Math 211
Quiz 14

R 25 Jul 2019

Your name : ________________________________

Exercise

(2 pt) Consider the 3rd-order linear ODE

$$y^{(3)} + 6y^{(2)} + 11y^{(1)} + 6y = 0, \tag{1}$$

where $y^{(n)}$ denotes the nth (ordinary) derivative $\frac{d^n y}{dt^n}$. Using the change of variables

$$x_n = y^{(n)},$$

for $n = 0, 1, 2$, translate the 3rd-order ODE (1) into a 1st-order linear system

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}' = A \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix},$$

where A is a 3×3 matrix of constants. *Hint:* The entries in the third row of A should be related to the coefficients of the original ODE (1); the entries in the first two rows of A should all be 0 or 1.

(Not required : Compute the characteristic polynomial of the coefficient matrix A, i.e. the polynomial $\det(A - \lambda I)$. (This is the polynomial we've met before, whose roots are the eigenvalues of A.) How does this compare to the equation we get by replacing each $y^{(n)}$ in the original ODE (1) with λ^n? How do the roots of these two polynomials compare?)

Solution: Computing the derivatives of the new variables x_n, we find

$$\begin{aligned} x_0' &= (y)' = y^{(1)} = x_1 \\ x_1' &= (y^{(1)})' = y^{(2)} = x_2 \\ x_2' &= (y^{(2)})' = y^{(3)} = -6y^{(2)} - 11y^{(1)} - 6 = -6x_2 - 11x_1 - 6x_0. \end{aligned}$$

Writing this system of 1st-order ODEs as a matrix equation, we have

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_0' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} & & x_1 & & \\ & & & & x_2 \\ -6x_0 & - & 11x_1 & - & 6x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}.$$

The characteristic polynomial of the coefficient matrix A is

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -6-\lambda \end{bmatrix} = -\lambda^3 - 6\lambda^2 - 11\lambda - 6.$$

This is the same as the polynomial we get if we replace each $y^{(n)}$ on the left side of our original linear ODE (1) with λ^n, then multiply everything by -1.

By definition, a root of a polynomial p is an input λ that makes $p(\lambda) = 0$. Multiplying both sides of this equation by -1, we get $-p(\lambda) = -0 = 0$. Thus, a root of a polynomial p is also a root of $-p$, and vice versa. That is, a polynomial and -1 times that polynomial have the same roots. In our context here, this says that the roots of the polynomial we get from (1) are exactly the eigenvalues of the matrix A.