Math 211
Quiz 17A

M 05 Aug 2019

Your name : ____________________________________

Exercise

(5 pt) For each of the following linear maps $T : \mathbf{R}^n \to \mathbf{R}^m$, where $\mathbf{R}^n, \mathbf{R}^m$ are viewed as vector spaces over $\mathbf{R}$,

(i) write a basis for the image $\mathrm{im}(T)$ and a basis for the kernel $\ker(T)$, and

(ii) confirm the rank–nullity theorem.

(a) (2.5 pt) The linear map T_1 given by

$$T_1 : \mathbf{R}^3 \to \mathbf{R}^5$$
$$\begin{bmatrix} x_1 \\ \vdots \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + x_3 \\ x_1 + x_2 - x_3 \\ x_2 - 2x_3 \\ x_2 - 2x_3 \\ -2x_2 + 4x_3 \end{bmatrix}.$$

Solution: We write the corresponding matrix A_1, then apply the row reduction algorithm (RRA) to get its reduced row echelon form (RREF):

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{RRA}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{RREF}(A_1).$$

The image is the span of the pivot columns of the original (!) matrix A_1

$$\mathrm{im}(T_1) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

These columns are linearly independent and hence form a basis for $\mathrm{im}(T_1)$.

Notice that the third column of A_1 is a linear combination of the first two. More precisely, letting $(A_1)_{\bullet,j}$ denote the jth column of A_1, we can check that $(A_1)_{\bullet,3} = (A_1)_{\bullet,1} - 2(A_1)_{\bullet,2}$. *(elaborate?)*

The nonpivot columns, in this case column 3, correspond to free variables, in this case x_3. The general vector in the kernel of T_1 can be read off from the RREF of A_1 — more precisely, from the augmented matrix $\left[\ \mathrm{RREF}(A) \mid \mathbf{0}\ \right]$:

$$\ker(T_1) = \left\{ \begin{bmatrix} -x_3 \\ 2x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbf{R} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

A set containing a single nonzero vector is always linearly independent. Thus any nonzero vector in $\ker(T_1)$ is a basis of $\ker(T_1)$.

The rank–nullity theorem writes as

$$\dim(\mathrm{domain}(T_1)) = 3 = 2 + 1 = \dim(\mathrm{im}(T_1)) + \dim(\ker(T_1)).$$

(b) (2.5 pt) The linear map T_2 given by

$$\begin{aligned} T_2 : \mathbf{R}^4 &\to \mathbf{R}^4 \\ \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} &\mapsto \begin{bmatrix} x_1 & + & & & x_3 & & \\ x_1 & + & x_2 & + & 2x_3 & + & 3x_4 \\ 2x_1 & + & & & 2x_3 & + & 2x_4 \\ & & & & & & 4x_4 \end{bmatrix}. \end{aligned}$$

Solution: We write the corresponding matrix A_2, then apply the RRA to get its RREF:

$$A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 3 \\ 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{\text{RRA}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{RREF} A_2.$$

By the same logic as in part (a), we find

$$\operatorname{im}(T_2) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 4 \end{bmatrix} \right\}$$

and

$$\ker(T_2) = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The rank–nullity theorem writes as

$$\dim(\operatorname{domain}(T_2)) = 4 = 3 + 1 = \dim(\operatorname{im}(T_2)) + \dim(\ker(T_2)).$$