Math 357
Expositional homework 04

Assigned: 2024–02–14 (W)
Due:

The goal of this homework is to understand modules—including their connection to and generalization of vector spaces—via a specific and important example: $F[t]$-modules. The exercises are adapted from Dummit & Foote, 3e, Sections 10.1 and 10.3.

Let F be a field, let t be an indeterminate, let V be an F-module (aka F-vector space), and let $T : V \to V$ be an F-module homomorphism (aka F-linear transformation).

(a) Use the linear transformation T to give V the structure of an $F[t]$-module. In particular, verify that the ring action of $F[t]$ on V that you define satisfies the module axioms. *Hint:* See p 340.

(b) Let $U \leqslant V$ be an F-submodule (aka F-subspace). U is T**-stable** (aka T**-invariant**) if for all $u \in U$, $T(u) \in U$.[1] Show that $W \subseteq V$ is an $F[t]$-submodule if and only if W is a T-stable F-subspace of V. *Hint:* See p 341.

(c) View V as an F-vector space. Recall that the **endomorphism ring of** V is the set $\mathrm{End}_F(V)$ of all F-linear transformations from V to itself, equipped with operations $+$ and $\times$ (see pp 346–7). There is a natural map

$$\begin{aligned} F &\to \mathrm{End}_F(V) \\ \alpha &\mapsto \alpha I \end{aligned}$$

where $I : V \to V$ is the identity map, and αI is the map

$$\begin{aligned} \alpha I : V &\to V \\ v &\mapsto \alpha \cdot I(v) \end{aligned}$$

The image of F in $\mathrm{End}_F(V)$ is called the **subring of scalar transformations**.[2] Prove or disprove the following statement: Let T be a scalar transformation. If V is a nonzero cyclic $F[t]$-module, then $\dim_F V = 1$. *Hint:* See p 352.

(d) Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let $T \in \mathrm{End}_F(V)$ be rotation by π radians around the origin. Show that every F-subspace of V is an $F[t]$-submodule for T. *Hint:* Classify all F-subspaces of V.

[1] One nice property of a T-stable submodule U is that the module homomorphism $T : V \to V$, restricted to U, again gives a module homomorphism $T|_U : U \to U$. Under favorable conditions, we can decompose $V = U \oplus W$ and understand the operation of T on V by studying the operation of its restrictions to the "smaller" spaces U and W.

[2] In the general setting of R-modules M, the analogous map $R \to \mathrm{End}_R(M)$ may not be injective, as it is when $R = F$ a field.