Math 112
Exam 04

2022–04–30 (S)

Your name: Solutions

Instructions

Number of exercises : 16
Permitted time : 3 hours
Permitted resources : None

Remarks:

- This exam has three sections, corresponding to the three midterm exams.
- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- Work hard, do your best, and have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/10	/2	/4	/4		
3	/10	/4	/4	/2		
4	/12	/2	/2	/4	/2	/2
5	/8	/2	/2	/4		
Part 1	/50					
6	/10	/2	/2	/2	/2	/2
7	/8	/4	/4			
8	/8	/4	/4			
9	/12	/4	/4	/4		
10	/12	/2	/4	/4	/2	
Part 2	/50					
11	/10	/2	/2	/2	/2	/2
12	/8	/4	/4			
13	/8	/4	/4			
14	/4					
15	/10	/4	/4	/2		
16	/10	/2	/4	/4		
Part 3	/50					
TOTAL	/150					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) Let $f : [0, +\infty) \to \mathbf{R}$ be given by $f(x) = x^2$, and let g be the inverse function to f. The domain of g' equals the domain of g.

true   false

(b) (2 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be a function. If $f'(a) = 0$, then $x = a$ is either a local minimum or a local maximum of f.

true   false

(c) (2 pt) The equation $x = -2$ describes a tangent line to the graph of $x^2 + y^2 = 4$.

true   false

$$= e^{\ln 2^3} \cdot e^{-a} e^{-b} e^{6c} \cdot \sqrt{e^{4a} e^{2b}}$$

$$= 2^3 \cdot e^{-a} e^{-b} e^{6c} \cdot e^{2a} e^{b}$$

$$= 8\, e^{a} \underbrace{(e^{c})^6}_{=(e^{3c})^2} = 8 \cdot 4 \cdot \frac{1}{8} = 4$$

(d) (2 pt) Let $e^a = 4$, $e^b = 2$, and $e^{3c} = \frac{1}{\sqrt{8}}$. Then

4

$$e^{3\ln 2} \cdot \frac{e^{5b+3c}}{e^{a+6b-3c}} \cdot \sqrt{\frac{e^{6a+b}}{e^{2a-b}}} = 1$$

true   false

exp 4   exp 6

$$= \ln\left(\frac{1}{\sqrt{e}}\right)^2 + \ln(a+b) - 3\ln c - \left[\ln b + \ln c + \ln(a+b) - 2\ln a\right]$$

(e) (2 pt) Let $\ln a = 3$, $\ln b = 5$, and $\ln c = -\frac{1}{4}$. Then

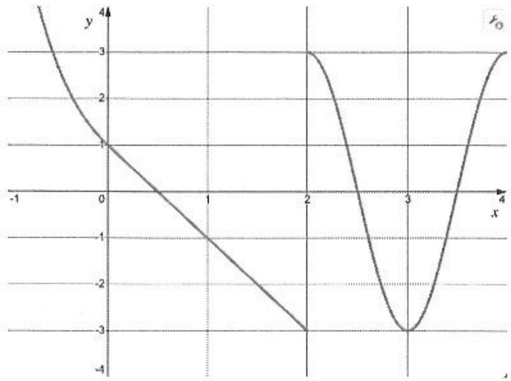
$$2\ln\left(\frac{1}{\sqrt{e}}\right) + \ln\left(\frac{a+b}{c^3}\right) - \ln\left(\frac{bc(a+b)}{a^2}\right) = 1$$

$$= \ln\frac{1}{e} + 2\ln a - \ln b - 4\ln c$$

true   false

$$= -1 + 6 - 5 - 4\left(-\frac{1}{4}\right)$$

$$= 1$$



Exercise 2

(10 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the piecewise function graphed above and given by

$$f(x) = \begin{cases} e^{-2x} & \text{if } x < 0 \\ -2x + 1 & \text{if } 0 \leqslant x \leqslant 2 \\ 3\cos(\pi x) & \text{if } x > 2 \end{cases}$$

(a) (2 pt) Using the graph, identify the value(s) of x at which f(x) is not continuous.

f is not continuous at $x = 2$

(b) (4 pt) Justify, algebraically, that f(x) is not continuous at the value(s) of x identified in part (a).

We compute

$$\lim_{x \uparrow 2} f(x) = \lim_{x \uparrow 2} (-2x+1) = -2(2)+1 = -3$$

$$\lim_{x \downarrow 2} f(x) = \lim_{x \downarrow 2} 3\cos(\pi x) = 3\cos(2\pi) = 3$$

Because $\lim_{x \uparrow 2} f(x) \neq \lim_{x \downarrow 2} f(x)$, $\lim_{x \to 2} f(x)$ does not exist, hence f(x) is not continuous at $x = 2$.

(c) (4 pt) Is the first-derivative function $f'(x)$ continuous at $x = 0$? Justify algebraically.

We compute

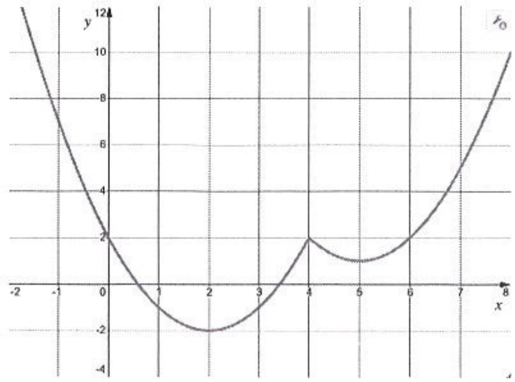
$$f'(x) = \begin{cases} -2e^{-2x} & \text{if } x < 0 \\ -2 & \text{if } 0 < x < 2 \\ -3\pi \sin(\pi x) & \text{if } x > 2 \end{cases}$$

and

$$\lim_{x \uparrow 0} f'(x) = \lim_{x \uparrow 0} -2e^{-2x} = -2e^{0} = -2$$

$$\lim_{x \downarrow 0} f'(x) = \lim_{x \downarrow 0} -2 = -2$$

This can be used to show $f'(x)$ is continuous at $x = 0$.



Exercise 3

(10 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the piecewise function graphed above and given by

$$f(x) = \begin{cases} x^2 - 4x + 2 & \text{if } x \leqslant 4 \\ x^2 - 10x + 26 & \text{if } x \geqslant 4 \end{cases}$$

(a) (4 pt) Find the intervals on which f is increasing and decreasing. Justify algebraically.

We compute

$$f'(x) = \begin{cases} 2x - 4 & \text{if } x \leq 4 \\ 2x - 10 & \text{if } x \geq 4 \end{cases}$$

So

$$x \leq 4 : \quad 2x - 4 = f'(x) \gtreqless 0 \iff x \gtreqless 2$$

$$x \geq 4 : \quad 2x - 10 = f'(x) \gtreqless 0 \iff x \gtreqless 5$$

We conclude that

- f increasing ($f'(x) > 0$) on $(2,4) \cup (5, +\infty)$
- f decreasing ($f'(x) < 0$) on $(-\infty, 2) \cup (4, 5)$

These results agree with the graph above.

(b) (4 pt) Find the x-coördinate of each local minimum and maximum of f. State whether each is a local minimum or maximum of f. Justify algebraically.

Local minima and maxima come from among the critical points of $f(x)$, which by definition are values of x such that $f'(x) = 0$ or $f'(x)$ is not defined. For the given function f, these are

$$x = 2 \qquad\qquad x = 5 \qquad\qquad x = 4$$

$$(f'(2)=0) \qquad\qquad (f'(5)=0) \qquad\qquad (f'(4) \text{ not defined})$$

~~By~~ Analysing the slopes on either side of these points, we find

$\ominus$ 0 $\oplus$ undef. $\ominus$ 0 $\oplus$ $f'(x)$

2 3 4 5 x

Thus $x = 2$ and $x = 5$ are local minima, and $x = 4$ is a local maximum. These results agree with the graph.

(c) (2 pt) Find the global minimum and maximum of f (x and y values). Justify algebraically.

Global minimum: ~~We~~ Comparing the y-coördinates of the two local minima, we find

$$f(2) = (2)^2 - 4(2) + 2 = -2 < 1 = (5)^2 - 10(5) + 26 = f(5)$$

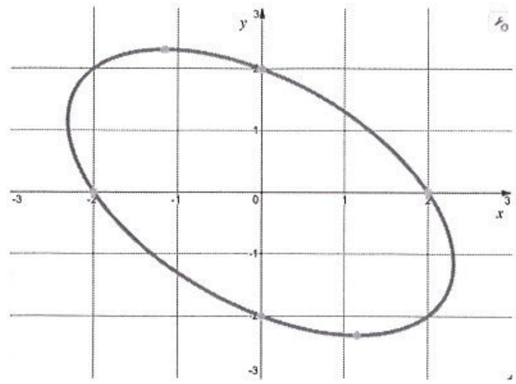
Thus $(2, -2)$ is the global minimum of f.

Global maximum: f has no global maximum, by the end behaviour computations.

End behaviour of f:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x^2 - 4x + 2) = +\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (x^2 - 10x + 26) = +\infty$$



Exercise 4

(12 pt) Consider the graph, above, of the ellipse given by the equation

$$x^2 + xy + y^2 = 4 \qquad (4.1)$$

(a) (2 pt) From the graph, the points $(x, y) = (-2, 2)$ and $(2, -2)$ appear to be on the ellipse. Prove this, algebraically.

We substitute these values of x and y into (4.1) and verify that the equation holds:

$(-2, 2)$: $(-2)^2 + (-2)(2) + (2)^2 \stackrel{?}{=} 4$

$4 - 4 + 4 \stackrel{?}{=} 4$

$4 = 4$ ✓ $\Rightarrow$ $(-2,2)$ is on the ellipse.

$(2, -2)$: $(2)^2 + (2)(-2) + (-2)^2 \stackrel{?}{=} 4$

$4 - 4 + 4 \stackrel{?}{=} 4$

$4 = 4$ ✓ $\Rightarrow$ $(2,-2)$ is on the ellipse.

(b) (2 pt) Using the graph, predict the slope of the tangent line to the graph at $(x, y) = (-2, 2)$.

The slope of the tangent line to the graph of the ellipse at $(x, y) = (-2, 2)$ looks equal to about 1.

(c) (4 pt) Compute the rule of assignment for y'. (Your answer will involve both x and y.)

Implicitly differentiating (4.1) with respect to x (and viewing y as an implicit function of x), we get

$$2x + \underbrace{y + xy'}_{\text{product rule}} + \underbrace{2yy'}_{\text{chain rule}} = 0$$

Solving for y', we find

$$(x+2y)\,y' = -2x - y$$

$$y' = -\frac{2x+y}{x+2y}$$

(d) (2 pt) Find an equation for the tangent line to the graph at the point $(x, y) = (-2, 2)$.

From part (c), the slope of the tangent line to the graph of the ellipse at $(x,y) = (-2,2)$ is

$$y'\big|_{(-2,2)} = -\frac{2(-2)+2}{-2+2(2)} = -\frac{-4+2}{-2+4} = -\frac{-2}{2} = 1$$

confirming our estimate in part (b). Thus an equation for the tangent line to the ellipse at $(-2,2)$ is

$$y - y_0 = m(x - x_0) \qquad y - 2 = x + 2$$

$$y - (2) = 1\,(x - (-2)) \qquad y = x + 4$$

(e) (2 pt) Find the x-coördinate of each point on the ellipse at which the tangent line is horizontal.

Horizontal tangent lines correspond to points (x,y) at which $y' = 0$. Using our result from part (c), we compute

$$0 \underset{\text{set}}{=} y' = -\frac{2x+y}{x+2y} \iff y = -2x$$

Substituting this into (4.1), we get

$$x^2 + x(-2x) + (-2x)^2 = 4$$

$$x^2 - 2x^2 + 4x^2 = 4$$

$$3x^2 = 4$$

$$x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$$

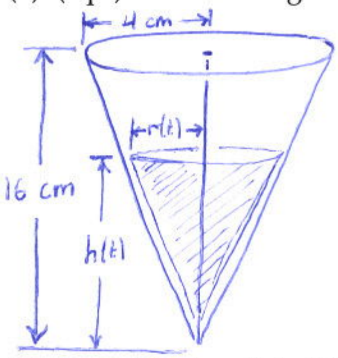
$$\approx \pm\frac{2}{1.7} \approx \pm 1.2$$

Note that, from the graph above, the tangent lines to the ellipse at $(x,y) \approx (-1.2, 2.2)$ and $(x,y) \approx (1.2, -2.2)$ look horizontal.

Exercise 5

(8 pt) You are pouring melted ice cream into an ice-cream cone at a steady rate of π cm^3 per second. The ice-cream cone has a height of 16 cm and a diameter of 8 cm. The volume V of a cone with height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$. This exercise explores how fast the level of liquid in the ice-cream cone is rising. Note that, as liquid builds up in the cone, it always forms a smaller cone with dimensions similar to (i.e. scaled down from) the ice-cream cone.

(a) (2 pt) Sketch a diagram and identify relevant variables.



Relevant variables:

- h : Height of liquid in cone
- r : Radius of liquid in cone
- V : Volume of liquid in cone

All three are functions of time t

(b) (2 pt) Use implicit differentiation to relate the rate of change of volume of liquid in the ice-cream cone to the rates of change of the radius and the height of the liquid cone.

~~Remem~~ Note that V, r, h are functions of time t. (Implicitly) differentiating $V = \frac{1}{3}\pi r^2 h$ with respect to t (which requires the product rule and chain rule), we get

$$\frac{dV}{dt} = \frac{1}{3}\pi\, 2r \frac{dr}{dt} h + \frac{1}{3}\pi r^2 \frac{dh}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) \qquad (5.1)$$

7

(c) (4 pt) How fast is the level of liquid in the cone rising when the ice-cream cone is half-full? *Hint:* Justify why, at all relevant times t, the radius r and height h of the liquid cone satisfy $h = 4r$.

The cone of liquid ice cream is similar to (that is, scaled down from) the ice-cream cone, so the ratios of corresponding measurements are equal:

$$\underbrace{r}_{\text{radius of liquid cone}} / \underbrace{R}_{\text{Radius of ice-cream cone}} = \underbrace{h}_{\text{height of liquid cone}} / \underbrace{H}_{\text{Height of ice-cream cone}} \quad \rightsquigarrow \quad \frac{r}{4\text{ cm}} = \frac{h}{16\text{ cm}} \iff 4r = h$$

Differentiating this relation with respect to time t gives $4\frac{dr}{dt} = \frac{dh}{dt}$ or equivalently $\frac{dr}{dt} = \frac{1}{4}\frac{dh}{dt}$. The ice-cream cone is half-full when $h = \frac{1}{2}H = \frac{1}{2}(16\text{ cm}) = 8$ cm, at which point $r = \frac{1}{4}h = 2$ cm.

Substituting all the known information into (5.1), we find

$$\cancel{\pi}^{1}\ \text{cm}^3/\text{s} = \frac{1}{3}\cancel{\pi}\left(2(2\text{cm})(\cancel{8}^{2}\text{ cm})\frac{1}{\cancel{4}}\frac{dh}{dt} + (2\text{cm})^2\frac{dh}{dt}\right)$$

$$1\ \text{cm}^3/\text{s} = \frac{1}{\cancel{3}}\left(\cancel{12}^{4}\ \text{cm}^2\frac{dh}{dt}\right) \iff \frac{dh}{dt} = \frac{1}{4}\ \text{cm/s}$$

Exercise 6

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) If direct evaluation of a limit gives an indeterminate form that is not $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then the limit does not exist.

true false

(b) (2 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Consider lower- and upper-sum approximations for $\int_a^b f(x)\, dx$. Any lower sum is less than or equal to any upper sum, even if we use different partitions for the lower sum and upper sum.

true false

(c) (2 pt) Let $f(x)$ be a continuous function, and let $F_1(x)$ and $F_2(x)$ be antiderivatives of $f(x)$. Then $F_1'(x) - F_2'(x) = 0$.

true false

For parts (d)–(e), let $f(x)$ and $g(x)$ be functions such that

$$\int_{-1}^{3} f(x)\, dx = 8 \qquad \int_{-1}^{3} g(x)\, dx = -4$$

(d) (2 pt) $\displaystyle\int_{-1}^{3} \left[\frac{1}{2}f(x) - 2g(x)\right] dx = \int_{-1}^{3} [f(x) - g(x)]\, dx$

true false

(e) (2 pt) The average value of $f(x) + g(x)$ on the interval $[-1, 3]$ equals 1.

true false

Exercise 7

(8 pt) Evaluate each limit to verify the result. Briefly but clearly justify your work.

(a) (4 pt) $\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}} = 0$

Direct evaluation gives $\frac{+\infty}{+\infty}$, an indeterminate form to which l'Hôpital's rule applies. We compute

$$\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to+\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to+\infty} \frac{2}{\sqrt{x}} = 0$$

(b) (4 pt) $\lim_{x\to1} \frac{3x-3}{-1+\sqrt{3x-2}} = 2$

Direct evaluation gives $\frac{3-3}{-1+\sqrt{1}} = \frac{0}{0}$, an indeterminate form to which l'Hôpital's rule applies. However, we try to evaluate the limit using conjugation:

$$\lim_{x\to1} \frac{3x-3}{-1+\sqrt{3x-2}} \cdot \frac{-1-\sqrt{3x-2}}{-1-\sqrt{3x-2}}$$

$$= \lim_{x\to1} \frac{(3x-3)(-1-\sqrt{3x-2})}{(-1)^2-(\sqrt{3x-2})^2}$$

$$= \lim_{x\to1} \frac{(3x-3)(-1)(1+\sqrt{3x-2})}{1-(3x-2)}$$

$$= \lim_{x\to1} \frac{-(3x-3)(1+\sqrt{3x-2})}{-3x+3}$$

$$= \lim_{x\to1} \frac{\overset{1}{\cancel{-(3x-3)}}(1+\sqrt{3x-2})}{\cancel{-(3x-3)}}$$

$$= \lim_{x\to1} 1+\sqrt{3x-2}$$

$$= 1+\sqrt{3(1)-2}$$

$$= 1+\sqrt{1}$$

$$= 2$$

Exercise 8

(8 pt) This exercise considers the limit

$$\lim_{x \to 0} \frac{6e^x - 6(x+1) - 3x^2}{2x^3} \tag{1}$$

(a) (4 pt) Evaluate the limit in (1) using the Taylor series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \qquad (8.1)$$

where the terms in "..." all involve x to the power 4 or higher.

Substituting (8.1) into (1), we get

$$\lim_{x \to 0} \frac{6\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)\right) - 6(x+1) - 3x^2}{2x^3}$$

$$= \lim_{x \to 0} \frac{\cancel{6} + \cancel{6x} + \cancel{3x^2} + x^3 + O(x^4) - \cancel{6x} - \cancel{6} - \cancel{3x^2}}{2x^3}$$

$$= \lim_{x \to 0} \frac{x^3 + O(x^4)}{2x^3} = \lim_{x \to 0} \frac{1 + O(x)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

(b) (4 pt) Evaluate the limit in (1) using l'Hôpital's rule.

At each step, we check direct evaluation (D.E.) of the limit; when it is $\frac{0}{0}$, we apply l'Hôpital's rule:

$$\lim_{x \to 0} \frac{6e^x - 6x - 6 - 3x^2}{2x^3} \overset{\text{D.E.}}{\rightsquigarrow} \frac{6 - 0 - 6 - 0}{0} = \frac{0}{0}$$

$$\overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{6e^x - 6 - 0 - 6x}{6x^2} \overset{\text{D.E.}}{\rightsquigarrow} \frac{6 - 6 - 0}{0} = \frac{0}{0}$$

$$\overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{6e^x - 0 - 6}{12x} \overset{\text{D.E.}}{\rightsquigarrow} \frac{6 - 6}{0} = \frac{0}{0}$$

$$\overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{6e^x}{12} \overset{\text{D.E.}}{=} \frac{6}{12} = \frac{1}{2}$$

as we found in part (a).

Exercise 9

(12 pt) Evaluate each indefinite integral. That is, find the most-general antiderivative of each integrand.

(a) (4 pt) $\int e^x + 2\sin x \, dx$

$$= \int e^x \, dx + 2\int \sin x \, dx$$

$$= e^x + C_1 + 2(-\cos x + C_2)$$

$$= e^x - 2\cos x + \underbrace{C_1 + 2C_2}_{\text{call it } C}$$

$$= e^x - 2\cos x + C$$

Check : $\frac{d}{dx}\left[e^x - 2\cos x + C\right] = e^x - 2(-\sin x) + 0$

$$= e^x + 2\sin x \quad \checkmark$$

(b) (4 pt) $\int \frac{x^4 - 1}{x^2} \, dx$

$$= \int \frac{x^4}{x^2} - \frac{1}{x^2} \, dx$$

$$= \int x^2 - x^{-2} \, dx$$

$$= \int x^2 \, dx - \int x^{-2} \, dx$$

$$= \frac{1}{3}x^3 - \frac{1}{-1}x^{-1} + C$$

$$= \frac{1}{3}x^3 + \frac{1}{x} + C$$

Check :

$\frac{d}{dx}\left[\frac{1}{3}x^3 + \frac{1}{x} + C\right]$

$$= \frac{1}{3}\cdot 3x^2 + (-1)x^{-2} + 0$$

$$= x^2 - x^{-2}$$

$$= x^2 - \frac{1}{x^2}$$

$$= \frac{x^4 - 1}{x^2} \quad \checkmark$$

(c) (4 pt) $\int (e^x + e^{-x})(e^x - e^{-x}) \, dx$

$$= \int (e^x)^2 + e^x(-e^{-x}) + e^{-x}e^x - (e^{-x})^2 \, dx$$

$$= \int e^{2x} - e^0 + e^0 - e^{-2x} \, dx$$

$$= \int e^{2x} - e^{-2x} \, dx$$

$$= \int e^{2x} \, dx - \int e^{-2x} \, dx$$

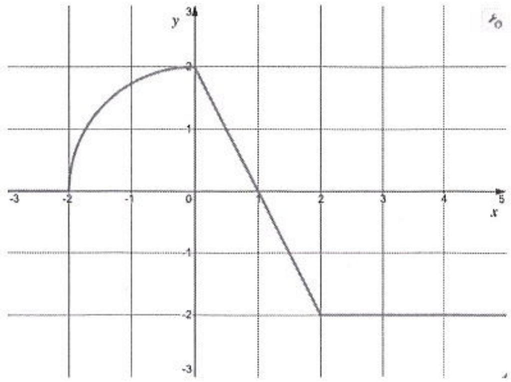
$$= \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} + C$$

Check :

$\frac{d}{dx}\left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} + C\right]$

$$= \frac{1}{2}\,2e^{2x} + \frac{1}{2}(-2)e^{-2x} + 0$$

$$= e^{2x} - e^{-2x} \quad \checkmark$$



Exercise 10

(12 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the piecewise function graphed above and given by

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant -2 \\ \sqrt{4 - x^2} & \text{if } -2 \leqslant x \leqslant 0 \\ 2 - 2x & \text{if } 0 \leqslant x \leqslant 2 \\ -2 & \text{if } x \geqslant 2 \end{cases}$$

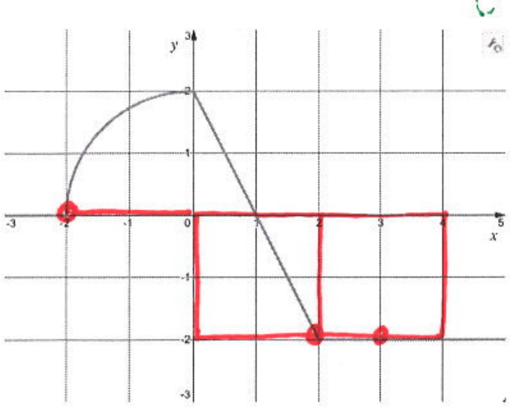
(a) (2 pt) Use finite geometry to show that $\int_{-2}^{4} f(x)\,dx = \pi - 4$.

$$\int_{-2}^{4} f(x)\,dx = \underbrace{\int_{-2}^{0} f(x)\,dx}_{\text{Quarter circle}} + \underbrace{\int_{0}^{1} f(x)\,dx}_{\text{Triangle}} + \underbrace{\int_{1}^{2} f(x)\,dx}_{\text{Triangle}} + \underbrace{\int_{2}^{4} f(x)\,dx}_{\text{Rectangle}}$$

$$= \tfrac{1}{4}\pi(2)^2 + \tfrac{1}{2}(1)(2) + \tfrac{1}{2}(1)(-2) + (2)(-2)$$

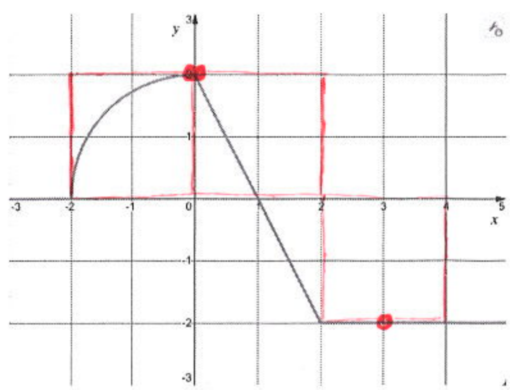
$$= \pi + 1 - 1 - 4 = \pi - 4$$

(b) (4 pt) On separate graphs below, draw a lower sum L_3 and an upper sum U_3, each with three subintervals of width 2, to estimate $\int_{-2}^{4} f(x)\,dx$. Compute the values of L_3 and U_3.



Lower sum (L_3)

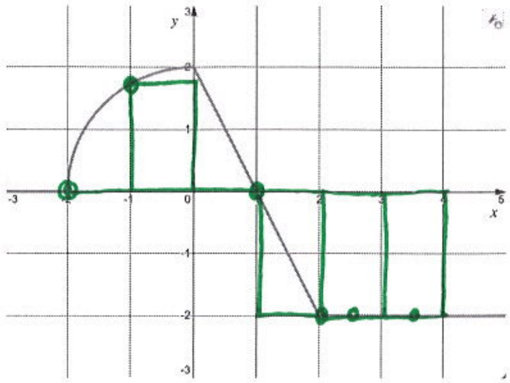
$L_3 = 0 + (-4) + (-4) = -8$



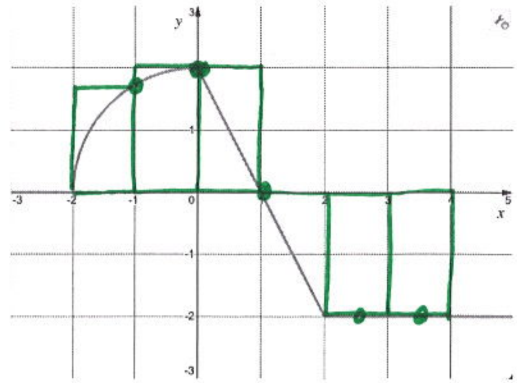
Upper sum (U_3)

$U_3 = 4 + 4 + (-4) = 4$

(c) (4 pt) On separate graphs below, draw a lower sum L_6 and an upper sum U_6, each with six subintervals of width 1, to estimate $\int_{-2}^{4} f(x)\,dx$. Compute the values of L_6 and U_6. You may leave your answer in terms of $\sqrt{3} \approx 1.7$.



Lower sum (L_6)



Upper sum (U_6)

$L_6 = 0 + \sqrt{3} + 0 + (-2) + (-2) + (-2)$
$= -6 + \sqrt{3} \approx -4.3$

$U_6 = \sqrt{3} + 2 + 2 + 0 + (-2) + (-2)$
$= \sqrt{3} \approx 1.7$

to estimate $\int_{-2}^{4} f(x)\,dx$

(d) (2 pt) You compute a lower sum L_{12} and an upper sum U_{12}, each with twelve subintervals of width $\frac{1}{2}$. You find

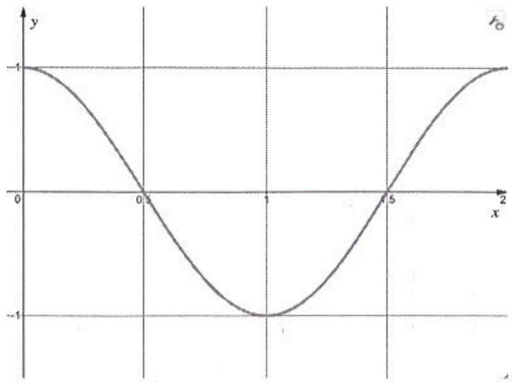
$$L_{12} = -6 \qquad\qquad U_{12} = -1$$

Explain why these cannot be the correct values of L_{12} and U_{12}.

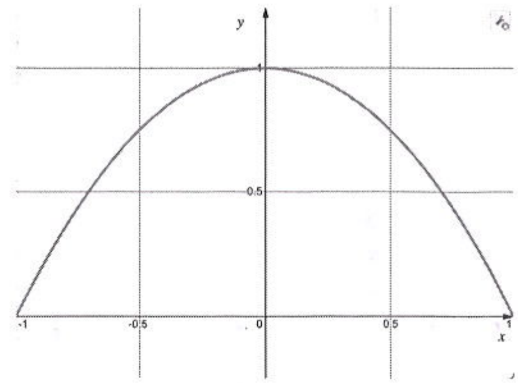
When we refine a partition — that is, when we subdivide a previous partition — the lower sum ~~stays~~ must increase (or stay the same), and the upper sum must decrease (or stay the same). Moreover, any lower sum is less than or equal to the exact area under the ~~curve~~ graph, and any upper sum is greater than or equal to the exact area under the graph. Putting all this together, we may write

$$\begin{array}{ccccccccccccc} L_3 & \le & L_6 & \le & L_{12} & \le & \int_{-2}^{4} f(x)\,dx & \le & U_{12} & \le & U_6 & \le & U_3 \\ \| & & \approx & & & & \| & & & & \approx & & \| \\ -8 & & -4.3 & & & & \pi - 4 \approx -0.9 & & & & 1.7 & & 4 \end{array}$$

Because $L_{12} \geq L_6 \approx -4.3$, we cannot have $L_{12} = -6$.
Because $U_{12} \geq \int_{-2}^{4} f(x)\,dx \approx -0.9$, we cannot have $U_{12} = -1$.



Graph of $g(x)$ for parts (a)–(b).



Graph of $F(x)$ for parts (c)–(e).

Figure 1: Graphs for Exercise 11.

Exercise 11

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

For parts (a)–(b), let $g : [0, 2] \to \mathbf{R}$ be the function given by $g(x) = \cos(\pi x)$, graphed in Figure 1.

(a) (2 pt) There are exactly three values b, with $0 \leqslant b \leqslant 2$, such that $\int_0^b g(x)\, dx = 0$.

true false

(b) (2 pt) There are no values b, with $0 \leqslant b \leqslant 2$, such that $\int_0^b g(x)\, dx < 0$.

true false

For parts (c)–(e), let $f : [-1, 1] \to \mathbf{R}$ be a continuous function, and let $F : [-1, 1] \to \mathbf{R}$ be the cumulative signed area function graphed in Figure 1, given by

$$F(x) = \int_{-1}^{x} f(t)\, dt \quad \overset{\text{FTC I.0}}{\Rightarrow} \quad F'(x) = \frac{d}{dx}\int_{-1}^{x} f(t)\,dt = f(x)$$

$$\Rightarrow \quad F''(x) = f'(x)$$

(c) (2 pt) For all x in the interval $[-1, 1]$, $f(x) < 0$.

true false

(d) (2 pt) For all x in the interval $[-1, 1]$, $f'(x) < 0$.

true false

(e) (2 pt) The average value of f on the interval $[-1, 0]$ equals $\frac{1}{2}$.

true false

$$\text{Avg Val}(f, [-1,0]) = \frac{1}{0-(-1)}\int_{-1}^{0} f(t)\, dt = \frac{1}{1} F(0) = 1 \cdot 1 = 1$$

Exercise 12

(8 pt) Evaluate each indefinite integral. Clearly communicate your approach.

(a) (4 pt) $\int 2t\left(\sin(t^2+1)\right)^4 \cos(t^2+1)\, dt$

We apply the change of variables (substitution)

$$u = \sin(t^2+1) \quad \Rightarrow \quad du = \cos(t^2+1)\cdot 2t\, dt$$

Then the original integral becomes

$$= \int u^4\, du = \frac{1}{5}u^5 + C$$
$$= \frac{1}{5}\left(\sin(t^2+1)\right)^5 + C$$

(b) (4 pt) $\int x^2 e^{-2x}\, dx$

Notation imprecise

We apply integration by parts, with

$$f(x) = x^2 \qquad g(x) = -\frac{1}{2}e^{-2x}$$
$$\Downarrow \qquad \Uparrow$$
$$f'(x) = 2x\, dx \qquad g'(x) = e^{-2x}\, dx$$

Thus

$$\int x^2 e^{-2x}\, dx = x^2\left(-\frac{1}{2}e^{-2x}\right) - \int -\frac{1}{2}e^{-2x}\, 2x\, dx$$
$$= -\frac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\, dx \qquad (12.1)$$

To evaluate this new integral, we do integration by parts again, with

$$f(x) = x \qquad g(x) = -\frac{1}{2}e^{-2x}$$
$$\Downarrow \qquad \Uparrow$$
$$f'(x) = 1\cdot dx \qquad g'(x) = e^{-2x}\, dx$$

Thus

$$\int x e^{-2x}\, dx = x\left(-\frac{1}{2}e^{-2x}\right) - \int -\frac{1}{2}e^{-2x}\, 1\, dx$$
$$= -\frac{1}{2}x e^{-2x} + \frac{1}{2}\int e^{-2x}\, dx$$
$$= -\frac{1}{2}x e^{-2x} + \frac{1}{2}\left(-\frac{1}{2}\right)e^{-2x} + C$$
$$= -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C$$

Substituting this result into (12.1), we conclude

$$\int x^2 e^{-2x}\, dx = -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C = -\frac{1}{4}e^{-2x}(2x^2+2x+1) + C$$

Exercise 13

to show the result

(8 pt) Evaluate each definite integral. Clearly communicate your approach.

(a) (4 pt) $\int_{-\frac{1}{2}}^{\frac{1}{2}} (1-2x)^3 \, dx$ $= 2$

We could multiply out the cube and integrate the resulting polynomial, but it will (probably) be faster and cleaner to do a change of variables:

$$u = 1-2x \quad \Rightarrow \quad du = -2\,dx \quad \Leftrightarrow \quad dx = -\tfrac{1}{2}\,du$$

Thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (1-2x)^3 dx = \int_{x=-\frac{1}{2}}^{x=\frac{1}{2}} u^3 \left(-\tfrac{1}{2}\,du\right) = -\frac{1}{2}\int_{x=-\frac{1}{2}}^{x=\frac{1}{2}} u^3\,du$$

$$= -\frac{1}{2}\left[\frac{1}{4}u^4\right]_{x=-\frac{1}{2}}^{x=\frac{1}{2}} = -\frac{1}{8}\left[(1-2x)^4\right]_{x=-\frac{1}{2}}^{x=\frac{1}{2}}$$

$$= -\frac{1}{8}\left[\left(1-2\left(\tfrac{1}{2}\right)\right)^4 - \left(1-2\left(-\tfrac{1}{2}\right)\right)^4\right] = -\frac{1}{8}\left[0-(2)^4\right]$$

$$= -\frac{1}{8}\left[-16\right] = 2$$

(b) (4 pt) $\int_0^1 (x^2+1)e^{x^3+3x} \, dx$ $= \dfrac{e^4-1}{3}$

We again use a change of variables:

$$u = x^3+3x \quad \Rightarrow \quad du = (3x^2+3)dx \quad \Rightarrow \quad \tfrac{1}{3}\,du = (x^2+1)\,dx$$

Thus

$$\int_0^1 (x^2+1)\,e^{x^3+3x}\,dx = \int_{x=0}^{x=1} e^u\left(\tfrac{1}{3}\,du\right)$$

$$= \frac{1}{3}\int_{x=0}^{x=1} e^u\,du = \frac{1}{3}\left[e^u\right]_{x=0}^{x=1} = \frac{1}{3}\left[e^{x^3+3x}\right]_{x=0}^{x=1}$$

$$= \frac{1}{3}\left[\left(e^{(1)^3+3(1)}\right) - \left(e^{(0)^3+3(0)}\right)\right] = \frac{1}{3}\left[e^4 - e^0\right]$$

$$= \frac{1}{3}\left(e^4-1\right)$$

Exercise 14

(4 pt) Use the fundamental theorem of calculus to compute the derivative. Assume $x \geqslant 0$.

$$\frac{d}{dx}\int_{4x^2}^{9x^2} e^{\sqrt{t}}\,dt$$

Let $F(t)$ be an antiderivative of $f(t) = e^{\sqrt{t}}$. Then by the fundamental theorem of calculus,

$$\int_{4x^2}^{9x^2} e^{\sqrt{t}}\,dt = F(9x^2) - F(4x^2)$$

so

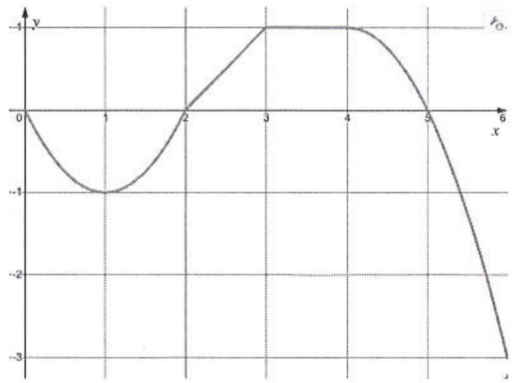
$$\frac{d}{dx}\int_{4x^2}^{9x^2} e^{\sqrt{t}}\,dt = \frac{d}{dx}\left[F(9x^2) - F(4x^2)\right]$$

$$= F'(9x^2)\cdot(18x) - F'(4x^2)\cdot(8x)$$

$$= f(9x^2)\cdot(18x) - f(4x^2)\cdot(8x)$$

$$= e^{\sqrt{9x^2}}\cdot 18x - e^{\sqrt{4x^2}}\cdot 8x$$

$$= 18x\,e^{3x} - 8x\,e^{2x}$$



Exercise 15

(10 pt) Let $f : [0, 6] \to \mathbf{R}$ be a piecewise function. A graph of $F(x) = \int_0^x f(t)\, dt$ is shown above.

(a) (4 pt) On which intervals is f positive? negative? equal to zero?

By the fundamental theorem of calculus (version 1.0),

$$F'(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \int_0^x f(t)\, dt = f(x) \qquad (15.1)$$

That is, the value of f at x equals the slope of the tangent line to the graph of F at x. Thus, from the graph of F, we see

- $f(x) > 0$ on $(1, 2) \cup (2, 3)$
- $f(x) < 0$ on $(0, 1) \cup (4, 6)$
- $f(x) = 0$ on $(3, 4) \cup \{1\}$

(b) (4 pt) On which intervals is f increasing? decreasing? constant?

Differentiating the relation in (15.1), we get

$$F''(x) = f'(x)$$

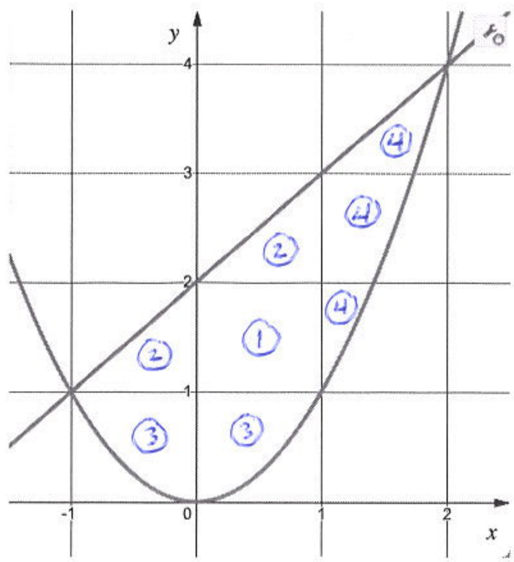
That is, the slope of the graph of f at x (i.e. whether f is increasing, decreasing, or constant at x) equals the concavity of the graph of F at x. Thus, from the graph of F, we see

- f increasing (i.e. $f'(x) > 0$) on $(0, 2)$
- f decreasing (i.e. $f'(x) < 0$) on $(4, 6)$
- f constant (i.e. $f'(x) = 0$) on $(2, 3) \cup (3, 4)$

(c) (2 pt) Find the average value of f on the interval $[0, 6]$.

By definition,

$$\begin{aligned} \text{AvgVal}(f, [0,6]) &= \frac{1}{6-0} \int_0^6 f(t)\, dt \\ &= \frac{1}{6} F(6) \\ &= \frac{1}{6}(-3) \\ &= -\frac{1}{2} \end{aligned}$$



Exercise 16

(10 pt) Consider the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = x^2 \qquad\qquad g(x) = x + 2$$

respectively. Graphs of f and g appear above.

(a) (2 pt) Using the graphs, write the two points (x, y) of intersection of f and g. Approximate the area between the graphs of f and g. Briefly explain the reasoning behind your approximation.

The graphs of f and g appear to intersect at
$(-1, 1)$ and $(2, 4)$

This can be verified by ~~substituting evaluating f(x) and g(x) at x = -1 and x = 2 and~~ showing ~~we~~ that $f(-1) = g(-1) = 1$ and $f(2) = g(2) = 4$.

"Counting boxes" between the graphs of f and g, we approximate the area to be ≈ 4, probably slightly greater than 4.

(b) (4 pt) Write and evaluate a single definite integral to find the area between the graphs of f and g.

We compute (note that the graph of the line g lies above the graph of the parabola f on the interval $[-1, 2]$):

$$\text{Area between } f \text{ and } g = \int_{-1}^{2} g(x) - f(x)\, dx$$

$$= \int_{-1}^{2} (x+2) - x^2 \, dx$$

$$= \int_{-1}^{2} -x^2 + x + 2 \, dx$$

$$= \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x \right]_{x=-1}^{x=2}$$

$$= \left(-\tfrac{1}{3}(2)^3 + \tfrac{1}{2}(2)^2 + 2(2)\right) - \left(-\tfrac{1}{3}(-1)^3 + \tfrac{1}{2}(-1)^2 + 2(-1)\right)$$

$$= \left(-\tfrac{8}{3} + 2 + 4\right) - \left(\tfrac{1}{3} + \tfrac{1}{2} - 2\right)$$

$$= 3\tfrac{1}{3} - \left(-1\tfrac{1}{6}\right) = \tfrac{10}{3} + \tfrac{7}{6}$$

$$= \frac{20+7}{6} = \frac{27}{6} = 4.5$$

Note that ^this is close to^ our estimate in part (a). ~~is reasonable~~.

(c) (4 pt) If we "tilt our heads to the right ninety degrees" and view the graphs of f and g as having input variable y instead of x—that is, solving $y = f(x)$ and $y = g(x)$ for x—we get

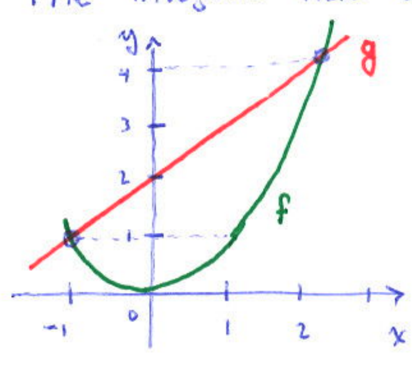
$$F(y) = \pm\sqrt{y} \qquad\qquad G(y) = y - 2$$

Using the graphs at the beginning of this exercise, explain how the following integrals compute the same area between the graphs that you computed in part (b):

7

$$\int_0^1 \sqrt{y} - (-\sqrt{y})\, dy + \int_1^{\cancel{3}\,4} \sqrt{y} - (y-2)\, dy$$

The integrals here are computing the area between the graphs of f and g using horizontal strips.



From $y=0$ to $y=1$ these strips start at the lower part of f ($F(y) = -\sqrt{y}$) and end at the upper part of f ($F(y) = \sqrt{y}$).

So the heights of these strips are

$\sqrt{y} - (-\sqrt{y})$

and the widths are dy.

From $y=1$ to $y=4$ the strips start at g ($G(y) = y-2$) and end at the upper part of f ($F(y) = \sqrt{y}$), so the heights of these strips are

$$\sqrt{y} - (y-2)$$

and the widths are dy.