Math 357
Expositional homework 06

Assigned: 2024–03–29 (F)
Due:

The goal of this homework is to strengthen our understanding of field theory, through proof and example.

Proofs

(a) Prove Proposition 13.12:[1] Let $K : K_0$ be a field extension, and let $\alpha \in K$. Then α is algebraic over K_0 if and only if $[K_0(\alpha) : K_0] < \infty$.

(b) Prove Theorem 13.25[2] on the existence of splitting fields: Let K_0 be a field, and let $f \in K_0[t]$. Then there exists an extension field $K : K_0$ such that f splits completely in $K[t]$.

(c) Let $K : K_0$ be a field extension such that $[K : K_0] < \infty$. Prove that $K : K_0$ is normal if and only if for all irreducible polynomials $f \in K_0[t]$, if there exists an $\alpha \in K$ such that $f(\alpha) = 0$, then f splits completely in $K[t]$.[3] (One may take this as the definition of a normal field extension, in which case one can prove the definition we gave in class as a proposition.)

(d) Let K_0 be a field, and let $f_1, f_2 \in K_0[t]$. Prove that the formal derivative D_t of a polynomial in $K_0[t]$ satisfies the following relations (as does the derivative operator from calculus):[4]

$$D_t(f_1 + f_2) = D_t f_1 + D_t f_2 \qquad\qquad D_t(f_1 f_2) = (D_t f_1) \cdot f_2 + f_1 \cdot (D_t f_2)$$

Examples

(e) Determine the degree over $\mathbf{Q}$ of $2 + \sqrt{3}$ and $1 + \sqrt[3]{2} + \sqrt[3]{4}$.[5]

(f) Let $K : K_0$ be a field extension of finite degree n, and let $\alpha \in K$.[6]

[1] See DF3e, p 521.
[2] See DF3e, p 536. If you are up for it, then you can also prove that any two splitting fields for f are isomorphic; see Corollary 13.28, p 542. This will take a little work, but the theory and proof are both accessible and rewarding.
[3] See DF3e, Exercise 13.4.5, p 545.
[4] See DF3e, Exercise 13.5.1, p 551.
[5] See DF3e, Exercise 13.2.4, p 530.
[6] See DF3e, Exercises 13.2.19(a) and 20, p 531.

(i) Prove that the map

$$\begin{aligned} T_\alpha : K &\to K \\ \beta &\mapsto \alpha\beta \end{aligned}$$

which is (left) multiplication by α, is a K_0-linear transformation of K.

Let $n \in \mathbf{Z}_{>0}$, let M be an $n \times n$ matrix, let I be the $n \times n$ identity matrix, and let t be an indeterminate. The **characteristic polynomial** of M is $\det(tI - A) = (-1)^n \det(A - tI)$.

(ii) Let $\mathcal{B}$ be a K_0-basis of K, and let $M_{\mathcal{B}}(T_\alpha)$ be the matrix of T_α with respect to $\mathcal{B}$. Prove that α is a zero of the characteristic polynomial of $M_{\mathcal{B}}(T_\alpha)$.

(iii) Use this technique to find monic polynomials in $\mathbf{Q}[t]$ of degree 3 satisfied by $\sqrt[3]{2}$ and by $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

(g) Let $p \in \mathbf{Z}_{>0}$ be prime, let $\mathbf{F}_p = \mathbf{Z}/(p)$ (a finite field of order p), let $a \in \mathbf{F}_p$ be nonzero, and let $f = t^p - t + a \in \mathbf{F}_p[t]$. Prove that f is irreducible in $\mathbf{F}_p[t]$ and separable.[7]

[7] See DF3e, Exercise 13.5.5, p 551.