Math 212
Quiz 21

W 19 Oct 2016

Your name: ______________________________

Exercise

(2 pt) Write (but do NOT evaluate) an integral to find the surface area of the part of the cylinder

$$S : x^2 + z^2 = 4$$

lying above (i.e. with $z \geqslant 0$) the square D in the xy-plane with vertices $(0,0), (2,0), (2,2), (0,2)$.

Solution: We present two methods to computing the surface area.

Method 1: Multivariable calculus. The part of the cylinder S lying above the xy-plane (i.e. with $z \geqslant 0$) can be described by solving the given equation for z:

$$z = \sqrt{4 - x^2}.$$

Viewing z as a function of x, y, we compute its partial derivatives to be

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4 - x^2}}, \qquad \frac{\partial z}{\partial y} = 0.$$

The surface area $A(S)$ of the part of the cylinder S lying above the square D is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint_D \sqrt{1 + \frac{x^2}{4 - x^2}}\, dA \\ &= \iint_D \frac{2}{\sqrt{4 - x^2}}\, dA. \end{aligned}$$

If desired, we can use Fubini's theorem to evaluate this double integral as an iterated integral. Using the change of variables indicated by the triangle below *(draw related triangle)*, we have

$$\sin u = \frac{\sqrt{4 - x^2}}{2}, \qquad \cos u = \frac{x}{2} \qquad \Rightarrow \qquad dx = -2 \sin u \, du,$$

so

$$\begin{aligned} A(S) &= \int_{y=0}^{y=2} \int_{x=0}^{x=2} \frac{2}{\sqrt{4 - x^2}}\, dx\, dy \\ &= \int_{y=0}^{y=2} dy \int_{x=0}^{x=2} \frac{1}{\sin u} (-2 \sin u \, du) \\ &= 2(-2) \int_{y=0}^{y=2} [u]_{x=0}^{x=2}\, dy \\ &= -4 \left[\arccos\left(\frac{x}{2}\right)\right]_{x=0}^{x=2} \\ &= -4 \left[0 - \frac{\pi}{2}\right] = 2\pi. \end{aligned}$$

Method 2: Geometry. The surface area of the part of the cylinder S lying above the square D is precisely $\frac{1}{4}$ of the surface area of the total cylinder S from $y = 0$ to $y = 2$. This is a cylinder of radius 2 and height 2, so

$$A(S) = \frac{1}{4} \left[\pi (2)^2 2\right] = 2\pi.$$