Math 112
LQuiz 09

2022–02–17 (R)

Your name: ________________________________

Exercise

(4 pt) Compute the following limits. Justify your steps.

(a) (2 pt) $\lim_{x \to 0} \frac{e^{3x^2} - 1}{x^2}$

Solution: Direct evaluation of the limit gives

$$\frac{e^{3(0)^2} - 1}{(0)^2} = \frac{e^0 - 1}{0} = \frac{0}{0}$$

an indeterminate form to which l'Hôpital's rule applies. Using it, we compute

$$\lim_{x \to 0} \frac{e^{3x^2} - 1}{x^2} = \lim_{x \to 0} \frac{6xe^{3x^2} - 0}{2x} = \lim_{x \to 0} \frac{3e^{3x^2} - 0}{1} = 3$$

(b) (2 pt) $\lim_{x \to 0} \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x^2}$

Solution: Direct evaluation of the limit gives

$$\frac{\sqrt{1 + (0)^2} - \sqrt{1 - (0)^2}}{(0)^2} = \frac{0}{0}$$

an indeterminate form to which l'Hôpital's rule applies. L'Hôpital's rule will work. However, it requires differentiating the numerator, which looks messy. Is there another approach?

Let's try multiplying numerator and denominator by the conjugate of the numerator:[1]

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x^2} &= \lim_{x \to 0} \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x^2} \cdot \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}} \\
&= \lim_{x \to 0} \frac{(1 + x^2) - (1 - x^2)}{x^2 \left(\sqrt{1 + x^2} + \sqrt{1 - x^2}\right)} \\
&= \lim_{x \to 0} \frac{2x^2}{x^2 \left(\sqrt{1 + x^2} + \sqrt{1 - x^2}\right)} \\
&= \lim_{x \to 0} \frac{2}{\left(\sqrt{1 + x^2} + \sqrt{1 - x^2}\right)} \\
&= \frac{2}{1 + 1} = 1
\end{aligned}$$

Try applying l'Hôpital's rule here. (It's messy, but not that messy.) You will get the same result. Compare the effort.

[1] Why might we think conjugates will work here? Multiplying an expression by its conjugate squares the two terms and puts a minus sign between them. The squares will cancel the square roots, and the minus sign will cancel the constants, leaving us with (some coefficient times) x^2. This, in turn, will cancel the x^2 in the denominator. That should be nice...