Math 211
Exam 03

W 31 Jul 2019

Your name : ______________________________

Start time : ____________________ End time : ____________________

Honor pledge :

Exam instructions

Number of exercises	: 6
Permitted time	: 90 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- You are well-trained. Do your best, work hard, have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/12	/2	/5	/5	X	X
3	/18	/4	/10	/2	/2	X
4	/21	/7	/7	/7	X	X
5	/25	/3	/6	/6	/6	/4
6	/14	/6	/4	/4	X	X
Total	/100					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary (though you may find it beneficial to check your intuition).

(a) (2 pt) Let V be a finite-dimensional vector space. The greatest number of linearly independent vectors in V always equals the minimum number of vectors needed to span V. *Hint:* If you're not sure, play with a few toy examples, e.g., $V = \mathbf{R}^2, \mathbf{R}^3$, etc. What do you find?

true false

(b) (2 pt) Let A be a square matrix, one of whose columns is a nontrivial linear combination of the others (i.e. not all coefficients in this linear combination are 0). Then 0 is an eigenvalue of A. *Hint:* Can we use the nontrivial linear combination to find an eigenvector for 0?

true false

(c) (2 pt) We can translate any nth-order linear ODE into an $n \times n$ 1st-order linear system.

true false

(d) (2 pt) Let $y_1(t)$ and $y_2(t)$ be solutions to the nonhomogeneous 3rd-order linear ODE $ty^{(3)} - e^t y' + y = \frac{t}{1+t}$. Then $y_2 - y_1$ is a solution to the corresponding homogeneous equation $ty^{(3)} - e^t y' + y = 0$. *Hint:* How do we check if something is a solution to an ODE?

true false

(e) (2 pt) Every ODE can be solved, i.e. we can always find a closed-form solution (e.g., an explicit equation for $y(t)$).

true false

Exercise 2

(12 pt) Consider the homogeneous 1st-order nonlinear ODE

$$e^{-t}y' - y^{\frac{1}{5}} = 0. \tag{1}$$

(a) (2 pt) Show that the ODE (1) has exactly one equilibrium solution. What is it? *Hint:* Recall that, by definition, an equilibrium solution is a solution $y(t)$ that does not depend on t.

(b) (5 pt) Find all solutions to the initial value problem given by the ODE (1) and the initial condition $y(0) = 0$. *Hint:* You should find at least one nonequilibrium solution.

(c) (5 pt) Does your result to part (b) contradict your result to part (a)? How do these results relate to the existence and uniqueness statements of Picard's theorem? *Hint:* Picard's theorem, as we learned it, applies to 1st-order ODEs in the form $y' = f(t, y)$. Put (1) in this form.

Exercise 3

(18 pt) Let T be the linear map[1]

$$T : \mathbf{R}^4 \to \mathbf{R}^3$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_2 - x_3 + x_4 \\ 2x_1 + 4x_2 - 2x_3 + 2x_4 \\ 3x_1 + 5x_2 - 4x_3 + 2x_4 \end{bmatrix}.$$

(a) (4 pt) Write the coefficient matrix A for T, i.e. the matrix such that $Ax = T(x)$, where x is the 4×1 column matrix with entries x_1, x_2, x_3, x_4. *Hint:* T must output a vector in $\mathbf{R}^3$, so Ax must be a 3×1 matrix. x is 4×1. What do these imply about the order (dimensions) of A?

(b) (10 pt) Find a basis for the image $\mathrm{im}(T)$. Find a basis for the kernel $\ker(T)$.

[1] Whenever we write a linear map between two vector spaces as a matrix, we are implicitly choosing a basis for each vector space. We can ignore this choice of bases for this problem. However, it's good to keep this general fact in mind.

(c) (2 pt) Find the dimensions $\dim(\mathrm{im}(\mathsf{T}))$ and $\dim(\ker(\mathsf{T}))$.

(d) (2 pt) Confirm the rank–nullity theorem:

$$\dim(\mathrm{domain}(\mathsf{T})) = \dim(\mathrm{im}(\mathsf{T})) + \dim(\ker(\mathsf{T})).$$

Exercise 4

(21 pt) For each of the following three homogeneous 1st-order 2×2 linear systems,

1. write the general solution, and
2. circle the number of its phase plane (shown on the next page, in the (x_1, x_2) plane).

Hint: For the phase planes, recall that the sign of (the real part of) each eigenvalue relates to whether solutions move toward or away from the equilibrium at the origin.

(a) (7 pt) Phase plane : (1) (2) (3)

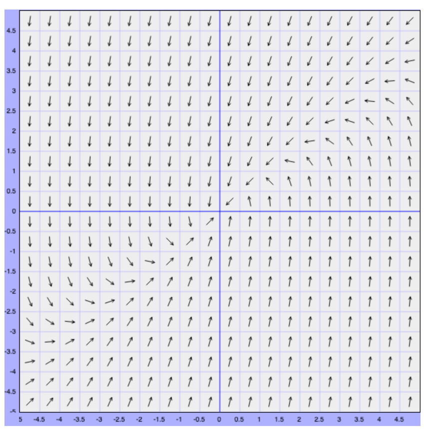
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) (7 pt) Phase plane : (1) (2) (3)

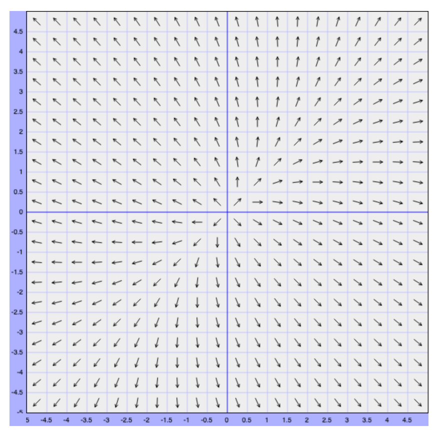
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(c) (7 pt) Phase plane : (1) (2) (3)

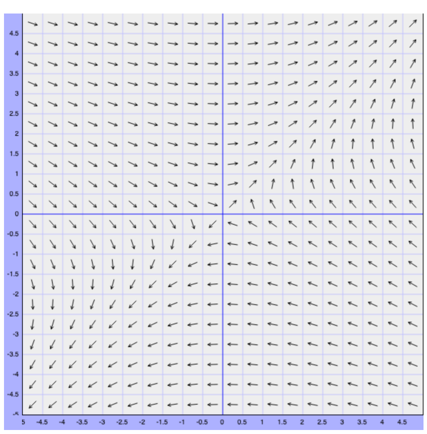
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -2 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



(1)



(2)



(3)

Figure 1: Phase planes for Exercise 4. Arrows indicate direction.

Exercise 5

(25 pt) Consider the homogeneous 2nd-order linear ODE

$$2y'' + 4y' + 2y = 0. \tag{2}$$

(a) (3 pt) Translate (2) into a homogeneous 1st-order 2×2 linear system. *Hint:* Use the change of variables $x_i = y^{(i)}$, as we've learned. Be careful — mind the coefficient on y'' in (2).

(b) (6 pt) Show that this ODE (either (2) or its equivalent 2×2 system you found in part (a)) has a repeated eigenvalue $\lambda = -1$, and that the associated eigenspace has dimension 1 (i.e. we can find at most one linearly independent eigenvector with eigenvalue -1). Write an eigenvector, call it v_1. *Hint:* You do not have to use part (a) to compute the eigenvalues; if preferred, you can do this straight from (2).

We say that -1 is an eigenvalue with **algebraic multiplicity** 2 and **geometric multiplicity** 1.

(c) (6 pt) Using the 2×2 coefficient matrix A from part (a), our eigenvalue $\lambda = -1$, and our eigenvector v_1 from part (b), find a vector $v_2 \in \mathbf{R}^2$ that solves

$$(A - \lambda I)v_2 = v_1. \tag{3}$$

Hint: View the two entries of the 2×1 vector v_2 as unknown variables, and view (3) as a system of equations. Solve this system.

The vector v_2 is called a **generalized eigenvector** of A associated to the eigenvalue -1.

(d) (6 pt) Using our eigenvalue $\lambda = -1$ and our vectors v_1, v_2, show that the 2×1 matrix functions

$$X_1(t) = e^{\lambda t} v_1 \qquad \text{and} \qquad X_2(t) = e^{\lambda t}(t v_1 + v_2)$$

are solutions to our linear system in part (a).

(e) (4 pt) Try to translate the general solution of our linear system in part (a), i.e. the linear combination

$$a_1 X_1(t) + a_2 X_2(t),$$

into the general solution $y(t)$ of our original 2nd-order ODE (2). *Hint:* Note that rows 1 and 2 of each $X_i(t)$ are x_0 and x_1, respectively. Consider our original change of variables. Note that we can check our proposed solution $y(t)$ by plugging it into the original ODE (2).

Exercise 6

(14 pt) Consider the nonhomogeneous 2nd-order linear ODE

$$y'' - 4y = 8\sin(2t) - 4. \tag{4}$$

(a) (6 pt) Write the corresponding homogeneous equation, and find the general solution $y_h(t)$.

(b) (4 pt) Show that

$$y_p(t) = -\sin(2t) + 1$$

is a particular solution to (4).

(c) (4 pt) Briefly justify why the nonhomogeneous principle applies to our ODE (4). Then use it, and our above results, to write the the general solution to (4).