# Math 211
# Quiz 16

M 29 Jul 2019

Your name : ______________________________

## Exercise

(5 pt) Consider the homogeneous 1st-order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} -7 & -18 \\ 3 & 8 \end{bmatrix}.$$

We have seen that the general solution to (1) is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c}$, where $\mathbf{c}$ is a $2 \times 1$ matrix of parameters.

Computing $e^{\mathbf{A}t}$ for a general square matrix $\mathbf{A}$ of constants can be hard. However, we have seen that for a diagonal matrix $\mathbf{D}$, the matrix exponential function is straightforward:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \Rightarrow \qquad e^{\mathbf{D}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}. \tag{2}$$

(a) (2 pt) Show the eigenvalues of $\mathbf{A}$ are $-1$ and $2$. For each eigenvalue, compute an eigenvector.

**Solution:** We can find the eigenvalues by solving

$$0 \underset{\text{set}}{=} \det(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda + 7 & 18 \\ -3 & \lambda - 8 \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \qquad \Leftrightarrow \qquad \lambda \in \{-1, 2\}.$$

We can find corresponding eigenvectors by finding nonzero vectors solving

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \qquad \Leftrightarrow \qquad (\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}.$$

For $\lambda = -1$,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} = \mathbf{A}\mathbf{v} = \begin{bmatrix} 6 & 18 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{v} = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

for any nonzero $c_1 \in \mathbf{R}$. For $\lambda = 2$,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} = \mathbf{A}\mathbf{v} \begin{bmatrix} 9 & 18 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{v} = c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

for any nonzero $c_2 \in \mathbf{R}$. Choosing $c_1, c_2 = 1$ gives us the eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

associated to the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, respectively.

N.B. We can check our work by computing, for $i = 1, 2$, $\mathbf{A}\mathbf{v}_i$ and confirming that it equals $\lambda_i \mathbf{v}_i$.

(b) (1 pt) Write your two eigenvectors as columns, side by side in a $2 \times 2$ matrix $\mathbf{P}$. Compute $\mathbf{P}^{-1}$, and confirm that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, where $\mathbf{D}$ is a $2 \times 2$ diagonal matrix with the eigenvalues of $\mathbf{A}$ on the diagonal. Thus $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

**Solution:** Writing our two eigenvectors side by side gives

$$\mathbf{P} = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}.$$

We can compute $\mathbf{P}^{-1}$ by applying the row reduction algorithm to the augmented matrix $\left[\begin{array}{c:c} \mathbf{P} & \mathbf{I} \end{array}\right]$:

$$\left[\begin{array}{c:c} \mathbf{P} & \mathbf{I} \end{array}\right] = \left[\begin{array}{cc:cc} 3 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \to \cdots \to \left[\begin{array}{cc:cc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \end{array}\right].$$

The left square in this RREF matrix is $\mathbf{I}$, so the right square is $\mathbf{P}^{-1}$:[1]

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}.$$

Note that we can check our work by computing $\mathbf{P}\mathbf{P}^{-1}$ (or $\mathbf{P}^{-1}\mathbf{P}$) and confirming that it equals the $2 \times 2$ identity matrix $\mathbf{I}$:

$$\mathbf{P}\mathbf{P}^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Got it.

As requested, we compute $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -7 & -18 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We indeed obtain a diagonal matrix, call it $\mathbf{D}$, with the eigenvalues of $A$ listed on the diagonal in the same order that we listed their corresponding eigenvectors in the matrix $\mathbf{P}$. Taking the equation

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

and multiplying both sides on the left by $\mathbf{P}$ and on the right by $\mathbf{P}^{-1}$, we get

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

(c) (2 pt) Use your result to part (b) to compute the $2 \times 2$ matrix $e^{\mathbf{A}t}$, by noting that

$$e^{\mathbf{A}t} = e^{t\mathbf{A}} = e^{t\mathbf{P}\mathbf{D}\mathbf{P}^{-1}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1}.$$

(Compute using this last expression.) The columns of this matrix form a basis for the solution space of our original ODE (1).

[1] If the left square in the RREF matrix contains a zero row, then we conclude $\mathbf{P}^{-1}$ does not exist, i.e. $\mathbf{P}$ is not invertible.

**Solution:** Using the definition of the matrix exponential (in terms of an infinite series), we have

$$e^{\mathbf{A}t} = e^{t\mathbf{A}} = e^{t(\mathbf{PDP}^{-1})} = \sum_{j=0}^{\infty} \frac{1}{j!} t^j \left(\mathbf{PDP}^{-1}\right)^j. \tag{3}$$

Using the fact that $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, we note that

$$\begin{aligned}
\left(\mathbf{PDP}^{-1}\right)^2 &= \mathbf{PDP}^{-1}\mathbf{PDP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1} \\
\left(\mathbf{PDP}^{-1}\right)^3 &= \mathbf{PDP}^{-1}\mathbf{PDP}^{-1}\mathbf{PDP}^{-1} = \mathbf{PD}^3\mathbf{P}^{-1} \\
&\vdots \\
\left(\mathbf{PDP}^{-1}\right)^j &= \mathbf{PD}^j\mathbf{P}^{-1}.
\end{aligned}$$

Substituting these results into (3), then factoring out from each term of the sum $\mathbf{P}$ on the left and $\mathbf{P}^{-1}$ on the right, we get

$$e^{\mathbf{A}t} = \mathbf{P}\left(\sum_{j=0}^{\infty} \frac{1}{j!} t^j \mathbf{D}^j\right)\mathbf{P}^{-1} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}.$$

Using our matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$, and the form (2) of the matrix exponential for a diagonal matrix, we get

$$e^{\mathbf{A}t} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 2e^{2t} & 6e^{-t} - 6e^{2t} \\ -e^{-t} + e^{2t} & -2e^{-t} + 3e^{2t} \end{bmatrix}.$$

In class we asserted that the columns of this matrix $e^{\mathbf{A}t}$ form a basis for the vector space of solutions to the homogeneous 1st-order linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we decompose these columns into separate pieces for each eigenvalue,[2] we get

$$\begin{bmatrix} 3e^{-t} - 2e^{2t} \\ -e^{-t} + e^{2t} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t}, \qquad \begin{bmatrix} 6e^{-t} - 6e^{2t} \\ -2e^{-t} + 3e^{2t} \end{bmatrix} = 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t}.$$

Analyzing this decomposition, we observe that the columns of $e^{\mathbf{A}t}$ — again, which form a basis for the vector space of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ — are linear combinations of expressions of the form

$$(\text{eigenvector}_j)e^{\text{eigenvalue}_j t}. \tag{4}$$

Here, the coefficients of this linear combination are given by the entries in the columns of $\mathbf{P}^{-1}$: $1, -1$ for column 1, and $2, -3$ for column 2.

This suggests that the basis coming from the columns of $e^{\mathbf{A}t}$ is itself built from basic building blocks of the form (4). That is, we can write another basis for the vector space of solutions, given by vectors of the form (4). If we suitably generalize the notion of eigenvector (to ensure that a basis of $k^n$, formed from $n$ generalized eigenvectors of a matrix, always exists), this is always true.

[2]More precisely, and more generally, we'll be interested in decomposing into separate pieces for each generalized eigenvector.