# Math 211
# Quiz 21

M 05 Aug 2019

Your name : ________________________________________

## Exercise

(5 pt) Consider the homogeneous 1st-order $2 \times 2$ linear system of ODEs

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} \mathbf{x}. \tag{1}$$

(a) (2 pt) Diagonalize the coefficient matrix (i.e. compute the eigenvalues and corresponding eigenvectors of the given matrix $\mathbf{A}$, and write it in the form $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$).

**Solution:** Denote the coefficient matrix in (1) by

$$\mathbf{A} = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}.$$

**Eigenvalues**. We compute

$$0 \underset{\text{set}}{=} \det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{bmatrix} -1-\lambda & -4 \\ 4 & -1-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 17.$$

Using the quadratic equation,

$$\lambda = \frac{-(2) \pm \sqrt{2^2 - 4(1)(17)}}{2(1)} = \frac{-2 \pm \sqrt{4 - 4(17)}}{2} = \frac{-2 \pm \sqrt{4(1-17)}}{2} = -1 \pm 4i.$$

**Eigenvectors.** Compute the eigenspace for $\lambda = -1 + 4i$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} \underset{\text{set}}{=} (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Form the corresponding augmented matrix and apply the row reduction algorithm:

$$\left[\begin{array}{cc:c} -4i & -4 & 0 \\ 4 & -4i & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - iR_1} \left[\begin{array}{cc:c} -4i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = -\frac{1}{4i}R_1} \left[\begin{array}{cc:c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

where in the last step we use $\frac{1}{i} = \frac{1 \cdot i}{i \cdot i} = \frac{i}{i^2} = \frac{i}{-1} = -i$. We conclude that

$$E(\mathbf{A}, -1+4i) = \operatorname{Span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}.$$

We can perform the analogous computations for the other eigenvalue $\lambda = -1 - 4i$, or we can note that because it is the complex conjugate of $-1 + 4i$, its eigenvectors are complex conjugate to the eigenvectors of $-1 + 4i$. Either way, we get

$$E(\mathbf{A}, -1-4i) = \operatorname{Span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}.$$

**Diagonalization.** Using these results, we conclude

$$\mathbf{A} = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}. \tag{2}$$

(b) (2 pt) Use this information to write the general real solution $\mathbf{x}(t)$ to (1). *Hint:* Recall that when the eigenvalues of a homogeneous $2 \times 2$ linear system are complex, we can compute one solution $e^{\lambda t}v$, just as we did with real eigenvalues; decompose it into real and imaginary parts; then use these parts as our basis for the solution space. The formulas

$$e^{a+ib} = e^a e^{ib}, \qquad e^{ib} = \cos b + i \sin b,$$

will be useful.

**Solution:** Call columns 1 and 2 of $\mathbf{P}$ $\mathbf{v}_+$ and $\mathbf{v}_-$, respectively. (They are eigenvectors corresponding to the eigenvalues $\lambda_+ = -1+4i$ and $\lambda_- = -1-4i$, respectively. Hence our choice of subscripts $+$ and $-$.) Our diagonalization (2) gives one basis for the vector space of complex (!) solutions to (1), namely, the vectors

$$\mathbf{x}_+(t) = e^{\lambda_+ t}\mathbf{v}_+ = e^{(-1+4i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}, \qquad \mathbf{x}_-(t) = e^{\lambda_- t}\mathbf{v}_- = e^{(-1-4i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

To describe the real (!) solutions to (1), we can decompose either one of these complex basis vectors into real and imaginary parts, $\mathbf{x}_{\mathrm{Re}}$ and $\mathbf{x}_{\mathrm{Im}}$, and take these as our basis vectors. Decomposing $\mathbf{x}_+(t)$, using the formulas in the hint, we find

$$\begin{aligned} \mathbf{x}_+(t) = e^{(-1+4i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} &= e^{-t}e^{i(4t)}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^{-t}(\cos(4t) + i\sin(4t))\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} -\sin(4t) + i\cos(4t) \\ \cos(4t) + i\sin(4t) \end{bmatrix} = \underbrace{e^{-t}\begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix}}_{\mathbf{x}_{\mathrm{Re}}} + i\,\underbrace{e^{-t}\begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}}_{\mathbf{x}_{\mathrm{Im}}}. \end{aligned}$$

(Note that our definition of $\mathbf{x}_{\mathrm{Im}}$ does not include the coefficient $i$.) The general real solution to (1) is the real linear combination of $\mathbf{x}_{\mathrm{Re}}$ and $\mathbf{x}_{\mathrm{Im}}$, i.e.

$$\mathbf{x}(t) = c_1\mathbf{x}_{\mathrm{Re}}(t) + c_2\mathbf{x}_{\mathrm{Im}}(t) = c_1 e^{-t}\begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix},$$

where $c_1, c_2 \in \mathbf{R}$.

(c) (1 pt) Quickly sketch the phase plane. Your sketch need not be exact; focus on whether trajectories move toward or away from the origin, and the direction of rotation.

**Solution:** The real part of both eigenvalues $\lambda_\pm$ is $-1$, which is negative; thus trajectories move toward the origin (as time increases). To determine the direction of rotation, we can use the original ODE (1) to compute the tangent vectors $\mathbf{x}'$ at a few points $(x_1, x_2)$:

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \qquad \mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}.$$

This shows that, at $(x_1, x_2) = (1, 1)$, tangent vectors point left ($x_1' = -5 < 0$) and up ($x_2' = 3 > 0$); at $(x_1, x_2) = (-1, 1)$, tangent vectors point left ($x_1' = -3 < 0$) and down ($x_2' = -5 < 0$). Thus trajectories rotate counterclockwise.

A phase plane for the linear system (1) is shown in Figure 1.

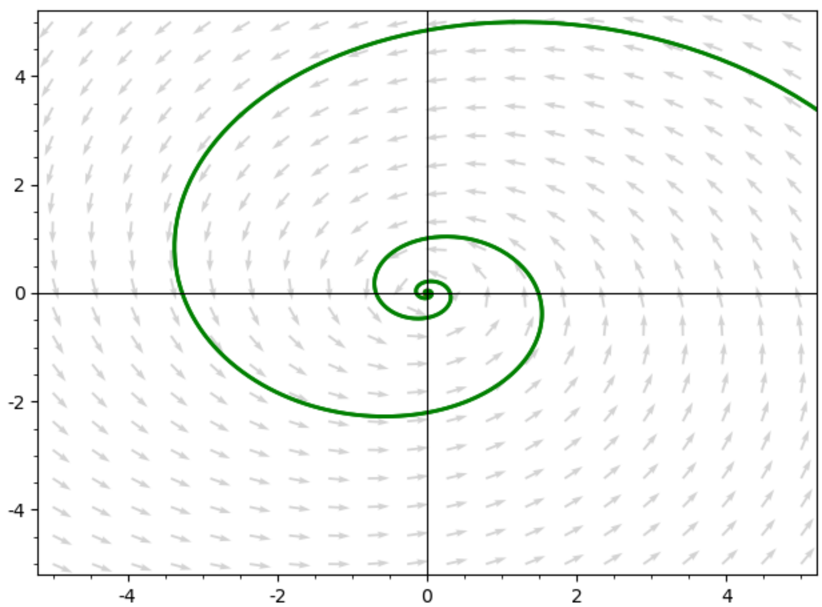


Figure 1: Phase plane, in the $(x_1, x_2)$ plane, for the homogeneous 1st-order linear system (1). Arrows point in the direction of increasing $t$.