Math 211
Quiz 01

M 08 Jul 2019

Your name: ________________________________

Exam instructions

Number of exercises	: 12
Permitted time	: 30 minutes
Permitted resources	: None

Instructor's note:

- BEFORE SOLVING any exercises, go through the entire quiz and write a "confidence number" 1–5 to the LEFT of each exercise, denoting how confident you are that you can solve the exercise (1 = "Not at all confident", 5 = "Very confident").

1.1	/2	2.1	/2	3.1	/2
1.2	/2	2.2	/2	3.2	/2
1.3	/2	2.3	/2	3.3	/2
1.4	/2	2.4	/2	3.4	/2
Total	/8		/8		/8

Single-Variable Calculus

1.1 Exercise 1.1

(2 pt) Let $a, s \in \mathbf{R}$ such that $s > a$. Evaluate the definite integral $\int_0^{+\infty} e^{-st} e^{at}\, dt$.

Solution: We compute

$$\begin{aligned}
\int_0^{+\infty} e^{-st} e^{at}\, dt &= \lim_{b \to +\infty} \int_0^b e^{-(s-a)t}\, dt \\
&= \lim_{b \to +\infty} \left[-\frac{1}{s-a} e^{-(s-a)t} \right]_{t=0}^{t=b} \\
&= -\frac{1}{s-a} \left[\lim_{b \to +\infty} e^{-(s-a)b} - 1 \right].
\end{aligned}$$

By hypothesis, $s - a > 0$, so

$$\lim_{b \to +\infty} e^{-(s-a)b} = 0,$$

and we are left with

$$\int_0^{+\infty} e^{-st} e^{at}\, dt = -\frac{1}{s-a} [0 - 1] = \frac{1}{s-a}.$$

1.2 Exercise 1.2

(2 pt) Let $p(t)$ be a continuous function of t, let $v(t)$ be a differentiable function of t, and let

$$y(t) = v(t) e^{-\int p(t)\, dt}.$$

Compute $y'(t)$.

Solution: We use the product rule, the chain rule, and the fundamental theorem of calculus:

$$y'(t) = v'(t) e^{-\int p(t)\, dt} + v(t) e^{-\int p(t)\, dt} (-p(t)) = e^{-\int p(t)\, dt} \left(v'(t) - p(t) v(t) \right).$$

1.3 Exercise 1.3

(2 pt) Evaluate the integral $\int \frac{x^2+3x+2}{(x-3)(x^2+1)}\,dx$.

Solution: We use partial fraction decomposition. First we find the decomposition:

$$\frac{A}{x-3}+\frac{Bx+C}{x^2+1}\underset{\text{set}}{=}\frac{x^2+3x+2}{(x-3)(x^2+1)}.$$

Clearing denominators and gathering like terms, we have

$$x^2+3x+2=A(x^2+1)+(Bx+C)(x-3)=(A+B)x^2+(-3B+C)x+(A-3C).$$

These polynomials are equal if and only if their coefficients for each x^n are equal. This yields a system of three equations in three unknowns:

$$A+B=1, \qquad -3B+C=3, \qquad A-3C=2.$$

This system has the unique solution

$$A=2, \qquad B=-1, \qquad C=0.$$

Thus

$$\begin{aligned}\int \frac{x^2+5}{(x-3)(x^2+1)}\,dx &= \int\left(\frac{2}{x-3}-\frac{x}{x^2+1}\right)dx\\ &= 2\int\frac{1}{x-3}\,dx-\frac{1}{2}\int\frac{2x}{x^2+1}\,dx\\ &= 2\ln|x-3|-\frac{1}{2}\ln\left|x^2+1\right|+c,\end{aligned}$$

where $c \in \mathbf{R}$.

1.4 Exercise 1.4

(2 pt) Evaluate the integral $3\int x^2\ln x\,dx$.

Solution: We use integration by parts (which is just the product rule, integrated): Let

$$u=\ln x \qquad \text{and} \qquad dv=x^2\,dx.$$

Then

$$du=\frac{1}{x}\,dx \qquad \text{and} \qquad v=\frac{1}{3}x^3,$$

so

$$\begin{aligned}3\int x^2\ln x\,dx &= 3\left(\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^3\frac{1}{x}\,dx\right)\\ &= x^3\ln x-\int x^2\,dx\\ &= x^3\ln x-\frac{1}{3}x^3+c,\end{aligned}$$

where $c \in \mathbf{R}$.

Algebra

2.1 Exercise 2.1

(2 pt) Let i satisfy $i^2 = -1$. Write the following in the form $A + iB$, for $A, B \in \mathbf{R}$:

$$(a + ib)e^{\alpha + i\beta}.$$

Solution: Recall **Euler's formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Using this, we compute

$$\begin{aligned}
(a+ib)e^{\alpha+i\beta} &= (a+ib)e^{\alpha}e^{i\beta} \\
&= e^{\alpha}(a+ib)(\cos\beta + i\sin\beta) \\
&= e^{\alpha}\left(a\cos\beta + i^2 b\sin\beta + ia\sin\beta + ib\cos\beta\right) \\
&= e^{\alpha}(a\cos\beta - b\sin\beta) + ie^{\alpha}(a\sin\beta + b\cos\beta).
\end{aligned}$$

2.2 Exercise 2.2

(2 pt) Let $f(x) = mx$, where $m \neq 0$. Let x_h be a solution to $f(x) = 0$, and let x_p be a solution to $f(x) = 1$. Let $a \in \mathbf{R}$. Show that $ax_h + x_p$ is also a solution to $f(x) = 1$.

Solution: We substitute $ax_h + x_p$ into $f(x) = mx$ and see what happens:

$$f(ax_h + x_p) = m(ax_h + x_p) = max_h + mx_p = af(x_h) + f(x_p) = a \cdot 0 + 1 = 1.$$

Thus $ax_h + x_p$ is a solution to $f(x) = 1$, as desired.

2.3 Exercise 2.3

(2 pt) Write the following system of equations as a matrix equation:

$$x_1 + x_2 - 2x_3 = b_1, \qquad -x_1 + 2x_2 + x_3 = b_2, \qquad x_2 - x_3 = b_3.$$

Solution: Coefficient matrix times variable vector equals constants vector:

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

2.4 Exercise 2.4

(2 pt) Show that for all b_1, b_2, b_3, there exists a unique solution to the system of equations in Exercise 2.3.

Solution: We can show this in many ways. One way is to treat the vector of constants b_i as parameters and explicitly solve the system (e.g., by substitution, by row reduction, by inverting the coefficient matrix, etc.). However, this is a bit of work. The exercise doesn't ask us to find the solution, only to show that it exists and is unique. This is true if the coefficient matrix is invertible, which is equivalent to the coefficient matrix having a nonzero determinant. Using expansion by minors down the first column, we compute

$$\begin{aligned} \det \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} &= (-1)^{1+1}(1) \det \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} + (-1)^{2+1}(-1) \det \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 0 \\ &= (1)(-2-1) + (1)(-1-(-2)) = -3 + 1 = -2 \neq 0. \end{aligned}$$

The determinant is nonzero, hence the coefficient matrix is invertible, hence the matrix equation has a unique solution for any value of the b_i, hence the system of equations has a unique solution for any value of the b_i. (All these are equivalent statements.)

Differential Equations

3.1 Exercise 3.1

(2 pt) For the following homogeneous 1st-order autonomous ODE, determine the equilibrium values and classify the stability of each.

$$y' = (y-1)(y-3)(y-5)^2.$$

Solution: By definition, an equilibrium is a solution y that does not change with time, i.e. it solves $y' = 0$, which is satisfied if and only if $y(t) \equiv c$ for $c \in \{1, 3, 5\}$. Analyzing the sign of each factor on the right side of the given ODE, we see that

$$y' > 0 \text{ on } (-\infty, 1) \cup (3, 5) \cup (5, +\infty) \qquad \text{and} \qquad y' < 0 \text{ on } (1, 3).$$

We conclude that this ODE has three equilibria:

- $y \equiv 1$: stable
- $y \equiv 3$: unstable
- $y \equiv 5$: semistable (stable from below, unstable from above)

3.2 Exercise 3.2

(2 pt) Find the general solution to the homogeneous 1st-order linear system of ODEs

$$\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix} \mathbf{x}.$$

Solution: Let $\mathbf{A}$ denote the 2×2 coefficient matrix. Compute the characteristic polynomial:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{bmatrix} -3-\lambda & 2 \\ -1 & -5-\lambda \end{bmatrix} = \lambda^2 + 8\lambda + 17.$$

Compute the roots of the characteristic polynomial, which are the eigenvalues:

$$\lambda = \frac{-8 \pm \sqrt{64 - 4(1)(17)}}{2(1)} = -4 \pm i.$$

The eigenvalues are complex, so it suffices to look at one of the eigenvalues and find the real and imaginary parts of its corresponding eigenvector. Consider $\lambda_+ = -4 + i$. Its eigenvectors $\mathbf{v}_+$ solve

$$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \mathbf{v}_+ = \mathbf{0}.$$

Solve this:

$$\mathbf{v}_+ = \begin{bmatrix} 1+i \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\mathbf{x}_{\text{Re}} + c_2\mathbf{x}_{\text{Im}},$$

where

$$\mathbf{x}_{\text{Re}}(t) = e^{-4t}\cos t \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-4t}\sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_{\text{Im}}(t) = e^{-4t}\sin t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-4t}\cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

3.3 Exercise 3.3

(2 pt) Find the general solution $y \in \mathbf{R}[t]$ to the homogeneous 4th-order ODE

$$y^{(4)} - 3y^{(2)} - 4y = 0.$$

Solution: The associated characteristic polynomial (obtained by replacing $y^{(n)}$ by λ^n) is

$$\lambda^4 - 3\lambda^2 - 4 = 0.$$

Factoring, we find

$$0 = (\lambda^2 + 1)(\lambda^2 - 4) = (\lambda + i)(\lambda - i)(\lambda + 2)(\lambda - 2).$$

This equation has four distinct roots, $\lambda = \pm i$ and $\lambda = \pm 2$, each with algebraic multiplicity 1. Hence the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 e^{-2t} + c_4 e^{2t},$$

where $c_1, \dots, c_4 \in \mathbf{R}$.

3.4 Exercise 3.4

(2 pt) Using the laplace transform, solve the following homogeneous 3rd-order ODE IVP:

$$y^{(3)} + 4y^{(2)} - 5y^{(1)} = 0, \qquad y(0) = 4, \qquad y'(0) = -7, \qquad y''(0) = 23.$$

Solution: Let $\mathcal{L}(y^{(n)})$ denote the laplace transform of $y^{(n)}$. Using the initial conditions, we compute

$$\begin{aligned}
\mathcal{L}(y') &= s\mathcal{L}(y) - y(0) = s\mathcal{L}(y) - 4 \\
\mathcal{L}(y^{(2)}) &= s\mathcal{L}(y') - y'(0) = s(s\mathcal{L}(y) - 4) - (-7) = s^2\mathcal{L}(y) - 4s + 7 \\
\mathcal{L}(y^{(3)}) &= s\mathcal{L}(y^{(2)}) - y^{(2)}(0) = s(s^2\mathcal{L}(y) - 4s + 7) - 23 = s^3\mathcal{L}(y) - 4s^2 + 7s - 23.
\end{aligned}$$

Apply the laplace transform $\mathcal{L}$ to both sides of the given ODE, and use (i) the fact that $\mathcal{L}$ is a linear operator and (ii) the above results:

$$0 = \mathcal{L}(0) = \mathcal{L}(y^{(3)}) + 4\mathcal{L}(y^{(2)}) - 5\mathcal{L}(y^{(1)}) = \mathcal{L}(y)(s^3 + 4s^2 - 5s) - 4s^2 - 9s + 25.$$

Solve for $\mathcal{L}(y)$, using partial fraction decomposition:

$$\mathcal{L}(y) = \frac{4s^2 + 9s - 25}{s^3 + 4s^2 - 5s} = \frac{5}{s} + \frac{1}{s+5} - \frac{2}{s-1}.$$

Now take the inverse laplace transform of both sides to give $y(t)$, using that $\mathcal{L}^{-1}$ is also linear:

$$\begin{aligned}
y(t) &= 5\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) \\
&= 5 + e^{-5t} - 2e^{t}.
\end{aligned}$$

One can (and should) check that this solution satisfies the original ODE and initial conditions.