Math 112
LQuiz 04

2022–01–20 (R)

Your name: ______________________________

Exercise

Consider the function

$$f : [3, +\infty) \to \mathbf{R} \qquad \text{given by} \qquad f(x) = \sqrt{x^2 - 9}$$

where as usual $\mathbf{R}$ denotes "all real numbers". Find the derivative function f'. Be sure to specify its domain and codomain, in addition to the rule(s) of assignment.

Solution: The rule of assignment for f involves a function within a function (namely, x^2-9 within the square root function), so when differentiating f, we must use the chain rule. We compute

$$f'(x) = \frac{1}{2}\left(x^2 - 9\right)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 - 9}}$$

This rule of assignment is invalid for x if and only if one of the following conditions holds:

1. The expression inside the square root is negative, i.e. $x^2 - 9 < 0$. This happens if and only if $x^2 < 9$, which happens if and only if $-3 < x < 3$, i.e. x is in the open interval $(-3, 3)$.

2. The denominator is zero. This happens if and only if $x = \pm 3$.

Combining these results, we conclude that the rule of assignment is invalid for all x in the closed interval $[-3, 3]$. Equivalently, the rule of assignment is valid for all x outside this interval, i.e. in $(-\infty, -3) \cup (3, +\infty)$.

However, this last expression is not the domain of f'. Why not? We must also consider the domain of the original function f, which is $[3, +\infty)$. The domain of f' is the set of points in the domain of the original function f for which the rule of assignment for f' is valid, i.e.

$$[3, +\infty) \cap ((-\infty, -3) \cup (3, +\infty)) = (3, +\infty)$$

where the symbol "$\cap$" means "intersection" (all the points in common). The codomain of f' we may take to be all real numbers. Thus we conclude that the derivative function for the given f is the function

$$f' : (3, +\infty) \to \mathbf{R} \qquad \text{given by} \qquad f(x) = \frac{x}{\sqrt{x^2 - 9}}$$