Vector Calculus

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Contents

Course Description

Teaching Philosophy

General Principles

- Dictionary between geometry and algebra
- Different coordinate systems as different ways to describe same space
- Curves, surfaces, etc. as zero sets
- What generalizes to $\mathbf{R}^n$, and what is specific to $\mathbf{R}^3$ (e.g., cross product)
- Reducing complicated objects (e.g., vector-valued functions, multiple integrals) to simpler objects that we better understand (e.g., real-valued functions, iterated single integrals)
- Generalizing approaches in single-variable calculus to multivariable settings
- Flavor for abstract thinking, definitions, structures
  - In particular (a few, carefully selected concepts): vector space, inner product
  - Taking key properties from concrete settings and using them as the definition in abstract setting
- Linear things are easy; reduce to linear case when possible

Chapter 12

Vectors and Geometry

Strangely enough for lecture notes on vector calculus, this first chapter contains no calculus. The role of this chapter is to set the stage where calculus will play its part. To that end, we explore

- the world in which our calculus will act, namely, euclidean space, and more specifically for us, $\mathbf{R}^2$ and $\mathbf{R}^3$;
- the objects involved, namely, vectors; and
- how these objects interact, which is in turn governed by the geometry of euclidean space.

12.1 Coordinates in $\mathbf{R}^3$

Key Ideas

- Coordinate systems in $\mathbf{R}^3$ (rectangular, cylindrical, spherical)
- Right-hand rule
- Function (domain, codomain, and rule of assignment)
- Distance formula (in $\mathbf{R}^2, \mathbf{R}^3, \mathbf{R}^n$)

12.1.1 Coordinate Axes

We assume familiarity with the standard coordinate axes in $\mathbf{R}^2$. *(draw picture)* In $\mathbf{R}^2$, one often calls the two coordinates x and y, and one adopts the convention that the horizontal axis measures x, with the positive direction oriented right, and the vertical axis measures y, with the positive direction oriented up.

In $\mathbf{R}^3$ there are three orthogonal directions, which one typically calls x, y, z. (Another common naming convention is x_1, x_2, x_3.) As in $\mathbf{R}^2$, we adopt a particular convention on the coordinate axes, known as the **right-hand rule** (for reasons of physical interpretation that will be explained shortly): The x- and y-axes are oriented as in $\mathbf{R}^2$, and the z-axis is oriented perpendicularly to the xy-plane. (Conventionally, the xy-plane in $\mathbf{R}^3$ is oriented horizontally; this means that the z-axis is vertical.) We still need to choose which direction the positive z-axis points. (Draw the xy-plane on the board, and act out the right-hand rule. Then rotate the picture into the conventional orientation.) Put your right hand along the x-axis, with fingers pointing in the positive x direction, and curl your fingers in the direction of the positive y-axis. Then your thumb points in the direction of the positive z-axis.

The right-hand rule gives a standard orientation to the x-, y-, and z-axes in $\mathbf{R}^3$. We will also use the right-hand rule to assign a standard direction to the cross product in Section 12.4.

The **coordinate planes** are the planes in $\mathbf{R}^3$ containing exactly two of the coordinate axes. Thus the xy-plane is the plane containing the x- and y-axes, etc. Note that the xy-plane is defined by the equation $z = 0$, etc.

Plotting points in $\mathbf{R}^3$ using rectangular coordinates follows analogous conventions to plotting points in $\mathbf{R}^2$.

In $\mathbf{R}^2$, we can take a point (x, y) and project it orthogonally onto the coordinate *axes*. For example, projecting (x, y) onto the y-axis, we obtain $(0, y)$. If we view the y-axis as $\mathbf{R}^1$, then this point corresponds to y.

Similarly, in $\mathbf{R}^3$, we can take a point (x, y, z) and project it orthogonally onto the coordinate *planes*. For example, projecting (x, y, z) orthogonally onto the xz-plane corresponds to setting $y = 0$; thus we obtain $(x, 0, z)$. If we view the xz-plane as $\mathbf{R}^2$ (with coordinates (x, z)), then this point corresponds to (x, z).

12.1.2 A Review of Polar Coordinates

In $\mathbf{R}^2$, polar coordinates (r, θ) provide an alternative to rectangular coordinates (x, y) as a way to specify points. In polar coordinates, θ is the counterclockwise angle (in radians) from the positive x-axis, and r is the distance from the origin in the direction θ. (If $r < 0$, then we move $|r|$ in the opposite direction of θ. That is, $(r, \theta) = (-r, \theta + \pi)$.) Recall that polar coordinates and rectangular coordinates are related by the equations

$$x = r\cos\theta, \qquad\qquad y = r\sin\theta.$$

Squaring these relations, adding, and using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$x^2 + y^2 = r^2.$$

So polar coordinates and rectangular coordinates are related by a nice right triangle. *(draw picture)*

In $\mathbf{R}^3$ there are two commonly used alternative coordinate systems to rectangular coordinates.

12.1.3 Cylindrical Coordinates

The first alternative system that we will study is a straightforward extension of polar coordinates in $\mathbf{R}^2$, where we simply append the z-value to the polar coordinates that we know and love. **Cylindrical coordinates** of a point $P \in \mathbf{R}^3$ have the form (r, θ, z), where

- r is the distance from the origin to the projection of P in the xy-plane,
- θ is the counterclockwise angle (in radians) in the xy-plane from the positive x-axis to the projection of P in the xy-plane, and
- z is the vertical distance from the xy-plane to P.

(draw picture) Cylindrical coordinates (r, θ, z) are related to rectangular coordinates (x, y, z) in $\mathbf{R}^3$ in the same way that polar coordinates are related to rectangular coordinates in $\mathbf{R}^2$:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

This is because all we've done is append z.

Example 12.1 (S6eQ15.07.02b,04b)**.** Convert $(1, \frac{3\pi}{2}, 2)$ from cylindrical to rectangular coordinates. Convert $(4, -3, 2)$ from rectangular to cylindrical coordinates.

Solution: We compute

$$\begin{aligned} x &= r\cos\theta = 1\cos\left(\frac{3\pi}{2}\right) = 0 \\ y &= r\sin\theta = 1\sin\left(\frac{3\pi}{2}\right) = -1 \\ z &= z = 2. \end{aligned}$$

Thus

$$\left(1, \frac{3\pi}{2}, 2\right) \leftrightarrow (0, -1, 2).$$

Similarly, we compute

$$\begin{aligned} r &= \sqrt{x^2+y^2} = \sqrt{(4)^2+(-3)^2} = 5 \\ \theta &= \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-3}{4}\right) \approx -.6435 \\ z &= z = 2. \end{aligned}$$

Thus

$$(4, -3, 2) \leftrightarrow (5, -.6435, 2).$$

(If you prefer $\theta \in [0, 2\pi)$, then note that $-.6435 + 2\pi \approx 5.6397$.)

12.1.4 Spherical Coordinates

The second commonly used coordinate system in $\mathbf{R}^3$ are spherical coordinates. **Spherical coordinates** have the form (ρ, θ, φ), where

- ρ is the distance from the origin to P,
- θ is the counterclockwise angle (in radians) in the xy-plane from the positive x-axis to the projection of P in the xy-plane, and
- φ is the angle (in radians) from the positive z-axis to the line through the origin containing P.

(draw picture) The relation between spherical coordinates (ρ, θ, φ) and rectangular coordinates (x, y, z) can be deduced from the picture:

$$x = \rho\sin\varphi\cos\theta, \qquad y = \rho\sin\varphi\sin\theta, \qquad z = \rho\cos\varphi.$$

Cylindrical and spherical coordinates are especially useful when we want to describe surfaces or volumes in $\mathbf{R}^3$ that have cylindrical or spherical symmetry. In particular, these coordinate systems will simplify integrals over such surfaces and volumes.

Example 12.2 (S6eQ15.08.02b,04b)**.** Convert $(4, \frac{3\pi}{4}, \frac{\pi}{3})$ from spherical to rectangular coordinates. Convert $(-1, 1, \sqrt{6})$ from rectangular to spherical coordinates.

Solution: We compute

$$\begin{aligned} x &= \rho \cos\varphi \cos\theta = 4 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{3\pi}{4}\right) = -\sqrt{6} \\ y &= \rho \cos\varphi \sin\theta = 4 \sin\left(\frac{\pi}{3}\right) \sin\left(\frac{3\pi}{4}\right) = \sqrt{6} \\ z &= \rho \cos\varphi = 4 \cos\left(\frac{\pi}{3}\right) = 2. \end{aligned}$$

Thus

$$\left(4, \frac{3\pi}{4}, \frac{\pi}{3}\right) \leftrightarrow \left(-\sqrt{6}, \sqrt{6}, 2\right).$$

Similarly, we compute

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{8} \\ \theta &= \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1}{-1}\right) = \frac{3\pi}{4} \\ \varphi &= \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{\sqrt{6}}{\sqrt{8}}\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}. \end{aligned}$$

Thus

$$(-1, 1, \sqrt{6}) \leftrightarrow \left(\sqrt{8}, \frac{3\pi}{4}, \frac{\pi}{6}\right).$$

12.1.5 Cartesian Product

Visualizing orthogonal coordinate axes works well for $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$, but doesn't work so well for higher dimensions. The notation of Cartesian products works well for general $\mathbf{R}^n$, for any $n \in \mathbf{Z}_{\geqslant 0}$. The Cartesian product notation for $\mathbf{R}^2$ and $\mathbf{R}^3$ is

$$\mathbf{R}^2 := \{(x, y) \mid x, y \in \mathbf{R}\}, \qquad \mathbf{R}^3 := \{(x, y, z) \mid x, y, z \in \mathbf{R}\}.$$

More generally,

$$\mathbf{R}^n := \{(x_1, \dots, x_n) \mid \forall i \in \{1, \dots, n\}, x_i \in \mathbf{R}\}.$$

This notation might look scary, but it's no more scary than Chinese.[1] Once you learn what this notation means (and it's arguably much easier to learn than Chinese), it provides a wonderfully compact and precise way of presenting information.

[1] As someone once quipped, Chinese can't be that hard — in China, even toddlers can speak it.

12.1.6 Equations and Graphs

Draw the graph of the equation $y = x$. Did you draw a line in $\mathbf{R}^2$? Why not a plane in $\mathbf{R}^3$? A given equation may describe a curve in $\mathbf{R}^2$, a surface in $\mathbf{R}^3$, etc. It depends on the context. More generally, an equation does not define a function! An equation is just a rule of assignment; it doesn't tell us what values to use as input, or where the output lives. To define a function, one needs to specify the domain and codomain.

A **function** consists of three pieces of information:

(1) domain: what are the valid inputs?

(2) codomain: what are the valid outputs?

(3) rule of assignment: for each valid input value, what is the corresponding output value?

The rule of assignment has to specify a *unique* output value for each input value. That is, each input value corresponds to one and only one output value.

Definition 12.3. Let X, Y be nonempty sets. A **function** from X to Y, written

$$f : X \to Y,$$

is a rule of assignment that assigns to each element $x \in X$ exactly one element $y \in Y$. The set X (i.e. the set of the input values) is called the **domain**. The set Y (i.e. the set of output values) is called the **codomain**.

Give example of rational functions in my work?

12.1.7 Distance Formula

In $\mathbf{R}^2$, we learn that the distance between two points $P := (x_1, x_2)$ and $Q := (y_1, y_2)$ is given by

$$d(P, Q) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

We justify this using the Pythagorean theorem.

Now consider the points $P := (x_1, x_2, x_3)$ and $Q := (y_1, y_2, y_3)$ in $\mathbf{R}^3$. We can use the Pythagorean theorem (twice) to justify the **distance formula in $\mathbf{R}^3$**:

$$d(P, Q) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}.$$

Alternatively, one can derive this distance formula from deeper structure on $\mathbf{R}^3$. (We will expound on this point briefly in Section 12.3.)

Note that the distance formulas in $\mathbf{R}^2$ and $\mathbf{R}^3$ can both be written as

$$d(P, Q) = \sqrt{(y_1 - x_1)^2 + \ldots + (y_n - x_n)^2}, \tag{12.1.1}$$

where $n = 2$ or 3, according to whether we are in $\mathbf{R}^2$ or $\mathbf{R}^3$, respectively. In fact, this formula defines the distance between two points $P := (x_1, \ldots, x_n)$ and $Q := (y_1, \ldots, y_n)$ in $\mathbf{R}^n$ for any $n \in \mathbf{Z}_{>0}$.

The equation of a sphere affords a pleasant application of the distance formula and one of our first examples of the dictionary between geometry and algebra. Write the equation of a sphere in $\mathbf{R}^3$ with center $O := (x_0, y_0, z_0)$ and radius $r \in \mathbf{R}_{\geqslant 0}$. Geometrically, what is this sphere in $\mathbf{R}^3$? It is the set of all points in $\mathbf{R}^3$ whose distance from O equals r. Use the geometric definition and the distance formula to derive the equation.

12.2 Vectors

Key Ideas

- Definition: vector (geometric and algebraic)
- How to add, subtract vectors (geometrically and algebraically)
- Scalar multiplication
- Norm (length) of a vector
- Standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbf{R}^3$
- How to make a unit vector

In this section we will work with another example of the complementary descriptions afforded by geometry and algebra.

12.2.1 Vectors

In courses such as this one, a **vector** is often defined as an object that has a magnitude (length) and a direction. This definition gives a good physical sense of many vectors we encounter in real life (velocity, torque, etc.). However, a vector can be defined more generally (and precisely) using the abstract notion of a vector space. We will stick with the physical interpretation of vectors for now, but we will simultaneously present an algebraic view of vectors that hints at the more powerful abstract notion.

Geometrically, a vector has an initial point and a terminal point; the vector is the directed line segment starting at the initial point and ending at the terminal point. Algebraically, a vector in $\mathbf{R}^n$ is an ordered n-tuple $\mathbf{v} = (v_1, \dots, v_n)$, where each $v_i \in \mathbf{R}$. (Note the similarity to coordinates in a cartesian product.) The ith entry v_i is called the ith **component** of the vector $\mathbf{v}$.

Two vectors are equal or **equivalent** if their defining properties are the same. Geometrically, two vectors are equal if they have the same length and direction. Algebraically, two vectors are equal if all components are equal.

The **zero vector** is what you think it is. Geometrically, the zero vector is the (unique) vector with length 0. By convention, the zero vector has no direction. (This oddity starts to suggest the shortcomings of the physical definition of vectors.) Algebraically, the zero vector in $\mathbf{R}^3$ is the vectors $(0, 0, 0)$. More generally, in $\mathbf{R}^n$ the zero vector is the vector whose components are all 0: $(0, \dots, 0)$.

12.2.2 Adding Vectors

Motivate definition of vector addition using Stewart's example: Consider a particle that starts at A, moves to B, then moves to C. The displacement vector is

$$\vec{AC} = \vec{AB} + \vec{BC}.$$

Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbf{R}^n$. (For the purposes of visualization, suppose these vectors are in $\mathbf{R}^2$ or $\mathbf{R}^3$.) We wish to define the sum $\mathbf{u} + \mathbf{v}$.

Geometrically, we can define vector addition using either the triangle law or the parallelogram law.

Triangle law: Place the initial point of the second vector $\mathbf{v}$ at the terminal point of the first vector $\mathbf{u}$. The sum $\mathbf{u} + \mathbf{v}$ is the vector whose initial point coincides with the initial point of the first vector $\mathbf{u}$ and whose terminal point coincides with the terminal point of the second vector $\mathbf{v}$. (This is what we called the displacement vector in our motivating example.)

Parallelogram law: Draw the parallelogram whose parallel sides are $\mathbf{u}$ and $\mathbf{u}$. The sum $\mathbf{u} + \mathbf{v}$ is the vector whose initial point coincides with the vertex of the parallelogram corresponding to the initial points of both $\mathbf{u}$ and $\mathbf{v}$ and whose terminal point coincides with the vertex of the parallelogram corresponding to the terminal points of both $\mathbf{u}$ and $\mathbf{v}$. This is most clearly seen in a picture. A picture is worth a thousand words.

Algebraically, the vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ can be written in terms of components:

$$\mathbf{u} = (u_1, \ldots, u_n), \qquad \mathbf{v} = (v_1, \ldots, v_n).$$

Vector addition is defined componentwise:

$$\mathbf{u} + \mathbf{v} := (u_1 + v_1, \ldots, u_n + v_n).$$

12.2.3 Scalar Multiplication

In an abstract vector space, vectors are allowed to be multiplied by scalars — in our case, by real numbers. Scalar multiplication has nice geometric and algebraic interpretations in $\mathbf{R}^n$. Let $\mathbf{v} \in \mathbf{R}^n$ be a vector, and let $c \in \mathbf{R}$.

Geometrically, the scalar multiple $c\mathbf{v}$ is defined to be the vector whose length is $|c|$ times the length of $\mathbf{v}$, and whose direction is the same as $\mathbf{v}$ if $c > 0$ and the opposite of $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Algebraically, $\mathbf{v}$ can be written in components $\mathbf{v} = (v_1, \ldots, v_n)$. The scalar multiple $c\mathbf{v}$ is defined by multiplying each component of $\mathbf{v}$ by c:

$$c\mathbf{v} := (cv_1, \ldots, cv_n).$$

Geometrically, parallel vectors point in the same or exact opposite direction. Algebraically, two vectors $\mathbf{u}, \mathbf{v}$ are **parallel** if they are scalar multiples of each other, i.e. if there exists some $c \in \mathbf{R}$ such that

$$\mathbf{u} = c\mathbf{v}. \tag{12.2.1}$$

This is an equality of vectors. Recall that, algebraically, two vectors are equal if and only if each of their corresponding components are equal. Thus (12.2.1) states that parallel vectors have components each of which are *the same* scalar multiple of each other: for all $i \in \{1, \dots, n\}$,

$$u_i = cv_i.$$

Note that the algebraic definition of parallel implies that the zero vector $\mathbf{0}$ is parallel to all vectors: For any vector $\mathbf{v}$, $\mathbf{0} = 0\mathbf{v}$, i.e. $\mathbf{0}$ is a scalar multiple of $\mathbf{v}$, taking the scalar $c = 0$.

Give an example, geometrically and algebraically.

The **negative** of the vector $\mathbf{v}$ is the scalar multiple $(-1)\mathbf{v}$. This vector is often denoted $-\mathbf{v}$. Geometrically, $-\mathbf{v}$ has the same length and the opposite direction as $\mathbf{v}$. Algebraically, $-\mathbf{v}$ is the vector each of whose components is the negative of the corresponding component in $\mathbf{v}$: If $\mathbf{v} = (v_1, \dots, v_n)$, then

$$-\mathbf{v} = (-v_1, \dots, -v_n).$$

Catchphrase

Algebraically, vector addition (and subtraction) and scalar multiplication are defined componentwise.

12.2.4 Vector Subtraction

Armed with vector addition and scalar multiplication, we can construct the difference of two vectors. Geometrically, we can do this two ways. Note that

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

That is, add the vectors $\mathbf{u}$ and $-\mathbf{v}$, for example using the triangle law.

Alternatively, using the fact that vector addition is associative and commutative, note that

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = (\mathbf{v} + (-\mathbf{v})) + \mathbf{u} = \mathbf{0} + \mathbf{u} = \mathbf{u}.$$

This equation says that $\mathbf{u} - \mathbf{v}$ is the vector that, when added to $\mathbf{v}$, gives $\mathbf{u}$. Thus, by definition of vector addition, drawing $\mathbf{v}$ and $\mathbf{u}$ so that their initial points coincide, $\mathbf{u} - \mathbf{v}$ is the vector that points from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$. *(draw pic)* (Pause to make sure students understand this.)

Algebraically, vector subtraction follows naturally from scalar multiplication and componentwise addition:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1, \dots, u_n) + (-v_1, \dots, -v_n) = (u_1 - v_1, \dots, u_n - v_n).$$

Vector subtraction affords a method to represent vectors between two specified points. Let A, B be points in $\mathbf{R}^n$, say with coordinates $A := (a_1, \ldots, a_n)$ and $B := (b_1, \ldots, b_n)$. Then the vector whose initial point is A and whose terminal point is B is

$$\vec{AB} = (b_1 - a_1, \ldots, b_n - a_n).$$

Note that this is the vector we obtain if we view the points A and B as vectors $\mathbf{a} := \vec{OA}$ and $\mathbf{b} := \vec{OB}$, respectively, and take the vector difference $\mathbf{b} - \mathbf{a}$.

(Do an example.)

12.2.5 Vector Length

Let $\mathbf{v}$ be a vector in $\mathbf{R}^n$. The **length** or **norm** of $\mathbf{v}$ is denoted $\|\mathbf{v}\|$.

Remark 12.4. Stewart (2008) uses the notation $|\mathbf{v}|$ to denote the length of $\mathbf{v}$. This notation is distasteful, because it conflates the norm (i.e. length) function with the absolute value function. *A norm is not an absolute value!* Norm and absolute value have similar properties, but they are defined on different structures (norms are defined on vector spaces, whereas absolute value functions are defined on fields).

Geometrically, a vector has the properties of length and direction, so $\|\mathbf{v}\|$ is just the length. Algebraically, a vector $\mathbf{v}$ in $\mathbf{R}^n$ can be written $(v_1, \ldots, v_n)$. (Here we envision placing $\mathbf{v}$ so that its initial point is at the origin.) Using the distance formula, we compute the norm (length) of $\mathbf{v}$ to be

$$\|\mathbf{v}\| = \sqrt{v_1^2 + \ldots + v_n^2}. \tag{12.2.2}$$

12.2.6 Properties of Vector Spaces

(Do not emphasize this section. It is not so important.)

Stewart (2008) lists several properties of vectors. These properties follow from the geometry (equivalently, algebra) used to defined $\mathbf{R}^n$. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^n$, and for all $c, d \in \mathbf{R}$,

1. Associative: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
2. Commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
4. Identity: $\mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{0} + \mathbf{u}$
5. Distributive 1: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. Distributive 2: $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. Identity for scalar multiplication: $1\mathbf{u} = \mathbf{u}$
8. Compatibility: $(cd)\mathbf{u} = c(d\mathbf{u})$

In abstract algebra, these key properties are taken as the *definition* of a vector space.

12.2.7 Standard Basis Vectors

In $\mathbf{R}^3$ we single out three vectors of particular importance:

$$\mathbf{i} := (1,0,0), \qquad \mathbf{j} := (0,1,0), \qquad \mathbf{k} := (0,0,1).$$

We call these vectors the **standard basis vectors**. The standard basis vectors are important because they (i) point in the coordinate directions, (ii) have length 1, and (iii) are mutually orthogonal (i.e. at right angles to each other). In fancy language, we say that these vectors form an orthonormal basis for $\mathbf{R}^3$.

Let $\mathbf{v} := (v_1, v_2, v_3)$ be any vector in $\mathbf{R}^3$. Note that

$$\begin{aligned} \mathbf{v} &= (v_1, v_2, v_3) \\ &= (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}. \end{aligned} \tag{12.2.3}$$

Thus any vector in $\mathbf{R}^3$ has a unique decomposition as a linear combination of the standard basis vectors. (More generally, given any basis of $\mathbf{R}^3$, any vector $\mathbf{v} \in \mathbf{R}^3$ has a unique decomposition as a linear combination of the basis vectors.)

Equation (12.2.3) is not deep. It simply gives us another way to express a vector $\mathbf{v} = (v_1, v_2, v_3)$ in terms of its components. Implicitly, the representation (v_1, v_2, v_3) assumes that the standard basis has been chosen.

More generally, in $\mathbf{R}^n$ the analogous standard basis vectors are

$$\mathbf{e}_i := (0, \ldots, 0, 1, 0, \ldots, 0),$$

where the unique nonzero entry 1 appears in the ith component. Note that in $\mathbf{R}^n$ there are n standard basis vectors.

12.2.8 Unit Vectors

A **unit vector** is a vector whose length is 1. Let $\mathbf{v} \in \mathbf{R}^n$ be any nonzero vector. Then the length of $\mathbf{v}$ is $\|\mathbf{v}\| \neq 0$ (why?). Therefore $\frac{1}{\|\mathbf{v}\|}$ exists. The vector

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

is the unit vector that points in the same direction as $\mathbf{v}$. To see this, note that $\mathbf{v} \neq \mathbf{0}$ implies that $\|\mathbf{v}\| > 0$. Therefore $\frac{1}{\|\mathbf{v}\|} > 0$, so $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a positive scalar multiple of $\mathbf{v}$, and hence points in the same direction as $\mathbf{v}$. Its length is

$$\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1,$$

so $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector. It is sometimes called the **normalized vector** in the direction of $\mathbf{v}$.

Catchphrase

To obtain a unit vector, scale the given (nonzero) vector by the inverse of its norm.

12.2.9 Application (Physics): Resultant Force

The **resultant force** of the forces acting on an object is simply the sum of the forces. Because forces are vectors (a force has magnitude and direction), the resultant force is a vector sum.

12.3 Inner Product

Key Ideas

- Algebraic definition of inner product
- Properties of inner product (positive-definite, symmetric, bilinear form)
- Relationship between inner product, length, and angle
- Orthogonality (what it means, how to measure)
- Vector, scalar projection

12.3.1 Definition

The (standard) **inner product** (also called **dot product** or **scalar product**) of two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^2$, denoted $\mathbf{u} \cdot \mathbf{v}$, is given by multiplying corresponding components of $\mathbf{u}$ and $\mathbf{v}$ and adding the results:

$$\mathbf{u} \cdot \mathbf{v} = (u_1, u_2) \cdot (v_1, v_2) := u_1 v_1 + u_2 v_2.$$

We can make an analogous definition in $\mathbf{R}^3$:

$$(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) := u_1 v_1 + u_2 v_2 + u_3 v_3,$$

and, more generally, in $\mathbf{R}^n$:

$$(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) := \sum_{i=1}^{n} u_i v_i. \tag{12.3.1}$$

Catchphrase

To compute (standard) inner product, multiply corresponding components and add.

If we consider vectors in $\mathbf{R}^n$ as column vectors (i.e. matrices with n rows and 1 column), then the inner product can be represented in matrix notation using the transpose: (have students propose this result)

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = \begin{pmatrix} u_1 & \ldots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \mathbf{u}^t \mathbf{v}. \tag{12.3.2}$$

The matrix representation is interesting for the following reason. Note that the computation in (12.3.2) doesn't change if we insert the identity matrix I between the two vectors:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^t \mathbf{v} = \mathbf{u}^t I \mathbf{v}.$$

This hints at a more general theory of a class of functions called bilinear forms. We can insert any $n \times n$ matrix A between the two vectors — i.e. $\mathbf{u}^t A \mathbf{v}$ — and the result will be a scalar (i.e. a real number, in our case). The entries of the matrix A effectively serve as "weights" in the product of the two vectors. Only certain matrices A will always yield results that we will find desirable (e.g., nonnegative, symmetric, etc.). More on this next.

12.3.2 Properties of the Inner Product

The standard inner product on $\mathbf{R}^n$ has the following properties:

(i) positive-definite: for all vectors $\mathbf{u} \in \mathbf{R}^n$,

$$\mathbf{u} \cdot \mathbf{u} \geqslant 0,$$

with equality if and only if $\mathbf{u} = \mathbf{0}$. (Thus, for all $\mathbf{u} \neq \mathbf{0}$, $\mathbf{u} \cdot \mathbf{u} > 0$.)

(ii) symmetric: for all vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$

(iii) linear in the first argument: for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^n$, and for all scalars $c \in \mathbf{R}$,

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \\ (c\mathbf{u}) \cdot \mathbf{v} &= c(\mathbf{u} \cdot \mathbf{v}).\end{aligned}$$

Note that symmetry and linearity in the first argument imply linearity in the second argument:

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= (\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} \\ &= \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}\end{aligned}$$

and

$$\mathbf{u} \cdot (c\mathbf{v}) = (c\mathbf{v}) \cdot \mathbf{u} = c(\mathbf{v} \cdot \mathbf{u}) = c(\mathbf{u} \cdot \mathbf{v}).$$

Fortunately, mathematicians have a term for a function that is linear in the first argument and linear in the second argument: such a function is called **bilinear**. Thus we can summarize the properties of inner product by saying that it is a nonnegative-definite, symmetric, bilinear form (where we can view "form" as a fancy word for function).

Catchphrase

The (standard) inner product on $\mathbf{R}^n$ is a positive-definite, symmetric, bilinear form.

12.3.3 Induced Structure

This section is to cultivate us as mathematicians.

Calculus textbooks sometimes write that the inner product on $\mathbf{R}^n$ satisfies the property that, for all $\mathbf{u} \in \mathbf{R}^n$,

$$\mathbf{u} \cdot \mathbf{u} = \|u\|^2 . \tag{12.3.3}$$

This is misleading: we've cooked the books so that this equality holds. Really what we've done is start with a strong geometric structure on $\mathbf{R}^n$, i.e. the inner product, and we've used it to *define* a notion of distance, i.e. the norm. That is, given the inner product (12.3.1) on $\mathbf{R}^n$, we *define* a norm on $\mathbf{R}^n$:

$$\|\mathbf{u}\| := \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + \ldots + u_n^2}, \tag{12.3.4}$$

which is how we *defined* norm in (12.2.2), before we knew about inner product. Note that squaring both sides of the first equality gives (12.3.3).

A norm (indeed, any norm) $\|\cdot\|$ on $\mathbf{R}^n$ can be used to define a metric (i.e. distance function) $d(\cdot, \cdot)$ as follows: for any $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$,

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| .$$

For the standard inner product (12.3.1), the induced norm is given by (12.3.4), so the induced metric (a.k.a. distance function) is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + \ldots + (u_n - v_n)^2}.$$

This agrees with the distance function that we defined in (12.1.1), before we new about inner product or norm.

We can take this induced structure one step further down the chain: A metric (distance function) induces a topology. Loosely speaking, **topology** is jargon for "the collection of all open sets". (Speaking more strictly, this collection of open sets needs to satisfy certain properties that we will not concern ourselves with here.) If we wrote out what all this means in the case of $\mathbf{R}^n$, we would find that the open balls that we are used to *(write out definition)* are indeed open sets in the topology induced by the metric induced by the norm induced by the standard inner product on $\mathbf{R}^n$. This is a fancy way of saying that the standard inner product on $\mathbf{R}^n$ gives us the geometry and continuity that we are used to.

Catchphrase

Inner product (dot product) induces norm (length) induces metric (distance) induces topology (open sets).

12.3.4 Geometry of Inner Product

The inner product is important because it gives us a notion of angles between vectors.

Theorem 12.5. *Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta. \tag{12.3.5}$$

The key to the proof is the law of cosines: Let ABC be a triangle with angles A, B, C and corresponding sides a, b, c (i.e. side a opposite angle A, etc.). Then

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

with analogous formulas for the other two angles.

Proof. *(draw picture! $\mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{v}$, and angle θ)*

Applying the law of cosines to the angle θ, we have

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta. \tag{12.3.6}$$

Using the relation (12.3.3) between the inner product and norm and the properties of inner product, we can rewrite the left side as

$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}.$$

Substituting this result into the left side of (12.3.6) and cancelling like terms, we obtain

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

□

If $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{u}\|, \|\mathbf{v}\| \neq 0$, and we can solve (12.3.5) for the angle θ between the two vectors, obtaining

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}. \tag{12.3.7}$$

Example 12.6 (S6eQ12.03.20). Find the angle between the vectors

$$\mathbf{a} := \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \qquad \mathbf{b} := 4\mathbf{i} - 3\mathbf{k}.$$

Solution: For convenience, let's write $\mathbf{a}, \mathbf{b}$ as

$$\mathbf{a} = (1, 2, -2), \qquad \mathbf{b} = (4, 0, -3).$$

Be careful to include a 0 component for omitted standard basis vectors! Let θ denote the angle between $\mathbf{a}, \mathbf{b}$. From (12.3.7),

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{(1, 2, -2) \cdot (4, 0, -3)}{(3)(5)} = \frac{2}{3},$$

so

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \approx .8411 \approx 48.1897^\circ.$$

12.3.5 Orthogonality

The notion of angle leads naturally to a notion of orthogonality, a generalization of perpendicularity.

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ are **orthogonal** (or **perpendicular**) if the angle between them is $\frac{\pi}{2}$. In this case,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\left(\frac{\pi}{2}\right) = \|\mathbf{u}\| \|\mathbf{v}\| \, 0 = 0.$$

As we have seen, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ if and only if $\|\mathbf{u}\|, \|\mathbf{v}\| \neq 0$, so in this case $\mathbf{u} \cdot \mathbf{v} = 0$ implies that $\theta = \frac{\pi}{2}$. By convention, the zero vector $\mathbf{0}$ is orthogonal to all vectors. Therefore we conclude the following.

Theorem 12.7. *Two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.*

The important property of inner product is that it allows us to define a notion of angle, which in turn yields a notion of orthogonality. In abstract algebra, we take the important properties as the definition: we *define* the angle θ between two vectors $\mathbf{u}, \mathbf{v}$ to be the value of θ satisfying (12.3.5), and we *define* $\mathbf{u}, \mathbf{v}$ to be orthogonal when $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 12.8 (S6eQ12.03.26)**.** For what values of b are the vectors

$$\mathbf{u} := (-6, b, 2), \qquad \mathbf{v} := (b, b^2, b)$$

orthogonal?

Solution: By Theorem 12.7, $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if

$$0 = \mathbf{u} \cdot \mathbf{v} = -6b + b^3 + 2b = b(b^2 - 4).$$

This equation has roots $b = 0, \pm 2$. Thus $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if $b \in \{-2, 0, 2\}$.

12.3.6 Interpretation of Inner Product

Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ be nonzero vectors. (Hence $\|\mathbf{u}\|, \|\mathbf{v}\| > 0$.) Using (12.3.5), we see that

- if $0 \leqslant \theta < \frac{\pi}{2}$, then $\cos\theta > 0$, so $\mathbf{u} \cdot \mathbf{v} > 0$; and
- if $\frac{\pi}{2} < \theta \leqslant \pi$, then $\cos\theta < 0$, so $\mathbf{u} \cdot \mathbf{v} < 0$.

Thus, roughly speaking, the inner product $\mathbf{u} \cdot \mathbf{v}$ measures the extent to which the vectors $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. The caveat "roughly speaking" is included because scaling the vectors by a positive scalar will affect their inner product (recall the property of bilinearity), but it does not affect their relative directions.

12.3.7 Direction Angles

(This section is not so important. Skip it.)

Let $\mathbf{v} := (v_1, v_2, v_3) \in \mathbf{R}^3$ be a nonzero vector. The **direction angles** of $\mathbf{v}$ are the angles α, β, γ that $\mathbf{v}$ makes with the positive x-, y-, and z-axes, respectively. To obtain the direction angles of $\mathbf{v}$, take the inner product of $\mathbf{v}$ with the relevant standard basis vector $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and solve for θ. For example,

$$\cos\alpha = \frac{\mathbf{v} \cdot \mathbf{i}}{\|\mathbf{v}\| \, \|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|}.$$

Similarly,

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \qquad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}.$$

Squaring the three direction angles and adding, we obtain

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{v_1^2 + v_2^2 + v_3^2}{\|\mathbf{v}\|^2} = \frac{\|\mathbf{v}\|^2}{\|\mathbf{v}\|^2} = 1.$$

Solving for v_1, v_2, v_3 in the preceding equations, note that

$$\mathbf{v} = (v_1, v_2, v_3) = (\|\mathbf{v}\| \cos\alpha, \|\mathbf{v}\| \cos\beta, \|v\| \cos\gamma) = \|\mathbf{v}\| (\cos\alpha, \cos\beta, \cos\gamma).$$

Therefore

$$(\cos\alpha, \cos\beta, \cos\gamma) = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

12.3.8 Projection

The notion of "projecting" one vector onto another will be crucial in our development of vector calculus. Intuitively, projection of a vector $\mathbf{v}$ onto a vector $\mathbf{u}$ measures how much of the vector $\mathbf{v}$ points in the direction of $\mathbf{u}$. We make this notion more precise in what follows.

Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ such that $\mathbf{u} \neq \mathbf{0}$.[2] Geometrically, the **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$, denoted $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$, is obtained as follows *(draw picture, maybe two: one with $\theta < \frac{\pi}{2}$, one with $\theta > \frac{\pi}{2}$)*: Position $\mathbf{u}$ and $\mathbf{v}$ so that their initial points coincide. Draw a line perpendicular to $\mathbf{u}$ passing through the terminal point of $\mathbf{v}$. The vector projection $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ is the vector whose initial point coincides with the initial points of $\mathbf{u}, \mathbf{v}$ and whose terminal point is the point where this perpendicular line intersects the line containing $\mathbf{u}$. Geometrically, the vector projection of $\mathbf{v}$ onto $\mathbf{u}$ can be viewed as the "shadow" of $\mathbf{v}$ on the line containing $\mathbf{u}$ (where the sun shines orthogonally to $\mathbf{u}$).

The **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$, denoted $\operatorname{comp}_{\mathbf{u}}(\mathbf{v})$, is the signed (!) magnitude of the vector projection. $\operatorname{comp}_{\mathbf{u}}(\mathbf{v})$ is positive if $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ points in the same direction as $\mathbf{u}$, and negative if $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ points in the opposite direction as $\mathbf{u}$. (Refer to picture!)

Let's use geometry and the relation (12.3.5) between inner product, angle, and length to obtain an algebraic expression for the vector projection of $\mathbf{v}$ onto $\mathbf{u}$. We consider two cases.

Case 1: $\mathbf{v} = \mathbf{0}$. In this case, for all vectors $\mathbf{u} \in \mathbf{R}^3$,

$$\operatorname{comp}_{\mathbf{u}}(\mathbf{v}) = 0, \qquad \operatorname{proj}_{\mathbf{u}}(\mathbf{0}) = \mathbf{0}.$$

These results are immediate if we think of the "shadow" interpretation of projection.

Case 2: $\mathbf{v} \neq \mathbf{0}$. (Draw the picture.) Using the geometry of right triangles, we have

$$\operatorname{comp}_{\mathbf{u}}(\mathbf{v}) = \|\mathbf{v}\| \cos\theta = \|\mathbf{v}\| \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \right) = \mathbf{v} \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|},$$

where the second equality uses (12.3.7), and the third equality uses the properties of inner product.

By definition, the vector projection $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ is the vector with the (signed) length $\operatorname{comp}_{\mathbf{u}}(\mathbf{v})$ that points in the direction of $\mathbf{u}$. How can we obtain this vector from the scalar projection we have? (pause) If we multiply the *unit vector* in the direction of $\mathbf{u}$ (recall that this is $\frac{\mathbf{u}}{\|\mathbf{u}\|}$) by the scalar $\operatorname{comp}_{\mathbf{u}}(\mathbf{v})$, then what we obtain is a vector in the direction of $\mathbf{u}$ with magnitude

$$\left\| \operatorname{comp}_{\mathbf{u}}(\mathbf{v}) \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = \left| \frac{\operatorname{comp}_{\mathbf{u}}(\mathbf{v})}{\|\mathbf{u}\|} \right| \|\mathbf{u}\| = \left| \operatorname{comp}_{\mathbf{u}}(\mathbf{v}) \right|,$$

(make logical step here explicit!) which is what we want.

[2] We require that $\mathbf{u} \neq \mathbf{0}$ because projection of $\mathbf{v}$ onto $\mathbf{u}$ measures how much $\mathbf{v}$ points in the direction of $\mathbf{u}$, and the zero vector $\mathbf{0}$ does not have a specified direction.

Catchphrase

Projections are shadows. Compute scalar projection using geometry of right triangles. Turn into vector by using unit vector.

Example 12.9 (S6eQ12.03.36)**.** Let

$$\mathbf{a} := (1, 2), \qquad \mathbf{b} := (-4, 1).$$

Find the scalar and vector projections of **b** onto **a**.

Solution: *(draw picture; educe rough answer)* The scalar projection of **b** onto **a** is

$$\operatorname{comp}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b} \cdot \frac{\mathbf{a}}{\|\mathbf{a}\|} = (-4, 1) \cdot \frac{1}{\sqrt{5}}(1, 2) = -\frac{2}{\sqrt{5}}.$$

The vector projection of **b** onto **a** is the vector with length $\operatorname{comp}_{\mathbf{a}}(\mathbf{b})$ in the direction of **a**:

$$\operatorname{proj}_{\mathbf{a}}(\mathbf{b}) = \operatorname{comp}_{\mathbf{a}}(\mathbf{b}) \frac{\mathbf{a}}{\|\mathbf{a}\|} = -\frac{2}{\sqrt{5}} \frac{1}{\sqrt{5}}(1, 2) = \left(-\frac{2}{5}, -\frac{4}{5}\right)$$

Note that this agrees with our geometric estimate from the picture.

12.3.9 Application (Physics): Work

.*(mention briefly)*

(skip this section)

12.4 Cross Product

Key Ideas

- Determinants
  - Computing 2×2, 3×3
  - Properties (normalized, n-linear, alternating)
- Cross product $\mathbf{u} \times \mathbf{v}$
  - Definition (in terms of determinant)
  - Geometric interpretation
    - Orthogonal to $\mathbf{u}, \mathbf{v}$
    - Length equal to area of parallelogram $\mathbf{u}, \mathbf{v}$
- Scalar triple product

Warning! The cross product of two vectors is defined only in $\mathbf{R}^3$.

In Section 12.3 we defined the inner product on $\mathbf{R}^n$, which takes as input two vectors and outputs a scalar (i.e. real number). In this section we define the cross product on $\mathbf{R}^3$ (heed the above warning!), which takes as input two vectors in $\mathbf{R}^3$ and outputs a vector in $\mathbf{R}^3$. The cross product is an important construction for two reasons:

1. The output vector is orthogonal to the two input vectors.
2. The norm of the output vector equals the area of the parallelogram defined by the two input vectors.

We'll explore both of these properties in more detail.

12.4.1 Determinants

The easiest way to define (and remember) the cross product is in terms of determinants. We remind ourselves of the important facts here.

Let A be a (square) $n \times n$ matrix. The determinant of A is denoted $\det A$.

The determinant of a 1×1 matrix $A := (a)$ is

$$\det A = \det (a) = a.$$

The determinant of a 2×2 matrix $A := \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ is

$$\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1.$$

The determinant of a 3×3 matrix is

$$\begin{aligned}\det A &= \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.\end{aligned}$$

The 3×3 case can be remembered by (i) using expansion by minors on any row or column *(illustrate; warn about sign)* or (ii) appending the first and second columns on the right of A to obtain a 3×5 matrix, then multiplying each of the three diagonals (with $+$) and three antidiagonals (with $-$).

Expansion by minors is the more useful technique, because it generalizes to $n \times n$ matrices. (Technique (ii) does not. *(comment about permutations?)*)

12.4.2 Abstract Determinant

The determinant as we defined it above has important properties. In abstract algebra, these properties are taken as the definition of determinant; then one shows that the function we called "determinant" satisfies the properties. In fact, one can show that there exists a unique function satisfying these properties. For our purposes the important properties are more important, so we list them here:

- $\det I = 1$
- linear in the rows (and columns) of the matrix *(explain)*
- alternating: if we swap any two rows (or any two columns) of the matrix, then the determinant changes sign

One can check that these properties hold in the 3×3 case. (Do it!)

In particular, the fact that determinant is alternating implies that if any two rows (or any two columns) of A are equal, then $\det A = 0$. (More generally, if any row (or any column) of A is a linear combination of the other rows (other columns), then $\det A = 0$.)

Catchphrase

Determinant is the unique normalized alternating n-linear form.

12.4.3 Computing the Cross Product

Definition 12.10. Let $\mathbf{u} := (u_1, u_2, u_3), \mathbf{v} := (v_1, v_2, v_3) \in \mathbf{R}^3$. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u} \times \mathbf{v}$, is[3]

$$\mathbf{u} \times \mathbf{v} := \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}. \tag{12.4.1}$$

Using expansion by minors along the first row, we can write

$$\mathbf{u} \times \mathbf{v} = \mathbf{i} \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - \mathbf{j} \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + \mathbf{k} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}.$$

This is an algebraic definition of the cross product. We can also give a geometric definition: The cross product $\mathbf{u} \times \mathbf{v}$ is the vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ whose

- direction is given by the right-hand rule (place your right hand along $\mathbf{u}$ with your fingers pointing in the direction of $\mathbf{u}$, and curl your fingers in the direction that $\mathbf{v}$ points; then your thumb points in the direction of $\mathbf{u} \times \mathbf{v}$)
- length (norm) equals the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$ (when their initial points are made to coincide).

Right now this geometric definition seems mysterious. We'll justify it in the next subsection.

12.4.4 Important Properties of Cross Product

We are now equipped to prove the properties of cross product that we mentioned at the beginning of this section. Let $\mathbf{u} := (u_1, u_2, u_3), \mathbf{v} := (v_1, v_2, v_3) \in \mathbf{R}^3$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

Theorem 12.11. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof. Recall that two vectors are orthogonal if and only if their inner product equals 0. So we want to show that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

[3] You may have seen the cross product $\mathbf{u} \times \mathbf{v}$ defined as

$$\mathbf{u} \times \mathbf{v} := \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix},$$

i.e. with the standard basis vectors along the bottom row of the matrix, rather than the top. Using the property that determinant is alternating, do you see why this definition is equivalent to (12.4.1)? (Alternatively, you could compute both determinants and observe that they are equal.)

We compute

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= (u_1,u_2,u_3)\cdot\left(\mathbf{i}\begin{pmatrix}u_2 & u_3\\ v_2 & v_3\end{pmatrix} - \mathbf{j}\begin{pmatrix}u_1 & u_3\\ v_1 & v_3\end{pmatrix} + \mathbf{k}\begin{pmatrix}u_1 & u_2\\ v_1 & v_2\end{pmatrix}\right)\\ &= \begin{pmatrix}u_1 & u_2 & u_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{pmatrix}\\ &= 0,\end{aligned}$$

where the final equality uses the property that determinant is alternating. (Alternatively, we can explicitly compute the dot product in terms of components.) An analogous computation shows that

$$\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0.$$

Therefore we conclude that $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. □

Theorem 12.12. *The norm of the cross product is*

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta. \tag{12.4.2}$$

Recall that the area of a parallelogram with side lengths u and v and angle θ between them equals $uv\sin\theta$. So geometrically, this result says that the norm of the cross product $\mathbf{u}\times\mathbf{v}$ equals the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ (when their initial points are made to coincide).

(hint at proof?)

Corollary 12.13. *The vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel if and only if*

$$\mathbf{u}\times\mathbf{v} = \mathbf{0}.$$

Proof. Recall that, by definition, two vectors are parallel if they point in the same direction or in exact opposite directions, i.e. if the angle between them is 0 or π. Because $\sin 0 = 0$ and $\sin\pi = 0$, we see that if $\mathbf{u}$ and $\mathbf{v}$ are parallel, then

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta = 0.$$

(make logic tight — address $\mathbf{u},\mathbf{v}=\mathbf{0}$*)* □

(cute application to area of triangle with given vertices?)

12.4.5 Cross Product of the Standard Basis Vectors

(Have students explain these to you geometrically.)

The cross products of the standard basis vectors are straightforward to determine, especially if we use the right-hand rule and (12.4.2). In particular, you needn't memorize these!

("recycle symbol" of cross products of standard basis vectors; reversing arrows gives negative (see below))

12.4.6 Warnings about Cross Products

Two properties of multiplication that we are used to do *not* hold for the cross product.

The cross product is not commutative, i.e. in general,

$$\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}.$$

In fact, the cross product is anticommutative:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

This follows immediately from the definition of cross product in terms of the determinant of a 3×3 matrix and the fact that determinant is alternating (so swapping two rows changes the sign).

The cross product is not associative, i.e. in general,

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

For example,

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \neq -\mathbf{j} = \mathbf{i} \times \mathbf{k} = \mathbf{i} \times (\mathbf{i} \times \mathbf{j}).$$

12.4.7 More Properties of Cross Product

(Skip this section?)

Theorem 12.4.8. Properties 1–4 follow immediately from properties of determinant. Properties 5–6 are defined in next subsection.

12.4.8 Scalar Triple Product and Vector Triple Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in $\mathbf{R}^3$. How can we use inner product and cross product to obtain (i) a scalar and (ii) a vector?

Definition 12.14. Their **scalar triple product** is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

Note that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Consider the parallelepiped (the three-dimensional version of a parallelogram) determined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$. *(draw picture)* The base, say the face determined by $\mathbf{v}, \mathbf{w}$, is a parallelogram, so its area is

$$\|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta = \|\mathbf{v} \times \mathbf{w}\|,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Let $\varphi \in [0, \frac{\pi}{2}]$ be the angle between $\mathbf{u}$ and the line containing $\mathbf{v} \times \mathbf{w}$. Then the height of the parallelepiped is $\|\mathbf{u}\| \cos \varphi$, so the volume of the parallelepiped is

$$\|\mathbf{v} \times \mathbf{w}\| \, \|\mathbf{u}\| \cos \varphi = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \,.$$

(explain absolute value, or revise definition of φ above) Thus the volume of the parallelepiped determined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the absolute value of their scalar triple product.

Definition 12.15. The **vector triple product** of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

12.4.9 Application (Physics): Torque

(Skip this section.)

12.4.10 Epilogue

Warning! The cross product of two vectors is defined only in $\mathbf{R}^3$.

12.5 Lines and Planes in $\mathbf{R}^3$

Key Ideas

- Parametric, vector equation of a line, line segment
  - Line is determined by point and direction vector
- Scalar, vector equation of a plane
  - Plane is determined by point and normal vector
- Computing intersections, parallel, skew

The ability to visualize curves and surfaces in $\mathbf{R}^3$ is essential to developing and applying the high-powered machinery of differential and integral multivariable calculus. Often, it is the most difficult part of the problem (the geometry, not the calculus!). We lay the foundation for this critical skill in this section and the next.

There are two ways of representing lines in $\mathbf{R}^n$ of particular interest to us: parametric equations and vector equations. In fact, these two representations are equivalent.

(emphasize interpretation)

- *Line: point (not vector!) and direction vector*
- *Plane: point (not vector!) and normal vector*

12.5.1 Equations of a Line in $\mathbf{R}^2$

Let us first consider the parametric representation of a line L in $\mathbf{R}^2$. Suppose that L passes through the point $P_0 := (x_0, y_0)$ and has slope v.

The slope of the line corresponds to a vector (in fact, infinitely many vectors) in $\mathbf{R}^2$: If we view $v = \frac{v_2}{v_1}$ as rise v_2 over run v_1, then v corresponds to the vector $\mathbf{v} := (v_1, v_2)$. In particular, we can consider $v = \frac{v}{1}$, so $\mathbf{v} = (1, v)$. In the case that the line L is vertical, then the slope corresponds to the vector $\mathbf{v} = (0, v_2)$ for any $v_2 \neq 0$; in particular, we can take $\mathbf{v} = (0, 1)$. In either case, a set of parametric equations for the line L is (have students propose this)

$$x = x_0 + v_1 t, \qquad y = y_0 + v_2 t. \tag{12.5.1}$$

Let $\mathbf{r} := (x, y)$ and $\mathbf{r}_0 := (x_0, y_0)$. Then we can write these parametric equations for the line L in vector form:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}. \tag{12.5.2}$$

Writing out the components of this vector equation, we have

$$(x, y) = (x_0, y_0) + t(v_1, v_2) = (x_0 + tv_1, y_0 + tv_2).$$

Thus the vector equation (12.5.2) simply takes the parametric equations (12.5.1) as its components.

Geometrically, we can interpret (12.5.2) as follows: Any point $\mathbf{r}$ on the line L can be obtained by starting at the point $\mathbf{r}_0$ and moving in the direction $\mathbf{v}$ an amount determined by t.

If neither component of $\mathbf{v}$ is zero (equivalently, if the slope ν of L is neither ∞ nor 0), then we can solve the parametric equations (12.5.1) for t and equate the results, obtaining

$$y - y_0 = \nu(x - x_0),$$

the point-slope form of a line. If either $v_1 = 0$ or $v_2 = 0$, then as noted above we can take the other component to be 1. In this case, the corresponding parametric equation in (12.5.1) becomes

$$x = x_0, \qquad\qquad y = y_0.$$

12.5.2 Equations of a Line in $\mathbf{R}^n$

The vector equation of a line given in (12.5.2) holds in $\mathbf{R}^n$. The geometric intuition is the same as in $\mathbf{R}^2$: start at the point $\mathbf{r}_0$, and move in the direction $\mathbf{v}$ an amount specified by t. Just as in $\mathbf{R}^2$, the components of the vector equation give a set of parametric equations for the line.

Warning: The vector equation and parametric equations of a line are not unique! Why not? (Have students answer this.) We can take any point on the line as the "base point" $\mathbf{r}_0$, and we can take any nonzero scalar multiple of $\mathbf{v}$ as the direction vector.

Eliminating the parameter t from the parametric equations yields what are called **symmetric equations** of the line. (Take care if $v_i = 0$.) *(explain this parenthetical in more detail)*

12.5.3 Equation of a Line Segment

Let P_0, P_1 be points in $\mathbf{R}^n$. How can we describe the (directed) line segment starting at P_0 and ending at P_1? (Have students motivate this.)

Let $\mathbf{r}_0, \mathbf{r}_1$ be the vectors with initial point the origin and terminal point P_0, P_1, respectively. Then the vector $\mathbf{v} := \mathbf{r}_1 - \mathbf{r}_0$ starts at P_0 and ends at P_1 *(draw picture)*. Scaling this vector by an appropriate $t \in [0, 1]$, we can hit any point on the line segment P_0P_1. Thus the directed line segment from P_0 to P_1 is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1,$$

where we restrict $t \in [0, 1]$.

Example 12.16 (S6eQ12.05.22)**.** Determine whether the lines

$$L_1 : \frac{x-1}{2} = \frac{y-3}{2} = \frac{z-2}{-1}, \qquad L_2 : \frac{x-2}{1} = \frac{y-6}{-1} = \frac{z+2}{3}$$

are parallel, skew, or intersecting. If they intersect, find the point of intersection.

Solution: Let's begin by translating these symmetric equations for L_1, L_2 into the (equivalent) vector equations. Remember that symmetric equations for a line in $\mathbf{R}^n$ are obtained by solving each component equation for the parameter t and equating the results. Reversing this process, let t_1, t_2 be the parameters for the lines L_1, L_2, respectively. Setting each expression in the symmetric equation equal to the parameter and solving for x, y, z, we find that a set of parametric equations for L_1 is

$$x = 1 + 2t_1, \qquad y = 3 + 2t_1, \qquad z = 2 - t_1,$$

and a set of parametric equations for L_2 is

$$x = 2 + t_2, \qquad y = 6 - t_2, \qquad z = -2 + 3t_2.$$

Translating these parametric equations into the corresponding vector equations, we have

$$\mathbf{r}_1(t_1) = (1, 3, 2) + t_1(2, 2, -1), \qquad \mathbf{r}_2(t_2) = (2, 6, -2) + t_2(1, -1, 3),$$

where $\mathbf{r}_i$ corresponds to the line L_i, $i \in \{1, 2\}$.

Note the following features of this translation: (i) the components of the "base point" of the vector equation are precisely the values subtracted in the numerator of the symmetric equations, and (ii) the components of the direction vector of the vector equation are precisely the values in the denominators of the symmetric equations. This is not magic: it follows immediately from how we translate between symmetric and vector equations. Noting this makes the translation process fast and easy. It also makes it easier to use the symmetric equations directly to answer whether the lines are parallel, intersect, etc.

Two lines are parallel if and only if their direction vectors are parallel, i.e. nonzero scalar multiples of one another. Thus L_1, L_2 are parallel if and only if there exists some $\lambda \in \mathbf{R}$ with $\lambda \neq 0$ such that

$$(2, 2, -1) = \lambda(1, -1, 3).$$

Writing this vector equality in terms of components, this is equivalent to

$$2 = \lambda, \qquad 2 = -\lambda, \qquad -1 = 3\lambda.$$

The first two equalities imply that $\lambda = 2$ and $\lambda = -2$, a contradiction. Therefore the lines L_1, L_2 are not parallel.

Two lines intersect if and only if they have a point in common. How does this geometric condition translate into algebra? It means that L_1, L_2 intersect if and only if there exist parameter values t_1, t_2 such that

$$\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2).$$

Note that the parameter values for the two lines are independent! Writing out this vector equality in terms of components, this is equivalent to

$$1 + 2t_1 = 2 + t_2, \qquad 3 + 2t_1 = 6 - t_2, \qquad 2 - t_1 = -2 + 3t_2.$$

Solving this system of three equations in two unknowns, we find

$$t_1 = 1, \qquad t_2 = 1$$

solves all three equations. Therefore L_1, L_2 intersect. At what point? At

$$\mathbf{r}_1(1) = (3, 5, 1) = \mathbf{r}_2(1).$$

We emphasize that the parameter values for the two lines are independent. We adopted the notation t_1, t_2 to emphasize this. As it happened to happen, $t_1 = t_2$ at the point of intersection. If we had parametrized the line L_2 by the equivalent parametrization

$$\mathbf{r}_2(t_2) = (3, 5, 1),$$

then we would have found that L_1, L_2 intersect when $t_1 = 1$ and $t_2 = 0$.

12.5.4 Planes in $\mathbf{R}^3$

We have seen that we can describe a line by specifying a point in the line and the direction of the line. Lines are pleasant because they are linear. The next linear object, one dimension up, is a plane. How can we describe a plane? Specifying a point in the plane and a direction doesn't seem to work, because there are infinitely many directions to travel in the plane. But note that all of these directions are orthogonal to a single line. We call any nonzero vector on this line a **normal** vector to the plane. Thus a plane can be described by a point in the plane and a nonzero normal vector.

Now let's describe a plane algebraically. Let $H \subseteq \mathbf{R}^3$ be a plane, let $\mathbf{n}$ be a nonzero normal vector to H, and let $P_0 := (x_0, y_0, z_0)$ and $P := (x, y, z)$ be points in H. We think of P_0 as being a fixed point (i.e. a point whose coordinates we know) and P as being a variable point. Let $\mathbf{r}_0, \mathbf{r}$ denote the vectors corresponding to P_0, P, respectively. That is, $\mathbf{r}_0$ points from the origin to P_0, and $\mathbf{r}$ points from the origin to P. Thus $\mathbf{r} - \mathbf{r}_0$ represents the vector $\vec{P_0P}$, which lies in the plane H. *(draw picture)* Because all vectors in H are orthogonal to the normal vector $\mathbf{n}$,

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0. \qquad (12.5.3)$$

This is called a **vector equation of a plane**. *(clean up logic, so iff clear)*

A **scalar equation of a plane** is obtained by writing the vector equation (12.5.3) explicitly in terms of components: Let $\mathbf{n} = (n_1, n_2, n_3)$. Then

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0.$$

Moving all scalar terms to the right side, we have

$$n_1 x + n_2 y + n_3 z = n_1 x_0 + n_2 y_0 + n_3 z_0.$$

An equation like this in which x, y, z appear with exponents only 1 (or 0) and no mixed terms (like xy) is called a **linear equation** in x, y, z. Note that the right side is a scalar (in practice, it will be a single number.) Note also that the components of the normal vector $\mathbf{n}$ are the coefficients of x, y, z.

As we saw with lines, vector and scalar equations of a plane are not unique, for the same reasons: We can choose different base points P_0 (corresponding to different $\mathbf{r}_0$), and we can choose different normal vectors $\mathbf{n}$ (multiply any nonzero normal vector by a nonzero scalar).

More generally, the method just described allows us to describe higher-dimensional analogs to planes, called hyperplanes, in $\mathbf{R}^n$. Because hyperplanes are determined by a single scalar equation, they have dimension $n - 1$. (Note that a line is a hyperplane in $\mathbf{R}^2$.)

Example 12.17 (S6eQ12.05.36)**.** Find an equation of the plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations $x = 2y = 3z$.

Solution: (Have students motivate solution.) Remember that a plane is determined by a point in the plane and a nonzero normal vector. If we have three nonlinear points P_0, P, Q in a plane, then a normal vector to the plane is given by the cross product $\vec{P_0P} \times \vec{P_0Q}$.

Let $P_0 := (1, -1, 1)$, let L be the line with symmetric equations $x = 2y = 3z$, and let H be the plane containing P_0 and L. Note that P_0 is not in the line L (why not?). Therefore we can take any two distinct points in L as our P, Q. Let's make our lives easy: Choose points that are easy to check, say $P := (0, 0, 0)$ and $Q := (6, 3, 2)$. Then

$$\vec{P_0P} = (0, 0, 0) - (1, -1, 1) = (-1, 1, -1), \quad \vec{P_0Q} = (6, 3, 2) - (1, -1, 1) = (5, 4, 1),$$

and

$$\vec{P_0P} \times \vec{P_0Q} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 5 & 4 & 1 \end{pmatrix} = (5, -4, -9).$$

Therefore a vector equation of the plane that contains P_0 and L is

$$(5, -4, -9) \cdot (x - 1, y + 1, z - 1) = 0.$$

The corresponding linear equation is

$$5x - 4y - 9z = 0. \tag{12.5.4}$$

In the equations above we chose P_0 as the base point. We could just as well choose P as the base point, in which case the vector equation of the plane writes as

$$(5, -4, -9) \cdot (x, y, z) = 0.$$

Note that the corresponding linear equation is identical to (12.5.4). This is not the case in general — it happens here because the plane contains the origin. (Think about what happens if we take $(x, y, z) = (0, 0, 0)$ and take any $P_0 \in H$.) How does the linear equation change if we choose a different normal vector?

Note that we can use the symmetric equations to write all variables in terms of z, say, i.e. $x = 3z$ and $y = \frac{3}{2}z$, and the substitute these expressions into the linear equation (12.5.4), obtaining

$$5(3z) - 4\left(\frac{3}{2}z\right) - 9z = 15z - 6z - 9z = 0.$$

Thus $L \subseteq H$, as required. Substituting the coordinates of P_0 into (12.5.4), we find

$$5(1) - 4(-1) - 9(1) = 0,$$

so $P_0 \in H$, as required.

We say that two planes are **parallel** if their normal vectors are parallel. *(clarify next bit)* Geometrically, this "indirect" definition makes sense: Think about what parallel planes would look like if we took an intuitive definition of the word "parallel", then realize that the direction of the normal vectors are completely specified and must be parallel.

Two distinct nonparallel planes intersect in a line. (Note the dimension drop here: The intersection of two distinct objects of codimension 1 has codimension 2.) The angle between two nonparallel planes is defined to be the angle $\theta \in [0, \frac{\pi}{2}]$ between their normal vectors.

Example 12.18 (S6eQ12.05.54)**.** Determine whether the planes

$$x + 2y + 2z = 1, \qquad\qquad 2x - y + 2z = 1$$

are parallel, perpendicular, or neither. If neither, then find the angle between them.

Solution: Let

$$H_1 := \{(x, y, z) \,|\, x + 2y + 2z - 1 = 0\}, \qquad H_2 := \{(x, y, z) \,|\, 2x - y + 2z - 1 = 0\}$$

be the two planes. Reading off the components of the normal vectors from the corresponding linear equations, we see that

$$\mathbf{n}_1 = (1, 2, 2), \qquad\qquad \mathbf{n}_2 = (2, -1, 2).$$

It is straightforward to check that these vectors are not parallel. Therefore, by definition, H_1, H_2 are not parallel. Let θ be the angle between $\mathbf{n}_1, \mathbf{n}_2$. Using the relation between dot product and angle, we have

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{1(2) + 2(-1) + 2(2)}{\sqrt{9}\sqrt{9}} = \frac{4}{9}.$$

Thus

$$\theta = \cos^{-1}\left(\frac{4}{9}\right) \approx 1.1102 \approx 63.6122^\circ.$$

12.5.5 Distance from a Point to a Plane

The distance from a point $P_1 := (x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

(explain derivation — geometrically intuitive!)

12.6 Cylinders and Quadric Surfaces in $\mathbf{R}^3$

Key Ideas

- Cylinder (definition and how to graph)
- Quadric surface (definition and how to graph)
- How to use level sets, sections, traces to help visualize a surface

In Section 12.1 we developed three coordinate systems on $\mathbf{R}^3$: rectangular, cylindrical, and spherical. In this section we use these coordinate systems to describe surfaces in $\mathbf{R}^3$.

Recall the earlier warning that an equation (or rule of assignment) does not specify a function or its graph — we also need to specify the domain and codomain. In this section, all equations are interpreted as functions $\mathbf{R}^3 \to \mathbf{R}$.

12.6.1 Cylinders

Definition 12.19. A **cylinder** is a surface equal to all lines, called **rulings**, that (i) are parallel to a given line and (ii) pass through a given curve.

Example 12.20 (S6eQ12.06.06)**.** Describe and sketch the surface $yz = 4$.

Solution: Let

$$H := \{(x, y, z) \mid yz - 4 = 0\}$$

be the surface. In the yz-plane, the equation $yz = 4$ describes a hyperbola. *(graph it)* This equation does not involve the variable x, so it is free. That is, given any values of y, z such that $yz = 4$, for any value of x, the point (x, y, z) satisfies $yz = 4$, and hence $(x, y, z) \in H$. *(extend previous graph to graph in $\mathbf{R}^3$)*

Note that the rulings are parallel to the x-axis, because the variable x is free.

Remember that a coordinate system is structure that we lay over a geometric setting. So we are allowed to orient our coordinate axes in the manner most convenient for us. If we are asked to describe a cylinder, it is easiest to orient the coordinate axes so that one of them is parallel to the rulings of the cylinder. It is also advantageous to place the coordinate axes so that the origin is located symmetrically.

Recall that cylindrical coordinates are triples (r, θ, z) where (r, θ) are polar coordinates in the xy-plane, and z is the height above the xy-plane. Cylindrical coordinates are particularly well-suited for describing surfaces possessing symmetry about a line.

Example 12.21. Describe the circular cylindrical surface with radius R. (draw picture)

Solution: Orient the coordinate axes so that (i) the origin lies at the center of (a cross-section of) the cylinder and (ii) the z-axis is parallel to the rulings of the cylinder. Recall that in polar coordinates, the equation for a circle with radius R centered at the origin is $r = R$. The variable θ is free. Under our chosen orientation, the variable z is also free. Therefore, with this choice of coordinates and orientation of coordinate axes, the equation of the circular cylindrical surface is simply

$$r = R.$$

We can also describe the circular cylindrical surface using rectangular coordinates. Orienting the coordinate axes as above, the cylinder intersects the xy-plane in a circle of radius R, which has the equation

$$x^2 + y^2 = R^2. \tag{12.6.1}$$

Because the z-axis is parallel to the rulings of the cylinder, the variable z is free. Thus (12.6.1) is an equation for the circular cylindrical surface in $\mathbf{R}^3$.

Example 12.22. Give an equation of an infinite cone, in cylindrical coordinates, with vertex angle $\frac{\pi}{4}$.

Solution: Position the cone on the coordinate axes of $\mathbf{R}^3$ so that the vertex is at the origin, the central axis of the cone coincides with the z-axis, and the cone opens down. The vertex angle of $\theta = \frac{\pi}{4}$ implies that if we travel down the z-axis a distance h (equivalently, if we look at the cross-section of the cylinder with the plane $z = -h$), then the radius of the cone at that point is

$$\tan\theta = \frac{r}{h} \qquad \Leftrightarrow \qquad r = h\tan\frac{\pi}{4} = h.$$

Thus we can describe the cone in cylindrical coordinates by

$$\{(r, \theta, z) \mid z \leqslant 0, r = -z\}.$$

Note that the variable θ is free.

Two remarks: First, if we adopt the convention that r can take negative values, then we can replace the condition $r = -z$ with $r = z$, i.e. the cone is given by

$$\{(r, \theta, z) \mid z \leqslant 0, r = z\}. \tag{12.6.2}$$

Second, when orienting the cone on the coordinate axes, we could just as well have oriented the cone so that it opened up. In this case, the representation in cylindrical coordinates would have been (12.6.2), but with $z \geqslant 0$ in place of $z \leqslant 0$.

12.6.2 Surfaces in Spherical Coordinates

We have seen that cylindrical coordinates yield particularly simple equations when a surface has "cylindrical symmetry", i.e. radial symmetry about an axis. Likewise, spherical coordinates yield particularly simple equations when a surface has spherical symmetry.

Example 12.23. Give an equation for a sphere of radius R. If the sphere were solid, and cut into equal eighths (e.g., place the sphere centered at the origin, and cut along the coordinate planes), describe the solid region that is one of these eighths.

Solution: Place the sphere on the coordinate axes so that the center of the sphere lies at the origin. Then by definition, the sphere of radius R consists of all points that are a distance R from the origin. The variables φ, θ can be anything. Thus in spherical coordinates, the sphere of radius R is given by

$$\{(\rho, \theta, \varphi) \mid \rho = R\}.$$

Now imagine cutting the (solid) sphere into equal eighths, along the coordinate planes. Consider the eighth corresponding to the orthant *(parenthetical description)* where $x, y, z \geqslant 0$. In spherical coordinates, this orthant corresponds to $\varphi \in [0, \frac{\pi}{2}]$ and $\theta \in [0, \frac{\pi}{2}]$. Moreover, because we are considering the sphere to be solid, we need to allow all values of $\rho \in [0, R]$. Thus we conclude that the solid eighth of the sphere can be described by

$$\left\{(\rho, \theta, \varphi) \mid \rho \in [0, R], \theta \in \left[0, \frac{\pi}{2}\right], \varphi \in \left[0, \frac{\pi}{2}\right]\right\}.$$

12.6.3 Quadric Surfaces

In studying $\mathbf{R}^2$, one encounters conic sections, which are curves in two variables defined by equations whose terms all have total degree 2 or smaller. The general equation of a conic section is

$$Ax^2 + By^2 + Cxy + D = 0.$$

(analogous standard forms under change of coordinates)

Quadric surfaces are generalizations of conic sections to $\mathbf{R}^n$.

Definition 12.24. A **quadric surface** is the graph of an equation whose terms all have total degree 2 or smaller.

In $\mathbf{R}^3$, there are three variables x, y, z, so the general equation of a quadric surface in $\mathbf{R}^3$ is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

One can show that, using a suitable change of coordinates (an algebraic operation that corresponds geometrically to translating, rotating, and dilating the coordinate axes), every quadric surface can be written in one of two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0, \qquad Ax^2 + By^2 + Iz = 0.$$

12.6.4 Level Sets and Sections

Visualizing surfaces and solids in $\mathbf{R}^3$ can be tricky, especially when the equations describing them are complicated. To assist with this visualization, we often consider the intersection of planes with the surface or solid.

Definition 12.25. A **section** of a surface is the intersection of the surface with a vertical plane.

A **trace** of a surface is a curve resulting from the intersection of the surface with a plane parallel to one of the coordinate planes. In particular, given a function $f : U \subseteq \mathbf{R}^n \to \mathbf{R}$ and a $c \in \mathbf{R}$, the **level set of** f **with value** c is the set of points $\mathbf{x} \in U$ such that $f(\mathbf{x}) = c$:

$$LS_f(c) := \{(x_1, \dots, x_n) \in U \mid f(x_1, \dots, x_n) = c\}.$$

(draw pictures to illustrate each of these terms)

Let's do some examples to make these definitions more concrete.

Example 12.26 (S6eE12.06.04)**.** Sketch the surface

$$z = 4x^2 + y^2.$$

Solution: Let

$$z = f(x, y) := 4x^2 + y^2.$$

Because the square of any real number is nonnegative, $z = f(x, y) \geqslant 0$ for all values of x, y. Thus the level sets of f with value c are empty when $c < 0$. When $c \geqslant 0$, the level set of f with value c is

$$c = f(x, y) = z = 4x^2 + y^2.$$

Focusing on the outer expressions, we recognize this as the equation of an ellipse. Thus the level sets with $c \geqslant 0$ are ellipses. (When $c = 0$, the ellipse degenerates to the point $(0, 0, 0)$.)

Now let's look at sections, and more particularly, at traces. Let $c \in \mathbf{R}$. Setting $x = c$, we have

$$z = 4c^2 + y^2,$$

which is a parabola in the yz-plane, opening in the positive z direction and shifted upward from the z-axis by $4c^2$. Similarly, setting $y = c$, we have

$$z = 4x^2 + c^2,$$

which is a parabola in the xz-plane, opening in the positive z direction and shifted upward from the z-axis by c^2.

Drawing a few of these level sets and traces, we get a sense of the graph of f.

(add another example with non-coordinate section, e.g., along $x = y$)

Chapter 13

Vector-Valued Functions

(short intro)

A common theme when dealing with vector-valued functions is that various operations (e.g., limit, derivative, integral) are defined componentwise.

13.1 Vector-Valued Functions

Key Ideas

- Definition: function
  - Maximum domain of definition
  - Vector-valued function, component functions
- Limit of a vector-valued function
- Continuity of a vector-valued function
- Vector notation for curves
- Visualizing space curves (computers, surfaces, intersections)

As we saw in Definition 12.3, a **function** is a rule of assignment that assigns to each point in a specified set of input values (the **domain**) exactly one point in a specified set of output values (the **codomain**). In general,

$$\begin{aligned} f : X &\to Y \\ x &\mapsto f(x). \end{aligned}$$

For example, if

$$X := \{x \in \mathbf{R} \mid |x| < 1\} \subseteq \mathbf{R}, \qquad Y := \mathbf{R}^3,$$

and the rule of assignment for f is to assign $x \in X$ to the vector $(\ln(1-x), \sqrt{1+x}, x) \in Y$, we write this

$$\begin{aligned} f : X &\to Y \\ x &\mapsto \left(\ln(1-x), \sqrt{1+x}, x\right). \end{aligned} \tag{13.1.1}$$

Remark 13.1. Stewart (2008) adopts the convention that the domain of a function, unless otherwise specified, is the maximum possible subset of the relevant space on which the rule of assignment is defined. For example, for the rule of assignment for f in (13.1.1), the domain (input values) is a subset of $\mathbf{R}$, and we see that the output (i.e. each component of the output vector) is defined if

$$1 - x > 0 \qquad \text{and} \qquad 1 + x \geqslant 0 \qquad \Leftrightarrow \qquad -1 \leqslant x < 1.$$

Note that this maximum domain is larger than the domain we specified for f. Strictly speaking, different domains (or different codomains) define different functions, even if the rule of assignment is the same.

Catchphrase

A function is defined by a domain, a codomain, and a rule of assignment.

13.1.1 Vector-Valued Functions

In this section we will focus on a specific type of function: vector-valued functions defined on (a subset of) the real numbers.

Definition 13.2. Let $U \subseteq \mathbf{R}$. A **vector-valued function** on U is a function

$$\begin{aligned} \mathbf{r} : U &\to \mathbf{R}^n \\ t &\mapsto (r_1(t), \ldots, r_n(t)) . \end{aligned}$$

Verbally, a vector-valued function on $\mathbf{R}$ is a function that accepts as input real numbers and that outputs vectors. The components of the output vector are real-valued functions $r_i : U \to \mathbf{R}$ of the input variable and are called the **component functions** of $\mathbf{r}$. That is,

$$\mathbf{r}(t) = (r_1(t), \ldots, r_n(t)).$$

13.1.2 Limits

In single-variable calculus we develop the notion of the limit of a function $f : U \subseteq \mathbf{R} \to \mathbf{R}$. We can use this notion to give a simple description of the limit of vector-valued functions. *(briefly explain the more general ε-δ definition, and how it implies (13.1.2))*

The **limit** of a vector-valued function $\mathbf{r}$ equals the vector of the limits of the component functions r_i, provided each of those limits exists:

$$\lim_{t \to a} \mathbf{r}(t) = \left(\lim_{t \to a} r_1(t), \ldots, \lim_{t \to a} r_n(t) \right) . \tag{13.1.2}$$

Catchphrase

The limit of a vector-valued function is determined componentwise.

Example 13.3 (S6eQ13.01.06)**.** Compute the limit

$$\lim_{t + \to \infty} \left(\arctan t, e^{-2t}, \frac{\ln t}{t} \right) .$$

Solution: Using l'Hôpital's rule to compute

$$\lim_{t \to +\infty} \frac{\ln t}{t} = \lim_{t \to +\infty} \frac{\frac{1}{t}}{1} = 0,$$

we compute

$$\begin{aligned}\lim_{t\to+\infty}\left(\arctan t, e^{-2t}, \frac{\ln t}{t}\right) &= \left(\lim_{t\to+\infty}\arctan t, \lim_{t\to+\infty} e^{-2t}, \lim_{t\to+\infty}\frac{\ln t}{t}\right)\\ &= \left(\frac{\pi}{2}, 0, 0\right).\end{aligned}$$

Limits of vector-valued functions behave as we expect:

(i) $\lim_{t\to a}(\mathbf{u}(t)+\mathbf{v}(t)) = \lim_{t\to a}\mathbf{u}(t) + \lim_{t\to a}\mathbf{v}(t)$

(ii) $\lim_{t\to a} c\mathbf{u}(t) = c\lim_{t\to a}\mathbf{u}(t)$

(iii) $\lim_{t\to a}(\mathbf{u}(t)\cdot\mathbf{v}(t)) = \lim_{t\to a}\mathbf{u}(t)\cdot\lim_{t\to a}\mathbf{v}(t)$

(iv) $\lim_{t\to a}\mathbf{u}(t)\times\mathbf{v}(t) = \lim_{t\to a}\mathbf{u}(t)\times\lim_{t\to a}\mathbf{v}(t)$,

provided the limits on the right exist.

The notion of limit for a vector-valued function — evaluate the limit componentwise — is simple, but its consequences are profound. It will yield the notions of continuity, derivative, and definite integral.

13.1.3 Continuity

(strike balance between precision — need $a \in U$, and U containing open set containing a — and intuition) In single-variable calculus, we define a function $f : U \subseteq \mathbf{R} \to \mathbf{R}$ to be continuous at the point $t = a$ in U if

$$\lim_{t\to a} f(t) = f(a).$$

Continuity of a vector-valued function is defined analogously:

Definition 13.4. The vector-valued function $\mathbf{r} : U \subseteq \mathbf{R} \to \mathbf{R}^n$ is **continuous at the point** $a \in U$ if

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a). \tag{13.1.3}$$

As in the single-variable case, we say that $\mathbf{r}$ is **continuous** if it is continuous at all points $a \in U$.

Writing the components of (13.1.3) and remembering that the limit of a vector-valued function equals the limit of the component functions, when the latter limits exist (see (13.1.2)), we have

$$\left(\lim_{t\to a} r_1(t), \dots, \lim_{t\to a} r_n(t)\right) = (r_1(a), \dots, r_n(a)).$$

Two vectors are equal if and only if corresponding components are equal. Therefore Definition (13.4) says that $\mathbf{r}$ is continuous at the point $t = a$ if and only if for each component function $r_i(t)$,

$$\lim_{t \to a} r_i(t) = r_i(a),$$

i.e. if and only if each component function of $\mathbf{r}(t)$ is continuous at the point $t = a$.

Catchphrase

A vector-valued function is continuous at a point if and only if all of its component functions are continuous there.

13.1.4 Vector Notation for Curves

When studying $\mathbf{R}^2$ we learn that we can represent curves in $\mathbf{R}^2$ by parametric equations

$$x := f(t), \qquad\qquad y := g(t),$$

where $t \in I \subseteq \mathbf{R}$ is the parameter. We have seen *(add ref)* that we can represent these parametric equations compactly using vector notation:

$$(x, y) = (f(t), g(t)).$$

This vector notation for functions generalizes immediately to $\mathbf{R}^n$ (including $n = 3$):

$$(x_1, \ldots, x_n) = (r_1(t), \ldots, r_n(t)).$$

Just as in $\mathbf{R}^2$, the component functions $r_i : I \to \mathbf{R}$ are called **parametric equations**, and t is called the **parameter**.

When the output vectors live in $\mathbf{R}^3$, we call the image of a continuous vector-valued function a space curve.

Definition 13.5. Let $I \subseteq \mathbf{R}$ be an interval (possibly infinite); and for $i = 1, 2, 3$, let $r_i : I \to \mathbf{R}$ be a continuous function. Then the set

$$C := \{(r_1(t), r_2(t), r_3(t)) \mid t \in I\}$$

is called a **space curve**.

13.1.5 Visualizing Space Curves

When we are dealing with space curves it is helpful to visualize them. The easiest way to do this is to use a computer (we live in the 21st century — let's avail ourselves of the technological benefits!). Two other methods that can complement (or take the place of) computer rendering are

- drawing the plane curve on a surface, and
- viewing the plane curve as the intersection of two surfaces.

The method of intersections is a powerful technique. *(elaborate, possibly in appendix)*

Example 13.6 (S6eQ13.01.40)**.** Sketch by hand the curve of intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid $x^2 + 4y^2 + 4z^2 = 16$. Then find parametric equations for this curve, and use these equations to have a computer graph the curve.

Solution: In the equation for the parabolic cylinder $y = x^2$, the variable z does not appear, so it is free. That is, the rulings are vertical lines (parallel to the z-axis) along the curve $y = x^2$ in the xy-plane. To help us draw the ellipsoid, we compute a few level sets. When $z = 0$, the equation becomes

$$\frac{x^2}{16} + \frac{y^2}{4} = 1,$$

an ellipse with center at the origin, semimajor axis length 4 along the x-axis, and semiminor axis length 2 along the y-axis. When $z = 1$, the equation becomes

$$\frac{x^2}{12} + \frac{y^2}{3} = 1.$$

When $z = 2$, the equation becomes

$$x^2 + 4y^2 = 0,$$

so in the plane $z = 2$ the ellipsoid degenerates to a point, $(0, 0, 2)$. This helps us sketch the two surfaces and their intersection. (draw the two surfaces and bold their intersection)

The intersection of the surfaces defined by $y = x^2$ and $x^2 + 4y^2 + 4z^2 = 16$ consists of all points $(x, y, z) \in \mathbf{R}^3$ that satisfy both equations. In particular, all points (x, y, z) in the intersection satisfy the equation for the parabolic cylinder, so we can substitute this relation into the equation for the ellipsoid, obtaining

$$x^2 + 4x^4 + 4z^2 = 16.$$

Solving this for z, we obtain

$$z = \frac{1}{2}\sqrt{16 - 4x^4 - x^2}. \tag{13.1.4}$$

(Note that taking square roots gives $\pm$ the right side. Because we are asked for the intersection of the parabolic cylinder with the top half of the ellipsoid, we restrict our attention to values of $z \geqslant 0$.) This gives z as a function of x. The equation for the parabolic cylinder gives y as a function of x. Thus we can take x as our parameter (or equivalently, take t as our parameter and $x = t$).

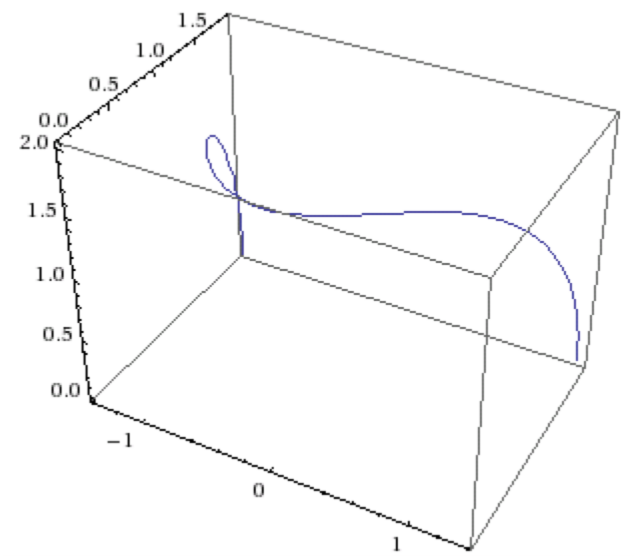

Figure 13.1.1: Computer plot for the intersection in Example 13.6.

Our sketch of the intersection suggests that x cannot take all values in $\mathbf{R}$; we need to determine the values of x, i.e. the **parameter space**, that will give us precisely the intersection of the two surfaces. For this, we consider the equation (13.1.4) defining z as a function of x. This relation is defined if and only if

$$0 \leqslant -4x^4 - x^2 + 16 \qquad \Leftrightarrow \qquad 0 \geqslant 4x^4 + x^2 - 16.$$

Using the quadratic formula (twice) or a computer algebra system, we find that the real solutions to this equation are

$$x \approx \pm 1.3707.$$

Thus we conclude that the intersection can be described by the parametric equations

$$x = t, \qquad y = t^2, \qquad z = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}, \qquad t \in [-1.3707, 1.3707].$$

Typing

```
plot(x=t,y=t^(2),z=(1/2)*sqrt(16-4*t^(4)-t^(2)),t=-1.3707
                       to t=1.3707)
```

into WolframAlpha yields the plot shown in Figure 13.1.1.

13.2 Derivatives and Integrals

Key Ideas

- Derivative of a vector-valued function
  - Defined as limit of difference quotient
  - Compute componentwise
- Tangent vector, unit tangent vector, tangent line to curve at point
- Higher-order derivatives
- Definite integral of a vector-valued function
  - Defined as limit of Riemann sums
  - Compute componentwise

In this section we explore how to differentiate and integrate vector-valued functions. The general rule is as easy as we could hope for: We differentiate and integrate componentwise.

13.2.1 Derivatives

Recall that the derivative f' of a real-valued function $f : \mathbf{R} \to \mathbf{R}$ is defined as the limit of a difference quotient:

$$f'(t) := \lim_{h\to 0} \frac{f(t+h) - f(t)}{h},$$

provided the limit exists. Geometrically, this limit is the limit of a sequence of secant lines through the point $P := f(t)$ and "nearby" points $Q_t := f(t+h)$. *(draw the picture for f)*

The derivative of a vector-valued function is defined analogously.

Definition 13.7. Let $\mathbf{r} : U \subseteq \mathbf{R} \to \mathbf{R}^n$ be a vector-valued function. The **derivative** of $\mathbf{r}$, denoted $\mathbf{r}'$ or $\frac{d\mathbf{r}}{dt}$, is

$$\mathbf{r}'(t) := \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

provided the limit exists.

(draw the picture for $\mathbf{f}$*)*

The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve C defined by $\mathbf{r}(t)$ at the point $P = \mathbf{r}(t)$. The picture *(add ref)* shows why this name is appropriate.

If $\mathbf{r}'(t) \neq \mathbf{0}$, then we can normalize its length to 1 by scaling it by the inverse of its norm, obtaining the **unit tangent vector** to C at P,

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

The line through the point P in the direction of the (unit) tangent vector is called (what do you think?) the **tangent line** to C at P.

The next result states that we can compute the derivative of a vector-valued function componentwise.

Theorem 13.8. *Let $\mathbf{r} : U \subseteq \mathbf{R} \to \mathbf{R}^n$ be a vector-valued function whose component functions $r_i : U \to \mathbf{R}$ are differentiable. Then*

$$\mathbf{r}'(t) = \left(r_1'(t), \ldots, r_n'(t)\right).$$

The key idea of the proof is that the limit of a vector-valued function is defined componentwise (recall (13.1.2)).

Proof. By definition,

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\
&= \lim_{h \to 0} \left(\frac{r_1(t+h) - r_1(t)}{h}, \ldots, \frac{r_n(t+h) - r_n(t)}{h} \right) \\
&= \left(\lim_{h \to 0} \frac{r_1(t+h) - r_1(t)}{h}, \ldots, \lim_{h \to 0} \frac{r_n(t+h) - r_n(t)}{h} \right) \\
&= \left(r_1'(t), \ldots, r_n'(t)\right).
\end{aligned}$$

□

Catchphrase

Differentiate vector-valued functions componentwise.

Example 13.9 (S6eQ13.02.08). Let

$$\mathbf{r}(t) := (1 + \cos t, 2 + \sin t).$$

Sketch the curve, compute $\mathbf{r}'(t)$, and sketch the position vector and tangent vector at $t = \frac{\pi}{6}$.

The given function $\mathbf{r}(t)$ defines a circle C of radius 1 around the point $(1, 2)$. We compute

$$\mathbf{r}'(t) = (-\sin t, \cos t).$$

At $t = \frac{\pi}{6}$, we have

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \left(\frac{2+\sqrt{3}}{2}, \frac{5}{2}\right), \qquad \mathbf{r}'\left(\frac{\pi}{6}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

(sketch graph)

Because differentiation of vector-valued functions is performed componentwise, it follows that higher-order derivatives are also computed componentwise. That is, the dth derivative of $\mathbf{r}$ is

$$\mathbf{r}^{(d)}(t) = \left(r_1^{(d)}(t), \dots, r_n^{(d)}(t)\right),$$

when the derivatives on the right all exist.

13.2.2 Rules for Differentiation

Derivatives of vector-valued functions behave as you expect them to from single-variable calculus.

Theorem 13.10. *Let $\mathbf{u}, \mathbf{v}$ be differentiable vector-valued functions, let f be a real-valued function, and let $c \in \mathbf{R}$.*

1. *The derivative is linear:*

   (a) $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$

   (b) $\dfrac{d}{dt}(c\mathbf{u}(t)) = c\mathbf{u}'(t)$

2. *The derivative satisfies various product rules:*

   (c) *Scalar product:* $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

   (d) *Inner product:* $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

   (e) *Cross product:* $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

3. *The derivative satisfies the chain rule:*

   (f) $\dfrac{d}{dt}\mathbf{u}(f(t)) = f'(t)\mathbf{u}'(f(t))$

It is straightforward to prove each of these results: Use the fact that differentiation of vector-valued functions is defined componentwise, then apply what we know from single-variable calculus.

This next result will be crucial in Section 13.3, when we develop a computationally friendly way to compute curvature.

Lemma 13.11 (S6eE13.02.04). *Let $\mathbf{r}(t)$ be a vector-valued function such that $\|\mathbf{r}(t)\|$ is constant. Then for all t in the domain of $\mathbf{r}$, $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.*

Geometrically, Lemma 13.11 says that if the curve traced out by $\mathbf{r}(t)$ always lies on a sphere (this is what it means for $\|\mathbf{r}\|$ to be constant), then for each t the tangent vector $\mathbf{r}'(t)$ is orthogonal to the position vector $\mathbf{r}(t)$. Intuitively, if $\mathbf{r}'(t)$ had a nonzero nonorthogonal component to $\mathbf{r}(t)$, then $\|\mathbf{r}(t)\|$ would change. We also emphasize that being orthogonal to $\mathbf{r}(t)$ is not the same as being orthogonal to the curve C traced out by $\mathbf{r}(t)$. The tangent vector $\mathbf{r}'(t)$ is always orthogonal to the curve C at the point $\mathbf{r}(t)$, for any function $\mathbf{r}(t)$. The tangent vector $\mathbf{r}'(t)$ being orthogonal to the vector $\mathbf{r}(t)$ is a different condition. *(clarify this discussion)*

Proof. By hypothesis, $\|\mathbf{r}(t)\| = c$ for some $c \in \mathbf{R}$. Squaring both sides of this equality, we obtain

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = c^2.$$

Differentiating (using the product rule on the left) and using the fact that inner product is symmetric, we have

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \qquad \Leftrightarrow \qquad \mathbf{r}'(t) \cdot \mathbf{r}(t) = 0.$$

This last equality holds if and only if $\mathbf{r}'(t), \mathbf{r}(t)$ are orthogonal. $\square$

13.2.3 Integrals

Recall that the definite integral of a real-valued function is defined as the limit of Riemann sums: *(remind split interval, choose points, take lim; not overly detailed)*

$$\int_a^b r(t)dt = \lim_{N\to\infty} \sum_{i=1}^N r(t_i)\Delta t_i.$$

We can define the definite integral of a vector-valued function in the same way: *(we don't need $\mathbf{r}$ to be continuous, just sufficiently nice)*

$$\begin{aligned}
\int_a^b \mathbf{r}(t)dt &= \lim_{N\to\infty} \sum_{i=1}^N \mathbf{r}(t_i)\Delta t_i \\
&= \lim_{N\to\infty} \left(\sum_{i=1}^N r_1(t_i)\Delta t_i, \ldots, \sum_{i=1}^N r_n(t_i)\Delta t_i \right) \\
&= \left(\lim_{N\to\infty} \sum_{i=1}^N r_1(t_i)\Delta t_i, \ldots, \lim_{N\to\infty} \sum_{i=1}^N r_n(t_i)\Delta t_i \right) \\
&= \left(\int_a^b r_1(t)dt, \ldots, \int_a^b r_n(t)dt \right),
\end{aligned}$$

where in the second equality we use the fact that the sum of vectors is defined componentwise (note that it is important that the sum is finite!), in the third equality we use the fact that the limit of a vector-valued function can be evaluated componentwise, and in the final equality we use the definition of the definite integral for real-valued functions.

Similarly, the indefinite integral of a vector-valued function is the indefinite integral of the component functions.

Note that the integral of a vector-valued function is a vector (whose components are scalars in the case of a definite integral, and single-variable functions in the case of an indefinite integral).

Catchphrase

Integrate vector-valued functions componentwise.

The fundamental theorem of calculus applies to continuous[1] vector-valued functions: Let $\mathbf{R}(t)$ be an antiderivative of $\mathbf{r}(t)$ (i.e. $\mathbf{R}'(t) = \mathbf{r}(t)$). Then

$$\int_a^b \mathbf{r}(t) = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

Example 13.12 (S6eQ13.02.36)**.** Evaluate the integral

$$\int \left(t^2\mathbf{i} + t\sqrt{t-1}\mathbf{j} + t\sin(\pi t)\mathbf{k}\right) dt.$$

Solution: Recall that the integral of a vector-valued function is computed componentwise. The integral of the first component is straightforward:

$$\int t^2 dt = \frac{1}{3}t^3 + C_1.$$

To compute the integral of the second component, we can use a substitution: Let

$$u := \sqrt{t-1}; \qquad \text{then} \qquad du = \frac{dt}{2\sqrt{t-1}}.$$

These equalities imply that

$$t = u^2 + 1, \qquad dt = 2\sqrt{t-1}du = 2udu.$$

[1]Recall the fundamental theorem of calculus for single-variable functions: Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Define

$$F(x) := \int_a^x f(t)dt. \tag{13.2.1}$$

Then (i) F is uniformly continuous on $[a, b]$; and (ii) F is differentiable on (a, b), with $F'(x) = f(x)$.

If f is (Riemann) integrable but not necessarily continuous, then the function F defined in (13.2.1) is continuous on $[a, b]$ but not necessarily differentiable.

Thus

$$\begin{aligned}\int t\sqrt{t-1}dt &= \int (u^2+1)u(2u)du \\ &= 2\int \left(u^4+u^2\right)du \\ &= \frac{2}{5}u^5+\frac{2}{3}u^3+C_2 \\ &= \frac{2}{5}(t-1)^{\frac{5}{2}}+\frac{2}{3}(t-1)^{\frac{3}{2}}+C_2.\end{aligned}$$

To compute the integral of the third component, we can use integration by parts:

$$\int t\sin(\pi t)dt = -\frac{1}{\pi}t\cos(\pi t)+\frac{1}{\pi}\int\cos(\pi t)dt = -\frac{1}{\pi}t\cos(\pi t)+\frac{1}{\pi^2}\sin(\pi t)+C_3.$$

Combining these results, we conclude that

$$\begin{aligned}&\int\left(t^2\mathbf{i}+t\sqrt{t-1}\mathbf{j}+t\sin(\pi t)\mathbf{k}\right)dt \\ &\quad= \left(\frac{1}{3}t^3+C_1, \frac{2}{5}(t-1)^{\frac{5}{2}}+\frac{2}{3}(t-1)^{\frac{3}{2}}+C_2, -\frac{1}{\pi}t\cos(\pi t)+\frac{1}{\pi^2}\sin(\pi t)+C_3\right).\end{aligned}$$

13.3 Arc Length, Curvature, TNB Frame

Key Ideas

- Arc length of a curve (in $\mathbf{R}^2, \mathbf{R}^n$)
- Arc-length function
- How to reparametrize a curve (e.g., with respect to arc length)
- Curvature (definition, how to compute in $\mathbf{R}^3$)
- Normal, binormal vectors
- TNB frame

(split this section in two, possibly three: arc length, curvature, TNB frame)

13.3.1 Arc Length in $\mathbf{R}^2$

Recall how we defined the arc length of a smooth curve C in $\mathbf{R}^2$ (see Stewart (2008), Section 10.2). Geometrically, the idea is to approximate the arc length by inscribed polygonal paths. *(draw picture)* (Mention application to GPS tracking?) The length of each linear piece of the polygonal path, say between points P_{i-1} and P_i, is given by the distance formula (in $\mathbf{R}^2$):

$$d(P_{i-1}, P_i) = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

Algebraically, we find a function $y = f(x)$ or a pair of parametric equations $x = r_1(t), y = r_2(t)$ whose image is the smooth curve C. Note that we can always express a function $y = f(x)$ as the pair of parametric equations $x = t, y = f(x) = f(t)$. Thus it suffices to consider the case of parametric equations. Because C is a smooth curve, we can take the functions r_1, r_2 to be smooth. *(be more precise but not pedantic about this point)* The arc length of C from P to Q corresponds to an interval $[a, b]$ of values for the parameter t, where $P = (r_1(a), r_2(a))$ and $Q = (r_1(b), r_2(b))$. The polygonal approximation to C is obtained by subdividing the interval $[a, b]$, say into d subintervals of equal length $\Delta t := \frac{b-a}{d}$. We then choose a point in each subinterval — say the right-most point, t_i — and look at its image point $P_i := (x_i, y_i) := (r_1(t_i), r_2(t_i))$ on the curve C. The length of the curve C can then be approximated by summing the distance between successive points, i.e.

$$\sum_{i=1}^{d} d(P_{i-1}, P_i) = \sum_{i=1}^{d} \sqrt{(r_1(t_i) - r_1(t_{i-1}))^2 + (r_2(t_i) - r_2(t_i))^2}. \qquad (13.3.1)$$

Because r_1, r_2 are differentiable, the mean-value theorem (see Stewart (2008), Section 4.2) implies that there exist values $t^*_{i,1}, t^*_{i,2} \in (t_{i-1}, t_i)$ such that

$$r_1'(t^*_{i,1}) = \frac{r_1(t_i) - r_1(t_{i-1})}{t_i - t_{i-1}}, \qquad r_2'(t^*_{i,2}) = \frac{r_2(t_i) - r_2(t_{i-1})}{t_i - t_{i-1}}.$$

Note that the numerators of the right side of these expressions are precisely the incremental changes in the distance formula (13.3.1). Solving for the numerators, letting $\Delta t_i := t_i - t_{i-1}$, and substituting the result into (13.3.1), we obtain

$$\sum_{i=1}^{d} d(P_{i-1}, P_i) = \sum_{i=1}^{d} \sqrt{(r_1'(t^*_{i,1}))^2 + (r_2'(t^*_{i,2}))^2}\Delta t_i.$$

One can show that, because r_1', r_2' are continuous, the fact that $t^*_{i,1} \neq t^*_{i,2}$ does not affect the limit of the sum as $d \to \infty$, so we obtain

$$L = \lim_{d\to\infty} \sum_{i=1}^{d} \sqrt{\left(r_1'(t^*_{i,1})\right)^2 + \left(r_2'(t^*_{i,2})\right)^2}\Delta t_i = \int_a^b \sqrt{\left(r_1'(t)\right)^2 + \left(r_2'(t)\right)^2}dt. \quad (13.3.2)$$

One can check that in the case of a curve C defined by a function $y = f(x)$, taking the parametric equations

$$x = r_1(t) =: t, \qquad y = r_2(t) =: f(t)$$

yields the familiar formula

$$L = \int_a^b \sqrt{1 + (f'(t))^2}dt,$$

as expected (see Stewart (2008), Section 8.1).

If the curve C is piecewise smooth, then just compute the length of each smooth piece, and add up the results. *(flesh out; draw picture; give example)*

(mention somewhere the caveat about the curve being traversed once for $t \in [a, b]$; distinction between curve C and parametrization of that curve)

Catchphrase

Estimate arc length using polygonal paths. (Computing the length of line segments is easy!)

13.3.2 Arc Length in $\mathbf{R}^n$

Let's look at the big picture of what we did to compute arc length of a smooth curve in $\mathbf{R}^2$. We started with a (potentially complicated) curve, we "linearized" the problem by considering polygonal paths, and we realized that in computing the length of each linear piece, the change in the x-variable and the change in the y-variable were separate *(be more precise?)*, so we could apply single-variable calculus (namely, the mean-value theorem) to each variable separately.

If we think about it, we realize that the fact that we were working in $\mathbf{R}^2$ was not important — this method generalizes to computing arc lengths of curves in $\mathbf{R}^n$. The resulting formula is the generalization of (13.3.2), i.e.

$$L = \int_a^b \sqrt{\left(r_1'(t)\right)^2 + \ldots + (r_n'(t))^2}dt = \int_a^b \left\|\mathbf{r}'(t)\right\| dt,$$

where in the final equality we let $\mathbf{r}(t) := (r_1(t), \ldots, r_n(t))$ and use the definition of norm on $\mathbf{R}^n$.

After this appetizer of theory, let's have a side of example.

Example 13.13 (S6eQ13.03.12)**.** Find the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

Solution: First let's get a geometric picture of the curve we're dealing with. The cylinder $4x^2 + y^2 = 4$ is an elliptical cylinder with central axis the z-axis and elliptical cross-sections parallel to the xy-plane. *(draw picture)* The plane $x+y+z = 2$ has normal vector $\mathbf{n} := (1, 1, 1)$. Viewing x, y as free parameters for the plane, we can easily specify a few points on the curve C of intersection: The pairs $(x, y) = (\pm 1, 0)$ and $(x, y) = (0, \pm 2)$ lie on the ellipse in the xy-plane, so the equation of the plane gives us the corresponding z-values:

$$(0, 2, 0), \qquad (0, -2, 4), \qquad (1, 0, 1), \qquad (-1, 0, 3).$$

This allows us to sketch the curve C of intersection.

All points of C lie on the cylinder, and we know how a parametrization of the ellipse:

$$x = r_1(t) := \cos t, \qquad y = r_2(t) := 2\sin t.$$

You should check that this is a valid parametrization of the cylinder. (Substitute these functions into the given equation for the cylinder.) Solving the equation for the plane for z gives us the corresponding parametric equation for z:

$$z = 2 - x - y = 2 - \cos t - 2\sin t =: r_3(t).$$

This gives us the necessary parametric equations to compute the arc length of C, but we still need to determine parameter values corresponding to traversing the curve C once. Well, $t \in [0, 2\pi]$ will do. We compute

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (2\cos t)^2 + (\sin t - 2\cos t)^2}dt \\ &= \int_0^{2\pi} \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\cos t}dt \\ &\approx 13.5191. \end{aligned}$$

As a check, let's find lower and upper bounds for the arc length.

Lower bound: The ellipse $4x^2 + y^2 = 4$ in the xy-plane has semimajor and semiminor axes of length 1 and 2, respectively. Thus its area equals

$$\pi(1)(2) = 2\pi \approx 6.2832.$$

The length of C should be greater than this, which it is.

The image of the rectangle tangent to the ellipse in the xy-plane with sides parallel to the x- and y-axes intersects the ellipse at $(\pm 1, 0), (0, \pm 2)$, and hence has vertices $(1,2), (-1,2), (-1,-2), (1,-2)$. The images of these points in the given plane can be found by substituting these values for (x, y) into $z = 2 - x - y$:

$$(1,2,-1), \qquad (-1,2,1), \qquad (-1,-2,5), \qquad (1,-2,3).$$

Using the distance formula, we can compute the side lengths of the rectangle in the plane defined by these four points. (We can simplify our work by noting that (i) one of the three components cancels in each distance computation, and (ii) because the figure is a rectangle, we only have to compute two (adjacent) sides.) Summing the four results, we obtain

$$2\left(2\sqrt{2} + 4\sqrt{2}\right) = 12\sqrt{2} \approx 16.9706.$$

The length of C should be less than this, which it is.

13.3.3 Arc Length and Parametrization

We have seen previously that the parametrization of a curve is not unique. When we defined the arc length of a curve above, we defined it with respect to a particular parametrization. This raises the natural question: If we choose a different parametrization, do we get a different arc length?

Let's think for a second what the answer to this question should be. Second's up. The answer is that we had better definitely not get a different arc length! The curve is a geometric object that exists independent of its parametrization. In particular, we can imagine lying a tape measure along a curve in $\mathbf{R}^3$ to measure it — arc length is independent of parametrization! Let's prove this.

(give proof (see Marsden & Tromba))

Definition 13.14. Let $C \subseteq \mathbf{R}^n$ be a smooth curve with a smooth parametrization

$$\begin{aligned} \mathbf{r} : I &\to \mathbf{R}^n \\ t &\mapsto (r_1(t), \ldots, r_n(t)). \end{aligned}$$

Fix a point $a \in I$. The **arc-length function** of C with base point $P_0 := \mathbf{r}(a)$ is

$$s(t) := \int_a^t \left\| \mathbf{r}'(u) \right\| du = \int_a^t \sqrt{\left(r_1'(u)\right)^2 + \ldots + \left(r_n'(u)\right)^2} du. \tag{13.3.3}$$

(take care about curve C being traversed only once in I?)

The arc-length function s measures the length of the curve from the fixed point $\mathbf{r}(a)$ to the point $\mathbf{r}(t)$.

Differentiating both sides of (13.3.3) with respect to t and using the fundamental theorem of calculus (note that (13.3.3) is a scalar equation), we obtain the important relation

$$\frac{ds}{dt} = s'(t) = \left\| \mathbf{r}'(t) \right\| . \tag{13.3.4}$$

Catchphrase

The norm of the tangent vector equals the derivative of the arc-length function.

As noted above, the arc length of a curve is an intrinsic property of the curve, independent of the choice of coordinate system and parametrization of the curve. Thus it is useful to parametrize a curve with respect to its arc length. Practically, we do this as follows:

1. Start with a parametrization $\mathbf{r}(t)$ of the curve.
2. Compute the arc-length function $s(t)$ explicitly as a function of the parameter t.
3. Solve for t as a function of s.
4. Substitute this result into the original parametrization.

Let's see an example.

Example 13.15 (S6eQ13.03.16)**.** Reparametrize the curve

$$\mathbf{r}(t) := \left(\frac{2}{t^2+1} - 1 \right) \mathbf{i} + \frac{2t}{t^2+1} \mathbf{j}$$

with respect to arc length measured from the point $(1,0)$ in the direction of increasing t. Express the reparametrization in its simplest form. What can you conclude about the curve?

Solution: The point $(1,0)$ corresponds to the parameter value $t = 0$. (Do you see why?) We compute

$$\begin{aligned} \left\| \mathbf{r}'(u) \right\| &= \sqrt{\left(-\frac{4u}{(u^2+1)^2} \right)^2 + \left(\frac{-2u^2+2}{(u^2+1)^2} \right)^2} \\ &= \frac{1}{(u^2+1)^2} \sqrt{16u^2 + 4u^4 - 8u^2 + 4} \\ &= \frac{1}{(u^2+1)^2} \sqrt{4(u^4 + 2u^2 + 1)} \\ &= \frac{2}{u^2+1}. \end{aligned}$$

Thus the arc-length function is

$$\begin{aligned} s(t) &= \int_0^t \|\mathbf{r}'(u)\| \, du = 2\int_0^t \frac{1}{u^2+1} du = \int_{u=0}^{u=t} \cos^2\theta \sec^2\theta d\theta \\ &= [\theta]_{u=0}^{u=t} = [\arctan u]_{u=0}^{u=t} = \arctan t, \end{aligned}$$

where to perform the integration we have used the trigonometric substitution $\tan\theta = u$. We can solve for the parameter t as a function of the arc length s by taking the tangent of both sides:

$$t = \tan s.$$

Substituting this into the original parametrization, we obtain

$$\begin{aligned} \mathbf{r}(s) &= \left(\frac{2}{(\tan s)^2+1} - 1\right)\mathbf{i} + \frac{2\tan s}{(\tan s)^2+1}\mathbf{j} \\ &= \left(\frac{2}{\sec^2 s} - 1\right)\mathbf{i} + \frac{2\tan s}{\sec^2 s}\mathbf{j} \\ &= \left(2\cos^2(s) - 1\right)\mathbf{i} + \left(2\sin s\cos s\right)\mathbf{j} \\ &= \cos(2s)\mathbf{i} + \sin(2s)\mathbf{j}. \end{aligned}$$

Note that for all values $s \in \mathbf{R}$,

$$\|\mathbf{r}(s)\| = \sqrt{\cos^2(2s) + \sin^2(2s)} = 1.$$

From this we conclude that the curve in $\mathbf{R}^2$ determined by $\mathbf{r}$ lies on the unit circle.

13.3.4 Curvature

(relocate this definition to Section 2, and make "smooth parametrization" a special case of "smooth vector-valued function")

Definition 13.16. A vector-valued function $\mathbf{r} : U \subseteq \mathbf{R} \to \mathbf{R}^n$ is **smooth** if

1. $\mathbf{r}'(t)$ exists and is continuous at all points in the domain U; and
2. for all $t \in U$, $\mathbf{r}'(t) \neq \mathbf{0}$.

Recall that a vector-valued function $\mathbf{r} : U \subseteq \mathbf{R} \to \mathbf{R}^n$ is **smooth** if (i) $\mathbf{r}'(t)$ exists and is continuous at all points in the domain U, and (ii) for all $t \in U$, $\mathbf{r}'(t) \neq \mathbf{0}$. A parametrization $\mathbf{r}(t)$ of a curve is a vector-valued function, so this definition gives the notion of a **smooth parametrization**. A curve is **smooth** if it has a smooth parametrization. (Note that a smooth curve has nonsmooth parametrizations! *(elaborate and move to remark)*) Geometrically, a smooth curve is a curve with no corners or cusps *(or internal stopping points)* — the tangent vector always changes continuously.

We would like to develop a notion of "curvature" of a curve. Geometrically, it is clear what curvature should be: a measure of how "curved" the curve is at a given point. If we consider a smooth parametrization of a smooth curve C, and the unit tangent vectors along the curve, we observe that the unit tangent vectors change direction quickly when the curve is relatively curved, and less quickly when the curve is relatively straight. This suggests that curvature is related to the rate of change of the unit tangent vectors. It turns out the way to formalize this is to define curvature as the rate of change of the unit tangent vectors with respect to arc length, so that curvature is independent of the parametrization.

Definition 13.17. Let C be a smooth curve, let $\mathbf{r}(s)$ be a parametrization of C with respect to arc length, and let $\mathbf{T}(s) := \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|}$ be the corresponding unit tangent vector function. The **curvature** of C at the point $\mathbf{r}(s_0)$ is

$$\kappa(s_0) := \left\| \frac{d\mathbf{T}}{ds}(s_0) \right\|.$$

Catchphrase

Curvature equals the norm of the rate of change of the unit tangent vector, when the curve is parametrized with respect to arc length.

We can use the chain rule *(add ref)* to express curvature in terms of the parameter t rather than the arc-length function s. More precisely, viewing the arc-length function s as a function of t, we have

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt},$$

so

$$\kappa(t) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}/dt}{ds/dt} \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}, \tag{13.3.5}$$

where in the final equality we have used the properties of norm (note that $\frac{ds}{dt}$ is a scalar-valued function) and the fact that $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$ (see (13.3.4)).

(example of circles; straight line as circle of infinite radius)

13.3.5 Computing Curvature in $\mathbf{R}^3$

The following result provides a useful way to compute the curvature of space curves (i.e. curves in $\mathbf{R}^3$).

Theorem 13.18. *Let C be a smooth curve, and let $\mathbf{r}(t)$ be a smooth parametrization of C. Then*

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \tag{13.3.6}$$

(remark about dropping parameter t for notational ease)

Proof. By definition of the unit tangent vector and result (13.3.4),

$$\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}, \qquad \|\mathbf{r}'\| = \frac{ds}{dt} = s'.$$

By hypothesis, $\mathbf{r}(t)$ is a smooth parametrization, so in particular for all values of t, $\mathbf{r}'(t) \neq 0$. Thus we can solve the equation defining the unit tangent vector for $\mathbf{r}'$:

$$\mathbf{r}' = \|\mathbf{r}'\| \, \mathbf{T} = s'\mathbf{T}.$$

Differentiating with respect to t (using the product rule on the right), we have

$$\mathbf{r}'' = s''\mathbf{T} + s'\mathbf{T}'.$$

Taking the cross product of these two expressions, we have

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= \left(s'\mathbf{T}\right) \times \left(s''\mathbf{T} + s'\mathbf{T}'\right) \\ &= s's'' \left(\mathbf{T} \times \mathbf{T}\right) + (s')^2 \left(\mathbf{T} \times \mathbf{T}'\right) \\ &= (s')^2 \left(\mathbf{T} \times \mathbf{T}'\right), \end{aligned} \tag{13.3.7}$$

where in the final equality we have used the fact that the cross product of any vector with itself equals $\mathbf{0}$. By definition, the unit tangent vector $\mathbf{T}$ always has constant norm equal to 1. Thus Lemma 13.11 implies that $\mathbf{T}$ is orthogonal to $\mathbf{T}'$. Thus

$$\|\mathbf{T} \times \mathbf{T}'\| = \|\mathbf{T}\| \, \|\mathbf{T}'\| \sin\left(\frac{\pi}{2}\right) = \|\mathbf{T}'\|.$$

Taking the norm of (13.3.7) and substituting in this result, we have

$$\|\mathbf{r}' \times \mathbf{r}''\| = \left(s'\right)^2 \|\mathbf{T}'\|.$$

Solving for $\|\mathbf{T}'\|$ and using the fact that $s' = \|\mathbf{r}'\|$,

$$\|\mathbf{T}'\| = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^2}.$$

Substituting this result into (13.3.5), we conclude that

$$\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}.$$

□

Consider the special case of a plane curve C given by $y = f(x)$ in the xy-plane. In this case, we can parametrize C (viewed as a curve in $\mathbf{R}^3$) by

$$\mathbf{r}(x) := (x, f(x), 0).$$

Thus

$$\mathbf{r}'(t) = \big(1, f'(x), 0\big)\,, \qquad\qquad \mathbf{r}''(t) = \big(0, f''(x), 0\big)\,,$$

so

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{pmatrix} = \big(0, 0, f''(x)\big)\,.$$

Hence by (13.3.6),

$$\kappa(x) = \frac{\|\mathbf{r}'(x) \times \mathbf{r}''(x)\|}{\|\mathbf{r}'(x)\|^3} = \frac{|f''(x)|}{\Big(1 + (f'(x))^2\Big)^{\frac{3}{2}}}.$$

13.3.6 Normal and Binormal Vectors

It is useful to have a standard way to put coordinate axes on points of a smooth curve in $\mathbf{R}^3$, in such a way that as we move along the curve, these coordinate axes rotate continuously. This leads us to develop the so-called **TNB frame** (also called the **Frenet–Serret frame** after its codiscoverers).

Our starting point is to note that every oriented curve comes with one natural direction at each point P of the curve: the tangent direction, given by the tangent vector at P. We can then look at the plane through the point P whose normal vector is the tangent vector at P; this plane is normal to the curve. If we're careful with our choice of tangent vector, then in fact we can obtain a standard normal vector. The tangent vector and normal vectors are orthogonal, so their cross product yields a third orthogonal direction, completing the construction of coordinate axes at P. We make this construction precise in what follows.

Let C be a smooth curve, and let $\mathbf{r}(t)$ of that curve be a smooth parametrization of C. Because $\mathbf{r}$ is smooth, for each parameter value t we have $\mathbf{r}'(t) \neq \mathbf{0}$. Therefore we can define a unit tangent vector $\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ at each point $\mathbf{r}(t)$ of the curve C. By definition, for each t we have $\|\mathbf{T}(t)\| = 1$. Thus Lemma 13.11 implies that for each t, $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. In general $\mathbf{T}'(t)$ is not a unit vector, but if $\mathbf{r}'$ is smooth then we can normalize $\mathbf{T}'(t)$:

$$\mathbf{N}(t) := \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

This unit vector $\mathbf{N}(t)$ is called the **unit normal vector** to C at $\mathbf{r}(t)$. By construction, $\mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$ (why?). Taking their cross product, we obtain a third unit vector orthogonal to $\mathbf{T}(t)$ and $\mathbf{N}(t)$, called the **binormal vector**:

$$\mathbf{B}(t) := \mathbf{T}(t) \times \mathbf{N}(t).$$

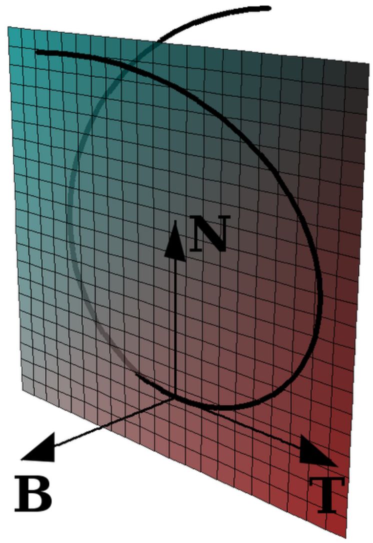

Figure 13.3.1: Illustration of the **TNB**-frame and the osculating plane at a point on a curve. Implicitly, the curve is parametrized in the right-hand direction (do you see why?). (Image obtained from Wikipedia.)

(Justify quickly to yourself why $\mathbf{B}(t)$ has these two properties.) At each point $\mathbf{r}(t)$ of C, the three vectors $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)$ give us analogs to the standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. See Figure 13.3.1. *(further comment about continuously varying)*

The plane determined by the unit normal vector $\mathbf{N}(t)$ and the binormal vector $\mathbf{B}(t)$ is called the **normal plane** to C at $\mathbf{r}(t)$. The normal plane contains all lines normal to the (unit) tangent vector $\mathbf{T}(t)$ of C at $\mathbf{r}(t)$.

The plane determined by the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ is called the **osculating plane** to C at $\mathbf{r}(t)$. The osculating plane is the best planar approximation to the curve C at $\mathbf{r}(t)$.

(mention osculating circle; not critical, good for culture)

(define torsion (see Exercise 13.03.53); give Frenet-Serret formulas (see Exercise 13.03.54))

Example 13.19 (S6eQ13.03.46)**.** Consider the curve C defined by the parametric equations

$$x := t, \qquad y := t^2, \qquad z = t^3,$$

called the **twisted cubic curve**. Find the vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ at the point $(1, 1, 1)$. Then give equations of the normal plane and osculating plane at this point.

Solution: Writing the parametric equations in vector form,

$$\mathbf{r}(t) := \left(t, t^2, t^3\right).$$

The point $(1, 1, 1)$ corresponds to the parameter value $t = 1$. (This can be read off the first component of $\mathbf{r}(t)$.) The tangent vector to C at $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \left(1, 2t, 3t^2\right),$$

so the unit tangent vector at $\mathbf{r}(t)$ is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\left(1, 2t, 3t^2\right)}{\sqrt{1 + 4t^2 + 9t^4}}.$$

In particular, the unit tangent vector to C at $\mathbf{r}(1)$ is

$$\mathbf{T}(1) = \frac{(1, 2, 3)}{\sqrt{14}}.$$

(COMPLETE THIS EXAMPLE! if too painful computationally, find simpler one — but twisted cubic is cool) To compute the unit normal vector to C, we first compute

$$\mathbf{T}'(t)$$

Thus the unit normal vector to C at $\mathbf{r}(t)$ is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

13.4 Velocity and Acceleration

Key Ideas

- Definitions: velocity, speed, acceleration
- Newton's second law of motion (vector equation)
- How to resolve a vector into tangential and normal components

In this section we consider the case that the vector-valued function $\mathbf{r}(t)$ describes the position of a moving object, e.g., a particle, over time. With this view, we can interpret the first and second derivatives of $\mathbf{r}(t)$ as the velocity and acceleration of the object. We can then apply the methods of vector calculus that we have been developing to tackle questions in the branch of physics known as classical mechanics.

Throughout this section, we let $\mathbf{r} : I \subseteq \mathbf{R} \to \mathbf{R}^n$ be a vector-valued function whose component functions have continuous first derivative. We interpret $\mathbf{r}(t)$ as the position vector of an object at time t.

13.4.1 Velocity, Speed, and Acceleration

Definition 13.20. Consider a moving object.

(i) Its **velocity** is the instantaneous rate of change of its position with respect to time.

(ii) Its **speed** is the magnitude of its velocity vector.

(iii) Its **acceleration** is the instantaneous rate of change of its velocity with respect to time.

The velocity $\mathbf{v}(t)$ of the object at time t is *(better motivation)*

$$\mathbf{v}(t) := \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t).$$

The speed $v(t)$ of the object at time t is

$$v(t) := \|\mathbf{v}(t)\| = \left\|\mathbf{r}'(t)\right\| = \frac{ds}{dt},$$

where $s(t)$ denotes the arc-length function. This last expression says that speed equals the rate of change of distance (i.e. s) with respect to time — precisely how we think of speed.

The acceleration $\mathbf{a}(t)$ of the object at time t is

$$\mathbf{a}(t) := \mathbf{v}'(t) = \mathbf{r}''(t).$$

We can relate the position $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$ of an object using the fact that differentiation and integration of vector-valued functions is done componentwise.

Example 13.21 (S6eQ13.04.28)**.** A batter hits a baseball 3 feet above the ground toward the center-field fence, which is 10 feet high and 400 feet from home plate. The ball leaves the bat with a speed of 115 feet/second at an angle of 50° above the horizontal. Is it a home run (in other words, does the ball clear the fence)? *Hint:* Assume that the only force experienced by the ball is the force of gravitation. Recall that the acceleration due to gravity is approximately -32.17 feet/second^2.

Solution: Overlay the coordinate axes of $\mathbf{R}^2$ on this situation so that home plate is at the origin, the line from home plate to the center field fence coincides with the positive x-axis, and the "upward" direction coincides with the positive y-axis.[2]

What information do we want to tease out of this situation? We want to know the height of the ball when it has travelled 400 feet. That is, we want to know the y-component of the position vector of the ball when its x-component is 400. However, what we know is the acceleration of the ball and its initial velocity. We will have to deduce the position vector from these.

By assumption, the only force experienced by the ball is the force of gravitation, so the acceleration (in ft/s^2) of the ball at any time t (while the ball is still in the air) is

$$\mathbf{a}(t) = (0, -32.17).$$

Integrating the acceleration with respect to time gives the velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = \left(c_1', -32.17t + c_2'\right), \tag{13.4.1}$$

where the c_i' are constants. The given information and our choice of coordinates imply that the initial velocity vector (in ft/s, immediately after the ball leaves the bat) is

$$\mathbf{v}(0) = (115\cos 50°, 115\sin 50°) \approx (73.9206, 88.0951).$$

Comparing this with (13.4.1) evaluated at $t = 0$, we find

$$c_1' = 115\cos 50°, \qquad c_2' = 115\sin 50°,$$

[2]We could have overlaid the coordinate axes of $\mathbf{R}^3$ on the situation, which might be our first inclination given that baseball is played in space (not in a plane). However, for the purposes of this exercise, we only care about two directions: the horizontal direction of the ball, and the vertical direction (i.e. the height of the ball above the ground). Thus we focus on the plane defined by these two directions. Had we worked in $\mathbf{R}^3$, we effectively would be keeping track of a third, irrelevant direction whose component in the acceleration, velocity, and position vectors are all zero (do you see why?).

so (13.4.1) writes as

$$\mathbf{v}(t) = \langle 115\cos 50^\circ, -32.17t + 115\sin 50^\circ \rangle.$$

Integrating the velocity with respect to time gives the position:

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \left\langle (115\cos 50^\circ)t + c_1, -\frac{32.17}{2}t^2 + (115\sin 50^\circ)t + c_2 \right\rangle. \quad (13.4.2)$$

The given information and our choice of coordinates imply that the initial position vector (in ft) is

$$\mathbf{r}(0) = \langle 0, 0 \rangle.$$

Comparing this with (13.4.2), we find

$$c_1 = 0, \qquad c_2 = 0,$$

so (13.4.2) writes as

$$\mathbf{r}(t) = \left\langle (115\cos 50^\circ)t, -\frac{32.17}{2}t^2 + (115\sin 50^\circ t) \right\rangle.$$

The x-coordinate of the position vector equals 400 when

$$(115\cos 50^\circ \text{ ft/s})t = 400 \text{ ft} \qquad \Leftrightarrow \qquad t = \frac{400 \text{ ft}}{115\cos 50^\circ \text{ ft/s}} \approx 4.5405 \text{ s}.$$

The y-coordinate at this time is

$$y(4.5405) = -\frac{32.17}{2} \text{ ft/s}^2 (4.5405 \text{ s})^2 + (115\sin 50^\circ \text{ ft/s})(4.5405 \text{ s}) \approx 68.3852 \text{ ft}.$$

We conclude that the ball clears the center-field fence with room to spare. Ho-o-ome run!

13.4.2 Newton's Second Law of Motion

Newton's second law of motion states that a force acting on an object produces an acceleration in the direction of the force whose magnitude is proportional to the force. More precisely, if m is the mass of the object, then

$$\mathbf{F}(t) = m\mathbf{a}(t).$$

Example 13.22 (S6eQ13.04.20)**.** Find the force required so that a particle of mass m has the position function

$$\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}.$$

Solution: The given position function corresponds to the acceleration function

$$\mathbf{a}(t) = \langle 6t, 2, 6t \rangle ,$$

obtained from $\mathbf{r}(t)$ by differentiating twice with respect to t. By Newton's second law of motion, for a particle of mass m, this acceleration vector corresponds to a force vector given by

$$\mathbf{F}(t) = m\mathbf{a}(t) = \langle 6mt, 2m, 6mt \rangle .$$

13.4.3 Tangential and Normal Components

When studying the motion of an object, it is useful to resolve the acceleration of the object (and sometimes the force acting on the object) into two components, one in the direction of motion of the object (i.e. the tangent direction), and the residual (which necessarily points in a direction normal to the motion of the object). *(draw Figure 13.04.07:* $\mathbf{r}, \mathbf{a}, \mathbf{T}, \mathbf{N}$*, and resolved acceleration vectors; be very clear with the geometric picture)* To this end, recall that at any point we have the orthonormal TNB frame *(elaborate)*.

Let $\mathbf{r}$ be the position function of an object moving in space, and assume that $\mathbf{r}$ is smooth. Then the unit tangent vector is

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}}{v}, \qquad \text{so} \qquad \mathbf{v} = v\mathbf{T}.$$

Differentiating both sides with respect to t (using the product rule on the right),

$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'. \tag{13.4.3}$$

The first term on the right is a tangential component. Our goal is to write the second term, in particular $\mathbf{T}'$, in terms of tangential and normal components.

In our discussion of curvature in Section 13.3, the expression (13.3.5) for curvature in terms of a particular parametrization of the curve involved $\mathbf{T}'$:

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{T}'(t)\|}{v(t)}, \qquad \text{so} \qquad \left\|\mathbf{T}'\right\| = \kappa v.$$

By definition, the unit normal vector is $\mathbf{N} := \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$. Solving for $\mathbf{T}'$ and substituting in the preceding expression,

$$\mathbf{T}' = \left\|\mathbf{T}'\right\| \mathbf{N} = \kappa v \mathbf{N}.$$

Substituting this into (13.4.3), we obtain

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}.$$

This is the desired resolution of the acceleration vector $\mathbf{a}$ into tangential and normal components. For convenience, let us denote their magnitudes

$$a_T := v', \qquad\qquad a_N := \kappa v^2.$$

In practice we often have just the position function $\mathbf{r}$ of the object, so ideally we'd like to express the tangential and normal components of acceleration in terms of $\mathbf{r}$. For the normal component, we can use the relation (13.3.6) and the definition of v to write

$$a_N = \kappa v^2 = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} \|\mathbf{r}'\|^2 = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|}.$$

For the tangential component, recall that (i) $\mathbf{v} = v\mathbf{T}$, and (ii) $\mathbf{T}, \mathbf{N}$ are orthogonal. Therefore

$$\mathbf{v} \cdot \mathbf{a} = (v\mathbf{T}) \cdot \left(v'\mathbf{T} + \kappa v^2 \mathbf{N}\right) = \left(vv'\right) \mathbf{T} \cdot \mathbf{T} + \left(\kappa v^3\right) \mathbf{T} \cdot \mathbf{N} = vv'.$$

Solving this equation for $a_T = v'$,

$$a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|}.$$

Example 13.23 (S6eQ13.04.36)**.** Let

$$\mathbf{r}(t) := t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}.$$

Find the tangential and normal components of the acceleration vector.

Solution: We have

$$\mathbf{r}(t) = \left(t, t^2, 3t\right), \qquad \mathbf{r}'(t) = (1, 2t, 3), \qquad \mathbf{r}''(t) = (0, 2, 0).$$

Thus $\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 10}$, and

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{4t}{\sqrt{4t^2 + 10}}$$
$$a_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|(-6, 0, 2)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{40}}{\sqrt{4t^2 + 10}}.$$

For example, when $t = 0$, we have

$$\mathbf{r}(0) = (0, 0, 0), \qquad \mathbf{a}(0) = \mathbf{r}''(0) = (0, 2, 0), \qquad a_T = 0, \qquad a_N = 2.$$

A plot of the position function $\mathbf{r}$ is shown in Figure 13.4.1.

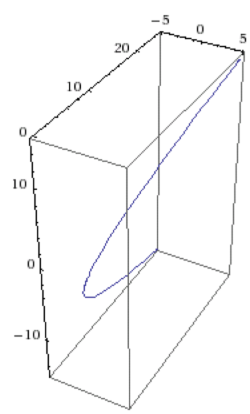

Figure 13.4.1: Plot of the position function **r** in Example 13.23.

Chapter 14

Partial Derivatives

In Chapter 13 we studied vector-valued functions defined on (subsets of) $\mathbf{R}$, i.e. functions that take as input a scalar (real number) and output a vector. In this chapter, we study functions that take as input a vector and output a scalar.

(discuss real-world applications of functions of several variables, i.e. motivate for the applied)

We begin with a general discussion of functions $f : \mathbf{R}^n \to \mathbf{R}$, their graphs, and their level sets. (Recall our discussion of graphs and level sets of surfaces in $\mathbf{R}^3$ in Section 12.6.) We then discuss limits of such functions. Limits of functions of several variables are more subtle than limits of functions of a single variable: In $\mathbf{R}$, $x \to a$ either from the left or from the right, but in $\mathbf{R}^n$ ($n > 1$), there are many possible paths of approach. *(clean up this discussion; make more accurate and more general)*

As with vector-valued functions, once we have defined a suitable notion of limit for functions of multiple variables, we can then define continuity, derivatives, and integrals of these functions. We investigate continuity and derivatives in this chapter; integrals are developed in Chapter 15.

The increased degrees of freedom in moving around in the domain enriches the possibilities for derivatives. Partial derivatives, i.e. derivatives in the directions of the coordinate axes, are discussed in Section 14.3. Partial derivatives are a special case of directional derivatives and are discussed in Section 14.6.

An important application of the differential calculus for functions of several variables is optimization, i.e. finding minima and maxima points of such a function. We develop optimization techniques in Sections 14.7 and 14.8.

14.1 Functions of Several Variables

Key Ideas

- Linear function
- Graph of a function of n variables (definition, how to visualize)
- Level sets

14.1.1 Functions of Several Variables

Recall (Definition 12.3) that a **function** is a rule of assignment from a specified domain to a specified codomain.

A **real-valued function of n variables** is a function from a subset $D \subseteq \mathbf{R}^n$ (the domain) to a subset $E \subseteq \mathbf{R}$ (the codomain).

The input variables are called **independent variables**. The output variables are called **dependent variables**, because their value depends on the input.

We adopt the convention discussed in Remark 13.1: Unless otherwise noted, the domain of a function will be the largest subset on which its rule of assignment is defined.

Example 14.1 (S6eQ14.01.14)**.** State (algebraically) and graph the maximum domain of definition for the rule of assignment

$$f(x, y) := \sqrt{y - x} \ln(y + x).$$

Solution: The square root function is defined for all input values greater than or equal to 0. The natural log function is defined for all input values greater than 0. Thus, $f(x, y)$ is defined for all $(x, y) \in \mathbf{R}^2$ such that

$$y - x \geqslant 0 \qquad \text{and} \qquad y + x > 0,$$

i.e.

$$y \geqslant x \qquad \text{and} \qquad y > -x.$$

Thus the maximum domain of f is

$$D := \left\{(x, y) \in \mathbf{R}^2 \,|\, y \geqslant x, y > -x\right\}.$$

A plot D in the xy-plane is shown in Figure 14.1.1 *(make better graph)*.

14.1.2 Graphs

Definition 14.2. Let $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ be a real-valued function of n variables. The **graph** of f is the set

$$\left\{(x_1, \dots, x_n, z) \in \mathbf{R}^{n+1} \,|\, (x_1, \dots, x_n, z) \in D, z = f(x_1, \dots, x_n)\right\}.$$

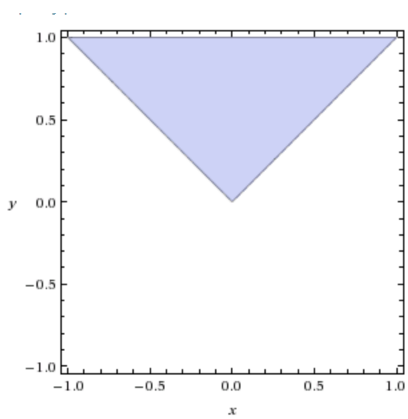

Figure 14.1.1: Maximum domain of definition for the function $f(x, y)$ in Example 14.1.

In the particular case $n = 1$, we have $f : D \subseteq \mathbf{R} \to \mathbf{R}$. Write the input and output variables as $y = f(x)$. In this case, the graph of f is a subset of the xy-plane, i.e. $\mathbf{R}^2$. To each input value x in the domain D we associate the (unique) point $(x, f(x))$. If f is continuous and D is connected, then the graph of f is a curve in $\mathbf{R}^2$, as is familiar from single-variable calculus.

In the particular case $n = 2$, we have $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$. Write the input and output variables as $z = f(x, y)$. In this case, the graph of f is a subset of xyz-space, i.e. $\mathbf{R}^3$. To each pair of input values (x, y) in the domain D we associate the (unique) point $(x, y, f(x, y))$. If f is continuous and D is connected, then the graph of f is a surface in $\mathbf{R}^3$.

We can visualize the graphs of functions $f : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R}^2 \to \mathbf{R}$. For $n \geqslant 3$, visualizing the graph of $f : \mathbf{R}^n \to \mathbf{R}$ directly becomes all but impossible. Techniques like plotting level sets or fixing all but one or two variables can help us visualize the graphs of these functions. Even when visualization is difficult, the algebraic techniques that we develop in this chapter apply to this general setting. This illustrates the power of the algebraic approach.

14.1.3 Linear Functions

In Section 13.3 we discussed the arc length of a continuous function $\mathbf{r} : \mathbf{R} \to \mathbf{R}^n$. The general shape of the curve corresponding to the graph of $\mathbf{r}$ can be complicated, so we estimated the arc length by inscribing line segments along the curve. Line segments are linear, so computing their length is easy; adding up the length of these inscribed line segments gave us an approximation to the length of the curve. In the limit as the number of line segments goes to infinity (and the length of any given line segment goes to zero), this approximation equals the length of the curve.

Catchphrase

Approximate general objects (hard) by linear objects (easy).

Arc length is one illustration of the power of linear approximation. We will see several more incarnations of this powerful technique *(cite explicitly? whet the palate)*. To this end, we discuss linear functions.

Definition 14.3. Let $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ be a real-valued function of n variables. f is **linear** if it has the form

$$f(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n$$

for some $a_1, \ldots, a_n \in \mathbf{R}$.

The general equation of a plane in $\mathbf{R}^3$ can be viewed as the level set of a linear function $f : \mathbf{R}^3 \to \mathbf{R}$. (Can you explain why?)

Remark 14.4. A linear function is linear with respect to its independent (i.e. input) variables. Confusion can arise when there are certain variables or "parameters" that are thought of as "fixed". For example, the function

$$f(x, y, z, t) := tx + t^2 y - e^t z$$

is linear with respect to the variables x, y, z, but not with respect to the variable t. For any fixed value of t, however, the function f is linear.

The graph of a linear function is a hyperplane. In the case $n = 2$, the equation for a linear function has the form

$$z = f(x, y) = ax + by + c,$$

so graph of a linear function has the form

$$ax + by - z + c = 0,$$

the familiar equation of a plane in $\mathbf{R}^3$. *(reconcile with above)*

14.1.4 Level Sets

Recall the concept of level sets, introduced in Section 12.6. Let $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ be a real-valued function of n variables. The **level set** of f with value c is the set of points in the domain of f such that the value of f at those points equals c:

$$LS_f(c) := \{(x_1, \ldots, x_n) \in D \mid f(x_1, \ldots, x_n) = c\}.$$

Note that level sets of f are traces of the graph of f in the plane $z = f(x_1, \ldots, x_n) = c$, projected onto the $x_1 \cdots x_n$-plane. *(clarify)*

Consider the case $n = 2$. In this case, the level sets of $f(x, y)$ are traces of the graph of f in the horizontal plane $z = c$, projected onto the xy-plane. We can reconstruct the graph of f by "lifting up" the level sets to their respective values c.

(topographic example — feel the level sets! hint at relation between closeness of level sets and steepness, i.e. rate of change)

(formal example in $\mathbf{R}^2, \mathbf{R}^3$; emphasize we've done this analysis before! ref particular section)

Example 14.5 (S6eQ14.01.42)**.** Draw a contour map of the function

$$f(x, y) := e^{\frac{y}{x}}.$$

Solution: The exponential function is defined at all real numbers, so the maximum domain of definition of f is all of $\mathbf{R}$. By definition, the general level set of f with value c is the set of points (x, y) such that

$$e^{\frac{x}{y}} = c. \tag{14.1.1}$$

The exponential function always outputs values strictly greater than 0 (for real input *(add footnote)*), so level sets of f for values $c \leqslant 0$ are empty. For $c > 0$, we can take the natural log of both sides of (14.1.1) and solve for y as a function of x:

$$y = \frac{1}{\ln c} x.$$

Thus the level sets of f with value $c > 0$ are lines in the xy-plane with slope $\frac{1}{\ln c}$.

(comment briefly on Stewart's "three views" of $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ (n variables, single "point variable", single "vector variable"); nothing scary)

14.2 Limits and Continuity

Key Ideas

- Limits of functions $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$
  - ε-δ definition
  - Infinite directions (paths)
  - How to compute (disprove and prove existence of limit)
- Continuity of functions $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$

14.2.1 Intuition

We encountered the notions of limit and continuity when we studied real-valued functions f of a single variable. In that setting, we saw that the limit

$$\lim_{x \to a} f(x)$$

could be interpreted geometrically as the function approaching the same value as x approaches (but never equals!) a from the left and from the right. We also saw that f is continuous at $x = a$ if and only if

$$\lim_{x \to a} f(x) = f(a),$$

i.e. if and only if (i) the limit of f exists at a, (ii) the function f is defined at a, and these two values (i) and (ii) are equal.

We would like to extend the notions of limit and continuity to real-valued functions of several variables. But even in the "simplest" extension — real-valued functions $f(x, y)$ of two variables — we encounter a difficulty with the geometric intuition provided by $f(x)$. Namely, when we go to define

$$\lim_{(x,y) \to (a,b)} f(x, y),$$

there aren't just two directions of $(x, y) \to (a, b)$ to consider; in $\mathbf{R}^2$, there are infinitely many! Even considering all straight-line directions to (a, b) doesn't yield a good definition of limit: There are (relatively simple) functions for which all straight-line paths to (a, b) yield the same limit, but still the limit fails to exist (see Example 14.02.03 in Stewart (2008)). It turns out that the correct notion is to consider all possible paths approaching the limit point (a, b) that stay in the domain of f.[1]

The paths-of-approach approach is useful to prove that a given limit of f does not exist: We can provide two paths along which f approaches different values.

[1]This last condition simply guarantees that f is defined along the path of approach.

This approach is less useful to prove that a given limit of f does exist. In this case, we must return to the ε-δ definition of limit (generalized to the case of real-valued functions of several variables), or use machinery like the squeeze theorem.

14.2.2 Limits

Definition 14.6. Let $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$, and let $(a, b) \in D$, and let D contain points distinct from (a, b) that are arbitrarily close to (a, b) *(simplify)*. If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 < d\left((x, y), (a, b)\right) < \delta \qquad \Rightarrow \qquad d(f(x, y), L) < \varepsilon,$$

then we say that the **limit** of $f(x, y)$ as (x, y) approaches (a, b) equals L.

Note that the condition

$$0 < d\left((x, y), (a, b)\right) < \delta$$

implies that $(x, y) \neq (a, b)$.

This definition generalizes immediately to higher dimensions, i.e. to functions $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$.

(write generalized definition)

The geometric picture in any dimension is the same: c is the limit of f as $x \to a$ if when we look at arbitrarily small neighborhoods of $c \in \mathbf{R}$ (i.e. points arbitrarily close to c), we can always find a deleted neighborhood around a in $D \subseteq \mathbf{R}^n$ not containing a (i.e. points sufficiently close to a but not equal to a), such that f maps every point of this deleted neighborhood of a to the corresponding neighborhood of c.

(draw a picture)

14.2.3 The "Correct" Definitions

The definitions of limit and continuity that we have given above make sense because we have a notion of distance in $\mathbf{R}^n$. That is, we can define (as we have done *in (add ref)*) a distance function (i.e. metric) d on $\mathbf{R}^n$. The space $\mathbf{R}^n$ equipped with the metric d is called a **metric space**.

It turns out that the notions of limit points and continuity hold more generally, even in spaces without a metric. To make sense of limit points and continuity in such spaces, the key concept is open sets. *(elaborate briefly; not too technical; give general flavor, encourage exploration (e.g., add hyperlinks))*

14.2.4 Computing Limits

In general, showing that a limit exists is relatively hard: We need to exhibit a $\delta > 0$ for every $\varepsilon > 0$ that does the trick in Definition *(add ref)*. Below we'll mention some results that make showing that a limit exists easier. The good news is that showing that a limit does not exist is relatively easy: We need only to exhibit two paths in the domain of f approaching the limit point that have different limits. The trick is finding these paths.

Catchphrase

If f approaches different limits along different paths leading to a point, then the limit of f does not exist at that point.

Example 14.7 (S6eQ14.02.18)**.** Find the following limit, or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}.$$

Solution: Let

$$f(x,y) := \frac{xy^4}{x^2+y^8}$$

be the function whose limit we are taking. Note that f is a rational function, i.e. the ratio of two polynomials. A rational function is defined everywhere where the denominator is nonzero. Thus f is defined at all points (x,y) such that

$$x^2+y^8 \neq 0.$$

For all $x, y \in \mathbf{R}$, $x^2 \geqslant 0$ and $y^8 \geqslant 0$, with equality if and only if $x = 0$ and $y = 0$, respectively. Therefore $x^2+y^8 = 0$ if and only if $(x,y) = (0,0)$. We conclude that the maximum domain of f is $\mathbf{R}^2 \setminus \{(0,0)\}$. Because we are computing the limit at the point $(0,0)$, all paths in $\mathbf{R}^2$ (not passing through the origin) are fair game.

It would simplify our analysis if we could reduce the given rational function in two variables to a rational function in one variable. We can do this if we choose our paths to $(0,0)$ wisely. In particular, consider the two paths

$$C_+ : \left\{(x,y) \,|\, y > 0, x = y^4\right\}, \qquad C_- : \left\{(x,y) \,|\, y > 0, x = -y^4\right\}.$$

Along C_+, the limit is

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{y\downarrow 0} \frac{y^8}{2y^8} = \frac{1}{2},$$

whereas along C_-, the limit is

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{y\downarrow 0} \frac{-y^8}{2y^8} = -\frac{1}{2}.$$

Because $\frac{1}{2} \neq -\frac{1}{2}$, we conclude that the limit does not exist.

To prove that a limit does exist, we can use analogs of the limit laws for single-variable functions.

- The limit of a sum equals the sum of the limits, provided that the individual limits exist.
- The limit of a product equals the product of the limits, provided that the individual limits exist.
- The limit of a quotient equals the quotient of the limits, provided that the individual limits exist and the limit of the denominator is nonzero.
- Limit of x, y, c. *(explain formally)*
- Squeeze theorem.

In particular, these results imply that the limit of a polynomial function exists at all points. This in turn can be used to show that the limit of a rational function (i.e. a ration of polynomial functions) exists at all points at which the denominator is defined.

Example 14.8 (S6eQ14.02.14)**.** Find the following limit, or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}.$$

Solution: Let

$$f(x,y) := \frac{x^4 - y^4}{x^2 + y^2}$$

be the function whose limit we are taking. Note that the maximum domain of f is $\mathbf{R}^2 \setminus \{(0,0)\}$. Because we are computing the limit at the point $(0,0)$, all paths in $\mathbf{R}^2$ (not passing through the origin) are fair game.

Note that

$$x^4 - y^4 = \left(x^2 + y^2\right)\left(x^2 - y^2\right).$$

Thus, if $x^2 + y^2 \neq 0$, then we can cancel the factors of $x^2 + y^2$ in the numerator and denominator of f. $x^2 + y^2 = 0$ if and only if $(x,y) = (0,0)$, which is the limit point we are considering. By definition of limit, we don't allow $(x,y) = (0,0)$ when computing the limit. Thus

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \left(x^2 - y^2\right).$$

Because the limit of a polynomial exists at every point (and equals the value of the polynomial at that point), we conclude that the limit of f exists and equals 0.

14.2.5 Continuity

Recall that a real-valued function $f(x)$ of a single variable is continuous at $x = a$ if and only if

$$\lim_{x\to a} f(x) = f(a).$$

This result generalizes to real-valued functions defined on (subsets of) $\mathbf{R}^n$.

Theorem 14.9. *Let* $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$*, and let* $a := (a_1, \dots, a_n) \in D$*.* f *is* ***continuous at*** $x = a$ *if*

$$\lim_{x\to a} f(x) = f(a).$$

f *is* ***continuous*** *if for all* $a \in D$*,* $f(x)$ *is continuous at* $x = a$*.*

By what we showed above, All polynomials are continuous, and all rational functions (i.e. ratios of polynomials) are continuous everywhere they are defined (i.e. everywhere the denominator is nonzero).

One can show that the composition of two continuous functions is continuous. This is a beautifully general fact, and follows immediately if we take the "right" view of continuity (as abstracted by elementary topology[2]).

Example 14.10 (S6eQ14.02.32)**.** Let

$$F(x, y) := e^{x^2 y} + \sqrt{x + y^2}.$$

Determine the set of points at which F is continuous.

Solution: We can view F as a composition of functions: Setting

$$f_1(x, y) := x^2 y, \qquad f_2(x, y) := x + y^2,$$

then

$$F(x, y) = e^{f_1(x,y)} + \sqrt{f_2(x, y)}.$$

The functions f_1, f_2 are polynomials, so they are continuous everywhere. The real-valued, single-variable function e in the first term of F is also continuous everywhere. The real-valued, single-variable square-root function in the second term of F is continuous everywhere it is defined, i.e. at all points $f_2(x, y) \geqslant 0$. Because the composition of continuous functions is continuous, we conclude that F is continuous at all points $(x, y) \in \mathbf{R}^2$ such that

$$f_2(x, y) \geqslant 0 \qquad \Leftrightarrow \qquad x \geqslant -y^2.$$

(add graph of set)

[2] As defined by elementary topology, a function $f : X \to Y$ is **continuous** if for every open set $V \subseteq Y$, its preimage

$$f^{-1}(V) = \{x \in X \,|\, f(x) \in V\},$$

i.e. the set of points in X that map to a point in V, is an open set in X. This definition assumes that we know which subsets of X and Y are "open" — precisely what a "topology" on a set tells us.

14.3 Partial Derivatives

Key Ideas

- How to compute partial derivatives
- Interpretations of partial derivatives
- Implicit differentiation
- Higher-order derivatives
- Clairaut's theorem (equality of mixed partials)

14.3.1 Introduction

As we saw in our study of limits in Section 14.2, when passing from $\mathbf{R}$ to $\mathbf{R}^n$, we pass from precisely two paths of approach (left and right) to infinitely many paths of approach. Because derivatives are defined in terms of limits (of difference quotients), it is no surprise that this same difficulty presents itself when we attempt to define the notion of derivative for functions $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$. Being good mathematicians, we deal with this difficulty by reducing to the case that we know how to deal with: Hold all variables except one constant, obtaining a function of one variable, and see how f changes as we change that one nonconstant variable. This notion is called "partial derivative".

Catchphrase

Partial derivatives are found by holding all variables except one constant.

14.3.2 Definitions

Let $f : \mathbf{R}^2 \to \mathbf{R}$. Fixing the value $y = b$, say, we can consider the function

$$g_b(x) := f(x, b).$$

Note that g_b is a real-valued function of a single variable, namely x. If g_b is differentiable, then we can compute its derivative using single-variable calculus. We call this derivative the **partial derivative of f with respect to x at (a, b)**:

$$\frac{\partial f}{\partial x}(a, b) := g_b'(a) = \lim_{h \to 0} \frac{g_b(a+h) - g_b(a)}{h} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}.$$

This partial derivative is also denoted $f_x(a, b)$. The subscript denotes the variable with respect to which the partial derivative is taken.

Fixing $x = a$, we can give an analogous argument to obtain the **partial derivative of f at with respect to y at (a, b)**:

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

If we think of the "fixed" variables as being allowed to take on any value (but after taking on this value, they remain fixed for the computation of the partial derivative), then we obtain **partial derivative functions**:

$$\frac{\partial f}{\partial x}(x, y), \qquad \frac{\partial f}{\partial y}(x, y).$$

Sometimes the alternative notation f_x and f_y are used for the partial derivatives of f with respect to x and y, respectively.

Because partial derivatives are really single-variable derivatives, the chain rule applies to computing partial derivatives just as it did in computing derivatives in single-variable calculus.

The takeaway from all this is that the notation is new, but the techniques are not: Computing partial derivatives is computing the derivative of a single-variable function. We reiterate the previous catchphrase:

Catchphrase

Partial derivatives are computed by holding all variables except one constant.

Looking back at the procedure that we used to define partial derivatives of functions on $\mathbf{R}^2$, we realize that there is nothing special about $\mathbf{R}^2$: The same approach yields notions of partial derivatives on $\mathbf{R}^n$, for any $n \in \mathbf{Z}_{>0}$.

Example 14.11 (S6eQ14.03.36)**.** Compute the first-order partial derivatives of the function

$$f(x, y, z, w) := \frac{xy^2}{w + 2z}.$$

Solution: f is a function of four variables, so it has four first-order partial derivatives. We compute

$$\frac{\partial f}{\partial x} = \frac{y^2}{w + 2z} \qquad \frac{\partial f}{\partial y} = \frac{2xy}{w + 2z}$$
$$\frac{\partial f}{\partial z} = -\frac{2xy^2}{(w + 2z)^2} \qquad \frac{\partial f}{\partial w} = -\frac{xy^2}{(w + 2z)^2}.$$

14.3.3 Interpretations of Partial Derivatives

(elaborate)

- Geometric: slopes of tangent lines (along planes)
- Physical: rates of change, holding all other variables fixed

These two interpretations are equivalent!

14.3.4 Implicit Differentiation

In single-variable calculus we learn the technique of implicit differentiation to differentiate expressions for which it is hard to isolate one of the variables, e.g.,

$$y^x = \sin(x+y).$$

The technique of implicit differentiation generalizes naturally to real-valued functions of several variables.

Example 14.12 (S6eQ14.03.46)**.** Let

$$yz = \ln(x+z),$$

where x, y are independent variables. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: Taking the partial derivative with respect to x, and using the chain rule on the right, we have

$$y\frac{\partial z}{\partial x} = \frac{1}{x+z}\left(1 + \frac{\partial z}{\partial x}\right).$$

Solving for $\frac{\partial z}{\partial x}$ gives

$$\frac{\partial z}{\partial x} = \frac{\frac{1}{x+z}}{y - \frac{1}{x+z}}.$$

Similarly, taking the partial derivative with respect to y, and using the product rule on the left and the chain rule on the right, we have

$$z + y\frac{\partial z}{\partial y} = \frac{1}{x+z}\left(\frac{\partial z}{\partial y}\right).$$

Solving for $\frac{\partial z}{\partial y}$ gives

$$\frac{\partial z}{\partial y} = \frac{-z}{y - \frac{1}{x+z}}.$$

14.3.5 Higher-Order Derivatives

We saw that partial derivatives, e.g., $\frac{\partial f}{\partial x}$, are themselves functions. If they are differentiable, then we can differentiate them, obtaining higher-order derivatives. For example, given $f(x, y)$, if we differentiate first with respect to x, then with respect to y, we obtain

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

This is also denoted f_{xy}.

We could also differentiate in the opposite order: First with respect to y, then with respect to x. It is natural to wonder when these two derivatives are equal.

(use vector notation for the point a?)

Theorem 14.13 (Clairaut–Schwarz Theorem)**.** *Let $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$, let $a \in D$, and suppose that there is an open ball B such that (i) $a \in B \subseteq D$ and (ii) $\frac{\partial^2 f}{\partial x_j \partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are both continuous on B.*[3] *Then these two second-order mixed partial derivatives are equal:*

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a).$$

(remark about symmetry implication for the matrix of second-order partial derivatives)

Example 14.14 (S6eQ14.03.58)**.** Let

$$u(x, y) := x^4 y^2 - 2xy^5.$$

Verify that the conclusion of the Clairaut–Schwarz theorem holds.

Solution: We compute

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(4x^3y^2 - 2y^5\right) = 8x^3 y - 10y^4,$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2x^4 y - 10xy^4\right) = 8x^3 y - 10y^4.$$

These two results are equal. Thus we conclude that at all points $(x, y) \in \mathbf{R}^2$,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y},$$

i.e. the conclusion of the Clairaut–Schwarz theorem holds.

In fact, it is straightforward to show that the Clairaut–Schwarz theorem (and not just its conclusion) holds: The function u is defined on all of $\mathbf{R}^2$, so for any point $a \in \mathbf{R}^2$ we can find an open ball B such that $a \in B \subseteq \mathbf{R}^2$. Moreover, the two second-order mixed partial derivatives are polynomial functions, which are continuous on all of $\mathbf{R}^2$ (and hence on the ball B). Thus the hypotheses of the Clairaut–Schwarz theorem are satisfied, so the conclusion must hold.

[3] It suffices to assume that all first-order partial derivatives are differentiable, a weaker condition than continuity of second-order partial derivatives.

Remark 14.15. For almost all functions that we will encounter, the hypothesis about continuity of second-order partial derivatives (or differentiability of first-order partial derivatives, if you prefer the weaker hypothesis) will be satisfied. However, there exist (relatively) simple functions for which the hypothesis fails; for these functions, the equality of second-order mixed partials need not hold. For example, one can check that the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x,y) := \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

is continuous on all of $\mathbf{R}^2$,[4] and that

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1 \neq 1 = \frac{\partial^2 f}{\partial x \partial y}(0,0).$$

That is, equality of mixed partials fails to hold at $(0,0)$. For details of this computation, see "Requirement of continuity" in the Wikipedia article "Symmetry of second derivatives".

[4]Recall that a rational function is continuous at all points at which it is defined. Thus f is continuous at all points $(x,y) \neq (0,0)$, so we need only verify continuity at the origin.

14.4 Tangent Planes

Key Ideas

- Linear things are easy
- Computing the tangent plane to a surface
  - Via cross product (in $\mathbf{R}^3$ only)
  - By deducing "slope" coefficients of tangent plane (general)
- Linear approximations
- Definition: differentiable function
- Differentials (small changes in the linear approximation)

(clean up, tighten; parallels!; standardize on notation for point of interest x_0 vs a)

14.4.1 Introduction

Why do we care about tangent lines and tangent planes? The answer is one of the Big Themes: Linear things are easy. When we consider the tangent line to the graph of a single-variable function $f(x)$, when we consider the tangent plane to the graph of a two-variable function $f(x, y)$, or when we consider higher-dimensional analogs (hyperplanes to hypersurfaces), we replace the complicated function f with a *linear approximation*, namely the tangent line, plane, or hyperplane. Close to the point of interest, this approximation is good, and we can study the function (hard) by studying its linearization (easy).

(consider rewriting the next two sections: open with general motivation, followed by two subapproaches: using cross product, or using general equation for linear hypersurface and deducing coefficients)

14.4.2 Tangent Planes via Cross Product

For real-valued functions $f(x)$ of one variable, given a point $x = a$ where $f(x)$ is differentiable, there exists a unique tangent line to $f(x)$ at $x = a$. In single-variable calculus, we learn how to compute the derivative of $f(x)$ at $x = a$. This derivative $f'(a)$ gives us the slope of the tangent line to the graph of $f(x)$ at $x = a$. We then learn that this tangent line is the best linear approximation to $f(x)$ near $x = a$. Graphically, as we "zoom in" on the point $(x, f(x))$, the graph of $f(x)$ looks more and more like the tangent line.

Analogous results hold in higher dimensions. For real-valued functions $f(x, y)$ of two variables, given a point $(x, y) = (a, b)$ where $f(x, y)$ is differentiable, there exists a unique tangent plane to the graph of $f(x, y)$ at $(x, y) = (a, b)$. We have learned (see Section 14.3) how to compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at a point $(x, y) = (a, b)$ (where f is differentiable). These partial derivatives give us the slopes of two tangent lines to $f(x, y)$ at $(x, y) = (a, b)$: $f_x(a, b)$ gives the slope of the tangent line in the plane $y = b$, and $f_y(a, b)$ gives the slope of the tangent line in the plane $x = a$.[5] The tangent plane to $f(x, y)$ at $(x, y) = (a, b)$ contains both of these lines, so provided that $f_x(a, b) \neq 0$ and $f_y(a, b) \neq 0$, these two tangent lines determine the tangent plane.

As in the single-variable case *(add this part in the preceding subsection)*, we can read off direction vectors for these two tangent lines:

$$\mathbf{v}_x := \left(1, 0, \frac{\partial f}{\partial x}(a, b)\right), \qquad \mathbf{v}_y := \left(0, 1, \frac{\partial f}{\partial y}(a, b)\right),$$

in the planes $y = b$ and $x = a$, respectively. The two tangent lines are contained in the tangent plane, so these two direction vectors (positioned so that their initial points are at $(a, b, f(a, b))$) lie in the tangent plane, and we can obtain a normal vector to the tangent plane by taking their cross product:

$$\mathbf{n} = \mathbf{v}_x \times \mathbf{v}_y = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(a, b) \\ 0 & 1 & \frac{\partial f}{\partial y}(a, b) \end{pmatrix} = \left(-\frac{\partial f}{\partial x}(a, b), -\frac{\partial f}{\partial y}(a, b), 1\right).$$

Thus an equation for the tangent plane to the graph of f at $(a, b, f(a, b))$ is

$$\begin{aligned} 0 &= \mathbf{n} \cdot (x - a, y - b, z - f(a, b)) \\ &= \left(-\frac{\partial f}{\partial x}(a, b)\right)(x - a) + \left(-\frac{\partial f}{\partial y}(a, b)\right)(y - b) + (z - f(a, b)), \end{aligned}$$

which we can rearrange as

$$z - f(a, b) = \left(\frac{\partial f}{\partial x}(a, b)\right)(x - a) + \left(\frac{\partial f}{\partial y}(a, b)\right)(y - b). \tag{14.4.1}$$

Note the analogy with the single-variable case: In each coordinate plane ($y = b$ or $x = a$), we take the direction vector (slope)

$$\mathbf{v}_x = \left(1, 0, \frac{\partial f}{\partial x}(a, b)\right) \qquad \text{and} \qquad \mathbf{v}_y = \left(0, 1, \frac{\partial f}{\partial y}(a, b)\right)$$

and rotate it 90 degrees counterclockwise, obtaining the vectors

$$\mathbf{v}_x^{\perp} = \left(-\frac{\partial f}{\partial x}(a, b), 0, 1\right) \qquad \text{and} \qquad \mathbf{v}_y^{\perp} = \left(0, -\frac{\partial f}{\partial y}(a, b), 1\right)$$

[5] Recall that to compute the partial derivative $\frac{\partial f}{\partial x}(a, b)$, we hold y constant at the value $y = b$, and compute the rate of change of the resulting single-variable function $g_b(x) = f(x, b)$ at $x = a$. The algebraic operation of setting $y = b$ corresponds to the geometric operation of intersecting the graph of $f(x, y)$ with the plane $y = b$. Similarly, the algebraic operation of computing the rate of change of $f(x, b)$ at $x = a$ corresponds to the geometric operation of finding the tangent line to the graph of $f(x, b)$ (in the plane $y = b$) at $x = a$. Analogous statements hold for $\frac{\partial f}{\partial y}(a, b)$.

orthogonal to $\mathbf{v}_x$ and $\mathbf{v}_y$, respectively. The normal vector $\mathbf{n}$ combines the information from both $\mathbf{v}_x^\perp$ and $\mathbf{v}_y^\perp$ *(be more precise?)*.

As in the single-variable case, the tangent plane is the best linear approximation to $f(x, y)$ near $(x, y) = (a, b)$. Graphically, as we "zoom in" on the point $(x, y, f(x, y))$, the graph of $f(x, y)$ looks more and more like the tangent plane.

(actually compute the tangent plane this way; give an example, same example as in next subsection; have students raise caveat: cross product won't generalize to higher dimensions)

14.4.3 Tangent Planes via Slopes

(mention somewhere — anywhere — that $z_0 = f(x_0, y_0)$)

In Subsection 14.4.2 we proposed a way to compute the tangent plane to $f(x, y)$ at a point (a, b): Find two particular tangent lines to $f(x, y)$ at (a, b), then find the plane containing these two lines, using the cross product. However, as you objected above, the cross product works only in $\mathbf{R}^3$, i.e. when f is a function of two input variables. In this subsection we develop a general technique that will work for any (differentiable) function $f : \mathbf{R}^n \to \mathbf{R}$. For concreteness, and to facilitate comparison with the preceding approach, we continue to consider the case $f : \mathbf{R}^2 \to \mathbf{R}$. When working through this section, vigilantly note how our argument would work more generally for $f : \mathbf{R}^n \to \mathbf{R}$.

Suppose that we know that $f(x, y)$ has a tangent plane at the point $(x, y) = (x_0, y_0)$. (This will be the case if $f(x, y)$ is has continuous first partial derivatives at (x_0, y_0).) *(draw picture!!! label tangent lines T_1, T_2 when they arise in the computation below)* How can we find the equation of this plane? Well, we have seen *(add ref)* that any plane in $\mathbf{R}^3$ can be expressed in the form

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \tag{14.4.2}$$

for some $n_1, n_2, n_3 \in \mathbf{R}$, where $\mathbf{n} := (n_1, n_2, n_3)$ is a normal vector to the given plane. If $n_3 \neq 0$, then we can divide (14.4.2) through by n_3, set $m_i := -\frac{n_i}{n_3}$ for $i = 1, 2$, and solve for $z - z_0$, obtaining an equivalent equation representing the tangent plane:

$$z - z_0 = m_1(x - x_0) + m_2(y - y_0). \tag{14.4.3}$$

Intersecting the tangent plane to $f(x, y)$ at (x_0, y_0) with the plane $y = y_0$ gives the tangent line to $f(x, y)$ at $(x, y) = (x_0, y_0)$ in the x-direction, call it T_1. Recall that the geometric operation of this intersection corresponds to the algebraic operation of setting $y = y_0$. Setting $y = y_0$ in (14.4.3), we obtain

$$z - z_0 = m_1(x - x_0).$$

This is the equation of a line in the xz-plane with slope m_1. The slope of the tangent line in the xz-plane ($y = y_0$) can also be obtained by fixing $y = y_0$ and differentiating the resulting function $f(x, y_0)$ (a function of one variable, x) with respect to x.

That is,

$$m_1 = \frac{\partial f}{\partial x}(x_0, y_0).$$

Similarly, intersecting the tangent plane to $f(x,y)$ at (x_0, y_0) with the plane $x = x_0$, we obtain

$$z - z_0 = m_2(y - y_0);$$

the slope m_2 of this line in the yz-plane ($x = x_0$) can be obtained via partial differentiation with respect to y:

$$m_2 = \frac{\partial f}{\partial y}(x_0, y_0).$$

Substituting these results into (14.4.3), we conclude that an equation for the tangent plane to $f(x,y)$ at the point (x_0, y_0) is

$$z - z_0 = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0). \tag{14.4.4}$$

Note that this agrees with equation (14.4.1) for the tangent plane that we obtained using the cross product.

Example 14.16 (S6eQ14.04.04)**.** Find an equation for the tangent plane to the surface in $\mathbf{R}^3$ defined by

$$z = y \ln x$$

at the point $(1, 4, 0)$.

Solution: Consider $z = z(x, y)$ as a function of x, y. We compute

$$\frac{\partial z}{\partial x} = \frac{y}{x}, \qquad \frac{\partial z}{\partial y} = \ln x.$$

Therefore

$$\frac{\partial z}{\partial x}(1,4) = 4, \qquad \frac{\partial z}{\partial y}(1,4) = 0.$$

Therefore an equation for the tangent plane to z at $(x, y) = (1, 4)$ is

$$z = z - 0 = \frac{\partial z}{\partial x}(1,4)(x-1) + \frac{\partial z}{\partial y}(1,4)(y-4) = 4(x-1).$$

14.4.4 Linear Approximations

The tangent plane to $f(x, y)$ at the point (x_0, y_0) is called the **linearization** of f at (x_0, y_0). Denoting the linearization $L_f(x_0, y_0)$ and using our result for the tangent plane in (14.4.4), we have

$$L_f(x_0, y_0) = z = z_0 + \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0).$$

Approximating $f(x_0, y_0)$ by its linearization (i.e. tangent plane) $L_f(x_0, y_0)$ is called **linearizing** f at (x_0, y_0).

(mention cautionary tale? see p 894)

(rewrite this next part; improve motivation)

Often we are interested in the behavior of a function near a point. Consider a real-valued function $y = f(x)$ of a single variable at the point x_0. If we change x_0 by a small amount Δx, then y changes by the corresponding amount

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

Let

$$\varepsilon := \frac{\Delta y}{\Delta x} - f'(x_0). \tag{14.4.5}$$

Note that ε is a function of x, because it depends on $\Delta x = x - x_0$. If f is differentiable at x_0, then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0),$$

so

$$\lim_{\Delta x \to 0} \varepsilon = 0. \tag{14.4.6}$$

Solving (14.4.5) for Δy,

$$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x.$$

If we ignore the second term on the right, this expresses the change in y, Δy, as a linear function of the change in x, Δx. The second term $\varepsilon \Delta x$ depends nonlinearly on Δx, because ε does. However, as $\Delta x \to 0$, (14.4.6) says that $\varepsilon \to 0$. Loosely speaking, in the limit as $\Delta x \to 0$, Δy is linear in Δx.

An analogous analysis for $f(x, y)$ *(elaborate)* yields the following.

Definition 14.17. Let $z := f(x, y)$. f is **differentiable** at (x_0, y_0) if there exist functions $\varepsilon_1(x, y), \varepsilon_2(x, y)$ satisfying

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_i(x, y) = 0$$

such that

$$\Delta z = \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

As in the case for real-valued functions of a single variable, this says that f is differentiable at the point (x_0, y_0) if the linear approximation (i.e. the tangent plane) to f at (x_0, y_0) is "good" when (x, y) is "close to" (x_0, y_0).

Theorem 14.18. *If the partial derivatives* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ *exist in a neighborhood of* (x_0, y_0) *and are continuous at* (x_0, y_0)*, then* f *is differentiable at* (x_0, y_0)*.*

14.4.5 Differentials

Differentials are the linear approximation to the change in a function near a point. *(draw picture!;* $dx = \Delta x, dy \neq \Delta y$*)*

Consider a real-valued function $f(x)$ of one variable, and let $y := f(x)$. The **differential** dx of the independent variable x is any real number, and the **differential** dy of the dependent variable y is

$$dy = f'(x_0)dx.$$

Now consider a real-valued function $f(x, y)$ of two variables, and let $z := f(x, y)$. The **differentials** dx, dy of the independent variables can be any real number, and the **(total) differential** dz of the dependent variable z is

$$dz = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right) dx + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right) dy.$$

Setting

$$dx := \Delta x = x - x_0, \qquad dy := \Delta y = y - y_0,$$

the (total) differential dz writes as

$$dz = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0).$$

Note that dz denotes the change in z from its starting value $z_0 = f(x_0, y_0)$, i.e. $dz = z - z_0$. In this view, the total differential gives us exactly the equation for the tangent plane to the graph of f at (x_0, y_0) that we derived in (14.4.1) and (14.4.4).

(draw geometric picture! see Figure 14.4.7 (p 897))

Everything we have done (save for our brief digression into the cross product) in this section generalizes to $\mathbf{R}^n$. *(work this into the individual subsections)*

14.5 The Chain Rule

Key Ideas

- Chain rule for single-variable functions
- Partial derivatives of multivariable functions are single-variable derivatives!
- Chain rules for multivariable functions
- Tree diagrams

14.5.1 Introduction

In several applications *(name a few explicitly)* we will wish to differentiate a multivariable function $f(x_1, \ldots, x_n)$ along a particular curve $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$. That is, we will wish to compute

$$\frac{d}{dt} f\left(x_1(t), \ldots, x_n(t)\right).$$

The analogous derivative for a single-variable function is

$$\frac{d}{dt} f(x(t)).$$

To evaluate this latter derivative, we can use the chain rule:

$$\frac{d}{dt} f(x(t)) = f'(x(t))x'(t).$$

To evaluate the multidimensional analog, we are led to extend the chain rule to functions of more than one variable.

14.5.2 The Chain Rule: $\mathbf{R}^1$

Consider the chain rule for a real-valued function of a single variable. Let $y = f(x)$ and $x = g(t)$ be differentiable functions. Then

$$y = f(x) = f(g(t)),$$

i.e. y is a (differentiable) function of t. The chain rule says that

$$\frac{dy}{dt} = f'(g(t))g'(t) = \frac{dy}{dx}\frac{dx}{dt}.$$

(reminder of proof)

14.5.3 The Chain Rule: Base Case

Theorem 14.19. *Let $z := f(x, y)$ be a differentiable function of two variables x, y, and let $x := f(t)$ and $y := g(t)$ be differentiable functions of one variable t. Then z is a differentiable function of t, and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

(present proof)

Example 14.20 (S6eQ14.05.04)**.** Let

$$z := \arctan\left(\frac{y}{x}\right), \qquad x := e^t, \qquad y := 1 - e^{-t}.$$

Use the chain rule to find $\frac{dz}{dt}$ at $t = 0$.

Solution: Let

$$z = f(x, y) := \arctan\left(\frac{y}{x}\right).$$

Recall that

$$\frac{d}{dt}\arctan t = \frac{1}{1 + t^2}.$$

Method 1: Chain rule.

$$\begin{aligned}\frac{dz}{dt} &= \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} \\ &= \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-x^{-2}y\right)e^t + \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(x^{-1}\right)e^{-t} \\ &= \frac{x^{-1}\left(-\frac{y}{x}e^t + e^{-t}\right)}{1 + \left(\frac{y}{x}\right)^2}.\end{aligned}$$

At $t = 0$,

$$x(0) = 1, \qquad y(0) = 1 - 1 = 0,$$

so

$$\frac{dz}{dt} = \frac{(0 + 1)}{1 + 0^2} = 1.$$

(clean up next note:) Note that we evaluated $x(t), y(t)$ directly at the desired value of t and substituted in the resulting values, rather than substituting the general expressions in terms of t.

Method 2: Direct substitution.

$$z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1 - e^{-t}}{e^t}\right) = \arctan\left(e^{-t} - e^{-2t}\right).$$

Thus

$$\frac{dz}{dt} = \frac{1}{1 + (e^{-t} - e^{-2t})^2} \left(-e^{-t} + 2e^{-2t}\right).$$

At $t = 0$,

$$\frac{dz}{dt} = \frac{1}{1 + (1-1)^2}(-1+2) = 1.$$

Again consider a differentiable function $z = f(x, y)$ of two variables. If the intermediate variables x, y are differentiable functions of more than one variable, say s, t, then $z = f(x, y)$ is a differentiable function of s, t. Hence we can compute the partial derivatives $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$. When computing the partial derivative with respect to a given variable, we hold all other variables constant, so we reduce to the case of the chain rule considered above *(clarify)*. Thus we obtain the following.

Theorem 14.21.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

independent, intermediate, dependent variables

Tree diagrams provide a convenient method for accounting when computing derivatives via the chain rule. *(illustrate)*

14.5.4 Chain Rule: General Version

Theorem 14.22. *Let z be a differentiable function of n variables $x_1, \dots, x_n$, and let each x_i be a differentiable function of m variables $t_1, \dots, t_m$. Then z is a differentiable function of $t_1, \dots, t_m$, and*

$$\frac{\partial z}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial t_j}.$$

Example 14.23 (S6eQ14.05.24)**.** Let

$$M := xe^{y-z^2}, \qquad x := 2uv, \qquad y := u - v, \qquad z := u + v.$$

Use the chain rule to find the partial derivatives $\frac{\partial M}{\partial u}, \frac{\partial M}{\partial v}$ when $(u, v) = (3, -1)$.

Solution: *(include tree diagram)* Note that M is a function of three intermediate variables x, y, z, each of which in turn is a function of two independent variables u, v. Thus

$$\begin{aligned}
\frac{\partial M}{\partial u} &= \frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u} \\
&= \left(e^{y-z^2}\right)(2v) + \left(xe^{y-z^2}\right)(1) + \left((-2z)xe^{y-z^2}\right)(1) \\
&= e^{y-z^2}\left(2v + x - 2xz\right),
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v} \\ &= \left(e^{y-z^2}\right)(2u) + \left(xe^{y-z^2}\right)(-1) + \left((-2z)xe^{y-z^2}\right)(1) \\ &= e^{y-z^2}\left(2u - x - 2xz\right).\end{aligned}$$

Note that the values of x, y, z at $(u, v) = (3, -1)$ are

$$x = -6, \qquad y = 4, \qquad z = 2.$$

Thus, evaluating the partial derivatives computed above at $(u, v) = (3, -1)$, we find

$$\begin{aligned}\frac{\partial M}{\partial u}(3, -1) &= e^{4-2^2}\left(2(-1) + (-6) - 2(-6)(2)\right) = 4, \\ \frac{\partial M}{\partial v}(3, -1) &= e^{4-2^2}\left(2(3) - (-6) - 2(-6)(2)\right) = 36.\end{aligned}$$

14.5.5 Matrix Version

Matrices present a compact and elegant way to keep track of the bookkeeping involved with the chain rule. We present the general theorem, followed immediately by examples that will clarify how things work.

Component functions go down the rows. For first-order partial derivatives, input variables go right along the columns.

Why this convention? View the input variables as column vectors, and consider linear functions; for concreteness,

$$f(x, y, z) := x + 2y + 3z.$$

Note that for a linear function, the coefficient of each variable represents the partial derivative of the function with respect to that variable. View this function as

$$f(x, y, z) = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Note that the first matrix is the matrix of partial derivatives of f.

For example, consider a function $f : \mathbf{R}^2 \to \mathbf{R}^3$, say with input variables x, y and component functions $f = (f_1, f_2, f_3)$. Then we view

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \\ f_3(x, y) \end{pmatrix}.$$

The derivative of f is the 3×2 matrix

$$\mathbf{D}f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix}.$$

Theorem 14.24. *Let $U \subseteq \mathbf{R}^m$ and $V \subseteq \mathbf{R}^n$ be open sets, and let*

$$g : U \subseteq \mathbf{R}^m \to \mathbf{R}^n, \qquad f : V \subseteq \mathbf{R}^n \to \mathbf{R}^p$$

be functions such that

(i) $g(U) \subseteq V$ *(so that $f \circ g$ is defined),*

(ii) g *is differentiable at $\mathbf{x}_0 \in U$, and*

(iii) f *is differentiable at $g(\mathbf{x}_0) \in V$.*

Then $f \circ g$ is differentiable at $\mathbf{x}_0$, and its derivative is

$$(\mathbf{D}(f \circ g))(\mathbf{x}_0) = \mathbf{D}f(g(\mathbf{x}_0))\mathbf{D}g(\mathbf{x}_0),$$

where the product of the derivatives on the right is a matrix product.

Example 14.25 (S6eQ14.05.04, matrix version)**.** Let

$$z := \arctan\left(\frac{y}{x}\right), \qquad x := e^t, \qquad y := 1 - e^{-t}.$$

Use the matrix version of the chain rule to find $\frac{dz}{dt}$ at $t = 0$.

Solution: Let's compute the desired derivative two ways: the chain rule, and direct substitution.

Method 1: Chain rule. First we need to determine the functions involved. One is easy:

$$\begin{aligned} z : \mathbf{R}^2 &\to \mathbf{R} \\ (x, y) &\mapsto \arctan\left(\frac{y}{x}\right). \end{aligned}$$

But there remain two functions, $x(t)$ and $y(t)$. How do these fit into the chain rule? The key paradigm shift is to note that both functions are functions of the same input variable t, so we can view them as the component functions of a vector-valued function g, namely

$$\begin{aligned} g : \mathbf{R} &\to \mathbf{R}^2 \\ t &\mapsto (x(t), y(t)) = \left(e^t, 1 - e^{-t}\right). \end{aligned}$$

We are now in a position to apply the chain rule. In this context, the chain rule says that

$$\mathbf{D}(z \circ g) = ((\mathbf{D}z)(g))\,(\mathbf{D}g).$$

Recall that

$$\frac{d}{dt}\arctan t = \frac{1}{1+t^2}.$$

We compute[6]

$$\mathbf{D}z = \begin{pmatrix} \frac{-yx^{-2}}{1+\left(\frac{y}{x}\right)^2} & \frac{x^{-1}}{1+\left(\frac{y}{x}\right)^2} \end{pmatrix},$$

and

$$\mathbf{D}g = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}.$$

At $t = 0$, we have

$$g(0) = (1, 0)$$

(recall that $g(t) = (x(t), y(t))$, so these are our values for x and y), so

$$(\mathbf{D}z)(g(0)) = (\mathbf{D}z)(1,0) = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad (\mathbf{D}g)(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore

$$(\mathbf{D}(z \circ g))(0) = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1.$$

Method 2: Direct substitution. Substituting the expressions for $x(t)$ and $y(t)$ into z, we obtain z as an explicit function of t:

$$z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1-e^{-t}}{e^t}\right) = \arctan\left(e^{-t} - e^{-2t}\right).$$

Thus we are back in the case of single-variable calculus, and we can differentiate this function using single-variable differentiation (remember the single-variable chain rule!):

$$\frac{dz}{dt} = \frac{1}{1+(e^{-t}-e^{-2t})^2}\left(-e^{-t} + 2e^{-2t}\right).$$

At $t = 0$, we have

$$\frac{dz}{dt}(0) = \frac{1}{1+(1-1)^2}(-1+2) = 1.$$

[6] Note that we do not have to substitute in the expressions for $x(t)$ and $y(t)$ here. Because we are interested in the derivative of z at a particular point (i.e. a particular value of t), it will be easier for us to find the corresponding values of x and y, and substitute these values into the expression for $\mathbf{D}z$ in terms of x and y.

Example 14.26 (MT5eE2.5.3). Let

$$g : \mathbf{R}^2 \to \mathbf{R}^2 \qquad f : \mathbf{R}^2 \to \mathbf{R}^3$$
$$(x, y) \mapsto (x^2 + 1, y^2) \qquad (u, v) \mapsto (u + v, u, v^2).$$

Using the chain rule, compute the derivative of $f \circ g$ at the point $(x, y) = (1, 1)$.

Solution: By the chain rule,

$$(\mathbf{D}(f \circ g))(1, 1) = ((\mathbf{D}f)(g(1, 1)))\,((\mathbf{D}g)(1, 1))\,. \tag{14.5.1}$$

Computing the derivatives of f and g, we find

$$\mathbf{D}f = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}, \qquad \mathbf{D}g = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}.$$

At the point $(x, y) = (1, 1)$, $g(1, 1) = 2, 1$, so

$$(\mathbf{D}f)(g(1, 1)) = (\mathbf{D}f)(2, 1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad (\mathbf{D}g)(1, 1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Substituting these results into (14.5.1), we obtain

$$(\mathbf{D}(f \circ g))(1, 1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

(comment on the meaning of each entry)

Example 14.27 (MT5eQ2.5.5(c)). Let $U = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 > 0\}$, and let

$$f : U \to \mathbf{R}, \qquad \mathbf{c} : \mathbf{R} \to \mathbf{R}^2$$
$$(x, y) \mapsto \left(x^2 + y^2\right) \log \sqrt{x^2 + y^2} \qquad t \mapsto \left(e^t, e^{-t}\right).$$

Verify the chain rule for the composition $f \circ \mathbf{c}$.

Solution: By the chain rule,

$$\mathbf{D}(f \circ \mathbf{c}) = ((\mathbf{D}f)(\mathbf{c}))\,\mathbf{D}\mathbf{c}.$$

We compute

$$\begin{aligned} \frac{\partial f}{\partial x} &= (2x) \log \sqrt{x^2 + y^2} + (x^2 + y^2) \left(\frac{1}{\sqrt{x^2 + y^2}} \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} (2x) \right) \\ &= x \log(x^2 + y^2) + x. \end{aligned}$$

By symmetry, the partial derivative $\frac{\partial f}{\partial y}$ is the same, except with x and y interchanged. Therefore

$$\mathbf{D}f = \begin{pmatrix} x\log(x^2+y^2)+x & y\log\left(x^2+y^2\right)+y \end{pmatrix}.$$

The derivative of $\mathbf{c}$ is straightforward (when writing its derivative, remember the convention that component functions go down the rows):

$$\mathbf{Dc} = \begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix}.$$

Thus

$$\begin{aligned}\mathbf{D}f(\mathbf{c}(t)) &= (\mathbf{D}f)(e^t, e^{-t}) \\ &= \begin{pmatrix} e^t\log\left(e^{2t}+e^{-2t}\right)+e^t & e^{-t}\log\left(e^{2t}+e^{-2t}\right)+e^{-t} \end{pmatrix},\end{aligned}$$

so

$$\begin{aligned}\mathbf{D}(f\circ\mathbf{c}) &= ((\mathbf{D}f)(\mathbf{c}))\,\mathbf{Dc} \\ &= \begin{pmatrix} e^t\log\left(e^{2t}+e^{-2t}\right)+e^t & e^{-t}\log\left(e^{2t}+e^{-2t}\right)+e^{-t} \end{pmatrix}\begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix} \\ &= e^{2t}\log\left(e^{2t}+e^{-2t}\right)+e^{2t}-e^{-2t}\log\left(e^{2t}+e^{-2t}\right)-e^{-2t} \\ &= \left(e^{2t}-e^{-2t}\right)\log\left(e^{2t}+e^{-2t}\right)+e^{2t}-e^{-2t}.\end{aligned}$$

Note that

$$\begin{aligned} f\circ\mathbf{c} : \mathbf{R} &\to \mathbf{R} \\ t &\mapsto \left(e^{2t}+e^{-2t}\right)\log\sqrt{e^{2t}+e^{-2t}},\end{aligned}$$

and

$$\mathbf{D}(f\circ\mathbf{c}) = \frac{d}{dt}(f\circ\mathbf{c}).$$

Thus we can also compute the desired derivative using single-variable calculus:

$$\begin{aligned}\mathbf{D}(f\circ\mathbf{c}) &= \left(2e^{2t}-2e^{-2t}\right)\log\sqrt{e^{2t}+e^{-2t}} \\ &\quad + \left(e^{2t}+e^{-2t}\right)\frac{1}{\sqrt{e^{2t}+e^{-2t}}}\frac{1}{2}\left(e^{2t}+e^{-2t}\right)^{-\frac{1}{2}}\left(2e^{2t}-2e^{-2t}\right) \\ &= \left(e^{2t}-e^{-2t}\right)\log\left(e^{2t}+e^{-2t}\right)+e^{2t}-e^{-2t},\end{aligned}$$

the same result that we obtained using the chain rule.

14.5.6 Implicit Differentiation

(brief account)

14.6 Directional Derivatives

Key Ideas

- Directional derivative
  - Geometric picture
  - Algebraic definition
- Directional derivative in terms of gradient (partial derivatives)
- Gradient vector
  - Physical interpretation (direction of steepest ascent)
  - Algebraic definition (vector of partial derivatives)
- At each point on a level set, the gradient vector is normal to the tangent plane

(break this section into two)

14.6.1 Introduction

(briefly pave way to view input to multivariable function as vector)

Let $f(x, y)$ be a differentiable real-valued function of two variables. Recall the partial derivatives of f at the point (x_0, y_0) are defined as the limit of difference quotients

$$\begin{aligned}\frac{\partial f}{\partial x}(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},\\ \frac{\partial f}{\partial y}(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}.\end{aligned}$$

(be precise about this next bit) These partial derivatives represent the rate of change in f in the x- and y-directions, respectively. If in the limits we consider $h \downarrow 0$, then we can view these partial derivatives as the change in f in the direction of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. That is, letting $\mathbf{r}_0 := (x_0, y_0)$, we can view these partial derivatives as

$$\begin{aligned}\frac{\partial f}{\partial x}(x_0, y_0) &= \lim_{h\downarrow 0} \frac{f(\mathbf{r}_0 + h\mathbf{i}) - f(\mathbf{r}_0)}{h},\\ \frac{\partial f}{\partial y}(x_0, y_0) &= \lim_{h\downarrow 0} \frac{f(\mathbf{r}_0 + h\mathbf{j}) - f(\mathbf{r}_0)}{h}.\end{aligned}$$

(tidy up all that follows)

Now suppose that we wish to find the rate of change of f at $P_0 := (x_0, y_0, f(x_0, y_0))$ in the direction of an arbitrary vector $\mathbf{v} := (v_1, v_2)$. Using the definition of derivative, we again look at the limit of difference quotients. Any point $P := (x, y, z)$

yields a vector $\vec{P_0P}$ in the xy-plane parallel to $\mathbf{v}$. Thus there exists a scalar $h \in \mathbf{R}$ such that

$$\vec{P_0P} = h\mathbf{v} = (hv_1, hv_2).$$

Thus the difference quotient writes as

$$\frac{f(x_0 + hv_1, y_0 + hv_2) - f(x_0, y_0)}{h} = \frac{f(\mathbf{r}_0 + h\mathbf{v}) - f(\mathbf{r}_0)}{h}.$$

Why unit vectors? Consider

$$z = f(x, y) := y, \qquad \mathbf{u} := \mathbf{j} = (0, 1), \qquad \mathbf{v} := 2\mathbf{j} = (0, 2).$$

What should the directional derivative be? What is it for $\mathbf{u}$ and $\mathbf{v}$?

Definition 14.28. Let $U \subseteq \mathbf{R}^n$ be an open set, let $\mathbf{a} := (a_1, \dots, a_n) \in U$, let $f : U \to \mathbf{R}$, and let $\mathbf{u} := (u_1, \dots, u_n) \in \mathbf{R}^n$ be a unit vector. The **directional derivative** of f at $\mathbf{a}$ in the direction of the unit vector $\mathbf{u}$ is

$$(D_{\mathbf{u}} f)(\mathbf{a}) = \lim_{h \downarrow 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h},$$

provided that the limit exists.

For example, if $f : \mathbf{R}^2 \to \mathbf{R}$ is everywhere differentiable, and $\mathbf{u} := (u_1, u_2)$ is a unit vector, then the directional derivative of f at any point (x_1, x_2) in the direction of $\mathbf{u}$ is

$$\begin{aligned}(D_{\mathbf{u}} f)(x_1, x_2) &= \lim_{h \downarrow 0} \frac{f((x_1, x_2) + h(u_1, u_2)) - f(x_1, x_2)}{h} \\ &= \lim_{h \downarrow 0} \frac{f(x_1 + hu_1, x_2 + hu_2) - f(x_1, x_2)}{h}.\end{aligned}$$

Theorem 14.29. *Let $f(x, y)$ be differentiable. Then for any unit vector $\mathbf{u}$, f has a directional derivative $D_{\mathbf{u}} f(x, y)$, and*

$$D_{\mathbf{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y) u_1 + \frac{\partial f}{\partial y}(x, y) u_2. \tag{14.6.1}$$

(proof)

Let θ be the angle measured counterclockwise from the positive x-axis to the unit vector $\mathbf{u}$. (This is the same θ as defined in polar, cylindrical, and spherical coordinates.) Then

$$\mathbf{u} = (\cos\theta, \sin\theta).$$

In particular, we can write (14.6.1) as

$$D_{\mathbf{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y) \cos\theta + \frac{\partial f}{\partial y}(x, y) \theta.$$

We can also write (14.6.1) as an inner product:

$$D_{\mathbf{u}} f(x, y) = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right) \cdot (u_1, u_2).$$

Definition 14.30. Let $f(x, y)$ be differentiable. The **gradient** of f is the vector

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

More generally, let $f(x_1, \ldots, x_n)$ be differentiable. The **gradient** of f is the vector

$$\nabla f := \left(\frac{\partial f}{\partial x_1}(x_1, \ldots, x_n), \ldots, \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n)\right).$$

Rewriting *(ref!)* with this notation, and generalizing to $\mathbf{R}^n$,

$$D_{\mathbf{u}} f(x_1, \ldots, x_n) = \nabla f \cdot \mathbf{u}.$$

Theorem 14.31. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable, and let $\mathbf{u} \in \mathbf{R}^n$ be a unit vector. Then the maximum value of the directional derivative $(D_{\mathbf{u}} f)(\mathbf{x})$ is $\|(\nabla f)(\mathbf{x})\|$. This maximum value occurs when $\mathbf{u}$ has the same direction as the gradient vector $(\nabla f)(\mathbf{x})$.*

Proof. We compute

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \, \|u\| \cos\theta = \|\nabla f\| \cos\theta,$$

where θ denotes the angle between ∇f and $\mathbf{u}$. Because the norm of a vector is always nonnegative, this value is maximized when $\cos\theta$ is maximized, i.e. when $\cos\theta = 1$, which is equivalent to $\theta = 0$. That is, $D_{\mathbf{u}} f$ is maximized when $\mathbf{u}$ points in the same direction as ∇f. In this case, the maximum value is $\|\nabla f\|$. □

(analogy in $\mathbf{R}^1$)
(allude to applications in optimization)

Example 14.32 (S6eQ14.06.29)**.** Find all points at which the direction of fastest change of the function

$$f(x, y) := x^2 + y^2 - 2x - 4y$$

is $\mathbf{i} + \mathbf{j}$.

Solution: By Theorem 14.31, the direction of fastest change of a differentiable function f is in the same direction as the gradient vector ∇f. For the given function f, we compute

$$\nabla f = (2x - 2, 2y - 4).$$

This gradient vector points in the same direction as the vector $\mathbf{i} + \mathbf{j} = (1, 1)$ if and only if there exists a positive scalar $t \in \mathbf{R}_{>0}$ such that

$$(2x - 2, 2y - 4) = \nabla f = t(1, 1) = (t, t).$$

Recall that two vectors are equal if and only if all components are equal, so

$$2x - 2 = t, \qquad 2y - 4 = t.$$

Thus ∇f points in the same direction as the vector $(1,1)$ if and only if

$$2x - 2 = t = 2y - 4 \qquad \Leftrightarrow \qquad y = x + 1 \qquad \text{and} \qquad t > 0.$$

The condition $t > 0$ corresponds to

$$\begin{aligned} 2x - 2 = t > 0 &\qquad \Rightarrow \qquad x > 1 \\ 2y - 4 = t > 0 &\qquad \Rightarrow \qquad y > 2. \end{aligned}$$

Thus we conclude that at all points (x, y) on the open half-line

$$\begin{aligned} L &:= \left\{(x,y) \in \mathbf{R}^2 \,|\, y = x + 1, x > 1, y > 2\right\} \\ &= \left\{\left(\frac{1}{2}t + 1, \frac{1}{2}t + 2\right) \,|\, t > 0\right\}, \end{aligned}$$

the direction of fastest change of f is $\mathbf{i} + \mathbf{j}$. *(add plot)*

14.6.2 Tangent Planes to Level Surfaces

(use $P_0 := (x_0, y_0, z_0)$*)*

Let $S \subseteq \mathbf{R}^3$ be a surface defined by the level set

$$F(x, y, z) = c$$

for some function $F : \mathbf{R}^3 \to \mathbf{R}$ and some $c \in \mathbf{R}$, and let C be a smooth curve that lies on S. Then there exists some continuous vector-valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$ whose image is C.

Because C lies on S, all points $P := (x(t), y(t), z(t)) \in C$ satisfy the defining equation for S:

$$F(\mathbf{r}(t)) = F(x(t), y(t), z(t)) = c.$$

Suppose that $x(t), y(t), z(t)$ are differentiable functions. Then we can differentiate both sides, using the chain rule on the left, to obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0. \tag{14.6.2}$$

Note that

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right), \qquad \mathbf{r}'(t) = \left(x'(t), y'(t), z'(t)\right) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

so we can rewrite (14.6.2) as the inner product

$$\nabla F \cdot \mathbf{r}'(t) = 0.$$

This equation says that at any point $P = \mathbf{r}(t)$ on the curve C, the gradient vector $\nabla F(x, y, z)$ at P is orthogonal to the tangent vector $\mathbf{r}'(t)$.

If we fix a particular point $P := (x_0, y_0, z_0)$ on the surface S and consider all smooth curves C on S that pass through P, we obtain this result for all of them. That is, the gradient vector ∇F is orthogonal to all tangent vectors to S at P. Recall that we can view the tangent plane to a surface S at P to be the collection of all lines tangent to S at P. Thus we can define the **tangent plane to the level surface** $F(x, y, z) = c$ **at the point** P to be the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Catchphrase

∇F is normal to the level sets of F.

This result provides geometric motivation for the method of Lagrange multipliers, discussed in Section 14.8.

The **normal line to** S **at** P is the line passing through P that is perpendicular to the tangent plane. We have seen that the gradient ∇F is perpendicular to the tangent plane to S at P, so an equation for the normal line to S at P is given by the initial point P and the direction vector $\nabla F(x_0, y_0, z_0)$.

(type special case of surfaces)

14.6.3 Physical Interpretation of the Gradient Vector

Direction of steepest ascent (fastest change).

Contour plots.

14.7 Maximum and Minimum Values

Key Ideas

- Definition: critical point
- Second-derivatives test
- Monotonic transformation
  - Definition
  - Application to optimization problems
- Extreme value theorem (hypotheses and conclusion)
- How to solve optimization problems

14.7.1 Introduction

Motivate by reminder: Extreme-value problems in single-variable calculus. Motivate by example: Draw cubic with local max, min in closed domain, one end point not min/max, other end point max on closed domain.

Definition 14.33. local max, local max value, local min, local min value

Theorem 14.34. *Let $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$ have a local minimum or local maximum at a point $(x_0, y_0) \in D$. If the first-order partial derivatives of f exist at (x_0, y_0), then*

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0, \qquad \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

(present proof; draw picture!)
(generalize to $\mathbf{R}^n$)

Definition 14.35. Let $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$. A point $\mathbf{x} \in D$ is a **critical point** of f if all first-order partial derivatives of f at $\mathbf{x}$ equal 0, or if one of these first-order partial derivatives does not exist.

Note carefully this last condition. It is the analog to the case with single-variable functions where a function can have a local minimum or maximum at a corner or cusp, where the derivative of $f(x)$ is not defined.

Also keep in mind that, just as in single-variable calculus, not all critical points are necessarily local minima or maxima.

(draw pictures for $f : \mathbf{R} \to \mathbf{R}$ to illustrate both of the preceding remarks)

Theorem 14.36 (Second-Derivatives Test). *Let $f : \Omega \subseteq \mathbf{R}^2 \to \mathbf{R}$, let $(x_0, y_0) \in \Omega$ such that both first-order partial derivatives of f equal 0 there (so (x_0, y_0) is a critical point of f),* *and let all second-order partial derivatives of* f *be continuous on an open disk with center* (x_0, y_0)*. Let*

$$D := D(x_0, y_0) := \det \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) \\ \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{pmatrix}.$$

(i) If $D > 0$ *and* $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$*, then* (x_0, y_0) *is a local minimum of* f*.*

(ii) If $D > 0$ *and* $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$*, then* (x_0, y_0) *is a local maximum of* f*.*

(iii) If $D < 0$*, then* (x_0, y_0) *is a saddle point of* f *— it is neither a local minimum nor a local maximum.*

(analogy to second-derivative test in single-variable calculus)

Note that under the hypotheses of the second-derivatives test, the Clairaut–Schwarz theorem (Theorem 14.13) implies that

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0).$$

At a saddle point, the graph of a function crosses its tangent plane. *(illustrate with graphic)*

If $D = 0$, the second-derivatives test gives no information.

In practice, one can use common sense as a check (look at dominant terms of the function; see p 926).

(present proof of case (ii) of the second-derivatives test)

In practice, finding minima and maxima can sometimes be made easier by taking a monotonic transformation of the given function. *(explain briefly; illustrate with next example)*

Example 14.37 (S6eQ14.07.42)**.** Find the points on the surface S defined by

$$y^2 = 9 + xz$$

that are closest to the origin.

Solution: Let $P := (x, y, z) \in \mathbf{R}^3$. The distance from P to the origin O is given by

$$d(O, P) = \sqrt{x^2 + y^2 + z^2}.$$

Because distance is always nonnegative, minimizing distance is equivalent to minimizing distance squared. (Note that for $t \geqslant 0$, the map $t \mapsto t^2$ is a monotonic transformation.) If $P \in S$, then (x, y, z) satisfy the defining equation for S. Thus we seek to solve

$$\min_{P \in S} x^2 + y^2 + z^2$$
$$\text{s.t. } y^2 = 9 + xz.$$

In this case, it is particularly convenient to substitute the (equality) constraint into the objective function. Doing so, we can write this minimization problem equivalently as

$$\min_{x,z\in\mathbf{R}} x^2 + xz + 9 + z^2.$$

Denote the objective function

$$f(x,z) := x^2 + xz + 9 + z^2.$$

Note that f is a polynomial, so its first-order partial derivatives exist everywhere. Therefore all critical points of f are points (x,z) such that

$$\frac{\partial f}{\partial x} = 2x + z = 0, \qquad \frac{\partial f}{\partial z} = x + 2z = 0.$$

Solving the first equation for z yields $z = -2x$. Substituting this into the second equation, we find

$$-3x = x - 4x = 0 \qquad \Leftrightarrow \qquad x = 0 \qquad \Leftrightarrow \qquad z = 0.$$

We need to check whether this critical point $(0,0)$ is a local minimum or local maximum or neither. Because (i) all critical points of f have all first-order partial derivatives equal to 0; and (ii) all second-order partial derivatives of f are again polynomials, and hence continuous; the second-derivatives test (Theorem 14.36) applies. We compute

$$D := \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3.$$

Because $D = 3 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 2 > 0$, the second-derivatives test implies that the point $(0,0)$ is a local minimum of f.

Substituting these results into the defining equation for S, we find

$$y^2 = 9 \qquad \Leftrightarrow \qquad y = \pm 3.$$

We conclude that the points $(0, \pm 3, 0)$ are the two points on S that are closest to the origin.

(present Jacobian and Hessian — more systematic method...)

14.7.2 Absolute Minima and Maxima

Definition 14.38. closed

Definition 14.39. bounded

Theorem 14.40 (Extreme Value Theorem)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be closed and bounded, and let $f : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}$ be continuous. Then f attains an absolute minimum and absolute maximum on D.*

(emphasize hypotheses; give examples where conclusion fails if each hypothesis does not hold in $\mathbf{R}^1$*)*

Theorem 14.34 implies that extreme values of f on D are either critical points of f or boundary points of D.

Cookbook Recipe for Finding Absolute Minima and Maxima

1. Find all critical points of f on D, and the corresponding values.
2. Find the extreme values of f on the boundary of D.
3. Find the smallest and largest values among those in the first two steps.

Example 14.41 (S6eQ14.07.34)**.** Find the absolute minimum and maximum values of

$$f(x,y) := xy^2$$

on the domain

$$\Omega := \left\{(x,y) \,|\, x \geqslant 0, y \geqslant 0, x^2+y^2 \leqslant 3\right\}.$$

Solution: Plotting the region Ω in the xy-plane, we see that it is the quarter disc of radius $\sqrt{3}$ in the first quadrant. In particular, Ω is closed and bounded.

The function f is a polynomial and therefore continuous, and the set Ω is closed and bounded. Thus by the extreme value theorem, f achieves a minimum and maximum on Ω.

Moreover, because f is a polynomial, its partial derivatives are defined everywhere, so the only critical points are points (x,y) such that $\nabla f(x,y) = \mathbf{0}$ (the zero vector). Computing the gradient of f, we find

$$\nabla f = \left(y^2, 2xy\right).$$

Thus

$$\nabla f = \mathbf{0} \qquad \Leftrightarrow \qquad y^2 = 0 \text{ and } 2xy = 0 \qquad \Leftrightarrow \qquad y = 0.$$

That is, the critical points of f have the form $(x,0)$, for any $x \in \mathbf{R}$. The value of $f(x,y) = xy^2$ at each of these critical points is 0. Note that these critical points (with $0 \leqslant x \leqslant \sqrt{3}$) lie on the boundary of Ω.

Because $x \geqslant 0$ and $y \geqslant 0$ on Ω, $f(x,y) = xy^2 \geqslant 0$ on Ω, with equality if and only if $x = 0$ or $y = 0$. Thus every critical point $(x,0) \in \Omega$ is a global minimum of f on Ω. Furthermore, every point $(0,y) \in \Omega$ is also a global minimum of f on Ω.

Because f has no interior critical points, it follows that the global maximum of f on Ω must lie on the boundary of Ω. We have shown that the boundary segments corresponding to $x = 0$ and $y = 0$ are global minima of f on Ω, so the global maximum must lie on the quarter-circle boundary segment corresponding to $x^2 + y^2 = 3$. This segment can be parametrized by

$$\mathbf{r}(t) := \left(\sqrt{3}\cos t, \sqrt{3}\sin t\right), \quad t \in \left[0, \frac{\pi}{2}\right].$$

(clarify next bit) Thus along the quarter-circle boundary segment,

$$f(\mathbf{r}(t)) = \left(\sqrt{3}\cos t\right)\left(\sqrt{3}\sin t\right)^2 = 3\sqrt{3}\sin^2 t\cos t = 3\sqrt{3}\cos t - 3\sqrt{3}\cos^3 t,$$

where in the last equality we use the identity $\sin^2 t = 1 - \cos^2 t$ (to simplify the derivative we're about to take).

This is a function of one variable, t, so the candidate minima and maxima are (i) the endpoints and (ii) the critical points. The endpoints of the quarter-circle boundary segment are $(\sqrt{3}, 0)$ and $(0, \sqrt{3})$, which we have already observed are global minima of f on Ω. The critical points are where the first derivative is zero or undefined. We compute

$$\frac{d}{dt} f(\mathbf{r}(t)) = -3\sqrt{3}\left(\sin t - 3\cos^2 t\sin t\right).$$

This derivative is defined for all $t \in [0, \frac{\pi}{2}]$ (indeed, for all $t \in \mathbf{R}$), so the critical points are found by setting this expression equal to 0:

$$\begin{aligned} \sin t\left(1 - 3\cos^2 t\right) = 0 \quad &\Leftrightarrow \quad \sin t = 0 \quad \text{or} \quad 1 - 3\cos^2 t = 0 \\ &\Leftrightarrow \quad t = 0 \quad \text{or} \quad \cos t = \frac{1}{\sqrt{3}}. \end{aligned}$$

(Recall that in our parametrization of the quarter-circle boundary segment, $0 \leqslant t \leqslant \frac{\pi}{2}$.) One can compute that $\cos t = \frac{1}{\sqrt{3}}$ corresponds to $t \approx .9553 \approx 54.7356°$, but as we'll see, we don't need this information.

We can determine whether each of these two critical points is a local minimum or local maximum by either (i) applying the second-derivative test (for single-variable functions), (ii) plugging in the values of t into $f(\mathbf{r}(t))$, or (iii) using logic. The last approach is probably easiest in this case. The parameter value $t = 0$ corresponds to the point $(\sqrt{3}, 0)$, which we have already determined is a global minimum of f on Ω. Above we noted that the extreme value theorem implies that f achieves a global maximum on Ω. It follows that the point corresponding to $\cos t = \frac{1}{\sqrt{3}}$, namely *(draw the right triangle)*

$$\left(\sqrt{3}\frac{1}{\sqrt{3}}, \sqrt{3}\frac{\sqrt{2}}{\sqrt{3}}\right) = \left(1, \sqrt{2}\right),$$

is this global maximum, with corresponding value

$$f\left(1, \sqrt{2}\right) = 1\left(\sqrt{2}\right)^2 = 2.$$

easy-on-eyes summary of results

14.8 Lagrange Multipliers

Key Ideas

- Method of Lagrange multipliers
  - Write objective function and equality constraints
  - Form auxiliary function (the Lagrangian)
  - Set gradient equal to zero and solve

14.8.1 Introduction

Optimization, and constrained optimization more particularly, is ubiquitous in fields such as economics, engineering, operations research, and statistics. The method of Lagrange multipliers is tool to help solve optimization problems.

14.8.2 Geometric Intuition

14.8.3 The Method

The good news is that the basic method of Lagrange multipliers is automated *(find a better adjective)*.

(modify method to use auxiliary function L*; cleaner approach)*

Cookbook method for Lagrange multipliers: Let $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ be differentiable functions, let $c \in \mathbf{R}$, and suppose that $\nabla g \neq \mathbf{0}$ on the level set $g(x_1, \ldots, x_n) = c$. If (!) extreme values of f subject to the constraint $g(x_1, \ldots, x_n) = c$ exist, then they can be found as follows:

1. Find all values $(x_1, \ldots, x_n)$ and λ such that

$$\nabla f(x_1, \ldots, x_n) = \lambda \nabla g(x_1, \ldots, x_n), \qquad g(x_1, \ldots, x_n) = c.$$

   Note that this is a system of $n+1$ (scalar) equations in $n+1$ unknowns. (The first equality is a vector equality; equating components yields n scalar equations.)

2. Evaluate f at each candidate point found in Step 1. The largest (respectively, smallest) value is the maximum (respectively, minimum) value of f subject to the constraint $g(x_1, \ldots, x_n) = c$.

The bad news is that the method of Lagrange multipliers often requires an ad hoc approach if we want a simple solution. It's important to adopt careful, organized analysis, and to keep track of cases.

Example 14.42 (S6eQ14.08.18)**.** Use the method of Lagrange multipliers to find the extreme values of the function

$$f(x,y) := 2x^2 + 3y^2 - 4x - 5$$

on the region

$$\Omega := \left\{(x,y) \in \mathbf{R}^2 \,|\, x^2 + y^2 \leqslant 16\right\}.$$

Solution: Note that (i) f is a polynomial, and therefore continuous; and (ii) Ω is closed and bounded. Thus the extreme value theorem implies that f achieves a minimum and maximum on Ω.

Any minima and maxima on the interior of Ω, i.e. on the set

$$\operatorname{int}\Omega = \left\{(x,y) \in \mathbf{R}^2 \,|\, x^2 + y^2 < 16\right\},$$

must be critical points of f. To find these critical points (if any), we compute

$$\nabla f = (2x - 4, 6y).$$

Setting $\nabla f = \mathbf{0}$ and solving, we find a unique critical point:

$$(x,y) = (2,0).$$

Note that this point lies in $\operatorname{int}\Omega$. We can determine whether it is a local minimum or maximum via the second-derivatives test. We compute

$$D(2,0) = \det\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} = 12.$$

Because $D(2,0) = 12 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 2 > 0$, we conclude that $(2,0)$ is a local minimum of f. The corresponding value is

$$f(2,0) = -5.$$

The extreme values of f on the boundary of Ω, i.e. on the set

$$\operatorname{bd}\Omega = \left\{(x,y) \in \mathbf{R}^2 \,|\, x^2 + y^2 = 16\right\},$$

can be found using the method of Lagrange multipliers. The constraint is *(better to set up all constraints as $= 0$? yes; see below)*

$$g(x,y) := x^2 + y^2 - 16 = 0.$$

So we solve

$$\nabla f = \lambda \nabla g \qquad \Leftrightarrow \qquad (2x - 4, 6y) = \lambda\,(2x, 2y).$$

Equating the first components,

$$2x - 4 = \lambda 2x \qquad \Leftrightarrow \qquad (\lambda - 1)\,x = -2. \tag{14.8.1}$$

Equating the second components,

$$6y = \lambda 2y \qquad \Leftrightarrow \qquad (\lambda - 3)\,y = 0 \qquad \Leftrightarrow \qquad \lambda = 3 \qquad \text{or} \qquad y = 0.$$

We analyze cases based on this second set of results.

Case 1: $y = 0$. Substituting this into the constraint $g(x, y) = 16$, we obtain $x = \pm 4$. Thus two candidate points are $(\pm 4, 0)$.

Case 2: $\lambda = 3$. Substituting this into (14.8.1), we obtain

$$2x = -2 \qquad \Leftrightarrow \qquad x = -1.$$

Substituting $x = -1$ into the constraint $g(x, y) = 16$, we obtain $y = \pm\sqrt{15}$. Thus two candidate points are $(-1, \pm\sqrt{15})$.

Computing the values of f at these four candidate points, we (or better, our friends the computers) find[7]

$$f(-4, 0) = 43, \qquad f(4, 0) = 11, \qquad f(-1, \pm\sqrt{15}) = 46.$$

Combining these results with those obtained for the interior critical points, we conclude that

- The minimum value of f on Ω is -5, which occurs at $(x, y) = (2, 0)$.
- The maximum value of f on Ω is 46, which occurs at $(x, y) = (-1, \pm\sqrt{15})$.

14.8.4 Two or More Constraints

(geometric intuition)

(clarify explanation!!!)

Suppose that we wish to find the minimum or maximum values of $f(x_1, \ldots, x_n)$ subject to several constraints

$$g_1(x_1, \ldots, x_n) = 0, \qquad \ldots, \qquad g_m(x_1, \ldots, x_n) = 0.$$

As before, if a constraint is given in the form $h_i(\mathbf{x}) = c_i$, define $g_i(\mathbf{x}) := h_i(\mathbf{x}) - c_i$, i.e. absorb the constant into the function.

Geometrically, the set of points $\mathbf{x} := (x_1, \ldots, x_n)$ simultaneously satisfying all the constraints $g_i(\mathbf{x}) = 0$ is the intersection of the hypersurfaces $g_i(\mathbf{x}) = 0$. Let C denote this intersection (note that C may be empty). As in the single-constraint case, if f has an extreme value at a point $P \in C$, then ∇f is orthogonal to C at P.

[7] Note that f is an even function of y (i.e. all appearances of y have even powers), so $f(x, -y) = f(x, y)$. In particular, $f(-1, -\sqrt{15}) = f(-1, \sqrt{15})$.

Note that C lies in each hypersurface $g_i = c_i$, so for each i, ∇g_i is orthogonal to C. It follows *(explain)* that at such a point P,

$$\nabla f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}) \qquad \Leftrightarrow \qquad \nabla f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}).$$

As before, consider the auxiliary function

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) := f(x_1, \ldots, x_n) - \sum_{i=1}^{m} \lambda_i g_i(x_1, \ldots, x_n),$$

obtained from the preceding condition by "dropping" the gradient operator. The critical points of this auxiliary function L are where $\nabla L = \mathbf{0}$, i.e. where the partial derivatives with respect to the variables $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$ all equal 0. The partial derivatives of L with respect to x_i and λ_i, respectively, yield

$$\frac{\partial}{\partial x_i} f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_i} g(\mathbf{x}) = 0, \qquad g_i(\mathbf{x}) = 0.$$

These are $n + m$ equations in $n + m$ unknowns.

Example 14.43 (S6eQ14.08.16)**.** Use the method of Lagrange multipliers to find the minimum and maximum values of the function

$$f(x, y, z) := 3x - y - 3z$$

subject to the constraints

$$x + y - z = 0, \qquad x^2 + 2z^2 = 1.$$

Solution: Define

$$g_1(x, y) := x + y - z, \qquad g_2(x, y) := x^2 + 2z^2 - 1.$$

Form the auxiliary function

$$\begin{aligned} L(x, y, z, \lambda_1, \lambda_2) &:= f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z) \\ &= (3x - y - 3z) - \lambda_1 (x + y - z) - \lambda_2 \left(x^2 + 2z^2 - 1\right). \end{aligned}$$

Note that all partial derivatives of L are defined everywhere (because L is a polynomial in the variables $x, y, z, \lambda_1, \lambda_2$), so its critical values are precisely the points $(x, y, z, \lambda_1, \lambda_2)$ where all partial derivatives of L equal 0. Computing these partial derivatives and setting each equal to 0, we have

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 3 - \lambda_1 - 2\lambda_2 x \underset{\text{set}}{=} 0 \\
\frac{\partial L}{\partial y} &= -1 - \lambda_1 \underset{\text{set}}{=} 0 \\
\frac{\partial L}{\partial z} &= -3 + \lambda_1 - 4\lambda_2 z \underset{\text{set}}{=} 0 \\
\frac{\partial L}{\partial \lambda_1} &= x + y - z \underset{\text{set}}{=} 0 \\
\frac{\partial L}{\partial \lambda_2} &= x^2 + 2z^2 - 1 \underset{\text{set}}{=} 0.
\end{aligned}$$

This is a system of five equations in five unknowns. The equation for $\frac{\partial L}{\partial y}$ immediately implies that

$$\lambda_1 = -1.$$

Substituting this into the equations for $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial z}$, we have

$$\lambda_2 x = 2, \qquad \lambda_2 z = -1.$$

These equations imply that $x, z, \lambda_2 \neq 0$. Thus we can solve these equations for λ_2 and set the resulting expressions equal:

$$\frac{2}{x} = \lambda_2 = -\frac{1}{z} \qquad \Leftrightarrow \qquad x = -2z, \qquad x, z \neq 0.$$

Substituting this result into the equation for $\frac{\partial L}{\partial \lambda_2}$ (i.e. the constraint g_2),

$$1 = x^2 + 2z^2 = 6z^2 \qquad \Leftrightarrow \qquad z = \pm\frac{1}{\sqrt{6}},$$

which implies that

$$x = -2z = \mp\frac{2}{\sqrt{6}}.$$

Substituting these results in to the equation for $\frac{\partial L}{\partial \lambda_1}$ (i.e. the constraint g_1),

$$y = -x + z = \pm\frac{3}{\sqrt{6}}.$$

Thus we have two candidate extreme points:

$$\mathbf{x}_- := \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{3}{\sqrt{6}}\right), \qquad \mathbf{x}_+ := \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{3}{\sqrt{6}}\right).$$

It remains to determine which is the minimum and which is the maximum, and their corresponding values. Evaluating f at these two points,[8]

$$f(\mathbf{x}_-) = -\frac{16}{\sqrt{6}}, \qquad f(\mathbf{x}_+) = \frac{16}{\sqrt{6}}.$$

We conclude that $\mathbf{x}_-$ is the minimum with value $-\frac{16}{\sqrt{6}}$, and $\mathbf{x}_+$ is the maximum with value $\frac{16}{\sqrt{6}}$.

[8]We can reduce computation by noting that, for the given function $f(x, y, z) = 3x - y - 3z$,

$$f(-\mathbf{x}) = -f(\mathbf{x}).$$

Because $\mathbf{x}_+ = -\mathbf{x}_-$, it follows that $f(\mathbf{x}_+) = -f(\mathbf{x}_-)$.

Chapter 15

Multiple Integrals

The key concept in integral calculus, whether of single-variable or multivariable functions, is the Riemann sum.

15.0.1 Single-Variable Functions

In single-variable calculus, we defined the definite integral of a real-valued function $f(x)$ over an interval $\Omega := [a, b]$ as follows: *(draw a picture!)*

1. Partition ("chop up") Ω into a finite number N of subintervals Ω_i, each with length $V(\Omega_i)$.

2. In each subinterval Ω_i, choose a point x_i^*.

3. On each subinterval Ω_i, pretend that f is constant, with value $f(x_i^*)$. Then the graph of f over the interval Ω looks like a collection of rectangles, each with base Ω_i and height $f(x_i^*)$.

4. Approximate the area under the graph of f by summing all the rectangles:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{N} V(\Omega_i)f(x_i^*).$$

   The sum on the right is called a **Riemann sum**.

5. In the limit as the number N of subintervals goes to infinity, this approximation equals the area under the curve, i.e.

$$\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{i=1}^{N} V(\Omega_i)f(x_i^*),$$

   provided that

   (i) this limit exists, and

(ii) we choose our partitions of Ω so that in the limit as $N \to \infty$, the length of the largest subinterval goes to 0; this is the case for example if we take all subintervals Ω_i to be the same length.

For a nonnegative function $f(x)$, we interpreted the definite integral $\int_a^b f(x)dx$ as the area under the graph of $f(x)$ (more precisely, between the graph of $f(x)$ and the x-axis) from $x = a$ to $x = b$. More generally, for an arbitrary function $f(x)$, we interpreted the definite integral $\int_a^b f(x)dx$ as the net signed area between the graph of $f(x)$ and the x-axis.

15.0.2 Multivariable Functions

Can we extend the concepts of the Riemann integral and Riemann sums to real-valued multivariable functions $f : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}$ *(change Ω so as not to conflict with region of integration)*? Yes! We take an approach is analogous to the one outlined above:

1. Partition ("chop up") the region Ω over which we wish to integrate f into a finite number N of subregions Ω_i (e.g., squares in $\mathbf{R}^2$, cubes in $\mathbf{R}^3$, hypercubes in $\mathbf{R}^n$).

2. Choose a point $\mathbf{x}_i^*$ in each subregion Ω_i.

3. On each subregion Ω_i, pretend that f is constant, with value $f(\mathbf{x}_i^*)$.

4. Form the Riemann sum
$$\sum_{i=1}^{N} f(\mathbf{x}_i^*)V(\Omega_i).$$

5. Take the limit as the number N of subregions goes to infinity, and define this to be the integral of $f(\mathbf{x})$ over Ω:
$$\int_\Omega f(\mathbf{x})dV := \lim_{N\to\infty} \sum_{i=1}^{N} f(\mathbf{x}_i^*)V(\Omega_i),$$
provided that (i) this limit exists and (ii) we choose our partitions such that $V(\Omega_i) \to 0$ for the "largest" subregion Ω_i.

Again we can interpret the definite integral as a net signed (hyper)volume, but it becomes exceedingly difficult if not impossible to visualize this for $n > 2$. This is where the abstract approach becomes powerful: To define the definite integral, cut up the region of integration into subregions Ω_i, compute volumes $V(\Omega_i)$, and multiply by $f(\mathbf{x}_i)$.

Key idea: Take multiple integral, reduce to iterated single-variable integrals. *(relate to partial derivatives, i.e. all variables but one fixed)*

15.0.3 The Geometry Strikes Back

We will see that the hardest part of computing many integrals is describing the region of integration. This is a geometric problem, not a calculus one.

There was really only one way to describe an interval in $\mathbf{R}^1$: specify its endpoints. This makes integration of single-variable functions (relatively) straightforward. For regions in $\mathbf{R}^2$, $\mathbf{R}^3$, and beyond, we have multiple coordinate systems at our disposal: rectangular and polar in $\mathbf{R}^2$; rectangular, cylindrical, and spherical in $\mathbf{R}^3$; etc. We will have to be careful to compute correct volumes of subregions Ω_i in these alternative coordinate systems. Once we do this, however, integration is nice.

Moreover, in Section 15.7 we will learn a powerful technique called change of variables. *(relate to u-substitution in single-variable calculus)*

15.0.4 Caveat

The concept of the Riemann integral works well for many (though not all!) functions. *(add Lebesgue teaser in down-to-earth terms)*

15.1 Double Integrals over Rectangles

Key Ideas

- Double Riemann sum
- Integrable function
- Average value
- Properties of multiple integrals: linear, monotonic

15.1.1 Volumes and Double Integrals

Special case first: $f(x, y) \geqslant 0$ on a closed rectangle R
S surface $f(x, y)$; compute volume of S
Chop up R, choose sample point, form 3D boxes, add volumes

Definition 15.1. The **double integral** of $f(x, y)$ on the closed rectangle R is

$$\iint_R f(x, y)\,dV := \lim_{N_1, N_2 \to \infty} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} f(x^*_{i,j}, y^*_{i,j})\Delta V,$$

if this limit exists.

double Riemann sum
integrable if the limit of the Riemann sums exists
Continuous functions are integrable. "Not too discontinuous" functions are integrable. In particular, if f is (i) bounded and (ii) continuous except on a finite number of smooth curves in R, then f is integrable on R.
Midpoint rule (not so important — just a method for choosing points in each subregion)
reminder: average value for single-variable function

Definition 15.2. The **average value** of $f(x, y)$ on a closed rectangle R is

$$\frac{1}{V(R)} \iint_R f(x, y)\,dV.$$

15.1.2 Properties of Double Integrals

Double integrals are defined as limits of double sums. It is relatively straightforward to show that this implies that double integrals have properties analogous to those of sums.

Proposition 15.3. *Let* $f, g : \mathbf{R}^2 \to \mathbf{R}$ *be integrable, and let* $R \subseteq \mathbf{R}^2$.

1. *linearity: Let* $c \in \mathbf{R}$. *Then*

$$\iint_R f(x,y) + g(x,y)\ dV = \iint_R f(x,y)\ dV + \iint_R g(x,y)\ dV$$
$$\iint_R cf(x,y)\ dV = c\iint_R f(x,y)\ dV.$$

2. *monotonicity: Suppose that for all* $(x,y) \in R$, $f(x,y) \geqslant g(x,y)$. *Then*

$$\iint_R f(x,y)dV \geqslant \iint_R g(x,y)dV.$$

15.2 Iterated Integrals

Key Ideas

- Iterated integrals
- Fubini's theorem (super important!)

The driving idea that we develop in this section is to reduce a double integral to two single integrals.

15.2.1 Iterated Integrals

Let $f(x, y)$ be integrable on a rectangle $R := [a_1, b_1] \times [a_2, b_2] \subseteq \mathbf{R}^2$. *(motivation: sweep out the entire rectangle* R*)*

We define **partial integration of** $f(x, y)$ **with respect to** x as follows: Fix y, and evaluate

$$\int_{a_1}^{b_1} f(x, y)dx.$$

The result is a single-variable function of y only (x has been integrated out), which we can then integrate:

$$\int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x, y)dx \right] dy.$$

Analogously, if we fix x, then we can define **partial integration of** $f(x, y)$ **with respect to** y:

$$\int_{a_2}^{b_2} f(x, y)dy.$$

The result is a single-variable function of y only (x has been integrated out), which we can then integrate:

$$\int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} f(x, y)dy \right] dx.$$

These two final expressions are called **iterated integrals**, because we perform one partial integral after another. Often the brackets are omitted.

Note the similarity of partial integration to partial differentiation.

Example 15.4 (S6eQ15.02.10)**.** Sketch the region of integration, then calculate the iterated integral

$$\int_0^1 \int_0^3 e^{x+3y} dxdy.$$

15.2.2 Fubini's Theorem

Theorem 15.5 (Fubini's Theorem). *Let* $R := [a_1, b_1] \times [a_2, b_2] \subseteq \mathbf{R}^2$ *be a rectangle, and let* $f(x, y)$ *be continuous on* R. *Then*

$$\iint_R f(x,y)dV = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y)dxdy = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)dydx.$$

(links to animation of Fubini's theorem:)

- *http://srjcstaff.santarosa.edu/~gsturr/StewartAnimations/movies/Sec15.2.1-2.html*

Fubini's theorem holds under weaker hypotheses: f is bounded on R and discontinuous on a finite number of smooth curves.

Note similarity of Fubini's theorem to Clairaut's theorem: change order of partial integration, change order of partial differentiation.

Fubini's theorem is important both theoretically and practically: practically because in many cases, evaluating an iterated integral in one order is much easier than another order.

Example 15.6 (S6eQ15.03.20). Let $R := [0, 1] \times [0, 1]$. Sketch the region of integration, then calculate the double integral

$$\iint_R \frac{x}{1+xy} dV.$$

Solution: Note that the integrand $\frac{x}{1+xy}$ is a rational function, and therefore continuous everywhere it is defined, i.e. at all points (x, y) such that $1 + xy \neq 0$. All points in the region $R = [0, 1] \times [0, 1]$ of integration satisfy this condition. Thus Fubini's theorem applies, and we can evaluate the double integral as an iterated integral in either order.

If we try integrating with respect to x first, then we have

$$\iint_R \frac{x}{1+xy}\, dV = \int_0^1\int_0^1 \frac{x}{1+xy}\, dx\, dy.$$

In the inner integral (with respect to x), y behaves as a constant. This integral is hard to evaluate.

If we try integrating with respect to y first, then we have

$$\iint_R \frac{x}{1+xy}\, dV = \int_0^1\int_0^1 \frac{x}{1+xy}\, dy\, dx.$$

In the inner integral (with respect to y), x behaves as a constant. Let

$$u := 1 + xy, \qquad \text{so} \qquad du = x\, dy.$$

(Remember that x behaves as a constant.) Making this substituting, we have

$$\begin{aligned}\iint_R \frac{x}{1+xy}\,dV &= \int_0^1 \int_{y=0}^{y=1} \frac{1}{u}\,du\,dx \\ &= \int_0^1 [\ln|u|]_{y=0}^{y=1}\,dx \\ &= \int_0^1 [\ln|1+xy|]_{y=0}^{y=1}\,dx \\ &= \int_0^1 [\ln|1+x| - \ln|1|]\,dx \\ &= \int_0^1 \ln|1+x|\,dx.\end{aligned}$$

Note that for $x \in [0,1]$, $1+x \geqslant 0$, so $|1+x| = 1+x$. Thus

$$\iint_R \frac{x}{1+xy}\,dV = \int_0^1 \ln(1+x)dx.$$

Using the substitution

$$v := 1+x, \qquad \text{so} \qquad dv = dx,$$

we evaluate this integral as[1]

$$\begin{aligned}\iint_R \frac{x}{1+xy}\,dV &= \int_{x=0}^{x=1} \ln v\,dv \\ &= [v\ln v - v]_{x=0}^{x=1} \\ &= [(1+x)\ln(1+x) - (1+x)]_{x=0}^{x=1} \\ &= (2\ln 2 - 2) - (-1) \\ &= \ln 4 - 1.\end{aligned}$$

Special case:

$$\iint_R g_1(x)g_2(y)dV = \int_{a_1}^{b_1} g_1(x)dx \int_{a_2}^{b_2} g_2(y)dy$$

Don't memorize this! It follows immediately from linearity. Understand the properties of the multiple integral (which follow from the properties of sums, because multiple integrals are defined as the limit of Riemann sums).

[1]To integrate $\int \ln v\,dv$, we can either perform integration by parts, or remember the general form of the integrated expression (i.e. $v\ln v \pm$something, then differentiate and set equal to $\ln v$ (the original integrand) to remind ourselves what "something" is.

15.3 Double Integrals over General Regions

Key Ideas

- Indicator function
- Double integral over a general bounded region
- Translating limits of integration to geometric sketch of region of integration, and vice versa
- How to compute double integrals

15.3.1 Extending the Double Integral to General Regions

In Section 15.2 we defined the double integral over rectangles in $\mathbf{R}^2$. That is, we know how to integrate functions $f(x, y)$ over rectangles. We would like to be able to integrate functions $f(x, y)$ over more general regions. *(finish thought)*

Definition 15.7. Let $D \subseteq \mathbf{R}^n$ be a set. The **indicator function of** D is the function

$$\begin{aligned} \mathbf{1}_D : \mathbf{R}^n &\to \mathbf{R} \\ \mathbf{x} &\mapsto \begin{cases} 1 & \text{if } \mathbf{x} \in D, \\ 0 & \text{if } \mathbf{x} \notin D. \end{cases} \end{aligned}$$

That is, the indicator function $\mathbf{1}_D(\mathbf{x})$ is the function that returns 1 if $\mathbf{x}$ belongs to D, and 0 if $\mathbf{x}$ does not belong to D.

The indicator function is useful is because it allows us to take a function f defined on an arbitrary (bounded) subset $D \subseteq \mathbf{R}^n$ and "extend" f to a function F defined on a rectangle $R \subseteq \mathbf{R}^n$.

$$F(x, y) := \mathbf{1}_D(x, y) f(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

double integral of f **on** D

$$\iint_D f(x, y)\, dV := \iint_R F(x, y)\, dV.$$

15.3.2 Describing Regions in $\mathbf{R}^2$

Region bounded vertically by two functions of x

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Region bounded horizontally by two functions of y

$$\int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$$

(two or three sketches of each type)

Always sketch the region of integration! Was easy for rectangles; for more general regions, will afford crucial insight into the geometry of the situation (and help us avoid mistakes!). Add direction arrow indicating direction of integration.

(explain two ways of "reading" region from integral, or describing region from sketch:

1. *describing in equations each boundary component, then identifying the relevant region*
2. *sweeping out)*

Sometimes can give integral straight from geometry!

Example 15.8 (S6eQ15.03.12). Evaluate the double integral

$$\iint_D x\sqrt{y^2 - x^2}\, dV,$$

where

$$D := \left\{(x,y) \in \mathbf{R}^2 \,|\, 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant y\right\}.$$

Solution: As usual, we begin by sketching the region of integration. To translate from the limits of integration to a geometric sketch of the region, we work our way in: y goes from 0 to 1; and for each fixed value of y, x goes from 0 to y. This yields the triangle *above the line $y = x$ (sketch the region of integration)*. Performing the integration, we find

$$\begin{aligned}
\int_0^1 \int_0^y x\sqrt{y^2 - x^2}\, dx\, dy &= -\frac{1}{2}\int_0^1 \int_0^y \sqrt{y^2 - x^2}(-2x)\, dx\, dy \\
&= -\frac{1}{2}\int_0^1 \left[\frac{2}{3}\left(y^2 - x^2\right)^{\frac{3}{2}}\right]_{x=0}^{x=y} dy \\
&= -\frac{1}{3}\int_0^1 y^3\, dy \\
&= -\frac{1}{12}\left[y^4\right]_{y=0}^{y=1} \\
&= -\frac{1}{12}.
\end{aligned}$$

N.B. If we had tried to integrate in the other order, i.e. first with respect to y, then with respect to x, we would have obtained the iterated integral

$$\int_0^1 \int_x^1 x\sqrt{y^2 - x^2}\, dy\, dx,$$

which we cannot evaluate directly. Suppose that the (iterated) integral had been given to us in this order. Then we would note that the integrand is continuous on the region of integration, and hence Fubini's theorem applies, allowing us to interchange the order of integration (taking care with the limits of integration!) and obtaining the iterated integral above that we can compute.

Example 15.9 (S6eQ15.03.26)**.** Find the volume of the solid bounded by the cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, and $z=0$ in the first octant (i.e. $x, y, z \geqslant 0$).

Solution: Perhaps the hardest part of this problem is the geometry: What exactly is this solid whose volume we are asked to find? It lies inside the cylinder, above the plane $z=0$, in front of the plane $x=0$, and on the positive y-axis side of the plane $x=2y$. This gives a cylindrical slice above a triangle in the xy-plane.

In the first octant (the region of interest), the equation of the cylinder can be solved for z as

$$z=\sqrt{4-y^2}.$$

Thus we can find the volume of the solid by integrating the cylindrical slice, whose height is given by the preceding equation, over the triangle in the xy-plane *(be more precise with equations for triangle)*. *(sketch the region of integration)*

Let's try to integrate in the order $dy\,dx$:

$$\iint_D z\,dA=\int_0^4\int_{\frac{1}{2}x}^2\sqrt{4-y^2}\,dy\,dx.$$

We cannot evaluate the inner integral directly. We could try a trig substitution, but remember that in multivariable calculus, we have another trick up our sleeve: Fubini. Because the function $\sqrt{4-y^2}$ is continuous on the region of integration, Fubini's theorem applies: exchanging the order of integration yields the same value. Integrating in the order $dx\,dy$, we find

$$\begin{aligned}\iint_D z\,dA&=\int_0^2\int_0^{2y}\sqrt{4-y^2}\,dx\,dy\\&=\int_0^2\sqrt{4-y^2}\,[x]_{x=0}^{x=2y}\,dy\\&=-\int_0^2\sqrt{4-y^2}(-2y)\,dy\\&=-\frac{2}{3}\left[\left(4-y^2\right)^{\frac{3}{2}}\right]_{y=0}^{y=2}\\&=-\frac{2}{3}\,[0-8]\\&=\frac{16}{3}.\end{aligned}$$

15.3.3 Properties of the Double Integral

(fill in equations, details)

linearity
monotonicity
partition of region of integration (additive?)
integral of $f(x, y) = 1$ is area
bounded

$$mV(D) \leqslant \iint_D f(x, y) dV \leqslant MV(D).$$

15.4 Double Integrals in Polar Coordinates

Key Ideas

- Area of polar rectangle
- Riemann sums over polar rectangles
- Double-angle identities ($\sin(2\theta), \cos(2\theta)$)
- How to compute double integrals using polar coordinates
  - Integration factor: $dx\,dy \rightsquigarrow r\,dr\,d\theta$

In our study of geometry in $\mathbf{R}^2$, we developed an alternative to rectangular coordinates: polar coordinates. We saw that some regions in $\mathbf{R}^2$ that would be complicated to describe in rectangular coordinates have simple descriptions in polar coordinates. In this section, we will see how to use this alternative coordinate system to simplify double integrals.

Although the specifics of integration in polar coordinates are important, also stay attuned to the general approach. We will follow an analogous approach to study integration in cylindrical coordinates and spherical coordinates in $\mathbf{R}^3$.

15.4.1 Definition

Recall that polar coordinates in $\mathbf{R}^2$ have the form (r, θ), where r is the distance from the origin to the given point, and θ is a counterclockwise angle from the positive x-axis to the ray from the origin containing the given point.

Suppose that we wish to integrate a function $f(x, y)$ over a polar rectangle

$$[r_1, r_2] \times [\theta_1, \theta_2]$$

in the xy-plane. We follow the same approach as before: Riemann sums. More precisely,

1. Partition the polar rectangle in the r- and θ-directions *(give graphic)*. Using geometry, we compute that each subregion has area

$$\Delta V_{i,j} = \frac{1}{2}\Delta\theta_j r_i^2 - \frac{1}{2}\Delta\theta_j r_{i-1}^2 = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j = r_i^* \Delta r_i \Delta\theta_j, \tag{15.4.1}$$

   where $r_i^* := \frac{1}{2}(r_i + r_{i-1})$ is the "midpoint" of the polar subrectangle in the r direction.

2. In each subregion, choose a point (r_i^*, θ_j^*).

3. On each subregion, pretend that f is constant, with value

$$f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*).$$

4. Form the Riemann sum:

$$\sum_{i,j} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j.$$

Carefully note that the area of the polar rectangle (computed in (15.4.1)) has the form $r\,\Delta r\,\Delta\theta$.

5. Take the limit *(be more precise)*. In the limit,

$$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Reminder of useful trigonometric identities:

$$\sin(2\theta) = 2\sin\theta\cos\theta, \qquad \begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta. \end{aligned}$$

We move between the equivalent expressions for $\cos(2\theta)$ by using the identity

$$\sin^2\theta + \cos^2\theta = 1.$$

Example 15.10 (S6eQ15.R.22)**.** Let $D \subseteq \mathbf{R}^2$ be the region in the first quadrant that lies between the two circles

$$x^2 + y^2 = 1, \qquad x^2 + y^2 = 2.$$

Evaluate the integral

$$\iint_D x\,dA.$$

Solution: *(sketch the region)* This region can be described by the polar rectangle

$$\left\{(r,\theta) \,|\, 1 \leqslant r \leqslant \sqrt{2}, 0 \leqslant \theta \leqslant \frac{\pi}{2}\right\}.$$

Thus in polar coordinates, the given integral writes as

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_1^{\sqrt{2}} r\cos\theta\ r\ dr\ d\theta &= \int_0^{\frac{\pi}{2}}\int_1^{\sqrt{2}} r^2\cos\theta\ dr\ d\theta \\ &= \int_0^{\frac{\pi}{2}}\cos\theta\int_1^{\sqrt{2}} r^2\ dr\ d\theta \\ &= \int_0^{\frac{\pi}{2}}\cos\theta\left[\frac{1}{3}r^3\right]_{r=1}^{r=\sqrt{2}} d\theta \\ &= \frac{2\sqrt{2}-1}{3}\int_0^{\frac{\pi}{2}}\cos\theta\ d\theta \\ &= \frac{2\sqrt{2}-1}{3}\left[\sin\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \frac{2\sqrt{2}-1}{3}.\end{aligned}$$

(clean up) Analogous to what we saw with double integrals over regions in $\mathbf{R}^2$ in rectangular coordinates, we can describe regions in polar coordinates in which one variable is a function of the other.

Example 15.11 (S6eQ15.04.14)**.** Let $D \subseteq \mathbf{R}^2$ be the region in the first quadrant that lies between the circles

$$C_0 : x^2 + y^2 = 1, \qquad\qquad C_1 : x^2 + y^2 = 2x.$$

Evaluate the integral

$$\iint_D x\ dA.$$

Solution: The circle C_0 is a circle of radius 1 centered at $(0,0)$. Completing the square for C_1, we find

$$(x-1)^2 + y^2 = 1,$$

so C_1 is a circle of radius 1 centered at $(1,0)$. *(sketch the region)* Note that the circles intersect at $\theta = \frac{\pi}{3}$. *(justify via inscribed equilateral triangle (quick), also by computing coordinates (x,y) of intersection (less quick, still not bad))*

It is natural (and useful for integration purposes) to decompose the region D in two, D_0 and D_1, depending on the value of θ. For $\theta \in [0, \frac{\pi}{3}]$, r goes from the origin to C_0, i.e. $r \in [0,1]$; denote this D_0. For $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$, r goes from the origin to C_1; denote this D_1. In polar coordinates, the equation for C_1 is

$$r^2 = 2r\cos\theta \qquad\Leftrightarrow\qquad r(r - 2\cos\theta) = 0;$$

because r is not identically 0 for C_1, we must have

$$r = 2\cos\theta.$$

Thus we conclude that for $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$, $r \in [0, 2\cos\theta]$.

Thus in polar coordinates, the original integral writes as

$$\begin{aligned}
\iint_D x\,dA &= \iint_{D_0} x\,dA + \iint_{D_1} x\,dA \\
&= \int_0^{\frac{\pi}{3}}\int_0^1 r\cos\theta\; r\,dr\,d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\cos\theta\; r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{3}} \cos\theta\,d\theta \int_0^1 r^2\,dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\theta \int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= [\sin\theta]_{\theta=0}^{\theta=\frac{\pi}{3}} \left[\frac{r^3}{3}\right]_{r=0}^{r=1} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\theta \frac{1}{3}(2\cos\theta)^3\,d\theta \\
&= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right) + \frac{2}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta \\
&= \frac{\sqrt{3}}{6} + \frac{2}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{8} + \frac{1}{8}\cos(4\theta)\right) d\theta \\
&= \frac{\sqrt{3}}{6} + \frac{2}{3}\left[\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{16}\sin(4\theta)\right]_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}} \\
&= \frac{\sqrt{3}}{6} + \frac{2}{3}\left[\frac{3}{8}\left(\frac{\pi}{6}\right) + \frac{1}{4}\left(0 - \frac{\sqrt{3}}{2}\right) + \frac{1}{16}\left(0 + \frac{\sqrt{3}}{2}\right)\right] \\
&= \frac{\sqrt{3}}{6} + \frac{\pi}{24} - \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{48} \\
&= \frac{5\sqrt{3}}{48} + \frac{\pi}{24} \approx .3113.
\end{aligned}$$

15.4.2 Volumes in Polar Coordinates

Example 15.12 (S6eQ15.04.26)**.** Find the volume of the solid bounded by the two paraboloids

$$z = 3x^2 + 3y^2, \qquad\qquad z = 4 - x^2 - y^2.$$

Solution: *(sketch surfaces)* Let z_1, z_2 denote the two paraboloids, respectively. The two paraboloids intersect when

$$3x^2 + 3y^2 = 4 - x^2 - y^2 \quad\Leftrightarrow\quad 4x^2 + 4y^2 = 4 \quad\Leftrightarrow\quad x^2 + y^2 = 1.$$

Thus the solid bounded by the two paraboloids lies above the disc $D : x^2 + y^2 \leqslant 1$. Its volume is given by

$$\begin{aligned}
\iint_D z_2 - z_1 \, dA &= 4\int_0^{2\pi}\int_0^2 \left(1 - x^2 - y^2\right) r \, dr \, d\theta \\
&= 4\int_0^{2\pi}\int_0^2 \left(1 - r^2\right) r \, dr \, d\theta \\
&= 4\int_0^{2\pi} d\theta \int_0^2 \left(r - r^3\right) dr \\
&= 4(2\pi)\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} \\
&= 2\pi.
\end{aligned}$$

15.5 Applications of Double Integrals

Key Ideas

- Integrating a density (e.g., charge per unit area) yields an amount (e.g., charge)
- Probability density functions

(overview)

15.5.1 Density

(brief discussion of theory)

In some cases we can treat an object as though it were a point mass at the center of mass with the same mass *(be more precise)*.

The **(first) moment about the** x**-axis** and y**-axis**, denoted M_x and M_y, are found by integrating (over the relevant region D) the given density function "weighted by" (i.e. multiplied by) the distance from the x-axis and y-axis, respectively, i.e. weighted by $y^1 = y$ and $x^1 = x$, respectively:

$$M_x := \iint_D y\rho(x,y)\,dA, \qquad M_y := \iint_D x\rho(x,y)\,dA.$$

Similarly, for any integer n we can define nth moments about the x- and y-axes are found by weighting the density function by y^n and x^n, respectively, and integrating over D.

Let m be the total mass of the lamina. The **center of mass** of the lamina is the point $(\bar{x}, \bar{y})$ such that

$$m\bar{x} = M_y = \iint_D x\rho(x,y)\,dA \qquad \text{and} \qquad m\bar{y} = M_x = \iint_D y\rho(x,y)\,dA.$$

These equations are telling us that the object behaves like *(be more precise, and explain)* its entire mass lies at its center of mass $(\bar{x}, \bar{y})$. Note that in these equations, the "barred" variable on the left agrees with the weighting variable in the integral for the moment M_* on the right.

This discussion can be generalized to densities in $\mathbf{R}^n$ for any positive integer n.

Example 15.13 (S6eQ15.05.08)**.** Consider a lamina (i.e. an infinitesimally thin plate) occupying the region $D \subseteq \mathbf{R}^2$ bounded by

$$y = \sqrt{x}, \qquad y = 0, \qquad x = 1$$

and with mass density function $\rho : \mathbf{R}^2 \to \mathbf{R}$ given by

$$\rho(x,y) = x.$$

Find the mass and center of mass of the lamina.

Solution: To find both the density and the center of mass, we'll need to compute integrals over the region D. As always, we start by sketching the region of integration. *(sketch the region of integration)*

The mass m of the lamina is found by integrating the mass density over the entire region:

$$m = \iint_D \rho(x,y)\, dA = \iint_D x\, dA.$$

The integrand $\rho(x,y) = x$ is continuous everywhere, and in particular on the region D. Thus Fubini's theorem applies, and we can evaluate the double integral as an iterated integral in either order. Using the order $dy\, dx$, we compute

$$\begin{aligned} m &= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{x}} x\, dy\, dx \\ &= \int_{x=0}^{x=1} x\, [y]_{y=0}^{y=\sqrt{x}}\, dx \\ &= \int_{x=0}^{x=1} x^{\frac{3}{2}}\, dx \\ &= \left[\frac{2}{5}x^{\frac{5}{2}}\right]_{x=0}^{x=1} \\ &= \frac{2}{5}. \end{aligned}$$

If instead we use the order $dx\, dy$, then we compute

$$\begin{aligned} m &= \int_{y=0}^{y=1} \int_{x=y^2}^{x=1} x\, dx\, dy \\ &= \int_{y=0}^{y=1} \left[\frac{1}{2}x^2\right]_{x=y^2}^{x=1} dy \\ &= \frac{1}{2}\int_{y=0}^{y=1} \left(1 - y^4\right) dy \\ &= \frac{1}{2}\left[y - \frac{1}{5}y^5\right]_{y=0}^{y=1} \\ &= \frac{1}{2}\left[\left(1 - \frac{1}{5}\right) - (0-0)\right] = \frac{2}{5}, \end{aligned}$$

the same result, as required by Fubini's theorem.

The center of mass $(\overline{x}, \overline{y})$ is given by

$$\begin{aligned} \overline{x} &= \frac{1}{m}\iint_D x\rho(x,y)\, dA = \frac{5}{2}\iint_D x^2\, dA, \\ \overline{y} &= \frac{1}{m}\iint_D y\rho(x,y)\, dA = \frac{5}{2}\iint_D xy\, dA. \end{aligned}$$

Using Fubini's theorem to evaluate these integrals, we find

$$\overline{x} = \frac{1}{3}, \qquad \overline{y} = \frac{5}{12}.$$

15.5.2 Probability

(brief discussion of theory)

We will use the following naive definitions. *(explain succinctly and unpretentiously why these definition are naive)*

Definition 15.14. A **probability density function (PDF)** on (an integrable) subset $D \subseteq \mathbf{R}^n$ is a function $f : D \to \mathbf{R}$ that satisfies the following two axioms.

(i) (nonnegative) For all points $\mathbf{x} \in D$, $f(\mathbf{x}) \geqslant 0$.

(ii) (total probability 1)

$$\int_D f \, dV = 1.$$

Definition 15.15. Let X, Y be random variables, and let $f(x, y)$ be their PDF, which we can assume to be defined over $\mathbf{R}^2$.[2] The **expected value of** X, denoted $\mathbf{E}[X]$, is the (first) moment

$$\mathbf{E}[X] := \iint_{\mathbf{R}^2} x f(x, y) \, dA.$$

Similarly, the **expected value of** Y is

$$\mathbf{E}[X] := \iint_{\mathbf{R}^2} y f(x, y) \, dA.$$

Similarly, we can define higher moments (which are useful in statistical applications, and therefore engineering, social sciences, etc.). Sometimes we are interested in centered moments, standardized moments, etc. *(short, crisp discussion with reference; see the Wikipedia article on moment)*

Example 15.16 (S6eQ15.05.28). Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x, y) := \begin{cases} 4xy & \text{if } 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that f is a joint density function.

[2] If $f(x, y)$ is defined on a subset of $\mathbf{R}^2$, then just define it to be 0 outside of this subset.

(b) Let X, Y be random variables whose joint density function is f. Find

$$P\left(X \geqslant \frac{1}{2}\right) \qquad \text{and} \qquad P\left(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2}\right).$$

(c) Find the expected values of X and Y.

Solution: *(standardize notation)*

We check the two defining axioms of a probability density function (PDF). Let $D_1 := [0,1] \times [0,1]$ denote the square where f is not identically 0, and let $D_2 := \mathbf{R}^2 \setminus D_1$ denote the rest of the plane $\mathbf{R}^2$ (i.e. the complement of D_1 in $\mathbf{R}^2$).

1. (nonnegative) Outside the square $D_1 = [0,1] \times [0,1]$, f is identically 0. For all points (x, y) in the square $D_1 = [0,1] \times [0,1]$, we have $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$, so $f(x, y) = 4xy \geqslant 0$. Thus f is nonnegative on its entire domain.

2. (total probability 1) Using Fubini's theorem to evaluate the integral over D_1, we compute

$$\begin{aligned}
\iint_{\mathbf{R}^2} f(x,y)\, dA &= \iint_{D_1} 4xy\, dA + \iint_{D_2} 0\, dA \\
&= 4\int_0^1 \int_0^1 xy\, dx\, dy + 0 \\
&= 4\int_0^1 \left[\frac{1}{2}x^2 y\right]_{x=0}^{x=1} dy \\
&= 2\int_0^1 y\, dy = 1.
\end{aligned}$$

Thus f is a valid PDF.

The probabilities are found by integrating the probability density over the relevant regions. Let

$$A := \left\{(x,y) \in \mathbf{R}^2 \,|\, x \geqslant \frac{1}{2}\right\}, \qquad B := \left\{(x,y) \in \mathbf{R}^2 \,|\, x \geqslant \frac{1}{2} \text{ and } y \leqslant \frac{1}{2}\right\}.$$

Then, using Fubini's theorem, we compute

$$\begin{aligned}
P\left(X \geqslant \frac{1}{2}\right) &= \iint_A f(x,y)\, dA \\
&= \iint_{A \cap D_1} 4xy\, dA + \iint_{A \setminus D_1} 0\, dA \\
&= 4\int_{y=0}^{y=1} \int_{x=\frac{1}{2}}^{x=1} xy\, dx\, dy \\
&= \frac{3}{2}\int_{y=0}^{y=1} y\, dy \\
&= \frac{3}{4}
\end{aligned}$$

and

$$\begin{aligned}
P\left(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2}\right) &= \iint_B f(x,y)\,dA \\
&= \iint_{B\cap D_1} 4xy\,dA + \iint_{B\setminus D_1} 0\,dA \\
&= 4\int_{y=0}^{y=\frac{1}{2}} \int_{x=\frac{1}{2}}^{x=1} xy\,dx\,dy \\
&= \frac{3}{2}\int_{y=0}^{y=\frac{1}{2}} y\,dy \\
&= \frac{3}{16}.
\end{aligned}$$

The expected values of X and Y are found by weighting the density by x and y, respectively, and integrating over the relevant region (in this case, the entire domain of f). For example, the expected value of X is

$$\begin{aligned}
\mathbf{E}[X] &= \iint_{\mathbf{R}^2} xf(x,y)\,dA \\
&= \iint_{D_1} 4x^2y\,dA + \iint_{\mathbf{R}^2\setminus D_1} 0\,dA \\
&= \dots \\
&= \frac{2}{3}.
\end{aligned}$$

(fill in steps of computation)

We could perform an analogous computation to compute the expected value of Y. Alternatively, we can use symmetry (!): The probability density function $f(x,y)$ is symmetric in x and y (it is important that the region on which f is nonzero is also symmetric in x and y!), and therefore the expected value of Y equals the expected value of X (changing the names of x and y in the integral for $\mathbf{E}[Y]$ gives exactly the integral for $\mathbf{E}[X]$ that we just computed!). Thus

$$\mathbf{E}[Y] = \frac{2}{3}.$$

15.6 Triple Integrals

Key Ideas

- Definition using Riemann sums
- How to compute triple integrals
  - Rectangular coordinates
  - Cylindrical coordinates: integration factor r (same as polar coordinates)
  - Spherical coordinates: integration factor $\rho^2 \sin\varphi$

Triple integrals are defined in the same way that we defined double integrals: as the limit of appropriate Riemann sums. *(emphasize generality to $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ — always defined the same way!)*

For the general case of a function f of n variables, Fubini's theorem states that if f is continuous (more generally, "not too discontinuous" *(explain in appendix)*), then we can write the multiple integral as an iterated integral in any order, and the result will be the same. For functions of three variables, there are ${}_3P_3 = \frac{3!}{(3-3)!} = 6$ different orders in which we can write the iterated integral. Which order we prefer will depend on the geometry of the region over which we integrate, and the function that we are integrating.

Example 15.17 (S6eE15.06.03)**.** Let $E \subseteq \mathbf{R}^3$ be the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Evaluate the integral

$$\iiint_E \sqrt{x^2+z^2}\, dV.$$

Solution: *(sketch the region)*

Attempt 1: Let's view z as a function of x, y:

$$z = \pm\sqrt{y - x^2}.$$

Obtain

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx.$$

Very difficult.

Attempt 2: Let's view y as a function of x, z:

$$y = x^2 + z^2.$$

Then the triple integral writes as

$$\iint_D \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dA.$$

In the partial integral with respect to y, the integrand $\sqrt{x^2+z^2}$ behaves as a constant, so we can pull it out of the innermost integral, obtaining

$$\iint_D \left(4 - x^2 - z^2\right) \sqrt{x^2+z^2}\, dA.$$

One can check that this integral is most easily evaluated by converting to polar coordinates in the xz-plane:

$$x = r\cos\theta, \qquad\qquad z = r\sin\theta.$$

In these coordinates, the integral becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^2 \left(4-r^2\right) r\, r\, dr\, d\theta &= \int_0^{2\pi} d\theta \int_0^2 \left(4r^2 - r^4\right) dr \\ &= 2\pi \left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_{r=0}^{r=2} \\ &= \frac{128\pi}{15}.\end{aligned}$$

15.6.1 Triple Integrals in Cylindrical Coordinates

Reminder of cylindrical coordinates (r, θ, z).

If we step back and consider our (successful) approach to evaluating the integral in Example 15.17, we observe that essentially what we have done is change our coordinate system to cylindrical coordinates (r, θ, y), i.e. (r, θ) in the xz-plane. Cylindrical coordinates are most useful when our integration is over regions in $\mathbf{R}^3$ that possess cylindrical symmetry.

First we need to see what form Riemann sums, and hence triple integrals, take in cylindrical coordinates. Suppose that we want to integrate a function $f(x, y, z)$ over a "cylindrical rectangle" $[r_1, r_2] \times [\theta_1, \theta_2] \times [z_1, z_2]$ in xyz-space. The infinitesimal volume element is *(draw picture)*

$$r\, dr\, d\theta\, dz.$$

Example 15.18 (S6eQ15.07.28)**.** Evaluate the integral

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx.$$

Solution: As usual, we first sketch the region of integration. *(sketch it)* The given integral would be difficult to evaluate directly. The region of integration possesses cylindrical symmetry, so we write the given integral in cylindrical coordinates and see what we obtain. From our sketch, the region of integration can be described by

$$\left\{ (r, \theta, z) \,|\, r \in [0, 3], \theta \in [0, \pi], z \in \left[0, 9 - r^2\right] \right\}.$$

Thus the integral writes as

$$\begin{aligned}\int_0^{\pi}\int_0^3\int_0^{9-r^2} r\,r\,dz\,dr\,d\theta &= \int_0^{\pi} d\theta \int_0^3 r^2 \int_0^{9-r^2} dz\,dr \\ &= \pi\int_0^3 r^2\left(9-r^2\right)dr \\ &= \pi\left[3r^3-\frac{1}{5}r^5\right]_{r=0}^{3} \\ &= \frac{272\pi}{5}.\end{aligned}$$

Example 15.19 (S6eQ15.07.20)**.** Let $E \subseteq \mathbf{R}^3$ be the region enclosed by the planes

$$x = 0, \qquad\qquad x = y + z + 5$$

and the cylinders

$$y^2 + z^2 = 4, \qquad\qquad y^2 + z^2 = 9.$$

Evaluate the triple integral

$$\iiint_E y\,dV.$$

The iterated integral is

$$\int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 5} r\cos\theta\; r\,dx\,dr\,d\theta$$

15.6.2 Triple Integrals in Spherical Coordinates

Reminder of spherical coordinates (ρ, θ, φ)

Volume element *(draw picture, or project it!)*

$$\Delta V \approx (\Delta\rho)(\rho_i\Delta\varphi)(\rho_i\sin\varphi_k\Delta\theta) = \rho_i^2\sin\varphi_k\,\Delta\rho\,\Delta\theta\,\Delta\varphi.$$

Using the mean value theorem, one can show that this approximation holds exactly for a suitable point $(\rho_i^*, \theta_j^*, \varphi_k^*)$ in the partition element. This yields a Riemann sum

write it out

Example 15.20 (S6eQ15.08.22)**.** Let $H \subseteq \mathbf{R}^3$ be the solid hemisphere

$$H : \left\{(x, y, z) \in \mathbf{R}^3 \,|\, x^2 + y^2 + z^2 \leqslant 9, z \geqslant 0\right\}.$$

Evaluate the triple integral

$$\iiint_H \left(9 - x^2 - y^2\right) dV.$$

Example 15.21 (S6eQ15.08.26)**.** Let $E \subseteq \mathbf{R}^3$ be the region between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\varphi = \frac{\pi}{3}$. Evaluate the triple integral

$$\iiint_E xyz \, dV.$$

15.7 Change of Variables

Key Ideas

- Change of variables: single-variable calculus
- Injective, surjective, bijective functions
- Jacobian matrix, Jacobian determinant
- Change-of-variables theorem: multivariable calculus
- How to apply a change of variables

(use MT as primary source? even MT not great...)

In single-variable calculus, we used change of variables ("u-substitution", "trig substitution", etc.) to simplify and evaluate integrals. For the same reasons, we can use change of variables in multivariable calculus. Often we'll effect a change of variables to simplify the algebraic description of the region of integration.

15.7.1 Single-Variable Calculus

Let's briefly recall how change of variables works in single-variable calculus.

Suppose that we have an integral $\int f(x)\,dx$ that is difficult to evaluate as given, for example,

$$\int_0^1 x^3\sqrt{x^2+1}\,dx. \tag{15.7.1}$$

If we can write x as a function of another variable u, say $x = g(u)$, then $dx = g'(u)du$, and we can write the given integral as

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du.$$

In our running example (15.7.1), we might set $u := x^2 + 1$. Then $x^2 = u - 1$ and $du = 2x\,dx$. Making these substitutions, and updating the limits of integration in terms of u, we obtain

$$\frac{1}{2}\int_1^2 (u-1)\sqrt{u}\,du = \frac{1}{2}\int_1^2 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du.$$

This is easier for us to (see how to) integrate, compared to the original integral.

Viewing our definition $u := x^2 + 1$ another way, we are setting

$$x = g(u) := \sqrt{u-1}.$$

Then

$$\frac{dx}{du} = g'(u), \qquad \Leftrightarrow \qquad dx = \frac{du}{2\sqrt{u-1}}.$$

Substituting these into the original integral (15.7.1), and updating the limits of integration in terms of u, we obtain

$$\begin{aligned}\int_1^2 (g(u))^3 \sqrt{(g(u))^2+1}\; g'(u)\; du &= \int_1^2 (u-1)^{\frac{3}{2}} \sqrt{u-1+1} \frac{du}{2\sqrt{u-1}} \\ &= \frac{1}{2}\int_1^2 (u-1)\sqrt{u}\; du,\end{aligned}$$

as before.

It is this last viewpoint that we seek to generalize to the case of integrals of functions of several variables.

15.7.2 Injective and Surjective Functions; Jacobians

Let X, Y be sets (e.g., vector spaces), and let $f : X \to Y$ be a function.

Definition 15.22. f is **injective** (or **one-to-one**) if for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \qquad \Rightarrow \qquad x_1 = x_2.$$

That is, f is injective if distinct points in the domain are mapped to distinct points in the codomain.

Definition 15.23. f is **surjective** (or **onto**) if for all $y \in Y$, there exists some $x \in X$ such that

$$f(x) = y.$$

That is, f is surjective if every point in the codomain is "hit" by some element in the domain.

(draw pictures!)

Definition 15.24. f is **bijective** if it is both injective and surjective.

Example 15.25. Consider the following functions.

1. $f : \mathbf{R} \to \mathbf{R}$, $f(x) = e^x$. Injective but not surjective.
2. $f : \mathbf{R} \to \mathbf{R}_{>0}$, $f(x) = e^x$. Injective and surjective.
3. $f : \mathbf{R} \to \mathbf{R}_{\geqslant 0}$, $f(x) = x^2$. Surjective but not injective.
4. $f : \mathbf{R} \to \mathbf{R}_{\geqslant 0}$, $f(x) = |x|$. Surjective but not injective.
5. $f : \mathbf{R} \to \mathbf{R}$, $f(x) = cx$. If $c \neq 0$, then injective and surjective. If $c = 0$, neither injective nor surjective.

Next we present the definition of the Jacobian matrix and determinant, in the case $n = 2$ (see the remark following the definition).

Definition 15.26. Let $U \subseteq \mathbf{R}^2$, let

$$\begin{aligned} T : U &\to \mathbf{R}^2 \\ (u, v) &\mapsto (x(u, v), y(u, v)) \end{aligned}$$

be a $\mathscr{C}^1$-transformation (i.e., the functions x and y are continuously differentiable). The **Jacobian matrix** of T is the derivative $\mathbf{D}T$ of T, i.e.

$$J_T = \mathbf{D}T = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

The **Jacobian determinant** of T is the determinant of this matrix, i.e.

$$\det J_T = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

The definition of the Jacobian matrix and Jacobian determinant generalizes in the way you think to higher dimensions, i.e. to $\mathscr{C}^1$-functions from (a subset of) $\mathbf{R}^n$ to (a subset of) $\mathbf{R}^n$. (In particular, write down for yourself these definitions for the case $n = 3$. This will be of particular importance to us in what follows.)

15.7.3 Multivariable Calculus

(geometric motivation; cross product in $\mathbf{R}^2$? misleading for higher dimensions; focus on Jacobian determinant instead?)

(briefly discuss images of maps)

Example 15.27. Consider the mapping

$$\begin{aligned} T : D^* := [0, +\infty) \times [0, 2\pi] &\to \mathbf{R}^2 \\ (r, \theta) &\mapsto (r \cos\theta, r \sin\theta). \end{aligned}$$

This is the usual transformation from polar coordinates to rectangular coordinates in $\mathbf{R}^2$. Every point $(x, y) \in \mathbf{R}^2$ can be described by a point $(r, \theta) \in D^*$, so the map T is surjective. Is T injective? No: For every point of the form $(0, \theta) \in D^*$, $T(0, \theta) = (0, 0)$. However, if we remove the line $\{(0, \theta) \mid \theta \in [0, 2\pi]\}$ from the domain D^*, obtaining the domain

$$D^{**} := (0, +\infty) \times [0, 2\pi],$$

then we can show that the map T restricted to D^{**} is injective. (However, it is not surjective; what point does it miss?)

Theorem 15.28. *Let* $U \subseteq \mathbf{R}^2$*, let*

$$\begin{aligned} \Phi : U &\to \mathbf{R}^2 \\ (u, v) &\mapsto (x(u, v), y(u, v)) \end{aligned}$$

be a $\mathscr{C}^1$*-transformation such that*

(i) $\det J_\Phi(u, v) \neq 0$ *for all points* $(u, v) \in U$*;*

(ii) Φ *maps* $S \subseteq \mathbf{R}^2$ *(in the* uv*-plane) to* $R \subseteq \mathbf{R}^2$ *(in the* xy*-plane);*

(iii) Φ *is injective except possibly on the boundary of* S*.*

Then if $f : R \to \mathbf{R}$ *is continuous,*

$$\iint_R f(x, y)\ dx\ dy = \iint_S f(\Phi(u, v))\ |\det J_\Phi(u, v)|\ du\ dv.$$

(pictures!)

Note that the change-of-variables integral includes the absolute value (!) of the Jacobian determinant. Practically, to change variables, we convert old variables into new variables, and we include the absolute value (!) of the Jacobian determinant.

Example 15.29. Consider the transformation Φ from polar coordinates (r, θ) to Cartesian coordinates (x, y) in $\mathbf{R}^2$:

$$x := r\cos\theta, \qquad\qquad y := r\sin\theta.$$

The Jacobian matrix for this transformation is

$$J_\Phi(r, \theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Its determinant (the Jacobian determinant) is

$$\det J_\Phi(r, \theta) = r\cos^2\theta + r\sin^2\theta = r.$$

Thus

$$\begin{aligned} \iint_D f(x, y)\ dx\ dy &= \iint_D f(\Phi(r, \theta))\ |\det J_\Phi(r, \theta)|\ dr\ d\theta \\ &= \iint_D f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta. \end{aligned}$$

This agrees with the result we obtained by applying the theory of Riemann sums directly to regions of $\mathbf{R}^2$ described using polar coordinates.

Example 15.30. Consider the transformation Φ from spherical coordinates (ρ, θ, φ) to Cartesian coordinates (x, y, z) in $\mathbf{R}^3$:

$$\begin{aligned} \Phi : [0, +\infty] \times [0, 2\pi] \times [0, \pi] &\to \mathbf{R}^3 \\ (\rho, \theta, \varphi) &\mapsto (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) . \end{aligned}$$

The Jacobian matrix for this transformation is

$$J_\Phi(\rho, \theta, \varphi) = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix} .$$

Expanding by minors across the bottom row, we compute the Jacobian determinant to be

$$\begin{aligned} \det J_\Phi(\rho, \theta, \varphi) &= (-1)^{3+1} \cos\varphi \det \begin{pmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \end{pmatrix} + 0 \\ &\quad - (-1)^{3+3} \rho\sin\varphi \det \begin{pmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{pmatrix} \\ &= \cos\varphi \left(-\rho^2 \sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right) \\ &\quad - \rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right) \\ &= -\rho^2\sin\varphi\cos\varphi - \rho^2\sin^3\varphi \\ &= -\rho^2\sin\varphi. \end{aligned}$$

Therefore

$$\begin{aligned} \iiint_E f(x, y, z)\, dx\, dy\, dz &= \iiint_E f(\Phi(\rho, \theta, \varphi))\, |\det J_\Phi(\rho, \theta, \varphi)|\, d\rho\, d\theta\, d\varphi \\ &= \iiint_E f(\Phi(\rho, \theta, \varphi))\, \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi. \end{aligned}$$

This agrees with the result we obtained by applying the theory of Riemann sums directly to regions of $\mathbf{R}^3$ described using spherical coordinates.

Example 15.31 (S6eQ15.09.20)**.** Let $R \subseteq \mathbf{R}^2$ be the rectangle enclosed by the lines

$$x - y = 0, \qquad x - y = 2, \qquad x + y = 0, \qquad x + y = 3.$$

Evaluate the integral

$$\iint_R (x + y)e^{x^2 - y^2}\, dA.$$

Solution: First note that we cannot evaluate the given integral directly. The form of the integral *(be more precise)* (and the given region) suggests that we try the change of variables

$$u := x + y, \qquad v := x - y.$$

Solving these equations for x, y in terms of u, v, we find

$$x = \frac{u+v}{2}, \qquad y = \frac{u-v}{2}.$$

Thus the corresponding transformation is

$$\Phi : \mathbf{R}^2 \to \mathbf{R}^2$$
$$(u, v) \mapsto \left(\frac{u+v}{2}, \frac{u-v}{2}\right).$$

The Jacobian matrix of the transformation Φ is

$$J_\Phi(u, v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

so

$$\det J_\Phi(u, v) = -\frac{1}{2}.$$

Therefore

$$\iint_R (x+y)e^{x^2-y^2}\, dx\, dy = \frac{1}{2}\iint_R ue^{uv}\, du\, dv. \tag{15.7.2}$$

Under the transformation Φ, the lines describing the region R write as

$$v = 0, \qquad v = 2, \qquad u = 0, \qquad u = 3.$$

By Fubini's theorem, we can write the double integral on the right in (15.7.2) as the iterated integral

$$\begin{aligned} \frac{1}{2}\int_0^3\int_0^2 ue^{uv}\, dv\, du &= \frac{1}{2}\int_0^2 \left[e^{uv}\right]_{v=0}^{v=2}\, du = \frac{1}{2}\int_0^2 \left(e^{2u}-1\right) du \\ &= \frac{1}{2}\left[\frac{1}{2}e^{2u} - u\right]_{u=0}^{u=3} = \frac{1}{2}\left[\left(\frac{1}{2}e^6 - 3\right) - \left(\frac{1}{2} - 0\right)\right] \\ &= \frac{e^6-7}{4}. \end{aligned}$$

In Example 15.31, the change of variables simplified not only the integrand, but also the limits of integration. Thus we see that we can gainfully employ the method of change of variables even if the integrand is not complicated.

Example 15.32 (MT5eQ06.03.32)**.** Let $R \subseteq \mathbf{R}^2$ denote the region inside $x^2 + y^2 = 1$ and outside $x^2 + y^2 = 2y$ satisfying $x, y \geqslant 0$. Consider the change of coordinates

$$u(x, y) := x^2 + y^2, \qquad v(x, y) := x^2 + y^2 - 2y.$$

Sketch the corresponding regions in the xy- and uv-planes. Then use the change of coordinates to compute the integral

$$\iint_R xe^y\, dx\, dy.$$

Solution: *(type up solution)*

Chapter 16

Integral Theorems of Vector Calculus

In this chapter we develop multivariable analogs to the fundamental theorem of calculus from the single-variable setting. *(say more)*

16.1 Vector Fields

Key Ideas

- Vector field
- Conservative (a.k.a. gradient) vector field; potential function

16.1.1 Vector Fields

Definition 16.1. Let $\Omega \subseteq \mathbf{R}^n$. A **vector field** on Ω is a function $F : \Omega \to \mathbf{R}^n$.

(show some plots of vector fields)
(provide link to online vector field plotter)

Example 16.2 (S6eQ16.01.08)**.** Sketch the vector field

$$\begin{aligned} \mathbf{F} : \mathbf{R}^2 &\to \mathbf{R}^2 \\ (x, y) &\mapsto -y\mathbf{k}. \end{aligned}$$

Solution: *(include graph, brief explanation)*

16.1.2 Gradient Fields

Let $\Omega \subset \mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a differentiable real-valued function. Recall that the **gradient** of f is the vector of partial derivatives:

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Note that the gradient ∇f is a function that takes n variables $(x_1, \ldots, x_n)$ as input and that outputs n variables (corresponding to the n different partial derivatives). That is,

$$\nabla f : \Omega \to \mathbf{R}^n,$$

i.e. the gradient of a function is a vector field (defined on the same domain as f).

Definition 16.3. Let $\Omega \subseteq \mathbf{R}^n$, and let $\mathbf{F} : \Omega \to \mathbf{R}^n$ be a vector field. $\mathbf{F}$ is a **gradient vector field** or **conservative vector field** if there exists some real-valued function $f : \Omega \to \mathbf{R}$ such that

$$\mathbf{F} = \nabla f.$$

In this case, the real-valued function f is called the **potential function** for $\mathbf{F}$.

Thus, given a differentiable real-valued function f, computing the gradient vector field of f means simply computing the gradient ∇f.
(remark about gradient vectors being perpendicular to level curves of f)

16.2 Line Integrals

Key Ideas

- Smooth curve
- Line integral of real-valued function
  - With respect to arc length
  - With respect to variable
- Orientation of a curve
- Line integral of a vector-valued function

(organize this section more cleanly)

Loosely speaking, a line integral is an integral of a function (real-valued or vector-valued) along a curve. In this section we develop an adequate definition that can handle both cases. (remark about "overloading" functions in computer science)

16.2.1 Smooth Curves

Recall (?) the following definition.

Definition 16.4. The curve $C \subseteq \mathbf{R}^n$ is *smooth* if there exists a parametrization $\mathbf{r} : I \to \mathbf{R}^n$ of C (where $I \subseteq \mathbf{R}$ is an interval) such that

(i) $\mathbf{r}'$ is continuous, and

(ii) for all parameter values $t \in I$, $\mathbf{r}'(t) \neq \mathbf{0}$.

16.2.2 Line Integrals of Real-Valued Functions

(intro comment about viewing line integral of real-valued function as Riemann sum of $f(x_i^*)$ "weighted" by aspect of C (e.g., length of segment, change in x, etc.))

With Respect to Curve Length

Consider first the line integral of a real-valued function $f(x, y)$ defined on a smooth curve $C \subseteq \mathbf{R}^2$, parametrized by the vector equation

$$\mathbf{r}(t) = (x(t), y(t)) .$$

Let P, Q be points on the curve C, corresponding to the parameter values t_P, t_Q, respectively. If we view the curve C is an interval twisted in $\mathbf{R}^2$, then the integral of f over C resembles the integral of a function over an interval in $\mathbf{R}^1$ — the usual integral in single-variable calculus. To compute the line integral of f over C, we set up a Riemann sum analogous to the single-variable case: *(draw picture!) (reword the following; standardize throughout text, to emphasize unity of approach)*

1. Partition the curve C from P to Q. (Let P_i denote the endpoints of the partition elements, and let Δs_i denote the length of C from P_{i-1} to P_i.)
2. Choose a point P_i^* in each partition. (Note that the point P_i^* corresponds to some parameter value t_i^*, i.e. $P_i^* = (x(t_i^*), y(t_i^*))$.)
3. Pretend f is constant with value $f(P_i^*)$ on the segment of C from P_{i-1} to P_i.
4. Compute the Riemann sum

$$\sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta s_i. \tag{16.2.1}$$

Just as in the single-variable case, we define the integral to be the limit of these Riemann sums as the number of partition elements of C goes to infinity (more precisely, as the length of the partition elements goes to zero).

Definition 16.5. Let $C \subseteq \mathbf{R}^n$ be a smooth curve, let $\Omega \subseteq \mathbf{R}^n$ be a region containing the curve C, and let $f : \Omega \to \mathbf{R}$ be a real-valued function. The **line integral of** f **along** C **with respect to curve length** is

$$\int_C f(x,y)\, ds = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta s_i,$$

if this limit exists.

We have seen *(add ref)* that the length of the curve C parametrized by $\mathbf{r}(t)$ on the parameter interval $t \in [a, b]$ is

$$\int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

If f is continuous *(not too discontinuous?)*, then

$$\begin{aligned}\int_C f(x,y)\, ds &= \int_{t_P}^{t_Q} f\left(x(t), y(t)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_{t_P}^{t_Q} f(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt. \end{aligned} \tag{16.2.2}$$

There is a potential issue with this computation: (have students raise this objection) We use a particular parametrization $\mathbf{r}$ of the curve C. The line integral of f along C should not depend on the parametrization. Indeed, one can show that it does not. *(provide proof)*

Consider the special case that C is the line interval from $P = (0, a)$ to $Q = (0, b)$, parametrized by

$$\mathbf{r}(t) = (t, 0), \qquad t \in [a, b].$$

Then

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + (0)^2} = \frac{dx}{dt},$$

so in this case (16.2.2) reduces to

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx.$$

This is the usual single-variable integral on the interval $[a, b]$ (view $f(x, 0) = g(x)$).

We defined the line integral of f along a smooth curve C. We can easily extend this definition to the line integral of f along a piecewise-smooth curve. (draw picture; have students propose approach)

Example 16.6 (S6eQ16.02.08)**.** Consider the curve $C \subseteq \mathbf{R}^2$ consisting of the top half of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and the line segment from $(-1, 0)$ to $(-2, 3)$. Evaluate the line integral

$$\int_C \sin x\, dx + \cos y\, dy.$$

Solution: *(draw picture; solve)*

With Respect to Variable

Let's return to the Riemann sum (16.2.1). Instead of weighting $f(x_i^*)$ by the length of the curve segment from P_{i-1} to P_i, we could weight it by just the change in x, or just the change in y. These give

$$\sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta x_i, \qquad \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta y_i,$$

respectively. Taking the limit as $N \to \infty$ (more precisely, as $\Delta x_i \to 0$ and $\Delta y_i \to 0$), if these limits exist, we obtain the **line integral of** f **along** C **with respect to** x and **with respect to** y, respectively:

$$\int_C f(x, y)\, dx := \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta x_i$$

$$\int_C f(x, y)\, dy := \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta y_i.$$

(line integrals with respect to different variables occurring together; expose students to (bad) notation)

We have seen that the line integral of f along C with respect to curve length does not depend on the parametrization of C. However, in general the line integral of f along C with respect to curve length does depend on the curve C, as illustrated in this next example.

Example 16.7 (S6eE16.02.04)**.** In this example we consider two curves in $\mathbf{R}^2$ from the point $P := (-5, -3)$ to the point $Q := (0, 2)$: Let C_1 be the line segment from P to Q, and let C_2 be the arc of the parabola $x = 4 - y^2$ from P to Q. Evaluate the line integral

$$\int_C y^2\,dx + x\,dy$$

for $C = C_1, C_2$.

Solution: *(draw picture!)* We consider each curve in turn.

C_1: We can parametrize the line segment by *(provide more-detailed computation)*

$$\mathbf{r}(t) := (-5 + 5t, -3 + 5t)\,, \qquad t \in [0, 1].$$

Thus

$$dx = 5dt, \qquad dy = 5dt,$$

and the given line integral writes as

$$\begin{aligned}
\int_{C_1} y^2\,dx + x\,dy &= \int_0^1 (-3 + 5t)^2 (5\,dt) + (-5 + 5t)(5\,dt) \\
&= 5 \int_0^1 \left(25t^2 - 25t + 4\right)\,dt \\
&= 5 \left[\frac{25}{3}t^3 - \frac{25}{2}t^2 + 4t\right]_{t=0}^{t=1} \\
&= -\frac{5}{6}.
\end{aligned}$$

C_2: The parabola $x = 4 - y^2$ gives x as a function of y, so we can use this for our parametrization of the curve C_2:

$$\mathbf{r}(y) := \left(4 - y^2, y\right)\,, \qquad y \in [-3, 2].$$

Thus

$$dx = -2y\,dy, \qquad dy = dy,$$

and the given line integral writes as *(verify final result)*

$$\begin{aligned}
\int_{C_1} y^2\,dx + x\,dy &= \int_{-3}^{2} y^2\,(-2y\,dy) + \left(4 - y^2\right)\,dy \\
&= \int_{-3}^{2} \left(-2y^3 - y^2 + 4\right)\,dy \\
&= \left[-\frac{1}{2}y^4 - \frac{1}{3}y^3 + 4y\right]_{y=-3}^{y=2} \\
&= \left(-4 - \frac{8}{3} + 8\right) - \left(-\frac{81}{2} + 9 - 12\right) \\
&= \frac{4}{3} - \left(-\frac{87}{2}\right) = \frac{269}{6}.
\end{aligned}$$

Note that nothing we have done in this section relies on the fact that our function f is defined on $\mathbf{R}^2$. Thus everything we have done — the approach with Riemann sums, (16.2.2) — generalizes immediately to any number of variables. That is, given a piecewise-smooth curve $C \subseteq \mathbf{R}^n$ and a real-valued function $f(x_1, \ldots, x_n)$ defined on C, we can define the **line integral of** f **along** C **with respect to curve length**

$$\int_C f(x_1, \ldots, x_n)\,ds := \lim_{N\to\infty} \sum_{i=1}^{N} f(x_{1,i}^*, \ldots, x_{n,i}^*)\Delta s_i,$$

and the **line integral of** f **along** C **with respect to** x_j

$$\int_C f(x_1, \ldots, x_n)\,dx_j := \lim_{N\to\infty} \sum_{i=1}^{N} f(x_{1,i}^*, \ldots, x_{n,i}^*)\Delta x_{j,i},$$

if these limit exist.

16.2.3 Oriented Curves

A parametrization of a curve determines an orientation. If C is a curve equipped with an orientation, then $-C$ denotes the same curve with the opposite orientation. *(draw picture)*

Line integral of a real-valued function f with respect to curve length does not depend on orientation of curve, i.e.

$$\int_{-C} f\,ds = \int_C f\,ds.$$

This is because curve length is always nonnegative. In contrast, the line integral of a real-valued function f with respect to a variable does depend on orientation:

$$\int_{-C} f\,dx_i = -\int_C f\,dx_i.$$

16.2.4 Line Integrals of Vector-Valued Functions

(motivation via physics: work; introduce with this motivation? because both S and MT do it; when in Rome...)

(key: reduce to case of line integral of real-valued function $\mathbf{F}\cdot\mathbf{T}$*, i.e.*

$$\begin{aligned}\int_C (\mathbf{F}\cdot\mathbf{T})\,ds &= \int_a^b \left(\mathbf{F}(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\right)\|\mathbf{r}'(t)\|\;dt\\ &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt.\end{aligned}$$

Note that we can cancel $\|\mathbf{r}'(t)\|$ *in the final equality because* $\mathbf{r}$ *is a smooth curve, therefore by definition* $\mathbf{r}'(t)\neq\mathbf{0}$*, and hence* $\|\mathbf{r}'(t)\|\neq 0$*, for all parameter values* t*.*

Let $\mathbf{F}:\mathbf{R}^n\to\mathbf{R}^n$ be a continuous *(is continuity required?)* vector field, and let $C\subseteq\mathbf{R}^m$ be a smooth curve with smooth parametrization $\mathbf{r}$. *(relocate following parenthetical)* (As with real-valued functions, we can extend the subsequent definition to piecewise-smooth curves by decomposing them into their smooth pieces.) Notice that the vectors input to and output by $\mathbf{F}$ live in the same space as the curve C. In this case, we can define the line integral of the vector-valued function $\mathbf{F}$ along C.

Let $I\subseteq\mathbf{R}$ be the interval of interest for the parametrization $\mathbf{r}$ of C. We set up the Riemann sum as follows:

1. Partition the curve C into a finite number N of subcurves C_i. This corresponds to partitioning the interval $I\subseteq\mathbf{R}$ into a finite number N of subintervals.

2. Choose a point P_i^* on each subcurve C_i. Note that the point P_i^* corresponds to some parameter value t_i^*, i.e. $\mathbf{r}(t_i^*)=P_i^*$.[1]

3. Pretend that $\mathbf{F}$ is constant with (vector) value $\mathbf{F}(\mathbf{r}(t_i^*))$ on the subcurve C_i. Moreover, pretend that the subcurve C_i is a line segment, with direction given by the tangent vector $\mathbf{T}(t_i^*)$ to $\mathbf{r}(t)$ at P_i^*.

4. Compute the Riemann sum

$$\sum_{i=1}^N \mathbf{F}(\mathbf{r}(t_i^*))\cdot\mathbf{T}(t_i^*)\,\Delta s_i.$$

We define the line integral of $\mathbf{F}$ along C to be the limit of this Riemann sum as the length of the partition elements goes to zero:

$$\int_C \mathbf{F}\cdot d\mathbf{r} := \lim_{N\to\infty}\sum_{i=1}^N \mathbf{F}(\mathbf{r}(t_i^*))\cdot\mathbf{T}(t_i^*)\,\Delta s_i,$$

[1] More precisely, $\mathbf{r}(t_i^*)$ is the vector such that, when its initial point is placed a the origin, its terminal point is P_i^*.

if this limit exists.

Note that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \text{and} \qquad ds = \left\|\mathbf{r}'(t)\right\| dt.$$

Thus

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t)\, ds = \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \left\|\mathbf{r}'(t)\right\| dt = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Therefore in practice, we can compute the line integral of $\mathbf{F}$ along C as follows:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

What object is this quantity? As noted previously, $\mathbf{F} \cdot \mathbf{r}'$ is a scalar, so the line integral of a vector field is actually an integral of a single-variable real-valued function: its value is a scalar (real number).

Example 16.8 (S6eQ16.02.40)**.** Find the work done by the force field

$$\mathbf{F}(x, y) := x \sin y\ \mathbf{i} + y\ \mathbf{j}$$

on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

Solution: In physics, work is defined as force in the direction of displacement times displacement. The inner product $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ gives us the component of the force vector in the direction the displacement vector at time t (recall that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\, \|\mathbf{v}\| \cos\theta$).

Let C denote the given parabolic curve. We can parametrize C with respect to x:

$$\mathbf{r}(x) := \left(x, x^2\right), \qquad x \in [-1, 2] \qquad \Rightarrow \qquad \mathbf{r}'(t) = (1, 2x).$$

Thus

$$\begin{aligned} W &= \int_C \mathbf{F}(\mathbf{r}(x)) \cdot \mathbf{r}'(x)\, dx \\ &= \int_{-1}^{2} \mathbf{F}(x, x^2) \cdot \mathbf{r}'(x)\, dx \\ &= \int_{-1}^{2} \left(x \sin x^2 + x^2 2x\right) dx \\ &= \left[-\frac{1}{2} \cos x^2 + \frac{1}{2} x^4\right]_{x=-1}^{2} \\ &= \frac{1}{2}(\cos 1 - \cos 4) + \frac{7}{2}. \end{aligned}$$

16.3 The Fundamental Theorem for Line Integrals

Key Ideas

- Fundamental theorem of line integrals
- Independence of path
- Closed curve
- Independence of path and line integral over all closed curves
- Open subset
- Path-connected subset
- Independence of path and conservative vector fields
- Conservative vector field and mixed partials
- Simple curve
- Simply connected region
- Sufficient conditions for vector field on simply connected subset of $\mathbf{R}^2$ to be conservative

16.3.1 Introduction

Recall the fundamental theorem of calculus in the setting of real-valued functions of a single variable: Let $[a, b] \subseteq \mathbf{R}$ be an interval, and let F' be a real-valued function that is continuous on $[a, b]$. Then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

In words, the fundamental theorem of calculus says that the integral of the derivative F' over the interval $I = [a, b]$ is related to F evaluated at the boundary (i.e. endpoints) of the region of integration I. It is this view that we generalize to several multivariable settings in the sections that follow.

16.3.2 The Fundamental Theorem for Line Integrals

(better motivation?) Let $f : \mathbf{R}^n \to \mathbf{R}$, and let $C \subseteq \mathbf{R}^n$ be a smooth curve from P to Q. Consider the line integral

$$\int_C \nabla f \cdot d\mathbf{r}.$$

Let $\mathbf{r} : I \to \mathbf{R}^n$ be a smooth parametrization of C, with $I := [a, b] \subseteq \mathbf{R}$, $\mathbf{r}(a) = P$, and $\mathbf{r}(b) = Q$. The "boundary" of the curve C are its endpoints, $P = \mathbf{r}(a)$ and $Q = \mathbf{r}(b)$. If we view the gradient ∇f as a derivative (by definition, ∇f is a vector of partial derivatives), then in light of the fundamental theorem of calculus for real-valued functions of a single variable, we might conjecture that

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Theorem 16.9. *Let $C \subseteq \mathbf{R}^n$ be a smooth curve, let $\mathbf{r} : I \to \mathbf{R}^n$ be a smooth parametrization of C, with $I := [a, b]$, and let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function such that ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. Let

$$\mathbf{r}(t) := (x_1(t), \ldots, x_n(t)).$$

Then

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt && \text{Definition of line integral} \\
&= \int_a^b \left(\frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} \right) dt && \text{Definition inner product} \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))dt && \text{Chain rule} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)). && \text{Fundamental theorem of calculus}
\end{aligned}$$

□

Theorem 16.9 generalizes immediately to line integrals over piecewise-smooth curves, by breaking such curves into their smooth pieces.

16.3.3 Independence of Path

Definition 16.10. Let $\Omega \subseteq \mathbf{R}^n$, let $A, B \in \Omega$ be two points, and let $\mathbf{F} : \Omega \to \mathbf{R}^n$ be a continuous vector field. The line integral of $\mathbf{F}$ from A to B is **independent of path** if for any two paths $C_1, C_2 \subseteq \Omega$ from A to B,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Let $\mathbf{F} : \Omega \to \mathbf{R}^n$ be a conservative vector field. Then by Theorem 16.9, for any path C from A to B,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F}(B) - \mathbf{F}(A).$$

Thus the line integral of a conservative vector field is independent of path.

Definition 16.11. A curve $C \subseteq \mathbf{R}^n$ is **closed** if the initial point of C and the terminal point of C coincide.

(draw examples; curve that passes through initial point multiple times can still be closed)

Theorem 16.12. *Let $\mathbf{F} : \Omega \to \mathbf{R}^n$ be a continuous vector field. For all points $A, B \in \Omega$, the line integral*

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path if and only if for every closed path $C \subseteq \Omega$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Proof. We prove each direction in turn.

($\Leftarrow$) Suppose that for every closed path $C \subseteq \Omega$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Let $A, B \in \Omega$ be any two points, and let $C_1, C_2 \subseteq \Omega$ be any two paths from A to B. Let C be the curve composed of C_1 (from A to B) followed by $-C_2$ (from B to A). Then $C \subseteq \Omega$ is a closed curve with basepoint A, so by the hypothesis, the line integral of $\mathbf{F}$ over C equals zero. Thus

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$

so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Because the points $A, B \in \Omega$ were arbitrary, and the two paths $C_1, C_2 \subseteq \Omega$ were arbitrary, we conclude that the result holds for any two paths in Ω between the same endpoints. That is, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

($\Rightarrow$) Suppose that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. *(better: integral over closed path C equals integral over trivial path, which equals 0)* Let $C \subseteq \Omega$ be an arbitrary closed path, and *(formally, need two cases: C consists of single point (then $d\mathbf{r} = 0$); otherwise)* let $A, B \in C$ be any two points on C. Let C_1 be the part of C from A to B, and let C_2 be the remaining part of C from B to A. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 \cup C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0,$$

where the last equality follows from the fact that C_1 and $-C_2$ have the same endpoints and the hypothesis that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. □

16.3.4 Identifying Conservative Vector Fields

The question we explore in this section is, Given a vector field $\mathbf{F}$, how can we determine whether $\mathbf{F}$ is conservative?

Definition 16.13. A subset $\Omega \subseteq \mathbf{R}^n$ is **open** if for every point $P \in \Omega$, there exists a ball $B_r(P)$ of sufficiently small radius $r > 0$ centered at P such that $B_r(P) \subseteq \Omega$.

Definition 16.14. A subset $\Omega \subseteq \mathbf{R}^n$ is **path-connected** if for any two points $A, B \in \Omega$, there exists a path $C \subseteq \Omega$ from A to B.

(draw pictures)

The following theorem gives us a sufficient condition for $\mathbf{F}$ to be conservative: If $\mathbf{F}$ is independent of path on an open, path-connected region, then $\mathbf{F}$ is conservative.

Theorem 16.15. *Let $\Omega \subseteq \mathbf{R}^n$ be open and path-connected, let $\mathbf{F} : \Omega \to \mathbf{R}^n$ be a vector field, and let $\int_C \mathbf{F} \cdot d\mathbf{r}$ be independent of path for all points $A, B \in \Omega$. Then $\mathbf{F}$ is a conservative vector field. That is, there exists a potential function $f : \Omega \to \mathbf{R}$ such that $\nabla f = \mathbf{F}$.*

(proof)

The following theorem gives a necessary condition for $\mathbf{F}$ to be conservative: If $\mathbf{F}$ is conservative, then its component functions must satisfy Clairaut's theorem.

Theorem 16.16. *Let*

$$\mathbf{F}(x, y) := (F_1(x, y), F_2(x, y))$$

be a conservative vector field such that F_1, F_2 have continuous partial derivatives on Ω. Then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

Proof. *(Clairaut)* □

Definition 16.17. A curve $C \subseteq \mathbf{R}^n$ is **simple** if it does not intersect itself except possibly at its endpoints. That is, there exists a parametrization $\mathbf{r} : I \to \mathbf{R}^n$, $I := [a, b] \subseteq \mathbf{R}$, such that for all distinct $t_1, t_2 \in I$, $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ unless $t_1 = a$ and $t_2 = b$, or vice versa.

In particular, note that a closed curve can be simple.

Definition 16.18. A subset $\Omega \subseteq \mathbf{R}^2$ is **simply connected** if it contains no holes. More precisely, every simple closed curve in Ω encloses only points in Ω.

Theorem 16.19. *Let $\Omega \subseteq \mathbf{R}^2$ be simply connected, and let $\mathbf{F} : \Omega \to \mathbf{R}^2$ be a vector field with component functions $\mathbf{F} = (F_1, F_2)$, and let the component functions F_1, F_2 have continuous first-order derivatives and satisfy*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

at all points in Ω. Then the vector field $\mathbf{F}$ is conservative.

Example 16.20. Determine whether each of the following vector fields, defined on all of $\mathbf{R}^2$, are conservative.

(a) $\mathbf{F}(x, y) := (xy, x^2 + y^2)$

(b) $\mathbf{F}(x, y) := (2xy, x^2 + 2y)$

Solution: No, Yes.

Example 16.21 (S6eQ16.03.14)**.** Consider the vector field $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\mathbf{F}(x, y) := \frac{y^2}{1 + x^2}\,\mathbf{i} + 2y \arctan x\,\mathbf{j}.$$

(a) Determine whether $\mathbf{F}$ is conservative. If so, find a potential function f for $\mathbf{F}$.

(b) Use the potential function to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve

$$C : \mathbf{r}(t) := t^2\,\mathbf{i} + 2t\,\mathbf{j}, \qquad\qquad t \in [0, 1].$$

Solution: Let

$$F_1(x, y) := \frac{y^2}{1 + x^2} \qquad\qquad F_2(x, y) := 2y \arctan x$$

denote the component functions of the given vector field $\mathbf{F}$. The domain of $\mathbf{F}$ is $\mathbf{R}^2$ which is simply connected, and the component functions F_1, F_2 have continuous partial derivatives, so Theorem 16.19 applies. Computing the relevant partial derivatives, we find

$$\frac{\partial F_1}{\partial y} = \frac{2y}{1 + x^2} = \frac{\partial F_2}{\partial x}.$$

Hence Theorem 16.19 implies that the vector field $\mathbf{F}$ is conservative. That is, there exists some potential function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $\nabla f = \mathbf{F}$. Writing out this last equality, this means that

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (F_1, F_2) \qquad \Leftrightarrow \qquad F_1 = \frac{\partial f}{\partial x} = \frac{y^2}{1 + x^2}, \qquad F_2 = \frac{\partial f}{\partial y} = 2y \arctan x.$$

Integrating the first relation with respect to x, we have

$$f(x, y) = \int \frac{\partial f}{\partial x}\,dx = \int \frac{y^2}{1 + x^2}\,dx = y^2 \arctan x + g(y).$$

Note that the "constant" of integration here is $g(y)$, a function of y. A function of y behaves as a constant with respect to partial differentiation by x, so this is the correct generalization of the "constant" of integration.

Computing the partial derivative of this expression for $f(x, y)$ with respect to y and comparing to $F_2(x, y)$, we find

$$2y \arctan x + g'(y) = \frac{\partial f}{\partial y} = F_2(x, y) = 2y \arctan x.$$

Hence $g'(y) = 0$, which implies that $g(y) = c$ a constant. Thus we conclude that any potential function $f(x, y)$ for the vector field $\mathbf{F}(x, y)$ has the form

$$f(x, y) = y^2 \arctan x + c.$$

Note that

$$\mathbf{r}(0) = (0, 0), \qquad \mathbf{r}(1) = (1, 2).$$

Taking the potential function f with $c = 0$, and using the fundamental theorem for line integrals (Theorem 16.9), we compute

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = 2^2 \arctan 1 - 0 = 2\pi.$$

16.3.5 Conservation of Energy

(lovely but not critical application; illustrates why we call f a "potential" function for $\mathbf{F}$)

16.4 Green's Theorem

Key Ideas

- Orientation (positive, negative) of simple closed curve in $\mathbf{R}^2$
- Green's theorem

Green's theorem relates the double integral of a function $f(x, y)$ over a region $\Omega \subseteq \mathbf{R}^2$ to the line integral of an appropriate "antiderivative" of f over the boundary of Ω, with appropriate qualifications.

16.4.1 Green's Theorem

Have students remind you what "simple" and "closed" mean for curves.

Definition 16.22. Let $C \subseteq \mathbf{R}^2$ be a simple closed curve. The **positive orientation** of C is the counterclockwise direction. The **negative orientation** of C is the clockwise direction.

(relate positive orientation to right-hand rule)

Theorem 16.23 (Green's Theorem)**.** *Let $\Omega \subseteq \mathbf{R}^2$ such that the boundary curve $\operatorname{bd}\Omega \subseteq \mathbf{R}^2$ is a piecewise-smooth, simple, closed curve; equip $\operatorname{bd}\Omega$ with its positive (counterclockwise) orientation; and let* (use better notation for domain) *$\mathbf{F} : \Omega' \to \mathbf{R}^2$ be a vector field (denote its component functions $\mathbf{F} = (F_1, F_2)$) such that F_1, F_2 have continuous partial derivatives on an open set containing Ω. Then*

$$\int_{\operatorname{bd}\Omega} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Omega} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

(proof)

Example 16.24 (S6eQ16.04.02)**.** Let $C \subseteq \mathbf{R}^2$ be the rectangle with vertices

$$(0, 0), \qquad (3, 0), \qquad (3, 1), \qquad (0, 1).$$

Evaluate the line integral

$$\int_C xy\, dx + x^2\, dy$$

in two ways: (a) directly, and (b) using Green's theorem.

Solution: As always, we start by sketching the region of integration, in this case, the curve C. *(include graphic)*

To evaluate the line integral directly, we would break the curve C into (at least) four piecewise-smooth curves (e.g., the sides of the rectangle), parametrize each curve, and evaluate the line integrals over these smooth pieces separately (by substituting in the values of x, y, dx, dy given by each parametrization, then evaluating the resulting single-variable integrals). Try this on your own.

Alternatively, we can use Green's theorem. Green's theorem relates the line integral of a function over a simple closed curve to the double integral of a "derivative" of that function over the region enclosed by the curve. In this example, what is $\mathbf{F}$? Answer:

$$\mathbf{F} = (F_1, F_2) = \left(xy, x^2\right).$$

The region Ω is the rectangular area enclosed by the given curve C (which, note, we can view as the boundary of Ω). In this example,

$$\Omega = \left\{(x, y) \in \mathbf{R}^2 \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 1\right\} = [0, 3] \times [0, 1].$$

So Green's theorem writes as

$$\begin{aligned}
\int_C xy\,dx + x^2\,dy &= \int_\Omega \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \\
&= \int_{x=0}^{x=3} \int_{y=0}^{y=1} (2x - x)\,dy\,dx \\
&= [y]_{y=0}^{y=1} \left[\frac{1}{2}x^2\right]_{x=0}^{x=3} = [1]\left[\frac{9}{2}\right] = \frac{9}{2}.
\end{aligned}$$

Note how Green's theorem simplified the computation considerably. Even though we passed from a single-integral (over a curve) to a double integral (over a region), the derivatives simplified the integrand, and moving to the double integral simplified the region of integration *(make more precise)*.

Green's theorem cuts both ways:

- It can be used to convert a line integral (over a piecewise-smooth, simple, closed curve) into a double integral (over region enclosed by the curve). Often this simplifies the integrand, because this conversion involves taking derivatives.

- It can be used to convert a double integral into a line integral (provided that the boundary of the region is a piecewise-smooth, simple, closed curve). This conversion is often useful if $\mathbf{F}$ has easy-to-analyze behavior (e.g., is constant) on the boundary curve.

A common application of the double-integral-to-line-integral conversion afforded by Green's theorem is in computing areas of regions in $\mathbf{R}^2$. Suppose that we wish to find the area of a region $\Omega \subseteq \mathbf{R}^2$. This is given by

$$\iint_\Omega 1\,dA.$$

If the boundary bd Ω of Ω satisfies the hypotheses of Green's theorem (piecewise-smooth, simple, closed curve), then Green's theorem writes as

$$\iint_{\Omega} 1 \, dA = \int_{\text{bd}\,\Omega} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (F_1, F_2)$ is any continuously differentiable vector field such that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1.$$

Some common choices for such a vector field $\mathbf{F}$ are

$$\mathbf{F}(x,y) := (0, x), \qquad \mathbf{F}(x,y) := (-y, 0), \qquad \mathbf{F}(x,y) := \left(-\frac{1}{2}y, \frac{1}{2}x\right).$$

In these cases, Green's theorem applied to $\iint_{\Omega} 1 \, dA$ yields, respectively,

$$\int_{\text{bd}\,\Omega} x \, dy, \qquad \int_{\text{bd}\,\Omega} -y \, dx, \qquad \frac{1}{2}\int_{\text{bd}\,\Omega} -y \, dx + x \, dy.$$

Example 16.25 (S6eE16.04.03)**.** Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution: Let $\Omega \subseteq \mathbf{R}^2$ be the area enclosed by the ellipse. We can parametrize the ellipse by $\mathbf{r}(t) := (x(t), y(t))$, where

$$x(t) := a\cos t, \qquad y(t) := b\sin t, \qquad t \in [0, 2\pi].$$

Note that this curve is a smooth, simple, closed curve. Thus Green's theorem implies that

$$\begin{aligned} A = \iint_{\Omega} 1 \, dA &= \frac{1}{2}\int_{\text{bd}\,\Omega} -y \, dx + x \, dy \\ &= \frac{1}{2}\int_{t=0}^{2\pi} -(b\sin t)(-a\sin t \, dt) + (a\cos t)(b\cos t \, dt) \\ &= \frac{1}{2}ab\int_{t=0}^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt \\ &= \frac{1}{2}ab\int_{t=0}^{2\pi} dt = \pi ab. \end{aligned}$$

Note that in the case $a = b$, corresponding to the circle centered at the origin with radius a, this computation yields area πa^2, as we expect.

16.4.2 Proof of Green's Theorem for More-General Regions

To prove Green's theorem when the region Ω enclosed by the curve is not simple, we can partition the region into a finite number of (nonoverlapping except on the boundary) simple subregions, then apply the proof give above. When we give each of the subregions its positive orientation, the line integrals along the added boundaries cancel, leaving us with just the line integral along the boundary of the original region, as stated in the conclusion of Green's theorem. This same approach can be used to prove Green's theorem in the case when the region Ω has holes.

16.5 Curl and Divergence

Key Ideas

- The operator ∇
- Curl (of a vector field on $\mathbf{R}^3$)
- $\operatorname{curl} \nabla f = \mathbf{0}$
- $\operatorname{curl} \mathbf{F} \neq \mathbf{0} \Rightarrow \mathbf{F}$ not conservative
- $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and domain of $\mathbf{F}$ simply connected $\Rightarrow \mathbf{F}$ conservative
- Physical interpretation of $\operatorname{curl} \mathbf{F}$ (evaluated at a point): counterclockwise rotation about axis given by $\operatorname{curl} \mathbf{F}$, speed of rotation given by $\|\operatorname{curl} \mathbf{F}\|$
- Divergence (of a vector field on $\mathbf{R}^n$)
- $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$

In this section we explore two operators that act on vector fields. The curl operator acts on vector fields defined on $\mathbf{R}^3$ and outputs another vector field on $\mathbf{R}^3$. The divergence operator acts on vector fields defined on $\mathbf{R}^n$ and outputs a scalar (real-valued) function. The physical motivation for curl and divergence comes from fluid mechanics: loosely speaking, curl measures the rotation in a fluid, and divergence measures the expansion or compression of a fluid.

Clairaut's theorem *(add ref)* plays a key role in the results of this section.

16.5.1 The Operator ∇

Recall the definition of the gradient vector: Given a $\mathbf{R}$-valued function $f : \mathbf{R}^n \to \mathbf{R}$, the **gradient** of f, denoted ∇f, is the vector of all first-order partial derivatives of f:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Let's look at the definition of the gradient of f in a new way: View ∇ as the **operator**

$$\nabla := \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right),$$

that acts on a function $f : \mathbf{R}^n \to \mathbf{R}$

Viewing ∇ as a vector allows us to create new constructions. More precisely, given $\Omega \subseteq \mathbf{R}^n$ and a vector field $\mathbf{F} : \Omega \to \mathbf{R}^n$, we can consider the inner product $\nabla \cdot \mathbf{F}$, called the **divergence** of $\mathbf{F}$. In the case $n = 3$, we can also consider the cross product $\nabla \times \mathbf{F}$, called the **curl** of $\mathbf{F}$. We explore each of these in more detail below.

16.5.2 Curl

Definition 16.26. Let $\Omega \subseteq \mathbf{R}^3$, let $\mathbf{F} : \Omega \to \mathbf{R}^3$ be a vector field with component functions $\mathbf{F} = (F_1, F_2, F_3)$, and suppose that the component functions F_i are differentiable. Then the **curl** of $\mathbf{F}$, denoted $\operatorname{curl}\mathbf{F}$, is the vector field on Ω defined by

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}.$$

On $\mathbf{R}^3$, the operator ∇ (pronounced "del") *(explain briefly about this operator?)* has the form

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

so

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Theorem 16.27. *Let $\Omega \subseteq \mathbf{R}^3$, and let $f : \Omega \to \mathbf{R}$ such that f has continuous second-order partial derivatives. Then*

$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

Proof. By hypothesis, f has continuous second-order partial derivatives, so Clairaut's theorem implies that second-order partial derivatives with respect to the same two variables are equal, regardless of the order of partial differentiation. Using this fact and the definitions of curl and gradient, we compute

$$\begin{aligned}\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\mathbf{k} \\ &= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0},\end{aligned}$$

where the penultimate equality follows from Clairaut's theorem (i.e. equality of mixed second-order partial derivatives). □

Recall that, by definition, a vector field $\mathbf{F}$ is conservative if there exists a scalar-valued function f (the potential function) such that $\nabla f = \mathbf{F}$. Thus Theorem 16.27 can be worded as follows:

Corollary 16.28. *Let* $\mathbf{F}$ *be a conservative vector field on* $\mathbf{R}^3$ *such that the component functions* F_1, F_2, F_3 *are* $\mathscr{C}^2$*. Then*

$$\operatorname{curl} \mathbf{F} = \mathbf{0}.$$

(streamline the following explanation; too wordy:) This rewording compactly expresses a result that we have encountered previously. In Theorem **??** *(add ref)* we saw that if $\mathbf{F}$ is a conservative vector field with $\mathscr{C}^2$ component functions, then Clairaut's theorem implies that for any two variables, the second-order partial derivatives with respect to the same two variables are equal. The components of $\operatorname{curl} \mathbf{F}$ vanish (i.e. equal 0) precisely when this occurs. Thus the text $\operatorname{curl} \mathbf{F} = \mathbf{0}$ provides a certificate for a vector field to *fail* to be conservative:

Corollary 16.29. *If* $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$*, then the vector field* $\mathbf{F}$ *is not conservative.*

Logically, Corollary 16.29 is the contrapositive of Corollary 16.28. They are logically equivalent.

In general, the converse of Corollary 16.28 is false: If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then the vector field $\mathbf{F}$ need not be conservative. However, the converse is true if we restrict to vector fields $\mathbf{F}$ defined on simply connected domains. (Recall that, loosely speaking, this means that the domain of $\mathbf{F}$ has no holes.) This is the analog of Theorem **??** *(add ref and brief explanation)*.

Theorem 16.30. *Let* $\Omega \subseteq \mathbf{R}^3$ *be simply connected, and let* $\mathbf{F} : \Omega \to \mathbf{R}^3$ *be a vector field such that (i) its component functions* F_1, F_2, F_3 *are* $\mathscr{C}^1$ *and (ii)* $\operatorname{curl} \mathbf{F} = \mathbf{0}$*. Then* $\mathbf{F}$ *is conservative.*

Example 16.31 (S6eQ16.05.14, modified)**.** Determine whether the vector field

$$\mathbf{F}(x, y, z) := xyz^2\, \mathbf{i} + x^2yz^2\, \mathbf{j} + x^2y^2z\, \mathbf{k}$$

is conservative. If so, find a function f such that $\nabla f = \mathbf{F}$.

Solution: We compute

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{pmatrix} \\ &= \left(2x^2yz - 2x^2yz\right) \mathbf{i} + \left(2xy^2z - 2xy^2z\right) \mathbf{j} + \left(2xyz^2 - 2xyz^2\right) \mathbf{k} \\ &= \mathbf{0}. \end{aligned}$$

The vector field $\mathbf{F}$ is defined on all of $\mathbf{R}^3$ (a simply connected domain), and its component functions are polynomials (and therefore $\mathscr{C}^\infty$, so certainly $\mathscr{C}^1$). Therefore Theorem 16.30 applies, and we conclude that $\mathbf{F}$ is conservative.

To find a potential function f for $\mathbf{F}$, we perform successive partial integrations, remembering that the constant of integration for a partial integration is a function of all other variables (other than the one with respect to which we are integrating). Starting with x, we compute

$$\frac{\partial f}{\partial x} = F_1 = xy^2z^2 \qquad \Leftrightarrow \qquad f = \frac{1}{2}x^2y^2z^2 + g(y,z).$$

Differentiating with respect to y,

$$x^2yz^2 + \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} = F_2 = x^2yz^2 \qquad \Rightarrow \qquad \frac{\partial g}{\partial y}0.$$

This last equation implies that $g(y,z) = h(z)$ is a function of z only, so

$$f = \frac{1}{2}x^2y^2z^2 + h(z).$$

Differentiating with respect to z,

$$x^2y^2z + h'(z) = \frac{\partial f}{\partial z} = F_3 = x^2y^2z \qquad \Rightarrow \qquad h'(z) = 0.$$

Thus $h(z)$ is a constant (i.e. a real number), call it c. We conclude that

$$f(x,y,z) = x^2y^2z^2 + c.$$

16.5.3 Curl: Geometric Intuition

Let $\mathbf{F}$ represent the velocity field of a fluid. Physically, particles near a given point $P := (x,y,z)$ tend to rotate counterclockwise (i.e. in the direction determined by the right-hand rule) about the axis given by $\operatorname{curl}\mathbf{F}(x,y,z)$. Then norm (magnitude) of $\operatorname{curl}\mathbf{F}(x,y,z)$ indicates the speed of rotation. If $\operatorname{curl}\mathbf{F}(x,y,z) = \mathbf{0}$, then the vector field $\mathbf{F}$ is said to be **irrotational** (i.e. free from rotations) at P. *(include physical mental image of tiny paddle wheel?)*

16.5.4 Divergence

Definition 16.32. Let $\Omega \subseteq \mathbf{R}^n$, let $\mathbf{F} : \Omega \to \mathbf{R}^n$ be a vector field, and suppose that the component functions F_i are differentiable. Then the **divergence** of $\mathbf{F}$, denoted $\operatorname{div}\mathbf{F}$, is the real-valued function on Ω defined by

$$\operatorname{div}\mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}.$$

Theorem 16.33. *Let $\Omega \subseteq \mathbf{R}^3$, and let $\mathbf{F} : \Omega \to \mathbf{R}^3$ be a vector field whose component functions F_i are $\mathscr{C}^2$. Then*

$$\operatorname{div}\operatorname{curl}\mathbf{F} = 0.$$

Proof. We compute

$$
\begin{aligned}
\operatorname{div}\operatorname{curl}\mathbf{F} &= \nabla\cdot(\nabla\times\mathbf{F})\\
&= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(F_1,F_2,F_2)\right)\\
&= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial F_2}{\partial z}-\frac{\partial F_3}{\partial y},-\frac{\partial F_1}{\partial z}+\frac{\partial F_3}{\partial x},\frac{\partial F_1}{\partial y}-\frac{\partial F_2}{\partial x}\right)\\
&= \frac{\partial^2 F_2}{\partial x\partial z}-\frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial^2 F_1}{\partial y\partial z}+\frac{\partial^2 F_3}{\partial y\partial x}+\frac{\partial^2 F_1}{\partial z\partial y}-\frac{\partial^2 F_2}{\partial z\partial x}\\
&= 0,
\end{aligned}
$$

where in the final step all terms cancel in pairs by Clairaut's theorem (equality of mixed second-order partial derivatives). □

Example 16.34 (S6eQ16.05.20)**.** Is there a vector field $\mathbf{G}$ on $\mathbf{R}^3$ such that

$$\operatorname{curl}\mathbf{G} = (xyz, -y^2z, yz^2)?$$

Solution: Suppose there were such a vector field $\mathbf{G}$. Then by Theorem 16.33,

$$0 = \operatorname{div}\operatorname{curl}\mathbf{G} = \operatorname{div}\left(xyz, -y^2z, yz^2\right) = yz - 2yz + 2yz = yz,$$

a contradiction. Thus we conclude that no such $\mathbf{G}$ exists.

Definition 16.35. The **Laplace operator**, denoted ∇^2, is

$$\nabla^2 := \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}.$$

(interpret Laplace as $\nabla\cdot\nabla$)
(vector forms of Green's theorem; they lay groundword for Stokes's theorem)

16.6 Parametric Surfaces

Key Ideas

- Parametrizing surfaces
- Smooth parametrization (of a surface)
- Tangent plane
- Surface area
  - Of a general surface
  - Of the graph of a function (special case)

16.6.1 Introduction

In Section 13.1 we explored how to parametrize curves in $\mathbf{R}^n$. A curve is a one-dimensional geometric object; a parametrization (recall that it is not unique) is an algebraic description of the curve that requires one parameter. Note that the number of parameters in the algebraic description is the same as the dimension of the geometric object.

In this section we explore how to parametrize two-dimensional surfaces in $\mathbf{R}^n$. (Our examples will be in $\mathbf{R}^3$, so that we can visualize the surface easily.) Our approach to surfaces will be analogous to our approach to curves: A surface is a two-dimension geometric object; a parametrization (again it will not be unique) is an algebraic description of the surface that requires two parameters. Again note that the number of parameters in the algebraic description is the same as the dimension of the geometric object.

16.6.2 Examples

Example 16.36 (S6eE16.06.03)**.** Find a vector-valued function that represents the plane that passes through the point P_0 (with associated position vector $\mathbf{r}_0$) and that contains two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$.

Solution: Because $\mathbf{u}, \mathbf{v}$ are nonparallel, any vector $\mathbf{w}$ in the plane can be represented as a linear combination

$$\mathbf{w} = a\mathbf{u} + b\mathbf{v},$$

for some $a, b \in \mathbf{R}$. Let $\mathbf{p}$ be the vector associated with the point P_0 (i.e. the vector $\vec{OP_0}$, where O denotes the origin). Then any vector $\mathbf{r}$ in the plane can be written as

$$\mathbf{r}(a, b) = \mathbf{p} + a\mathbf{u} + b\mathbf{v}.$$

This defines a function $\mathbf{r} : \mathbf{R}^2 \to \mathbf{R}^n$ whose image is the desired plane.

Let $\mathbf{u}, \mathbf{v}, \mathbf{p}$ have components

$$\mathbf{u} = (u_1, \ldots, u_n), \qquad \mathbf{v} = (v_1, \ldots, v_n), \qquad \mathbf{p} = (p_1, \ldots, p_n).$$

Then the function $\mathbf{r} : \mathbf{R}^2 \to \mathbf{R}^n$ can be written as

$$\mathbf{r}(a, b) = (p_1 + au_1 + bv_1, \ldots, p_n + au_n + bu_n),$$

where $x_i(a, b) := p_i + au_i + bv_i$, $i \in \{1, \ldots, n\}$, are the component functions of $\mathbf{r}$.

Example 16.37 (S6eE16.06.04)**.** Find a parametric representation for the sphere of radius 1 centered at the origin in $\mathbf{R}^3$.

Solution: Our first inclination might be to use rectangular coordinates (x, y, z), but we quickly realize that doing so would require us to consider at least two cases (because we have to treat $+$ and $-$ square roots differently). We can achieve a single parametric description of the sphere using spherical coordinates. More precisely, any point on the unit sphere has the form $(1, \theta, \varphi)$. Thus a parametrization of the unit sphere is

$$\begin{aligned} \mathbf{r} : [0, 2\pi] \times [0, \pi] &\to \mathbf{R}^3 \\ (\theta, \varphi) &\mapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi). \end{aligned}$$

Note that (i) the components of the output of $\mathbf{r}$ are precisely x, y, z in spherical coordinates, with $\rho = 1$; and (ii) this parametrization is surjective (all points on the unit sphere are hit by some input point (θ, φ)) but not injective (some points on the unit sphere are hit by different input points (θ, φ), e.g., $\mathbf{r}(0, \varphi) = \mathbf{r}(2\pi, \varphi)$ for all $\varphi \in [0, \pi]$). We could make $\mathbf{r}$ injective (in fact, bijective) by restricting the domain to $[0, 2\pi) \times [0, \pi]$.

Example 16.38 (S6eE16.06.05)**.** Find a parametric representation for the part of the cylinder $x^2 + y^2 = 4$ such that $x, y, z \geqslant 0$.

Solution: Geometrically, the given equation describes the circular cylinder of radius 2 centered around the z-axis. Because we want $x, y, z \geqslant 0$, we could describe the surface in rectangular coordinates: At all points (x, y, z) on the surface, x and y are related by $x^2 + y^2 = 4$. Solving for one in terms of the other (e.g., solving for y in terms of x) — it is here that we leverage the restriction that $y \geqslant 0$ to consider only the positive square root — and treating z as a free variable, we obtain the parametrization

$$\begin{aligned} \mathbf{r}_1 : [0, 2] \times \mathbf{R}_{\geqslant 0} &\to \mathbf{R}^3 \\ (x, z) &\mapsto \left(x, \sqrt{4 - y^2}, z\right). \end{aligned}$$

Note that $x \in [0, 2]$ because (i) the restriction $x \geqslant 0$ and (ii) the condition $x^2 + y^2 = 4$ implies that $x^2 = 4 - y^2 \leqslant 4$, so $x \leqslant 2$.

It is arguably more convenient to use cylindrical coordinates:

$$\begin{aligned}\mathbf{r}_2 : \left[0, \frac{\pi}{2}\right] \times \mathbf{R}_{\geqslant 0} &\to \mathbf{R}^3 \\ (\theta, z) &\mapsto (2\cos\theta, 2\sin\theta, z)\,.\end{aligned}$$

Both parametrizations $\mathbf{r}_1, \mathbf{r}_2$ are bijective.

Example 16.39 (S6eE16.06.06)**.** Find a parametric representation of the elliptic paraboloid $y = x^2 + 2z^2$.

Solution: The equation for the elliptic paraboloid expresses one variable, y, as a function of the other two, x, z. The elliptic paraboloid is the graph of $y(x, z)$. Thus

$$\begin{aligned}\mathbf{r} : \mathbf{R}^2 &\to \mathbf{R}^3 \\ (x, z) &\mapsto \left(x, x^2 + 2z^2, z\right)\end{aligned}$$

is a parametrization of the surface.

Note that the approach we took in Example 16.39 works for any surface in $\mathbf{R}^3$ such that we can express one variable as a function of the other two.

Example 16.40 (S6eE16.06.08)**.** Find parametric equations for the surface generated by rotating the curve $y = \sin x$, for $0 \leqslant x \leqslant 2\pi$, about the x-axis.

Solution: *(motivate, explain)*

$$\begin{aligned}\mathbf{r} : [0, 2\pi] \times [0, 2\pi] &\to \mathbf{R}^3 \\ (x, \theta) &\mapsto (x, \sin x \cos\theta, \sin x \sin\theta)\,.\end{aligned}$$

16.6.3 Tangent Planes

Let $S \subseteq \mathbf{R}^3$ be a surface with parametric representation

$$\mathbf{r}(u, v) := (x(u, v), y(u, v), z(u, v))\,,$$

and let P_0 be a point on S with corresponding position vector $\mathbf{r}(u_0, v_0)$ for some parameter pair (u_0, v_0). Let's find an equation for the tangent plane to S at P_0.

If we hold u constant at $u = u_0$ and let v vary, then we obtain a (vertical) line through P_0 in the uv-plane that corresponds to a grid curve C_{u_0} on S. The tangent vector to C_{u_0} at P_0 is given by the partial derivative

$$\mathbf{r}_v := \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0) = \left(\frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0)\right).$$

Similarly, if we hold v constant at $v = v_0$ and let u vary, then we obtain a (horizontal) line through P_0 in the uv-plane that corresponds to a grid curve C_{v_0} on S. The tangent vector to C_{v_0} at P_0 is given by the partial derivative

$$\mathbf{r}_u := \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) = \left(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\right).$$

If $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$, then $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent, and they span the tangent plane to S at P_0. Their cross product is a normal vector to this tangent plane, and the point P_0 lies in this tangent plane, completing our description of the tangent plane to S at P_0.

If for all parameter pairs (u, v), the cross product $\mathbf{r}_u \times \mathbf{r}_v$ of these two tangent vectors is nonzero, then $\mathbf{r}$ is a **smooth parametrization** of S. The surface S is **smooth** if it admits a smooth parametrization.

(draw Figure 16.06.11)

Example 16.41 (S6eQ16.06.36)**.** Find an equation of the tangent plane to the parametric surface

$$\mathbf{r}(u, v) := uv\,\mathbf{i} + u\sin v\,\mathbf{j} + v\cos u\,\mathbf{k}$$

at the point where $u = 0$ and $v = \pi$.

Solution: We compute

$$\mathbf{r}_u := (v, \sin v, -v\sin u)\,, \qquad \mathbf{r}_v := (u, u\cos v, \cos u)\,.$$

so

$$\mathbf{r}_u(0, \pi) = (\pi, 0, \pi), \qquad \mathbf{r}_v(0, \pi) = (0, 0, 1).$$

The cross product of these two tangent vectors is

$$\mathbf{n} := \mathbf{r}_u(0, \pi) \times \mathbf{r}_v(0, \pi) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \pi & 0 & \pi \\ 0 & 0 & 1 \end{pmatrix} = (0, -\pi, 0)\,,$$

which is nonzero. The point P_0 has corresponding position vector

$$\mathbf{r}(0, \pi) = (0, 0, \pi)\,.$$

Thus an equation of the tangent plane to S at P_0 is

$$0 = \mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = (0, -\pi, 0)\cdot(x, y, z - \pi) = -\pi y.$$

Note that we could have also computed the normal vector at the general point $\mathbf{r}(u, v)$:

$$\begin{aligned} \mathbf{n} &:= \mathbf{r}_u \times \mathbf{r}_v \\ &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & \sin v & -v\sin u \\ u & u\cos v & \cos u \end{pmatrix} \\ &= (\cos u \sin v + uv\sin u\cos v, -v\cos u - uv\sin u, uv\cos v - u\sin v)\,. \end{aligned}$$

16.6.4 Surface Area

(provide geometric motivation)

Definition 16.42. Let $S \subseteq \mathbf{R}^3$ be a smooth surface with parametric representation

$$\mathbf{r}(u, v) := (x(u, v), y(u, v), z(u, v)),$$

for (u, v) in some region $D \subseteq \mathbf{R}^2$, and suppose that S is covered precisely once as (u, v) varies over D. Then the **surface area** of S is

$$A(S) = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA. \tag{16.6.1}$$

Example 16.43 (S6eE16.06.10)**.** Using (16.6.1), find the surface area of a sphere of radius a.

Solution: Before we embark on the calculus, let's put on our geometry caps and ask ourselves what result we expect to get. Hopefully the answer is $4\pi a^2$.

Now to the calculus. A parametrization of the sphere S of radius a centered at the origin in $\mathbf{R}^3$ is

$$\mathbf{r}(\theta, \varphi) := (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a\cos\varphi), \qquad \theta \in [0, 2\pi], \qquad \varphi \in [0, \pi],$$

i.e. the domain D in the $\theta\varphi$-plane is

$$D := \{(\theta, \varphi) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \varphi \leqslant \pi\}.$$

We compute

$$\begin{aligned}
\mathbf{r}_\theta &= (-a \sin\varphi \sin\theta, a \sin\varphi \cos\theta, 0) \\
\mathbf{r}_\varphi &= (a \cos\varphi \cos\theta, a\cos\varphi \sin\theta, -a \sin\varphi).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{r}_\theta \times \mathbf{r}_\varphi &= \left(-a^2 \sin^2\varphi \cos\theta, -a^2 \sin^2\varphi \sin\theta, -a^2 \sin\varphi \cos\varphi \left(\sin^2\theta + \cos^2\theta\right)\right) \\
&= \left(-a^2 \sin^2\varphi \cos\theta, -a^2 \sin^2\varphi \sin\theta, -a^2 \sin\varphi\cos\varphi\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathbf{r}_\theta \times \mathbf{r}_\varphi\| &= \sqrt{a^4 \sin^4\varphi \cos^2\theta + a^4 \sin^4\varphi \sin^2\theta + a^4 \sin^2\varphi \cos^2\varphi} \\
&= a^2 \sqrt{\sin^2\varphi \left(\sin^2\varphi + \cos^2\varphi\right)} \\
&= a^2 \sin\varphi,
\end{aligned}$$

so using Fubini's theorem *(be more precise about the hypotheses)*, the surface area of the sphere S is

$$\begin{aligned}
A(S) &= \iint_D \|\mathbf{r}_\theta \times \mathbf{r}_\varphi\| \, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\varphi \, d\varphi \, d\theta \\
&= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi = a^2 (2\pi) \left[-\cos\varphi\right]_{\varphi=0}^{\pi} \\
&= 4\pi a^2.
\end{aligned}$$

16.6.5 Surface Area of the Graph of a Function

(special case of the above)

Let S be the surface given by the graph of a function $f(x, y)$. Then we can parametrize the surface S by

$$\mathbf{r}(x, y) := (x, y, f(x, y)) .$$

In this case, the two tangent vectors are

$$\mathbf{r}_x = \left(1, 0, \frac{\partial f}{\partial x}\right), \qquad \mathbf{r}_y = \left(0, 1, \frac{\partial f}{\partial y}\right).$$

Thus

$$\mathbf{r}_x \times \mathbf{r}_y = \left(-\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}, 1\right),$$

so

$$A(S) = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA.$$

Example 16.44 (S8eQ15.05.12)**.** Find the area of the part of the sphere

$$S : x^2 + y^2 + z^2 = 4z$$

that lies inside the paraboloid

$$P : z = x^2 + y^2.$$

Solution: First let's get a sense for the geometry of the situation. Writing the equation of the sphere S in standard form, we find

$$x^2 + y^2 + (z - 2)^2 = 4, \tag{16.6.2}$$

which describes a sphere of radius 2 centered at the point $(0, 0, 2)$. The paraboloid P has its vertex point at the origin, opens upward (in the direction of the positive z-axis), and has circular cross-sections parallel to the xy-plane. *(sketch)* The surface area that we want to find is the surface area of the "cap" S' that the sphere puts on the paraboloid.

Our sketch suggests that S' can be viewed as the graph of a function over a region D of the xy-plane. More precisely, the relevant region D lies inside the projection of the intersection $S \cap P$ onto the xy-plane. Algebraically, the intersection of these geometric objects is given by the set of point (x, y, z) that simultaneously satisfy the equations for both S and P. In particular, at a point of intersection the z-values of these two equations are equal, so we can substitute the equation for P into the equation for S, obtaining

$$x^2 + y^2 + (x^2 + y^2)^2 = 4(x^2 + y^2).$$

Translating to polar coordinates, this expression becomes

$$r^2 + r^4 = 4r^2 \qquad \Leftrightarrow \qquad r^2\left(r^2 - 3\right) = 0,$$

so

$$r = 0 \qquad \text{or} \qquad r = \sqrt{3}.$$

We conclude that the part S' of the sphere whose surface area we wish to compute lies above the region

$$D := \left\{(r, \theta) \,|\, 0 \leqslant r \leqslant \sqrt{3}, 0 \leqslant \theta \leqslant 2\pi\right\} \subseteq \mathbf{R}^2.$$

An equation for this part of the sphere is given by the equation (16.6.2) for S, solved for z:

$$z = 2 + \sqrt{4 - x^2 - y^2}.$$

Viewing z as a function of x, y, we compute the partial derivatives

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4 - x^2 - y^2}}.$$

Thus

$$\begin{aligned} A(S) &= \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \\ &= \iint_D \sqrt{\frac{x^2 + y^2}{4 - x^2 - y^2} + 1}\, dA \\ &= \iint_D \sqrt{\frac{4}{4 - x^2 - y^2}}\, dA. \end{aligned}$$

Using Fubini's theorem to write the double integral as an iterated integral, and converting the integral to polar coordinates (remember the integration factor of r), we compute

$$\begin{aligned} A(S) &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{3}} \frac{2}{\sqrt{4 - r^2}}\, r\, dr\, d\theta \\ &= (2\pi)\left[-2\sqrt{4 - r^2}\right]_{r=0}^{r=\sqrt{3}} \\ &= -4\pi\,[1 - 2] = 4\pi. \end{aligned}$$

16.7 Surface Integrals

Key Ideas

- Surface integral of a $\mathbf{R}$-valued function
- Orientable surface, oriented surface, positive orientation
- Closed surface
- Surface integral (a.k.a. flux) of a vector field

16.7.1 Introduction

In Section 16.2 we explored how to define the line integral of real-valued functions and vector-valued functions. *(explain at length the two cases; a partition of the curve induces a partition of the interval of parametrization)*

In this section, we follow an analogous approach to define the surface integral of real-valued functions and vector-valued functions. This will be another application of the general idea of the Riemann sum.

16.7.2 Surface Integral of Real-Valued Functions

Let $S \subseteq \mathbf{R}^3$ be a surface parametrized by the equation

$$\mathbf{r}(u,v) := (x(u,v), y(u,v), z(u,v)), \qquad (u,v) \in D \subseteq \mathbf{R}^2.$$

Partition D, which induces a partition on S; choose a point $P^*_{i,j}$ in each subregion $D_{i,j}$; form Riemann sum.

Definition 16.45. Let $S \subseteq \mathbf{R}^3$ be a surface, let $\Omega \subseteq \mathbf{R}^3$ such that $S \subseteq \Omega$ (so, in particular, f is defined at all points of S), and let $f : \Omega \to \mathbf{R}$. The **surface integral of** f **over** S is

$$\iint_S f\, dS := \lim_{m_1, m_2 \to \infty} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} f(P^*_{i_1,i_2})\, \Delta S_{i_1,i_2},$$

if the limit exists.

Recall that

$$\Delta S_{i,j} \approx \|\mathbf{r}_u \times \mathbf{r}_v\|\ \Delta u\, \Delta v,$$

where

$$\mathbf{r}_u := \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \qquad \mathbf{r}_v := \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

(be more precise about this next part) In the limit as $\Delta u, \Delta v \to 0$,

$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv. \tag{16.7.1}$$

If the surface S is covered exactly once as (u, v) varies over D, then

$$\iint_S f \, dS = \iint_D f(\mathbf{r}(u, v)) \, \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA.$$

One can show *(do it)* that the value of the surface integral does not depend on the choice of parametrization.

As we have seen with multiple integrals, the surface area of a surface S is given by integrating the constant function 1 over S:

$$\iint_S 1 \, dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA = A(S).$$

16.7.3 Graphs

(special case of the above)

16.7.4 Oriented Surfaces

Definition 16.46. *(define formally)* **orientable**, **orientation**

Let $S \subseteq \mathbf{R}^3$ be a smooth *(smooth not necessary?)* orientable surface with parametrization $\mathbf{r}(u, v)$. A unit normal vector to S at any point $\mathbf{r}(u, v)$ is given by

$$\mathbf{n} := \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}. \tag{16.7.2}$$

Definition 16.47. A surface $S \subseteq \mathbf{R}^3$ is **closed** if it is the boundary of a solid region *(give better definition)*.

By convention, for a closed surface S, the **positive orientation** of S is one for which the normal vectors point outward from the bounded solid region.

16.7.5 Surface Integrals of Vector Fields

Definition 16.48. Let $\Omega \subseteq \mathbf{R}^n$, let $\mathbf{F} : \Omega \to \mathbf{R}^n$ be a vector field, and let $S \subseteq \Omega$ be an oriented surface with unit normal vector $\mathbf{n}$ (a function of the point $(x_1, \ldots, x_n) \in S$). The **surface integral of F over** S, also called the **flux of F across S**, is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

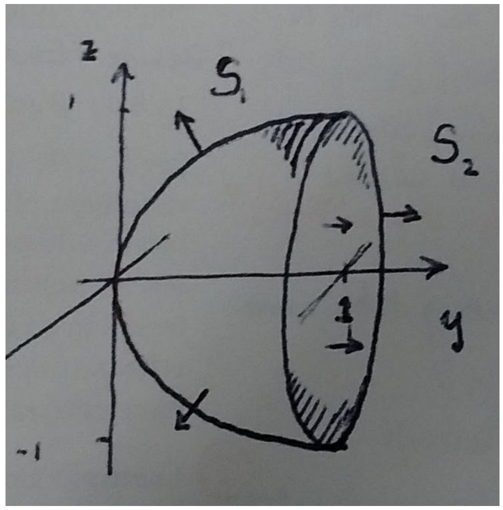

Figure 16.7.1: Surface S of integration for Example 16.49.

Using expressions (16.7.2) for $\mathbf{n}$ and (16.7.1) for dS, we observe that

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, dS \\ &= \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA \\ &= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.\end{aligned}$$

(emphasize analogy to line integral, perhaps before the preceding computation)

Example 16.49 (S6eQ16.07.25)**.** Let $S \subseteq \mathbf{R}^3$ be the surface consisting of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1, y = 1$, and equip S with its positive (outward) orientation; and let

$$\mathbf{F}(x, y, z) := y\,\mathbf{j} - z\,\mathbf{k}.$$

Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

In other words, find the flux of $\mathbf{F}$ across S.

Solution: Let S_1 denote the component of the surface S formed by the paraboloid, and let S_2 denote the component of S formed by the disk. (See Figure 16.7.1. *(beautify figure)*) S_1 and S_2 together form the entire surface S (mathematically, $S = S_1 \cup S_2$, and S_1 and S_2 intersect only along their boundaries), so we may decompose the surface integral of $\mathbf{F}$ over all of S into the surface integral of $\mathbf{F}$ over S_1 plus the surface integral of $\mathbf{F}$ over S_2:[2]

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

The benefit of doing this is that the components S_1 and S_2 have natural (and relatively straightforward) parametrizations. Let's compute the surface integral over each.

We can view the surface S_1 as the graph of the function $y(x,z) = x^2 + z^2$, as described in the problem. This yields the parametrization

$$\begin{aligned} \mathbf{r} &: D \to \mathbf{R}^3 \\ (x,z) &\mapsto \left(x, x^2 + z^2, z\right), \end{aligned}$$

where

$$D := \left\{(x,z) \in \mathbf{R}^2 \mid x^2 + z^2 \leqslant 1\right\} \tag{16.7.3}$$

is the closed unit disk. (See Remark 16.50 below for an alternative parametrization of S_1 in polar coordinates directly, rather than using a change of coordinates, as we do here.) For this parametrization, the partial derivatives are

$$\mathbf{r}_x := \frac{\partial \mathbf{r}}{\partial x} = (1, 2x, 0), \qquad \mathbf{r}_z := \frac{\partial \mathbf{r}}{\partial z} = (0, 2z, 1),$$

so

$$\mathbf{r}_x \times \mathbf{r}_z = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 2z & 1 \end{pmatrix} = (2x, -1, 2z). \tag{16.7.4}$$

Note that the y-component of this normal vector is -1, so it points outward from the solid bounded by S, consistent with the required positive orientation on S. (If the normal vector that we compute pointed inward, then we would simply multiply it by the scalar -1 to reverse its direction and obtain the outward-pointing normal vector.) Thus

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(x,z)) \cdot (\mathbf{r}_x \times \mathbf{r}_z)\, dA \\ &= \iint_D \left(0, x^2 + z^2, -z\right) \cdot (2x, -1, 2z)\, dA \\ &= \iint_D \left(-(x^2 + z^2) - 2z^2\right) dA. \end{aligned}$$

[2]This is a multidimensional analog of the corresponding property from single-variable calculus, e.g.,

$$\int_0^2 f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx.$$

The integrand is a polynomial and therefore continuous everywhere, so by Fubini's theorem we may write the double integral as an iterated integral in any valid manner that we like. Converting to polar coordinates (in the xz-plane, i.e. $x = r\cos\theta$, $z = r\sin\theta$, and $r^2 = x^2 + z^2$), the region of integration becomes the rectangle

$$E = \{(r,\theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}, \tag{16.7.5}$$

and the integral writes as (forget not the integration factor r!)

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left(-r^2 - 2r^2\sin^2\theta\right) r\, dr\, d\theta \\
&= -\int_{\theta=0}^{2\pi} \left(1 + 2\sin^2\theta\right) d\theta \int_{r=0}^{1} r^3\, dr \\
&= -\int_{\theta=0}^{2\pi} \left(1 + 2\left(\frac{1}{2}(1-\cos(2\theta))\right)\right) d\theta \left[\frac{1}{4}r^4\right]_{r=0}^{1} \\
&= -\frac{1}{4}\int_{\theta=0}^{2\pi} (2 - \cos(2\theta))\, d\theta \\
&= -\frac{1}{4}\left[2\theta - \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{2\pi} \\
&= -\pi.
\end{aligned} \tag{16.7.6}$$

The surface S_2 can be parametrized by

$$\begin{aligned}
\mathbf{r} : D &\to \mathbf{R}^3 \\
(x,z) &\mapsto (x,1,z),
\end{aligned}$$

where D is the same parametrization domain as in (16.7.3). Moreover, as we see from a sketch of the surface S, the outward-pointing unit normal vector at each point of S_2 is $\mathbf{j}$. Thus

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_D (x,1,z)\cdot(0,1,0)\, dA = \iint_D dA = \pi.$$

(For this last equality, we could evaluate the double integral, or we could note that the double integral yields the surface area of D, a disc of radius 1.)

We conclude that the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = -\pi + \pi = 0.$$

Remark 16.50. As an alternative parametrization of the surface S_1, we can use polar coordinates directly:

$$\begin{aligned}
\mathbf{r} : E &\to \mathbf{R}^3 \\
(r,\theta) &\mapsto \left(r\cos\theta, r^2, r\sin\theta\right),
\end{aligned}$$

where E is the rectangle given in (16.7.5). (Convince yourself that this parametrization indeed describes the surface S_1!) For this parametrization, the partial derivatives are

$$\mathbf{r}_r := \frac{\partial \mathbf{r}}{\partial r} = (\cos\theta, 2r, \sin\theta)\,, \qquad \mathbf{r}_\theta := \frac{\partial \mathbf{r}}{\partial \theta} = (-r\sin\theta, 0, r\cos\theta)\,,$$

so[3]

$$\begin{aligned}
\mathbf{r}_r \times \mathbf{r}_\theta &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & 2r & \sin\theta \\ -r\sin\theta & 0 & r\cos\theta \end{pmatrix} \\
&= \left(2r^2\cos\theta, -r\sin^2\theta - r\cos^2\theta, 2r^2\sin\theta\right) \\
&= r\,(2r\cos\theta, -1, 2r\sin\theta)\,. \qquad (16.7.7)
\end{aligned}$$

With this parametrization of S_1, the surface integral of $\mathbf{F}$ over S_1 writes as[4]

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \iint_E \mathbf{F}(\mathbf{r}(r,\theta))\cdot(\mathbf{r}_r(r,\theta)\times\mathbf{r}_\theta(r,\theta))\ dA \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left(0, r^2, -r\sin\theta\right)\cdot r\,(2r\cos\theta, -1, 2r\sin\theta)\ dr\,d\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\left(-r^2 - 2r^2\sin^2\theta\right)\ dr\,d\theta.
\end{aligned}$$

This is exactly the same integral we obtained in (16.7.6) after changing from rectangular to polar coordinates above, so by the same computations, we get

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = -\pi.$$

As you can see from Example 16.49, computing surface integrals are often quite involved. In Section 16.9 we will learn an alternative method for computing surface integrals such as this one. Let's preview the method here. The key idea is what we will call our "guiding light":

Remark 16.51 (Guiding Light)**.** Integrating a derivative over a region is related to evaluating the original function on the boundary of that region.

---

[3]Note that if we convert the final result (16.7.7), which is in polar coordinates, to rectangular coordinates, using $x = r\cos\theta$ and $z = r\sin\theta$ as above, then we obtain

$$\mathbf{r}_r \times \mathbf{r}_\theta = r\,(2x, -1, 2z)\,.$$

This is precisely the cross-product $\mathbf{r}_x \times \mathbf{r}_z$ we obtained in (16.7.4), but with the integration factor r already "built in". *(expound)*

[4]Note that no integration factor r is needed, because we are not changing coordinates! Rather, we are starting with a different parametrization.

We've actually been following this guiding light all our calculus lives. In single-variable calculus, we learned about the fundamental theorem of calculus, one incarnation of which states that (under appropriate conditions)

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

On the left, we integrate the derivative of f over the region (i.e. interval) $[a, b]$. On the right, we evaluate the original function f on the points b and a — precisely the boundary points of the region $[a, b]$. The fundamental theorem of calculus relates these two.

Let's see how our guiding light applies to Example 16.49. Let V denote the solid volume enclosed by the surface S. Then S is the boundary of V. I claim that

$$\iiint_V \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}. \tag{16.7.8}$$

Here, $\operatorname{div} \mathbf{F}$ is the divergence of $\mathbf{F}$, also written $\nabla \cdot \mathbf{F}$. The divergence is a type of derivative. So on the left side of (16.7.8), we're integrating a derivative of $\mathbf{F}$ over the region V. On the right side of (16.7.8), we're evaluating (via integration) the original function $\mathbf{F}$ over the boundary of that region V, namely S. So (16.7.8) fits into the framework of our guiding light.

If we compute the divergence of $\mathbf{F}$, we find that $\operatorname{div} \mathbf{F} = 0$. More precisely,

$$\operatorname{div} \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (0, y, -z) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-z) = 0 + 1 - 1 = 0.$$

Substituting this into (16.7.8), we conclude that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \operatorname{div} \mathbf{F}\,dV = \iiint_V 0\,dV = 0.$$

This is the same answer that we computed via surface integrals, but with our guiding light, we've done a lot less work.

(surface integral of a vector field over surface given by graph — special case of the preceding)

16.8 Stokes's Theorem

Key Ideas

- Induced orientations
- Stokes's theorem (statement, how to apply, geometric intuition)
- How to choose a convenient surface for the surface integral

16.8.1 The Theorem

In Section 16.4 we developed Green's theorem, which relates the double integral over a region in $\mathbf{R}^2$ to the line integral along its boundary. Stokes's theorem (in $\mathbf{R}^3$) is a generalization of Green's theorem to $\mathbf{R}^3$, relating the surface integral over an oriented surface in $\mathbf{R}^3$ to the line integral along its boundary.

Definition 16.52. *(reword the following for greater clarity)* Let $S \subseteq \mathbf{R}^3$ be an oriented surface whose boundary $\operatorname{bd} S$ is a piecewise-smooth simple closed curve. The **positive orientation** induced $\operatorname{bd} S$ by the orientation on S is the orientation on $\operatorname{bd} S$ such that the right-hand rule yields the direction of the unit normal vectors on S.

Theorem 16.53 (Stokes's Theorem)**.** *Let $S \subseteq \mathbf{R}^3$ be an oriented piecewise-smooth surface whose boundary $\operatorname{bd} S$ is a piecewise-smooth simple closed curve, equipped with the positive orientation induced by S, and let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ be a vector field whose component functions F_1, F_2, F_3 are $\mathscr{C}^1$ on an open set containing S.*[5] *Then*

$$\int_{\operatorname{bd} S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}. \tag{16.8.1}$$

16.8.2 Geometric Intuition

Chop up the surface S, integrate a bunch of little curls in each square, internal sides of squares cancel, leaving only the curl dot (tangent along) boundary.

16.8.3 Special Case

Let $\mathbf{F} = (F_1, F_2, F_3)$. In the special case that the surface $S \subseteq \mathbf{R}^3$ lies in the xy-plane with upward-pointing unit normal vectors, these vectors are $\mathbf{k}$, the surface S can be taken as its own parametrization, and Stokes's theorem writes as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA,$$

which is Green's theorem.

[5] Recall that a function is $\mathscr{C}^1$ if it is differentiable and its derivatives are continuous.

16.8.4 Remark

Let $C \subseteq \mathbf{R}^3$ be an oriented piecewise-smooth simple closed curve. Note that the statement of Stokes's theorem implies that *any* orientable piecewise-smooth surface S whose boundary is C satisfies (16.8.1) (provided that we equip S with the orientation induced by C). Thus, in situations where we wish to convert a line integral to a surface integral via Stokes's theorem, we have many choices regarding what surface to take. In practice, what surface is best will depend on (i) the ease of parametrization of the candidate surface S and (ii) the given vector field $\mathbf{F}$.

Example 16.54 (S6eE16.08.01)**.** Let

$$\mathbf{F}(x, y, z) := -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k},$$

and let $C \subseteq \mathbf{R}^3$ be the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Equip C with the counterclockwise orientation when viewed from above. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution: It's possible to compute the line integral directly, but it will turn out to be easier to compute via Stokes's theorem.

As noted above, we have many choices for the surface S — any oriented piecewise-smooth surface whose boundary (with induced orientation) is the given oriented curve C will do. In this example, a particularly easy surface S to parametrize is the elliptical region bounded by the curve C in the plane $y + z = 2$. Taking x, y as our parameters, a parametrization of S is

$$\begin{aligned} \mathbf{r} : D &\to \mathbf{R}^3 \\ (x, y) &\mapsto (x, y, 2 - y)\,, \end{aligned} \tag{16.8.2}$$

where

$$\begin{aligned} D &:= \left\{(x, y) \in \mathbf{R}^2 \,\middle|\, x^2 + y^2 \leqslant 1\right\} \\ &= \{(r, \theta) \,|\, r \in [0, 1], \theta \in [0, 2\pi]\}\,. \end{aligned}$$

We seek to compute the integral on the right side of (16.8.1). Given a parametrization $\mathbf{r} : D \to \mathbf{R}^3$ of S, this surface integral writes as

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA. \tag{16.8.3}$$

Using the given vector field $\mathbf{F}$ and the parametrization $\mathbf{r}$ of S in (16.8.2), we compute

$$\operatorname{curl} \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{pmatrix} = (0, 0, 1 + 2y)\,, \tag{16.8.4}$$

and

$$\mathbf{r}_x := \frac{\partial \mathbf{r}}{\partial x} = (1, 0, 0), \qquad \mathbf{r}_y := \frac{\partial \mathbf{r}}{\partial y} = (0, 1, -1),$$

so

$$\mathbf{r}_x \times \mathbf{r}_y = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} = (0, 1, 1). \tag{16.8.5}$$

Note that the direction of this normal vector $\mathbf{r}_x \times \mathbf{r}_y$ agrees with the orientation induced on S by C.

Substituting (16.8.4) and (16.8.5) into (16.8.3), we find

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \iint_D (0, 0, 1 + 2y) \cdot (0, 1, 1) dA = \iint_D (1 + 2y)\, dA.$$

Converting this double integral to polar coordinates (forget not the integration factor of r!), we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1 + 2r \sin\theta)\, r\, dr\, d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(r + 2r^2 \sin\theta\right) dr\, d\theta \\ &= \int_{\theta=0}^{2\pi} \left[\frac{1}{2} r^2 + \frac{2}{3} r^3 \sin\theta\right]_{r=0}^{1} d\theta \\ &= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta\right) d\theta \\ &= \left[\frac{1}{2}\theta - \frac{2}{3}\cos\theta\right]_{\theta=0}^{2\pi} \\ &= \pi. \end{aligned}$$

Remark 16.55. *(add discussion addressing cancelling of norm of normal vector (see Section 16.7)* The orientation induced on S by C is the upward-pointing unit normal. Because S lies in the plane $y + z = 2$, its (upward-pointing) unit normal is the same as the (upward-pointing) unit normal vector of the plane. An upward-pointing normal vector of the plane can be read from the coefficients of its defining equation: $(0, 1, 1)$. The corresponding (upward-pointing) unit normal vector is therefore

$$\frac{1}{\|(0, 1, 1)\|}(0, 1, 1) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

16.9 Divergence Theorem

Key Ideas

- The divergence theorem (statement, how to apply, geometric intuition)

16.9.1 The Theorem

Theorem 16.56. *Let* $E \subseteq \mathbf{R}^3$ *be a simple solid region, equip the surface* bd E *with its positive (outward-pointing) orientation, and let* $\mathbf{F}$ *be a vector field whose component functions* F_1, F_2, F_3 *are* $\mathscr{C}^1$ *on an open region containing* E. *Then*

$$\iint_{\text{bd } E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

16.9.2 Geometric Intuition

Chop up the solid E, *integrate a bunch of little divergences in each cube, internal sides of cubes cancel, leaving only the flux through the boundary.*

16.9.3 Examples

Example 16.57 (S6eE16.09.02)**.** Let $S \subseteq \mathbf{R}^3$ be the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the three planes $z = 0$, $y = 0$, and $y + z = 2$, and let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) := xy\ \mathbf{i} + \left(y^2 + e^{xz^2}\right) \mathbf{j} + \sin(xy)\ \mathbf{k}.$$

Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Solution: *(motivation: why not evaluate directly)*
We compute

$$\operatorname{div} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(xy, y^2 + e^{xz^2}, \sin(xy)\right) = y + 2y + 0 = 3y.$$

(sketch the region E*)* By the divergence theorem,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV \\
&= \iiint_{E_1} \operatorname{div}\mathbf{F}\,dV + \iiint_{E_2} \operatorname{div}\mathbf{F}\,dV \\
&= \int_{x=-1}^{x=1}\int_{z=0}^{z=1-x^2}\int_{y=0}^{y=2-z} 3y\,dy\,dz\,dx \\
&= 3\int_{x=-1}^{x=1}\int_{z=0}^{z=1-x^2}\left[\frac{1}{2}y^2\right]_{y=0}^{y=2-z} dz\,dx \\
&= \frac{3}{2}\int_{x=-1}^{x=1}\int_{z=0}^{z=1-x^2}\left(4-4z+z^2\right)dz\,dx \\
&= \frac{3}{2}\int_{x=-1}^{x=1}\left[4z-2z^2+\frac{1}{3}z^3\right]_{z=0}^{z=1-x^2} dx \\
&= \frac{1}{2}\int_{x=-1}^{x=1}\left(12\left(1-x^2\right)-6\left(1-x^2\right)^2+\left(1-x^2\right)^3\right)dx \\
&= \frac{1}{2}\int_{x=-1}^{x=1}\left(-x^6-3x^4-3x^2+7\right)dx.
\end{aligned}$$

Because the integrand is an even function of x, we can rewrite this last integral as

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \frac{1}{2}\left[2\int_{x=0}^{x=1}\left(-x^6-3x^4-3x^2+7\right)dx\right] \\
&= \left[-\frac{1}{7}x^7-\frac{3}{5}x^5-x^3+7x\right]_{x=0}^{x=1} \\
&= -\frac{1}{7}-\frac{3}{5}-1+7 = 5\frac{9}{35} = \frac{184}{35}.
\end{aligned}$$

(verify result)

Example 16.58. Let us return to Example 16.49, which asked us to compute the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

for the closed surface $S \subseteq \mathbf{R}^3$ given by the paraboloid $y = x^2 + z^2$ with $0 \leqslant y \leqslant 1$ and the disk $x^2 + z^2 = 1, y = 1$, equipped with the outward-pointing unit normal vectors; and the vector field

$$\mathbf{F}(x, y, z) := y\,\mathbf{j} - z\,\mathbf{k}.$$

Solution: We compute the divergence of $\mathbf{F}$ to be

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(0, y, -z) = 0.$$

Thus the divergence theorem affords a particularly attractive way to compute the surface integral: No matter what simple region $E \subseteq \mathbf{R}^3$ we integrate $\operatorname{div} \mathbf{F}$ over, the triple integral equals 0. In particular, this is true for the simple region E bounded by S. That is,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 0 \, dV = 0.$$

Comparing this to our solution in Example 16.49, we see that in this case, using the divergence theorem requires much less work than evaluating the surface integral directly!

Bibliography

Stewart, James. 2008. *Calculus: Early Transcendentals*. 6 edn. Thomson Brooks/Cole.