Math 211
Exam 01

W 17 Jul 2019

Your name : ______________________________

Start time : ________________ End time : ________________

Honor pledge :

Exam instructions

Number of exercises	: 6
Permitted time	: 90 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- You are well-trained. Do your best!

Exercise	Total	(a)	(b)	(c)	(d)	(e)	(f)
1	/15	/3	/3	/3	/3	/3	X
2	/15	/2.5	/2.5	/2.5	/2.5	/2.5	/2.5
3	/20	/10	/10	X	X	X	X
4	/24	/10	/10	/4	X	X	X
5	/16	/8	/8	X	X	X	X
6	/10	X	X	X	X	X	X
Total	/100						

Exercise 1

(15 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) Let y_1 and y_2 be solutions to the first-order ODE

$$ty' + e^t y = t. \tag{1}$$

Then any linear combination $a_1 y_1 + a_2 y_2$ is also a solution to (1).

true false

(b) The function $y(t) = e^{4t}$ solves the ODE $y'' - 3y' - 4y = 0$.

true false

(c) The first-order ODE $y' = \frac{t}{t^2+1} y^2 + \cos(ty)$ has a solution for any initial condition $y(t_0) = y_0$.

true false

(d) Let $y_1(t)$ be a solution to the IVP $y' = e^{ty} + t, y(0) = 1$, and let $y_2(t)$ be a solution to the IVP $y' = e^{ty} + 2t, y(0) = 1$. Then for any $t \in \mathbf{R}_{>0}$ (i.e. for any $t > 0$), $y_1(t) < y_2(t)$.

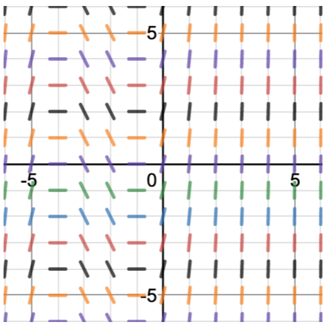
true false

(e) Every ODE has a closed-form solution (e.g., an explicit equation for $y(t)$).

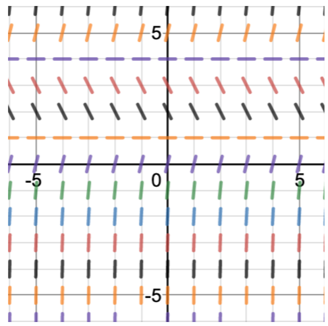
true false

Exercise 2

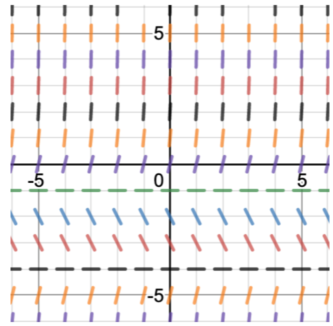
(15 pt) Matching. Write the number of each slope field next to its corresponding ODE.



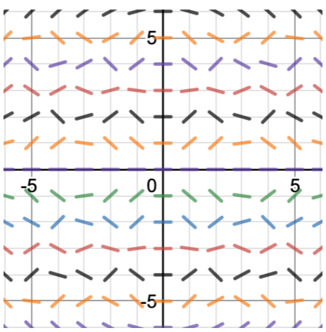
(1)



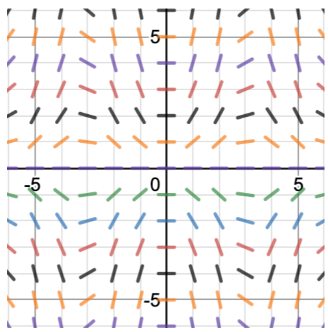
(2)



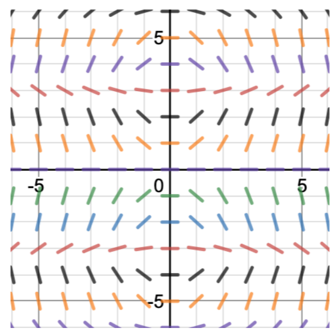
(3)



(4)



(5)



(6)

____ (a) $\dfrac{dy}{dt} = y^2 - 5y + 4$

____ (b) $\dfrac{dy}{dt} = y^2 + 5y + 4$

____ (c) $\dfrac{dy}{dt} = t^2 + 5t + 4$

____ (d) $\dfrac{dy}{dt} = t \sin y$

____ (e) $\dfrac{dy}{dt} = y \sin t$

____ (f) $\dfrac{dy}{dt} = \sin(ty)$

Exercise 3

(20 pt) For each of the following first-order ODEs, find the general solution.

(a) $\dfrac{dy}{dt} = \dfrac{2t+1}{4y^3+4y}$

(b) $(t^2+9)y' + ty = 0$

Exercise 4

(24 pt) Consider the following first-order nonhomogeneous linear ODE:

$$y' - 4y = te^{6t}. \tag{2}$$

(a) (10 pt) Find the general solution to the corresponding homogeneous ODE.

(b) (10 pt) Find the general solution to the nonhomogeneous ODE (2).

(c) (4 pt) Find the particular solution of the solution family you gave in part (b) that satisfies the initial condition $y(0) = 1$.

Exercise 5

(16 pt) Consider the one-parameter family of first-order nonlinear ODEs

$$\frac{dy}{dt} = y^3 + \alpha y, \tag{3}$$

where the parameter α is allowed to take any value in $\mathbf{R}$.

(a) (8 pt) Show that for $\alpha \geqslant 0$, the ODE (3) has a unique equilibrium, and that it is unstable.

(b) (8 pt) Show that for $\alpha < 0$, the ODE (3) has three equilibria, at 0 and $\pm\sqrt{-\alpha}$. Classify the stability of each equilibrium.

Exercise 6

(10 pt) In Class 5 (Friday 12 July) we argued that the first-order nonhomogeneous linear ODE

$$y' - y = t \tag{4}$$

has the general solution

$$y(t) = -t - 1 + ce^{t}, \tag{5}$$

where $c \in \mathbf{R}$. The question was posed: How do we know (5) captures *all* the solutions to (4)?

We gave one argument:

> The solutions to the corresponding homogeneous ODE form a one-dimensional vector space over **R** (as we will see), and the nonhomogeneous principle guarantees that any solution to (4) has the form $y = y_p + y_h$.

Give another argument, using Picard's theorem about existence and uniqueness of solutions. *Hint:* Consider any point (t_0, y_0). Is there a solution to (4) of the form (5) that passes through (t_0, y_0)? It may help to sketch some solutions ce^t to the corresponding homogeneous ODE.