Math 211
Quiz 10A

T 23 Jul 2019

Your name : ______________________________

Exercise

(5 pt) Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 1 & -1 & 2 \\ 0 & -2 & 4 \end{bmatrix}.$$

(a) (2 pt) Show that $\det A = 4$.

Solution: Using expansion by minors along column 1, we compute

$$\begin{aligned} \det A &= (-1)^{1+1}(-1)\det\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} + (-1)^{1+1}(1)\det\begin{bmatrix} 0 & -2 \\ -2 & 4 \end{bmatrix} + 0 \\ &= (1)(-1)(-4-(-4)) + (-1)(1)(0-4) = 0+4 = 4. \end{aligned}$$

(b) (2 pt) Apply the row reduction algorithm to $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{I}_3 \end{array}\right]$ to show that

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 4 & -2 \\ -4 & -4 & 0 \\ -2 & -2 & 1 \end{bmatrix}.$$

Solution: We compute

$$\begin{aligned} \left[\begin{array}{c:c} \mathbf{A} & \mathbf{I}_3 \end{array}\right] &= \left[\begin{array}{ccc:ccc} -1 & 0 & -2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \\ &\xrightarrow{R_1=-R_1} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \\ &\xrightarrow{R_2=R_2-R_1} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \\ &\xrightarrow{R_2=-R_2} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \\ &\xrightarrow{R_3=R_3+2R_2} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 4 & -2 & -2 & 1 \end{array}\right] \\ &\xrightarrow{R_3=\frac{1}{4}R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \end{array}\right] \\ &\xrightarrow{R_1=R_1-2R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \end{array}\right]. \end{aligned}$$

The 3×3 matrix to the right of the dashed line is A^{-1}. Factoring out $\frac{1}{4}$ from all of the entries of this matrix, we conclude that

$$A^{-1} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 4 & -2 \\ -4 & -4 & 0 \\ -2 & -2 & 1 \end{bmatrix},$$

as claimed.

(c) (1 pt) What is $\det A^{-1}$? *Hint:* You can answer this without computing $\det A^{-1}$ directly. How?

Solution: Recall that, for any square matrices A, B of the same order,

$$\det(AB) = (\det A)(\det B).$$

In particular, if A is invertible, then we may take $B = A^{-1}$, giving

$$1 = \det I = \det(AA^{-1}) = (\det A)(\det(A^{-1})).$$

Because A is invertible, $\det A \neq 0$, so we may solve this equation for $\det A^{-1}$:

$$\det(A^{-1}) = (\det A)^{-1}.$$

For our particular matrix A, we conclude that

$$\det(A^{-1}) = \frac{1}{4}.$$

One can check that this agrees with the result we get via computation of $\det A^{-1}$ directly. (Try it! To simplify computation, use the multilinearity property of the determinant: For an $n \times n$ matrix B, and for any scalar c, $\det(cB) = c^n \det B$. In particular, to compute the determinant of our matrix A^{-1}, we may compute the determinant of the matrix of integers (we should find it equals 16), and multiply the final result by $(\frac{1}{4})^3 = \frac{1}{64}$.)