Math 211
Quiz 20

F 02 Aug 2019

Your name : ____

Exercise

(2 pt) Match each of the homogeneous 1st-order 2×2 linear systems with its corresponding phase plane in Figure 1. (N.B. In each ODE, $\mathbf{x} = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ is a 2×1 matrix of scalar-valued functions.) *Hint:* (Some of the) distinguishing features of the phase plane are associated with

- eigenvalues and their corresponding eigenvectors of the coefficient matrix;
- nullclines, i.e. lines in the phase plane along which $x_1' = 0$ or $x_2' = 0$; and
- evaluating the original ODE at points (x_1, x_2).

Hint: Recall that in the decomposition $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, if $\mathbf{D}$ is a diagonal matrix, then column j of the matrix $\mathbf{P}$ is an eigenvector corresponding to (the eigenvalue on) the jth diagonal entry of $\mathbf{D}$.

(a) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1} \mathbf{x}$

(b) $\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ -4 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \mathbf{x}$

(c) $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}$

(d) $\mathbf{x}' = \begin{bmatrix} 1 & -10 \\ 10 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+10i & 0 \\ 0 & 1-10i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}$

(e) $\mathbf{x}' = \begin{bmatrix} -4 & 6 \\ -3 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1-i & 1+i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1+3i & 0 \\ 0 & -1-3i \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}$

(f) $\mathbf{x}' = \begin{bmatrix} 2 & -6 \\ 3 & -4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1+3i & 0 \\ 0 & -1-3i \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}$

Solution: Focusing on the features in the hint, we find

(a) = (1) ; (b) = (2) ; (c) = (5) ; (d) = (6) ; (e) = (4) ; (f) = (3).

More precisely, we can reason as follows. Based on their diagonalization decomposition $\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, we group similar ODEs:

- (a) and (b),
- (c) and (c),
- (e) and (f).

Reading the eigenvalues from the diagonal matrices D in the decomposition, (a) and (b) have real eigenvalues, one positive and one negative; hence the equilibrium (at the origin) is a saddle. The phase portraits illustrating a saddle equilibrium are (1) and (2). In (1), trajectories move toward the equilibrium (i.e. the origin) in the direction of the vector $(1, -2)$, and away from the equilibrium in the direction of the vector $(1, 2)$. In (2), these directions are reversed. Comparing this to the eigenvectors (i.e. the columns) in the $\mathbf{P}$ matrix in the ODEs (a) and (b), this behavior allows us to match (a) with (1) and (b) with (2).

(c)–(f) have complex eigenvalues; hence their phase planes will display rotation around the equilibrium (at the origin). The eigenvalues for (c) and (d) have positive real part, both 1; hence trajectories (solution curves) spiral away from the equilibrium. The eigenvalues for (e) and (f) have negative real part, both -1; hence trajectories spiral toward the origin. This allows us to group (c) and (d) with phase planes (5) and (6), and (e) and (f) with phase planes (3) and (4).

Nullclines are one way to distinguish between (5) and (6) for ODEs (c) and (d). Writing out (c),

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}.$$

This matrix equation represents the system of equations

$$\begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= x_1 + x_2. \end{aligned}$$

By definition, x_1-nullclines satisfy

$$0 \underset{\text{set}}{=} x_1' = x_1 - x_2 \qquad \Leftrightarrow \qquad x_2 = x_1.$$

This tells us that, along the line $x_2 = x_1$ in the (x_1, x_2) plane, any direction of change lies entirely in the x_2 direction, i.e. trajectories pass either up or down when passing through the line $x_2 = x_1$. Whether it is up or down can be determined by evaluating the original ODE at a few points on the x_1-nullcline: At any value t_0 such that $(x_1(t_0), x_2(t_0)) = (1, 1)$,

$$\mathbf{x}'(t_0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Note that the first entry $x_1'(t_0) = 0$, as required for points on the x_1-nullcline. The second entry $x_2'(t_0) = 2 > 0$, which says that $x_2(t)$ is increasing at the point $(x_1, x_2) = (1, 1)$, i.e. trajectories are moving up at this point. Similarly, for t_0 such that $(x_1(t_0), x_2(t_0)) = (-1, -1)$,

$$\mathbf{x}'(t_0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Here $x_2'(t_0) = -2 < 0$, so $x_2(t)$ is decreasing at this point $(x_1, x_2) = (1, 1)$, i.e. trajectories are moving down.

Similarly, by definition, x_2-nullclines satisfy

$$0 \underset{\text{set}}{=} x_2' = x_1 + x_2 \qquad \Leftrightarrow \qquad x_2 = -x_1.$$

Trajectories pass either left or right when passing through the line $x_2 = -x_1$.

All of this is consistent with the phase plane (5). Hence we conclude that the system (c) corresponds to phase plane (5).

Playing the same game with the ODE (d), we get nullclines

$$x_1\text{-nullcline} : x_2 = \frac{1}{10} x_1, \qquad x_2\text{-nullcline} : x_2 = -10 x_1.$$

This says that, graphically, trajectories have purely vertical tangents (velocities) along $x_2 = \frac{1}{10} x_1$ (a relatively flat line, close to the x_1-axis), and purely horizontal tangents (velocities) along $x_2 = -10x_1$ (a relatively steep line, close to the x_2-axis). This is consistent with phase plane (6).

These same approaches (i.e. analysis of nullclines, evaluating the original system at a few points) allow us to distinguish between (e) and (f), matching them to phase planes (4) and (3), respectively.

N.B. In all cases, we can confirm the direction of vectors in the phase planes at a few points $(x_1, x_2) \in \mathbf{R}^2$ by plugging $\mathbf{x} = (x_1, x_2)$ into the right side of the original ODEs.

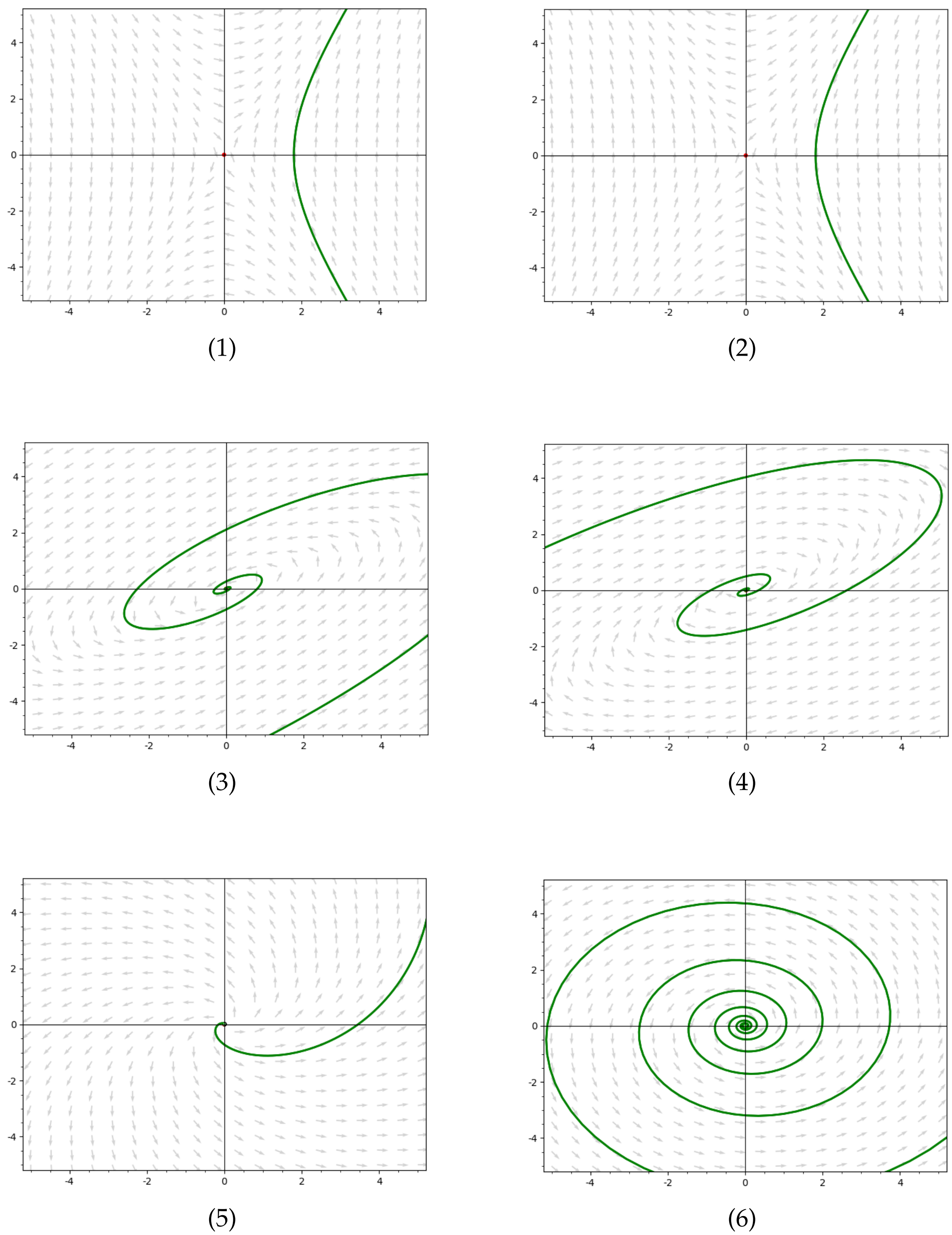

Figure 1: Phase planes for Quiz 20, in the (x_1, x_2) plane.