Math 211
Quiz 16

M 29 Jul 2019

Your name : ______________________________

Exercise

(5 pt) Consider the homogeneous 1st-order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} -7 & -18 \\ 3 & 8 \end{bmatrix}.$$

We have seen that the general solution to (1) is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c}$, where $\mathbf{c}$ is a 2×1 matrix of parameters.

Computing $e^{\mathbf{A}t}$ for a general square matrix $\mathbf{A}$ of constants can be hard. However, we have seen that for a diagonal matrix $\mathbf{D}$, the matrix exponential function is straightforward:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \Rightarrow \qquad e^{\mathbf{D}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}.$$

(a) (2 pt) Show the eigenvalues of $\mathbf{A}$ are -1 and 2. For each eigenvalue, compute an eigenvector.

(b) (1 pt) Write your two eigenvectors as columns, side by side in a 2×2 matrix $\mathbf{P}$. Compute $\mathbf{P}^{-1}$, and confirm that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, where $\mathbf{D}$ is a 2×2 diagonal matrix with the eigenvalues of $\mathbf{A}$ on the diagonal. Thus $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

(c) (2 pt) Use your result to part (b) to compute the 2×2 matrix $e^{\mathbf{A}t}$, by noting that

$$e^{\mathbf{A}t} = e^{t\mathbf{A}} = e^{t\mathbf{P}\mathbf{D}\mathbf{P}^{-1}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1}.$$

(Compute using this last expression.) The columns of this matrix form a basis for the solution space of our original ODE (1).