Math 211
Exam 02

W 24 Jul 2019

Your name : ______________________________

Start time : ________________ End time : ________________

Honor pledge :

Exam instructions

Number of exercises	: 6
Permitted time	: 90 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- You are well-trained. Do your best!

Exercise	Total	(a)	(b)	(c)	(d)	(e)	(f)
1	/10	/2	/2	/2	/2	/2	X
2	/9	/3	/3	/3	X	X	X
3	/12	X	X	X	X	X	X
4	/20	/4	/8	/8	X	X	X
5	/17	/2	/5	/8	/2	X	X
6	/32	/8	/8	/2	/8	/4	/2
Total	/100						

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary (though you may find it beneficial to check your intuition).

(a) (2 pt) Let y_1 and y_2 be solutions to the 1st-order ODE

$$(\sin t)y^{-1}y' = te^t. \tag{1}$$

Then any linear combination $a_1y_1 + a_2y_2$ is also a solution to (1). *Hint:* Can you rearrange the equation? Can you justify your answer?

true false

(b) (2 pt) We can find four linearly independent vectors in $\mathbf{R}^3$, the vector space of 3×1 matrices with real entries.

true false

(c) (2 pt) We can find four linearly independent vectors in $\mathcal{P}^3(\mathbf{R})$, the vector space of polynomial functions in one variable, of degree less than or equal to 3, and with real coefficients.

true false

(d) (2 pt) Let A be an $n \times m$ matrix with entries in $\mathbf{R}$, and let b_1, b_2 be two $n \times 1$ matrices. If there exists a solution to the matrix equation $Ax = b_1$ and a (possibly different) solution to the matrix equation $Ax = b_2$, then there exists a solution to the matrix equation $Ax = b_1 + b_2$.

true false

(e) (2 pt) Every ODE can be solved, i.e. we can always find a closed-form solution (e.g., an explicit equation for $y(t)$).

true false

Exercise 2

(9 pt) For each of the following, indicate whether the subset $W \subseteq V$ described is a subspace of the given vector space V. If it is not, indicate whether it fails to be nonempty, fails to be closed under the vector space operations (equivalently, fails to be closed under linear combinations), or both.

(a) (3 pt) $V =$ the space of all continuously differentiable functions $f : \mathbf{R} \to \mathbf{R}$, $W =$ the set of solutions to the 1st-order homogeneous linear ODE $a_1(t)y' + a_2(t)y = 0$, where $a_1(t), a_2(t)$ are continuous on all of $\mathbf{R}$.

yes, subspace — no, fails to be nonempty — no, fails to be closed

(b) (3 pt) $V = \mathbf{R}^3$, $W =$ the set of solutions to the system of linear equations represented by the augmented matrix

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 4 & 2 & 1 \end{array}\right].$$

Hint: Row reduce.

yes, subspace — no, fails to be nonempty — no, fails to be closed

(c) (3 pt) $V = \mathbf{R}^3$, $W =$ the set of solutions to the system of linear equations represented by the augmented matrix

$$\left[\begin{array}{ccc:c} 3 & 1 & -2 & a \\ 2 & 0 & -1 & b \\ 1 & 0 & 1 & c \end{array}\right],$$

where $a, b, c \in \mathbf{R}$ are not all 0. *Hint:* This is all we need to know about a, b, c. Why?

yes, subspace — no, fails to be nonempty — no, fails to be closed

Exercise 3

(12 pt) Solve (i.e. give the explicit equation of the solution $y(t)$ to) the following 1st-order nonhomogeneous linear ODE IVP:

$$y' + 2y = 2t^2, \qquad\qquad y(0) = -\frac{3}{2}.$$

Exercise 4

(20 pt) Consider the following system of three linear equations in three unknowns:

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 6 \\ x_2 - x_3 &= -2 \\ x_1 + x_3 &= 6. \end{aligned} \tag{2}$$

(a) (4 pt) Write the linear system (2) as a matrix equation. *Hint:* Double-check that your matrices indeed multiply to yield the given system (2) before you proceed!

(b) (8 pt) Compute the determinant of the 3×3 matrix of coefficients in part (a). (You should get 6.) What does this tell us about existence and uniqueness of solutions to the linear system (2)?

(c) (8 pt) Find all solutions to the linear system (2). Briefly justify why this is all solutions.

Exercise 5

(17 pt) Consider a linear map $T : \mathbf{R}^5 \to \mathbf{R}^3$ which (for a choice of basis for $\mathbf{R}^5$ and $\mathbf{R}^3$) is represented by the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

That is, given a vector $v \in \mathbf{R}^5$ (which we can view as a 5×1 column matrix), $T(v)$ is given by the matrix multiplication Av. (Note that the product of the 3×5 matrix A times the 5×1 matrix v is a 3×1 matrix, i.e. a vector in $\mathbf{R}^3$, as required by the definition of T.)

(a) (2 pt) State the dimension of the domain of T, i.e. $\mathbf{R}^5$, and specify a basis.

(b) (5 pt) By definition, the image of T is the set of vectors $w \in \mathbf{R}^3$ that T outputs for some input $v \in \mathbf{R}^5$:

$$\operatorname{im} T = \left\{ w \in \mathbf{R}^3 \,|\, \text{for some } v \in \mathbf{R}^5, T(v) = w \right\}.$$

If we unwind this definition, it says that $\operatorname{im} T$ is the span of the columns of A, where each column is viewed as a 3×1 matrix. Using this latter definition, specify $\operatorname{im} T$ (note that it must be a vector subspace of $\mathbf{R}^3$), and show it has dimension $\dim \operatorname{im} T = 3$. *Hint:* Focus on the pivot columns of A.

(c) (8 pt) By definition, the kernel of T is the set of vectors $v \in \mathbf{R}^5$ that T maps to $0 \in \mathbf{R}^3$:

$$\ker \mathsf{T} = \left\{ v \in \mathbf{R}^5 \,|\, \mathsf{T}(v) = 0 \right\}.$$

If we unwind this definition, it says that $\ker \mathsf{T}$ is the set of solutions $v \in \mathbf{R}^5$ to $Av = 0$ (where 0 is the 3×1 zero matrix). Using this latter definition, specify $\ker \mathsf{T}$ (note that it must be a vector subspace of $\mathbf{R}^5$), and show it has dimension $\dim \ker \mathsf{T} = 2$. *Hint:* View $Av = 0$ as a linear system of equations. Note that A is already in reduced row echelon form.

(d) (2 pt) Using your results to parts (a), (b), and (c), make a brief conjecture about the relationship among the three dimensions: (i) the dimension of the domain of T, i.e. $\mathbf{R}^5$; (ii) the dimension of the image of T; and (iii) the dimension of the kernel of T.

Exercise 6

(32 pt) Consider the linear map $T : \mathbf{R}^2 \to \mathbf{R}^2$ which (for a choice of basis for $\mathbf{R}^2$) is represented by the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}.$$

(For the purposes of this problem, we can focus on the matrix A and not worry about T.)

(a) (8 pt) By definition, the eigenvalues of A are scalars $\lambda \in \mathbf{R}$ such that there exists a nonzero (!) vector $v \in \mathbf{R}^2$ such that $Av = \lambda v$. We saw that this is equivalent to the condition $\det(A - \lambda I) = 0$. Using this latter condition, show that the eigenvalues of A are $\lambda = 2, 3$.

(b) (8 pt) For each eigenvalue in part (a), find an associated eigenvector. *Hint:* Recall we can find eigenvectors by solving the linear system of equations given by the matrix equation $(A - \lambda I)v = 0$, where in this case 0 is the 2×1 zero matrix. Note that we once we find an eigenvector v, we can check our work by computing Av and confirming it equals λv.

(c) (2 pt) Write the 2×2 matrix P you get by writing your two eigenvectors from part (b) side by side, as the two 2×1 columns in the matrix P.

(d) (8 pt) Compute P^{-1}. *Hint:* One way to do this is to row reduce the augmented matrix $\left[\begin{array}{c:c} A & I \end{array}\right]$. Note that we can check our work by computing PP^{-1} and confirming it equals the identity matrix I.

(e) (4 pt) Compute the matrix product $P^{-1}AP$. (You should get a diagonal matrix.) What do you notice about the diagonal entries?

(f) (2 pt) Based on your results above, make a brief conjecture about what you've discovered.