Math 211
Quiz 11

M 22 Jul 2019

Your name : ________________________________

Exercise

(2 pt) Fix the following notation:

- Let $I \subseteq \mathbf{R}$ denote the closed interval $[0,1]$.
- Let $\mathscr{C}^0(I)$ denote the vector space over $\mathbf{R}$ of continuous functions $f : I \to \mathbf{R}$.
- Let $\mathbf{R}^3$ denote the vector space over $\mathbf{R}$ of 3×1 matrices whose entries are real numbers.

Circle the corresponding letter if the subset $W \subseteq V$ described is a subspace of the given vector space V.

(a) $V = \mathbf{R}^3$, W is the set of matrices $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \in V$ such that $x_1 + x_2 + x_3 = 0$.

(b) $V = \mathbf{R}^3$, W is the set of matrices $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \in V$ such that $x_1 x_2 x_3 = 0$.

(c) $V = \mathscr{C}^0(I)$, W is the set of functions $f \in V$ such that $f(1) = 0$.

(d) $V = \mathscr{C}^0(I)$, W is the set of functions $f \in V$ such that $f(0) = 1$.

Solution: We analyze each subset in turn. Each time, we check the two defining properties of a subspace: (i) nonempty, and (ii) closed under linear combinations.

(a) is a subspace. Nonempty : The 3×1 zero matrix $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ satisfies the condition that its entries sum to 0, so $W \neq \varnothing$. Closure : Let $w_1, w_2 \in W$, and let $a_1, a_2 \in \mathbf{R}$. The linear combination

$$a_1 w_1 + a_2 w_2 = \begin{bmatrix} a_1 w_{1,1} + a_2 w_{2,1} \\ a_1 w_{1,2} + a_2 w_{2,2} \\ a_1 w_{1,3} + a_2 w_{2,3} \end{bmatrix}.$$

The sum of the entries of this linear combination is

$$\begin{aligned} &(a_1 w_{1,1} + a_2 w_{2,1}) + (a_1 w_{1,2} + a_2 w_{2,2}) + (a_1 w_{1,3} + a_2 w_{2,3}) \\ &= a_1(w_{1,1} + w_{1,2} + w_{1,3}) + a_2(w_{2,1} + w_{2,2} + w_{2,3}) \\ &= a_1(0) + a_2(0) = 0, \end{aligned}$$

where in the last line we use the hypothesis that $w_1, w_2 \in W$.

(b) is not a subspace. It is not closed under linear combinations. For example, $w_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $w_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ are both in W, but their sum $w_1 + w_2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ is not.

(c) is a subspace. Nonempty : The zero function $f(t) \equiv 0$ satisfies $f(1) = 0$, so $W \neq \varnothing$. Closure : Let $f_1, f_2 \in W$, and let $a_1, a_2 \in \mathbf{R}$. The linear combination $a_1 f_1 + a_2 f_2$ satisfies

$$\begin{aligned} (a_1 f_1 + a_2 f_2)(1) &= a_1 \cdot f_1(1) + a_2 \cdot (f_2(1) \\ &= a_1 \cdot 0 + a_2 \cdot 0 = 0, \end{aligned}$$

where in the last line we use the hypothesis that $f_1, f_2 \in W$.

(d) is not a subspace. It is not closed under linear combinations. For example, $f(t) \equiv 1$ is in W, but $2f$ is not (because $(2f)(1) = 2f(1) = 2 \cdot 1 = 2 \neq 1$).