Math 112
LQuiz 01

2022–01–11 (T)

Your name: ____________________________

Instructions

Number of exercises	: 12
Permitted time	: 30 minutes
Permitted resources	: None

Instructor's notes:

- BEFORE SOLVING any exercises, please go through the entire quiz and write a "confidence number" 1–5 to the LEFT of each exercise, denoting how confident you are that you can correctly solve the exercise (1 = "Not at all confident", 5 = "Very confident"). Then, have fun solving!
- Sections 1 and 2 review concepts you have engaged previously. Section 3 previews concepts we will engage this semester. While concepts in Section 3 may be novel to you (for now), I encourage you to try to say something insightful on these exercises, time permitting.
- AFTER you are done working with this quiz, please scan or photograph your work and upload it to the Assignments page on Canvas, titled "Quiz : 2022_01_11".

1.1	/4	2.1	/4	3.1	/4
1.2	/4	2.2	/4	3.2	/4
1.3	/4	2.3	/4	3.3	/4
1.4	/4	2.4	/4	3.4	/4
Total	/16		/16		/16

Precalculus

1.1 Exercise 1.1

(4 pt) Let

$$f(x) = \frac{3x+8}{x^2+4x+2} \qquad\qquad g(x) = x - 2$$

(Assume the domain and codomain of f and g are the largest possible subsets of the real numbers, $\mathbf{R}$, for which the above rules of assignment make sense.) Find the composition $(f \circ g)(x)$, presented as simply as possible, and state its domain (i.e. the allowed values of x).

Solution: We compute

$$(f \circ g)(x) = f(g(x)) = f(x-2) = \frac{3(x-2)+8}{(x-2)^2+4(x-2)+2} = \frac{3x+2}{x^2-2}$$

This final expression for $f \circ g$ is not defined if and only if the denominator equals 0, i.e.

$$x^2 - 2 = 0 \qquad \Leftrightarrow \qquad x = \pm\sqrt{2}$$

Hence the domain of $f \circ g$ is all real numbers except $\pm\sqrt{2}$. We can also write this as

$$\mathbf{R} \setminus \{\pm\sqrt{2}\} \qquad \text{or} \qquad \left(-\infty, -\sqrt{2}\right) \cup \left(-\sqrt{2}, \sqrt{2}\right) \cup \left(\sqrt{2}, +\infty\right)$$

1.2 Exercise 1.2

(4 pt) Show that the following expression simplifies to a single trigonometric function. State any disallowed values of θ.

$$(\sec\theta - \cos\theta)\csc\theta$$

Solution: Let's begin by distributing, then rewriting $\csc\theta$ and $\sec\theta$ in terms of $\cos\theta$ and $\sin\theta$:

$$(\sec\theta - \cos\theta)\csc\theta = \sec\theta\csc\theta - \cos\theta\csc\theta = \frac{1}{\cos\theta\sin\theta} - \frac{\cos\theta}{\sin\theta}$$

To head further in the direction of a single trigonometric function, let's write the difference as a single fraction:

$$\cdots = \frac{1}{\cos\theta\sin\theta} - \frac{\cos\theta}{\cos\theta}\frac{\cos\theta}{\sin\theta} = \frac{1-\cos^2\theta}{\cos\theta\sin\theta}$$

Recall the identity $\cos^2\theta + \sin^2\theta = 1$. Using this, we can rewrite the expression as

$$\cdots = \frac{\sin^2\theta}{\cos\theta\sin\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

We can only cancel a factor of $\sin\theta$ from numerator and denominator when $\sin\theta \neq 0$, i.e. when θ is not an integer multiple of π. Otherwise, the denominator is 0; thus these values of θ are not allowed. Moreover, $\tan\theta$ is not defined when θ is an odd-integer multiple of $\frac{\pi}{2}$. These are the only disallowed values of θ.

1.3 Exercise 1.3

(4 pt) Solve the following equation exactly. If this is not possible, or no solution exists, state so.

$$\ln \sqrt{x+3} = 2$$

Solution: Let's begin by exponentiating both sides and simplifying:

$$\sqrt{x+3} = e^{\ln \sqrt{x+3}} = e^2$$

Because the exponential function[1] is injective (i.e. one-to-one), this transformation does not change the solutions to the original equation. Now we may square both sides and solve for x:

$$x = \left(e^2\right)^2 - 3 = e^4 - 3$$

The "square" function is not injective, so it is possible that we added spurious solutions at that step.[2] However, we can check directly that the solution $x = e^4 - 3$ indeed solves the original equation.

1.4 Exercise 1.4

(4 pt) Compute the following.

$$\sum_{m=1}^{19} (5m+2) - \sum_{n=-2}^{4} n^2$$

Solution: Let's tackle the two sums separately. For the first sum, we compute

$$\sum_{m=1}^{19} (5m+2) = 5 \sum_{m=1}^{19} m + \sum_{m=1}^{19} 2 = 5\frac{19(19+1)}{2} + 19(2) = 5(190) + 38 = 950 + 38$$

For the second sum, there is a formula when the summation index starts at 1 (or 0),[3] and we could partition the sum into two subsums to use it. However, because this sum has only 7 relatively small terms, I find it easier to compute them directly:

$$\sum_{n=-2}^{4} n^2 = (-2)^2 + (-1)^2 + \ldots + 3^2 + 4^2 = 4 + 1 + \ldots + 9 + 16 = 35$$

Putting our two results together, we conclude that

$$\sum_{m=1}^{19} (5m+2) - \sum_{n=-2}^{4} n^2 = (950 + 38) - (35) = 953$$

[1] The inverse of the exponential function — the natural logarithm function — is also injective.

[2] Consider the equation $x = 1$. Square both sides, then solve. What do you find? Is the result you found false?

[3] The formula, which we can prove by induction, is $\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$. (Can you prove this?)

Differential calculus

2.1 Exercise 2.1

(4 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = x^3 + 3x^2 - 9x + 13$$

Find all points $(x, f(x))$ where the tangent line to the graph of f is horizontal.

Solution: Horizontal tangents to the graph of f occurs at values of x where $f'(x) = 0$. We compute

$$0 \underset{\text{set}}{=} f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x+3)(x-1)$$

This equation has two solutions: $x = -3$ and $x = 1$. For each solution, we evaluate f at that value of x to find the corresponding point $(x, f(x))$ on the graph of f. This gives us the two points

$$(-3, 40) \qquad \text{and} \qquad (1, 8)$$

(Can you determine whether each point is a local minimum or a local maximum of f? Do you need to use calculus to do this?)

2.2 Exercise 2.2

(4 pt) Compute the first derivative of the following function.

$$v(t) = t^2 \ln(2t^3) + \arctan(5t) - \sin\left(\frac{\pi}{12}\right)$$

(Assume the domain and codomain of v are the largest possible subsets of the real numbers, $\mathbf{R}$, for which the above rule of assignment make sense.)

Solution: The derivative is linear, so we can differential v term-by-term.

The first term of v is a product of (nontrivial) functions of t, so we must use the product rule:

$$\begin{aligned}
\frac{d}{dt}\left(t^2 \ln(2t^3)\right) &= \frac{d}{dt}\left(t^2\right) \ln(2t^3) + t^2 \frac{d}{dt}\left(\ln(2t^3)\right) \\
&= (2t) \ln(2t^3) + t^2 \left(\frac{1}{2t^3}(6t^2)\right) \\
&= 2t \ln(2t^3) + 3t
\end{aligned}$$

where in the second equality we use the chain rule on the second term.

The second term of v also requires the chain rule (why?). Frankly, I don't remember the derivative of $y = \arctan x$, so I rewrite it as $\tan y = x$ and use implicit differentiation and elementary trigonometry, obtaining $y' = \frac{1}{x^2+1}$. Thus

$$\frac{d}{dt}\left(\arctan(5t)\right) = \frac{1}{(5t)^2 + 1}(5) = \frac{5}{25t^2 + 1}$$

The third term of v is a constant (no variable t!), so its derivative is 0.

Combining these results, we conclude that

$$v'(t) = 2t \ln(2t^3) + 3t + \frac{5}{25t^2 + 1}$$

2.3 Exercise 2.3

(4 pt) Let $f : [2, +\infty) \to \mathbf{R}$ be the function defined by

$$f(x) = \sqrt{2x - 4}$$

Using the limit definition of the derivative (!), compute $f'(x)$.

Solution: Fix $x \in [2, +\infty)$. Using the limit definition of the derivative (i.e. the limit of a difference quotient), we compute

$$f'(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \downarrow 0} \frac{\sqrt{2x + 2h - 4} - \sqrt{2x - 4}}{h}$$

Direct evaluation of this limit gives $\frac{0}{0}$, an indeterminate form. Let's rationalize the numerator to see if that helps:[4]

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{\sqrt{2x + 2h - 4} - \sqrt{2x - 4}}{h} &= \lim_{h \downarrow 0} \frac{\sqrt{2x + 2h - 4} - \sqrt{2x - 4}}{h} \cdot \frac{\sqrt{2x + 2h - 4} + \sqrt{2x - 4}}{\sqrt{2x + 2h - 4} + \sqrt{2x - 4}} \\
&= \lim_{h \downarrow 0} \frac{2h}{h\left(\sqrt{2x + 2h - 4} + \sqrt{2x - 4}\right)} \\
&= \lim_{h \downarrow 0} \frac{2}{\left(\sqrt{2x + 2h - 4} + \sqrt{2x - 4}\right)} = \frac{2}{2\sqrt{2x - 4}} = \frac{1}{\sqrt{2x - 4}}
\end{aligned}$$

(Can you show we get the same result if we compute $f'(x)$ using the "rules" for derivatives?)

2.4 Exercise 2.4

(4 pt) Compute the following limit.

$$\lim_{x \to +\infty} \frac{3e^x + 5}{5e^x + x + 1}$$

Solution: Direct evaluation of the limit gives $\frac{+\infty}{+\infty}$, an indeterminate form. Divide both numerator and denominator by e^x, or equivalently, multiply both numerator and denominator by e^{-x}:

$$\lim_{x \to +\infty} \frac{3e^x + 5}{5e^x + x + 1} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \to +\infty} \frac{3 + 5e^{-x}}{5 + xe^{-x} + e^{-x}} = \frac{\lim_{x \to +\infty} \left(3 + 5e^{-x}\right)}{\lim_{x \to +\infty} \left(5 + xe^{-x} + e^{-x}\right)}$$

where the final equality uses limit laws (which? what is the exact statement?). In the numerator, direct evaluation gives

$$\lim_{x \to +\infty} \left(3 + 5e^{-x}\right) = 3 + 5e^{-\infty} = 3 + 5(0) = 3$$

In the denominator, another use of limit laws (which?) and one application of l'Hôpital's rule give

$$\lim_{x \to +\infty} \left(5 + xe^{-x} + e^{-x}\right) = 5 + \lim_{x \to +\infty} \frac{1}{e^x} + 0 = 5 + 0 + 0 = 5$$

which we note is nonzero. (Why might we note that?) We conclude that

$$\lim_{x \to +\infty} \frac{3e^x + 5}{5e^x + x + 1} = \frac{3}{5}$$

[4]Can we apply l'Hôpital's rule here? What do we need to "know" to do the computations l'Hôpital's rule requires?

Integral calculus

3.1 Exercise 3.1

(4 pt) Solve the following initial value problem (i.e. find f satisfying the following conditions).

$$f'(x) = x^2 + \sqrt{x} \qquad \text{such that} \qquad f(0) = 2$$

Solution: First we integrate $f'(x)$ to get a family of functions:

$$f(x) = \int f'(x)\,dx = \int x^2 + \sqrt{x}\,dx = \frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} + C$$

where $C \in \mathbf{R}$. This is the general solution, ignoring the initial value (aka boundary condition) $f(0) = 2$. To obtain the particular solution, we apply the initial value:

$$2 \underset{\text{set}}{=} f(0) = \frac{1}{3}0^3 + \frac{2}{3}0^{\frac{3}{2}} + C = C$$

Thus, the particular solution to the initial value problem is

$$f(x) = \frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} + 2$$

(Can you confirm that this function satisfies all parts of the initial value problem? What about uniqueness: Is this solution unique?)

3.2 Exercise 3.2

(4 pt) Let $f : [-2, 2] \to \mathbf{R}$ be the function defined by

$$f(x) = 2 - \sqrt{4 - x^2}$$

Find the average value of f on the interval $[0, 2]$. Then find a value x_0 of x on this interval such that $f(x_0)$ equals this average value. (Is such an x_0 guaranteed to exist? Why or why not? Is it unique?)

Solution: The average value (aka mean value) of f on the interval $[0, 2]$ is given by

$$\mathrm{MV}(f, [0, 2]) = \frac{1}{2-0}\int_0^2 f(x)\,dx$$

The integral is messy to compute algebraically...but a breeze to compute geometrically! (Why? Sketch the graph of f...) Elementary euclidean geometry gives us

$$\mathrm{MV}(f, [0, 2]) = \frac{1}{2}\left(2 \cdot 2 - \frac{1}{4}\pi \cdot 2^2\right) = 2 - \frac{\pi}{2}$$

We find the requested x_0 — guaranteed to exist by the mean value theorem — by solving

$$2 - \frac{\pi}{2} = \mathrm{MV}(f, [0, 2]) \underset{\text{set}}{=} f(x_0) = 2 - \sqrt{4 - x_0^2} \qquad \Rightarrow \qquad x_0 = \pm\sqrt{4 - \frac{\pi^2}{4}}$$

Only the positive square root lies in the interval $[0, 2]$ (can you show this? without a calculator?), so $x_0 = \sqrt{4 - \frac{\pi^2}{4}}$.

3.3 Exercise 3.3

(4 pt) Compute the following derivative.

$$\frac{d}{dx}\int_0^{\ln x} e^{2t}\,dt$$

Solution: By the fundamental theorem of calculus,[5]

$$\frac{d}{dx}\int_0^{\ln x} e^{2t}\,dt = e^{2(\ln x)}\frac{d}{dx}\ln x - e^{2(0)}\frac{d}{dx}0 = e^{\ln(x^2)}\frac{1}{x} - 0 = x$$

(Can you show we get the same result, if we first compute the definite integral (as a function of x), then compute its derivative?)

Note that the original integral is defined only for $x \in (0, +\infty)$, so our final answer has the same constraint. That is, a more precise answer is

$$\frac{d}{dx}\int_0^{\ln x} e^{2t}\,dt = x, \quad x \in (0, +\infty)$$

(Can you explain this constraint, geometrically?)

3.4 Exercise 3.4

(4 pt) Compute the following indefinite integral.

$$\int x^2 \sin x\,dx$$

Solution: One application of integration by parts (with $u = x^2$ and $dv = \sin x\,dx$) gives

$$\int x^2 \sin x\,dx = -x^2\cos x - \int -\cos x(2x)\,dx = -x^2\cos x + 2\int x\cos x\,dx \tag{1}$$

Another integration by parts, on the second integral (with $u = x$ and $dv = \cos x\,dx$), gives

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C_0$$

Substituting this result into (1), we obtain

$$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$$

(Can you show that differentiating our answer indeed gives the original integrand?)

[5] The integrand, e^{2t}, is continuous on all of $\mathbf{R}$, so the fundamental theorem of calculus applies, at least on a subset of $(0, +\infty)$, the lower bound 0 coming from the lower limit of integration of the original integral.