Math 112
MockExam 01

2022–02–01 (T)

Your name: ______________________________

Instructions

Number of exercises	: 6
Permitted time	: 75 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- Work hard, do your best, and have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/8	/4	/4			
3	/12	/4	/4	/4		
4	/16	/4	/4	/4	/4	
5	/12	/4	/4	/4		
6	/12					
Total	/70					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) The natural logarithm $\ln a$ of a real number a can be negative.

true false

Solution: True. For example, let $a = e^{-1}$. Then $\ln a = \ln(e^{-1}) = -1 < 0$.

(b) (2 pt) The exponential e^a of a real number a can be negative.

true false

Solution: False. The image (aka range) of the exponential function is $(0, +\infty)$. Although the input to the (real) exponential function can be negative, the output is always positive.

(c) (2 pt) Let f be a function. The domain (aka set of inputs) of its first-derivative function f' includes all points in the domain of f.

true false

Solution: False. For example, let $f : [0, +\infty) \to \mathbf{R}$ be given by $f(x) = \sqrt{x} = x^{\frac{1}{2}}$. The rule of assignment for the first-derivative function is $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. The domain of f' is $(0, +\infty)$. In particular, the domain of f' does not include $x = 0$, which is in the domain of f.

For parts (d) and (e), let f be a function defined on an open set containing a point a.

(d) (2 pt) If f is continuous at $x = a$, then $\lim_{x \to a} f(x)$ exists.

true false

Solution: True. By definition, the function f is continuous at $x = a$ if (1) f is defined at $x = a$; (2) $\lim_{x \to a} f(x)$ exists; and (3) these two values are equal, that is, $\lim_{x \to a} f(x) = f(a)$.

(e) (2 pt) If $\lim_{x \to a} f(x)$ exists, then f is continuous at $x = a$.

true false

Solution: False. For example, let $a = 0$, and let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Then $\lim_{x \to 0} f(x)$ exists and equals 0, but f is not continuous at $x = 0$. (Can you justify these three statements?)

Exercise 2

(8 pt) Compute the following. (The answers are integers.)

(a) (4 pt) Let

$$e^{a} = 4\pi \qquad e^{b} = 6\pi \qquad e^{c} = 9\pi$$

Compute

$$e^{-3a-b-c} \cdot \frac{e^{5a-b+4c}}{\left(e^{2}\right)^{b} e^{c}}$$

Solution: We compute

$$\begin{aligned}
e^{-3a-b-c} \cdot \frac{e^{5a-b+4c}}{\left(e^{2}\right)^{b} e^{c}} &= \left(e^{-3a}e^{-b}e^{-c}\right)\left(e^{5a}e^{-b}e^{4c}\right)\left(e^{-2b}e^{-c}\right) \\
&= e^{2a}e^{-4b}e^{2c} \\
&= (e^{a})^{2}\left(e^{b}\right)^{-4}(e^{c})^{2} \\
&= 4^{2}\pi^{2} \cdot 6^{-4}\pi^{-4} \cdot 9^{2}\pi^{2} \\
&= 2^{4} \cdot (2^{-4}3^{-4})3^{4} \\
&= 1
\end{aligned}$$

(b) (4 pt) Let

$$\ln a = \frac{1}{3} \qquad \ln b = 3 \qquad \ln c = \frac{1}{2}$$

Compute

$$\ln\left(\frac{a^{15}b^{21}c^{6}}{a^{12}b^{22}c^{2}}\right) - \ln\left(a^{2}-b^{2}\right) + \ln\left(\frac{a+b}{c}\right) + \ln(ac-bc)$$

Solution: We compute

$$\begin{aligned}
&\ln\left(\frac{a^{15}b^{21}c^{6}}{a^{12}b^{22}c^{2}}\right) - \ln\left(a^{2}-b^{2}\right) + \ln\left(\frac{a+b}{c}\right) + \ln(ac-bc) \\
&\quad = \ln\left(a^{3}b^{-1}c^{4}\right) + \ln\left(\frac{1}{a^{2}-b^{2}} \cdot \frac{a+b}{c} \cdot \frac{(a-b)c}{1}\right) \\
&\quad = 3\ln a - \ln b + 4\ln c + \ln 1 \\
&\quad = 3 \cdot \frac{1}{3} - 3 + 4 \cdot \frac{1}{2} + 0 \\
&\quad = 1 - 3 + 2 \\
&\quad = 0
\end{aligned}$$

Exercise 3

(12 pt) Consider the piecewise function $f : \mathbf{R} \to \mathbf{R}$ whose rule of assignment is

$$f(x) = \begin{cases} x^3 - 6x^2 + 12x - 11 & \text{if } x < 1 \\ x^2 - 5 & \text{if } x \geqslant 1 \end{cases}$$

(a) (4 pt) Find $\lim_{x \to 1} f(x)$. If the limit does not exist, explain. In either case, show your work.

Solution: To compute the limit of f from the left as x approaches 1, we use the "$x < 1$" rule of assignment. We can directly evaluate the resulting limit:

$$\lim_{x \uparrow 1} f(x) = \lim_{x \uparrow 1} \left(x^3 - 6x^2 + 12x - 11\right) = 1^3 - 6(1)^2 + 12(1) - 11 = 1 - 6 + 12 - 11 = -4$$

To compute the limit of f from the right as x approaches 1, we use the "$x > 1$" rule of assignment. (Why do we omit $x = 1$ here?) Again, we can directly evaluate the resulting limit:

$$\lim_{x \downarrow 1} f(x) = \lim_{x \downarrow 1} \left(x^2 - 5\right) = (1)^2 - 5 = -4$$

Because the limits of f from the left and from the right as x approaches 1 are equal, we conclude that the given (two-sided) limit exists, and

$$\lim_{x \to 1} f(x) = -4$$

(b) (4 pt) Is f continuous at $x = 1$? Justify.

Solution: Using the relevant rule of assignment at $x = 1$, we compute

$$f(1) = (1)^2 - 5 = -4$$

The value $f(1)$ equals $\lim_{x \to 1} f(x)$, which we computed in part (b). Thus by definition of continuous, f is continuous at $x = 1$.

(c) (4 pt) Is the first-derivative function f' continuous at $x = 1$? Justify.

Solution: First we compute the rules of assignment for f', by differentiating those given for f:

$$f'(x) = \begin{cases} 3x^2 - 12x + 12 & \text{if } x < 1 \\ 2x & \text{if } x > 1 \end{cases}$$

Note that we don't (yet) include the "breakpoint" $x = 1$ in the domain of f', as we don't (yet) know whether f is differentiable there. We compute

$$\lim_{x \uparrow 1} f'(x) = 3 \neq 2 = \lim_{x \downarrow 1} f'(x)$$

Because the limits of $f'(x)$ from the left and right as x approaches 1 do not agree, it follows that $\lim_{x \to 1} f'(x)$ does not exist. Thus $f'(x)$ cannot be continuous at $x = 1$.

Exercise 4

(16 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = -3x^4 + 4x^3 + 12x^2 - 10$$

(a) (4 pt) Find the interval(s) on which f is increasing and decreasing.

Solution: We compute

$$f'(x) = -12x^3 + 12x^2 + 24x = -12x(x^2 - x - 2) = -12x(x+1)(x-2)$$

It follows that $f'(x) = 0$ if and only if $x = -1, 0, 2$. These are the "critical points" of f. By using the sign and degree of the leading term of the polynomial f, or by determining the sign of $f'(x)$ at values of x between and outside these critical points (for example, at $x = -2, -\frac{1}{2}, 1, 3$), we conclude that f is

- increasing on $(-\infty, -1) \cup (0, 2)$ and
- decreasing on $(-1, 0) \cup (2, +\infty)$.

(b) (4 pt) Find the (x, y)-coördinates of each local minimum and maximum of f. State whether each is a local minimum or maximum of f.

Solution: The critical points of f are the candidate local extrema.[1] Evaluating f at the three critical points of f found in part (a), we find

$$f(-1) = -5 \qquad f(0) = -10 \qquad f(2) = 22$$

Thus the three candidate local extrema for f are

$$(-1, -5) \qquad (0, -10) \qquad (2, 22)$$

The sign and degree of the leading term of the polynomial f are sufficient to deduce that $(-1, -5)$ and $(2, 22)$ are local maxima, and $(0, -10)$ is a local minimum. (Can you justify this?)

We can also arrive at this conclusion using the second-derivative test — that is, by analyzing the concavity of f. We compute the second-derivative function of f to be

$$f'' : \mathbf{R} \to \mathbf{R} \qquad \text{given by} \qquad f''(x) = -36x^2 + 24x + 24$$

Evaluating $f''(x)$ at the three critical points of f, we find

$$f''(-1) = -36 \qquad f''(0) = 24 \qquad f''(2) = -72$$

Recall that a negative second derivative at a point indicates the function is concave down there, whereas a positive second derivative at a point indicates the function is concave up there. Thus, f is concave down at $x = -1$ and $x = 2$, so these are local maxima; and f is concave up at $x = 0$, so this is a local minimum.

[1] If we were analyzing the function on a domain with boundary points, then the boundary points would also be candidate local extrema. For example, if the domain were the closed interval $[-3, 3]$, then the boundary points ± 3 would also be candidate local extrema.

(c) (4 pt) Find the global minimum and maximum of f.

Solution: By looking at the rule of assignment of $f(x)$, we see that as $|x|$ becomes arbitrarily large, $f(x)$ becomes arbitrarily small.[2] Thus f has no global minimum, on its full domain $\mathbf{R}$. As another result of this "end behavior" of f, the global maximum of f must be one of its local maxima, that is, either $(-1, -5)$ or $(2, 22)$. Comparing the output values (that is, y-values) of these two local maxima, we conclude that $(2, 22)$ is the global maximum of f.

(d) (4 pt) Find the x-coördinate of each inflection point of f.

Solution: By definition, an inflection point of $f(x)$ is a value of x at which the concavity of f changes sign. Recall that the concavity of f is captured by the sign of the second-derivative function f''. For the concavity of f to change sign at x, we must have $f''(x) = 0$. Values of x that satisfy $f''(x) = 0$ are the candidate inflection points.

In part (b) we found that

$$f''(x) = -36x^2 + 24x + 24$$

Setting $f''(x)$ equal to 0 and solving for x (for example, using the quadratic formula), we find

$$x = \frac{2 \pm \sqrt{4 + 24}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

Denote these two solutions $a_\pm$, that is,

$$a_- = \frac{1 - \sqrt{7}}{3} \approx -0.5486 \qquad\qquad a_+ = \frac{1 + \sqrt{7}}{3} \approx 1.2153$$

These are the *candidate* inflection points. We must certify that the concavity of f indeed changes sign at these values. This follows by analyzing the second-derivative function $f''(x)$: It is negative on $(-\infty, a_-) \cup (a_+, +\infty)$, and positive on (a_-, a_+). In particular, $f''(x)$ changes sign at $x = a_-$ and $x = a_+$.

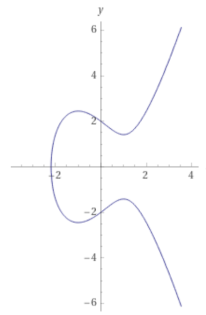
[2] As $|x|$ becomes large, the largest-degree term of $f(x)$, which is $-3x^4$, dominates the behavior of $f(x)$. That is, as $|x|$ becomes large, the behavior of $f(x)$ resembles that of $-3x^4$.

Exercise 5

(12 pt) The graph of the equation

$$y^2 = x^3 - 3x + 4 \tag{1}$$

shown below, is an example of an **elliptic curve**.[3]



(a) (4 pt) Compute the rule of assignment for y'.

Solution: Differentiating the given equation with respect to x, we get[4]

$$2yy' = 3x^2 - 3 \qquad \Leftrightarrow \qquad y' = \frac{3x^2 - 3}{2y}$$

(b) (4 pt) The graph suggests that the point $(0, 2)$ is on the elliptic curve, and that the slope of the tangent line there is negative. Show, algebraically, that these statements are true.

Solution: Geometrically, a point P is on a graph if and only if, algebraically, the coördinates of P satisfy the equation defining the graph. Evaluating Equation (1) at $(x, y) = (0, 2)$, we get

$$2^2 = 0^3 - 3(0) + 4 \qquad \Leftrightarrow \qquad 4 = 4$$

This statement is true, so the point $(0, 2)$ satisfies Equation (1). Thus this point is on the graph.

The slope of the tangent line to the graph at the point $(x, y) = (0, 2)$ is given by evaluating the first-derivative function at this point. Using our rule of assignment for y' in part (a), we compute

$$y'(0, 2) = \frac{3(0)^2 - 3}{2(2)} = -\frac{3}{4}$$

This value is negative, as suggested by the graph.

(c) (4 pt) Find the linearization to the elliptic curve at the point $(0, 2)$.

Solution: The linearization L to the curve at the point (x_0, y_0) is the function

$$L : \mathbf{R} \to \mathbf{R} \qquad \text{given by} \qquad L(x) - y_0 = y'(x_0, y_0) \cdot (x - x_0)$$

Solving this rule of assignment for $L(x)$, and substituting in the values $(x_0, y_0) = (0, 2)$ and $y'(0, 2) = -\frac{3}{4}$, we get

$$L(x) = 2 - \frac{3}{4}(x - 0) = -\frac{3}{4}x + 2$$

[3]Elliptic curves have important applications in digital security (cryptography).

[4]We use implicit differentiation and the chain rule to differentiate the left side of Equation (1).

Exercise 6

(12 pt) The base of a triangle is shrinking at a rate of 1 cm/s, and the height of the triangle is increasing at the rate of 5 cm/s. Find the rate at which the area of the triangle changes when the base is 10 cm and the height is 22 cm.

Solution: Let b denote the length of the base of the triangle, let h denote the length of the height of the triangle, and let A denote the area of the triangle. Recall from planar geometry that

$$A = \frac{1}{2}bh \tag{2}$$

Note that the statement of the exercise implies that b and h, and hence A, are implicitly functions of time, t. If we wish, we may rewrite Equation (2) as

$$A(t) = \frac{1}{2}b(t)h(t) \tag{3}$$

Moreover, the exercise tells us that

$$b'(t) = \frac{db}{dt} = -1 \text{ cm/s} \qquad\qquad h'(t) = \frac{dh}{dt} = 5 \text{ cm/s}$$

To find how the area of the triangle changes with time, we implicitly differentiate Equation (3) (or, equivalently, Equation (2)) with respect to t:

$$A'(t) = \frac{1}{2}b'(t)h(t) + \frac{1}{2}b(t)h'(t) \tag{4}$$

This relation holds for all relevant times t.

We are asked to find $A'(t)$ when the base is 10 and the height is 22. At this point, we might pause, because we don't know the value of t when the base and height have these values. Indeed, the exercise gives us no way to determine this! However, let's ask ourselves: Do we need to know this value of t, explicitly? The answer is no. Whatever the magic value of t is, we know all the values on the right side of Equation (4):

$$b(t) = 10 \text{ cm} \qquad h(t) = 22 \text{ cm} \qquad b'(t) = -1 \text{ cm/s} \qquad h'(t) = 5 \text{ cm/s}$$

Substituting these values into Equation (4), we find

$$A'(t) = \frac{1}{2}(-1 \text{ cm/s})(22 \text{ cm}) + \frac{1}{2}(10 \text{ cm})(5 \text{ cm/s}) = 14 \text{ cm}^2\text{/s}$$