Math 112
LQuiz 11

2022–03–01 (T)

Your name: ______________________________

Exercise

(4 pt) Take as given the following "infinite polynomial" expression for $\cos x$:

$$\begin{aligned} \cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \dots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6) \end{aligned} \tag{1}$$

(Recall that $O(x^6)$ means "terms involving x to powers 6 and higher".) Consider the limit

$$\lim_{x\to 0} \frac{1 - \cos x - \frac{1}{2}x^2}{x^4} \tag{2}$$

(a) (2 pt) Compute the limit in (2) by substituting (1) for $\cos x$, simplifying, and evaluating.

Solution: Substituting (1) for $\cos x$ in the limit in (2), we get

$$\begin{aligned} &\lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)\right) - \frac{1}{2}x^2}{x^4} \\ &= \lim_{x\to 0} \frac{1 - 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 - O(x^6) - \frac{1}{2}x^2}{x^4} \\ &= \lim_{x\to 0} \frac{-\frac{1}{24}x^4 - O(x^6)}{x^4} \\ &= \lim_{x\to 0} \left[-\frac{1}{24} - O(x^2)\right] \\ &= -\frac{1}{24} \end{aligned}$$

In the penultimate (next-to-last) equality we have canceled x^4 from each term in the numerator. In particular, every term in $O(x^6)$ involves x to the power 6 or higher. When we divide these terms by x^4, every term now involves x to the power $6 - 4 = 2$ or higher. Hence we can collect these terms into what we call $O(x^2)$.

For the final equality, every term in $O(x^2)$ involves x to the power 2 or higher, so direct evaluation of the limit (as x approaches 0) makes all these terms 0, leaving only $-\frac{1}{24}$.

(b) (2 pt) Compute the limit in (2) by iteratively applying l'Hôpital's rule. (You should apply l'Hôpital's rule four times. Briefly show it applies each time you use it!) Confirm you get the same result you got in part (a).

Solution: For each equality below, we (i) can check that direct evaluation of the limit on the left gives $\frac{0}{0}$, an indeterminate form to which l'Hôpital's rule applies; and (ii) apply l'Hôpital's rule to get the limit on the right.

$$\lim_{x\to 0} \frac{1 - \cos x - \frac{1}{2}x^2}{x^4} = \lim_{x\to 0} \frac{\sin x - x}{4x^3} = \lim_{x\to 0} \frac{\cos x - 1}{12x^2} = \lim_{x\to 0} \frac{-\sin x}{24x} = \lim_{x\to 0} \frac{-\cos x}{24}$$

Direct evaluation of this last limit gives $-\frac{1}{24}$. This agrees with our answer to part (a).