Math 357
Long quiz 04B

2024–02–26 (M)

Your name: ______________________________

Let $\mathbf{Q}$ denote the field of rational numbers; given a prime $p \in \mathbf{Z}_{>0}$, let $\mathbf{F}_p \cong \mathbf{Z}/(p)$ denote the finite field with p elements; and let t be an indeterminate. For each of the quotient rings below, characterize its algebraic structure as "field", "integral domain but not field", or "ring but not integral domain". Justify your characterization.

$$R_1 = \mathbf{F}_3[t]/(t^4 + t^3 + t^2 + 1)$$
$$R_2 = \mathbf{Q}[t]/(3t^3 - 6t^2 + 7t + 8)$$
$$R_3 = \mathbf{Q}[t]/(t^4 - 4t^3 + 6t^2 - t + 28)$$

Hint: If you feel inclined to do a lot of computation, then I invite you to first check with me.

Solution: R_1 : Field. Let $f_1 = t^4 + t^3 + t^2 + 1$. Direct computation shows that f_1 has no zeros in $\mathbf{F}_3$. It remains to check for factors of degree 2. Without loss of generality, we may restrict our attention to monic irreducible factors of degree 2 (why?). A polynomial of degree 2 is reducible if and only if it has a linear factor, so by taking all possible products of the three linear polynomials in $\mathbf{F}_3[t]$, we may enumerate the reducible monic polynomials in $\mathbf{F}_3[t]$. These are

$$t^2 + t + 1, t^2 - t, t^2 - 1, t^2, t^2 + t, t^2 - t + 1$$

Removing these from the list of the nine monic polynomials of degree 2 in $\mathbf{F}_3[t]$, we are left with

$$t^2 - t - 1, t^2 + 1, t^2 + t - 1$$

For each of these three polynomials g, we perform polynomial division in $\mathbf{F}_3[t]$ on f_1 by g. In each case, we obtain a nonzero remainder, so we conclude that f_1 is irreducible. Hence $R_1 = \mathbf{F}_3[t]/(f_1)$ is a field.

R_2 : Field. Let $f_2 = 3t^3 - 6t^2 + 7t + 8 \in \mathbf{Q}[t]$. Analyze $f_3 \in \mathbf{R}[t]$; $f_2(-1) = -1 < 0$ and $f_2(0) = 8 > 0$, so f_2 has a zero in $\mathbf{R}$ in the interval $[-1, 0]$. Analyze the first derivative: For all $t \in \mathbf{R}$, $f_2'(t) > 0$, thus $[-1, 0]$ is the only interval in which a rational zero can possibly occur. Check the rational zeros in this interval consistent with the divisibility conditions implied by the coefficients of f_2: $-\frac{2}{3}, -\frac{1}{3}$. Neither is a zero of f_2, so f_2 is irreducible. Hence $\mathbf{Q}[t]/(f_2)$ is a field.

R_3 : Field. Let $f_3 = t^4 - 4t^3 + 6t^2 - t + 28 \in \mathbf{Q}[t]$. Apply the Eistenstein–Schönemann criterion with $p = 3$ to $f_3(t + 1) = t^4 + 3t + 30$ to conclude that f_3 is irreducible. Hence $\mathbf{Q}[t]/(f_3)$ is a field.