Math 112
Exam 04

2022–04–30 (S)

Your name: ______________________________

Instructions

Number of exercises	: 16
Permitted time	: 3 hours
Permitted resources	: None

Remarks:

- This exam has three sections, corresponding to the three midterm exams.
- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- Work hard, do your best, and have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/10	/2	/4	/4		
3	/10	/4	/4	/2		
4	/12	/2	/2	/4	/2	/2
5	/8	/2	/2	/4		
Part 1	/50					
6	/10	/2	/2	/2	/2	/2
7	/8	/4	/4			
8	/8	/4	/4			
9	/12	/4	/4	/4		
10	/12	/2	/4	/4	/2	
Part 2	/50					
11	/10	/2	/2	/2	/2	/2
12	/8	/4	/4			
13	/8	/4	/4			
14	/4					
15	/10	/4	/4	/2		
16	/10	/2	/4	/4		
Part 3	/50					
TOTAL	/150					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) Let $f : [0, +\infty) \to \mathbf{R}$ be given by $f(x) = x^2$, and let g be the inverse function to f. The domain of g' equals the domain of g.

true false

(b) (2 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be a function. If $f'(a) = 0$, then $x = a$ is either a local minimum or a local maximum of f.

true false

(c) (2 pt) The equation $x = -2$ describes a tangent line to the graph of $x^2 + y^2 = 4$.

true false

(d) (2 pt) Let $e^a = 4$, $e^b = 2$, and $e^{3c} = \frac{1}{\sqrt{8}}$. Then

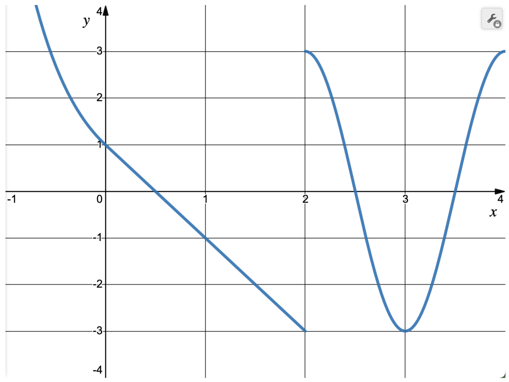
$$e^{3\ln 2} \cdot \frac{e^{5b+3c}}{e^{a+6b-3c}} \cdot \sqrt{\frac{e^{4a+b}}{e^{2a-b}}} = 1$$

true false

(e) (2 pt) Let $\ln a = 3$, $\ln b = 5$, and $\ln c = -\frac{1}{4}$. Then

$$2\ln\left(\frac{1}{\sqrt{e}}\right) + \ln\left(\frac{a+b}{c^3}\right) - \ln\left(\frac{bc(a+b)}{a^2}\right) = 1$$

true false



Exercise 2

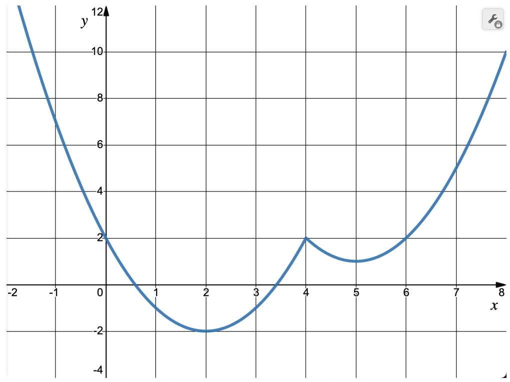
(10 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the piecewise function graphed above and given by

$$f(x) = \begin{cases} e^{-2x} & \text{if } x < 0 \\ -2x + 1 & \text{if } 0 \leqslant x \leqslant 2 \\ 3\cos(\pi x) & \text{if } x > 2 \end{cases}$$

(a) (2 pt) Using the graph, identify the value(s) of x at which $f(x)$ is not continuous.

(b) (4 pt) Justify, algebraically, that $f(x)$ is not continuous at the value(s) of x identified in part (a).

(c) (4 pt) Is the first-derivative function $f'(x)$ continuous at $x = 0$? Justify algebraically.



Exercise 3

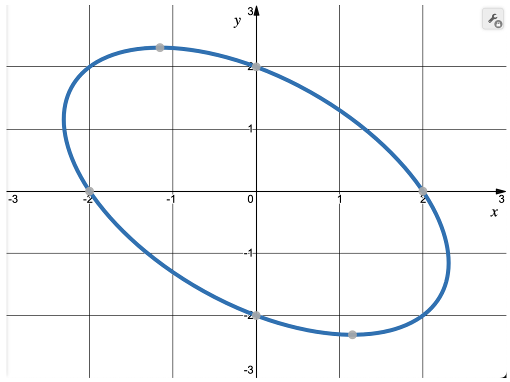
(10 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the piecewise function graphed above and given by

$$f(x) = \begin{cases} x^2 - 4x + 2 & \text{if } x \leqslant 4 \\ x^2 - 10x + 26 & \text{if } x \geqslant 4 \end{cases}$$

(a) (4 pt) Find the intervals on which f is increasing and decreasing. Justify algebraically.

(b) (4 pt) Find the x-coördinate of each local minimum and maximum of f. State whether each is a local minimum or maximum of f. Justify algebraically.

(c) (2 pt) Find the global minimum and maximum of f (x and y values). Justify algebraically.



Exercise 4

(12 pt) Consider the graph, above, of the ellipse given by the equation

$$x^2 + xy + y^2 = 4$$

(a) (2 pt) From the graph, the points $(x, y) = (-2, 2)$ and $(2, -2)$ appear to be on the ellipse. Prove this, algebraically.

(b) (2 pt) Using the graph, predict the slope of the tangent line to the graph at $(x, y) = (-2, 2)$.

(c) (4 pt) Compute the rule of assignment for y'. (Your answer will involve both x and y.)

(d) (2 pt) Find an equation for the tangent line to the graph at the point $(x, y) = (-2, 2)$.

(e) (2 pt) Find the x-coördinate of each point on the ellipse at which the tangent line is horizontal.

Exercise 5

(8 pt) You are pouring melted ice cream into an ice-cream cone at a steady rate of π cm^3 per second. The ice-cream cone has a height of 16 cm and a diameter of 8 cm. The volume V of a cone with height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$. This exercise explores how fast the level of liquid in the ice-cream cone is rising. Note that, as liquid builds up in the cone, it always forms a smaller cone with dimensions similar to (i.e. scaled down from) the ice-cream cone.

(a) (2 pt) Sketch a diagram and identify relevant variables.

(b) (2 pt) Use implicit differentiation to relate the rate of change of volume of liquid in the ice-cream cone to the rates of change of the radius and the height of the liquid cone.

(c) (4 pt) How fast is the level of liquid in the ice-cream cone rising when the cone is half-full? *Hint:* Justify why, at all relevant times t, the radius r and height h of the liquid cone satisfy $h = 4r$.

Exercise 6

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) If direct evaluation of a limit gives an indeterminate form that is not $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then the limit does not exist.

true false

(b) (2 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Consider lower- and upper-sum approximations for $\displaystyle\int_a^b f(x)\ dx$. Any lower sum is less than or equal to any upper sum, even if we use different partitions for the lower sum and upper sum.

true false

(c) (2 pt) Let $f(x)$ be a continuous function, and let $F_1(x)$ and $F_2(x)$ be antiderivatives of $f(x)$. Then $F_1'(x) - F_2'(x) = 0$.

true false

For parts (d)–(e), let $f(x)$ and $g(x)$ be functions such that

$$\int_{-1}^{3} f(x)\ dx = 8 \qquad \int_{-1}^{3} g(x)\ dx = -4$$

(d) (2 pt) $\displaystyle\int_{-1}^{3} \left[\frac{1}{2}f(x) - 2g(x)\right]\ dx = \int_{-1}^{3} [f(x) - g(x)]\ dx$

true false

(e) (2 pt) The average value of $f(x) + g(x)$ on the interval $[-1, 3]$ equals 1.

true false

Exercise 7

(8 pt) Evaluate each limit to verify the result. Briefly but clearly justify your work.

(a) (4 pt) $\lim_{x \to +\infty} \dfrac{\ln x}{\sqrt{x}} = 0$

(b) (4 pt) $\lim_{x \to 1} \dfrac{3x - 3}{-1 + \sqrt{3x - 2}} = 2$

Exercise 8

(8 pt) This exercise considers the limit

$$\lim_{x \to 0} \frac{6e^x - 6(x+1) - 3x^2}{2x^3} \tag{1}$$

(a) (4 pt) Evaluate the limit in (1) using the Taylor series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

where the terms in "..." all involve x to the power 4 or higher.

(b) (4 pt) Evaluate the limit in (1) using l'Hôpital's rule.

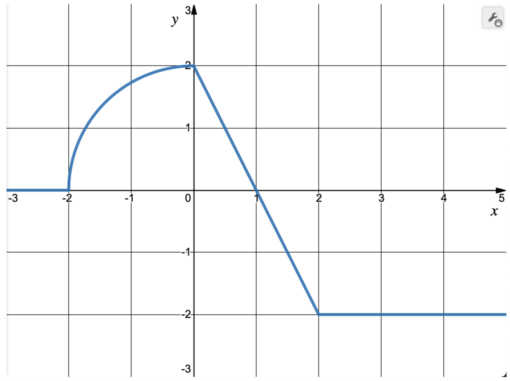
Exercise 9

(12 pt) Evaluate each indefinite integral. That is, find the most-general antiderivative of each integrand.

(a) (4 pt) $\int e^x + 2\sin x \, dx$

(b) (4 pt) $\int \frac{x^4 - 1}{x^2} \, dx$

(c) (4 pt) $\int \left(e^x + e^{-x}\right)\left(e^x - e^{-x}\right) \, dx$



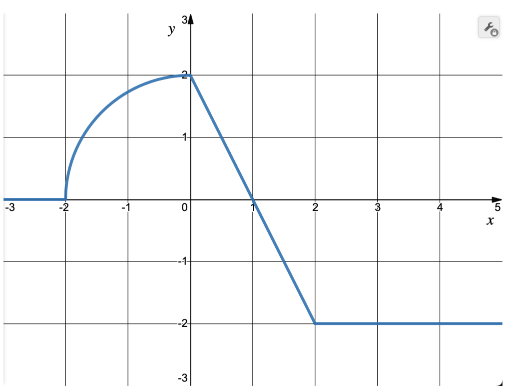
Exercise 10

(12 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the piecewise function graphed above and given by

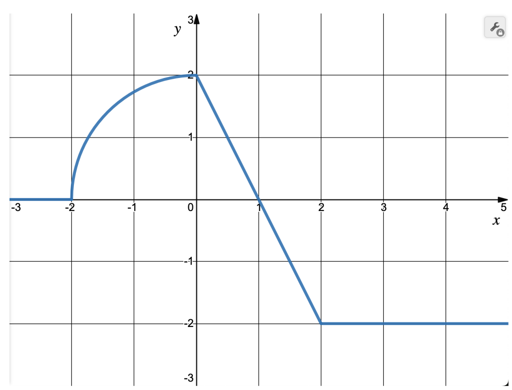
$$f(x) = \begin{cases} 0 & \text{if } x \leqslant -2 \\ \sqrt{4-x^2} & \text{if } -2 \leqslant x \leqslant 0 \\ 2-2x & \text{if } 0 \leqslant x \leqslant 2 \\ -2 & \text{if } x \geqslant 2 \end{cases}$$

(a) (2 pt) Use finite geometry to show that $\displaystyle\int_{-2}^{4} f(x)\,dx = \pi - 4$.

(b) (4 pt) On separate graphs below, draw a lower sum L_3 and an upper sum U_3, each with three subintervals of width 2, to estimate $\displaystyle\int_{-2}^{4} f(x)\,dx$. Compute the values of L_3 and U_3.

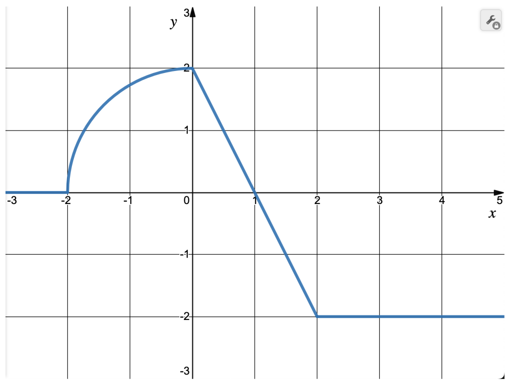


Lower sum (L_3)

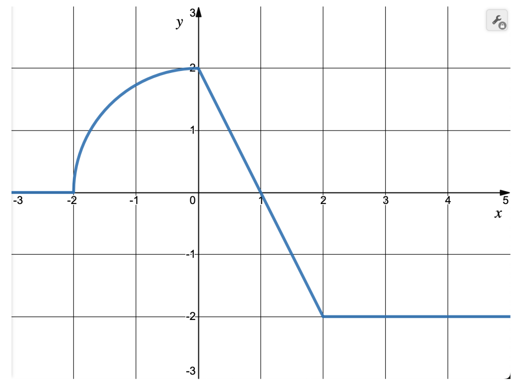


Upper sum (U_3)

(c) (4 pt) On separate graphs below, draw a lower sum L_6 and an upper sum U_6, each with six subintervals of width 1, to estimate $\int_{-2}^{4} f(x)\,dx$. Compute the values of L_6 and U_6. You may leave your answer in terms of $\sqrt{3} \approx 1.7$.



Lower sum (L_6)

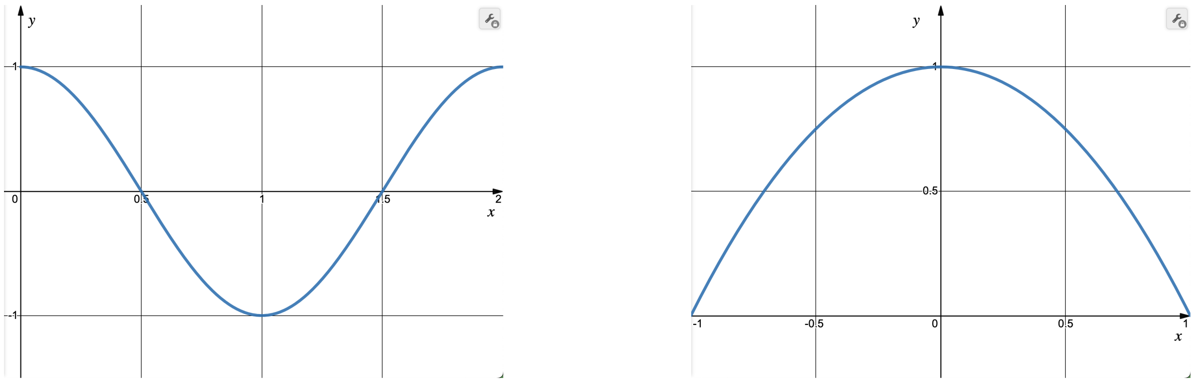


Upper sum (U_6)

(d) (2 pt) You compute a lower sum L_{12} and an upper sum U_{12}, each with twelve subintervals of width $\frac{1}{2}$, to estimate $\int_{-2}^{4} f(x)\,dx$. You find

$$L_{12} = -6 \qquad\qquad U_{12} = -1$$

Explain why these cannot be the correct values of L_{12} and U_{12}.



Graph of $g(x)$ for parts (a)–(b). Graph of $F(x)$ for parts (c)–(e).

Figure 1: Graphs for Exercise 11.

Exercise 11

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.
For parts (a)–(b), let $g : [0, 2] \to \mathbf{R}$ be the function given by $g(x) = \cos(\pi x)$, graphed in Figure 1.

(a) (2 pt) There are exactly three values b, with $0 \leqslant b \leqslant 2$, such that $\displaystyle\int_0^b g(x)\,dx = 0$.

true false

(b) (2 pt) There are no values b, with $0 \leqslant b \leqslant 2$, such that $\displaystyle\int_0^b g(x)\,dx < 0$.

true false

For parts (c)–(e), let $f : [-1, 1] \to \mathbf{R}$ be a continuous function, and let $F : [-1, 1] \to \mathbf{R}$ be the cumulative signed area function graphed in Figure 1, given by

$$F(x) = \int_{-1}^{x} f(t)\,dt$$

(c) (2 pt) For all x in the interval $[-1, 1]$, $f(x) < 0$.

true false

(d) (2 pt) For all x in the interval $[-1, 1]$, $f'(x) < 0$.

true false

(e) (2 pt) The average value of f on the interval $[-1, 0]$ equals $\frac{1}{2}$.

true false

Exercise 12

(8 pt) Evaluate each indefinite integral. Clearly communicate your approach.

(a) (4 pt) $\int 2t\left(\sin(t^2+1)\right)^4 \cos(t^2+1)\, dt$

(b) (4 pt) $\int x^2 e^{-2x}\, dx$

Exercise 13

(8 pt) Evaluate each definite integral to verify the result. Clearly communicate your approach.

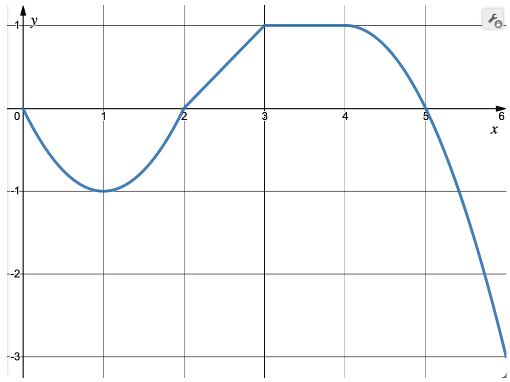
(a) (4 pt) $\int_{-\frac{1}{2}}^{\frac{1}{2}} (1-2x)^3 \, dx = 2$

(b) (4 pt) $\int_0^1 (x^2+1)e^{x^3+3x} \, dx = \dfrac{e^4-1}{3}$

Exercise 14

(4 pt) Use the fundamental theorem of calculus to compute the derivative. Assume $x \geqslant 0$.

$$\frac{d}{dx} \int_{4x^2}^{9x^2} e^{\sqrt{t}}\, dt$$



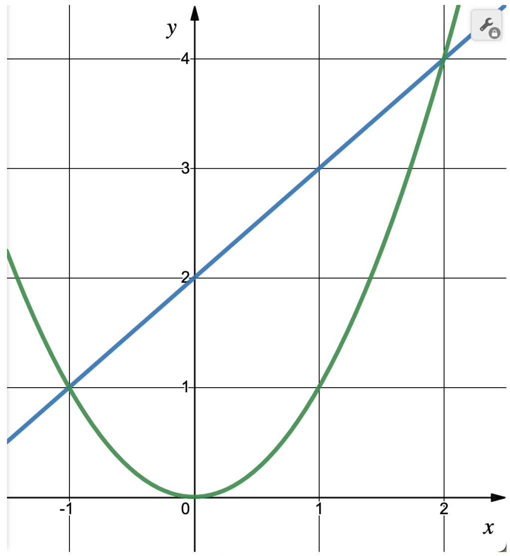
Exercise 15

(10 pt) Let $f : [0, 6] \to \mathbf{R}$ be a piecewise function. A graph of $F(x) = \int_0^x f(t)\, dt$ is shown above.

(a) (4 pt) On which intervals is f positive? negative? equal to zero?

(b) (4 pt) On which intervals is f increasing? decreasing? constant?

(c) (2 pt) Find the average value of f on the interval $[0, 6]$.



Exercise 16

(10 pt) Consider the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = x^2 \qquad\qquad g(x) = x + 2$$

respectively. Graphs of f and g appear above.

(a) (2 pt) Using the graphs, write the two points (x, y) of intersection of f and g. Approximate the area between the graphs of f and g. Briefly explain the reasoning behind your approximation.

(b) (4 pt) Write and evaluate a single definite integral to find the area between the graphs of f and g.

(c) (4 pt) If we "tilt our heads to the right ninety degrees" and view the graphs of f and g as having input variable y instead of x—that is, solving $y = f(x)$ and $y = g(x)$ for x—we get

$$F(y) = \pm\sqrt{y} \qquad\qquad G(y) = y - 2$$

Using the graphs at the beginning of this exercise, explain how the following integrals compute the same area between the graphs that you computed in part (b):

$$\int_0^1 \sqrt{y} - (-\sqrt{y})\, dy + \int_1^4 \sqrt{y} - (y-2)\, dy$$