Math 212
Quiz 17

F 07 Oct 2016

Your name: ____________________________________

Exercise

(5 pt) Find the global minimum and maximum values of the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x,y) = x^2y + xy^2 - xy$$

on the closed set $D \subseteq \mathbf{R}^2$ given by

$$D = \left\{(x,y) \in \mathbf{R}^2 \,|\, x \geqslant 0, y \geqslant 0, x+y \leqslant 2\right\},$$

a 45-45-90 right triangle with side length 2 in the first quadrant of the xy-plane. We'll do this in steps.

(a) (1 pt) Justify why a global minimum and maximum exist in this case. *Hint:* Name that theorem, and validate its hypotheses.

Solution: The function f is continuous (polynomials, and more generally, rational functions, are continuous everywhere they are defined), and the set D is closed and bounded. Thus by the extreme value theorem, f achieves a (global) minimum and maximum on D.

(b) (2 pt) Find all critical points on the interior of D. *Hint:* As in single-variable optimization, do this by setting the appropriate notion of "derivative of f" equal to (the appropriate notion of) zero. Note that in the interior of D, $x \neq 0$ and $y \neq 0$. The derivative equal to zero gives a system of two equations, which will yield a unique solution — our critical point.

Solution: Because f is differentiable everywhere, all critical points (x,y) of f satisfy the condition $(\nabla f)(x,y) = \mathbf{0}$. Computing the gradient,[1] we find

$$(0,0) = \mathbf{0} = (\nabla f)(x,y) = \left(2xy + y^2 - y, 2xy + x^2 - x\right).$$

This vector equality is equivalent to the following system of equations:

$$0 = y(2x+y-1) \qquad \text{and} \qquad 0 = x(2y+x-1).$$

These equations have the following solutions:

$$y = 0 \quad \text{or} \quad 2x+y-1=0, \qquad \text{and} \qquad x=0 \quad \text{or} \quad 2y+x-1=0.$$

On the interior of D, $x > 0$ and $y > 0$ (do you see why this is true, geometrically?). Hence we must have

$$2x+y-1=0 \qquad \text{and} \qquad 2y+x-1=0,$$

a system of two equations in two unknowns with the unique solution

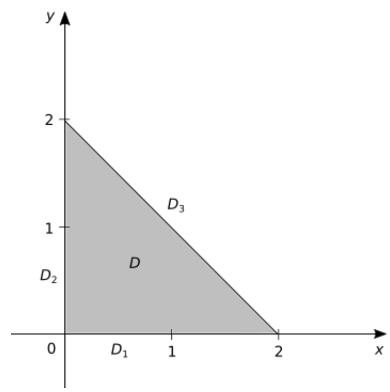
$$(x,y) = \left(\frac{1}{3}, \frac{1}{3}\right).$$

[1] Note that the function f is symmetric in x and y, i.e. $f(x,y) = f(y,x)$. This implies that $\frac{\partial f}{\partial y}$ can be obtained from $\frac{\partial f}{\partial x}$ by interchanging x and y.

(c) (1 pt) Find all critical points on the boundary of D. *Hint:* Note that $f(x, y) = 0$ along the boundary components of D where $x = 0$ or $y = 0$. Thus we need only consider the boundary component $x + y = 2$. Solve for y as a function of x (or vice versa), substitute into f to obtain a function of a single variable, and optimize this using single-variable calculus. Again you should find a unique critical point.

Solution: The boundary of D has three natural components:

1. $D_1 = \{(x, y) \in \mathbf{R}^2 \mid 0 \leqslant x \leqslant 2, y = 0\}$
2. $D_2 = \{(x, y) \in \mathbf{R}^2 \mid x = 0, 0 \leqslant y \leqslant 2\}$
3. $D_3 = \{(x, y) \in \mathbf{R}^2 \mid x + y = 2, x \geqslant 0, y \geqslant 0\}$



As noted in the hint, $f(x, y) = 0$ along the boundary components D_1 and D_2, because

$$f(x, y) = xy\,(x + y - 1)\,, \tag{1}$$

so $f(x, y) = 0$ if $x = 0$ or $y = 0$. It remains to analyze critical points of the boundary component D_3. The endpoints $(2, 0)$ and $(0, 2)$ of D_3 lie in D_1 and D_2, respectively, so we know the value of f is 0 at these points. Along D_3, we have $x + y = 2$, or equivalently,

$$y = 2 - x.$$

Substituting this into the expression (1) for f, we obtain

$$g(x) = f(x, 2 - x) = x(2 - x)(x + (2 - x) - 1) = 2x - x^2,$$

a function of the single variable x. This function is a polynomial, so it is differentiable everywhere, and hence all critical points x of g satisfy the condition $g'(x) = 0$. We compute

$$0 = g'(x) = 2 - 2x \qquad \Leftrightarrow \qquad x = 1.$$

Thus the unique critical point along the boundary component D_3 is

$$(x, y) = (1, 1).$$

We should check that this point indeed lies in D_3. It does.

(d) (1 pt) State the global minimum and maximum values of f on D. *Hint:* Compare values of f at points from (b) and (c).

Solution: We compute the values of f at the candidate extremal points found in (b) and (c):

$$f\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3} + \frac{1}{3} - 1\right) = -\frac{1}{27}, \qquad f(1, 1) = 1 + 1 - 1 = 1.$$

We conclude that f has a unique global minimum and maximum on D, as summarized below.[2]

Point	Value	Type
$(\frac{1}{3}, \frac{1}{3})$	$-\frac{1}{27}$	global min
$(1, 1)$	1	global max

[2]The points in D_1 and D_2 are also critical points. The value of f at each of these points is 0, and $f(\frac{1}{3}, \frac{1}{3}) < 0 < f(1, 1)$, so none of these points is a global minimum or maximum.