Math 357
Expositional homework 02

Assigned: 2024–01–22 (M)
Due:

The goal of this homework is to recall ideas from general ring theory and engage them in the specific setting of polynomial rings. The exercises are adapted from Dummit & Foote, 3e, Exercises 9.2.1–5.

Let F be a field, let t be an indeterminate over F, and let $f \in F[t]$.

(a) Let $\deg f = n \geqslant 1$. For each $g \in F[t]$, let $\overline{g}$ denote the residue of g under the natural projection map $\varphi : F[t] \to F[t]/(f)$. Prove that for each $\overline{g} \in F[t]/(f)$ there exists a unique polynomial $g_0 \in F[t]$ such that $\deg g_0 \leqslant n-1$ and $\overline{g}_0 = \overline{g}$.

(b) Prove that $F[t]/(f)$ is a field if and only if f is irreducible.

(c) Let $f = \prod_i p_i$ be a factorization of f into irreducible elements. Describe all ideals in the ring $F[t]/(f)$, in terms of the p_i.

(d) Prove that $F[t]$ has infinitely many prime elements. *Hint:* Analyze the cases F infinite and F finite separately. See Exercise 9.2.4 (p 301).

(e) Further assume that F is a finite field, of order q. Let $\deg f = n \geqslant 1$. Prove that $F[t]/(f)$ has exactly q^n elements. *Hint:* Explain how to view the result of Exercise (a) in the framework of vector spaces. See Exercise 9.2.1 (p 301).