Math 357
Long quiz 01A

2024–02–05 (M)

Your name: ________________________________

Let $(R, +, \times)$ be a commutative ring with a (multiplicative) identity $1 \neq 0$, and let $I \trianglelefteq R$ be an ideal. Prove the following.

(a) $I = R$ if and only if I contains a unit.

Solution: ($\Rightarrow$) Let $I = R$. By hypothesis, there exists $1 \in R$. By definition of a (multiplicative) identity, 1 is a unit: $1 \times 1 = 1$. (Is using 1 our only option?)

($\Leftarrow$) Let I contain a unit, denote it u. By definition of a unit, there exists some $v \in R$ such that $uv = 1$ and $vu = 1$.[1] By definition of an ideal, I is closed under (left- and right-) multiplication by elements of R. In particular, $v \in R$ and $u \in I$, so $1 = vu \in I$. Using the same logic with $1 \in I$ and arbitrary $r \in R$, we conclude that $I = R$.

(b) R is a field if and only if its only ideals are (0) and (1).

Solution: ($\Rightarrow$) Let R be a field, and let $I \trianglelefteq R$ be an ideal. Case 1: $I = (0)$. We are done. Case 2: $I \neq (0)$. In this case there exists an element $a \in I$ such that $a \neq 0$. Because R is a field, $a \in R^\times$; that is, a is a unit. By part (a), $I = R = (1)$.

($\Leftarrow$) Let (0) and (1) be the only ideals of R. To show that R is a field, we need to show that every nonzero element of R is a unit. Let $r \in R$ such that $r \neq 0$. Consider the (principal) ideal (r). Because $r \neq 0$ and $r \in (r)$, it follows that $(r) \neq (0)$. Hence $(r) = (1)$. In particular, $1 \in (1) = (r)$, so there exists some $s \in R$ such that $sr = 1$. (Do we have to worry whether this s satisfies $rs = 1$, too?) That is, r is a unit.

[1] In this exercise, we're assuming that R is commutative, so these two conditions are equivalent.