Math 357
Exam 00

2024–01–08 (M)

Your name: ______________________________

Instructions

1. In the space above, please legibly write your name and the Rice Honor Pledge, then sign.
2. Next, please read each exercise and, in the left margin beside each, write an integer 0–4 indicating how confident you feel with that exercise, with 0 indicating "not yet confident" and 4 indicating "very confident".
3. Please use the remaining time to work on exercises of your choice.

Exercise	Total	(a)	(b)	(c)
1	/4			
2	/6	/2	/2	/2
3	/6	/2	/4	
4	/4			
5	/6	/2	/4	
6	/6	/2	/4	
7	/6	/2	/4	
8	/4			
9	/4			
10	/4			

Linear algebra

Exercise 1

(4 pt) State and prove the rank–nullity theorem, which relates dimensions of certain subspaces arising from a linear map.

Exercise 2

Let **R** denote the field of real numbers, let **C** denote the field of complex numbers, and let $V = \{p \in \mathbf{C}[t] \mid \deg(p) \leqslant 2\}$ denote the set of univariate polynomials of degree at most 2 with complex coefficients.

(a) (2 pt) View V as a vector space over **C**, and view **C** as a vector space over **R**, with addition and scalar multiplication defined as usual. Give the dimension of each vector space.

(b) (2 pt) Can we view V as a vector space over **R**? If so, what is its dimension?

(c) (2 pt) Use the above to make a conjecture.

Group theory

Exercise 3

(a) (2 pt) Define a normal subgroup. Explain its importance.

(b) (4 pt) From the same "parent" group of your choice, give two examples of proper, nontrivial subgroups, one of which is normal, the other of which is not. Justify your assertions.

Exercise 4

(4 pt) From the symmetric group S_4, give an example of a cyclic subgroup of order 4 and, separately, a noncyclic subgroup of order 4. Justify your assertions.

Ring theory

Exercise 5

(a) (2 pt) Define “integral domain” and, separately, “field”.

(b) (4 pt) Prove that every finite integral domain is a field.

Exercise 6

(a) (2 pt) Define “euclidean domain” and, separately, “principal ideal domain”.

(b) (4 pt) Prove that a euclidean domain is a principal ideal domain.

Exercise 7

Let A be an integral domain.

(a) (2 pt) Define what it means for an element of A to be prime and, separately, irreducible.

(a) (4 pt) Relate, as fully as possible, the notions of prime and irreducible elements. Provide proof or counterexample for your assertions.

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Exercise 8

(4 pt) Let $p \in \mathbf{Z}_{>0}$ be prime. Prove that the polynomial $f(t) = \sum_{j=0}^{p-1} t^j$ is irreducible in $\mathbf{Q}[t]$.

Exercise 9

(4 pt) Construct a character table for the symmetric group S_3.

Exercise 10

(4 pt) Let $p(t) = t^3 - 2 \in \mathbf{Q}[t]$, and let K denote the splitting field of p over $\mathbf{Q}$. Draw a diagram of intermediate fields of $K|\mathbf{Q}$ and a diagram of subgroups of the galois group $\mathrm{Gal}(K|\mathbf{Q})$. In your diagrams, indicate all normal subgroups and all galois extensions of $\mathbf{Q}$.