Math 212
Quiz 22

F 21 Oct 2016

Your name: ____________________________________

Exercise

(5 pt) A thin washer (i.e. O-shaped piece of material) is described by the region $D \subseteq \mathbf{R}^2$ lying between the circles

$$C_1 : x^2 + y^2 = 1, \qquad C_2 : x^2 + y^2 = 4.$$

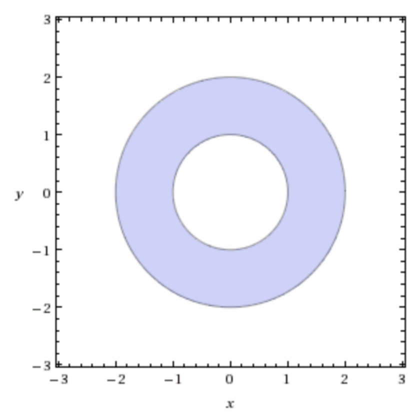
The charge density of the washer is given by the function $\sigma : D \to \mathbf{R}$ defined by

$$\sigma(x, y) = \frac{2xy}{x^2 + y^2}.$$

We want to find the total (net) charge of the washer.

(a) (1 pt) Recall that we recover a quantity (e.g., mass, charge, etc.) by integrating a density. Sketch the relevant region of integration.

Solution: The relevant region of integration is D, the region occupied by the washer.



(b) (3 pt) Set up an iterated (!) integral that gives the total (net) charge Q of the washer. *Hint:* Use polar coordinates. Mind the integration factor.

Solution: The region D has a particularly simple description in polar coordinates:

$$D = \{(r, \theta) \,|\, 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant 2\pi\}.$$

The total charge Q of the washer is given by integrating the charge density $\sigma(x, y)$ over the region D occupied by the washer. Because $\sigma(x, y)$ is continuous on D, Fubini's theorem allows us to write this double integral as an iterated integral. Using polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ (note the integration factor of r). Thus

$$\begin{aligned} Q &= \iint_D \sigma(x, y)\, dA \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \frac{2(r\cos\theta)(r\sin\theta)}{r^2}\, r\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} r\sin(2\theta)\, dr\, d\theta, \end{aligned}$$

where in the final step we have used the trigonometric identity

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

(c) (1 pt) Evaluate the integral in part (b) to show that the total (net) charge $Q = 0$. *Hint:* Recall that $\sin(2\theta) = 2\sin\theta\cos\theta$.

Solution: Evaluating the integral from part (b), we find

$$\begin{aligned} m &= \int_{\theta=0}^{\theta=2\pi} \sin(2\theta)\, d\theta \int_{r=1}^{r=2} r\, dr \\ &= \left[-\frac{1}{2}\cos(2\theta)\right]_{\theta=0}^{\theta=2\pi} \left[\frac{1}{2}r^2\right]_{r=1}^{r=2} \\ &= 0, \end{aligned}$$

because the first integral evaluates to 0.