Math 211
Quiz 12

T 23 Jul 2019

Your name : ____

Exercise

(5 pt) Let $V = \mathbf{R}^3$ be the vector space of 3×1 matrices with entries in $\mathbf{R}$, equipped with the usual operations of matrix addition and scalar multiplication. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) (2 pt) Row reduce the matrix $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ to show that the set $\{v_1, v_2, v_3\}$ is linearly independent. *Hint:* Is every column in the row-reduced matrix a pivot column?

Solution: By definition, the set $\{v_1, v_2, v_3\}$ is linearly independent if their only linear combination that equals the zero vector is the trivial linear combination with all coefficients equal to 0:

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \qquad \Leftrightarrow \qquad \text{all } a_i = 0.$$

This is equivalent to the statement that the corresponding system of linear equations

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

has the unique solution $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$. If we row reduce the augmented matrix associated to this system, we obtain the 3×3 identity matrix I on the left, and (necessarily — why?) a column of 0s in the added column 4. The RREF augmented matrix corresponds to the linear system

$$a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0.$$

This shows that if a linear combination of v_1, v_2, v_3 equals the zero vector, then all the coefficients a_i are zero. Hence the set $\{v_1, v_2, v_3\}$ is linearly independent.

(b) (2 pt) Compute the determinant of the 3×3 matrix $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$. *Hint:* Use expansion by minors along row 3.

Solution: We compute

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = (-1)^{3+1}(1) \det \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + 0 + 0 = (1)(1)(2 - (-1)) = 3.$$

(c) (1 pt) How does our computation in part (b) give us another solution to part (a)? How is the row-reduction approach in part (a) more general than the determinant approach in part (b)?

Solution: In part (b), we found that the determinant of $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ is nonzero. Thus there exists a (unique) inverse matrix, $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1}$. Left-multiplying both sides of (1) by this inverse matrix, we get

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is, all $a_i = 0$, so the set $\{v_1, v_2, v_3\}$ is linearly independent.

Note that we can apply the row-reduction approach to any number of vectors; this approach says the vectors are linearly independent if and only if every column is a pivot column. The determinant is defined only for square matrices. Thus, the determinant approach works only for n vectors in $\mathbf{R}^n$; this approach says that n vectors in $\mathbf{R}^n$ are linearly independent if and only if the determinant of the corresponding $n \times n$ matrix is nonzero. (Does the order of the vectors in this matrix matter? Why or why not?)