

Math 211

Exam Oral

F 09 Aug 2019

Your name : _____

Exercise 1: Diagonalization

Are all square matrices diagonalizable? Can we give an example of a 2×2 matrix that is not diagonalizable? Can we prove that this matrix is not diagonalizable?

Exercise 2 : Linearization

Consider the general nonhomogeneous autonomous (i.e. no t dependence) 1st-order nonlinear system of ODEs

$$\begin{aligned}x_1'(t) &= f_1(x_1, x_2) \\ x_2'(t) &= f_2(x_1, x_2).\end{aligned}$$

Write the 1st-order linear system that provides the best linear approximation to the nonlinear system at an equilibrium point (i.e. jacobian matrix).

2.1 What is linearization doing?

(BDH p 457) Consider the nonlinear system

$$\begin{aligned}x_1' &= e^{x_2} - 1 \\ x_2' &= -x_2 - \sin x_1.\end{aligned}$$

Show that $(0,0)$ is an equilibrium. Jacobian matrix

$$J(x_1, x_2) = \begin{bmatrix} 0 & e^{x_2} \\ -\cos x_1 & -1 \end{bmatrix}$$

At $(0,0)$:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Gives linear system

$$\mathbf{x}' = J(0,0)\mathbf{x} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$

Another approach : Write everything as a power series centered at $(0,0)$:

$$\begin{aligned}x_1' &= \left(1 + x_2 + \frac{1}{2!}x_2^2 + \dots\right) - 1 \\ x_2' &= -x_2 - \left(x_1 - \frac{1}{3!}x_1^3 + \dots\right)\end{aligned}$$

Why do polynomials stay the same? (Polynomials are already (finite) power series. However, if we expand around a point other than $(0,0)$, they will change shape, not degree.) This simplifies to

$$\begin{aligned}x_1' &= x_2 + \frac{1}{2!}x_2^2 + \dots \\ x_2' &= -x_2 - x_1 + \frac{1}{3!}x_1^3 - \dots\end{aligned}$$

When (x_1, x_2) is close (but not equal) to $(0,0)$, the higher-order terms are much smaller than the linear ones. So the system behaves like the linear part:

$$\begin{aligned}x_1' &= x_2 \\ x_2' &= -x_2 - x_1.\end{aligned}$$

Same as we get with jacobian matrix.

(FHMW p 433) Let (a_1, a_2) be an equilibrium solution. What does this mean? (By definition of equilibrium, $x'_i(a_1, a_2) = 0$, for $i = 1, 2$.) Write both functions x_1, x_2 as power series centered at (a_1, a_2) . Similar to single-variable Taylor series, just now multiple variables. General form

$$f_i(x_1, x_2) = f_i(a_1, a_2) + (x_1 - a_1) \frac{\partial f_i}{\partial x_1}(a_1, a_2) + (x_2 - a_2) \frac{\partial f_i}{\partial x_2}(a_1, a_2) + R_i(x_1, x_2),$$

where the remainder term $R_i(x_1, x_2)$ is second-order, hence relatively small sufficiently close to the equilibrium. (a_1, a_2) an equilibrium implies the first term $f_i(a_1, a_2) = 0$, in both expansions. Staying sufficiently close to the equilibrium point (a_1, a_2) makes the remainder term $R_i(x_1, x_2)$ arbitrarily small compared to the linear terms.

2.2 Stability

(BDH p 464) Consider the nonlinear 1st-order system

$$\begin{aligned} x'_1(t) &= x_2 + (x_1^2 + x_2^2)x_1 \\ x'_2(t) &= -x_1 + (x_1^2 + x_2^2)x_2. \end{aligned}$$

Show that this system has an equilibrium at the origin, that the jacobian matrix $J(0,0)$ has eigenvalues $\pm i$, and thus this system describes a (stable) center, with trajectories circles centered at the origin. Claim : In the original nonlinear system, the origin is unstable, a spiral source. Why? View the system as describing a vector field, and decompose the vector field into two parts, the linear and nonlinear parts:

$$\mathbf{V}_1(x_1, x_2) = (-x_2, x_1), \quad \mathbf{V}_2(x_1, x_2) = ((x_1^2 + x_2^2)x_1, (x_1^2 + x_2^2)x_2).$$

\mathbf{V}_1 corresponds to the linear system — trajectories are circles centered at the origin. \mathbf{V}_2 always points away from the origin. Thus $\mathbf{V}_1 + \mathbf{V}_2$ always has a positive radial component (explain ; sketch vector field). This will cause trajectories to spiral away from the equilibrium. Thus the origin is a spiral source (unstable), not a center (stable).

Exercise 3 : Linearity

Consider the nonhomogeneous 1st-order 2×2 linear system of ODEs

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -10e^{3t} + \sin(2t) \\ -\cos(2t) \end{bmatrix}. \quad (1)$$

Find the general solution to the corresponding homogeneous equation. Describe a strategy to find a particular solution. (If using linearity, decompose nonhomogeneous “ \mathbf{f} ” matrix into two parts.) What is the idea behind this decomposition? What can we do with the parameter α ? (We can treat α as a scalar in the linear combination.) If using undetermined coefficients, give an augmented matrix for a system of equations that will give us the coefficients. Call the particular solutions to the subproblems $\mathbf{x}_{1,p}$ and $\mathbf{x}_{2,p}$. How to find the general solution to the nonhomogeneous linear system

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5e^{3t} - 3\sin(2t) \\ 3\cos(2t) \end{bmatrix}. \quad (2)$$

Exercise 4 : Phase planes

Sketch a particular vector field (phase plane). What is being represented? (How component functions move in phase space over time.) Can trajectories cross? Yes. (See FHMW p 413.) Illustrative example : Nonhomogeneous system. E.g.,

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t-2 \\ 4t-1 \end{bmatrix}. \quad (3)$$

In this case, direction of vector field changes with time. So trajectories passing through the same point (x_1, x_2) at different times t may point in different directions. Give explicit example : Consider the point $(x_1, x_2) = (0, 0)$:

$$t = 0 \rightarrow \mathbf{x}' = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad t = 4 \rightarrow \mathbf{x}' = \begin{bmatrix} 2 \\ 15 \end{bmatrix}.$$

Make sense of following remarks (FHMW p 354):

- For a linear system of differential equations in \mathbf{R}^n , trajectories do not cross in (t, x_1, \dots, x_n) space.
- For an autonomous linear system in \mathbf{R}^n , trajectories do not cross in (x_1, \dots, x_n) space, either.

Exercise 5 : Laplace transform

State definition of laplace transform of a “suitable” function f . Claim : We can deduce transform–inverse-transform pairs for 0 , 1 , e^{at} , and $\sin(bt)$ and $\cos(bt)$ from just

$$\begin{aligned}\mathcal{L}\{e^{at}\cos(bt)\} &= \frac{s-a}{(s-a)^2+b^2}, & s > a; \\ \mathcal{L}\{e^{at}\sin(bt)\} &= \frac{b}{(s-a)^2+b^2}, & s > a.\end{aligned}$$

Work through Quiz 23Y : Solve the following IVP:

$$y'' + y' + y = 1, \quad y(0) = 0, \quad y'(0) = 0. \quad (4)$$

Exercise 6 : Kernel

Let $T : V \rightarrow W$ be a linear map. Give definition of kernel. So one interpretation : What gets sent to $\mathbf{0} \in W$. Another : Suppose $T(v) = w$. Then for any $v_0 \in \ker T$, what can we say about $T(v + v_0)$? (Also w .) So another interpretation : The space of our degree of freedom of adjusting input to a linear map and getting same output. Explain how this fits with linear ODEs? Linearity (“nonhomogeneous principle”) said if we had one particular solution to a linear ODE (more generally, any linear equation), then all particular solutions have form $x_p + x_h$. This x_h had some parameters — our degrees of freedom. $x_p + x_h$ still solution to same ODE as x_p solved.

Exercise 7 : Fixed Point Theorems

Consider the sequence $\{x_n\}_{n=1}^{\infty}$ in \mathbf{R} defined by

$$x_1 = 0, \quad x_{n+1} = 1 + \frac{5}{2x_n + 1}, \quad n \geq 1.$$

Find $\lim_{n \rightarrow \infty} x_n$. (Think in terms of fixed points. Distinguish which using initial condition, monotonicity, etc.)

7.1 Solution

We present two solutions, which use the following results.¹

Proposition 7.1. *Let (X, d) be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X . If $\{x_n\}_{n=1}^{\infty}$ is contractive, then the sequence is Cauchy.*

Proposition 7.2. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in \mathbf{R} . Then $\{x_n\}_{n=1}^{\infty}$ converges if and only if the sequence is Cauchy.*

Proposition 7.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in \mathbf{R} . If $\{x_n\}_{n=1}^{\infty}$ converges, say to x , then all subsequences of this sequence converge to the same limit x .*

Theorem 7.4 (Monotone Convergence Theorem). *Let $\{x_n\}_{n=1}^{\infty}$ be a monotone sequence of real numbers. Then $\{x_n\}_{n=1}^{\infty}$ converges if and only if the sequence is bounded. Moreover, this limit equals $\sup x_n$ if the sequence is increasing, and $\inf x_n$ if the sequence is decreasing.*

Theorem 7.5 (Squeeze Theorem). *Let $\{a_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$, and $\{b_n\}_{n=1}^{\infty}$ be sequences such that $a_n \leq x_n \leq b_n$ for all $n \geq 1$. If the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to the same limit x , then $\{x_n\}_{n=1}^{\infty}$ converges to x as well.*

Both of our solutions use the fact that, although the sequence $\{x_n\}_{n=1}^{\infty}$ is not monotonic,² the subsequence of odd terms and the subsequence of even terms are both monotonic. This is suggested by computing the first few terms of the sequence $\{x_n\}_{n=1}^{\infty}$ (see (8)); we now prove it rigorously by comparing the terms x_n and x_{n+2} .

¹Note that Proposition 7.1 is the generalized result of Exercise ??.

²Rigorously, we can show that $\{x_n\}_{n=1}^{\infty}$ is not monotone, no matter how far out in the sequence we go, as follows. For convenience, rewrite the definition of x_{n+1} in the equivalent form

$$x_{n+1} = 1 + \frac{5}{2x_n + 1} = \frac{2x_n + 6}{2x_n + 1}.$$

Then

$$x_{n+1} > 2 \quad \Leftrightarrow \quad 2x_n + 6 > 4x_n + 2 \quad \Leftrightarrow \quad x_n < 2,$$

and

$$0 \leq x_{n+1} < 2 \quad \Leftrightarrow \quad 2x_n + 6 < 4x_n + 2 \quad \Leftrightarrow \quad x_n > 2.$$

Thus when $x_1 = 0$, the sequence $\{x_n\}_{n=1}^{\infty}$ oscillates forever around 2; more precisely, if x_n is less than (greater than) 2, then x_{n+1} is greater than (less than) 2. (Recall that we have shown that $x_n \geq 0$ for all n .)

Note that our analysis yields the following corollary for free: The subsequence $\{x_{2n-1}\}_{n=1}^{\infty}$ of odd terms is bounded above by 2, and the subsequence $\{x_{2n}\}_{n=1}^{\infty}$ of even terms is bounded below by 2. This corollary follows from the results of the previous paragraph, the observations that $x_1 = 0 < 2$ and $x_2 = 6 > 2$, and induction. As another corollary, $x_n \neq 2$, for all $n \geq 1$.

First, note that given the initial term $x_1 = 0$, all terms in the sequence $\{x_n\}_{n=1}^{\infty}$ are nonnegative. This follows from the fact that, in the definition of x_{n+1} , all terms are positive and added, so once a positive value of x_n appears, x_{n+1} will be positive thereafter. Next, using the recursive definition to write x_{n+2} in terms of x_n , we have

$$x_{n+2} = 1 + \frac{5}{2x_{n+1} + 1} = 1 + \frac{5}{2\left(1 + \frac{5}{2x_n + 1}\right) + 1} = 1 + \frac{5(2x_n + 1)}{6x_n + 13} = \frac{16x_n + 18}{6x_n + 13}.$$

Thus

$$x_{n+2} \begin{matrix} \geq \\ \leq \end{matrix} x_n \quad \Leftrightarrow \quad \frac{16x_n + 18}{6x_n + 13} \begin{matrix} \geq \\ \leq \end{matrix} x_n \quad \Leftrightarrow \quad 0 \begin{matrix} \geq \\ \leq \end{matrix} 6x_n^2 - 3x_n - 18 = 3(2x_n + 3)(x_n - 2). \quad (5)$$

(Note that the final equivalence assumes that $x_n > -\frac{13}{6}$, so that the direction of the inequality does not reverse when clearing the denominator. Because all terms in $\{x_n\}_{n=1}^{\infty}$ are nonnegative, this indeed holds.) Analyzing the linear factors in the right-most expression, we find

$$2x_n + 3 \begin{matrix} \geq \\ \leq \end{matrix} 0 \quad \Leftrightarrow \quad x_n \begin{matrix} \geq \\ \leq \end{matrix} -\frac{3}{2}, \quad x_n - 2 \begin{matrix} \geq \\ \leq \end{matrix} 0 \quad \Leftrightarrow \quad x_n \begin{matrix} \geq \\ \leq \end{matrix} 2;$$

therefore (see Figure ??) the quadratic satisfies

$$3(2x_n + 3)(x_n - 2) < 0 \quad \Leftrightarrow \quad x_n > -\frac{3}{2} \text{ and } x_n < 2 \quad (6)$$

and

$$3(2x_n + 3)(x_n - 2) > 0 \quad \Leftrightarrow \quad x_n < -\frac{3}{2} \text{ or } x_n > 2. \quad (7)$$

From (5), if x_n satisfies (6) then $x_{n+2} > x_n$, and if x_n satisfies (7) then $x_{n+2} < x_n$. Because $-\frac{3}{2} < x_1 = 0 < 2$ and $x_2 = 6 > 2$, these results and induction imply that the subsequence $\{x_{2n-1}\}_{n=1}^{\infty}$ of odd terms is monotonically increasing, and the subsequence $\{x_{2n}\}_{n=1}^{\infty}$ of even terms is monotonically decreasing.

Armed with these facts about our sequence — that all terms are nonnegative, that the subsequence of odd terms is monotonically increasing and bounded above by 2, that the subsequence of even terms is monotonically decreasing and bounded below by 2, and that $x_n \neq 2$ for all $n \geq 1$ — we are prepared to attack the main proofs.

Method 1: Contractive Sequence The approach of this method is as follows: If we can show that $\{x_n\}_{n=1}^{\infty}$ is contractive, then it is Cauchy, hence it converges to some point x , and all subsequences converge to this same point x . In particular, the subsequence of odd terms and the subsequence of even terms both converge to x . Because the former is bounded above by 2 and the latter is bounded below by 2, it follows that $x = 2$.

To show that $\{x_n\}_{n=1}^{\infty}$ is contractive, we must give a constant $c \in [0, 1)$ such that $d(x_{n+2}, x_{n+1}) \leq cd(x_{n+1}, x_n)$ for all $n \geq 1$. Thus we are led to consider the two distances in this last inequality. Using the recursive definition of the sequence to write x_{n+1} and x_{n+2} in terms of x_n , we have

$$x_{n+1} = \frac{2x_n + 6}{2x_n + 1}, \quad x_{n+2} = \frac{16x_n + 18}{6x_n + 13}.$$

Therefore

$$d(x_{n+2}, x_{n+1}) = \left| \frac{10(2x_n + 3)(x_n - 2)}{(6x_n + 13)(2x_n + 1)} \right|$$

and

$$d(x_{n+1}, x_n) = \left| \frac{(2x_n + 3)(x_n - 2)}{2x_n + 1} \right|.$$

Using the facts that $x_n \geq 0$ and $x_n \neq 2$ for all $n \geq 1$, we have $d(x_{n+1}, x_n) \neq 0$ for all $n \geq 1$, so we can form the ratio

$$\frac{d(x_{n+2}, x_{n+1})}{d(x_{n+1}, x_n)} = \left| \frac{10}{6x_n + 13} \right|.$$

Note that, for $x_n > -\frac{13}{6}$, this ratio is decreasing in x_n . We have shown that all terms in $\{x_n\}_{n=1}^{\infty}$ are nonnegative, so the minimum value of x_n is 0, and hence the maximum value of the ratio is $\frac{10}{13} < 1$. Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is contractive, with (one possible choice of contractive constant) $c := \frac{10}{13}$.

From Proposition 7.1 it follows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy, and thus it converges by Proposition 7.2; denote the point of convergence by x . By Proposition 7.3 all subsequences of $\{x_n\}_{n=1}^{\infty}$ converge to this same x . In particular, the subsequence of odd terms and the subsequence of even terms both converge to x . Because the former is bounded above by 2 and the latter is bounded below by 2, it follows that $x = 2$.³

Method 2: Squeeze Theorem If the given sequence $\{x_n\}_{n=1}^{\infty}$ converges, then Proposition 7.2 implies that it is Cauchy. In other words, if the given sequence converges, then its terms must become arbitrarily close together. This suggests that we set $x_{n+1} = x_n = x$ in the definition of x_{n+1} to find the candidate limit(s) x :

$$x_n \underset{\text{set}}{=} x_{n+1} = 1 + \frac{5}{2x_n + 1} \quad \Leftrightarrow \quad 2x^2 - x - 6 = 0 \quad \Leftrightarrow \quad x = -\frac{3}{2} \text{ or } x = 2.$$

This procedure of setting $x_{n+1} = x_n = x$ and solving for x is sometimes referred to as “finding the fixed points” of the sequence $\{x_n\}_{n=1}^{\infty}$.

Thus we have two candidate limits: $x = -\frac{3}{2}$ and $x = 2$. Which candidate to consider can be informed either (i) by recalling that we have shown that all terms in the sequence are nonnegative, or (ii) by computing the first few terms of the sequence,

$$0, 6, \frac{18}{13} \approx 1.3846, \frac{114}{49} \approx 2.3265, \frac{522}{277} \approx 1.8845, \frac{2706}{1321} \approx 2.0484, \dots, \quad (8)$$

suggesting that the terms are nonnegative and oscillate around 2. Either way, we are led to consider $x = 2$ as our candidate point of convergence.

We now prove that $\{x_n\}_{n=1}^{\infty}$ converges to 2. We cannot apply Theorem 7.4 directly to the sequence $\{x_n\}_{n=1}^{\infty}$, because it is not monotone, as suggested by our computation of the first few terms in (8). So we adopt the following approach:

³If $x < 2$ then all terms in the subsequence of even terms would be bounded away from x by a distance of at least $2 - x$. Similarly, if $x > 2$ then all terms in the subsequence of odd terms would be bounded away from x by a distance of $x - 2$. In either case one of the subsequences could not converge to x , a contradiction.

1. Use the given sequence $\{x_n\}_{n=1}^{\infty}$ to construct monotonic sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that (i) $a_n \leq x_n \leq b_n$ for all $n \geq 1$ and (ii) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge, then they converge to the same limit x .
2. Apply Theorem 7.4 to $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$.
3. Apply Theorem 7.5 to conclude that $\{x_n\}_{n=1}^{\infty}$ converges to x .

This approach may look daunting, but in practice it is relatively painless. Let's see how it plays out.

The "problem" with the given sequence $\{x_n\}_{n=1}^{\infty}$ is that it is not monotonic. But as we have shown, the subsequences consisting of just the odd terms or just the even terms are monotonic. By Proposition 7.3, if the given sequence $\{x_n\}_{n=1}^{\infty}$ converges, say to x , then these two subsequences also converge to x . So now our task becomes modifying these two subsequences so that we can apply Theorem 7.5.

To this end, define sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as follows:

$$a_n := \begin{cases} x_n & \text{if } n \text{ is odd,} \\ x_{n-1} & \text{if } n \text{ is even;} \end{cases} \quad b_n := \begin{cases} x_{n+1} & \text{if } n \text{ is odd,} \\ x_n & \text{if } n \text{ is even,} \end{cases}$$

for all $n \geq 1$. Note that

$$\{a_n\}_{n=1}^{\infty} = x_1, x_1, x_3, x_3, \dots, \quad \{b_n\}_{n=1}^{\infty} = x_2, x_2, x_4, x_4, \dots$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is monotonically increasing and bounded within $[0, 2]$; similarly, the sequence $\{b_n\}_{n=1}^{\infty}$ is monotonically decreasing and bounded within $[2, 6]$. Thus we can apply Theorem 7.4 to conclude that $\lim_{n \rightarrow \infty} a_n = 2 = \lim_{n \rightarrow \infty} b_n$.

Next, note that $a_n \leq 2 \leq b_n$ for all n , and

$$x_n = \begin{cases} a_n & \text{if } n \text{ is odd,} \\ b_n & \text{if } n \text{ is even,} \end{cases}$$

so that $a_n \leq x_n \leq b_n$ for all $n \geq 1$. Thus Theorem 7.5 implies that $\{x_n\}_{n=1}^{\infty}$ converges to 2, as claimed.