Math 112
MockExam 02

2022–03–03 (R)

Your name: ______________________________

Instructions

Number of exercises	: 5
Permitted time	: 75 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- Work hard, do your best, and have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/20	/4	/4	/4	/4	/4
3	/16	/4	/4	/4	/4	
4	/16	/4	/4	/4	/4	
5	/18	/4	/4	/2	/4	/4
Total	/80					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) If direct evaluation of a limit gives an indeterminate form, then we can always apply l'Hôpital's rule, even though other methods may be faster.

true   false

(b) (2 pt) Let $f(x)$ be a function, and let $F(x)$ be an antiderivative of $f(x)$. Then $(F(x))^2$ is an antiderivative of $(f(x))^2$.

true   false

(c) (2 pt) Let $f(x)$ be a function, and let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then the function $F(x) - G(x)$ is always a constant function.

true   false

For parts (d)–(e), let f and g be functions such that

$$\int_{-1}^{3} f(x)\, dx = -2 \qquad \int_{-1}^{3} g(x)\, dx = 4$$

(d) (2 pt) $\displaystyle\int_{-1}^{3} [f(x) + g(x)]\, dx = 2$

true   false

(e) (2 pt) $\displaystyle\int_{-1}^{0} f(x)\, dx + \int_{0}^{3} f(x)\, dx = -2$

true   false

Exercise 2

(20 pt) Evaluate each of the following limits. Briefly but clearly justify your work.

(a) (4 pt) $\lim_{x \to 0} \frac{x + \cos x}{-1 + \sin x}$

(b) (4 pt) $\lim_{x \to -\infty} \frac{6x^3 - x^2 + 5x + 5}{2x^3 + 2x}$

(c) (4 pt) Use the Taylor series

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots$$

to evaluate

$$\lim_{x \to 0} \frac{x^5}{\sin(x) - x + \frac{1}{6}x^3}$$

(d) (4 pt) Use l'Hôpital's rule to evaluate

$$\lim_{x\to 0} \frac{x^5}{\sin(x) - x + \frac{1}{6}x^3}$$

(Note that this is the same limit as in part (c).)

(e) (4 pt) $\lim_{x\downarrow 0} (1+x)^{\frac{1}{x}}$ (Recall that $x \downarrow 0$ means the same as $x \to 0^+$.)

Exercise 3

(16 pt) Evaluate the indefinite integrals. (That is, find the most-general antiderivative $F(x)$ of the integrand $f(x)$ in the following integrals $\int f(x)\, dx$.)

(a) (4 pt) $\displaystyle\int 4x^3 - 2x + 1\, dx$

(b) (4 pt) $\displaystyle\int e^{2x} - e^{-x}\, dx$

(c) (4 pt) $\displaystyle\int \frac{x^2 - 1}{\sqrt{x}}\, dx$

(d) (4 pt) $\displaystyle\int (x^2 - 1)(4x + 3)\, dx$

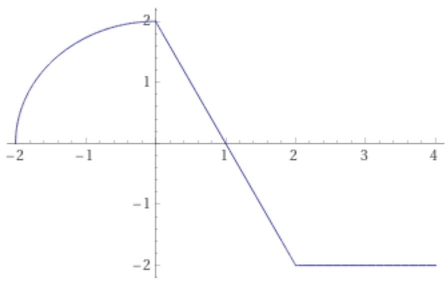
Exercise 4

(16 pt) Consider the piecewise function $f : \mathbf{R} \to \mathbf{R}$ given by

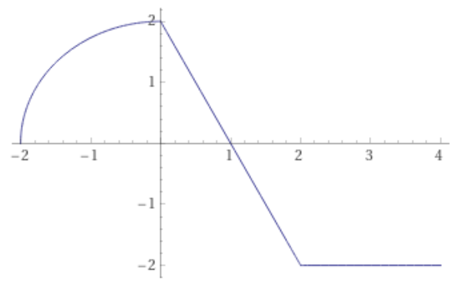
$$f(x) = \begin{cases} 0 & \text{if } x \leqslant -2 \\ \sqrt{4-x^2} & \text{if } -2 \leqslant x \leqslant 0 \\ 2-2x & \text{if } 0 \leqslant x \leqslant 2 \\ -2 & \text{if } x \geqslant 2 \end{cases}$$

Graphs of f are included in parts (a) and (b).

(a) (4 pt) Draw and compute a lower- and upper-sum estimate (call them L_3 and U_3, respectively) for $\int_{-2}^{4} f(x)\,dx$ by partitioning $[-2, 4]$ into three subintervals, each of width 2.

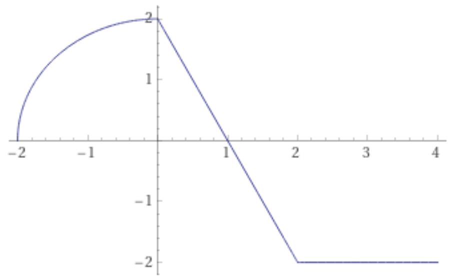


Lower sum (L_3)

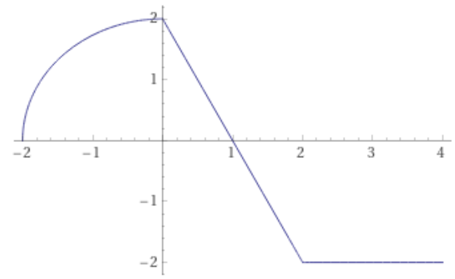


Upper sum (U_3)

(b) (4 pt) Draw and compute a lower- and upper-sum estimate (call them L_6 and U_6, respectively) for $\int_{-2}^{4} f(x)\,dx$ by partitioning $[-2, 4]$ into six subintervals, each of width 1. (Note: $f(-1) = \sqrt{3} \approx 1.73$.)



Lower sum (L_6)



Upper sum (U_6)

(c) (4 pt) Use geometry to compute the exact value of $\int_{-2}^{4} f(x)\,dx$.

(d) (4 pt) Order all your results, from parts (a)–(c), in increasing order. Make a conjecture about where lower- and upper-sum estimates L_{12} and U_{12}, with twelve subintervals, each of width $\frac{1}{2}$, would go in your order.

Exercise 5

(18 pt) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

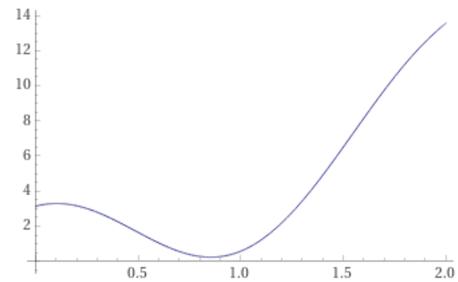
$$f(x) = e^x + \pi \cos(\pi x) + 2x - 1$$

(a) (4 pt) Find an antiderivative $F(x)$ of $f(x)$. Verify that it is indeed an antiderivative.

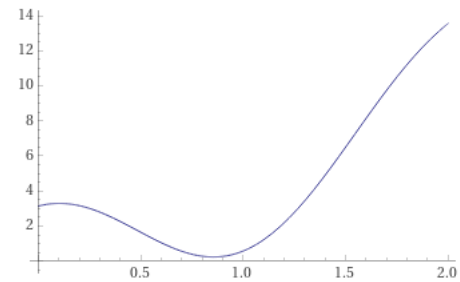
(b) (4 pt) Using your antiderivative $F(x)$ from part (a), show that $\int_0^2 f(x)\, dx = e^2 + 1$ (approximately 8.3890).

(c) (2 pt) Find the average value of $f(x)$ on the interval $[0, 2]$. (You have already done almost all the work!)

(d) (4 pt) On the graphs of f below, draw a lower- and upper-sum approximation to the definite integral $\int_0^2 f(x)\,dx$. Partition the interval $[0, 2]$ into four subintervals, each of width $\frac{1}{2}$.



Lower sum



Upper sum

(e) (4 pt) Using the values $f(x)$ below, compute the upper- and lower-sum approximations you sketched in part (d). Show that these approximations bound your value of the definite integral in part (b).

- $f(0.0) \approx 3.14$
- $f(0.1) \approx 3.29$ (a local maximum)
- $f(0.5) \approx 1.65$
- $f(0.9) \approx 0.24$ (a local minimum)
- $f(1.0) \approx 0.58$
- $f(1.5) \approx 6.48$
- $f(2.0) \approx 13.53$