Math 112
Exam 02

2022–03–10 (R)

Your name: ______________________________

Instructions

Number of exercises	: 5
Permitted time	: 75 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- Work hard, do your best, and have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/16	/4	/4	/4	/4	
3	/12	/4	/4	/4		
4	/10	/2	/4	/4		
5	/18	/4	/2	/4	/4	/4
Total	/66					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

For parts (a)–(b), let $f(x)$ be a function, and let $F_1(x)$ and $F_2(x)$ be antiderivatives of $f(x)$.

(a) (2 pt) The function $2F_1(x)$ is an antiderivative of $2f(x)$.

true false

(b) (2 pt) The function $F_1(x) + F_2(x)$ is an antiderivative of $2f(x)$.

true false

For parts (c)–(d), let $f(x)$ and $g(x)$ be functions such that

$$\int_0^2 f(x)\,dx = 1 \qquad \int_0^2 g(x)\,dx = -4$$

(c) (2 pt) $\displaystyle\int_0^2 \left[3f(x) + \frac{1}{2}g(x)\right]\,dx = 1$

true false

(d) (2 pt) $\displaystyle\int_0^1 g(x)\,dx + \int_1^2 g(x)\,dx = -2$

true false

(e) (2 pt) If the definite integral of a function is zero, then the function must be the zero function.

true false

Exercise 2

(16 pt) Evaluate each of the following limits. Briefly but clearly justify your work.

(a) (4 pt) Use the Taylor series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \tag{1}$$

where the terms in "…" all involve x to the power 4 or higher, to evaluate

$$\lim_{x\to 0} \frac{x^2 + x - \ln(1+x)}{x^2}$$

(b) (4 pt) Use l'Hôpital's rule to evaluate

$$\lim_{x\to 0} \frac{x^2 + x - \ln(1+x)}{x^2}$$

(Note that this is the same limit as in part (a).)

(c) (4 pt) $\lim_{x \to 2} \frac{1 - \sqrt{5 - 2x}}{x - 2}$

(d) (4 pt) $\lim_{x \to +\infty} x^3 e^{-x}$

Exercise 3

(12 pt) Evaluate the indefinite integrals. (That is, find the most-general antiderivative $F(x)$ of the integrand $f(x)$ in the following integrals $\int f(x)\,dx$.)

(a) (4 pt) $\displaystyle\int 4x^3 - 3x^2 + 1\,dx$

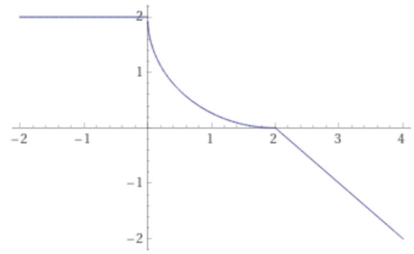
(b) (4 pt) $\displaystyle\int 6e^{3x} - 25\cos(5x)\,dx$

(c) (4 pt) $\displaystyle\int \frac{1 - x\sin x}{x}\,dx$

Exercise 4

(10 pt) Consider the piecewise function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 2 & \text{if } x \leqslant 0 \\ 2 - \sqrt{4x - x^2} & \text{if } -2 \leqslant x \leqslant 0 \\ 2 - x & \text{if } x \geqslant 2 \end{cases}$$



A graph of f is shown at right.

(a) (2 pt) Consider the rule of assignment for f on the interval $[0, 2]$, namely,

$$y = 2 - \sqrt{4x - x^2}$$

Use algebra to massage this equation into the form $(x - a)^2 + (y - b)^2 = r^2$, thus showing that the graph of f on the interval $[0, 2]$ is part of a circle of radius $r = 2$.

(b) (4 pt) Use geometry to compute the exact value of $\int_{-2}^{4} f(x)\, dx$. You may leave your answer in terms of π; if you prefer, you may use that $\pi \approx 3.14$. (Hint: Use the conclusion from part (a) to help find the area on the interval $[0, 2]$. The area of a full circle of radius r is πr^2. Remember, area is signed $(+/-)$!)

(c) (4 pt) Is the definite integral $\int_{0}^{4} f(x)\, dx$ positive, negative, or zero? Justify. (Hint: You need not give an exact value, though you may. It is enough to justify your answer using the geometry of the graph of $f(x)$.)

Exercise 5

(18 pt) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = 4x^3 - 12x + 8 \tag{2}$$

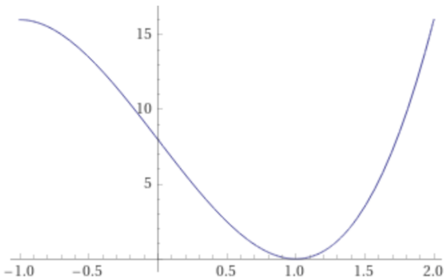
(a) (4 pt) Find an antiderivative $F(x)$ of $f(x)$. Verify that it is indeed an antiderivative.

(b) (2 pt) Using your antiderivative $F(x)$ from part (a), show that

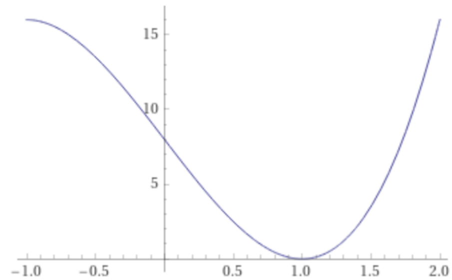
$$\int_{-1}^{2} f(x)\, dx = F(2) - F(-1) = 21$$

Take the first equality as given. (It is the fundamental theorem of calculus.) Just show that $F(2) - F(-1) = 21$.

(c) (4 pt) On the graphs of f below, draw a lower- and upper-sum approximation to the definite integral $\int_{-1}^{2} f(x)\, dx$. Partition the interval $[-1, 2]$ into three subintervals, each of width 1.



Lower sum



Upper sum

(d) (4 pt) Compute the upper- and lower-sum approximations you sketched in part (c). (Use the rule of assignment for $f(x)$, given in Equation (2) at the start of this exercise, to help find the heights of your rectangles.) Show that these approximations bound your value of the definite integral computed in part (b).

(e) (4 pt) Your friend computes lower- and upper-sum approximations for $\int_{-1}^{2} f(x)\, dx$ with 300 subintervals, each of width $\frac{1}{100}$. Your friend reports the results are $L_{300} \approx -1.2723$ and $U_{300} \approx 20.8603$. Explain to your friend why neither result can be correct. (Be kind—your friend just did a lot of work!)