Math 357
Long quiz 04A

2024–02–19 (M)

Your name: ______________________________

Let $\mathbf{Q}$ denote the field of rational numbers; given a prime $p \in \mathbf{Z}_{>0}$, let $\mathbf{F}_p \cong \mathbf{Z}/(p)$ denote the finite field with p elements; and let t be an indeterminate. For each of the quotient rings below, characterize its algebraic structure as "field", "integral domain but not field", or "ring but not integral domain". Justify your characterization.

$$R_1 = \mathbf{F}_2[t]/(t^4 + t^2 + 1) \qquad R_2 = \mathbf{Q}[t]/(t^3 + t^2 - t + 1) \qquad R_3 = \mathbf{Q}[t]/(3t^3 + 4t^2 + 2t - 4)$$

Hint: If you feel inclined to do a lot of computation, then I invite you to first check with me.

Solution: We analyze each quotient ring in turn.

R_1 : Ring but not integral domain. Let $f_1 = t^4 + t^2 + 1 \in \mathbf{F}_2[t]$. If f_1 is reducible, then it has a factor of degree 1 or degree 2. The polynomial $f_1 \in \mathbf{F}_2[t]$ has a factor of degree 1 if and only if the function $f_1 : \mathbf{F}_2 \to \mathbf{F}_2$ has a zero, which direct computation shows is not the case. There are only four polynomials in $\mathbf{F}_2[t]$ with degree 2 (why?), and three of them factor into linear factors, which we checked for in the previous case. Thus it remains only to check whether

$$f_1 = (t^2 + t + 1)^2$$

which direct computation shows is a valid equation. Thus $f_1 \in \mathbf{F}_2[t]$ is reducible, so $\mathbf{F}_2[t]/(f_1)$ is a ring but not an integral domain.

R_2 : Field. Let $f_2 = t^3 + t^2 - t + 1 \in \mathbf{Z}[t]$. If we apply the reduction homomorphism corresponding to the ideal $(3) \triangleleft \mathbf{Z}$ to f_2, then we get the polynomial $\overline{f}_2 \in \mathbf{F}_3[t]$, which we may express with the same coefficients as f_2 (viewed in $\mathbf{F}_3$, rather than in $\mathbf{Z}$ or $\mathbf{Q}$). Because $\deg \overline{f}_2 = 3$, the polynomial $\overline{f}_2 \in \mathbf{F}_3[t]$ is reducible if and only if the function $\overline{f}_2 : \mathbf{F}_3 \to \mathbf{F}_3$ has a zero. Direct computation shows that for all $\alpha \in \mathbf{F}_3$, $\overline{f}_2(\alpha) \neq 0$, so $\overline{f}_2$ is irreducible. Because f_2 is nonconstant and monic, and $(3) \triangleleft \mathbf{Z}$ is proper, this implies that $f_2 \in \mathbf{Z}[t]$ is irreducible. Thus Gauß's lemma implies that $f_2 \in \mathbf{Q}[t]$ is irreducible. Hence $\mathbf{Q}[t]/(f_2)$ is a field.

Note that we cannot apply the Eisenstein–Schönemann criterion directly to f_2 (why not?), nor may we (correctly) argue that f_2 must have a zero because $\deg f_2$ is odd (why not?).

R_3 : Ring but not integral domain. Let $f_3 = 3t^3 + 4t^2 + 2t - 4 \in \mathbf{Z}[t]$. Because $\deg f_3 = 3$, the polynomial $f_3 \in \mathbf{Q}[t]$ is reducible if and only if the function $f_3 : \mathbf{Q} \to \mathbf{Q}$ has a zero. If $\frac{a}{b} \in \mathbf{Q}$ is a zero of f_3, and if $\gcd(a, b) = 1$, then we have seen that, in $\mathbf{Z}$, $a \mid -4$ (the constant term of f_3) and $b \mid 3$ (the leading coefficient of f_3). We have finitely many possibilities for $\frac{a}{b}$—24 in this case (why?)—but we may reduce our work significantly if we view $f_3 \in \mathbf{R}[t]$ and note that

$$f_3(0) = -4 < 0 \qquad\qquad f_3(1) = 5 > 0$$

Because the function induced by a polynomial is continuous, the intermediate value theorem implies that f_3 has a zero (in $\mathbf{R}$) on the interval $(0, 1)$. The only values of $\frac{a}{b}$ in this interval, and consistent with the divisibility requirements on a and b, are $\frac{1}{3}$ and $\frac{2}{3}$. Checking these possibilities, we find $f_3(\frac{2}{3}) = 0$. Hence f_3 is reducible in $\mathbf{Q}[t]$, so $\mathbf{Q}[t]/(f_3)$ is a ring but not an integral domain.

Note that we cannot apply the Eisenstein–Schönemann criterion directly to f_3 (why not?).