

Math 212

HalfExam 02

N 30 Oct 2016

Your name: _____

Exam instructions

Number of exercises : 11
Permitted time : 120 minutes
Permitted resources : None

Instructor's note:

- Hints to each exercise are provided on the final page. Work through the entire exam prior to looking at the hints, then use the hints to help you with the challenging exercises.
- Manage your time deliberately.
- An effort has been made to make the exercises as clear and unambiguous as possible. If the statement of an exercise seems unclear, briefly (one sentence) write your understanding of the exercise, then proceed.

Exercise	Total
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
11	/10
Total	/110

Exercise 1

Let $D = \{(x, y, z) \in \mathbf{R}^3 \mid z \neq 0\}$, and let $f : D \rightarrow \mathbf{R}$ be given by

$$f(x, y, z) = \frac{x + y}{z}.$$

Find the maximum rate of change of f at the point $(1, 1, -1)$ and the direction in which it occurs.

Solution: The maximum rate of change of a function f occurs in the direction of its gradient vector ∇f , and its value equals the norm $\|\nabla f\|$.

We compute

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{1}{z}, \frac{1}{z}, -\frac{x + y}{z^2} \right).$$

At the given point $(1, 1, -1)$, we have

$$(\nabla f)(1, 1, -1) = (-1, -1, -2), \quad \|(\nabla f)(1, 1, -1)\| = \sqrt{6}.$$

Thus the maximum rate of change of f at the point $(1, 1, -1)$ is $\sqrt{6}$, and it occurs in the direction of the vector $(-1, -1, -2)$.

Exercise 2

The ideal gas law states that

$$PV = nRT, \quad (1)$$

where P is pressure (in kPa), V is volume (in L), n is number of moles, $R = 8.3144 \text{ J}/(\text{mol K})$, and T is temperature (in K). (You may ignore units; they are chosen so that they work out.¹)

The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s , and its temperature is increasing at a rate of 0.15 K/s . Find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K .

Solution: Note that R is a constant. In this exercise, n (the number of moles of gas) is also constant. The remaining three variables P, V, T are functions of time t . We present two solutions.

Method 1: Chain rule. Solving (1) for V as a function of P, T , we find

$$V(P, T) = V = \frac{nRT}{P}.$$

Differentiating this expression with respect to time t , using the chain rule, we have

$$\frac{dV}{dt} = \frac{\partial V}{\partial P} \frac{dP}{dt} + \frac{\partial V}{\partial T} \frac{dT}{dt} = \left(-\frac{nRT}{P^2} \right) \frac{dP}{dt} + \left(\frac{nR}{P} \right) \frac{dT}{dt}. \quad (2)$$

Substituting in the given values, we compute

$$\frac{dV}{dt} = \left(-\frac{(1)(8.3144)(320)}{(20)^2} \right) (.05) + \left(\frac{(1)(8.3144)}{(20)} \right) (.15) \approx -.2702 \text{ L/s}.$$

Method 2: Direct. Differentiating both sides of (1) with respect to t (using the product rule on the left, because both P and V are functions of T), we have

$$\frac{dP}{dt} V + P \frac{dV}{dt} = nR \frac{dT}{dt}.$$

Solving for the desired quantity $\frac{dV}{dt}$, we find²

$$\frac{dV}{dt} = P^{-1} \left(nR \frac{dT}{dt} - V \frac{dP}{dt} \right) = \left(\frac{nR}{P} \right) \frac{dT}{dt} - \left(\frac{nRT}{P^2} \right) \frac{dP}{dt},$$

where in the second equality we have used the ideal gas law (1) to write $V = \frac{nRT}{P}$. Substituting in the given values yields $\frac{dV}{dt} \approx -.2702 \text{ L/s}$.

¹For those of us who mind our units, note that

$$1 \text{ J} = 1 \text{ Pa m}^3 = 1000 \text{ Pa L} = 1 \text{ kPa L}.$$

²Note that the final expression is equal to the final expression in (2).

Exercise 3

Let $D \subseteq \mathbf{R}^2$ be the closed disc

$$D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 2\},$$

and let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be the function given by

$$f(x, y) = x^2 + y^2 - x - y + 1.$$

Does f achieve a global minimum and maximum on D ? Explain. If yes, find the corresponding values.

Solution: The function f is a polynomial, hence continuous on its entire domain \mathbf{R}^2 , and in particular on D . The disc D is a closed and bounded set. Hence by the extreme value theorem, f achieves a global minimum and maximum on D .

To find the corresponding global minimum and maximum values, we find all critical points on the interior of D and on the boundary of D separately, then compare values.

Interior: Because f is a polynomial, it is everywhere differentiable (i.e. its gradient ∇f is defined everywhere), so all critical points on the interior of D are given by $\nabla f = \mathbf{0}$. We compute

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x - 1, 2y - 1).$$

Thus

$$\nabla f = \mathbf{0} \quad \Leftrightarrow \quad (x, y) = \left(\frac{1}{2}, \frac{1}{2} \right).$$

Note that this point is indeed in D . Thus $(\frac{1}{2}, \frac{1}{2})$ is the unique critical point on the interior of D .

Boundary: The boundary of D is the circle of radius $\sqrt{2}$ in \mathbf{R}^2 , which we can parametrize by

$$t \mapsto (\sqrt{2} \cos t, \sqrt{2} \sin t), \quad t \in [0, 2\pi].$$

Substituting $x = \sqrt{2} \cos t, y = \sqrt{2} \sin t$ into f , we have

$$f(x, y) = f(\cos t, \sin t) = 2 - \sqrt{2} \cos t - \sqrt{2} \sin t + 1 = 3 - \sqrt{2} \cos t + \sqrt{2} \sin t,$$

a real-valued function of a single variable t ; denote this function by $g(t)$. Note that $g(t)$ is everywhere differentiable, so all critical points of g are given by $g'(t) = 0$. We compute

$$g'(t) = \sqrt{2} \sin t - \sqrt{2} \cos t.$$

Thus

$$g'(t) = 0 \quad \Leftrightarrow \quad \tan t = 1 \quad \Leftrightarrow \quad t = \frac{\pi}{4} \text{ or } t = \frac{5\pi}{4}.$$

The points (x, y) corresponding to these values of t are

$$\frac{\pi}{4} : (1, 1) \quad \frac{5\pi}{4} : (-1, -1).$$

Computing and comparing the values of f at these three critical points, we find

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}, \quad f(1, 1) = 1, \quad f(-1, -1) = 5.$$

Thus the global minimum value of f on D is $\frac{1}{2}$, and the global maximum value of f on D is 5.

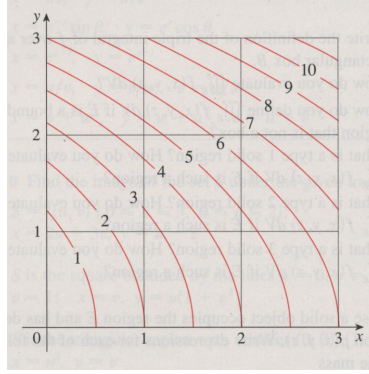


Figure 1: Level sets for the function $f : \mathbf{R} \rightarrow \mathbf{R}$ in Exercise 4.

Exercise 4

Let $R = [0, 3] \times [0, 3] \subseteq \mathbf{R}^2$, and let $f : \mathbf{R} \rightarrow \mathbf{R}$ have the level sets shown in Figure 1. Use a Riemann sum with nine terms to estimate the value of the double integral

$$\iint_R f(x, y) \, dA.$$

Take the sample points of each subregion to be the upper-right corner of the subregion.

Solution: From the level sets in Figure 1, we estimate the values of f on the upper-right points of each subregion to be

$$\begin{array}{lll} f(1, 3) \approx 8, & f(2, 3) \approx 10, & f(3, 3) \approx 12, \\ f(1, 2) \approx 4.5, & f(2, 2) \approx 6.5, & f(3, 2) \approx 8.5, \\ f(1, 1) \approx 2.5, & f(2, 1) \approx 4.5, & f(3, 1) \approx 6.5. \end{array}$$

Note that each subregion of the partition has area 1. Thus the Riemann sum with nine terms, taking the upper-right corner of each subregion as the labeled point, is

$$\sum_{i,j=1}^3 f(i, j) = 2.5 + 4.5 + 6.5 + 4.5 + 6.5 + 8.5 + 8 + 10 + 12 = 61.$$

Exercise 5

Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be a continuous function. Rewrite the following sum of iterated integrals as a single iterated integral, by changing the order of integration:

$$\int_{-4}^0 \int_0^{4+x} f(x, y) \, dy \, dx + \int_0^4 \int_0^{4-x} f(x, y) \, dy \, dx.$$

Solution: Let R_1 and R_2 denote the region of integration corresponding to the first and second integral, respectively. Sketching these regions of integration, we find that R_1 is the triangle in the xy -plane with vertices $(-4, 0)$, $(0, 0)$, $(0, 4)$, and R_2 is the triangle in the xy -plane with vertices $(0, 0)$, $(4, 0)$, $(0, 4)$. *(add graphic)* Because these regions R_1 and R_2 overlap only on their boundaries, additivity of the (Riemann) integral implies that we can write the sum of the given two integrals as a single double integral over the region $R = R_1 \cup R_2$, i.e. the triangle in the xy -plane with vertices $(-4, 0)$, $(4, 0)$, $(0, 4)$.

By hypothesis, f is continuous on all of \mathbf{R}^2 , and hence in particular on R . Hence by Fubini's theorem, we may write the double integral $\iint_R f(x, y) \, dA$ as an iterated integral in any order. As per the instructions of this exercise, we are interested in the order $dx \, dy$. Referring to the sketch of the region R , we see that with this order of integration, the double integral writes as

$$\iint_D f(x, y) \, dA = \int_{y=0}^{y=4} \int_{x=y-4}^{x=-y+4} f(x, y) \, dx \, dy.$$

Exercise 6

Calculate the value of the iterated integral

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dx dy.$$

Solution: The region of integration R is the area above the line $y = 0$ (i.e. the x -axis), left of the curve $x = \sqrt{y}$ (i.e. the right half of $y = x^2$), and left of the line $x = 1$. Note that the integrand $f(x, y) = \frac{ye^{x^2}}{x^3}$ is continuous everywhere on R except at the point $(0, 0)$; in this situation, Fubini's theorem still applies.³ Changing the order of integration (more precisely, writing the double integral $\iint_R f(x, y) dA$ as an iterated integral in the order $dy dx$), we have

$$\begin{aligned} \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \frac{ye^{x^2}}{x^3} dy dx &= \int_{x=0}^{x=1} \frac{e^{x^2}}{x^3} \left[\int_{y=0}^{y=x^2} y dy \right] dx \\ &= \int_{x=0}^{x=1} \frac{e^{x^2}}{x^3} \left[\frac{1}{2}x^4 \right] dx \\ &= \frac{1}{4} \int_{x=0}^{x=1} e^{x^2} 2x dx \\ &= \frac{1}{4} \left[e^{x^2} \right]_{x=0}^{x=1} \\ &= \frac{1}{4}(e - 1). \end{aligned}$$

³Loosely speaking, integrating f over all of R or over R without a point yields the same result.

Exercise 7

Let $T \subseteq \mathbf{R}^3$ be the solid tetrahedron with vertices

$$P_0 = (0, 0, 0), \quad P_1 = \left(\frac{1}{3}, 0, 0\right), \quad P_2 = (0, 1, 0), \quad P_3 = (0, 0, 1).$$

Find the value of the triple integral

$$\iiint_T xy \, dV.$$

Solution: Note that the integrand xy is a polynomial, hence continuous everywhere on \mathbf{R}^3 , and in particular on the region of integration T . Hence by Fubini's theorem we may write the triple integral as an iterated integral using any order of integration.

The solid tetrahedron T is the volume of \mathbf{R}^3 bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $3x + y + z = 1$. (This last plane is determined by P_1, P_2, P_3 ; recall that three distinct points in \mathbf{R}^3 determine a plane. In this case, an equation of the plane can be deduced by looking at the points.) Sketching the region T , we see that its projection onto the yz -plane is the triangle with vertices $(0, 0), (1, 0), (0, 1)$. Using this, and writing the plane through P_1, P_2, P_3 as

$$x = \frac{1}{3}(1 - y - z),$$

we obtain a convenient algebraic description of the region T , which we can use to write the triple integral as the iterated integral in the order $dx \, dz \, dy$:

$$\begin{aligned} \iiint_T xy \, dV &= \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \int_{x=0}^{x=\frac{1}{3}(1-y-z)} xy \, dx \, dz \, dy \\ &= \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \left[\frac{1}{2} x^2 y \right]_{x=0}^{x=\frac{1}{3}(1-y-z)} dz \, dy \\ &= \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \frac{1}{18} (1-y-z)^2 y \, dz \, dy \\ &= \frac{1}{18} \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} (y - 2y^2 - 2yz + 2y^2z + y^3 + yz^2) \, dz \, dy \\ &= \frac{1}{18} \int_{y=0}^{y=1} \left[yz - 2y^2z - yz^2 + y^2z^2 + y^3z + \frac{1}{3}yz^3 \right]_{z=0}^{z=1-y} dy \\ &= \frac{1}{18} \int_{y=0}^{y=1} \left(y(1-y) - 2y^2(1-y) - y(1-y)^2 + y^2(1-y)^2 + y^3(1-y) + \frac{1}{3}y(1-y)^3 \right) dy \\ &= \frac{1}{18} \int_{y=0}^{y=1} \left(\frac{1}{3}y - y^2 + y^3 - \frac{1}{3}y^4 \right) dy \\ &= \frac{1}{18} \left[\frac{1}{6}y^2 - \frac{1}{3}y^3 + \frac{1}{4}y^4 - \frac{1}{15}y^5 \right]_{y=0}^{y=1} \\ &= \frac{1}{18} \left[\frac{1}{6} - \frac{1}{3} + \frac{1}{4} - \frac{1}{15} \right] = \frac{1}{1080}. \end{aligned}$$

Exercise 8

Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$.

Solution: We present two solutions.

Method 1: Calculus. The region of integration V suggests that cylindrical coordinates will be a convenient choice of coordinates for this integral. With this choice of coordinates (remember the integration factor r), we can write the volume of the solid as

$$\begin{aligned} V &= \iiint_V 1 \, dV = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=3-r \sin \theta} r \, dz \, dr \, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (3r - r^2 \sin \theta) \, dr \, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \left[\frac{3}{2} r^2 - \frac{1}{3} r^3 \sin \theta \right]_{r=0}^{r=2} d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \left(6 - \frac{8}{3} \sin \theta \right) d\theta \\ &= \left[6\theta + \frac{8}{3} \cos \theta \right]_{\theta=0}^{\theta=2\pi} \\ &= 12\pi. \end{aligned}$$

Method 2: Geometry. The solid whose volume we wish to compute is a cylinder cut by a plane. More precisely, the base of the cylinder is a disc of radius 2, and the plane cutting the cylinder leaves the base intact. Close your eyes (after reading this sentence), and envision cutting this solid along the plane parallel to the base through the point where the central axis of symmetry of the cylinder intersects the plane cutting the top of the solid. This cuts off an angled piece that we can flip over and place on the “missing” angled piece, forming a complete cylinder with height equal to the height of the point where we made the cut. Because the solid in this exercise has the center of its base at the origin (we can always translate such a solid so that this is the case), this height equals the z -value where the plane cutting the top of the solid intersects the z -axis: This is found by setting $x, y = 0$ in the equation of the plane, yielding $z = 3$. Hence the volume of the solid is

$$V = \pi r^2 h = \pi(2)^2 3 = 12\pi.$$

Exercise 9

Evaluate the integral

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} dz dx dy.$$

Solution: You should have two reactions to this integral: (i) barf, and (ii) spherical coordinates (look at all those squares and square roots! the Clue Fairy is flying low indeed!)

The integrand is continuous on all of \mathbf{R}^3 , hence in particular on the region of integration. Thus Fubini's theorem applies, and we may compute the triple integral associated to the given iterated integral in any order of integration.

Let $V \subseteq \mathbf{R}^3$ denote the region of integration, and let $f(x, y, z)$ denote the integrand. Using the limits of integration to sketch V , we see that V is half of the solid sphere of radius 2 centered at the origin, namely the half with $x \geq 0$. Thus, converting the given iterated integral to spherical coordinates (remember the integration factor $\rho^2 \sin \varphi$), we have

$$\begin{aligned} \iiint_V f(x, y, z) dV &= \int_{\varphi=0}^{\varphi=\pi} \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} (\rho \sin \varphi \sin \theta)^2 \rho (\rho^2 \sin \varphi) d\rho d\theta d\varphi \\ &= \int_{\varphi=0}^{\varphi=\pi} \sin^3 \varphi d\varphi \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \sin^2 \theta d\theta \int_{\rho=0}^{\rho=2} \rho^5 d\rho. \end{aligned} \quad (3)$$

Because none of the limits of integration depend on any variables, we may evaluate each of these single integrals separately. For φ , we use the identity $\sin^2 \varphi + \cos^2 \varphi = 1$ to compute

$$\begin{aligned} \int_0^{\pi} \sin^3 \varphi d\varphi &= \int_0^{\pi} (1 - \cos^2 \varphi) \sin \varphi d\varphi \\ &= \int_0^{\pi} \sin \varphi d\varphi + \int_0^{\pi} \cos^2 \varphi (-\sin \varphi) d\varphi \\ &= [-\cos \varphi]_{\varphi=0}^{\varphi=\pi} + \left[\frac{1}{3} \cos^3 \varphi \right]_{\varphi=0}^{\varphi=\pi} \\ &= [1 - (-1)] + \frac{1}{3}[-1 - (1)] = \frac{4}{3}. \end{aligned}$$

For θ , we use the identity $\cos(2\theta) = 1 - 2\sin^2 \theta$ to compute

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta d\theta &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos(2\theta)) d\theta \\ &= \frac{1}{2} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta - \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta) 2 d\theta \right) \\ &= \frac{1}{2} \left[\theta - \frac{1}{2} \sin(2\theta) \right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \\ &= \frac{\pi}{2}. \end{aligned}$$

For ρ , we compute

$$\int_0^2 \rho^5 \, d\rho = \frac{1}{6} [\rho^6]_{\rho=0}^{\rho=2} = \frac{32}{3}.$$

Substituting these results into (3), we conclude that

$$\iiint_V f(x, y, z) \, dV = \frac{64\pi}{9} \approx 22.3402.$$

Exercise 10

Let X, Y be random variables with probability density function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ given by

$$f(x, y) = \begin{cases} C(x + y) & \text{if } 0 \leq x \leq 3, 0 \leq y \leq 2; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant C .

Solution: By definition, a probability density function (pdf) must (i) be everywhere nonnegative and (ii) integrate to 1. The given function f satisfies condition (i) if $C \geq 0$. The given function f satisfies condition (ii) if

$$\begin{aligned} 1 &= \int_{x=0}^{x=3} \int_{y=0}^{y=2} C(x + y) \, dy \, dx \\ &= C \int_{x=0}^{x=3} \left[xy + \frac{1}{2}y^2 \right]_{y=0}^{y=2} dx \\ &= C \int_{x=0}^{x=3} (2x + 2) \, dx \\ &= C \left[x^2 + 2x \right]_{x=0}^{x=3} \\ &= 15C, \end{aligned}$$

equivalently, if $C = \frac{1}{15}$. Because $\frac{1}{15} \geq 0$, this value of C makes f a valid pdf.

(b) Find $\text{Prob}(X \leq 2, Y \geq 1)$.

Solution: We find probability by integrating the probability density function f over the relevant region. Keeping in mind that $f(x, y) = 0$ outside the rectangle $[0, 3] \times [0, 2]$, we compute

$$\begin{aligned} \text{Prob}(X \leq 2, Y \geq 1) &= \int_{x=-\infty}^{x=2} \int_{y=1}^{y=+\infty} f(x, y) \, dy \, dx \\ &= \frac{1}{15} \int_{x=0}^{x=2} \int_{y=1}^{y=2} (x + y) \, dy \, dx \\ &= \frac{1}{15} \int_{x=0}^{x=2} \left[xy + \frac{1}{2}y^2 \right]_{y=1}^{y=2} dx \\ &= \frac{1}{15} \int_{x=0}^{x=2} \left(x + \frac{3}{2} \right) dx \\ &= \frac{1}{15} \left[\frac{1}{2}x^2 + \frac{3}{2}x \right]_{x=0}^{x=2} \\ &= \frac{1}{3}. \end{aligned}$$

(c) Find $\text{Prob}(X + Y \leq 1)$.

Solution: With the same comments as in part (b), we compute

$$\begin{aligned}\text{Prob}(X + Y \leq 1) &= \int_{x=-\infty}^{x=+\infty} \int_{y=-\infty}^{y=1-x} f(x, y) \, dy \, dx \\&= \frac{1}{15} \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (x + y) \, dy \, dx \\&= \frac{1}{15} \int_{x=0}^{x=1} \left[xy + \frac{1}{2}y^2 \right]_{y=0}^{y=1-x} dx \\&= \frac{1}{15} \int_{x=0}^{x=1} \left(x(1-x) + \frac{1}{2}(1-x)^2 \right) dx \\&= \frac{1}{15} \int_{x=0}^{x=1} \left(-\frac{1}{2}x^2 + \frac{1}{2} \right) dx \\&= \frac{1}{15} \left[-\frac{1}{6}x^3 + \frac{1}{2}x \right]_{x=0}^{x=1} \\&= \frac{1}{15} \left(\frac{1}{3} \right) = \frac{1}{45}.\end{aligned}$$

Exercise 11

Let $R \subseteq \mathbf{R}^2$ be the square with vertices

$$(0, 2), \quad (1, 1), \quad (2, 2), \quad (1, 3).$$

Evaluate the double integral

$$\iint_R \frac{x-y}{x+y} dA.$$

Solution: Sketching the region of integration R , we see that it is a square in the xy -plane. Its sides are translates of the lines $y = x$ and $y = -x$, more precisely, the lines

$$y - x = 0, \quad y - x = 2, \quad y + x = 2, \quad y + x = 4. \quad (4)$$

The given integral seems difficult to evaluate directly, for (at least) two reasons: (i) the integrand is relatively complicated, and (ii) the region of integration would have to be subdivided in order to evaluate the double integral as (two or more) iterated integrals. These reasons suggest that we try a change of variables, and the form of the integrand (as well as the region of integration) suggests natural candidates for new variables u, v :

$$u = y - x, \quad v = y + x. \quad (5)$$

To do a change of variables, we need to express the given variables x, y in terms of the new variables u, v . That is, we want a function $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ of the form

$$T(u, v) = (x(u, v), y(u, v)).$$

From the definitions of u, v in (5), we compute

$$u + v = 2y, \quad -u + v = 2x,$$

so

$$x = -\frac{1}{2}u + \frac{1}{2}v, \quad y = \frac{1}{2}u + \frac{1}{2}v.$$

Thus the Jacobian matrix and Jacobian determinant of the transformation T are

$$J_T(u, v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \det(J_T(u, v)) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2},$$

respectively.

In terms of the new variables u, v , the curves in (4) defining the boundary of the region R in the xy -plane write as

$$u = 0, \quad u = 2, \quad v = 2, \quad v = 4.$$

Because the transformation T is linear, it follows that the region

$$S := \{(u, v) \in \mathbf{R}^2 \mid 0 \leq u \leq 2, 2 \leq v \leq 4\}$$

in the uv -plane is mapped bijectively to the region R in the xy -plane. Thus we can use the change of variables given by the transformation T to write the given double integral as

$$\iint_R \frac{x-y}{x+y} dA = \iint_S -\frac{u}{v} |\det J_T(u, v)| dA = -\frac{1}{2} \iint_S \frac{u}{v} dA.$$

Because the integrand is continuous on the region S of integration, Fubini's theorem states that the value of the double integral equals the value of an iterated integral, integrated in either order. Choosing the order $dv du$, we compute⁴

$$\begin{aligned} -\frac{1}{2} \iint_S \frac{u}{v} dA &= -\frac{1}{2} \int_{u=0}^{u=2} \int_{v=2}^{v=4} \frac{u}{v} dv du \\ &= -\frac{1}{2} \int_{u=0}^{u=2} u du \int_{v=2}^{v=4} v^{-1} dv \\ &= -\frac{1}{2} [2] [\ln 4 - \ln 2] = -\ln 2. \end{aligned}$$

We conclude that

$$\iint_R \frac{x-y}{x+y} dA = -\ln 2.$$

Remark 11.1. As a rough check, note that the region R in the xy -plane lies above the line $y = x$, so on this region, $y \geq x$, or equivalently, $x - y \leq 0$. Hence the integrand is nonpositive on this region (in fact, negative on R except on the lower right boundary), so we expect the value of the integral to be negative.

⁴Because all limits of integration for the variables u, v are constants, the iterated integral in the order $du dv$ is evaluated identically.

Hints

The following numbers correspond to the exercise numbers above.

1. Consider ∇f .
2. This exercise involves rates of change, i.e. derivatives, each with respect to time. View each variable as a function of time, and use the chain rule to relate the various rates of change.
3. Appeal to the EVT. Parametrize the boundary of D using polar coordinates.
4. Use the definition of a Riemann sum.
5. Sketch the region(s) of integration.
6. Justify why you can change the order of integration.
7. Sketch the region of integration.
8. Choose a convenient coordinate system, based on the region of integration.
9. Choose a convenient coordinate system, based on the region of integration.
10. Use the definition of a probability density function.
11. Apply a change of variables. It may be easier to first write the new variables u, v as functions of the given variables x, y , then invert to find x, y as functions of u, v . Remember the Jacobian determinant.