Math 357
Expositional homework 03

Assigned: 2024–02–07 (W)
Due:

The goal of this homework is to practice and extend our irreducibility–detector toolkit. The exercises are adapted from Dummit & Foote, 3e, Section 9.4.

Let t be an indeterminate; let $\mathbf{Z}$ denote the ring of integers; and for $p \in \mathbf{Z}$ prime, let $\mathbf{F}_p = \mathbf{Z}/(p)$ (a finite field with p elements).

(a) Prove the Eisenstein–Schönemann criterion for $\mathbf{Z}$, as we stated it in class: Let $p \in \mathbf{Z}$ be prime; let $f = a_n t^n + \ldots + a_0 \in \mathbf{Z}[t]$, with $n = \deg f \geqslant 1$; and let $p \nmid a_n$, $p \mid a_{n-1}, \ldots, a_0$, and $p^2 \nmid a_0$. Then f is irreducible in $\mathbf{Q}[t]$. Moreover, if $\gcd(a_n, \ldots, a_0) = 1$, then f is irreducible in $\mathbf{Z}[t]$.

(b) For each of the following polynomials, determine whether it is irreducible or reducible in the indicated polynomial ring. If it is reducible, then give its factorization into irreducibles.

$$\begin{aligned} f(t) &= t^6 + 30t^5 - 15t^3 + 6t - 120 \in \mathbf{Z}[t] \\ g(t) &= t^3 + t + 1 \in \mathbf{Z}[t] \\ h(t) &= t^3 + t + 1 \in \mathbf{F}_3[t] \end{aligned}$$

(c) Let $n \in \mathbf{Z}_{>0}$, and consider the polynomial

$$f(t) = 1 + \prod_{i=1}^{n} (t - i) \in \mathbf{Z}[t]$$

Show that f is irreducible for all $n \neq 4$.

(d) Let $p \in \mathbf{Z}$ be prime, and consider the cyclotomic polynomial[1]

$$\Phi_p(t) = t^{p-1} + \ldots + 1 \in \mathbf{Z}[t]$$

Show that Φ_p is irreducible. Explain the technique you use. *Hint:* See p 310.

(e) Let F be a field, let $f \in F[t]$, and let $n = \deg f$. The **reverse** of f is the polynomial $t^n f(t^{-1})$. Justify why this construction gives a valid polynomial in $F[t]$ (even though t^{-1} is not an element of $F[t]$). Give an example of a polynomial and its reverse that clearly illustrates why this name is apt for this construction. Prove that if $f(0) \neq 0$, then f is irreducible if and only if its reverse is irreducible.

[1] Note that we can view Φ_p as the quotient of $t^p - 1$ when (evenly) divided by $t - 1$; that is, $\Phi_p(t) = \frac{t^p - 1}{t - 1}$.