Math 112
Exam 01

2022–02–08 (T)

Your name: ______________________________

Instructions

Number of exercises	: 6
Permitted time	: 75 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- Work hard, do your best, and have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)	(f)
1	/12	/2	/2	/2	/2	/2	/2
2	/8	/4	/4				
3	/18	/2	/4	/4	/4	/4	
4	/16	/4	/4	/4	/4		
5	/14	/2	/4	/4	/4		
6	/12	/4	/4	/4			
Total	/80						

Exercise 1

(12 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = 2x - 2 \qquad\qquad g(x) = \frac{1}{2}x + 1$$

The functions f and g are inverse functions.

true false

Solution: True. We can check that, for all $x \in \mathbf{R}$,

$$f(g(x)) = f\left(\frac{1}{2}x + 1\right) = 2\left(\frac{1}{2}x + 1\right) - 2 = x$$

and

$$g(f(x)) = g\,(2x - 2) = \frac{1}{2}\,(2x - 2) + 1 = x$$

Thus f and g are inverse functions.

(b) (2 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be a function. If $f'(a) = 0$ for some input $a \in \mathbf{R}$, then $x = a$ is either a local minimum or a local maximum of f.

true false

Solution: False. As a counterexample, consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$. Then $f'(0) = 0$, but $x = 0$ is neither a local minimum nor a local maximum of f.

(c) (2 pt) For the graph of the unit circle $x^2 + y^2 = 1$, we cannot give an equation of the tangent line at every point on the circle.

true false

Solution: False. The two points we might wonder about are $(-1, 0)$ and $(1, 0)$, where the tangent line is vertical. However, we can give an equation for vertical lines. These lines are described by the equations $x = -1$ and $x = 1$, respectively.

(d) (2 pt) Let f be a function. Suppose that the domain (aka the set of inputs) of the second-derivative function f'' equals the domain of f. Then the domain of the first-derivative function f' also equals the domain of f.

true false

Solution: True. We can view the second-derivative function f'' as the first-derivative function of f', that is, $f'' = (f')'$. Thus the domain of f'' sits inside the domain of f', which in turn sits inside the domain of f. We can express this compactly as

$$\text{domain}(f'') \subseteq \text{domain}(f') \subseteq \text{domain}(f)$$

where the symbol "$\subseteq$" means "sits inside" or (more precisely) "is a subset of". We are told that $\text{domain}(f'') = \text{domain}(f)$. This forces the "middle" set $\text{domain}(f')$ to be equal to both of them, because $\text{domain}(f')$ contains $\text{domain}(f'')$ and is contained in $\text{domain}(f)$.

For parts (e) and (f), let f be a function defined on an open set containing a point a.

(e) (2 pt) If $\lim\limits_{x \uparrow a} f(x)$ (that is, the limit from the left) and $\lim\limits_{x \downarrow a} f(x)$ (that is, the limit from the right) exist, then $\lim\limits_{x \to a} f(x)$ exists.

true false

Solution: False. If both one-sided limits of $f(x)$ exist as x approaches a, and these one-sided limits are not equal, then the two-sided limit of $f(x)$ does not exist as x approaches a.

(f) (2 pt) If $\lim\limits_{x \to a} f(x)$ exists, then $\lim\limits_{x \uparrow a} f(x)$ (that is, the limit from the left) and $\lim\limits_{x \downarrow a} f(x)$ (that is, the limit from the right) exist.

true false

Solution: True. If the two-sided limit of $f(x)$ exists as x approaches a, then both one-sided limits of $f(x)$ exist as x approaches a (and, moreover, these one-sided limits are equal).

Exercise 2

(8 pt) Compute the following. (The answers are integers.)

(a) (4 pt) Let

$$e^a = 2 \qquad e^b = 3 \qquad e^c = 16$$

Compute

$$\sqrt{\frac{e^{4a+c}}{e^{2b}}} \cdot \frac{e^{6a-b+c}}{e^{4a+2c}} \cdot e^{2\ln 3}$$

Solution: We compute

$$\begin{aligned}\sqrt{\frac{e^{4a+c}}{e^{2b}}} \cdot \frac{e^{6a-b+c}}{e^{4a+2c}} \cdot e^{2\ln 3} &= e^{2a}e^{\frac{1}{2}c}e^{-b} \cdot e^{2a}e^{-b}e^{-c} \cdot 3^2 \\ &= 9\left(e^a\right)^4\left(e^b\right)^{-2}\left(e^c\right)^{-\frac{1}{2}} \\ &= 9(2)^4(3)^{-2}(16)^{-\frac{1}{2}} \\ &= 4\end{aligned}$$

(b) (4 pt) Let

$$\ln a = \frac{1}{3} \qquad \ln b = \frac{1}{5} \qquad \ln c = -\frac{1}{2}$$

Compute

$$\ln\left((ac+bc)^2\right) - \ln\left(a^2+2ab+b^2\right) + \ln\left(\frac{a^3b^5}{c^{-2}}\right)$$

Solution: We compute

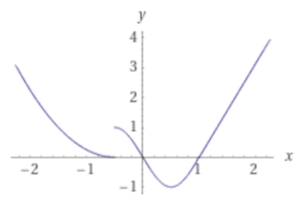
$$\begin{aligned}&\ln\left((ac+bc)^2\right) - \ln\left(a^2+2ab+b^2\right) + \ln\left(\frac{a^3b^5}{c^{-2}}\right) \\ &\quad = 2\ln c + 2\ln(a+b) - \ln\left((a+b)^2\right) + 3\ln a + 5\ln b + 2\ln c \\ &\quad = 3\ln a + 5\ln b + 4\ln c \\ &\quad = 0\end{aligned}$$

Exercise 3

(18 pt) Consider the piecewise function $f : \mathbf{R} \to \mathbf{R}$ whose rule of assignment is

$$f(x) = \begin{cases} \dfrac{1}{4}(2x+1)^2 & \text{if } x < -\frac{1}{2} \\ -\sin(\pi x) & \text{if } -\frac{1}{2} \leqslant x \leqslant 1 \\ \pi x - \pi & \text{if } x > 1 \end{cases}$$

A graph of f is shown below.



(a) (2 pt) Using the graph, identify the values of x at which $f(x)$ is not continuous.

Solution: From the graph, the only value of x at which $f(x)$ appears to be not continuous is $x = -\frac{1}{2}$. (It's hard to tell from the graph whether this value of x is exactly $-\frac{1}{2}$, but we can assert this with confidence using the algebraic rules of assignment for $f(x)$. Can you explain?)

(b) (4 pt) Justify, algebraically, that $f(x)$ is not continuous at the value(s) of x you identified in part (a).

Solution: The limit from the left of $f(x)$ as x approaches $-\frac{1}{2}$ is

$$\lim_{x \uparrow -\frac{1}{2}} f(x) = \lim_{x \uparrow -\frac{1}{2}} \left(\frac{1}{4}(2x+1)^2 \right) = \frac{1}{4} \left(2\left(-\frac{1}{2}\right) + 1 \right)^2 = 0$$

consistent with the graph. The limit from the right of $f(x)$ as x approaches $-\frac{1}{2}$ is

$$\lim_{x \downarrow -\frac{1}{2}} f(x) = \lim_{x \downarrow -\frac{1}{2}} (-\sin(\pi x)) = -\sin\left(-\frac{\pi}{2}\right) = 1$$

again consistent with the graph. Because the one-sided limits are not equal, we conclude that $f(x)$ is not continuous at $x = -\frac{1}{2}$.

(c) (4 pt) Is the first-derivative function $f'(x)$ continuous at $x = 1$? Justify.

Solution: Graphically, $f(x)$ looks "smooth" at $x = 1$. This suggests that the slope of the tangent line to the graph of $f(x)$ at $x = 1$ may be well defined — that is, that the first-derivative function $f'(x)$ may be continuous at $x = 1$. Let's investigate this, algebraically.

Using the rules of assignment for the given function f, we compute the rules of assignment for the first-derivative function f':[1]

$$f'(x) = \begin{cases} 2x+1 & \text{if } x < -\frac{1}{2} \\ -\pi\cos(\pi x) & \text{if } -\frac{1}{2} < x < 1 \\ \pi & \text{if } x > 1 \end{cases} \tag{1}$$

Note that we don't (yet) define the first-derivative function $f'(x)$ at the "break points" $x = -\frac{1}{2}$ and $x = 1$ of $f(x)$.

Because $f(x)$ is continuous at $x = 1$, we can determine whether $f'(x)$ is continuous at $x = 1$ by analyzing the one-sided limits of $f'(x)$ there. (Think about this! To see why we need $f(x)$ to be continuous at x for this to be valid, see part (e).) We compute

$$\lim_{x\uparrow 1} f'(x) = \lim_{x\uparrow 1} (-\pi\cos(\pi x)) = -\pi\cos(pi(1)) = \pi$$

and

$$\lim_{x\downarrow 1} f'(x) = \lim_{x\downarrow 1} pi = \pi$$

Thus the one-sided limits of $f'(x)$ at $x = 1$ are equal. Because $f(x)$ is continuous at $x = 1$, we conclude that the limit of $f'(x)$ as x approaches 1 exists, that $f'(x)$ is defined at $x = 1$, and that these values are equal. Hence $f'(x)$ is continuous at $x = 1$.

(d) (4 pt) Is the second-derivative function $f''(x)$ continuous at $x = 1$? Justify.

Solution: We find the rules of assignment for the second-derivative function f'' by differentiating the rules of assignment (1) for the first-derivative function f' that we computed in part (c):

$$f''(x) = \begin{cases} 2 & \text{if } x < -\frac{1}{2} \\ \pi^2\sin(\pi x) & \text{if } -\frac{1}{2} < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

As we did for f', we don't (yet) define the second-derivative function $f''(x)$ at the "break points" $x = -\frac{1}{2}$ and $x = 1$ of $f'(x)$.

Because $f(x)$ is continuous at $x = 1$, we can determine whether $f'(x)$ is continuous at $x = 1$ by analyzing the one-sided limits of $f'(x)$ there. We compute

$$\lim_{x\uparrow 1} f''(x) = \lim_{x\uparrow 1} \pi^2\sin(\pi) = 0 \qquad \text{and} \qquad \lim_{x\downarrow 1} f''(x) = \lim_{x\uparrow 1} 0 = 0$$

Thus the one-sided limits of $f''(x)$ at $x = 1$ are equal. Because $f'(x)$ is continuous at $x = 1$, we conclude (as in part (c)) that $f''(x)$ is continuous at $x = 1$.

[1] In differentiating the rules of assignment when $x < -\frac{1}{2}$ and when $-\frac{1}{2} \leqslant x \leqslant 1$, we use the chain rule. (If preferred, for the rule of assignment when $x < -\frac{1}{2}$, we can expand the square, then differentiate term by term.)

(e) (4 pt) Is the first-derivative function $f'(x)$ continuous at $x = -\frac{1}{2}$? Justify.

Solution: For $f'(x)$ to be continuous at $x = -\frac{1}{2}$, $f'(x)$ must be defined at $x = -\frac{1}{2}$. That is, $f(x)$ must be differentiable at $x = -\frac{1}{2}$. If $f(x)$ is differentiable at $x = -\frac{1}{2}$, then it is continuous at $x = -\frac{1}{2}$. However, we showed in part (b) that $f(x)$ is not continuous at $x = -\frac{1}{2}$. Thus $f'(x)$ cannot be continuous at $x = -\frac{1}{2}$.

Remark. If we try to compute the one-sided limits of $f'(x)$ as x approaches $-\frac{1}{2}$ by evaluating the two relevant rules of assignment for $f'(x)$ at $x = -\frac{1}{2}$, we find that both equal 0. The failure of $f'(x)$ to be continuous at $x = -\frac{1}{2}$ is coming from the fact that $f'(x)$ is not defined at $x = -\frac{1}{2}$. (This also explains why our limit approach is invalid here: It assumes that $f'(x)$ is defined at $x = -\frac{1}{2}$.)

Exercise 4

(16 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = 2x^3 - x^2 - 4x + 2$$

(a) (4 pt) Find the interval(s) on which f is increasing and decreasing.

Solution: We compute

$$f'(x) = 6x^2 - 2x - 4 = 2(3x^2 - x - 2) = 2(3x+2)(x-1)$$

Thus $f'(x) = 0$ if and only if $x = -\frac{2}{3}$ or $x = 1$. These are the "critical points" of f. By using the degree and sign of the leading term of the polynomial f, or by determining the sign of $f'(x)$ at values of x between and outside these critical points (for example, at $x = -1, 0, 2$), we conclude that

- f is increasing on $\left(-\infty, -\frac{2}{3}\right) \cup (1, +\infty)$ and
- f is decreasing on $\left(-\frac{2}{3}, 1\right)$.

(b) (4 pt) Find the x-coördinate of each local minimum and maximum of f. State whether each is a local minimum or maximum of f. Justify.

Solution: The critical points of f are the candidate local extrema,[2] so the x-coöordinates are $x = -\frac{2}{3}$ and $x = 1$. To determine whether each is a local minimum or maximum of f, we analyze the concavity of f. The second-derivative function of f is

$$f'' : \mathbf{R} \to \mathbf{R} \qquad \text{given by} \qquad f''(x) = (f'(x))' = 12x - 2$$

Evaluating $f''(x)$ at the two critical points of f, we find

$$f''\left(-\frac{2}{3}\right) = -10 < 0 \qquad\qquad f''(1) = 10 > 0$$

A negative second derivative at a point indicates the function is concave down there. A positive second derivative at a point indicates the function is concave up there. Thus, f is concave down at $x = -\frac{2}{3}$, so this is a local maximum; and f is concave up at $x = 1$, so this is a local minimum.

[2] If we were analyzing the function on a domain with boundary points, then the boundary points would also be candidate local extrema. For example, if the domain were the closed interval $[-3, 3]$, then the boundary points ± 3 would also be candidate local extrema.

(c) (4 pt) Find the global minimum and maximum of f.

Solution: The function f is a polynomial of odd degree. Thus one branch of the graph goes to $-\infty$, and the other branch of the graph goes to $+\infty$. (We can see this by evaluating $f(x)$ at values of x with large absolute value.) Thus f has no global minimum or global maximum.

(d) (4 pt) Find the x-coördinate of each inflection point of f.

Solution: By definition, an inflection point of $f(x)$ is a value of x at which the concavity of f changes sign. The concavity of f is captured by the sign of the second-derivative function f''. For the concavity of f to change sign at x, we must have $f''(x) = 0$. Values of x that satisfy $f''(x) = 0$ are the candidate inflection points.

In part (b) we found that

$$f''(x) = 12x - 2$$

Setting $f''(x)$ equal to 0 and solving for x, we find

$$0 \underset{\text{set}}{=} f''(x) = 12x - 2 \qquad \Leftrightarrow \qquad x = \frac{1}{6}$$

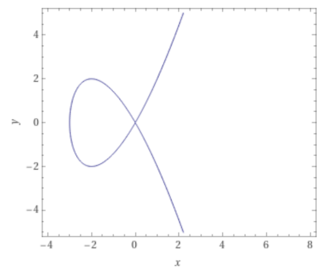
The sign of $f''(x)$ indeed changes, from negative to positive, as the value of x passes through $x = \frac{1}{6}$ (why?). Thus $x = \frac{1}{6}$ is the x-coördinate of an inflection point of f.

Exercise 5

(14 pt) The graph of the equation

$$y^2 = x^3 + 3x^2 \tag{2}$$

shown below, is an elliptic curve. It is said to be "singular" (due to its behavior at the origin).



(a) (2 pt) Using the graph, estimate the (x, y)-coördinates of the two points on the graph at which the tangent line to the graph is horizontal.

Solution: From the graph, it looks like the curve has two points at which the tangent line to the curve is horizontal. These points are roughly $(-2, \pm 2)$.

(b) (4 pt) Compute the rule of assignment for y'.

Solution: Differentiating the given equation with respect to x, we get[3]

$$2yy' = 3x^2 + 6x \qquad \Leftrightarrow \qquad y' = \frac{3x^2 + 6x}{2y}$$

(c) (4 pt) Using the rule of assignment you computed in part (b), determine, algebraically, the points (x, y) on the graph at which the tangent line is horizontal. Compare these results to your estimates in part (a).

Solution: The slope of the tangent line to a graph at a point (geometry) equals the first derivative of the function evaluated at that point (algebra). A horizontal line has slope 0. Thus, the graph has a horizontal tangent line at the point (x_0, y_0) on the graph if and only if $y'(x_0, y_0) = 0$. Using our expression for y' found in part (b), we set

$$0 \underset{\text{set}}{=} y' = \frac{3x^2 + 6x}{2y}$$

If $y \neq 0$ (we'll return to the case when $y = 0$ later), then we can multiply both sides of this equation by $2y$, getting

$$0 = 3x^2 + 6x = 3x(x + 2)$$

This equation has the solutions $x = 0$ or $x = -2$.

[3]We use implicit differentiation and the chain rule to differentiate the left side of Equation (2).

When $x = -2$, we can substitute this into the original equation (2) describing the graph. Solving for y, we find

$$y^2 = (-2)^3 + 3(-2)^2 = 4 \qquad \Leftrightarrow \qquad y = \pm 2$$

This agrees with our estimates in part (a) of where the graph has horizontal tangent lines.

When $x = 0$, we can do the same computation. Here we obtain $y = 0$. The point $(x, y) = (0, 0)$ is a point on the graph, but our analysis of y' assumed that $y \neq 0$. Thus we can't use that analysis to conclude anything (valid) about the tangent lines to graph when $y = 0$. From the graph, we can see there's some funny intersection business going on at $(0, 0)$, and neither branch of the curve there has a horizontal tangent.

(d) (4 pt) Find an equation for the tangent line to the graph at the point $(6, -18)$.

Solution: We quickly check that the point $(x_0, y_0) = (6, -18)$ is indeed on the graph:

$$y_0^2 = (-18)^2 = 324 = 216 + 3(36) = 6^3 + 3(6)^2 = x_0^3 + 3x_0^2$$

The equation is satisfied at (x_0, y_0), so this point is indeed on the graph.

The linearization L to the curve at the point (x_0, y_0) is the function

$$L : \mathbf{R} \to \mathbf{R} \qquad \text{given by} \qquad L(x) - y_0 = y'(x_0, y_0) \cdot (x - x_0)$$

Solving this rule of assignment for $L(x)$, and substituting in the values $(x_0, y_0) = (6, -18)$ and

$$y'(6, -18) = \frac{3(6)^2 + 6(6)}{2(-18)} = \frac{4(36)}{-36} = -4$$

we get

$$L(x) = -18 - 4(x - 6) = -4x + 6$$

Exercise 6

(12 pt) Two cyclists, training for Beer Bike, start at the intersection by the Rec Center, rear wheels touching.[4] They start pedaling simultaneously, one cyclist hurtling southwest (toward Rice Stadium) at 9 meters per second (about 21 miles per hour), the other cyclist flying southeast (toward Tudor Fieldhouse and the track, ish) at 12 meters per second (about 28 miles per hour).[5]

(a) (4 pt) Sketch a diagram. Label relevant information.

Solution: One possible diagram: Right triangle, one leg 90 meters long with velocity vector 9 m/s pointing away from the right angle, the other leg 120 meters long with velocity vector 12 m/s pointing away from the right angle, hypotenuse (at that point in time) 150 meters.

(b) (4 pt) Write an equation that relates relevant variables and does not (!) involve rates. Justify briefly why this equation is true. Using implicit differentiation, differentiate the equation.

Solution: At any time t after the cyclists start, their two current positions and their starting point form the three vertices of a right triangle. Thus by the Pythagorean theorem, at any time t,

$$(a(t))^2 + (b(t))^2 = (c(t))^2$$

Differentiating this equation with respect to time t, we get

$$2a(t)a'(t) + 2b(t)b'(t) = 2c(t)c'(t)$$

which we can simplify to

$$a(t)a'(t) + b(t)b'(t) = c(t)c'(t) \tag{3}$$

(c) (4 pt) How fast (in meters per second) are the cyclists moving apart 10 seconds after they started? Measure "moving apart" along the straight line connecting the two cyclists at that point in time.

Solution: Ten seconds after the cyclists start, they have moved

$$a(10\text{ s}) = (9\text{ m/s})(10\text{ s}) = 90\text{m} \qquad \text{and} \qquad b(10\text{ s}) = (12\text{ m/s})(10\text{ s}) = 120\text{ m}$$

We can find the distance between them after 10 seconds by substituting these values into the Pythagorean theorem and solving for $c(10\text{ s})$ (or by noting that the right triangle formed is always a scalar multiple of a 3, 4, 5 right triangle, with the scalar given by the time in seconds since the start):

$$c(10\text{ s}) = 150\text{ m}$$

Solving the related rates equation (3) for $c'(t)$, we get

$$c'(t) = \frac{a(t)a'(t) + b(t)b'(t)}{c(t)}$$

Substituting in these values and the cyclists' speeds $a'(t) = 9$ m/s and $b'(t) = 12$ m/s, we get

$$c'(10\text{ s}) = \frac{(90\text{ m})(9\text{ m/s}) + (120\text{ m})(12\text{ m/s})}{150\text{ m}} = 15\text{ m/s}$$

[4] Yes — somehow, their colleges convinced RUPD to close off campus roads to give us (and them) this exercise.

[5] Yes — somehow, these cyclists are able to accelerate instantaneously to racing speed, and hold it perfectly.