Math 212
Quiz 32

F 18 Nov 2016

Your name: ______________________________

Exercise

(5 pt) In this quiz we prove that the curl of any gradient is zero. More precisely, let $f : \mathbf{R}^3 \to \mathbf{R}$ be $\mathscr{C}^2$ (i.e. twice continuously differentiable). Then

$$\operatorname{curl}(\nabla f) = \mathbf{0}. \tag{1}$$

(a) (1 pt) What kind of object is the $\mathbf{0}$ in (1)?

Solution: It is a vector field. More precisely, it is the vector field defined on the same domain as f, such that for all points (x, y, z) in this domain, it outputs the zero vector $\mathbf{0} \in \mathbf{R}^3$.

(b) (3 pt) Prove (1), justifying your steps. *Hint:* Definitions, compute, Clairaut.

Solution: We compute

$$\begin{aligned}
\operatorname{curl}(\nabla f) &= \nabla \times (\nabla f) \\
&= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y\, \partial z} - \frac{\partial^2 f}{\partial z\, \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z\, \partial x} - \frac{\partial^2 f}{\partial x\, \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x\, \partial y} - \frac{\partial^2 f}{\partial y\, \partial x} \right) \mathbf{k} \\
&= 0\, \mathbf{i} + 0\, \mathbf{j} + 0\, \mathbf{k} = \mathbf{0},
\end{aligned}$$

where the equalities are justified as follows:

1. Equivalent representation of curl
2. Definition of curl, gradient, cross product
3. Compute the determinant
4. Clairaut–Schwarz theorem (equality of mixed partials)

(c) (1 pt) Application: Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ be a vector field, and suppose that $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$. An unenlightened colleague from Math 212 (in another section, of course) asks you to find a potential function for $\mathbf{F}$, i.e. some $f : \mathbf{R}^3 \to \mathbf{R}$ such that $\nabla f = \mathbf{F}$. State why you refuse, with logical justification. *Hint:* Use (1).

Solution: We refuse because a potential function cannot exist for the given vector field $\mathbf{F}$. Suppose for the sake of contradiction that $\mathbf{F}$ had a potential function, i.e. $\mathbf{F}$ is conservative. Then

$$\mathbf{0} \neq \operatorname{curl}(\mathbf{F}) = \operatorname{curl}(\nabla f) = \mathbf{0},$$

where (i) the first inequality is by hypothesis; (ii) the middle equality assumes that $\mathbf{F}$ is conservative, so $\mathbf{F} = \nabla f$ for some f; and (iii) the final equality uses (1)). Thus $\mathbf{0} \neq \mathbf{0}$, a contradiction. We conclude that $\mathbf{F}$ cannot have a potential function.