Math 212
Quiz 11

F 16 Sep 2016

Your name: ______________________________

Exercise

(5 pt) This exercise investigates limits of functions $f : D \subseteq \mathbf{R}^2 \to \mathbf{R}$. For each function f appearing in the following limits, let D be its maximum domain of definition.

(a) (2.5 pt) Prove that the following limit exists, and find it.

$$\lim_{(x,y)\to(0,0)} \frac{y^3 - x^3}{y - x}$$

1.1 Solution

The numerator factors:

$$y^3 - x^3 = (y - x)(y^2 + yx + x^2).$$

Substituting this result into the given limit, we would like to cancel the factor $y - x$ in the numerator and denominator. We can do this provided that $y - x \neq 0$. This inequality holds for all $(x, y) \in D$ (when $y - x = 0$, the rational function $\frac{y^3-x^3}{y-x}$ is not defined). We conclude that

$$\lim_{(x,y)\to(0,0)} \frac{y^3 - x^3}{y - x} = \lim_{(x,y)\to(0,0)} \left(y^2 + yx + x^2\right) = 0 + 0 + 0 = 0.$$

(Recall that the limit of a rational function, of which a polynomial function is a special case, is defined at all points where that rational function is defined, and equals its value at that point.)

(b) (2.5 pt) Prove that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

1.2 Solution

To prove that the limit does not exist, we exhibit two paths of approach to the origin along which the given function has different limits.

Along any line $y = mx$ of approach to the origin, the given function writes as

$$f(x, mx) = \frac{x^2(mx)}{x^4 + (mx)^2} = \frac{mx^3}{x^2(x^2 + m^2)}.$$

If $m = 0$, then $y = 0$, and this function is identically zero, so its limit is 0. If $m \neq 0$, then the limit also evaluates to

$$\lim_{x\to 0} \frac{mx^3}{x^2(x^2 + m^2)} = \lim_{x\to 0} \frac{mx}{x^2 + m^2} = \frac{0}{m^2} = 0.$$

Now consider approaching the origin along the path $y = x^2$. Then

$$f(x, x^2) = \frac{x^2(x^2)}{x^4 + (x^2)^2} = \frac{x^4}{2x^4}.$$

Because $x \neq 0$ when evaluating the limit along this path ($x = 0 \Rightarrow y = x^2 = 0$; by definition of limit, we must consider points (x, y) not equal to the limiting point $(0, 0)$), the limit evaluates to

$$\lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

Because different paths of approach yield different limiting values, the limit does not exist.