Math 357
Expositional homework 07

Assigned: 2024–04–12 (F)
Due: 2024–04–19 (F)

The goal of this homework is to work with elementary theory and examples in galois theory to better understand the essential building blocks.

(a) Let $K : K_0$ be a field extension; and let $\mathrm{Aut}(K : K_0)$ be the automorphisms of K that fix K_0, a group under composition of functions. Let K_i be an intermediate field of $K : K_0$, let H_i be a subgroup of $\mathrm{Aut}(K : K_0)$, and let $\mathcal{F}(H_i)$ denote the fixed field of H_i:

$$\mathcal{F}(H_i) = \{\alpha \in K \mid \forall \sigma \in H_i, \sigma(\alpha) = \alpha\}$$

Prove that the associations

$$K_i \mapsto \mathrm{Aut}(K : K_i) \qquad\qquad H_i \mapsto \mathcal{F}(H_i)$$

are inclusion-reversing, that is, if $K_1 \subseteq K_2$ and $H_1 \subseteq H_2$, then

$$\mathrm{Aut}(K : K_1) \supseteq \mathrm{Aut}(K : K_2) \qquad\qquad \mathcal{F}(H_1) \supseteq \mathcal{F}(H_2)$$

(b) Prove Proposition 14.5:[1] Let K_0 be a field, let $f \in K_0[t]$, and let $\tilde{K}_{0,f} : K_0$ be a splitting field for f over K_0. Then

$$\# \mathrm{Aut}(\tilde{K}_{0,f} : K_0) \leqslant [\tilde{K}_{0,f} : K_0]$$

with equality if f is separable. You may take as your starting point our diagram from class (see Classes 35 and 36).

(c) Let $\alpha = \sqrt{2} + \sqrt{5} \in \mathbf{C}$.

(i) Find the minimal polynomial $m_{\alpha,\mathbf{Q}}$ for α over $\mathbf{Q}$.
(ii) Prove that $\mathbf{Q}(\alpha) = \mathbf{Q}(\sqrt{2}, \sqrt{5})$.
(iii) Prove that $m_{\alpha,\mathbf{Q}}$ splits completely in $\mathbf{Q}(\alpha)$. *Hint:* Use part (ii).
(iv) Specify all automorphisms in the galois group $\mathrm{Gal}(\mathbf{Q}(\alpha) : \mathbf{Q})$. State a (more common) group isomorphic to $\mathrm{Gal}(\mathbf{Q}(\alpha) : \mathbf{Q})$, and draw its subgroup lattice.
(v) Use part (iv) and the fundamental theorem of galois theory to draw the lattice of intermediate fields for $\mathbf{Q}(\alpha) : \mathbf{Q}$.

[1] See DF3e, pp 561–2.