Math 211
Quiz 15A

T 30 Jul 2019

Your name : ________________________________

Exercise

(5 pt) Solve the initial value problem given by the homogeneous 3nd-order ODE

$$y''' + y'' - 6y' = 0$$

and the initial conditions

$$y(0) = 6, \qquad y'(0) = 3, \qquad y''(0) = 21.$$

Your final answer should be an explicit equation for $y(t)$.

Solution: The corresponding characteristic polynomial[1] is (up to sign)

$$p(\lambda) = \lambda^3 + \lambda^2 - 6\lambda = \lambda(\lambda+3)(\lambda-2),$$

which has roots (the eigenvalues)

$$\lambda \in \{-3, 0, 2\}.$$

Thus the general solution $y(t)$ to the ODE is

$$y(t) = c_1 e^{-3t} + c_2 + c_3 e^{2t}.$$

To solve the IVP, need to apply the given initial conditions. First we compute the derivatives:

$$\begin{aligned} y'(t) &= -3c_1 e^{-3t} + 2c_3 e^{2t} \\ y''(t) &= 9c_1 e^{-3t} + 4c_3 e^{2t}. \end{aligned}$$

Then we apply the initial conditions:

$$\begin{array}{ccccccccc} 6 & = & y(0) & = & c_1 & + & c_2 & + & c_3 \\ 3 & = & y'(0) & = & -3c_1 & + & & & 2c_3 \\ 21 & = & y''(0) & = & 9c_1 & + & & & 4c_3. \end{array}$$

This is a system of equations in the unknowns c_1, c_2, c_3. Writing the corresponding augmented matrix and applying the row reduction algorithm, we get

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ -3 & 0 & 2 & 3 \\ 9 & 0 & 4 & 21 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right],$$

from which we can read off the solution

$$c_1 = 1, \qquad c_2 = 2, \qquad c_3 = 3.$$

We conclude that the solution to the IVP is

$$y(t) = e^{-3t} + 2 + 3e^{2t}.$$

Note that we can check our solution by (i) plugging it into the original ODE and confirming that the equation holds, and (ii) evaluating $y(t)$ and its derivatives at 0 and confirming that we get the values given by the initial conditions.

[1]We can obtain the characteristic polynomial by (i) using the change of variables $x_i = y^{(i)}$ to rewrite the given higher-order linear ODE as a 1st-order linear system, and computing the eigenvalues of the coefficient matrix A via $\det(\lambda I - A) = 0$; or (ii) replacing $y^{(i)}$ with λ^i — this is precisely the polynomial we get using approach (i) — and computing the roots.