Math 212
Quiz 28

F 04 Nov

Your name: ____________________________________

Exercise

(5 pt) Let $R \subseteq \mathbf{R}^2$ be the square bounded by the lines

$$x+y=-1, \qquad x+y=1, \qquad x-y=-1, \qquad x-y=1.$$

Show that

$$\iint_R e^{x+y}\,dA = e - \frac{1}{e}.$$

Hint: Apply a change of variables. More precisely, let the equations of the boundary of the region R and the integrand guide your definition of new variables u, v as functions of the given variables x, y. Solve for x, y as functions of u, v. Remember the Jacobian determinant.

Solution: The equations defining the region R (and the integrand) suggest the change of variables

$$u = x+y, \qquad v = x-y.$$

Solving this system of equations for x, y in terms of u, v, we find

$$x = \frac{1}{2}u + \frac{1}{2}v, \qquad y = \frac{1}{2}u - \frac{1}{2}v.$$

Thus the relevant transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$, from the uv-plane to the xy-plane, is

$$T(u,v) = (x(u,v), y(u,v)) = \left(\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u - \frac{1}{2}v\right),$$

with Jacobian matrix and Jacobian determinant

$$J_T(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad \text{and} \qquad \det J_T = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

respectively.

The region S of the uv-plane that maps to the region R of the xy-plane can be found by rewriting the equations defining R in terms of u, v. Doing this, we find

$$S = \left\{(u,v) \in \mathbf{R}^2 \mid -1 \leqslant u \leqslant 1, -1 \leqslant v \leqslant 1\right\}.$$

Applying the change-of-variables theorem to the given integral using the transformation T gives

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \iint_S f(T(u,v))\,|\det J_T|\,dA \\
&= \frac{1}{2}\int_{v=-1}^{v=1}\int_{u=-1}^{u=1} e^u\,du\,dv \\
&= \frac{1}{2}\int_{v=-1}^{v=1} dv \int_{u=-1}^{u=1} e^u\,du \\
&= \frac{1}{2}[2]\left[e^1 - e^{-1}\right] = e - \frac{1}{e}.
\end{aligned}$$