Math 112
ReQuiz 04A

2022–02–01 (T)

Your name: ______________________________

Exercise

(4 pt) Consider the function

$$f : (-2, 2) \to \mathbf{R} \qquad \text{given by} \qquad f(x) = \cos\left(\frac{x}{\sqrt{4-x^2}}\right)$$

where as usual $\mathbf{R}$ denotes "all real numbers". Find the derivative function f'. Be sure to specify its domain and codomain, in addition to the rule(s) of assignment.

Solution: If desired, we may write the rule of assignment for $f(x)$ equivalently as[1]

$$f(x) = \cos\left(x\left(4-x^2\right)^{-\frac{1}{2}}\right)$$

Using the chain rule (several times) and the product rule (or quotient rule), we compute[2]

$$\begin{aligned}
f'(x) &= -\sin\left(\frac{x}{\sqrt{4-x^2}}\right)\left(\left(4-x^2\right)^{-\frac{1}{2}} + x\left(-\frac{1}{2}\right)\left(4-x^2\right)^{-\frac{3}{2}}(-2x)\right) \\
&= -\sin\left(\frac{x}{\sqrt{4-x^2}}\right)\left(\left(4-x^2\right)^{-\frac{1}{2}} + x^2\left(4-x^2\right)^{-\frac{3}{2}}\right) \\
&= -\sin\left(\frac{x}{\sqrt{4-x^2}}\right)\frac{\left(4-x^2\right)+x^2}{(4-x^2)^{\frac{3}{2}}} \\
&= -\sin\left(\frac{x}{\sqrt{4-x^2}}\right)\frac{4}{(4-x^2)^{\frac{3}{2}}} \\
&= -\frac{4\sin\left(\frac{x}{\sqrt{4-x^2}}\right)}{(4-x^2)^{\frac{3}{2}}}
\end{aligned}$$

This rule of assignment is undefined if and only if either

(i) $4-x^2 < 0$, for then we would take the square root of a negative number, which is not possible in the real numbers; or

(ii) $4-x^2 = 0$, for then we would divide by zero, which is not possible.

These two conditions imply the rule of assignment is defined if and only if $-2 < x < 2$, that is, on the interval $(-2, 2)$. This is the same as the domain of f. Thus we conclude that the domain of f' is $(-2, 2)$. The codomain of f' we may take to be $\mathbf{R}$ (all real numbers).

[1] I find rewriting quotients as products, and rewriting roots as fractional exponents, helps me compute derivatives.

[2] Any of these expressions for $f'(x)$ are "correct". The last two may be preferred, as they present the rule of assignment simply, without extraneous computation (for example, the "$-x^2+x^2$" appearing in line 3).