Math 112
LQuiz 10

2022–02–24 (R)

Your name: ______________________________

Exercise

(4 pt) Compute the following indefinite integrals (aka antiderivatives). Check your answers by differentiating. (What should you get?)

(a) (2 pt) $\int 3x^2 + 6x - 1 \, dx$

Solution: Our goal is to find the most general function $F(x)$ such that its derivative $F'(x)$ equals the given integrand, $3x^2 + 6x - 1$.

With experience, we can often "guess and check" relatively simple indefinite integrals. However, we'll present a more systematic approach here.

Using the linearity of the indefinite integral, we have

$$\int 3x^2 + 6x - 1 \, dx = 3\int x^2 \, dx + 6\int x \, dx - \int 1 \, dx \tag{1}$$

Each of these simpler integrals can be solved by thinking of the power rule for differentiation, run in reverse. More precisely, the power rule for differentiation gives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

If $n \neq 0$, we can rearrange this as

$$\frac{d}{dx}\left(\frac{1}{n}x^n\right) = \frac{1}{n}\frac{d}{dx}(x^n) = x^{n-1}$$

Using the definition of an antiderivative, we can view this last statement as saying

$$\frac{1}{n}x^n \text{ is an antiderivative of } x^{n-1}$$

for $n \neq 0$. To get the most general antiderivative from a particular antiderivative, simply add an arbitrary constant $C \in \mathbf{R}$ to the particular antiderivative. Thus we conclude that

$$\frac{1}{n}x^n + C \text{ is the most general antiderivative of } x^{n-1}$$

or equivalently,

$$\frac{1}{n}x^n + C = \int x^{n-1} \, dx$$

Applying these results to the three integrals in (1), we get

$$\int x^2 \, dx = \frac{1}{3}x^3 \qquad \int x \, dx = \frac{1}{2}x^2 \qquad \int 1 \, dx = \int x^0 \, dx = x$$

Note that we've left off all the "+C"s for the moment — what we have written is one particular antiderivative for each integral. Substituting these into (1), we get

$$\int 3x^2 + 6x - 1 \, dx = 3\left(\frac{1}{3}x^3\right) + 6\left(\frac{1}{2}x^2\right) - x = x^3 + 3x^2 - x$$

This equation says that $x^3 + 3x^2 - x$ is one particular antiderivative of $3x^2 + 6x - 1$. To get the most general antiderivative, we just add C to the particular antiderivative:

$$\int 3x^2 + 6x - 1 \, dx = x^3 + 3x^2 - x + C$$

To check our result, we differentiate it, to ensure that we get the original integrand:

$$\frac{d}{dx}\left(x^3 + 3x^2 - x + C\right) = 3x^2 + 6x - 1$$

(b) (2 pt) $\int e^{2x} - \cos x \, dx$

Solution: Using the linearity of the integral, we have

$$\int e^{2x} - \cos x \, dx = \int e^{2x} \, dx - \int \cos x \, dx \tag{2}$$

The power rule for differentiation doesn't apply to these integrands. We go back to the definition of an antiderivative.

For the first integral, what is a function whose derivative equals e^{2x}? Well, the derivative of $e^{\text{something}}$ is $e^{\text{something}}$. So let's try $F(x) = e^{2x}$ as our antiderivative of $f(x) = e^{2x}$. Using the chain rule, we compute

$$F'(x) = \frac{d}{dx} e^{2x} = 2e^{2x} = 2f(x)$$

So $F'(x) \neq f(x)$ — we're off by a factor of 2. However, note that we can move that factor of 2 to the left side of our equation, getting

$$\left(\frac{1}{2}F(x)\right)' = \frac{1}{2}F'(x) = f(x)$$

This equation tells us that the antiderivative we seek isn't $F(x)$, but $\frac{1}{2}F(x)$, that is, $\frac{1}{2}e^{2x}$:

$$\int e^{2x} \, dx = \frac{1}{2}e^{2x}$$

For the second integral, what is a function whose derivative is $\cos x$? The function $\sin x$ satisfies $\frac{d}{dx}(\sin x) = \cos x$. Thus by definition of antiderivative,

$$\int \cos x \, dx = \sin x$$

Substituting these two results into (2), we get

$$\int e^{2x} - \cos x \, dx = \frac{1}{2}e^{2x} - \sin x$$

This equation says that $\frac{1}{2}e^{2x} - \sin x$ is one particular antiderivative of $e^{2x} - \cos x$. To get the most general antiderivative, we just add C to the particular antiderivative:

$$\int e^{2x} - \cos x \, dx = \frac{1}{2}e^{2x} - \sin x + C$$

To check our result, we differentiate it, to ensure that we get the original integrand:

$$\frac{d}{dx}\left(\frac{1}{2}e^{2x} - \sin x + C\right) = \frac{1}{2}\left(2e^{2x}\right) - \cos x + 0 = e^{2x} - \cos x$$