Math 357
Exam 02

2024–04–19 (F)

Your name: ______________________________

Honor pledge:

Instructions

1. In the space above, please legibly write your name and the Rice Honor Pledge, then sign.
2. Full time for this exam is exactly 50 minutes. No resources are allowed.
3. Your reasoning—correctness and clarity—is more important than your "answer".
4. If you think there is ambiguity or error in an exercise, then briefly (!) write your understanding of the exercise and any additional hypotheses you are making, then proceed.
5. Note: The definition parts of Exercises 1–3 will be graded as Exercise 0.

This exam is an imperfect measure of my understanding at a particular point in time. It is not a measure of who I am or who I will be.

Exercise	Total	(a)	(b)	(c)	(d)
0	/4	/4	/4	/4	/4
1	/4	—	—	/4	/4
2	/4	—	/4	/4	/4
3	/4	—	/4	/4	/4
4	/4	/4	/4	/4	/4
Total	/20				

Exercise 1

Let K_0 be a field, let $n \in \mathbf{Z}_{\geqslant 0}$, and let $f = a_n t^n + \ldots + a_0 \in K_0[t]$. For your definitions, clearly introduce any additional objects you use and the hypotheses you make.

(a) Define what it means for f to be (i) irreducible and (ii) separable.

(b) Define the formal derivative of f. As relevant to your definition, explain what multiplication by an integer means if $\operatorname{char} K_0 \neq 0$ (in which case, K_0 does not contain an isomorphic copy of $\mathbf{Z}$).

(4 pt) For the remaining parts of this exercise, let $\operatorname{char} K_0 = 0$.

(c) Prove that if f is irreducible, then f is separable.

(d) Give a counterexample that illustrates that the converse to the statement in part (c) is false. That is, give a polynomial f that is separable and reducible.

Exercise 2

For your definitions, clearly introduce the objects you use and the hypotheses you make.

(a) Define "minimal polynomial".

(4 pt) For the remaining parts of this exercise, let $\alpha = \sqrt[3]{5 - 3\sqrt{-1}} \in \mathbf{C}$.

(b) Find the minimal polynomial $m_{\alpha,\mathbf{Q}}$ of α over $\mathbf{Q}$. Demonstrate that it satisfies the axioms (i.e. defining properties) in your definition in part (a).

(c) Prove that $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 6$.

(d) Let $f \in \mathbf{Q}[t]$ such that $\deg f = 4$ and f has no zeros in $\mathbf{Q}$, and let $\beta \in \mathbf{C}$ satisfy $f(\beta) = 0$. Can $\beta \in \mathbf{Q}(\alpha)$? Justify.

Exercise 3

For your definitions, clearly introduce the objects you use and the hypotheses you make.

(a) Define "splitting field".

(4 pt) Let $p, q \in \mathbf{Z}_{>0}$ be prime; let

$$f = t^p - q \qquad\qquad g = \sum_{j=0}^{p-1} t^j = t^{p-1} + \ldots + t + 1$$

be polynomials in $\mathbf{Q}[t]$; fix a splitting field K for fg, the product of f and g, over $\mathbf{Q}$; let $\alpha \in K$ be a zero of f; let $\zeta \in K$ be a zero of g; and let K_f be the splitting field for f in $K : \mathbf{Q}$.

(b) Prove that f and g are irreducible in $\mathbf{Q}[t]$. *Hint:* For g, let $\tilde{g}(t) = (t-1)g = t^p - 1$, and consider $\tilde{g}(t+1)$.

(c) Prove that for each integer $k \in \{0, \ldots, p-1\}$, ζ^k is a zero of $\tilde{g}$, and $\zeta^k \alpha$ is a zero of f. Deduce that $K = K_f$.

(d) Prove that $[K : \mathbf{Q}] = p^2 - p$. *Hint:* Use a field diagram.

Exercise 4

(4 pt) Let $K : K_0$ be a field extension.

(a) Define $\mathrm{Aut}(K : K_0)$.

(b) Let H be a subgroup of $\mathrm{Aut}(K : K_0)$. Define the fixed field of H.

(c) State what it means for $K : K_0$ to be galois. You may use any characterization of a galois extension that we have discussed in class.

(d) When $K : K_0$ is galois, the fundamental theorem of galois theory gives an inclusion-reversing, bijective correspondence between subfields (intermediate fields) of $K : K_0$ and subgroups of the galois group $\mathrm{Gal}(K : K_0)$. Define the map from subfields to subgroups, and the map from subgroups to subfields.