Math 211
Quiz 17

T 30 Jul 2019

Your name : ______________________________

Exercise

(5 pt) For each of the following linear maps $T : \mathbf{R}^n \to \mathbf{R}^m$, where $\mathbf{R}^n, \mathbf{R}^m$ are viewed as vector spaces over $\mathbf{R}$,

(i) write a basis for the image $\text{im}(T)$ and a basis for the kernel $\ker(T)$,

(ii) find the dimensions $\dim(\text{im}(T))$ and $\dim(\ker(T))$, and

(iii) confirm the rank–nullity theorem:

$$\dim(\text{domain}(T)) = \dim(\text{im}(T)) + \dim(\ker(T)).$$

Hint: For each linear map T, write a corresponding matrix representation A, so that $T(x) = y$ corresponds to the matrix equation $Ax = y$. Then focus on the pivot or nonpivot columns of A.

(a) (2.5 pt) The linear map T_1 given by

$$T_1 : \mathbf{R}^4 \to \mathbf{R}^3$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 & + & & & x_3 \\ & & x_2 & - & x_3 \\ & & 0 & & \end{bmatrix}.$$

Solution: The corresponding matrix A_1 is

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is already in reduced row echelon form (RREF).

Image. The pivot columns are columns 1 and 2. They form a basis for the image of T_1:

$$\text{basis}(\text{im}(T_1)) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right),$$

The number of pivot columns equals the dimension of the image:

$$\dim(\text{im}(T_1)) = 2.$$

Kernel. The other two columns, 3 and 4, correspond to free variables, namely x_3 and x_4. The number of free variables equals the dimension of the kernel:

$$\dim(\ker(T_1)) = 2.$$

Solving the system of equations represented by the matrix equation $A_1 x = 0$, where x is a 4×1 matrix of variables x_i and 0 is the 3×1 zero matrix, gives the kernel of T_1:

$$\ker(T_1) = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} \mid x_3, x_4 \in \mathbf{R} \right\} = \left\{ x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbf{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The two vectors in this last expression for $\ker(T_1)$ are linearly independent, and hence they form a basis for $\ker(T_1)$.

Rank–nullity theorem. For T_1, we have

$$\dim(\mathrm{domain}(T_1)) = \dim(\mathbf{R}^4) = 4 = 2 + 2 = \dim(\mathrm{im}(T_1)) + \dim(\ker(T_1)),$$

confirming that the rank–nullity theorem holds for T_1.

(b) (2.5 pt) The linear map T_2 given by

$$T_2 : \mathbf{R}^4 \to \mathbf{R}^3$$
$$\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 & + & x_2 & + & 2x_3 & + & x_4 \\ 3x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 \end{bmatrix}.$$

Solution: The corresponding matrix A_2 is

$$A_2 = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Applying the row reduction algorithm to this matrix, we get

$$\mathrm{RREF}(A_2) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Image. The pivot columns are columns 1, 2, and 3. These columns, in the original matrix A_2 (!), form a basis for the image of T_2:

$$\mathrm{basis}(\mathrm{im}(T_2)) = \left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right).$$

The number of pivot columns equals the dimension of the image of T_2:

$$\dim(\mathrm{im}(T_2)) = 3.$$

N.B. Because $\mathrm{im}(T_2)$ is a subspace of the vector space $\mathbf{R}^3$, and their dimensions are equal — $\dim(\mathrm{im}(T_2)) = \dim(\mathbf{R}^3)$ — this subspace must be the entire vector space. That is,

$$\mathrm{im}(T_2) = \mathbf{R}^3.$$

Kernel. The nonpivot column, column 4, corresponds to the free variable, namely x_4. The number of free variables equals the dimension of the kernel:

$$\dim(\ker(T_2)) = 1.$$

The set of solutions to the system of equations represented by the matrix equation $A_2x = 0$ is equivalent to the set of solutions to the system of equations represented by the matrix equation

$\mathrm{RREF}(A_2)x = 0$; in both equations, x is a 4×1 matrix of variables x_i and 0 is the 3×1 zero matrix. This set of solutions is the kernel of T_2:

$$\ker(T_2) = \left\{ \begin{bmatrix} x_4 \\ -4x_4 \\ x_4 \\ x_4 \end{bmatrix} \mid x_4 \in \mathbf{R} \right\} = \left\{ x_4 \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \end{bmatrix} \mid x_4 \in \mathbf{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

The vector in this last expression for $\ker(T_2)$ is a basis for $\ker(T_2)$.

Rank–nullity theorem. For T_2, we have

$$\dim(\mathrm{domain}(T_2)) = \dim(\mathbf{R}^4) = 4 = 3 + 1 = \dim(\mathrm{im}(T_2)) + \dim(\ker(T_2)),$$

confirming that the rank–nullity theorem holds for T_2.