Math 112
MockExam 03

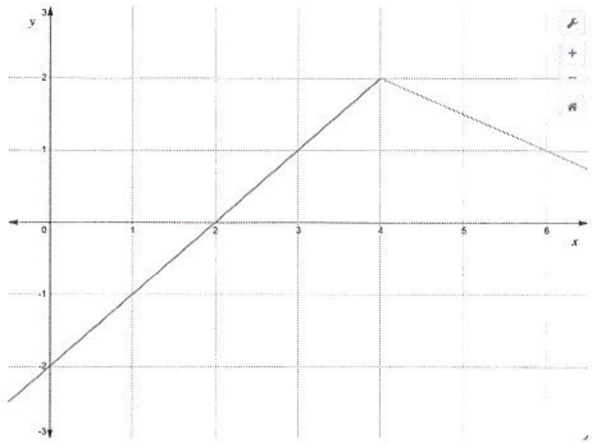
2022–04–09 (S)

Your name: Grader's Solutions

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

Parts (a)–(b) concern the piecewise-linear function $f : \mathbf{R} \to \mathbf{R}$ graphed below.



(a) (2 pt) The definite integral $\int_0^3 f(x)\, dx$ is positive.

true   false

$\int_0^3 f(x) = \int_0^2 f(x) + \int_2^3 f(x)$
$= -2 + \frac{1}{2} = -1.5$

(b) (2 pt) The definite integral $\int_0^6 f(x)\, dx$ is positive.

$\int_0^6 f(x) = \int_0^4 f(x) + \int_4^6 f(x)$

true   false $= 0 +$ (some positive #)

(c) (2 pt) The integration procedure of change of variables (aka substitution) is related to the product rule from differential calculus, by viewing the latter in terms of antiderivatives.

true   false

For parts (d)–(e), let $f : [0,1] \to \mathbf{R}$ be a function such that for all $x \in [0,1]$, $f(x) > 0$; and let $F : [0,1] \to \mathbf{R}$ be the cumulative signed area function given by $F(x) = \int_0^x f(t)\, dt$.

(d) (2 pt) There exists no $x \in [0,1]$ such that $F(x)$ is negative.

true   false

(e) (2 pt) The average value of $f(x)$ on $[0,1]$ is less than $f(1)$.

let $f(x) = 1$ for all $x \in [0,1]$

true   false

Exercise 2

(12 pt) Evaluate each indefinite integral. Clearly communicate your approach.

(a) (4 pt) $\int 4x^3 - 4x^2 + 6x + 3 \, dx$

$= \boxed{x^4 - \frac{4}{3}x^3 + 3x^2 + 3x + C}$

(b) (4 pt) $\int e^{(x^2-2x)^2}(x^2 - 2x)(x - 1) \, dx$

$u = x^2 - 2x$

$du = 2x - 2\, dx$

$du = 2(x-1)\, dx$

$\frac{du}{2(x-1)} = dx$

$= \frac{1}{2}\int e^{u^2} u \, du$

$= \frac{1}{2}\left(\frac{1}{2}e^{u^2}\right) + C$

$= \boxed{\frac{1}{4}e^{(x^2-2x)^2} + C}$

(c) (4 pt) $\int z^2 e^{2z} \, dz$

$u = z^2 \quad du = 2z\, dz$

$dv = e^{2z} dz \quad v = \frac{1}{2}e^{2z}$

$= \frac{1}{2}z^2 e^{2z} - \int e^{2z} z \, dz$

Exercise 3

(16 pt) Evaluate each definite integral. Clearly communicate your approach.

(a) (4 pt) $\int_0^\pi \sin\theta - \cos(2\theta)\, d\theta$

$= \int_0^\pi \sin\theta d\theta - \int_0^\pi \cos(2\theta)\, d\theta$

$= -\cos\theta - \frac{1}{2}\sin(2\theta)\Big]_0^\pi = -\cos(\pi) - \frac{1}{2}\sin(2\pi) + \cos(0) + \frac{1}{2}\sin(0)$

$= 1 + 0 + 1 + 0$

$= \boxed{2}$

(b) (4 pt) $\int_0^2 (1-2t)^2 - 1\, dt$

$u = 1-2t$ $\quad u = 1-2(0) = 1$

$du = -2dt$ $\quad v = 1-2(2) = -3$

$-\frac{1}{2}du = dt$

$= -\frac{1}{2}\int_{t=0}^{t=2} v^2 - 1\, dv$

$= -\frac{1}{2}\int_{v=1}^{v=-3} v^2 - 1\, dv$

$= -\frac{1}{2}\left(\frac{1}{3}v^3 - v\right)\Big]_1^{-3} = -\frac{1}{2}(-9+3) + \frac{1}{2}\left(\frac{1}{3} - 1\right) = \boxed{\frac{8}{3}}$

(c) (4 pt) $\int_{-3}^3 \sqrt{9-x^2}\, dx$

Let $y = \sqrt{9-x^2} \Rightarrow x^2 + y^2 = 9$

$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2 = \boxed{\frac{9\pi}{2}}$

(d) (4 pt) $\int_{-1}^1 16x^3(1+x^4)^3\, dx$

$u = 1 + x^4$

$du = 4x^3\, dx$

$\frac{1}{4x^3}du = dx$

$= \int_{x=-1}^{x=1} 4u^3\, du$

$= \int_{u=2}^{u=2} 4u^3\, du = \boxed{0}$

Exercise 4

(8 pt) Use the fundamental theorem of calculus to compute the following derivatives.

(a) (4 pt) $\frac{d}{dx}\int_{-2}^{x}\cos\left(e^{\sin\sqrt{t}}\right)\,dt$

$= \cos(e^{\sin\sqrt{x}})$

(b) (4 pt) $\frac{d}{dx}\int_{\sqrt{x}}^{x^2}\frac{t^2}{1+t^2}\,dt$

$f(t) = \frac{t^2}{1+t^2}$

$\frac{d}{dx}\int_{\sqrt{x}}^{x^2} f(t)\,dt = \frac{d}{dx}\left[F(x^2) - F(\sqrt{x})\right]$

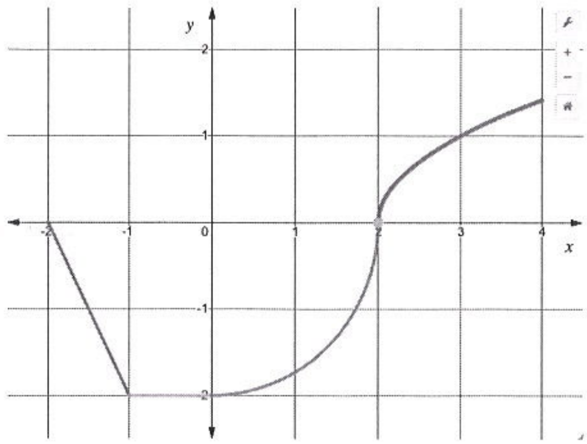
$= 2xF'(x^2) - \frac{1}{2\sqrt{x}}F'(\sqrt{x})$

$= 2xf(x^2) - \frac{1}{2\sqrt{x}}f(\sqrt{x})$

$\boxed{= (2x)\frac{x^4}{1+x^4} - \left(\frac{1}{2\sqrt{x}}\right)\frac{x}{1+x}}$

Exercise 5

(18 pt) Let $f : [-2, 4] \to \mathbf{R}$ be a piecewise function. A graph of $F(x) = \int_{-2}^{x} f(t)\, dt$ is shown below.



(a) (4 pt) Over which intervals is f positive? negative? equal to zero?

$F'(x) = f(x)$

f positive on $(0,4)$

f equal to zero on $(-1,0)$

f negative on $(-2,-1)$

(b) (4 pt) Over which intervals is f increasing? decreasing? constant?

concave up   concave down   neither

$F''(x) = f'(x)$

f increasing on $(2,4)$

f decreasing on $(0,2)$

f constant on $(-2,0)$

(c) (2 pt) What are the maximum and minimum values of f?

maximum at $x = 2$

minimum at $x = -2$

(d) (2 pt) What is the average value of f on the interval $[-2, 4]$? **If it helps, you may assume that $F(4) = \sqrt{2}$.**

$$\frac{1}{4-(-2)} \int_{-2}^{4} f(t)\,dt$$

$$= \frac{1}{6} F(4) + \cancel{\frac{1}{6} F(-2)}$$

$$= \boxed{\frac{\sqrt{2}}{6}}$$

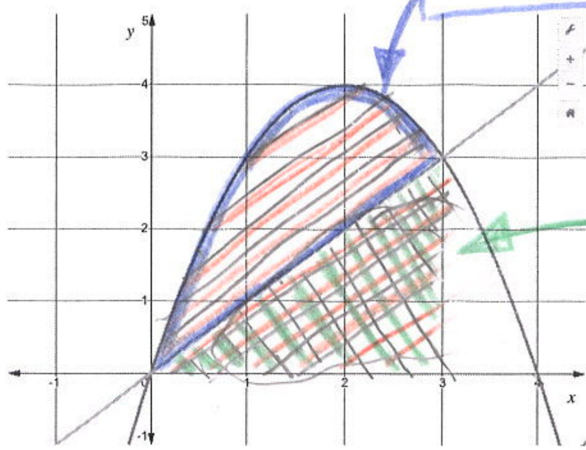
Exercise 6

(12 pt) Consider the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = -x^2 + 4x \qquad\qquad g(x) = x$$

respectively. A graph of f and g appears below.

area that remains



area being subtracted

This exercise explores the definite integral

$$\int_0^3 f(x) - g(x)\ dx$$

(a) (4 pt) Evaluate $\int_0^3 f(x)\ dx$. That is, find the area between the graph of f and the x-axis, from $x = 0$ to $x = 3$.

$$\int_0^3 -x^2 + 4x\, dx$$

$$= -\frac{1}{3}x^3 + 2x^2 \Big]_0^3 = -9 + 18 = \boxed{9}$$

(b) (4 pt) Evaluate $\int_0^3 g(x)\ dx$. That is, find the area between the graph of g and the x-axis, from $x = 0$ to $x = 3$.

$$\int_0^3 x\, dx = \frac{1}{2}x^2 \Big]_0^3 = \boxed{\frac{9}{2}}$$

(c) (4 pt) Recall that linearity of the integral implies that

$$\int_0^3 f(x) - g(x)\, dx = \int_0^3 f(x)\, dx - \int_0^3 g(x)\, dx$$

Use this to help explain, geometrically, why the area between the graphs of $f(x)$ and $g(x)$ equals $\int_0^3 f(x) - g(x)\, dx$.

check graph