Math 212
Quiz 31

W 16 Nov 2016

Your name: ______________________________

Exercise

(2 pt) Let $C \subseteq \mathbf{R}^2$ be the circle of radius 1 centered at the origin, oriented counterclockwise. Use Green's theorem to rewrite the line integral

$$\int_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

as an iterated (!) integral over the appropriate region $D \subseteq \mathbf{R}^2$. *Hint:* Green's theorem changes (i) the integrand (think curl) and (ii) the region of integration (view C as the boundary of D). The iterated integral is most easily expressed in polar coordinates (mind the integration factor!).

Solution: The unit circle C encloses the unit disc D, which is described in polar coordinates by

$$D = \{(r, \theta) \,|\, 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}.$$

The unit circle C is a smooth, simple, closed curve; and the component functions of

$$\mathbf{F} = (F_1, F_2) = \left(3y - e^{\sin x}, 7x + \sqrt{y^4 + 1}\right)$$

in the integrand of the line integral are $\mathscr{C}^1$ (continuously differentiable). Hence by Green's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (3 - 7) r \, dr \, d\theta = -4 \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r \, dr \, d\theta.$$

Remark 1.1. Suppose that we wanted to evaluate this integral. The region D has area $\pi(1)^2 = \pi$, so $\iint_D 1 \, dA = \pi$, and hence (using Green's theorem as above)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -4 \iint_D dA = -4\pi.$$

Note that, because the integrand was a constant (and the area of the region D is easily computable via geometry), we did not even have to set up and evaluate an iterated integral.