Math 211
Exam 03

W 31 Jul 2019

Your name : ______________________

Start time : ______________ End time : ______________

Honor pledge :

Exam instructions

Number of exercises	: 6
Permitted time	: 90 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- You are well-trained. Do your best, work hard, have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/12	/2	/5	/5	X	X
3	/18	/4	/10	/2	/2	X
4	/21	/7	/7	/7	X	X
5	/25	/3	/6	/6	/6	/4
6	/14	/6	/4	/4	X	X
Total	/100					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary (though you may find it beneficial to check your intuition).

(a) (2 pt) Let V be a finite-dimensional vector space. The greatest number of linearly independent vectors in V always equals the minimum number of vectors needed to span V. *Hint:* If you're not sure, play with a few toy examples, e.g., $V = \mathbf{R}^2, \mathbf{R}^3$, etc. What do you find?

true false

Solution: True. By definition, a basis of a vector space is a set of linearly independent vectors that spans the vector space. That is, a set of basis vectors is both linearly independent and a spanning set. If we add another vector to a basis, then the new set is no longer linearly independent; if we remove a vector from a basis, then the new set no longer spans the vector space. That is, a basis is both a maximal set of linearly independent vectors, and a minimal spanning set. Thus, the number of vectors in a basis gives both the greatest number of linearly independent vectors, and the minimum number of spanning vectors.

(b) (2 pt) Let A be a square matrix, one of whose columns is a nontrivial linear combination of the others (i.e. not all coefficients in this linear combination are 0). Then 0 is an eigenvalue of A. *Hint:* Can we use the nontrivial linear combination to find an eigenvector for 0?

true false

Solution: True. View the columns of A as column vectors $v_1, \ldots, v_n$. By hypothesis, one of these columns, say column n, is a linear combination of the others, say $v_n = a_1 v_1 + \ldots + a_{n-1} v_{n-1}$. Then

$$A \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ -1 \end{bmatrix} = \begin{bmatrix} | & & | & | \\ v_1 & \ldots & v_{n-1} & v_n \\ | & & | & | \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ -1 \end{bmatrix} = a_1 v_1 + \ldots + a_{n-1} v_{n-1} - v_n = \mathbf{0} = 0 \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ -1 \end{bmatrix},$$

where $\mathbf{0}$ denotes the $n \times 1$ zero vector. This shows that $\begin{bmatrix} a_1 & \ldots & a_{n-1} & -1 \end{bmatrix}^T$ is an eigenvector of A with associated eigenvalue 0. (Note that the last entry of this vector is $-1 \neq 0$, so $\begin{bmatrix} a_1 & \ldots & a_{n-1} & -1 \end{bmatrix}^T$ is not the zero vector, hence a valid eigenvector.)

For example, in the 3×3 (square) matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

column 3 is a linear combination of columns 1 and 2, namely (using the above notation),

$$v_3 = v_1 + v_2.$$

We compute

$$A\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}1\\1\\-1\end{bmatrix},$$

so $\begin{bmatrix}1&1&-1\end{bmatrix}^T$ is an eigenvector of A with associated eigenvalue 0.

(c) (2 pt) We can translate any nth-order linear ODE into an $n\times n$ 1st-order linear system.

true false

Solution: True. If the original nth-order linear ODE is

$$a_n(t)y^{(n)}+\ldots+a_0(t)y=f(t), \tag{1}$$

then we can make the change of variables $x_i=y^{(i)}$, for $i=0,\ldots,n-1$. Then one can check that (1) can be written as the 1st-order $n\times n$ linear system

$$\begin{bmatrix}x_0\\ \vdots\\ x_{n-1}\end{bmatrix}'=\begin{bmatrix}0&1&0&\ldots&0\\0&0&1&\ldots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&1\\-a_0(t)&-a_1(t)&-a_2(t)&\ldots&-a_n(t)\end{bmatrix}\begin{bmatrix}x_0\\ \vdots\\ x_{n-1}\end{bmatrix}.$$

(d) (2 pt) Let $y_1(t)$ and $y_2(t)$ be solutions to the nonhomogeneous 3rd-order linear ODE $ty^{(3)}-e^ty'+y=\frac{t}{1+t}$. Then y_2-y_1 is a solution to the corresponding homogeneous equation $ty^{(3)}-e^ty'+y=0$. *Hint:* How do we check if something is a solution to an ODE?

true false

Solution: True. We can check that something is a solution to an ODE by plugging it into the ODE. Doing this for y_2-y_1, we compute

$$\begin{aligned}e^t(y_2-y_1)'+(y_2-y_1)&=e^t(y_2'-y_1')+(y_2-y_1)\\&=\left(e^ty_2'+y_2\right)-\left(e^ty_1'+y_1\right)\\&=\frac{t}{1+t}-\frac{t}{1+t}=0,\end{aligned}$$

where in the first equality, we use linearity of the derivative; in the second equality, we rearrange terms; and in the third equality, we use the hypothesis that y_1 and y_2 are solutions to the given nonhomogeneous ODE.

(e) (2 pt) Every ODE can be solved, i.e. we can always find a closed-form solution (e.g., an explicit equation for $y(t)$).

true false

Solution: False.

Exercise 2

(12 pt) Consider the homogeneous 1st-order nonlinear ODE

$$e^{-t}y' - y^{\frac{1}{5}} = 0. \tag{2}$$

(a) (2 pt) Show that the ODE (2) has exactly one equilibrium solution. What is it? *Hint:* Recall that, by definition, an equilibrium solution is a solution $y(t)$ that does not depend on t.

Solution: By definition, an equilibrium solution is a solution to the ODE (the "solution" part of "equilibrium solution") that does not change over time (the "equilibrium" part of "equilibrium solution"), i.e. a constant function $y(t) \equiv a$ for some $a \in \mathbf{R}$. For our ODE (2), an equilibrium solution satisfies

$$0 - a^{\frac{1}{5}} = 0 \qquad \Leftrightarrow \qquad a = 0.$$

We conclude that (2) has a unique equilibrium solution, $y(t) \equiv 0$.

(b) (5 pt) Find all solutions to the initial value problem given by the ODE (2) and the initial condition $y(0) = 0$. *Hint:* You should find at least one nonequilibrium solution.

Solution: The ODE (2) is separable. Separating variables and solving, we have

$$y^{-\frac{1}{5}}y' = e^t \qquad \Leftrightarrow \qquad \frac{5}{4}y^{\frac{4}{5}} = e^t + c_1 \qquad \Rightarrow \qquad y = \pm\left(\frac{4}{5}e^t + c_2\right)^{\frac{5}{4}},$$

where $c_1, c_2 \in \mathbf{R}$.[1] Applying the initial condition to this general solution, we find

$$0 = y(0) = \pm\left(\frac{4}{5}e^0 + c_2\right)^{\frac{5}{4}} = \pm\left(\frac{4}{5} + c_2\right)^{\frac{5}{4}} \qquad \Leftrightarrow \qquad c_2 = -\frac{4}{5}.$$

Thus there are two (nonequilibrium) solutions to the given initial value problem (IVP):

$$y(t) = \pm\left(\frac{4}{5}\left(e^t - 1\right)\right)^{\frac{5}{4}}.$$

Note that our equilibrium solution $y(t) \equiv 0$ from part (a) also solves the IVP.

(c) (5 pt) Does your result to part (b) contradict your result to part (a)? How do these results relate to the existence and uniqueness statements of Picard's theorem? *Hint:* Picard's theorem, as we learned it, applies to 1st-order ODEs in the form $y' = f(t, y)$. Put (2) in this form.

[1] If we need to be careful, we'll note that $e^t + c_1 \geqslant 0$ for all relevant values of t, so that the second equality makes sense. Hence $c_2 = \frac{4}{5}c_1 \geqslant -\frac{4}{5}e^t$, again for all relevant values of t. If t is allowed to be any real number, than $c_1, c_2 \geqslant 0$.

Solution: Why didn't the equilibrium solution we found in part (a) appear in our analysis in part (b)? When we separated variables, we divided both sides of the given ODE (2) by $y^{\frac{1}{5}}$. This division is a problem when $y(t) \equiv 0$ — it is literally division by zero. So our analysis in part (b) implicitly assumed that y is not the zero function — precisely the "missing" solution.

We can rewrite the original ODE (2) in the form $y' = f(t, y)$:

$$y' = e^t y^{\frac{1}{5}}.$$

(Note that there are no division by zero issues with this rearrangement.) The right side of this equation is the product of two continuous functions, e^t and $y^{\frac{1}{5}}$, hence is continuous, at all points (t, y). Thus Picard's theorem ensures that a solution to any IVP exists. The partial derivative with respect to y of the right side is

$$\frac{\partial}{\partial y}\left(e^t y^{\frac{1}{5}}\right) = \frac{1}{5} e^t y^{-\frac{4}{5}},$$

which is not continuous at any point (t, y) with $y = 0$. Thus the statement about uniqueness of solutions in Picard's theorem does not apply here: Solutions to our IVP may or may not be unique. In this case, our work in part (b) shows that they are not.

Exercise 3

(18 pt) Let T be the linear map[2]

$$T : \mathbf{R}^4 \to \mathbf{R}^3$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 & + & 2x_2 & - & x_3 & + & x_4 \\ 2x_1 & + & 4x_2 & - & 2x_3 & + & 2x_4 \\ 3x_1 & + & 5x_2 & - & 4x_3 & + & 2x_4 \end{bmatrix}.$$

(a) (4 pt) Write the coefficient matrix A for T, i.e. the matrix such that $Ax = T(x)$, where x is the 4×1 column matrix with entries x_1, x_2, x_3, x_4. *Hint:* T must output a vector in $\mathbf{R}^3$, so Ax must be a 3×1 matrix. x is 4×1. What do these imply about the order (dimensions) of A?

Solution: The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 5 & -4 & 2 \end{bmatrix}.$$

Check : We can compute that Ax gives the 3×1 matrix for $T(x)$, given in the statement of this exercise.

(b) (10 pt) Find a basis for the image $\operatorname{im}(T)$. Find a basis for the kernel $\ker(T)$.

Solution: Row reducing our matrix A from part (a), we get

$$\operatorname{RREF}(A) = \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Image. There are exactly two pivot columns, columns 1 and 2. (The number of pivot columns equals the dimension of the image.) The corresponding columns in the original matrix A form a basis for the image:

$$\operatorname{basis}(\operatorname{im}(T)) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right\}.$$

By definition, the image is a subspace of the codomain, in this case $\mathbf{R}^3$. Note that the basis vectors listed here are 3×1 matrices, hence live in $\mathbf{R}^3$, as required.

Kernel. By definition, $\ker(T)$ is the set of vectors that T maps to 0. Translating this definition into matrices, $\ker(T)$ is the solution space of the matrix equation $Ax = 0$, where Ax comes from our representation of $T(x)$ in part (a); x is a 4×1 column matrix of variables x_1, x_2, x_3, x_4; and 0 is the 3×1 zero matrix. The solution space of $Ax = 0$ is equal to the solution space of $\operatorname{RREF}(A)x = 0$

[2]Whenever we write a linear map between two vector spaces as a matrix, we are implicitly choosing a basis for each vector space. We can ignore this choice of bases for this problem. However, it's good to keep this general fact in mind.

(because the elementary row operations yield equivalent systems). We can read the solutions to this last equation from RREF(A) in (3):

$$\ker(T) = \left\{ \begin{bmatrix} 3x_3 & + & x_4 \\ -x_3 & - & x_4 \\ x_3 & & \\ & & x_4 \end{bmatrix} \mid x_3, x_4 \in \mathbf{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

By construction, the two vectors listed in this span are linearly independent, a fact we can also check directly.

(c) (2 pt) Find the dimensions $\dim(\operatorname{im}(T))$ and $\dim(\ker(T))$.

Solution: In part (b) we found a basis for $\operatorname{im}(T)$, consisting of 2 vectors. Thus

$$\dim(\operatorname{im}(T)) = 2.$$

Check : The number of pivot columns of A (equivalently, of RREF(A)) equals the dimension of the image. Our RREF of A had two pivots, confirming that $\dim(\operatorname{im}(T)) = 2$.

In part (b) we found a basis for $\ker(T)$, consisting of 2 vectors. Thus

$$\dim(\ker(T)) = 2.$$

Check : The number of nonpivot columns of A (equivalently, of RREF(A)), corresponding to the number of free variables in the equation $Ax = b$, equals the dimension of the kernel. Our RREF of A had two nonpivot columns, confirming that $\dim(\ker(T)) = 2$.

(d) (2 pt) Confirm the rank–nullity theorem:

$$\dim(\operatorname{domain}(T)) = \dim(\operatorname{im}(T)) + \dim(\ker(T)).$$

Solution: The domain of T is the input space, i.e. $\mathbf{R}^4$, so

$$\dim(\operatorname{domain}(T)) = \dim(\mathbf{R}^4) = 4.$$

Applying our results above, we find

$$\dim(\operatorname{domain}(T)) = 4 = 2 + 2 = \dim(\operatorname{im}(T)) + \dim(\ker(T)),$$

confirming the rank–nullity theorem holds for T.

Exercise 4

(21 pt) For each of the following three homogeneous 1st-order 2×2 linear systems,

1. write the general solution, and
2. circle the number of its phase plane (shown on the next page, in the (x_1, x_2) plane).

Hint: For the phase planes, recall that the sign of (the real part of) each eigenvalue relates to whether solutions move toward or away from the equilibrium at the origin.

(a) (7 pt) Phase plane : (1) (2) (3)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution: Let A denote the 2×2 coefficient matrix. Computing the eigenvalues, we find

$$0 \underset{\text{set}}{=} \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4),$$

so our two eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$. By solving the matrix equation $0 \underset{\text{set}}{=} (A - \lambda I)v$ for v, we compute corresponding eigenvectors to be

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus the general solution is

$$x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where $c_1, c_2 \in \mathbf{R}$.

To describe the phase plane, we note the following.

- Both eigenvalues are real and positive, so the equilibrium (which is at the origin) is a source (unstable) — all solutions move away from the equilibrium as time increases.
- The two eigenspaces are $\mathrm{Span}\{\begin{bmatrix} 1 & 1 \end{bmatrix}^T\}$ and $\mathrm{Span}\{\begin{bmatrix} 1 & -1 \end{bmatrix}^T\}$. These correspond to lines in the phase plane, corresponding to the particular solutions with $c_2 = 0$ and $c_1 = 0$, respectively. (*(clarify the following)* These lines are a graphical incarnation of the fact that if a solution starts it an eigenspace, it will forever remain in that eigenspace.)
- As $t \uparrow +\infty$, the general solution more and more resembles the dominant term, $e^{4t} \begin{bmatrix} 1 & -1 \end{bmatrix}^T$; that is, as $t \uparrow +\infty$, all trajectories in the phase plane become parallel to $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

All this is consistent with the phase plane (2).

N.B. One can also evaluate (the right side of) the ODE at various points $(x_1, x_2) \in \mathbf{R}^2$ to determine the phase plane.

(b) (7 pt) Phase plane : (1) (2) (3)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution: We compute the eigenvalues and eigenvectors to be

$$\lambda_1 = -2, \quad v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \qquad\qquad \lambda_2 = 1, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence the general solution is

$$x(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $c_1, c_2 \in \mathbf{R}$. Both eigenvalues are real, one negative and one positive, so the equilibrium (which is at the origin) is a saddle. This corresponds to the phase plane (3).

Note how the lines corresponding to the eigenspaces appear in the phase plane.

(c) (7 pt) Phase plane : (1) (2) (3)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -2 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution: We compute the eigenvalues and eigenvectors to be

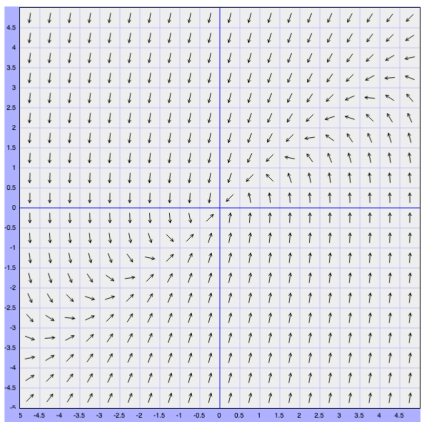
$$\lambda_1 = -6, \quad v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad\qquad \lambda_2 = -2, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence the general solution is

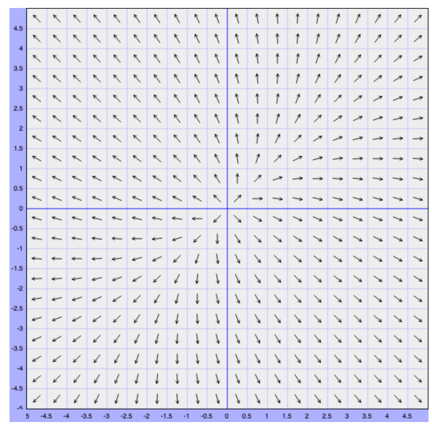
$$x(t) = c_1 e^{-6t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $c_1, c_2 \in \mathbf{R}$. Both eigenvalues are real and negative, so the equilibrium (which is at the origin) is a sink (stable) — all solutions approach the origin as time increases. This corresponds to phase plane (1).

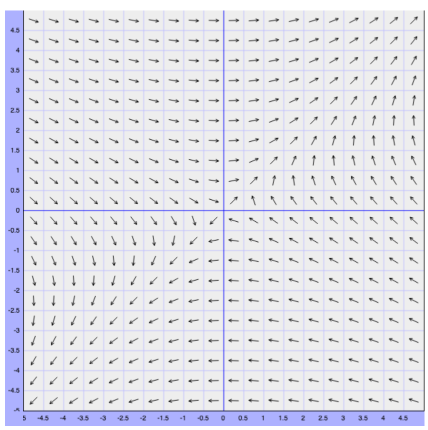
Note how the lines corresponding to the eigenspaces appear in the phase plane. As $t \uparrow +\infty$, the term of the general solution with e^{-6t} goes to zero (more precisely, to the zero vector) faster than the term with e^{-2t}. Thus, as $t \uparrow +\infty$, all trajectories in the phase plane become parallel to the eigenspace associated with the second term, i.e. parallel to $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$. This is indeed the behavior we observe in phase plane (1).



(1)



(2)



(3)

Figure 1: Phase planes for Exercise 4. Arrows indicate direction.

Exercise 5

(25 pt) Consider the homogeneous 2nd-order linear ODE

$$2y'' + 4y' + 2y = 0. \tag{4}$$

(a) (3 pt) Translate (4) into a homogeneous 1st-order 2×2 linear system. *Hint:* Use the change of variables $x_i = y^{(i)}$, as we've learned. Be careful — mind the coefficient on y'' in (4).

Solution: Let

$$x_0 = y, \qquad\qquad x_1 = y'.$$

Then, using the definition of our new variables x_i and the original ODE (4), we compute

$$x_0' = y' = x_1, \qquad\qquad x_1' = y'' = -y - 2y' = -x_0 - 2x_1,$$

which we can write as the matrix equation

$$\begin{bmatrix} x_0' \\ x_1' \end{bmatrix} = \begin{bmatrix} & & x_1 \\ -x_0 & - & 2x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

This is our desired homogeneous 1st-order 2×2 linear system.

Let $\mathbf{A}$ denote the 2×2 matrix of coefficients, and let $\mathbf{x} = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^T$.

(b) (6 pt) Show that this ODE (either (4) or its equivalent 2×2 system you found in part (a)) has a repeated eigenvalue $\lambda = -1$, and that the associated eigenspace has dimension 1 (i.e. we can find at most one linearly independent eigenvector with eigenvalue -1). Write an eigenvector, call it $\mathbf{v}_1$. *Hint:* You do not have to use part (a) to compute the eigenvalues; if preferred, you can do this straight from (4).

We say that -1 is an eigenvalue with **algebraic multiplicity** 2 and **geometric multiplicity** 1.

Solution: Either by substituting λ^i for $y^{(i)}$ in the original ODE (4) and dividing the equation by 2, or by computing $\det(\lambda\mathbf{I} - \mathbf{A})$ for the coefficient matrix $\mathbf{A}$ in part (a), we get the characteristic polynomial

$$p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2.$$

The two roots of $p(\lambda)$ are both -1, i.e. -1 is an eigenvalue with algebraic multiplicity 2.

To compute the eigenspace $E(\mathbf{A}, -1)$ of $\mathbf{A}$ associated to the eigenvalue -1, we find the solution set of the matrix equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \qquad \Leftrightarrow \qquad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

We can find the solution set by applying the row reduction algorithm to the augmented matrix

$$\left[\begin{array}{c:c} \mathbf{A} - \lambda\mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cc:c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

From this reduced row echelon form (RREF) matrix, we can read off the solution space:[3]

$$E(\mathbf{A}, -1) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

We see that the solution set $E(\mathbf{A}, -1)$ is a subspace (of $\mathbf{R}^2$) of dimension 1. The dimension tells us that within this subspace, the largest set of linearly independent (eigen)vectors we can build has 1 vector. Let's choose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(c) (6 pt) Using the 2×2 coefficient matrix $\mathbf{A}$ from part (a), our eigenvalue $\lambda = -1$, and our eigenvector $\mathbf{v}_1$ from part (b), find a vector $\mathbf{v}_2 \in \mathbf{R}^2$ that solves

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1. \tag{5}$$

Hint: View the two entries of the 2×1 vector $\mathbf{v}_2$ as unknown variables, and view (5) as a system of equations. Solve this system.

The vector $\mathbf{v}_2$ is called a **generalized eigenvector** of $\mathbf{A}$ associated to the eigenvalue -1.

Solution: Writing out the matrix equation (5), with the 2×1 matrix $\mathbf{v}_2$ as our unknown, we have

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (\mathbf{A} - \lambda \mathbf{I})\, \mathbf{v}_2 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can find the solution set to this system of equations by applying the row reduction algorithm to the augmented matrix:

$$\left[\begin{array}{c:c} \mathbf{A} - \lambda \mathbf{I} & \mathbf{v}_1 \end{array}\right] = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right] \xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

From this RREF matrix, we read off the solution set to be

$$\left\{ \begin{bmatrix} 1 - v_2 \\ v_2 \end{bmatrix} \mid v_2 \in \mathbf{R} \right\} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + E(\mathbf{A}, -1).$$

Any vector from this set will do. Let's choose

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(d) (6 pt) Using our eigenvalue $\lambda = -1$ and our vectors $\mathbf{v}_1, \mathbf{v}_2$, show that the 2×1 matrix functions

$$\mathbf{X}_1(t) = e^{\lambda t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{X}_2(t) = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$$

are solutions to our linear system in part (a).

[3] Note that this solution set $E(\mathbf{A}, -1)$ is the kernel of the matrix $\mathbf{A} - \lambda \mathbf{I}$, with $\lambda = -1$.

Solution: First let's define these two 2×1 matrix functions explicitly:

$$\mathbf{X}_1(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{X}_2(t) = e^{-t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} t \\ -t+1 \end{bmatrix}. \tag{6}$$

To show that these are solutions to our linear system in part (a), we plug them into that linear system, and verify the equation holds. We compute

$$\mathbf{X}_1'(t) = -e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and (doing the matrix multiplication, noting that e^{-t} is a scalar, or more precisely, a scalar-valued function)

$$\mathbf{A}\mathbf{X}_1(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \left(e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We find that $\mathbf{X}_1'(t) = \mathbf{A}\mathbf{X}_1(t)$, so by definition, $\mathbf{X}_1(t)$ is a solution to our system in part (a).

Similarly, for $\mathbf{X}_2(t)$ we compute (using the product rule)

$$\mathbf{X}_2'(t) = \begin{bmatrix} te^{-t} \\ (-t+1)e^{-t} \end{bmatrix}' = \begin{bmatrix} e^{-t} - te^{-t} \\ -e^{-t} - (-t+1)e^{-t} \end{bmatrix} = e^{-t} \begin{bmatrix} 1-t \\ -1-(-t+1) \end{bmatrix} = e^{-t} \begin{bmatrix} 1-t \\ -2+t \end{bmatrix}$$

and (doing the matrix multiplication, again noting that e^{-t} is a scalar)

$$\mathbf{A}\mathbf{X}_2(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \left(e^{-t} \begin{bmatrix} t \\ -t+1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} -t+1 \\ -t-2(-t+1) \end{bmatrix} = e^{-t} \begin{bmatrix} -t+1 \\ t-2 \end{bmatrix}.$$

We find that $\mathbf{X}_2'(t) = \mathbf{A}\mathbf{X}_2(t)$, so by definition, $\mathbf{X}_2(t)$ is a solution to our system in part (a).

(e) (4 pt) Try to translate the general solution of our linear system in part (a), i.e. the linear combination

$$a_1\mathbf{X}_1(t) + a_2\mathbf{X}_2(t),$$

into the general solution $y(t)$ of our original 2nd-order ODE (4). *Hint:* Note that rows 1 and 2 of each $\mathbf{X}_i(t)$ are x_0 and x_1, respectively. Consider our original change of variables. Note that we can check our proposed solution $y(t)$ by plugging it into the original ODE (4).

Solution: Our homogeneous 1st-order linear system in part (a) is 2×2, so its solution set (of $\mathbf{R}$-valued solutions) is a vector space over $\mathbf{R}$ of dimension 2. One can show that the solutions $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ we found in part (d) are linearly independent. (What happens when we write these 2×1 matrices side-by-side and take the determinant of the resulting 2×2 matrix?) Thus the general solution to our linear system in part (a) is

$$a_1\mathbf{X}_1(t) + a_2\mathbf{X}_2(t) = a_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_2 e^{-t} \begin{bmatrix} t \\ -t+1 \end{bmatrix} = \begin{bmatrix} a_1e^{-t} + a_2te^{-t} \\ -a_1e^{-t} - a_2te^{-t} + a_2e^{-t} \end{bmatrix}. \tag{7}$$

By definition, the entries in rows 1 and 2 of our 2×1 matrices $\mathbf{x}(t)$ — including $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ — are y and y', respectively. So the first row of the general solution (7) is the general solution $y(t)$ to our original 2nd-order ODE (4):

$$y(t) = a_1 e^{-t} + a_2 t e^{-t}. \tag{8}$$

The second row in (7) is supposed to be $y'(t)$. Let's differentiate (8) and check:

$$y'(t) = -a_1 e^{-t} + a_2 e^{-t} - a_2 t e^{-t},$$

the same as the entry in row 2 of (7).

Remark 5.1. Note that each step in our work above depended on choices we made in the previous step: Our computation of $\mathbf{v}_2$ depended on the eigenvector $\mathbf{v}_1$ we chose in part (b), the 2×1 matrices $\mathbf{X}_1(t), \mathbf{X}_2(t)$ depended on our choices of $\mathbf{v}_1$ and $\mathbf{v}_2$, the general solution depended on our definition of $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$. The general solution to our 1st-order linear system is a property of the system, which exists independent of our choices. That is, the solution set should always be the same, regardless of our choices. Is it? Let's see what happens if we make different choices.

Suppose in part (b) we take our eigenvector to be

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then in part (c) the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$ defining $\mathbf{v}_2$ writes as

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

which we can compute has the solution set

$$\left\{ \begin{bmatrix} -1 - v_2 \\ v_2 \end{bmatrix} \mid v_2 \in \mathbf{R} \right\} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + E(\mathbf{A}, -1).$$

Any vector from this set will do. Let's choose

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

In this case, our 2×1 matrices $\mathbf{X}_1(t), \mathbf{X}_2(t)$ write as

$$\mathbf{X}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{X}_2(t) = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) = e^{-t} \begin{bmatrix} -t - 1 \\ t \end{bmatrix}.$$

These are different matrices than those we computed in (6), with our other choice of $\mathbf{v}_1, \mathbf{v}_2$. If we take the linear combination of our $\mathbf{X}_1(t), \mathbf{X}_2(t)$ here, we find

$$b_1\mathbf{X}_1(t) + b_2\mathbf{X}_2(t) = b_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b_2 e^{-t} \begin{bmatrix} -t - 1 \\ t \end{bmatrix} = \begin{bmatrix} (-b_1 - b_2)e^{-t} - b_2 t e^{-t} \\ b_1 e^{-t} + b_2 t e^{-t} \end{bmatrix}. \tag{9}$$

Comparing this to (7), equating coefficients of like terms in each entry, we get

$$a_1 = -b_1 - b_2, \qquad a_2 = -b_2,$$

which we can write as a matrix equation

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{10}$$

Call the 2×2 coefficient matrix $\mathbf{B}$.

Given a general solution (9), we can get the general solution (7) by using these values of a_1, a_2, i.e. $\begin{bmatrix} a_1 & a_2 \end{bmatrix}^T = \mathbf{B} \begin{bmatrix} b_1 & b_2 \end{bmatrix}^T$. Note that the coefficient matrix $\mathbf{B}$ has determinant 1; hence it is invertible. Thus, conversely, given a general solution (7), we can get the general solution (9) by settiing $\begin{bmatrix} b_1 & b_2 \end{bmatrix}^T = \mathbf{B}^{-1} \begin{bmatrix} a_1 & a_2 \end{bmatrix}^T$. Explicitly, given a_1, a_2, take b_1, b_2 to be

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_1 + a_2 \\ -a_2 \end{bmatrix}.$$

(clarify this conclusion) What we have found is that the $\mathbf{X}_1(t), \mathbf{X}_2(t)$ we computed here, and the $\mathbf{X}_1(t), \mathbf{X}_2(t)$ we computed in part (d), are different bases of the same solution space. The 2×2 coefficient matrix $\mathbf{B}$ explicitly relates these two bases. That is, the set of solutions to our original ODE can be described in different ways, but all these different descriptions are equivalent.

Exercise 6

(14 pt) Consider the nonhomogeneous 2nd-order linear ODE

$$y'' - 4y = 8\sin(2t) - 4. \tag{11}$$

(a) (6 pt) Write the corresponding homogeneous equation, and find the general solution $y_h(t)$.

Solution: The corresponding homogeneous equation is

$$y'' - 4y = 0. \tag{12}$$

The characteristic polynomial (which we can find by either substituting λ^i for $y^{(i)}$ in (12); or by translating (12) into a homogeneous 1st-order 2×2 linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then computing $\det(\lambda\mathbf{I} - \mathbf{A})$) is

$$p(\lambda) = \lambda^2 - 4,$$

which has two distinct real roots, $\lambda = \pm 2$. Thus the general solution to (12) is

$$y(t) = c_1 e^{-2t} + c_2 e^{2t},$$

where $c_1, c_2 \in \mathbf{R}$.

Note that we can check that $y(t)$ is indeed a solution by plugging it into (12) and confirming that the equation is true.

(b) (4 pt) Show that

$$y_p(t) = -\sin(2t) + 1$$

is a particular solution to (11).

Solution: As usual, to show that something is a solution, we plug it in:

$$y_p'' - 4y_p = (-\sin(2t) + 1)'' - 4(-\sin(2t) + 1) = 4\sin(2t) + 4\sin(2t) - 4 = 8\sin(2t) - 4,$$

which is the right side of (11). Hence $y_p(t)$ is indeed a particular solution to (11).

(c) (4 pt) Briefly justify why the nonhomogeneous principle applies to our ODE (11). Then use it, and our above results, to write the the general solution to (11).

Solution: The nonhomogeneous ODE (11) is linear, hence the nonhomogeneous principle (another way to say "linearity") applies. By this principle, the general solution $y(t)$ to (11) is

$$y(t) = y_p(t) + y_h(t),$$

where $y_h(t)$ is the general solution to the corresponding homogeneous equation, and $y_p(t)$ is a particular solution. Using our results for $y_h(t)$ and $y_p(t)$ in parts (a) and (b) above, we conclude that the general solution to (11) is

$$y(t) = -\sin(2t) + 1 + c_1 e^{-2t} + c_2 e^{2t}.$$