Math 357
Short quiz 12

2024–04–10 (W)

Your name: ______________________________

Let $K : K_0$ be a field extension, let $\alpha \in K$ be algebraic over K_0, and let $\sigma \in \mathrm{Aut}(K : K_0)$. Characterize, as precisely and as fully as possible, the polynomials in $K_0[t]$ for which α or $\sigma(\alpha)$ is a zero.

Solution: By hypothesis, $\alpha \in K$ is algebraic over K_0, so there exists a (unique) minimal polynomial m_{α,K_0}, which satisfies the three defining axioms:

(i) $m_{\alpha,K_0}(\alpha) = 0_K$ (view m_{α,K_0} as a function $K \to K$).

(ii) m_{α,K_0} is irreducible. (Equivalently, for all nonzero $f \in K_0[t]$ such that $f(\alpha) = 0_K$, $\deg m_{\alpha,K_0} \leqslant \deg f$.)

(iii) $\mathrm{LC}(m_{\alpha,K_0}) = 1_{K_0}$.

Because $\sigma \in \mathrm{Aut}(K : K_0)$ and $m_{\alpha,K_0} \in K_0[t]$, we have seen that

$$m_{\alpha,K_0}(\alpha) = 0_K \qquad \Leftrightarrow \qquad m_{\alpha,K_0}(\sigma(\alpha)) = 0_K$$

It follows that m_{α,K_0} satisfies the three defining axioms to be the minimal polynomial for $\sigma(\alpha)$ over K_0, that is,

$$m_{\sigma(\alpha),K_0} = m_{\alpha,K_0}$$

We have seen that if $f \in K_0[t]$ has an algebraic element as a zero, then f is divisible by the minimal polynomial of that element over K_0. We conclude that the polynomials in $K_0[t]$ for which α is a zero are the same as those for which $\sigma(\alpha)$ is a zero, and they are precisely the polynomials in the ideal generated by m_{α,K_0} in $K_0[t]$.