a scalar, so the left side is a scalar in the single unknown λ, called the **characteristic polynomial** of $\mathbf{A}$. The equation (3.9.1) is called the **characteristic equation** of $\mathbf{A}$.

To find the eigenvalues and eigenvectors of $\mathbf{A}$,

1. Write the characteristic equation.
2. Solve the characteristic equation for the eigenvalues.
3. For each eigenvalue λ, solve the system

$$(\mathbf{A} - \lambda\mathbf{I}_n)\,\mathbf{v} = \mathbf{0}$$

   to find the corresponding eigenvector(s).

Example 3.56 (Example 5.3.2)**.** Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

Solution. The characteristic equation is

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}_2) = \det\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3.$$

This factors as

$$(\lambda + 1)(\lambda - 3) = 0.$$

This has two solutions, $\lambda = -1$ and $\lambda = 3$. These are the eigenvalues.

Let's find corresponding eigenvectors. For $\lambda = -1$, we solve

$$\mathbf{0} = (\mathbf{A} - (-1)\mathbf{I}_2)\,\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Forming the augmented matrix for this system, then applying the Gauss–Jordan reduction algorithm, we obtain

$$\left[\begin{array}{cc:c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

From the RREF matrix, we see that v_2 is a free variable, and $v_1 = -\frac{1}{2}v_2$. Setting $v_2 = 2$ gives the eigenvector

$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(Any nonzero scalar multiple of this vector, corresponding to different choices of v_2 above, is also an eigenvector for $\lambda = -1$.) One can check that

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = (-1)\mathbf{v},$$

as required.

Similarly, for $\lambda = 3$, we solve

$$\mathbf{0} = (\mathbf{A} - 3\mathbf{I}_2)\,\mathbf{v} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Form the augmented matrix and apply Gauss–Jordan:

$$\left[\begin{array}{cc:c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Again v_2 is a free variable, and $v_1 = \frac{1}{2}v_2$. Setting $v_2 = 2$ gives the eigenvector $\mathbf{v} = (1, 2)$.

3.9.2 Results

Some results.

Theorem 3.57. *Let $\mathbf{A}$ be an upper or lower triangular $n \times n$ matrix. Then the eigenvalues of $\mathbf{A}$ are the diagonal entries.*

Theorem 3.58. *Let $\mathbf{A}$ be an $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_p$ be distinct (!) eigenvalues. If, for each i, $\mathbf{v}_i$ is an eigenvector associated to λ_i, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.*

Definition 3.59. Let k be a field, let V be a vector space over k, let $T : V \to V$ be a linear transformation, and let λ be an eigenvalue of T. The **eigenspace** of T associated to λ, denoted $E(T, \lambda)$ is the set of eigenvectors of T associated to λ, along with the zero vector:

$$E(T, \lambda) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda \mathbf{v}\}.$$

Theorem 3.60. *Each eigenspace is a subspace of* V.

3.9.3 Characteristic Equation for 2×2 Matrices

The 2×2 case will be important to us in our study of linear systems of ODEs. The eigenvalues of a 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are found by solving the characteristic equation

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda \mathbf{I}_2) &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + ad - bc \\ &= \lambda^2 - (\operatorname{tr} \mathbf{A})\lambda + \det \mathbf{A}. \end{aligned}$$

3.9.4 Repeated Eigenvalues

The eigenvalues of a linear operator or matrix need not be distinct. The **algebraic multiplicity** of an eigenvalue is its algebraic multiplicity as a root of the characteristic equation. The **geometric multiplicity** of an eigenvalue is the dimension of its corresponding eigenspace.

Example 3.61 (Jordan normal form)**.** Determine the algebraic and geometric multiplicities of the eigenvalues for

$$\mathbf{A} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}.$$

Solution. All three matrices are upper triangular. The eigenvalues of an upper triangular matrix are exactly the diagonal entries. Hence for all three matrices, 1 is an eigenvalue of algebraic multiplicity 3.

Because $\mathbf{A}$ is the identity, all nonzero vectors in $\mathbf{R}^3$ are eigenvectors (with eigenvalue 1). Thus $E(\mathbf{A}, 1) = \mathbf{R}^3$, which has dimension 3. Hence 1 has geometric multiplicity 3. (One can confirm this by writing out the characteristic polynomials explicitly.)

Eigenvectors $\mathbf{v}$ for $\mathbf{B}$ solve the system

$$\mathbf{0} = (\mathbf{B} - (1)\mathbf{I})\,\mathbf{v} = \det \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

We see that $v_3 = 0$, and v_1, v_2 are free variables. Thus

$$E(\mathbf{B}, 1) = \operatorname{Span}\{\mathbf{e}_1, \mathbf{e}_2\},$$

so in this case, the eigenvalue 1 has geometric multiplicity 2.

Eigenvectors $\mathbf{v}$ for $\mathbf{C}$ solve the system

$$\mathbf{0} = (\mathbf{C} - (1)\mathbf{I})\,\mathbf{v} = \det \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

We see that $v_2, v_3 = 0$, and v_1 is a free variable. Thus

$$E(\mathbf{C}, 1) = \operatorname{Span}\{\mathbf{e}_1\},$$

so in this case, the eigenvalue 1 has geometric multiplicity 1.

(add general discussion of generalized eigenvectors and jordan normal form)

Let's preview how we'll use this.

Example 3.62 (Example 5.3.9)**.** Consider the homogeneous linear second-order ODE

$$y'' - y' - 2y = 0. \tag{3.9.2}$$

We can convert this to a linear system by applying the change of variables

$$x_1 = y, \qquad\qquad x_2 = y'.$$

From the definition of our new variables,

$$x_1' = y' = x_2,$$

and from the original ODE,

$$x_2' = y'' = y' + 2y = x_2 + 2x_1.$$

Thus our change of variables yields the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_2 + 2x_1, \end{aligned}$$

which we can write in matrix form as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The characteristic equation for the coefficient matrix $\mathbf{A}$ is

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2).$$

Thus $\mathbf{A}$ has two (distinct) eigenvalues, $\lambda = -1$ and $\lambda = 2$. One can check that putting these eigenvalues as the parameter k in the function $y(t) = e^{kt}$ yields linearly independent solutions to our original ODE. Specifically, $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ are linearly independent (e.g., via the Wronskian) solutions (e.g., by substitution into (3.9.2)). Thus the general solution to (3.9.2) is

$$y(t) = c_1 e^{-t} + c_2 e^{2t},$$

for $c_1, c_2 \in \mathbf{R}$.

The solution $y = y_1(t) = e^{-t}$ corresponds, in our change of variables, to

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}.$$

One can check that

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = (-1)\mathbf{x}_1.$$

This shows that $\mathbf{x}_1$ is an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue -1.

Similarly, the solution $y = y_2(t) = e^{2t}$ corresponds to

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix},$$

and

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix} = 2\mathbf{x}_2.$$

This shows that $\mathbf{x}_2$ is an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue 2.

Take-aways *(check if homogeneous matters)*:

- The characteristic roots *(add ref)* of a second-order linear ODE are the eigenvalues of the corresponding first-order linear system.
- The solutions of a second-order linear ODE correspond to linear combinations of the eigenvectors of the corresponding first-order linear system (more precisely, to any entry of the resulting vector) *(clarify this)*.

3.10 Coordinates and Diagonalization

Key Ideas

- Coordinate vector
- How to apply a change of basis to coordinate vectors
- How to diagonalize a matrix

Let k be a field, let V be a finite dimensional vector space over k, say $\dim V = n$, and let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis for V. By definition of basis, for any $\mathbf{v} \in V$, there exist unique scalars $\beta_1, \ldots, \beta_n \in k$ such that

$$\mathbf{v} = \beta_1 \mathbf{b}_1 + \ldots + \beta_n \mathbf{b}_n.$$

The vector

$$\mathbf{v}_B = (\beta_1, \ldots, \beta_n)$$

of these scalars is called the **coordinate vector** for $\mathbf{v}$ with respect to the basis B.

3.10.1 Change of Basis

(try to clean up to clarify)

Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be bases for V. By definition of basis, there exist unique scalars $\alpha_i, \beta_i \in k$ such that

$$\sum_{i=1}^{n} \alpha_i \mathbf{a}_i = \mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{b}_i.$$

Writing this equation in matrix form,

$$\begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{v} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix},$$

or

$$A\mathbf{v}_A = \mathbf{v} = B\mathbf{v}_B. \tag{3.10.1}$$

The matrices A and B are $1 \times n$ matrices of (abstract) vectors. Each $\mathbf{b}_i$ is a vector in V, and $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a basis for V. Hence by definition of basis, each basis vector $\mathbf{b}_i \in B$ has unique coordinates $\beta_{j,i} \in k$ (equivalently, a unique coordinate vector $(\mathbf{b}_i)_A \in k^n$):

$$\mathbf{b}_i = \sum_{j=1}^{n} \beta_{j,i} \mathbf{a}_i = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \beta_{1,i} \\ \vdots \\ \beta_{n,i} \end{bmatrix} = A(\mathbf{b}_i)_A.$$

Writing all of this information, for all the basis vectors $\mathbf{b}_i$, in a single a matrix equation,

$$B = \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \beta_{1,1} & \dots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \dots & \beta_{n,n} \end{bmatrix},$$

or even more simply,

$$B = A\beta,$$

where β is the $n \times n$ matrix of coordinates $\beta_{j,i}$ for each $\mathbf{b}_i$ in terms of the basis A. Substituting this into (3.10.1),

$$A\mathbf{v}_A = (A\beta)\mathbf{v}_B = A(\beta\mathbf{v}_B),$$

where in the last equality we use that matrix multiplication is associative.

What are $\mathbf{v}_A$ and $\beta\mathbf{v}_B$? Both are $n \times 1$ coefficient vectors for the vector $\mathbf{v}$ in terms of the basis A. Because a vector has a unique (!) coordinates for any basis, we must have

$$\mathbf{v}_A = \beta\mathbf{v}_B.$$

That is, the $n \times n$ matrix β — which, remember, gives the coordinates of the basis vectors in B with respect to the basis A — takes coordinate vectors in B coordinates and outputs coordinate vectors in A coordinates.

The boxed equations in Farlow *et al.* (2018) (page 329) are special cases when $A = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the standard basis, in which case β is the $n \times n$ matrix whose columns are the coordinate vectors $\beta_{\bullet,i}$ of the basis vectors $\mathbf{b}_i$, i.e.

$$\beta = \begin{bmatrix} | & & | \\ \beta_{\bullet,1} & \dots & \beta_{\bullet,n} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ (\mathbf{b}_1)_A & \dots & (\mathbf{b}_n)_A \\ | & & | \end{bmatrix}.$$

Example 3.63 (Example 5.4.4)**.** Let V be a 2-dimensional vector space over $\mathbf{Q}$, and let $\mathbf{v} \in V$ have coordinate vector

$$\mathbf{u}_A = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

with respect to some basis A of V. Let $B = (\mathbf{b}_1, \mathbf{b}_2)$ be another basis of V such that

$$(\mathbf{b}_1)_A = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad (\mathbf{b}_2)_A = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find the coordinate vector of $\mathbf{v}$ with respect to the basis B.

Solution. From our discussion above,

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \mathbf{v}_A = \begin{bmatrix} | & | \\ (\mathbf{b}_1)_A & (\mathbf{b}_2)_A \\ | & | \end{bmatrix} \mathbf{v}_B = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix} \mathbf{v}_B.$$

We can solve for $\mathbf{v}_B$ in various ways, e.g., (1) by inverting the coefficient matrix β on the right, (2) by setting up and solving the augmented matrix corresponding to this matrix equation, etc. Taking approach (2),

$$\left[\begin{array}{cc:c} -2 & 3 & 3 \\ 1 & 2 & -5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c} 1 & 2 & -5 \\ -2 & 3 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c} 1 & 2 & -5 \\ 0 & 7 & -7 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{cc:c} 1 & 2 & -5 \\ 0 & 1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c} 1 & 0 & -3 \\ 0 & 1 & -1 \end{array}\right].$$

We conclude that

$$\mathbf{v}_B = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

3.10.2 Diagonalizing Matrices

Catchphrase

Eigenvectors make good basis vectors.

Let $\mathbf{A}$ be an $n \times n$ matrix with n linearly independent (!) eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. (Note: The eigenvectors must (!) be linearly independent. The eigenvectors, however, need not be distinct.) Use these eigenvectors and eigenvalues to define the following matrices:

$$\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

By definition, for each i,

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

Also note that, for each standard coordinate vector $\mathbf{e}_i$, and for any $m \times n$ matrix $\mathbf{M}$,

$$\mathbf{M}\mathbf{e}_i = \mathbf{M}_{i,\bullet},$$

the ith column of $\mathbf{M}$. We compute

$$(\mathbf{A}\mathbf{P})\mathbf{e}_i = \mathbf{A}(\mathbf{P}\mathbf{e}_i) = \mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i,$$

and

$$(\mathbf{PD})\mathbf{e}_i = \mathbf{P}(\mathbf{D}\mathbf{e}_i) = \lambda_i \mathbf{P}\mathbf{e}_i = \lambda_i \mathbf{v}_i.$$

These two computations show that the ith column of $\mathbf{AP}$ equals the ith column of $\mathbf{PD}$. Since this is true for each i, the two matrices are equal:

$$\mathbf{AP} = \mathbf{PD}.$$

(clarify this next bit ; right now it is loose with matrix of abstract vectors vs matrix of coordinate vectors ; $\mathbf{P}$ is an n-vector of abstract vectors ; argue why it is "invertible") Inverting $\mathbf{P}$, we can rewrite this as

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$$

By **conjugating** the given matrix $\mathbf{A}$ by a matrix of linearly independent eigenvectors, we transform $\mathbf{A}$ into the diagonal matrix $\mathbf{D}$. We say that **$\mathbf{P}$ diagonalizes $\mathbf{A}$**, and that $\mathbf{A}$ is **diagonalizable**.

Remark 3.64. In general, if a basis of linearly independent eigenvectors of $\mathbf{A}$ exists, then there are a multitude of matrices $\mathbf{P}$ that will diagonalize $\mathbf{A}$. Recall that any nonzero scalar multiple of an eigenvector is again an eigenvector for the same eigenvalue. Moreover, we can reorder the vectors in a basis, and it will still be a basis.

Theorem 3.65. *Let k be a field, and let $\mathbf{A}$ be an $n \times n$ matrix with entries in k. The following are equivalent:*

(a) *$\mathbf{A}$ is diagonalizable.*

(b) *$\mathbf{A}$ has n linearly independent eigenvectors.*

(c) *The sum of the dimensions of the eigenspaces of $\mathbf{A}$ equals n.*

Definition 3.66. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ matrices with entries in k. $\mathbf{B}$ is **similar** to $\mathbf{A}$ if there exists an invertible $n \times n$ matrix $\mathbf{P}$ that conjugates $\mathbf{A}$ into $\mathbf{B}$, i.e.

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}.$$

Remark 3.67. Being "similar to" is an equivalence relation among square matrices. That is, $\sim$ is

1. symmetric: for all square matrices $\mathbf{A}$, $\mathbf{A} \sim \mathbf{A}$
2. reflexive: if $\mathbf{A} \sim \mathbf{B}$, then $\mathbf{B} \sim \mathbf{A}$
3. transitive: if $\mathbf{A} \sim \mathbf{B}$ and $\mathbf{B} \sim \mathbf{C}$, then $\mathbf{A} \sim \mathbf{C}$

Prove this for yourself. (Hint: Keep it simple.)

Diagonalizing matrices is useful for the following purposes:

- factoring a matrix. In the setting of ODEs, $\mathbf{P}$ gives the eigenvectors, $\mathbf{D}$ gives the eigenvalues, and $\mathbf{P}^{-1}$ gives the constants in an IVP.
- providing natural coordinates. Geometrically, diagonalization corresponds to changing bases to a coordinate system where the eigenvectors are the coordinate axes.
- decoupling a linear system of ODEs. We'll see this. *(add ref)*
- taking powers of a matrix. If $\mathbf{A}$ is diagonalizable, say as $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, then $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$. Prove this for yourself. (Hint: Keep it simple.)

Chapter 4

Linear Systems and Higher-Order ODEs

4.1 Solution Methods for Higher-Order ODEs

Key Ideas

- Characteristic polynomial
- Method of undetermined coefficients
- Variation of parameters (for 2nd-order ODEs)

4.1.1 Higher-Order ODEs

Let $n \in \mathbf{Z}_{>0}$. An n**th-order ODE** in the single dependent variable y and the single dependent variable t is an ODE involving y and its (ordinary) derivatives up to order n, where all coefficients in the ODE are functions of t. The nth-order ODE is **linear** if it has the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = f(t),$$

where all functions $a_i(t)$ and $f(t)$ are assumed to be defined over a common t-interval (denote it I). To simplify notation, we often denote the entire left side of this last equation by a linear differential operator, e.g.,

$$\begin{aligned} L : \mathscr{C}^n(I) &\to \mathscr{C}^0(I) \\ y &\mapsto a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y, \end{aligned}$$

or using the differential operator D notation,

$$L = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0.$$

4.1.2 Linear ODEs

(emphasize where vector spaces appear in all this)

As we saw in our study of 1st-order linear ODEs, the property of linearity has important consequences.

Theorem 4.1 (Superposition Principle)**.** *Let* L *be a linear differential operator. For* $i = 1, \dots, m$*, let* $y_i(t)$ *be a solution to the linear ODE* $L(y) = f_i(t)$*, and let* a_i *be a scalar. Then*

$$y(t) = a_1 y_1(t) + \ldots + a_m y_m(t) \tag{4.1.1}$$

is a solution to the linear ODE

$$L(y) = a_1 f_1 + \ldots + a_m f_m. \tag{4.1.2}$$

Proof. This follows immediately by plugging the proposed solution (4.1.1) into the ODE (4.1.2) and using linearity and the fact that y_i solves $L(y) = f_i$. □

Theorem 4.2 (Nonhomogeneous Principle)**.** *Let* L *be a linear differential operator. The general solution to the nonhomogeneous ODE*

$$L(y) = f$$

has the form

$$y = y_p + y_h,$$

where y_p *is a particular solution to* $L(y) = f$*, and* y_h *is the general solution to the corresponding homogeneous equation* $L(y) = 0$*.*

Proof. The proof is a straightforward generalization of the one given for 1st-order linear ODEs in Section *(add ref)*. Nothing in our proof there used the fact that the ODE was of order 1. (In contrast, we heavily used the fact that our ODE was linear!) □

4.1.3 Applications

The superposition and nonhomogeneous principles are useful to solving linear ODEs.

Example 4.3 (4.4.1 & 4.4.2)**.** Find the general solution to the following nonhomogeneous 2nd-order linear ODEs.

(a) The ODE

$$y'' - y' - 2y = 2t + 1 - 2e^t. \tag{4.1.3}$$

(b) The ODE

$$y'' - y' - 2y = t + \frac{1}{2} + 8e^t. \tag{4.1.4}$$

Solution. (a) : To find a particular solution to this ODE, we apply the superposition principle in reverse. The left side of (4.1.3) is a linear differential operator, call it $L = D^2 - D - 2$, applied to y; view the right side as two separate terms

$$f_1(t) = 2t + 1, \qquad f_2(t) = e^t.$$

By the superposition principle, if we can find particular solutions y_i to the ODEs $L(y) = f_i$, then the linear combination $y_1 - 2y_2$ is a particular solution to $L(y) = f_1 - 2f_2$, i.e. to (4.1.3). Using the techniques of this section, we find particular solutions, e.g.,

$$y_1(t) = -t, \qquad y_2(t) = e^t.$$

Thus by the superposition principle,

$$y_p(t) = y_1(t) - 2y_2(t) = -t - 2e^t$$

is a particular solution to (4.1.3). Explicitly, using linearity, we have

$$L(y_p) = L(y_1 - 2y_2) = L(y_1) - 2L(y_2) = f_1 - 2f_2,$$

so y_p is indeed a solution of (4.1.3).

The corresponding homogeneous equation is

$$L(y) = y'' - y' - 2y = 0.$$

The corresponding characteristic equation is

$$\lambda^2 - \lambda - 2 = 0,$$

which has roots

$$\lambda = -1, \qquad \lambda = 2.$$

Hence the general solution of the corresponding homogeneous equation is

$$y_h(t) = c_1 e^{2t} + c_2 e^{-t}.$$

Therefore, by the nonhomogeneous principle, the general solution y to our original ODE (4.1.3) is

$$y(t) = y_p(t) + y_h(t) = -t - 2e^t + c_1 e^{2t} + c_2 e^{-t}.$$

Again, one can use linearity to check that y solves (4.1.3), for any choice of c_1, c_2.

(b) : We could go through the same procedure as in part (a) to compute the general solution. Alternatively, if we note that the left side of (4.1.4) is $\frac{1}{2}f_1 - 4f_2$, then we can use the superposition principle and our solutions y_1, y_2, y_h from part (a) to immediately conclude that the general solution to (4.1.4) is

$$y(t) = \frac{1}{2}y_1(t) - 4y_2(t) + y_h(t) = -\frac{1}{2}t - 4e^t + c_1 e^{2t} + c_2 e^{-t}.$$

4.1.4 Characteristic Polynomial

(preface with note this analysis is better understood in the framework of linear systems of ODEs and linear algebra)

(rewrite in terms of defining the characteristic polynomial and setting it equal to zero ? more consistent with future procedure for finding eigenvalues of matrices)

In our discussion below of solution methods (undetermined coefficients and variation of parameters) for higher-order ODEs, we'll want to know the general solution to the corresponding homogeneous equation. In the case of homogeneous linear ODEs with constant coefficients, the general solution can be found by computing the roots of the **characteristic equation**, i.e. the equation obtained by replacing the derivatives $y^{(n)}$ in the homogeneous ODE with λ^n, where $\lambda \in \mathbf{R}$ (or more generally, in $\mathbf{C}$). For example, given the homogeneous linear ODE

$$y'' - 4y' + 3y = 0,$$

the characteristic equation is

$$\lambda^2 - 4\lambda + 3 = 0.$$

(Note that y corresponds to the zeroth derivative, $y^{(0)}$, and hence becomes $\lambda^0 = 1$ in the characteristic equation.) The roots of this characteristic equation, $\lambda = 1, 3$, are called **eigenvalues** or **characteristic roots** of the ODE. The general solution to the homogeneous linear ODE is then found by taking the linear combination of $e^{\lambda t}$ for these eigenvalues λ:

$$y_h(t) = c_1 e^t + c_2 e^{3t}.$$

If the characteristic polynomial has a root λ of algebraic multiplicity μ_λ, then the general solution contains a term for each of $e^{\lambda t}, te^{\lambda t}, \ldots, t^{\mu-1}e^{\lambda t}$. *(verify ; simplify the following explanatory parenthetical)* (It turns out that, when we form the corresponding linear system and compute the eigenvalues of the coefficient matrix, the geometric multiplicity $g_\lambda = 1$, in this case.) *(relate to generalized eigenvalues)*

We'll study the theory behind this in the next section *(add ref)*. For now, note that if we guess a solution of the form $y(t) = e^{\lambda t}$, substituting this into the general homogeneous linear ODE

$$a_n y^{(n)} + \ldots + a_0 y = 0$$

yields

$$0 = a_n \lambda^n e^{\lambda t} + \ldots + a_0 e^{\lambda t} = e^{\lambda t}\left(a_n \lambda^n + \ldots + 1\right).$$

Because $e^{\lambda t}$ never equals 0, our test function $y^{\lambda t}$ is a solution if and only if λ is a root of the second factor, i.e. the characteristic polynomial.

4.1.5 Method of Guess and Check

Sometimes we can spot a likely solution and check that it indeed works.

(add constant example, e.g., Example 4.4.3(b))

4.1.6 Method of Undetermined Coefficients

The method of undetermined coefficients works only for linear ODEs with constant coefficients, and for only certain families of "right-side" functions $f(t)$.

The key idea is that certain families of functions are closed under the operation of differentiation. For example, taking the derivative of an exponential function ce^{at}, where a and c are constants, gives another such exponential function:

$$\frac{d}{dt}\left(ce^{at}\right) = (ac)e^{at}.$$

Some common families closed under differentiation in this way are

- polynomials in t. If $f(t)$ is a polynomial of degree n, then the solution function we try is also a polynomial of degree n. (Why? Hint: What does the derivative do to the degree of our trial solution polynomial?)
- exponentials e^{at}
- sine and cosine functions together (!) $c_1 \sin(a_1 t) + c_2 \cos(a_2 t)$ We must always consider sine and cosine together. (Why? Hint: What does differentiation do to each?)
- Any finite sum or product of the above.

Example 4.4 (4.4.8)**.** Find the general solution to the 2nd-order linear ODE

$$y'' - y' - 2y = t^2 e^t. \tag{4.1.5}$$

Solution. In this example, the right side involves a polynomial of degree 2, and an exponential e^t. So for a particular solution, we try a function of the form

$$y_p = (a_2 t^2 + a_1 t + a_0)e^t, \tag{4.1.6}$$

where the a_i are constants that we must determine. Using

$$\begin{aligned} y_p' &= (2a_2 t + a_1)e^t + (a_2 t^2 + a_1 t + a_0)e^t \\ &= \left(a_2 t^2 + (2a_2 + a_1)t + (a_1 + a_0)\right) e^t \end{aligned}$$

and

$$\begin{aligned} y_p'' &= (2a_2 t + (2a_2 + a_1))\, e^t + \left(a_2 t^2 + (2a_2 + a_1)t + (a_1 + a_0)\right) e^t \\ &= \left(a_2 t^2 + (4a_2 + a_1)t + (2a_2 + 2a_1 + a_0)\right) e^t, \end{aligned}$$

we get that substituting this y_p into (4.1.5) and grouping like terms gives

$$\left(-2a_2t^2 + (2a_2 - 2a_1)t + (2a_2 + a_1 - 2a_0)\right) e^t = t^2e^2.$$

Equating the coefficients of the various t^ne^t terms on the left and right (for $n = 0, 1, 2$), we get

$$-2a_2 = 1, \qquad 2a_2 - 2a_1 = 0, \qquad 2a_2 + a_1 - 2a_0 = 0.$$

Solving this system of three equations for the three unknowns, we find

$$a_2 = -\frac{1}{2}, \qquad a_1 = -\frac{1}{2}, \qquad a_0 = -\frac{3}{4}.$$

Plugging these results into our y_p in (4.1.6), we have

$$y_p(t) = -\frac{1}{4}\left(2t^2 + 2t + 3\right) e^t.$$

In Example 4.3 we found that the general solution y_h to the corresponding homogeneous equation here is

$$y_h(t) = c_1e^{2t} + c_2e^{-t}.$$

Thus, by the nonhomogeneous principle, we conclude that the general solution to (4.1.5) is

$$y(t) = y_p + y_h = -\frac{1}{4}\left(2t^2 + 2t + 3\right) e^t + c_1e^{2t} + c_2e^{-t}.$$

Warning : If the right side of our ODE coincides with one or more of the terms in the general solution to the corresponding homogeneous equation, then we'll find that this procedure produces a contradiction. In this case, keep multiplying the test function for the particular solution by t until it works. *(explain WHY this happens, and WHY this works)*

A prudent approach is to always compute the general solution to the corresponding homogeneous equation first, so that we know what functions may cause us problems when trying to compute a particular solution to the nonhomogeneous equation.

Example 4.5 (4.4.9)**.** Find the general solution to the nonhomogeneous 2nd-order linear ODE

$$y'' - 2y' + y = 3e^t. \tag{4.1.7}$$

Solution. The corresponding homogeneous equation is

$$y'' - 2y' + y = 0,$$

which has the general solution

$$y_h(t) = c_1e^t + c_2te^t.$$

So we see that e^t on the right side of the nonhomogeneous ODE will cause us problems (twice).

Indeed, if we try the test function

$$y_p(t) = ae^t,$$

where $a \in \mathbf{R}$ is an undetermined coefficient, then plugging it into the original nonhomogeneous ODE yields

$$0 = (a - 2a + a)e^t = 3e^t,$$

a contradiction. Our first attempt at a fix,

$$y_p(t) = ate^t,$$

yields (using that $y_p' = a(t+1)e^t$ and $y_p'' = a(t+2)e^t$)

$$0 = a(t + 2 - 2(t+1) + t)e^t = a(t+2)e^t - 2a(t+1)e^t + ate^t = 3e^t,$$

again a contradiction. One can check that our second attempt at a fix,

$$y_p(t) = at^2e^t$$

gives (using that $y_p' = a(t^2 + 2t)e^t$ and $y_p'' = a(t^2 + 4t + 2)e^t$)

$$a(2)e^t = 3e^t.$$

Solving for the undetermined coefficient a, we find

$$a = \frac{3}{2}.$$

Thus a particular solution to (4.1.7) is

$$y_p(t) = \frac{3}{2}t^2e^t,$$

and the general solution y to (4.1.7) is

$$y(t) = y_p + y_h = \frac{3}{2}t^2e^t + c_1e^t + c_2te^t,$$

where $c_1, c_2 \in \mathbf{R}$.

4.1.7 Variation of Parameters

The solution method of variation of parameters is more general than the method of undetermined coefficients. Variation of parameters applies to linear ODEs with general (i.e. nonconstant) coefficients, and it can handle "right-side" functions not in the list given above for undetermined coefficients.

(include general theory)

Example 4.6 (4.5.4)**.** Find the general solution to the nonhomogeneous 2nd-order linear ODE

$$t^2y'' - 2ty' + 2y = t\ln t, \tag{4.1.8}$$

where we restrict $t \in \mathbf{R}_{>0}$. Hint: First check that $y_1(t) = t$ and $y_2(t) = t^2$ solve the corresponding homogeneous equation.

Solution. *(start by normalizing the given ODE so y'' term has coefficient 1)* The corresponding homogeneous equation is

$$t^2y'' - 2ty' + 2y = 0.$$

It is straightforward to check that the given functions y_1 and y_2 are solutions to this homogeneous ODE. Thus we seek to solve the system of ODEs

$$tv_1' + t^2v_2' = 0 \tag{4.1.9}$$

$$v_1' + 2tv_2' = \frac{\ln t}{t} \tag{4.1.10}$$

for v_1 and v_2. (Note that the right side of this second equation is $\frac{\ln t}{t}$, not $t\ln t$, because our general work above assumed that the coefficient of the y'' term is 1.) From (4.1.9),

$$v_1' = -tv_2'.$$

Substituting this into (4.1.10), we get

$$tv_2' = \frac{\ln t}{t} \qquad \Leftrightarrow \qquad v_2' = t^{-2}\ln t,$$

so (e.g., using integration by parts, with $u = \ln t$ and $dv = t^{-2}\,dt$)

$$v_2 = -t^{-1}(1 + \ln t).$$

We also compute

$$v_1' = -tv_2' = -t^{-1}\ln t.$$

Again using integration by parts (e.g., with $u = \ln t$ and $dv = -t^{-1}\,dt$), we get

$$v_1 = -\int t^{-1}\ln t\,dt = -(\ln t)^2 + \int t^{-1}\ln t\,dt$$

so

$$v_1 = -\frac{1}{2}(\ln t)^2.$$

Thus

$$y_p(t) = v_1 y_1 + v_2 y_2 = -\frac{1}{2}t(\ln t)^2 - t(1 + \ln t).$$

(add comment about cramer–wronskian method) Thus the general solution to (4.1.8) is

$$\begin{aligned} y(t) = y_p + y_h &= -\frac{1}{2}t(\ln t)^2 - t(1 + \ln t) + c_0 t + c_2 t^2 \\ &= -\frac{1}{2}t(\ln t)^2 - t\ln t + c_1 t + c_2 t^2, \end{aligned}$$

where $c_1, c_2 \in \mathbf{R}$.

4.2 First-Order Linear Systems

Key Ideas

- Existence and uniqueness of solutions to 1st-order linear systems
- Solution space theorem
- Wronskian of solutions to 1st-order linear system; fundamental matrix
- Phase space, phase plane, trajectories

A quick reminder of motivation to study linear systems. Most real-world phenomena are not linear. However, many are approximately linear, especially "close" to points of interest.

(include summary of key ideas?)

(if not overly complicating, present more general definition, for fields other than $\mathbf{R}$*)*

(can we relax these hypotheses on $\mathbf{A}$ *and* $\mathbf{f}$ *— e.g., not require the component functions be continuous? do we include them just to guarantee existence and uniqueness?)*

Definition 4.7. An n-dimensional **1st-order linear system of ODEs** on an open interval $I \subseteq \mathbf{R}$ can be written as a matrix equation

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t),$$

where

- $\mathbf{A}(t)$ is an $n \times n$ matrix of functions $a_{i,j}(t)$, each of which is continuous on I.
- $\mathbf{f}(t)$ is an $n \times 1$ vector of functions $f_i(t)$, each of which is continuous on I.
- $\mathbf{x}(t)$ is an $n \times 1$ vector of functions $x_i(t)$, each of which is differentiable on I.

If $\mathbf{f}(t) = \mathbf{0}$, then the system reduces to

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t),$$

and we call it **homogeneous**.

An **initial value problem (IVP)** is the combination of a linear system and an initial value vector:

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Remark 4.8. The equation for a 1st-order linear system of ODEs is the matrix analog of a 1st-order linear ODE. This is perhaps more evident if we move the term $\mathbf{A}(t)\mathbf{x}(t)$ to the left, and dispense with the "function of t" reminder:

$$\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{f}.$$

This form also makes our definition of a homogeneous system analogous to our earlier one for a single ODE.

4.2.1 Existence and Uniqueness

By definition, a 1st-order linear system is a collection of n 1st-order linear ODEs:

$$x_i' = a_{i,1}(t)x_1 + \ldots + a_{i,n}x_n + f_i(t) = F_i(x_1, \ldots, x_n, t).$$

In Section 1.5, we studied existence and uniqueness in the case of one variable (i.e. $n = 1$). Loosely speaking, we saw that existence is guaranteed (on some subinterval) if F_i is continuous, and uniqueness is guaranteed (on some subinterval) if $\frac{\partial F_i}{\partial x_i}$ is continuous. For a linear system, these partial derivatives are

$$\frac{\partial F_i}{\partial x_i} = a_{i,i}(t).$$

Under the hypotheses that all the functions in the coefficient matrix $\mathbf{A}(t)$ and the vector $\mathbf{f}(t)$ are continuous on I, these conditions are satisfied.

Theorem 4.9 (Existence and Uniqueness)**.** *Let $\mathbf{A}(t)$ be an $n \times n$ matrix, let $\mathbf{f}(t)$ be an $n \times 1$ vector, and let the entries of both be continuous on an open interval I containing some point t_0. Then for any $\mathbf{x}_0 \in \mathbf{R}^n$, there exists a unique solution vector $\mathbf{x}(t)$ satisfying the IVP*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}, \qquad\qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

After we've developed a bit more language, we'll return to discuss what existence and uniqueness implies about graphs of solutions. (See Remark 4.17.)

4.2.2 Structure of Solutions

Solutions to homogeneous linear ODEs form a vector space (over the underlying field *(be more precise without being prolix)*).

Theorem 4.10 (Solution Space Theorem)**.** *Let $\mathbf{A}(t)$ be an $n \times n$ matrix with entries in $\mathbf{R}(t)$. Then the solution set to the homogeneous 1st-order linear system*

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

is a vector space over $\mathbf{R}$ of dimension n.

This has the following two consequences.

First, we construct the general solution to a 1st-order linear system as before: Find the general solution $\mathbf{x}_h$ to the corresponding homogeneous equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

find a particular solution $\mathbf{x}_p$ to the IVP

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f},$$

and take their sum

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p.$$

Second, because vector spaces are closed under the vector space operations (vector addition and scalar multiplication), we get the superposition principle.

Corollary 4.11. *Consider the homogeneous 1st-order linear system*

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{4.2.1}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_m$ *be solution vectors to* (4.2.1) *on an interval* $I \subseteq \mathbf{R}$. *Then any linear combination of these* $\mathbf{x}_i$ *is a solution. That is, for all* $a_1, \ldots, a_m \in \mathbf{R}$,

$$\mathbf{x} = a_1\mathbf{x}_1 + \ldots + a_m\mathbf{x}_m$$

is a solution to (4.2.1) *on* I.

The solution space theorem (Theorem 4.10) says that any basis of the solution space has n vectors. Thus, if we find n linearly independent solutions, then we have a basis.

Definition 4.12. Let $\mathbf{x}' = \mathbf{A}\mathbf{x}$ be an n-dimensional homogeneous 1st-order linear system of ODEs, let $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ be solutions on an interval $I \subseteq \mathbf{R}$, and let

$$\mathbf{X} = \begin{bmatrix} | & & | \\ \mathbf{x}_1(t) & \ldots & \mathbf{x}_n(t) \\ | & & | \end{bmatrix}.$$

The **wronskian** of these solutions is

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \det \mathbf{X}(t).$$

Example 4.13 (Examples 6.1.1–4)**.** Consider the homogeneous 1st-order linear system

$$x_1' = 3x_1 - 2x_2 \tag{4.2.2}$$
$$x_2' = x_1 \tag{4.2.3}$$
$$x_3' = -x_1 + x_2 + 3x_3. \tag{4.2.4}$$

Let

$$\mathbf{A} = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

With this notation, we can write the given system as a single matrix equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

One can check (and we will see how to deduce) that

$$\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$$

are solutions to (4.2.4) on the interval $I = \mathbf{R}$. Thus the superposition principle guarantees that for all $a_1, a_2, a_3 \in \mathbf{R}$,

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3$$

is also a solution to (4.2.4) on $\mathbf{R}$. We can write this linear combination as

$$\begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Writing out the vectors $\mathbf{x}_i$ explicitly, the general solution has the form

$$\begin{bmatrix} e^t & 2e^{2t} & 0 \\ e^t & e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{4.2.5}$$

We claim the solutions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. To show this, it suffices[1] to find a $t \in I$ (recall in this example, $I = \mathbf{R}$) such that the vectors $\mathbf{x}_i(t)$ are linearly independent; which is equivalent to the 3×3 matrix in (4.2.5), evaluated at t, having nonzero determinant. Trying $t = 0$, we obtain

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

which has determinant

$$0 + 0 + (-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = -1 \neq 0.$$

This certifies that the three vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)$ are linearly independent.

Alternatively, we can certify linear independence using the wronskian:

$$\begin{aligned} W(\mathbf{x}_1, \ldots, \mathbf{x}_3) &= \det \begin{bmatrix} e^t & 2e^{2t} & 0 \\ e^t & e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \end{bmatrix} \\ &= 0 + 0 + (-1)^{3+3} \det \begin{bmatrix} e^t & 2e^{2t} \\ e^t & e^{2t} \end{bmatrix} \\ &= e^{3t} - 2e^{3t} = -e^{3t}, \end{aligned}$$

[1] In fact, this condition is necessary, too. However, it may not be easy to find such a t.

which is not the zero function (!). This certifies that $\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)$ are linearly independent.

By the solution space theorem (Theorem 4.10), the solution space of our linear system (4.2.4) is a vector space over $\mathbf{R}$ of dimension 3. Thus three linearly independent vectors form a basis. In particular, $(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t))$ is a basis of the solution space.

Definition 4.14. Let $\mathbf{x}' = \mathbf{A}\mathbf{x}$ be an n-dimensional homogeneous 1st-order linear system of ODEs. A **fundamental matrix** for this system is a matrix of the form

$$\begin{bmatrix} | & & | \\ \mathbf{x}_1(t) & \dots & \mathbf{x}_n(t) \\ | & & | \end{bmatrix},$$

where $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is a basis for the solution space.

Remark 4.15. A fundamental matrix is not unique. Why? In general, we can choose different linearly independent vectors to form a basis, and reordering the elements of basis rearranges the columns of the matrix.

4.2.3 Phase Space

Consider an n-dimensional 1st-order linear system of ODEs

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f},$$

and let $\mathbf{x}(t)$ be a solution.

Definition 4.16. The graph of $\mathbf{x}(t)$ in the (t, x_i)-plane is called a **component graph** (a.k.a. **solution graph**, **time series**). The space $\mathbf{R}^n$ with coordinates $(x_1, \dots, x_n)$ is called **phase space**.

In the case $n = 2$, the phase space $\mathbf{R}^2$ (with coordinates (x_1, x_2)) is called the **phase plane**. The parametric curves

$$(x_1(t), x_2(t))$$

are called **trajectories**. The plot of trajectories on a phase plane is called a **phase portrait**.

Remark 4.17 (Warning!)**.** What does existence and uniqueness say about graphs of solutions? Consider an n-dimensional 1st-order linear system of ODEs that satisfies the hypotheses for existence and uniqueness of solutions. It has solution curves that do not intersect *in the space of all variables.* For an n-dimensional system, "all variables" means $x_1, \dots, x_n$,...and t. So solutions do not intersect in the space $\mathbf{R}^{n+1}$ with these coordinates. However, solutions (i.e. trajectories) plotted in the phase space ($\mathbf{R}^n$ with coordinates $(x_1, \dots, x_n)$) may intersect. This is not a contradiction: Points of intersection in phase space have all corresponding x_i coordinates equal, but their corresponding t coordinates may differ (indeed, must differ, if the hypotheses for existence and uniqueness are satisfied).

It follows *(expound?)* that if our linear system is autonomous (i.e. if all the entries in $\mathbf{A}$ and $\mathbf{f}$ do not depend on t) and satisfies the hypotheses of existence and uniqueness, then trajectories in phase space also cannot cross. (Can you argue why?)

4.3 Solving 1st-Order Linear Systems

Key Ideas

- Motivation for general form of solutions to homogeneous 1st-order linear system
- How to write nth-order linear ODE as 1st-order linear system

4.3.1 Solving the Homogeneous 1st-Order Linear System with Constant Coefficients

Consider an n-dimensional homogeneous 1st-order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{4.3.1}$$

and suppose that the coefficient matrix $\mathbf{A}$ is constant (i.e. all entries $a_{i,j}$ are constants). This resembles the homogeneous 1st-order ODE

$$x' = ax,$$

a separable ODE with general solution

$$x(t) = ce^{at}. \tag{4.3.2}$$

Motivated by this, we guess a solution of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v} \tag{4.3.3}$$

for a vector $\mathbf{v} \in \mathbf{R}^n$. (In particular, $\mathbf{v}$ is a vector of constants, and does not depend on t.) Note that in the analogy to the solution to (4.3.2), the constant vector $\mathbf{v}$ plays the role of the constant coefficient c. In both cases, it is the e^{at} or $e^{\lambda t}$ factor that varies with t.

Substituting our guess (4.3.3) into our original linear system (4.3.1), we have

$$\lambda e^{\lambda t}\mathbf{v} = \mathbf{A}\left(e^{\lambda t}\mathbf{v}\right),$$

which we can rearrange as

$$e^{\lambda t}\left(\mathbf{A} - \lambda \mathbf{I}_n\right)\mathbf{v} = \mathbf{0}.$$

For any λ, and for all $t \in \mathbf{R}$, $e^{\lambda t} \neq 0$. Thus for this matrix equation to hold, we must have

$$\left(\mathbf{A} - \lambda \mathbf{I}_n\right)\mathbf{v} = \mathbf{0}.$$

A scalar λ and vector $\mathbf{v}$ satisfying this equation are precisely an eigenvalue and eigenvector of the matrix $\mathbf{A}$.

We have seen *(add ref)* that the solution space for an n-dimensional homogeneous 1st-order system has dimension n. If the coefficient matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, then the vectors

$$\mathbf{x}_i = e^{\lambda_i t}\mathbf{v}_i \tag{4.3.4}$$

(in any order) form a basis for the solution space. In particular, if the eigenvalues λ_i of $\mathbf{A}$ are distinct, then (as we have seen) the corresponding eigenvectors $\mathbf{v}_i$ are linearly independent in $\mathbf{R}^n$, hence *(does this need explanation? maybe add a footnote)* the solution vectors $\mathbf{x}_i$ defined in (4.3.4) are linearly independent, and, because there are n of them, they form a basis for the solution space.

Remark 4.18. As we have seen in our study of eigenthings, an $n \times n$ matrix need not have n distinct eigenvectors. In this case, we will need additional techniques to determine a basis for the solution space. See Subsection *(add ref)*.

4.3.2 Higher-Order ODEs as 1st-Order Systems

Definition 4.19. An n**th-order linear ODE** can be written in the form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = f(t). \tag{4.3.5}$$

Consider the class of homogeneous nth-order linear ODEs with constant coefficients, i.e. such that $a_i(t)$ is a constant for all coefficient functions a_i. Conventionally, one is taught to solve this class of ODEs by forming its associated **characteristic equation**, i.e. the polynomial obtained by replacing $y^{(n)}$ by λ^n:[2]

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0.$$

Then, one computes the roots λ_i of this polynomial, called the **characteristic roots**, over the complex numbers $\mathbf{C}$. (Factoring over $\mathbf{C}$ guarantees that this polynomial has exactly n roots.[3]) If the roots are all distinct, then the general solution is

$$y(t) = c_1e^{\lambda_1 t} + \ldots + c_ne^{\lambda_n t}.$$

[2]There is good motivation for considering this polynomial. Let's guess a solution of the form $y(t) = e^{\lambda t}$. Plugging this into (4.3.5) (with $f(t) = 0$), we obtain

$$0 = a_n\lambda^n e^{\lambda t} + a_{n-1}\lambda^{n-1}e^{\lambda t} + \ldots + a_1\lambda e^{\lambda t} + a_0 = e^{\lambda t}\left(a_n\lambda^n + \ldots + a_0\right).$$

Note that, for any λ, for all $t \in \mathbf{R}$, $e^{\lambda t} \neq 0$. Thus this last equation holds if and only if

$$a_n\lambda^n + \ldots + a_0 = 0.$$

The solutions λ to this equation are the characteristic roots.

[3]This is the content of the fundamental theorem of algebra. One says that the field $\mathbf{C}$, or any other field satisfying this condition, is algebraically closed.

If some of the roots λ_i are repeated (i.e. have algebraic multiplicity greater than 1) — say $\lambda_{i+1} = \lambda_i$ — then one includes expressions like $c_{i+1}te^{\lambda_i t}$. *(verify generalizes to $n > 2$)*

(make concise) However, this presentation is inefficient and out of context. nth-order linear ODEs can be viewed as a special case of 1st-order linear systems, via a change of variables. So everything one does in the above special setting, one can do in the more general setting of linear systems, and one can also do more. If one wants to use results specific to the special setting, knowing the general theory makes it easy to derive.

For example, consider the homogeneous 2nd-order linear ODE

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0.$$

Define

$$x_0(t) = y(t), \qquad\qquad x_1(t) = y'(t).$$

Then

$$\begin{aligned} x_0'(t) &= y'(t) = x_1(t), \\ x_1'(t) &= y''(t) = \alpha_1(t)y' + \alpha_0(t)y = \alpha_1(t)x_1(t) + \alpha_0(t)x_0(t), \end{aligned}$$

where[4]

$$\alpha_i(t) = -\frac{a_i(t)}{a_2(t)}.$$

We can write this system in matrix notation

$$\mathbf{x}' = \begin{bmatrix} x_0'(t) \\ x_1'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha_0 & \alpha_1 \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} = \mathbf{A}\mathbf{x}.$$

a first-order linear system in 2 variables.

Suppose that the original a_i coefficient functions are constants. Then the $\alpha_i = -\frac{a_i}{a_2}$ are constants. If we compute the characteristic polynomial of the coefficient matrix $\mathbf{A}$, we find

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} -\lambda & 1 \\ \alpha_0 & \alpha_1 - \lambda \end{bmatrix} = \lambda^2 - \alpha_1\lambda - \alpha_0.$$

Multiplying through by a_2, we obtain

$$a_2\lambda^2 + a_1\lambda + a_0.$$

Thus the characteristic equation associated to the homogeneous 2nd-order linear ODE with constant coefficients is the same (up to multiplication by a nonzero

[4]By hypothesis, our original ODE was 2nd-order, so $a_2(t)$ cannot be the zero function, and hence we may divide by $a_2(t)$.

scalar, which does not change the roots of the equation) as the characteristic polynomial of the associated 1st-order linear system. The characteristic roots of the characteristic equation are precisely the eigenvalues of the associated linear system.

Similarly, the homogeneous 3rd-order linear ODE

$$a_3(t)y^{(3)} + \ldots + a_0(t)y = 0$$

can be written as a first-order linear system in 3 variables,

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \end{bmatrix} \mathbf{x}.$$

Likewise for the general homogeneous nth-order linear ODE.

4.3.3 Examples

We study several examples of 2-dimensional homogeneous linear systems, because trajectories in their phase space (namely, $\mathbf{R}^2$) are relatively easy for us to analyze.

Example 4.20 (Example 6.2.3)**.** Find the general solution to the homogeneous 1st-order linear system

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}.$$

Solution. One can check that the eigenvalues and eigenvectors for the coefficient matrix are

$$\lambda_1 = -1, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \qquad\qquad \lambda_2 = -3, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Because they belong to different eigenvalues, these eigenvectors are linearly independent, and hence so are the corresponding solution vectors $\mathbf{x}_i = e^{\lambda_i t}\mathbf{v}_i$. We conclude that the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

for $c_1, c_2 \in \mathbf{R}$.

Plotting the eigenvectors in the (x_1, x_2)-plane and labelling them with the direction the trajectory follows with increasing t, we have *(add graphic!!!)*. This equilibrium $(0,0)$ is called a **stable equilibrium**: Both eigenvalues are negative, and all trajectories move toward the equilibrium.

Example 4.21 (Example 6.2.2)**.** Find the general solution to the homogeneous 1st-order linear system

$$\mathbf{x}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Solution. One can check that the eigenvalues and eigenvectors for the coefficient matrix are

$$\lambda_1 = 4, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \qquad\qquad \lambda_2 = 1, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Because they belong to different eigenvalues, these eigenvectors are linearly independent, and hence so are the corresponding solution vectors $\mathbf{x}_i = e^{\lambda_i t}\mathbf{v}_i$. We conclude that the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

for $c_1, c_2 \in \mathbf{R}$.

Plotting the eigenvectors in the (x_1, x_2)-plane and labelling them with the direction the trajectory follows with increasing t, we have *(add graphic!!!)*. This equilibrium $(0, 0)$ is called an **unstable equilibrium**: Both eigenvalues are positive, and all trajectories move away from the equilibrium.

Example 4.22 (Example 6.2.1)**.** Find the general solution to the homogeneous 1st-order linear system

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Solution. One can check that the eigenvalues and eigenvectors for the coefficient matrix are

$$\lambda_1 = -1, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \qquad\qquad \lambda_2 = 3, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Because they belong to different eigenvalues, these eigenvectors are linearly independent, and hence so are the corresponding solution vectors $\mathbf{x}_i = e^{\lambda_i t}\mathbf{v}_i$. We conclude that the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

for $c_1, c_2 \in \mathbf{R}$.

Plotting the eigenvectors in the (x_1, x_2)-plane and labelling them with the direction the trajectory follows with increasing t, we have *(add graphic!!!)*. This equilibrium $(0, 0)$ is called a **saddle equilibrium**: One positive and one negative eigenvalue.

Let's pause to summarize the commonalities in our examples thus far. (All of our examples so far have been autonomous, homogeneous 2-dimensional linear systems.)

- Eigenvectors and their corresponding eigenvalues help us draw the phase portrait.
- There is a unique equilibrium, at the origin $(0, 0)$.
- Trajectories move toward or away from this equilibrium according to the sign of the eigenvalue (negative or positive, respectively) associated to each eigenvector.
- Eigenvectors define four unique trajectories (each called a **separatrix**), in the four directions $\pm\mathbf{v}_i$ (two choices for the sign, two choices for $i \in \{1, 2\}$, so four directions total (recall that, by definition, an eigenvector cannot be the zero vector!)). These separatrices separate all other trajectories. (This is an example of a uniqueness result mentioned last section *(add ref)*: For autonomous linear systems, trajectories do not cross in phase space. (Can you explain why trajectories to the origin in the case of stable and saddle equilibria do not contradict this?)

We also make some remarks about the speed and shape of trajectories.

- The relative speed along a trajectory depends on the relative absolute value of the eigenvalues. The larger the absolute value of the eigenvalue, the faster the speed along the corresponding eigenvector.

The relative speed helps us figure out which separatrices trajectories approach. *(add Figure 6.2.5 and explain)*

4.3.4 Generalized Eigenvectors

(reconcile this section with whatever discussion is added in the preceding chapter(s) on linear algebra)

(delete the following, give general presentation based on Wikipedia : Generalized eigenvector : jordan form, jordan chains, etc. ; then give several small examples (again, see Wikipedia) ; this is last topic in Section 6.2)

Consider a homogeneous n-dimensional linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

let $\mathbf{A}$ be constant, and suppose that $\mathbf{A}$ has an eigenvalue λ such that

$$\operatorname{AlgMult}\lambda > \operatorname{GeoMult}\lambda.$$

(In the case $n = 2$, this means that $\mathbf{A}$ has a repeated eigenvalue and only one eigenvector.) In this case, a basis for the eigenspace $E\lambda$ can be computed as follows.

1. Find all linearly independent eigenvectors $\mathbf{v}$ for λ. (At least one eigenvector always exists.)

4.4 Matrix Exponential

Key Ideas

•

4.4.1

In our study of 1st-order ODEs, we found that the homogeneous linear ODE

$$y' = ay,$$

where a is a constant, has the general solution

$$y(t) = ce^{at}, \tag{4.4.1}$$

where $c \in \mathbf{R}$. In our study of systems of 1st-order linear ODEs, we encounter the analogous homogeneous linear ODE

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $n \times n$ matrix of constants. (In this equation, $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$.) For this matrix ODE, the solution analogous to (4.4.1) would seem to be

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{c},$$

where $\mathbf{c}$ is an $n \times 1$ matrix of constants. (In this analogy, we put the "constant" $\mathbf{c}$ after the exponential, so that the order of the resulting matrix agrees with the order of $\mathbf{x}$, which is $n \times 1$.) Is it possible to define the exponential of a (square) matrix, so that this analogy is true? It turns out the answer is yes. We explore how to do so.

Our motivation for the definition of the matrix exponential comes from the Taylor series expansion of e^t, centered at $t = 0$:

$$e^t = \sum_{j=0}^{\infty} \frac{1}{j!} t^j = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \dots. \tag{4.4.2}$$

Recall that one can show (e.g., via the ratio test) that this series converges for all $t \in \mathbf{R}$. (In fact, this series converges for all $t \in \mathbf{C}$.[5])

What is the corresponding series for matrices? Let $\mathbf{A}$ be an $n \times n$ matrix of constants. Then

$$e^{\mathbf{A}} = \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{A}^j = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{6}\mathbf{A}^3 + \dots. \tag{4.4.3}$$

(add comment about convergence)

[5] This can also be shown by the ratio test. *(give ref)*

Definition 4.23. Let $\mathbf{A}$ be an $n \times n$ matrix. $\mathbf{A}$ is **nilpotent** if there exists some (positive) integer m such that

$$\mathbf{A}^m = \mathbf{0}.$$

There are two cases of the matrix exponential that will be of particular interest to us.

- $\mathbf{A}$ is diagonal. Say

$$\mathbf{A} = \begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix}.$$

Then one can check (e.g., via induction) that for any $j \in \mathbf{Z}$,

$$\mathbf{A}^j = \begin{bmatrix} a_1^j & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n^j \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{\mathbf{A}} = \sum_{j=0}^{\infty} \frac{1}{j!}\mathbf{A}^j &= \sum_{j=0}^{\infty} \frac{1}{j!} \begin{bmatrix} a_1^j & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n^j \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=0}^{\infty} \frac{1}{j!} a_1^j & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sum_{j=0}^{\infty} \frac{1}{j!} a_n^j \end{bmatrix} \\ &= \begin{bmatrix} e^{a_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{a_n} \end{bmatrix}, \end{aligned}$$

where in the second line we have used the properties of scalar multiplication and matrix addition, and in the third line we have used the Taylor series expansion of the (scalar) exponential.

- $\mathbf{A}$ is nilpotent. By definition of nilpotent, there exists some $m \in \mathbf{Z}_{>0}$ such that $\mathbf{A}^m = \mathbf{0}$. It follows that for all $j \geqslant m$, $\mathbf{A}^j = \mathbf{0}$. Hence when applying the definition of the matrix exponential to a nilpotent matrix $\mathbf{A}$, all terms in the series defining $e^{\mathbf{A}}$ become $\mathbf{0}$ from term $j = m$ on. That is, in this case, the series is a finite sum.

Example 4.24 (Examples 6.6.1,6.6.2)**.** Find the matrix exponential of the following two matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution. The matrix $\mathbf{A}$ is diagonal. Thus

$$e^{\mathbf{A}} = \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix}.$$

The matrix $\mathbf{B}$ is not diagonal (nor can it be diagonalized — why?). One computes

$$\mathbf{B}^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and hence, for all integers $j \geqslant 3$,

$$\mathbf{B}^j = \mathbf{0},$$

where $\mathbf{0}$ denotes the 3×3 zero matrix. Thus

$$e^{\mathbf{B}} = \mathbf{I} + \mathbf{B} + \frac{1}{2}\mathbf{B}^2 + \mathbf{0} + \mathbf{0} + \ldots = \begin{bmatrix} 1 & -1 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark 4.25. One can show that, in general, a strictly upper triangular (square) matrix — i.e. a matrix with 0s on and below the diagonal — is nilpotent.

Proposition 4.26 (Properties of the matrix exponential)**.** *Let* $\mathbf{A}, \mathbf{B}$ *be* $n \times n$ *matrices with entries in a field* k.

(a) $e^{\mathbf{0}} = \mathbf{I}$, *where* $\mathbf{0}$ *denotes the* $n \times n$ *zero matrix, and* $\mathbf{I}$ *denotes the* $n \times n$ *identity matrix.*

(b) $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$.

(c) If $\mathbf{A}$ *and* $\mathbf{B}$ *commute, i.e. if* $\mathbf{AB} = \mathbf{BA}$, *then*

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}.$$

Remark 4.27. Some remarks.

1. In general, two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ will not commute. In a special case of interest, when $\mathbf{A}$ and $\mathbf{B}$ are both powers of the same matrix, then they commute. (Why?)

2. Property (b) says that, for any (!) square matrix $\mathbf{A}$, the matrix exponential $e^{\mathbf{A}}$ is invertible. In particular, this implies that the columns of $e^{\mathbf{A}}$ are linearly independent, when viewed as vectors in $\mathbf{R}^n$.

Let $\mathbf{A}$ be a square matrix with entries in a field k. If t is a scalar variable (i.e. a variable that takes various values in k), then for any value of t, $t\mathbf{A}$ is also a square matrix with entries in k, so its exponential is given by formula (4.4.2):

$$e^{\mathbf{A}t} = \sum_{j=0}^{\infty} \frac{1}{j!}(\mathbf{A}t)^j = \mathbf{I} + t\mathbf{A} + \frac{1}{2}t^2\mathbf{A}^2 + \frac{1}{6}t^3\mathbf{A}^3 + \dots.$$

This is sometimes called the **matrix exponential function**, because its value is a function of t.

Remark 4.28. Some remarks.

1. One can show *(give key intuition, cite)* that, for all square matrices $\mathbf{A}$, and for all values of t, the series defining $e^{\mathbf{A}t}$ converges.
2. Note that the jth term in the series for the matrix exponential function is the same as the jth term in the series for the matrix exponential, just with a factor t^j. In particular, if we have computed powers of our matrix to compute the matrix exponential, we can reuse those results to compute the matrix exponential function.
3. Evaluating the matrix exponential function at $t = 1$ gives the matrix exponential: $e^{\mathbf{A}(1)} = e^{\mathbf{A}}$.

Example 4.29 (Example 6.6.3)**.** Consider the nilpotent matrix $\mathbf{B}$ from Example 4.24. Its matrix exponential function is

$$e^{\mathbf{B}t} = \mathbf{I} + t\mathbf{B} + \frac{1}{2}t^2\mathbf{B}^2 = \begin{bmatrix} 1 & -t & -\frac{1}{2}t^2 + 2t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 4.30. *Let $\mathbf{A}$ be a square matrix with entries in a field k, and let t be a scalar variable taking values in k. Then*

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}.$$

Proof. Differentiate the power series defining the matrix exponential function, term by term. *(include proof)* □

Remark 4.31. Some remarks.

1. Note that the product on the right is the product of two $n \times n$ matrices, hence it is defined.
2. *(state this next bit more eloquently . this is the grand result !)* This theorem explains why exponentials naturally appear in the solution to $n \times n$ homogeneous 1st-order linear systems of ODEs. Practically, this theorem lets us use the matrix exponential function to solve such systems.

(enhance exposition of this next result . again, grand result ! proof is guess-and-check ; can we provide intuition, avoid "guessing" ?)

Theorem 4.32. *Let* $\mathbf{A}$ *be an* $n \times n$ *(i.e. square) matrix with (constant) entries in a field* k*, and let* $\mathbf{x} = \begin{bmatrix} x_1(t) & \dots & x_n(t) \end{bmatrix}^T$ *be an* $n \times 1$ *matrix of continuously differentiable functions of* t*, where* t *is a scalar variable taking values in the same field* k*. Then the general solution to the 1st-order linear system*

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c},$$

where $\mathbf{c}$ *is an* $n \times 1$ *column matrix of arbitrary parameters in* k *(the "coefficients").*

If the linear system also has an initial condition $\mathbf{x}(0) = \mathbf{x}_0$ *(note that* $\mathbf{x}_0$ *is an* $n \times 1$ *column matrix of scalars; also, it is important that* $t = 0$ *in the initial condition, for this result* (as such, maybe downplay this second part of the theorem ?)*), then the solution to the corresponding IVP is*

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0.$$

Proof. Check the solution as we learned as DE infants: Plug the proposed solution into the ODE, and verify the equation holds.

Applying the initial condition, we get

$$\mathbf{x}_0 = \mathbf{x}(0) = e^{\mathbf{A}(0)}\mathbf{c} = \mathbf{I}\mathbf{c} = \mathbf{c}.$$

Thus the parameter values $\mathbf{c}$ for the solution to the IVP are given by the initial condition matrix $\mathbf{x}_0$. □

(include alternative views of the matrix exponential ; see pp 406–408)

4.4.2 Special Case: A Diagonalizable

Consider the $n \times n$ homogeneous 1st-order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

Let $\mathbf{A}$ be a diagonalizable. This implies *(add ref to earlier result, in section on linear algebra)* that $\mathbf{A}$ has n linearly independent eigenvectors. Let $\mathbf{D}$ be the diagonal matrix of eigenvalues of $\mathbf{A}$, and let $\mathbf{P}$ be the matrix whose columns are the eigenvectors of $\mathbf{A}$, listed in the same order as their corresponding eigenvalues are listed along the diagonal of $\mathbf{D}$. As we have seen *(add same ref)*,

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

(finish this analysis — see p 409)

Bibliography

Farlow, Jerry, Hall, James E., McDill, Jean Marie, & West, Beverly H. 2018. *Differential Equations and Linear Algebra*. 2 edn. Pearson.