Math 212
HalfExam 01

F 09 Sep 2016

Your name: ______________________________

Exam instructions

Number of exercises	: 3
Permitted time	: 60 minutes
Permitted resources	: None

Instructor's note:

- Manage your time deliberately.
- An effort has been made to make the exercises as clear and unambiguous as possible. If the statement of an exercise seems unclear, briefly (one sentence) write your understanding of the exercise, then proceed.

Exercise	Total	(a)	(b)	(c)
1	/20			
2	/10			
3	/20	/5	/10	/5
Total	/50			

Exercise 1

(20 pt) Find an equation for the plane containing both (i) the point of intersection of the lines

$$L_1 : x = \frac{y+1}{3} = 1 - z, \qquad\qquad L_2 : 3 - x = \frac{y+2}{2} = \frac{z+4}{2}$$

and (ii) the line of intersection of the planes

$$H_1 : x + y + z = 6, \qquad\qquad H_2 : 2y - x = 0.$$

Hint: Break it down. First find the coordinates of the point P_0 of intersection of the lines L_1 and L_2. Also find an equation for the line L of intersection of the planes H_1 and H_2. Then put it all together to get an equation for the plane containing P_0 and L. Think geometrically!

1.1 Solution

We know an easy way to write an equation for a plane in $\mathbf{R}^3$:[1] Given a point $P_0 = (x_0, y_0, z_0)$ in the plane and a normal vector $\mathbf{n} = (n_1, n_2, n_3)$ to the plane, any point $P = (x, y, z)$ in the plane satisfies

$$0 = \mathbf{n} \cdot \overrightarrow{P_0P} = (n_1, n_2, n_3) \cdot (x - x_0, y - y_0, z - z_0)\,. \tag{1.1}$$

Let's use the given information to find a P_0 and an $\mathbf{n}$.

1.1.1 Point P_0 of intersection of L_1 and L_2

By definition, the point of intersection of the lines L_1 and L_2 is a point $P_0 = (x_0, y_0, z_0)$ whose coordinates satisfy the equation for both L_1 and L_2. We can find this point in at least two ways.

Method 1: Using the symmetric equations The key idea is that *the same point* $P_0 = (x_0, y_0, z_0)$ satisfies both sets of symmetric equations. Thus we can equate two expressions for the same variable (one expression from each set of symmetric equations) and then solve. For the variables x, y, we have

$$\frac{y_0+1}{3} = x_0 = 3 - \frac{y_0+2}{2} \qquad \Leftrightarrow \qquad 2y_0 + 2 = 18 - 3y_0 - 6 \qquad \Leftrightarrow \qquad y_0 = 2.$$

We can now use any of the equalities to find the remaining coordinates x_0, z_0. From the symmetric equations for L_1, for example,

$$x_0 = \frac{y_0+1}{3} = 1, \qquad\qquad z_0 = 1 - x_0 = 0.$$

We conclude that $P_0 = (1, 2, 0)$.

[1] This approach generalizes nicely to hyperplanes in $\mathbf{R}^n$. Do you see how?

Method 2: Using the parametric equations Recall that the symmetric equations of a line are obtained by eliminating the parameter, call it t, Thus all the expressions in the symmetric equations equal t. Solving for x, y, z as functions of t, we find

$$\mathbf{r}_1(t) = (t, -1+3t, 1-t)\,, \qquad \mathbf{r}_2(t) = (3-t, -2+2t, -4+2t)\,,$$

where $\mathbf{r}_i$ is a vector equation for the line L_i, for $i \in \{1,2\}$. The key idea is that, to say that L_1 and L_2 have a point of intersection (i.e. a point in common) is equivalent to saying that there exist parameter values t_1 and t_2 such that

$$\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2).$$

Writing out what this vector equality means in terms of components, we obtain a system of three equations (one for each component) in two unknowns (the parameters t_1, t_2):

$$t_1 = 3 - t_2, \qquad -1+3t_1 = -2+2t_2, \qquad 1-t_1 = -4+2t_2. \tag{1.2}$$

Substituting the first equation into the second, we find that if a solution to this system exists (i.e. if L_1 and L_2 have a point of intersection), then

$$-1+3(3-t_2) = -2+2t_2 \qquad \Leftrightarrow \qquad 10 = 5t_2 \qquad \Leftrightarrow \qquad t_2 = 2.$$

This is the parameter value for L_2 corresponding to the candidate intersection point. Substituting this into the first equation, we find the corresponding parameter value for L_1:

$$t_1 = 3 - t_2 = 1.$$

To verify that these parameter values indeed correspond to a point of intersection, we need to check that they satisfy the remaining inequalities — in this case, just the third equality (corresponding to the z-coordinate):

$$1 - t_1 = 0 = -4 + 2t_2.$$

Thus all three equalities in (1.2) are satisfied when $t_1 = 1$ and $t_2 = 2$, i.e. $\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2)$, and we have a point of intersection. Its coordinates are found by evaluating $\mathbf{r}_i(t_i)$ for either value of i:

$$P_0 = \mathbf{r}_1(t_1) = (1, 2, 0).$$

1.1.2 Line L of intersection of H_1 and H_2

Close your eyes and view the geometric picture of two different planes intersecting. You will see that they intersect in a line. Algebraically, we can describe that line by giving a point on the line and a direction vector.

Any point $P_0' = (x_0', y_0', z_0')$ on the line of intersection of H_1 and H_2 will do. Analogous to our discussion for the point of intersection of two lines, a point on the line of intersection of two planes satisfies the equation describing both planes. This gives us two equations (one for each plane) in three unknowns (x_0', y_0', z_0'). We can choose an arbitrary value for one of these variables,[2] say $z_0 = 0$. This reduces the system of equations to

$$x_0 + y_0 = 6, \qquad 2y_0 - x_0 = 0.$$

[2] *(discuss subtlety, e.g., a line whose z values are constant)*

Solving the second equation for x_0 and substituting the result into the first, we have

$$(2y_0) + y_0 = 6 \qquad \Leftrightarrow \qquad y_0 = 2.$$

Hence $x_0 = 2y_0 = 4$, so

$$P_0' = (4, 2, 0).$$

A direction vector of the line of intersection is even easier to find. Again, think geometrically: The line of intersection lies in both planes, so in particular, a (nonzero) direction vector of that line lies in both planes. Because a normal vector to a plane is perpendicular to all vectors in the plane, we conclude that the direction vector of the line of intersection of two planes must be normal to *both* normal vectors. And we know a way, if we are given two nonparallel vectors, to construct a vector normal (a.k.a. perpendicular) to both: take their cross product.

The components of normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ of the planes H_1 and H_2, respectively, can be read off from the corresponding coefficients of the variables x, y, z in their defining equations:

$$\mathbf{n}_1 = (1, 1, 1), \qquad \mathbf{n}_2 = (-1, 2, 0).$$

By the preceding discussion, a direction vector $\mathbf{v}$ for the line of intersection is

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix} = (-2, -1, 3).$$

1.1.3 Putting it all together

At this point we have (i) the point

$$P_0 = (1, 2, 0)$$

of intersection of the lines L_1 and L_2 and (ii) an equation

$$\mathbf{r}(t) = (4, 2, 0) + t(-2, -1, 3)$$

for the line L of intersection of the planes H_1 and H_2. We want an equation for the plane containing both P_0 and L. Note that P_0 does not lie in L.[3] Thus the plane we are asked to find indeed exists.

We already have an explicit point in this plane (in fact, two: P_0 and P_0'), so all that remains is to find a normal vector to the plane. We also have an explicit vector in this plane, namely, the direction vector $\mathbf{v} = (-2, -1, 3)$ of the line L. If we can find a second vector in the plane that is not parallel to $\mathbf{v}$, then we can take a cross product and be done. Think geometrically about our situation: We have a line L in the desired plane, and a point P_0 in the desired plane, with P_0 not in L. If we take any point P_0' in L, and form the vector $\overrightarrow{P_0'P_0}$, then this vector cannot be parallel to the

[3]You can check this by substituting the coordinates of P_0 for $\mathbf{r}(t)$ and trying to solve the resulting vector equation

$$(1, 2, 0) = (2, 1, 0) + t(-2, -1, 3)$$

for t. (Keep in mind that this vector equation can be viewed as a system of three equations (one for each component) in one unknown (the parameter t).) You will find that there is no solution.

direction vector $\mathbf{v}$ of the line L (if it were, then P_0 would be in L). This gives us our desired second vector:

$$\overrightarrow{P_0'P_0} = (-3, 0, 0).$$

Hence a normal vector $\mathbf{n}$ to the plane H containing P_0 and L is

$$\mathbf{n} = \overrightarrow{P_0'P_0} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 0 & 0 \\ -2 & -1 & 3 \end{pmatrix} = (0, 9, 3).$$

Any nonzero scalar multiple of a normal vector is also a normal vector (do you see why, geometrically?), so we can also take the vector

$$\mathbf{n}' = \frac{1}{3}\mathbf{n} = (0, 3, 1)$$

as a normal vector to the plane H.[4] Substituting $\mathbf{n}'$ and P_0 into (1.1), we conclude that an equation for the plane L containing P_0 and L is

$$0 = (0, 3, 1) \cdot (x - 1, y - 2, z - 0) = 3y - 6 + z,$$

or equivalently,

$$3y + z = 6.$$

[4]Using $\mathbf{n}'$ just makes the subsequent computation a bit simpler. Using $\mathbf{n}$ yields an equivalent equation for the plane, which we can reduce to the equation obtained from using $\mathbf{n}'$ by dividing by 3. Do you see why?

Exercise 2

(10 pt) Find the point P^* in the line

$$L : \mathbf{r}(t) = (2t + 4, -2t, t - 4)$$

that is closest to the point $Q = (12, 6, 1)$. *Hint:* Think geometrically! Use vectors.

2.1 Solution

As usual, we start by thinking geometrically, to acquire some intuition for how to approach this problem. The line L and the point Q are in $\mathbf{R}^3$, but we can restrict our attention to a plane containing L and Q.[5] That is, we can think geometrically in two dimensions. So imagine L and Q in $\mathbf{R}^2$. We want the point P^* in L that is closest to Q. Visualizing this situation, you will convince yourself that this point P^* is unique and is the intersection of L with the line through Q that is perpendicular to L *(draw it!)*.

For each point P in the line L, we can associate the vector $\vec{PQ}$ from P to Q. More precisely, to say that $\mathbf{r}(t)$ describes the line L means that any point P in L has coordinates

$$\mathbf{r}(t) = (2t + 4, -2t, t - 4)$$

for some $t \in \mathbf{R}$. Thus for each point P in L,

$$\vec{PQ} = (8 - 2t, 6 + 2t, 5 - t)\,. \tag{2.1}$$

In particular, $\vec{P^*Q}$ has this form.

The geometric intuition that we developed above now provides us with at least two ways to find the coordinates of P^*.

Using inner product In our geometric picture, we see that the vector $\vec{P^*Q}$ corresponding to the special point P^* is orthogonal to the direction vector $\mathbf{v}$ of the line L. We can read off the components of the direction vector from the corresponding coefficients of the parameter t in the equation defining $\mathbf{r}(t)$:

$$\mathbf{v} = (2, -2, 1).$$

Because $\vec{P^*Q}$ is orthogonal to $\mathbf{v}$, their inner product equals zero:

$$0 = \vec{P^*Q} \cdot \mathbf{v} = (8 - 2t, 6 + 2t, 5 - t) \cdot (2, -2, 1) = (16 - 4t) + (-12 - 4t) + (5 - t) = 9 - 9t,$$

which yields the solution

$$t = 1.$$

Thus

$$P^* = \mathbf{r}(1) = (6, -2, -3).$$

[5]If the point Q is not in the line L — as is the case here — then this plane is unique. If the point Q is in the line L, then there are infinitely many planes containing Q and L, but for the purposes of our problem, we would be done: Q itself would be point in L closest to Q.

The point Q lies in L if and only if there exists a parameter value t such that $\mathbf{r}(t) = Q$. This vector equality is equivalent to a system of three equations (one for each component) in one unknown (the parameter t). It is straightforward to show that this system of equations has no solution (do it).

Using norm In our geometric picture, we see that the vector $\vec{P^*Q}$ corresponding to the special point P^* is the shortest vector of the form (2.1). That is, the point P^* corresponds to the parameter value t that minimizes

$$\begin{aligned}\left\|\vec{PQ}\right\| &= \sqrt{(8-2t)^2+(6+2t)^2+(5-t)^2} \\ &= \sqrt{(64-32t+4t^2)+(36+24t+4t^2)+(25-10t+t^2)} \\ &= \sqrt{9t^2-18t+125} = \sqrt{9(t^2-2t+16)} \\ &= 3\sqrt{(t-1)^2+15}.\end{aligned}$$

This expression is minimized when the expression in the square root is minimized, which in turn occurs when $t = 1$.[6] Thus

$$P^* = \mathbf{r}(1) = (6, -2, -3).$$

[6]The expression in the square root describes an upward-opening parabola with vertex $(1, 15)$.

Exercise 3

(20 pt) At time $t = 0$, you espy a flying cockroach whizzing around your bedroom. Always quick on your feet, you apply your mad skillz of vector calculus to determine that the cockroach's position vector is given by

$$\mathbf{r}(t) = \left(\frac{4\sqrt{2}}{5}t^{\frac{5}{2}} - 1, t^2, \frac{2}{3}t^3 + 1\right),$$

where the component functions are measured in meters, and t is measured in seconds. At time $t = 3$, you squash the sucker.

(a) (5 pt) Find the cockroach's velocity function $\mathbf{v}(t)$. What are the units of the component functions?

3.1 Solution to part (a)

The velocity vector is the derivative of the position vector with respect to time. Remembering that derivatives of vector-valued functions are computed componentwise, we have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left(2\sqrt{2}t^{\frac{3}{2}}, 2t, 2t^2\right).$$

The component functions are measured in meters/second.

(b) (10 pt) Determine how far the cockroach travels from the time you spot it to the time you squash it.

3.2 Solution to part (b)

The distance s traveled by the cockroach is the arc length of $\mathbf{r}(t)$ from $t = 0$ to $t = 3$:

$$s = \int_0^3 \left\|\mathbf{r}'(t)\right\| \, dt. \tag{3.1}$$

The derivative $\mathbf{r}'(t)$ is precisely the velocity function that we found in part (a). Thus

$$\left\|\mathbf{r}'(t)\right\| = \sqrt{(8t^3) + (4t^2) + (4t^4)} = \sqrt{4t^2\left(t^2 + 2t + 1\right)} = \sqrt{(2t)^2(t+1)^2} = 2t(t+1),$$

where in the final equality we have used the fact that $\mathbf{r}(t)$ (and hence $\mathbf{r}'(t)$) is defined only for $t \geqslant 0$ (do you see why?). Substituting this expression into (3.1), we find

$$s = 2\int_0^3 \left(t^2 + t\right) dt = 2\left[\frac{1}{3}t^3 + \frac{1}{2}t^2\right]_{t=0}^{t=3} = 2\left[\left(9 + \frac{9}{2}\right) - 0\right] = 27.$$

That is, the cockroach travels 27 meters from the time you see it to the time you squash it.

(c) (5 pt) You suspect that the cockroach was flying to its nest. In one (!) sentence, explain how you can (use something about vectors to) locate the nest.

3.3 Solution to part (c)

If our suspicion is correct, then following the curve described by $\mathbf{r}(t)$ for $t > 3$ will lead us to the nest.