Math 357
Exam 01M

2024–02–23 (F)

Your name: ______________________________________

Instructions

1. In the space above, please legibly write your name and the Rice Honor Pledge, then sign.
2. You have exactly 50 minutes for this exam. No resources are allowed.

Exercise	Total	(a)	(b)	(c)
1	/4	/4	/4	/4
2	/4	/4	/4	
3	/4	/4	/4	/4
4	/4	/4	/4	
5	/4			
Total	/20			

Exercise 1

(4 pt) Let R be an integral domain. For each of the following statements, state whether it is true or false, and prove it (if true) or give a counterexample (if false).

(a) The product of two units is a unit.

(b) If $r \in R$ is prime, then it is irreducible.

(c) If $r \in R$ is irreducible, then it is prime.

Exercise 2

(4 pt) Let F be a field, let t be an indeterminate, and let $I \trianglelefteq F[t]$ be an ideal.

(a) Characterize I as completely as possible.

(b) For each of the following algebraic structures, state whether there exists an ideal $I \trianglelefteq F[t]$ such that the quotient ring $F[t]/I$ has that structure. If yes, then give an example. If no, then justify why.

(i) field (ii) integral domain but not field (iii) ring but not integral domain

Exercise 3

(4 pt) Let $\mathbf{Q}$ denote the field of rational numbers; given a prime $p \in \mathbf{Z}_{>0}$, let $\mathbb{F}_p \cong \mathbf{Z}/(p)$ denote the finite field with p elements; and let t be an indeterminate. For each of the following polynomials, state whether it is irreducible or reducible. Justify your assertions.

(a) $f_1 = t^4 - 2t^3 + t + 1 \in \mathbb{F}_5[t]$

(b) $f_2 = 6t^4 + 11t^3 + 8t^2 - 6t - 4 \in \mathbf{Q}[t]$

(c) $f_3 = t^4 + 4t^3 + t + 16 \in \mathbf{Q}[t]$

Exercise 4

(4 pt) Let F be a field, let G be a finite group, and let V be an F-vector space.

(a) Given a representation $\rho : G \to GL(V)$, describe how to define an FG-module structure on V that extends the F-action of the F-vector space V and that affords the representation ρ.

(b) Given an FG-module V, describe how to recover a representation ρ of G on V.

Exercise 5

(4 pt) Consider three matrix representations $\rho_j : \mathbf{Z}/4\mathbf{Z} \to \mathrm{GL}_2(\mathbf{C})$, $j \in \{1, 2, 3\}$, defined by[1]

$$\rho_1(m) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^m \qquad \rho_2(m) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^m \qquad \rho_3(m) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^m$$

(Here $i \in \mathbf{C}$ denotes a square root of -1.) State which of these matrix representations are faithful, which are irreducible, and which are equivalent. Briefly justify your assertions.

[1] More precisely, we should write $\rho_j(m + 4\mathbf{Z})$. To keep notation simple, I have written what I have written.