Math 212
Quiz 01

M 22 Aug 2016

Your name: ______________________________

Honor pledge:

Exam instructions

Number of exercises	: 12
Permitted time	: 30 minutes
Permitted resources	: None

Instructor's note:

- BEFORE SOLVING any exercises, go through the entire quiz and write a "confidence number" 1–5 to the LEFT of each exercise, denoting how confident you are that you can solve the exercise (1 = "Not at all confident", 5 = "Very confident").

1.1	/2	2.1	/2	3.1	/2
1.2	/2	2.2	/2	3.2	/2
1.3	/2	2.3	/2	3.3	/2
1.4	/2	2.4	/2	3.4	/2
Total	/8		/8		/8

Geometry

1.1 Exercise 1.1

(2 pt) Compute the distance between the points $(-2,-1)$ and $(4,7)$ in $\mathbf{R}^2$.

Solution: The distance is

$$d = \sqrt{(4-(-2))^2 + (7-(-1))^2} = \sqrt{6^2+8^2} = 10.$$

1.2 Exercise 1.2

(2 pt) Write the area of a circular sector with radius r and angle θ, where $0 \leqslant \theta \leqslant 2\pi$.

Solution: Note that a sector of a circle is a fraction of the circle, more precisely, the fraction $\frac{\theta}{2\pi}$. Thus the area of the sector is given by

$$\frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}\theta r^2.$$

1.3 Exercise 1.3

(2 pt) Write the area of a parallelogram with adjacent side lengths u, v and enclosed angle θ.

Solution: Imagine dropping a perpendicular from one side, say a side of length u, to the opposite parallel side (also of length a); cutting off the resulting right triangle (with hypotenuse v); and pasting this triangle to the other side of the parallelogram, lining up the sides of length v. This yields a rectangle with side lengths u and $v\sin\theta$. Thus the area of the parallelogram is

$$uv\sin\theta.$$

1.4 Exercise 1.4

(2 pt) Convert the point $(4, \frac{7\pi}{6})$, given in polar coordinates (r,θ), to rectangular coordinates (x,y).

Solution: The point $(4, \frac{7\pi}{6})$ in polar coordinates lies in the third quadrant (negative x, negative y) of the euclidean plane. The corresponding rectangular coordinates are (x,y), where

$$x = r\cos\theta = 4\cos\frac{7\pi}{6} = -2\sqrt{3}, \qquad y = r\sin\theta = 4\sin\frac{7\pi}{6} = -2.$$

Single-Variable Calculus

2.1 Exercise 2.1

(2 pt) Evaluate the integral $\int \cos^2\theta \, d\theta$.

Solution: Recall the trigonometric identity

$$\cos 2\theta = 2\cos^2\theta - 1 \qquad \Leftrightarrow \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta).$$

Substituting this into the given integral we find

$$\begin{aligned}\int \cos^2\theta \, d\theta &= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta \\ &= \frac{1}{2}\int d\theta + \frac{1}{4}\int \cos 2\theta \, 2d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C,\end{aligned}$$

where C is a constant of integration.

2.2 Exercise 2.2

(2 pt) Evaluate the integral $\int \sqrt{2x+1}\, dx$.

Solution: We cannot evaluate the given integral directly. Consider the change of variables (a.k.a. u-substitution)

$$u := 2x+1 \qquad \Rightarrow \qquad du = 2dx \qquad \Leftrightarrow \qquad dx = \frac{1}{2}du.$$

When we make this substitution, the given integral becomes

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C,$$

where C is a constant of integration.

2.3 Exercise 2.3

(2 pt) Let f be a real-valued function defined on the closed interval $[a, b]$, and let f' be continuous on $[a, b]$. State the length of the curve $f(x)$ from $x = a$ to $x = b$.

Solution: The length of the curve is

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx. \tag{1}$$

One can arrive at this result as follows. Choose $n + 1$ (distinct) points $(x_i, f(x_i))$ on the graph of the function f on the closed interval $[a, b]$, with $(x_0, f(x_0)) = (a, f(a))$ and $(x_n, f(x_n)) = (b, f(b))$. Then "connect the dots", i.e. draw a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$, for each $i \in \{1, \ldots, n\}$. This yields a polygonal approximation to the graph of f. Let

$$\Delta x_i := x_i - x_{i-1}, \qquad \Delta y_i := f(x_i) - f(x_{i-1});$$

these are the changes in x and y, respectively, over the line segment connecting $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$. Thus by Pythagorean's theorem, the length of this line segment, denote it Δs_i, is

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\Delta x_i,$$

and the length of the polygonal approximation to the length s of the curve $f(x)$ is

$$s \approx \sum_{i=1}^{n} \Delta s_i = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\Delta x_i. \tag{2}$$

If we choose more and more points along the curve (i.e. as the number of points n approaches infinity),[1] then this polygonal approximation becomes better and better. In fact, as $n \to \infty$, $\frac{\Delta y_i}{\Delta x_i} \to f'(x_i)$, and the sum (2) becomes the integral (1). *(draw the picture)*

2.4 Exercise 2.4

(2 pt) Let f be a real-valued function defined on a closed interval $[a, b]$ with $a < b$. Draw a picture depicting a Riemann sum corresponding to the definite integral

$$\int_a^b f(x)dx.$$

Solution: See Sections 5.1 and 5.2 of Stewart. *(include sketch; make f noncontinuous, changing sign)*

Vector Calculus

3.1 Exercise 3.1

(2 pt) Let $\mathbf{u} := (2, 0, 1)$ and $\mathbf{v} := (0, -1, 1)$ be vectors in $\mathbf{R}^3$.

(a) (1 pt) Compute the inner product (a.k.a. dot product) $\mathbf{u} \cdot \mathbf{v}$.

Solution: We compute

$$\mathbf{u} \cdot \mathbf{v} = (2, 0, 1) \cdot (0, -1, 1) = 2 \cdot 0 + 0 \cdot (-1) + 1 \cdot 1 = 1.$$

(b) (1 pt) Compute the cross product $\mathbf{u} \times \mathbf{v}$.

Solution: We compute

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \\ &= (-1)^{1+1} \det \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{i} + (-1)^{1+2} \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{j} + (-1)^{1+3} \det \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{k} \\ &= \mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = (1, -2, -2). \end{aligned}$$

3.2 Exercise 3.2

(2 pt) Find the absolute maximum and minimum values of the function

$$f(x, y) := x^2 - 2xy + 2y$$

on the closed rectangle

$$D := \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}.$$

3.3 Exercise 3.3

(2 pt) Evaluate the integral

$$\iint_R e^{\frac{x+y}{x-y}}\, dA,$$

where R is the trapezoidal region in $\mathbf{R}^2$ with vertices $(1,0), (2,0), (0,-2), (0,-1)$.

3.4 Exercise 3.4

(2 pt) Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where

$$\mathbf{F}(x, y, z) := \left(xy, y^2 + e^{xz^2}, \sin(xy)\right),$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.