Math 357
Short quiz 08

2024–02–16 (F)

Your name: ______________________________

By popular demand, you are sharing knowledge of modules with your friends while you all share a meal together. “Indeed!” says one, “Why, module homomorphisms are exactly like ring homomorphisms: They both respect addition and multiplication.” Respond.

Solution: “Indeed!” say I. “Why, ‘exactly’ being such a strong word, in a careful response to its use in casual conversation, one almost always finds a ‘no’, or at least a ‘not exactly’.” (“Which is ‘exactly’ why nonmathematicians can find casual conversation with mathematicians tedious,” jokes another friend, barely sotto voce. I acknowledge the bait with a grinning frown and continue with my response.) Let me offer three justifications for this “not exactly”. First, the multiplication in rings and on modules differs: In ring, multiplication is a binary operation; whereas on modules, (scalar) multiplication is a ring action. This may be clearer in writing,” I remark, and taking a handy paper napkin and a loitering pen, I write

In a ring R	On an R-module M
$\times : R \times R \to R$	$\cdot : R \times M \to M$

“This different algebraic structure leads to different treatment by the respective homomorphisms,” I continue, appending the following rows to the table on the napkin (on the table):

$\varphi(r_1 \times_R r_2) = \varphi(r_1) \times_S \varphi(r_2)$	$\varphi(r \cdot_M m) = r \cdot_N \varphi(m)$
for $\varphi : R \to S$	for $\varphi : M \to N$

“Now, one could consider the case $M = R$, with scalar multiplication on the R-module R given by the multiplication in the ring R. In this case, the domain and codomain of the maps $\times$ and $\cdot$ (I point to the table) would be the same, but notice that the homomorphisms would still treat the two multiplications differently. This allows there to be ring homomorphisms that are not R-module homomorphisms, and vice versa,” I observe solemnly, appending two final rows to the napkin table:

$\varphi : \mathbf{Z} \to \mathbf{Z}$	$\varphi : \mathbf{F}[t] \to \mathbf{F}[t]$
$\varphi(a) = 2a$	$\varphi(f(t)) = f(t^2)$

“Which,” I conclude with a smile, “would be hard to accomplish if these two homomorphisms were exactly alike.” My friends joining me in general agreement, I join them in enjoying our shared meal.