Math 211
Quiz 10

F 19 Jul 2019

Your name : ______________________________

Exercise

(5 pt) Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & -2 & 1 \end{bmatrix}.$$

(a) (2 pt) Show that $\det A = 2$. *Hint:* Use expansion by minors along row 1. (Why?)

Solution: Using expansion by minors along row 1 (which we choose because row 1 contains many zeros, which will simplify our work), we compute

$$\det A = 0 + 0 + (-1)^{1+3}(1)\det \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix} = (1)(1)(1(-2) - 2(-2)) = 2,$$

where we use the fact that the determinant of a 2×2 matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Note: Expansion by minors along any row or column will yield this same result. To illustrate, consider expansion by minors along column 2. We get

$$\begin{aligned} \det A &= 0 + (-1)^{2+2}(2)\det \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} + (-1)^{3+2}(-2)\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= 0 + (1)(2)(0(1) - 1(-2)) + (-1)(-2)(0(0) - 1(1)) = 2(2) + 2(-1) = 2. \end{aligned}$$

(b) (2 pt) Apply the row reduction algorithm to $\left[\begin{array}{c:c} \mathbf{A} & \mathbf{I}_3 \end{array}\right]$ to show that

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 & -2 \\ -1 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Hint: Recall that one of the three elementary row operations lets us swap any two rows.

Solution: We compute

$$
\left[\, \mathbf{A} \;\vdots\; \mathbf{I}_3 \,\right] = \left[\begin{array}{ccc:ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_3 = R_3 + 2R_1} \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_3 = \frac{1}{2}R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 1 & \frac{1}{2} \end{array}\right]
$$

$$
\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]
$$

$$
\xrightarrow{R_1 = R_1 + R_3,\, R_2 = R_2 - \frac{1}{2}R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right].
$$

The 3×3 matrix to the right of the dashed lines is A^{-1}. Factoring out $\frac{1}{2}$ from each of the entries of this matrix, we conclude that

$$
A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -2 & -2 \\ -1 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix},
$$

as claimed.

(c) (1 pt) Explain how existence of A^{-1} in part (b) is consistent with our answer in part (a).

Solution: A square matrix is invertible if and only if its determinant is nonzero. In part (a), we computed $\det A = 2 \neq 0$, so we know A^{-1} exists (and hence our computations in part (b) to find it are not in vain).