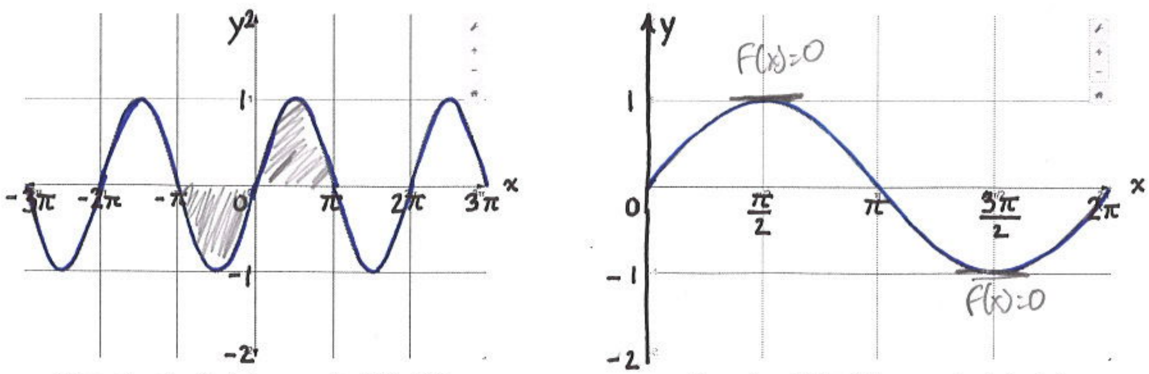
Math 112
Exam 03

2022–04–14 (R)

Your name: Grader's Solutions



Graph of $g(x)$ for parts (a)–(b).

Graph of $F(x)$ for parts (c)–(e).

Figure 1: Graphs for Exercise 1.

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

For parts (a)–(b), let $g : \mathbf{R} \to \mathbf{R}$ be the function given by $g(x) = \sin x$, graphed in Figure 1.

(a) (2 pt) $\int_{-\pi}^{\pi} g(x)\, dx = 0$ Check graph $\int \sin(x)$

true false

(b) (2 pt) For every positive real number a, $\int_{-a}^{a} g(x)\, dx = 0$.

$\int_{-a}^{a} \sin(x)\,dx$

$-\cos(x)\Big]_{-a}^{a}$

true false

$-\cos(a) + \cos(-a) =$

$-\cos(a)$

$+\cos(a)$

$= \boxed{0}$

For parts (c)–(e), let $f : [0, 2\pi] \to \mathbf{R}$ be a continuous function, and let $F : [0, 2\pi] \to \mathbf{R}$ be the cumulative signed area function graphed in Figure 1, given by

$$F(x) = \int_0^x f(t)\, dt$$

$F'(x) = f(x)$

Slope is negative for $x \in (\frac{\pi}{2}, \pi)$

(c) (2 pt) For all x in the range $0 \leqslant x \leqslant \pi$, $f(x) > 0$.

true false

(d) (2 pt) On the interval $[0, 2\pi]$, $f(x) = 0$ at exactly two points.

true false Check graph

(e) (2 pt) The average value of f on the interval $[0, 2\pi]$ equals 0.

$\frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt$

true false

$0 = \cancel{\frac{1}{2\pi}F(2\pi)} - \cancel{\frac{1}{2\pi}F(0)}$

Exercise 2

(12 pt) Evaluate each indefinite integral. Clearly communicate your approach.

(a) (4 pt) $\int e^{2x} + 3x^2 - 4x \, dx$

$$= \boxed{\frac{1}{2}e^{2x} + x^3 - 2x^2 + C}$$

(b) (4 pt) $\int t^2(\sin(t^3))^2 \cos(t^3)\, dt$

$u = \sin(t^3)$

$du = 3t^2\cos(t^3)\,dt$

$\frac{1}{3}du = t^2\cos(t^3)\,dt$

$$= \int t^2\cos(t^3)(\sin(t^3))^2\,dt$$

$$= \frac{1}{3}\int u^2\,du$$

$$= \frac{1}{9}u^3 + C$$

$$= \boxed{\frac{1}{9}(\sin(t^3))^3 + C}$$

(c) (4 pt) $\int x^2 \cos x \, dx$

$u = x^2$ $\quad du = 2x\,dx$

$dv = \cos x\,dx$ $\quad v = \sin x$

$$= x^2\sin x - \int 2x\sin x\,dx$$

$u = 2x$ $\quad du = 2dx$

$dv = \sin x\,dx$ $\quad v = -\cos x\,dx$

$$= x^2\sin x + 2x\cos x - 2\int \cos x\,dx$$

$$= \boxed{x^2\sin x + 2x\cos x - 2\sin x + C}$$

Exercise 3

$U = 2(0) - 1 = -1$

$U = 2(1) - 1 = 1$

(12 pt) Evaluate each definite integral. Clearly communicate your approach.

(a) (4 pt) $\int_0^1 (2x-1)^3\,dx$

$U = 2x - 1$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$= \frac{1}{2}\int_{x=0}^{x=1} U^3\,du$

$= \frac{1}{2}\int_{U=-1}^{U=1} U^3\,du$

$= \frac{1}{8}U^4\Big]_{-1}^{1} = \frac{1}{8} - \frac{1}{8} = \boxed{0}$

(b) (4 pt) $\int_0^4 \sqrt{4x - x^2}\,dx$

Hint: Set the integrand equal to y. Massage this equation into the form $(x-a)^2+(y-b)^2 = r^2$, an equation of a circle with center (a, b) and radius r. Use geometry to deduce the value of the integral.

$y = \sqrt{4x - x^2} \Rightarrow x^2 - 4x + y^2 = 0$

$\Rightarrow (x-2)^2 + y^2 = 2^2$

$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = \boxed{2\pi}$

Circle w/ radius = 2
centered at (2, 0)
positive square root =
upper area of circle

(c) (4 pt) $\int_0^2 (x^2-1)(x^3-3x)^3\,dx$

$U = (0)^3 - 3(0) = 0$

$U = (2)^3 - 3(2) = 2$

$U = x^3 - 3x$

$du = 3x^2 - 3\,dx$

$du = 3(x^2-1)\,dx$

$\frac{1}{3}du = x^2 - 1\,dx$

$= \frac{1}{3}\int_{x=0}^{x=2} U^3\,du$

$= \frac{1}{3}\int_{U=0}^{U=2} U^3\,du$

$= \frac{1}{12}U^4\Big]_0^2 = \frac{16}{12} = \boxed{\frac{4}{3}}$

Exercise 4

(8 pt) Use the fundamental theorem of calculus to compute each derivative. Assume $x \geqslant 0$.

(a) (4 pt) $\frac{d}{dx}\int_0^x e^{-t^2}\,dt$

$= \boxed{e^{-x^2}}$

(b) (4 pt) $\frac{d}{dx}\int_{x^2}^{4x^2} \sin(\sqrt{t})\,dt$

$= \frac{d}{dx}\left[F(4x^2) - F(x^2)\right]$

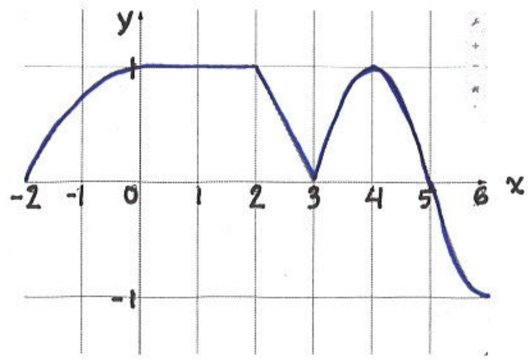
$= 8x f(4x^2) - 2x f(x^2)$

$= 8x \sin(\sqrt{4x^2}) - 2x \sin(\sqrt{x^2})$

$= \boxed{8x \sin(2x) - 2x \sin(x)}$

Exercise 5

(10 pt) Let $f : [-2, 6] \to \mathbf{R}$ be a piecewise function. A graph of $F(x) = \int_{-2}^{x} f(t)\, dt$ is shown below.



(a) (4 pt) On which intervals is f positive? negative? equal to zero?

$F'(x) = f(x)$

f positive on $(-2,0)$ and $(3,4)$

f negative on $(2,3)$ and $(4,6)$

f equal to zero on $(0,2)$

(b) (4 pt) On which intervals is f increasing? decreasing? constant?

$F''(x) = f'(x)$ Must check concavity of $F(x)$

f increasing on $(5,6)$

f decreasing on $(-2,0)$ and $(3,5)$

f constant on $(0,2)$ and $(2,3)$

(c) (2 pt) What is the average value of f on the interval $[-2, 6]$?

$$\frac{1}{6+2}\int_{-2}^{6} f(t)\, dt = \frac{1}{8}\left[F(6) - \cancel{F(-2)}\right]$$

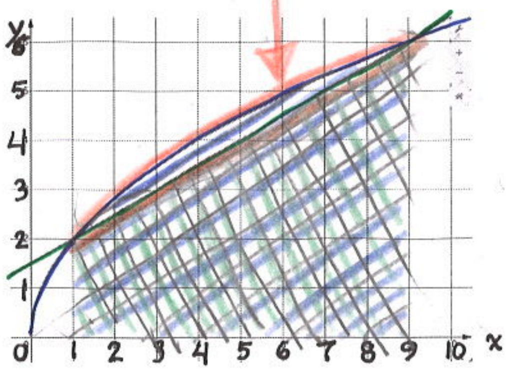
$$= \boxed{-\frac{1}{8}}$$

Exercise 6

(8 pt) Consider the functions $f : [0, +\infty) \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = 2\sqrt{x} \qquad\qquad g(x) = 2 + \frac{1}{2}(x-1)$$

respectively. Graphs of f and g appear below. Area being Calculated



(a) (4 pt) Using the graph, write the two points (x, y) of intersection of f and g. Using the equations for f and g, show that, for each point (x, y) of intersection, $f(x) = y$ and $g(x) = y$. That is, the intersection points (x, y) are on the graphs of both f and g.

intersection points at $(1,2)$ and $(9,6)$

$f(1) = 2\sqrt{1} = 2$ $\qquad$ $f(9) = 2\sqrt{9} = 6$

$g(1) = 2 + \frac{1}{2}(1-1) = 2$ $\qquad$ $g(9) = 2 + \frac{1}{2}(9-1) = 6$

(b) (4 pt) Recall that linearity of the integral implies that

$$\int_a^b f(x) - g(x)\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

Use this to help explain, geometrically, why the area between the graphs of $f(x)$ and $g(x)$ equals $\int_1^9 f(x) - g(x)\, dx$.

Check graph