Math 212
Quiz 33

M 21 Nov 2016

Your name: ________________________________

Exercise

(2 pt) For each of the following vector fields $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$, write "Conservative" if $\mathbf{F}$ is conservative, and "Not conservative" otherwise. Justify your answer. *Hint:*

> Conservative? Liberal? Head in a whirl —
> For vector-field politics, compute the ________.

(a) (1 pt) $\mathbf{F}(x, y, z) = (e^z, 1, xe^z)$

Solution: Conservative. We compute

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^z & 1 & xe^z\end{pmatrix} \\ &= \mathbf{i}\,(0-0) - \mathbf{j}\,(e^z - e^z) + \mathbf{k}\,(0-0) \\ &= \mathbf{0},\end{aligned}$$

the zero vector field. Because the domain of $\mathbf{F}$, i.e. $\mathbf{R}^3$, is simply connected and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, it follows that $\mathbf{F}$ is conservative. (For practice, show that a potential function $f : \mathbf{R}^3 \to \mathbf{R}$ for $\mathbf{F}$ always has the form $f(x, y, z) = xe^z + y + C$ for $C \in \mathbf{R}$.)

(b) (1 pt) $\mathbf{F}(x, y, z) = (ye^{-x}, e^{-x}, 2z)$

Solution: Not conservative. We compute

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{-x} & e^{-x} & 2z\end{pmatrix} \\ &= \mathbf{i}\,(0-0) - \mathbf{j}\,(0-0) + \mathbf{k}\left(-e^{-x} - e^{-x}\right) \\ &= \left(0, 0, -2e^{-x}\right).\end{aligned}$$

Because $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$,[1] it follows that $\mathbf{F}$ is not conservative (recall that $\operatorname{curl}(\nabla f) = \mathbf{0}$).

[1]Note that $-2e^{-x} \neq 0$ on $\mathbf{R}^3$; e.g., at $(x, y, z) = (0, 0, 0)$, $-2e^{-x} = -2 \neq 0$.