Math 212
Quiz 14

F 23 Sep 2016

Your name: ______________________________

Exercise

(5 pt) Consider the unit sphere S^2 in $\mathbf{R}^3$ centered at the origin. Find an equation for the tangent plane to S^2 at the point $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$. *Hint:* Recall that an equation for the unit sphere is

$$x^2 + y^2 + z^2 = 1.$$

Solve for z as a function of x, y, minding signs (!), then take partial derivatives. As an alternative to all this, think geometrically.

Solution 1 (algebraic): Solving for z as a function of x, y, we obtain

$$z = \pm\sqrt{1 - x^2 - y^2}. \tag{1}$$

We are interested in the tangent plane at the point $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$, for which the z-coordinate is negative. Thus we take the negative square root in (1). Computing the first-order partial derivatives of z (note that we can use symmetry of z in x and y to make just one computation), we find

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{1 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{1 - x^2 - y^2}}.$$

Evaluating these partial derivatives at the point $(x, y) = (\frac{1}{2}, \frac{1}{2})$ (again we can use symmetry in x and y to make just one computation), we have

$$\frac{\partial z}{\partial x}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\sqrt{2}}{2}, \qquad \frac{\partial z}{\partial y}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\sqrt{2}}{2}.$$

We conclude that an equation for the tangent plane to S^2 at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$ is

$$\begin{aligned} z - z_0 &= \left(\frac{\partial z}{\partial x}\left(\frac{1}{2}, \frac{1}{2}\right)\right)(x - x_0) + \left(\frac{\partial z}{\partial y}\left(\frac{1}{2}, \frac{1}{2}\right)\right)(y - y_0) \\ z + \frac{1}{\sqrt{2}} &= \frac{\sqrt{2}}{2}\left(x - \frac{1}{2}\right) + \frac{\sqrt{2}}{2}\left(y - \frac{1}{2}\right). \end{aligned} \tag{2}$$

Solution 2 (geometric): Close your eyes and see the unit sphere (in fact, any sphere) in $\mathbf{R}^3$ centered at the origin, and also see a tangent plane to this sphere, at any point $P_0 = (x_0, y_0, z_0)$ on the sphere. Note that any normal vector to this plane (positioned so that its initial point is at P_0) lies in the line containing the radius from the origin O to P_0, i.e. is parallel to the vector

$$\vec{OP_0} = (x_0, y_0, z_0).$$

In particular, we can take the given point P_0 as the normal vector to the tangent plane. Doing so, we obtain the following equation for the tangent plane:

$$0 = (x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0).$$

In particular, an equation for the tangent plane to the unit sphere at the point $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$ is

$$0 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right) \cdot \left(x - \frac{1}{2}, y - \frac{1}{2}, z + \frac{1}{\sqrt{2}}\right) = \frac{1}{2}\left(x - \frac{1}{2}\right) + \frac{1}{2}\left(y - \frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(z + \frac{1}{\sqrt{2}}\right).$$

Note that isolating the $z + \frac{1}{\sqrt{2}}$ term yields precisely equation (2), found algebraically.