

Math 357

Expositional homework 01

Assigned: 2024-01-08 (M)

Due: 2024-01-22 (M)

The goal of this homework is to expand and explore our library of examples related to ring structure. In particular, we will explore concrete examples related to the following chain of (one-way) implications:

$$\text{field} \Rightarrow \text{ED} \Rightarrow \text{PID} \Rightarrow \text{UFD} \Rightarrow \text{ID} \tag{1}$$

where ED denotes euclidean domain, PID denotes principal ideal domain, UFD denotes unique factorization domain, and ID denotes integral domain.

Let \mathbf{Z} denote the ring of integers.

- (a) Define each of the algebraic structures in (1).
- (b) Give an example of a ring that is not an integral domain. Justify your assertion.
- (c) Show that $\mathbf{Z}[\sqrt{-5}]$ is an integral domain that is not a unique factorization domain.
- (d) Let t be an indeterminate. Show that $\mathbf{Z}[t]$ is a unique factorization domain that is not a principal ideal domain.
- (e) Show that $\mathbf{Z}[(1 + \sqrt{-19})/2]$ is a principal ideal domain that is not a euclidean domain. *Hint:* See Dummit & Foote, 3e, pages 277 and 281–282. You may also find helpful discussions on pages 227 (Example (5)) and 229–230.
- (f) Give an example of a euclidean domain that is not a field. Justify your assertion.