Math 357
Long quiz 01B

2024–03–22 (F)

Your name: ______________________________

(a) Let R be a ring with a multiplicative identity 1_R. R is **boolean** if, for all $a \in R$, $a^2 = a$. Prove that a boolean ring is commutative.

Solution: Note that the ring axioms imply that $-1_R \in R$, and that it satisfies

$$-1_R = (-1_R)^2 = 1_R \tag{1}$$

where for the first equality we have used the hypothesis that R is boolean.

Case 1: $1_R = 0_R$. In this case, R is the zero ring, which satisfies the axioms of a boolean ring. There is only one element $a \in R$ (namely 0_R), and any element commutes with itself. Hence R is commutative.

Case 2: $1_R \neq 0_R$. Let $a, b \in R$ be arbitrary, and consider the element $a + b \in R$. Then

$$a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$$

where in the first equality, we use the hypothesis that R is boolean; in the second, the left- and right-distributivity axioms of a ring; and in the third, the hypothesis that R is boolean. Because R is a ring, by definition it contains the additive inverses $-a$ and $-b$. Adding these to both sides of the last equation, we get

$$ab + ba = 0 \qquad \Leftrightarrow \qquad ab = -ba$$

Applying the ring axioms and equation (1) to this last equality, we conclude that

$$ab = -ba = -1_R(ba) = 1_R(ba) = ba$$

as desired.

(b) Let R be a ring; and let I_1, I_2 be ideals of R. Recall that

$$I_1 + I_2 = \{a_1 + a_2 \mid a_i \in I_i\} \qquad I_1 I_2 = \left\{ \sum_{j=1}^{n} a_{1,j} a_{2,j} \mid n \in \mathbf{Z}_{>0}; \forall j, a_{i,j} \in I_i \right\}$$

are ideals. (In particular, note that $I_1 I_2$ comprises all finite sums of terms of the form $a_1 a_2$ with $a_i \in I_i$.) Prove that if R is a commutative ring with a multiplicative identity, and if $I_1 + I_2 = R$, then $I_1 \cap I_2 = I_1 I_2$.

Solution: The inclusion $I_1 I_2 \subseteq I_1 \cap I_2$ holds for arbitrary rings, by definition of an ideal (namely, closure under addition and "strong closure" under multiplication by ring elements). To show the reverse inclusion, $I_1 \cap I_2 \subseteq I_1 I_2$, let $a \in I_1 \cap I_2$. By hypothesis, $I_1 + I_2 = R$, and there exists a multiplicative identity element $1_R \in R$; so in particular, there exist $e_1 \in I_1$ and $e_2 \in I_2$ such that $e_1 + e_2 = 1_R$. Then

$$a = a1_R = a(e_1 + e_2) = ae_1 + ae_2 = e_1 a + ae_2$$

where in the last equality we have used the hypothesis that R is commutative. The last expression is a sum of elements, each with the form (element of I_1) times (element of I_2). Thus $a \in I_1 I_2$, as desired.