Math 357
Long quiz 01

2024–01–19 (F)

Your name: ______________________________

(a) Let R be a ring; let $a, b, c \in R$; and suppose that a is not a zero divisor. Prove the left-cancellation law: If $ab = ac$, then $a = 0$ or $b = c$.

Solution: Using the ring axioms, we have[1]

$$ab = ac \qquad \Leftrightarrow \qquad ab - ac = 0 \qquad \Leftrightarrow \qquad a(b - c) = 0 \tag{1}$$

Case 1: $a = 0$. The conclusion holds. Case 2: $a \neq 0$. By hypothesis, a is not a zero divisor, so in this case we must have $b - c = 0$, which is equivalent to $b = c$.

(b) Let R and S be commutative rings with (multiplicative) identity, let $a \in R$ be a zero divisor, and let $f : R \to S$ be a ring homomorphism such that $f(a) \in S^\times$. Show that f is not injective.

Solution: For clarity, let 0_R and 0_S denote the additive identities of R and S, respectively. Similarly, let 1_S denote the multiplicative identity of S.

By hypothesis, $a \in R$ is a zero divisor, so by definition there exists a $b \in R - \{0_R\}$ such that $ab = 0_R$. Applying the ring homomorphism f to this equation, we have[2]

$$0_S = f(0_R) = f(ab) = f(a)f(b)$$

By hypothesis, $f(a) \in S^\times$, so by definition there exists an $s \in S$ such that $sf(a) = 1_S$. Left-multiplying both sides of $0_S = f(a)f(b)$ by this s, we get[3]

$$0_S = s0_S = s(f(a)f(b)) = (sf(a))f(b) = 1_S f(b) = f(b)$$

That is, $b \in \ker f$. Because $b \neq 0_R$, we conclude that f is not injective.

[1]Justify each equivalence. Note that the second equivalence in (1) assumes that $-(ac) = a(-c)$. Why is this true? Does this require that R have a (multiplicative) identity?

[2]Justify each equality.

[3]Justify each equality.