Math 357
Short quiz 03

2024–01–22 (M)

Your name: ______________________________

(a) Let $(R, +, \times)$ be a commutative (!) ring. Define what it means for a subset $I \subseteq R$ to be an ideal. Define what it means for an ideal to be principal.

Solution: A subset $I \subseteq R$ is an **ideal** of R if the following two conditions are satisfied:

(i) (subring) $(I, +|_I, \times|_I)$ is a subring of $(R, +, \times)$.

(ii) (strongly closed under $\times_R$) For all $r \in R$, for all $a \in I$, $ra \in I$.

An ideal I is **principal** if it is generated by a single element; that is, there exists an $a \in R$ such that $I = (a) = \{\sum_{i=1}^{n} r_i a_i \,|\, n \in \mathbf{Z}_{>0}; \forall i, r_i \in R, a_i \in I\}$. For commutative rings, this is equivalent to saying that there exist an $a \in I$ such that for all $b \in I$, there exists an $r \in R$ such that $b = ra$.[1] That is, for commutative rings, all elements of a principal ideal are R-multiples of a generator.

(b) Are the following ideals principal? Answer both; briefly justify at least one.

$$(2, t) \text{ as an ideal of } \mathbf{Z}[t] \qquad\qquad (2, t) \text{ as an ideal of } \mathbf{Q}[t]$$

Solution: The ideal $(2, t)$ in the ring $\mathbf{Z}[t]$ is not principal. We prove this in our responses to Expositional Homework 01.[2] The ideal $(2, t)$ in the ring $\mathbf{Q}[t]$ is principal. We can see this as follows: $2 \in (2, t)$; therefore by strong closure, $1 = \frac{1}{2} \times 2 \in (2, t)$; therefore again by strong closure, for all $r \in \mathbf{Q}[t]$, $r \times 1 \in (2, t)$. That is, $(2, t) = (1)$, which is all of $\mathbf{Q}[t]$.

[1] Can you prove this? On which ring axioms does this rely?

[2] Alternatively, see Dummit & Foote, 3e, p 252, Example (3).