Math 211
Quiz 23

W 07 Aug 2019

Your name : ________________________________

Exercise

(5 pt) Solve the following nonhomogeneous 1st-order linear initial value problem, using the laplace transform:

$$y' + y = e^{-t}, \qquad y(0) = -1. \tag{1}$$

Hint: Recall that, from the definition of the lapace transform,

$$\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\} - y(0).$$

The following transform–inverse-transform pairs may be useful:

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \qquad s > a; \qquad\qquad \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}, \qquad s > a.$$

Solution: Applying the laplace transform to both sides of the ODE in (1), using the fact that the laplace transform is linear, we get

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{y' + y\} = \mathcal{L}\{e^{-t}\}. \tag{2}$$

Denote

$$Y(s) = \mathcal{L}\{y\}.$$

From the definition of the laplace transform, and using the given initial condition, we compute

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = sY(s) - (-1) = sY(s) + 1.$$

Using the transform–inverse-transform dictionary, we see that

$$\mathcal{L}\{e^{-t}\} = \mathcal{L}\{e^{(-1)t}\} = \frac{1}{s-(-1)} = \frac{1}{s+1}.$$

Substituting these results into (2), we get

$$(sY(s) + 1) + Y(s) = \frac{1}{s+1}.$$

Solving this equation for $Y(s)$, we find

$$Y(s) = \frac{\frac{1}{s+1} - 1}{s+1} = \frac{1}{(s+1)^2} - \frac{1}{s+1}. \tag{3}$$

We can't do partial fraction decomposition on fractions of this form. Nor do we need to. These transforms appear directly in our transform–inverse-transform dictionary: The first is the laplace transform of $t^n e^{at}$, with $n = 1$ and $a = -1$; the second is the laplace transform of e^{at} with $a = -1$. Applying the inverse laplace transform $\mathcal{L}^{-1}$ to both sides of (3), and using the fact that $\mathcal{L}^{-1}$ is linear, we find

$$\begin{aligned} y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} - \frac{1}{s+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = te^{-t} - e^{-t}. \end{aligned}$$

One can check that this function $y(t)$ indeed solves the IVP (1), by plugging it in and verifying that the equation is true. One can also check that we obtain the same solution $y(t)$ if we solve (1) by other means, e.g., using an integrating factor, variation of parameters, or linearity combined with guessing a particular solution (in this case, one would have to guess $y_p(t) = te^{-t}$).