Math 212
Quiz 25

F 28 Oct 2016

Your name: ______________________________

Exercise

(5 pt) Let $E \subseteq \mathbf{R}^3$ be the region inside the cylinder $x^2 + y^2 = 1$, above the xy-plane, and below (!) the cone $z = \sqrt{x^2 + y^2}$. We seek to evaluate the triple integral

$$\iiint_E z \, dV.$$

(a) (1 pt) Sketch the region E. *Hint:* Where does the cone intersect the cylinder?

Solution: *(include graphic)*

(a) (2 pt) State your choice of coordinate system. Write the corresponding differential dV, and give an algebraic description of the region E (i.e. lower and upper limits on the variables) in these coordinates. *Hint:* One variable will have limits that depend on another variable.

Solution: Because the region E of integration is defined (in part) by a (bona fide) cylinder, cylindrical coordinates will be easiest. In cylindrical coordinates,

- region of integration:

$$\begin{aligned} E &= \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant 1, 0 \leqslant z \leqslant r\} \\ &= \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant z \leqslant 1, z \leqslant r \leqslant 1\}, \end{aligned}$$

- integrand: $z = z$,
- differential: $dV = r \, dr \, d\theta \, dz$.

(b) (2 pt) Show that $\iiint_E z \, dV = \frac{\pi}{4}$ (i.e. evaluate the triple integral).

Solution: The integrand $f(x, y, z) = z$ is continuous everywhere on $\mathbf{R}^3$; in particular, it is continuous on the region of integration E. Hence by Fubini's theorem, we may evaluate the triple integral as an iterated integral using any order of integration.

With the order $dr \, d\theta \, dz$, the triple integral writes as

$$\begin{aligned} \iiint_E z \, dV &= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} \int_{r=z}^{r=1} z \, r \, dr \, d\theta \, dz \\ &= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{z=0}^{z=1} z \int_{r=z}^{r=1} r \, dr \, dz \\ &= (2\pi) \int_{z=0}^{z=1} z \left[\frac{1}{2} r^2 \right]_{r=z}^{r=1} dz \\ &= \pi \int_{z=0}^{z=1} \left(z - z^3 \right) dz \\ &= \pi \left[\frac{1}{2} z^2 - \frac{1}{4} z^4 \right]_{z=0}^{z=1} \\ &= \frac{\pi}{4}. \end{aligned}$$

With the order $\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r$, the triple integral writes as

$$\begin{aligned}
\iiint_E z\,\mathrm{d}V &= \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=r} z\,r\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_{\theta=0}^{\theta=2\pi}\mathrm{d}\theta\int_{r=0}^{r=1} r\int_{z=0}^{z=r} z\,\mathrm{d}z\,\mathrm{d}r \\
&= (2\pi)\int_{r=0}^{r=1} r\left[\frac{1}{2}z^2\right]_{z=0}^{z=r}\mathrm{d}r \\
&= \pi\int_{r=0}^{r=1} r^3\,\mathrm{d}r \\
&= \pi\left[\frac{1}{4}r^4\right]_{r=0}^{r=1} \\
&= \frac{\pi}{4}.
\end{aligned}$$

N.B. Because the variable θ appears neither in the integrand nor in any of the limits of integration, we may always pull the integral with respect to θ out on its own, so any order of integration is equivalent to (the second line of) one of the above two analyses.