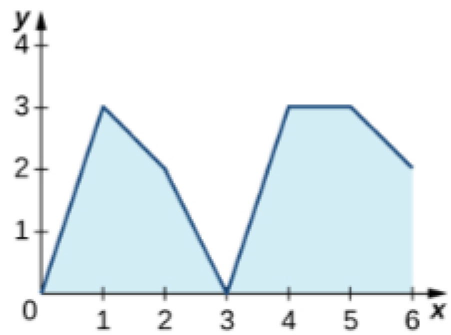
Math 112
LQuiz 15

2022–03–31 (R)

Your name: ____________________________________

Exercise

(4 pt) Let $f : [0, 6] \to \mathbf{R}$ be a piecewise constant function. A graph of $y = \int_0^x f(t)\ dt$ is shown below.



(a) (2 pt) Over which intervals is f positive? negative? equal to zero?

Solution: View y as a function of x, the cumulative signed area under the graph of f from $t = 0$ to $t = x$. If f is positive on some interval, then the signed area under the graph of f is increasing on that interval. If f is negative on some interval, then the signed area under the graph of f is decreasing on that interval. If f is zero on some interval, then the signed area under the graph of f is not changing on that interval. The logical converse holds, too.

The fundamental theorem of calculus neatly captures this discussion, and more:

$$y' = \frac{d}{dx}y = \frac{d}{dx}\int_0^x f(t)\ dt = f(x) \tag{1}$$

This equation says that the *slope* of the cumulative signed area function y at any point x equals the *value* of the function f at x.

Analyzing the given graph, we see that $y' > 0$ on the interval $(0, 1)$, so $f(x) = y' > 0$ on this interval. Similarly, $f(x) = y' < 0$ on the intervals $(1, 2)$, $(2, 3)$, and $(5, 6)$; and $f(x) = y' = 0$ on the interval $(4, 5)$.

Challenge: Why don't we include the endpoints of these intervals? Hint: Graph the cumulative signed area functions for the functions $f_1 : [0, 2] \to \mathbf{R}$ and $f_2 : [0, 2] \to \mathbf{R}$ given by

$$f_1(x) = \begin{cases} 3 & \text{if } x \in [0, 1] \\ -1 & \text{if } x \in (1, 2] \end{cases} \qquad f_2(x) = \begin{cases} 3 & \text{if } x \in [0, 1) \\ -1 & \text{if } x \in [1, 2] \end{cases}$$

What do you observe? Try to explain why this makes sense, in light of our theory.)

(b) (1 pt) What are the maximum and minimum values of f?

Solution: Equation (1), which we obtained by a straightforward application of the fundamental theorem of calculus, tells us that the value of $f(x)$ equals the slope of the cumulative signed area function y at the input x. Thus the maximum (respectively, minimum) value of $f(x)$ equals the maximum (respectively, minimum) slope of y. From the graph, we see that the maximum value of y', and hence of $f(x)$, is 3 (take any point x on the intervals $(0, 1)$ or $(3, 4)$); and the minimum value of y', and hence of $f(x)$, is -2 (take any point x on the interval $(2, 3)$).

(c) (1 pt) What is the average value of f on the interval $[0, 6]$?

Solution: By definition, the average value of f on the interval $[0, 6]$ is

$$\frac{1}{6-0}\int_0^6 f(x)\,dx$$

By definition of y, the value of y at $x = 6$ is (the value of) the definite integral $\int_0^6 f(x)\,dx$. We can read this value off the graph of y, namely, $y(6) = 2$. Therefore the average value of f on the interval $[0, 6]$ is

$$\frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}(2) = \frac{1}{3}$$

Challenge: We can say more! By definition, the average value of f on the interval $[0, x]$ is

$$\frac{1}{x-0}\int_0^x f(t)\,dt$$

This is the same definition we used above, except (i) we have replaced the right endpoint 6 above with the variable x here, because we analyzed the interval $[0, 6]$ above, whereas here we analyze the interval $[0, x]$; and (ii) we use the variable of integration t rather than x here, because we're using x as our upper endpoint on the integral.

I claim that if we compute this average value *function*, we get

$$A(x) = \begin{cases} 3 & \text{if } x \in [0,1] \\ \frac{4}{x} - 1 & \text{if } x \in [1,2] \\ \frac{6}{x} - 2 & \text{if } x \in [2,3] \\ 3 - \frac{9}{x} & \text{if } x \in [3,4] \\ \frac{3}{x} & \text{if } x \in [4,5] \\ \frac{8}{x} - 1 & \text{if } x \in [5,6] \end{cases}$$

which is continuous. Can you validate this? Can you use the graph of y above to check the values of this function, for example, on the integer values of x? How does our answer above, for the average value of f on the interval $[0, 6]$, fit into the framework of this average value function?