Math 112
LQuiz 03

2022–01–18 (T)

Your name: ______________________________

Exercise

Consider the piecewise function $f : \mathbf{R} \to \mathbf{R}$ whose rule of assignment is

$$f(x) = \begin{cases} -x^2 & \text{if } x \leqslant 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

(a) (1 pt) Find $\lim_{x \uparrow 0} f(x)$ (i.e. the limit from the left). Justify briefly.

Solution: To compute the limit of f from the left at $x = 0$, we consider input values x that are less than (i.e. "to the left of") the value of x that the limit is approaching — here, $x_0 = 0$ — and we see whether the corresponding output values $f(x)$ get "closer and closer" to some fixed value L as these input values x get "closer and closer" to x_0. When $x < 0$, our function has the rule of assignment $f(x) = -x^2$. As we take input values x "closer and closer" to 0 (and always less than 0), the corresponding output values $f(x)$ get "closer and closer" to 0. Thus we conclude that

$$\lim_{x \uparrow 0} f(x) = 0$$

(b) (1 pt) Is f continuous from the left at $x = 0$? Justify briefly.

Solution: Recall that continuity of a function f at a point x_0 requires three things:

1. $f(x_0)$ is defined.
2. $\lim_{x \to x_0} f(x)$ exists.
3. These two values are equal, i.e. $\lim_{x \to x_0} f(x) = f(x_0)$.

If we are interested in one-sided continuity, as we are here, then we replace the two-sided limits in the above definition with the appropriate one-sided limit.

Our function f is defined at $x_0 = 0$: Using the rule of assignment for $x \leqslant 0$, we find $f(0) = 0$. This equals the limit of f from the left at $x = 0$, which we computed in part (a). Thus we conclude that yes, f is continuous from the left at $x = 0$.

(c) (1 pt) Find $\lim_{x \downarrow 0} f(x)$ (i.e. the limit from the right). Justify briefly.

Solution: Using reasoning analogous to that in our solution to part (a), we find

$$\lim_{x \downarrow 0} f(x) = 0$$

(d) (1 pt) Is f continuous from the right at $x = 0$? Justify briefly.

Solution: Using the same reasoning as we did in our solution to part (b), we find that yes, f is continuous from the right at $x = 0$.

N.B. "But," we might object, "the rule of assignment for f is split at the value $x = 0$, and the "right-side" rule of assignment (i.e. for $x > 0$) doesn't include the input value $x = 0$." More precisely, the rule of assignment splits the domain $\mathbf{R}$ (all real numbers) into two intervals, $(-\infty, 0]$ and $(0, +\infty)$, and the right interval doesn't include $x = 0$. This is true, but it does not necessarily mean f is not continuous from the right at $x = 0$. The definition of continuity tells us to think in two "separate" parts: (2) What is happening to $f(x)$ when x "gets close" to x_0 (possibly from one side or the other, as we have here)? and (1) What happens to $f(x)$ when x equals x_0? If our answers to these two questions are the same value, then f is continuous at x_0 (one-sided or two-sided continuity, depending on how we defined "gets close" in question (2)).

For this exercise (and much of single-variable calculus), sketching the graph of f helps guide our analysis. Try it! You should get a graph that looks like (but is not (!) the same as) the graph of the function $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = x^3$.