Math 357
Exam 02M

2024–04–13 (S)

Your name: ______________________________

Honor pledge:

Instructions

1. In the space above, please legibly write your name and the Rice Honor Pledge, then sign.
2. You have exactly 50 minutes for this exam. No resources are allowed.

Exercise	Total	(a)	(b)	(c)	(d)
1	/4	/4	/4	/4	
2	/4	/4	/4		
3	/4	/4	/4	/4	/4
4	/4	/4	/4		
Total	/20				

Exercise 1

(4 pt) Let $K : K_0$ be a field extension, and let $\alpha \in K$ be algebraic over K_0.

(a) Define (axiomatically) the minimal polynomial m_{α,K_0} for α over K_0.

In addition, let K_1 be an intermediate field of $K : K_0$; that is, let $K : K_1 : K_0$.

(b) Prove that m_{α,K_1} divides m_{α,K_0}. In what polynomial ring(s) does this divisibility apply?

(c) Give an example in which $[K_1 : K_0] > 1$ and $m_{\alpha,K_1} = m_{\alpha,K_0}$ has degree greater than 1.

Exercise 2

(4 pt) Let

$$f = t^4 - 6t^3 + 21t^2 - 36t + 36 \in \mathbf{Q}[t]$$

(a) Compute the formal derivative $D_t f$ of f.

(b) You apply the euclidean algorithm to f and $D_t f$ and find

$$\begin{aligned} f &= q_1 D_t f + r_1 \\ D_t f &= q_2 r_1 + 0 \end{aligned}$$

where $q_1, q_2, r_1 \in \mathbf{Q}[t]$, and

$$\deg q_1 = 1 \qquad \deg q_2 = 1 \qquad \deg r_1 = 2$$

From this, what can we conclude about the separability of f? about the irreducibility of f? Explain.

Exercise 3

(4 pt) Consider $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ as a subfield of $\mathbf{C}$.

(a) Prove that $[\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q}] = 4$. Give a basis for $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ as a $\mathbf{Q}$-vector space.

(b) Specify the elements of $\mathrm{Aut}(\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q})$. *Hint:* Recall that to specify a $\sigma \in \mathrm{Aut}(\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q})$, it suffices to specify $\sigma(\sqrt{2})$ and $\sigma(\sqrt{3})$.

(c) List the subgroups of $\mathrm{Aut}(\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q})$, and find the fixed field of each.

(d) Prove that the field extension $\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q}$ is galois.

Exercise 4

(4 pt) Let $K : K_0$ be a field extension.

(a) Using the definitions from the theory we developed in class, define what it means for $K : K_0$ to be finite, normal, separable, and galois. (That is, "$K : K_0$ is finite if...", "$K : K_0$ is normal if...", etc.)

(b) One can prove that $K : K_0$ is galois if and only if it is finite, normal, and separable. Discuss where, in your definition of a galois extension, each of these latter three concepts appears.