Math 357
Long quiz 03A

2024–02–05 (M)

Your name: ______________________________________

Let R be an integral domain, and let t be an indeterminate. Consider the polynomial ring $R[t]$.

(a) Prove that $(R[t])^\times \cong R^\times$. That is, we may view the units of $R[t]$ to be exactly the units of R. *Hint:* $\deg(pq) = \dots$

Solution: For the isomorphism (of R as a subring of $R[t]$ more generally), identify each $r \in R$ with the constant polynomial $r \in R[t]$. This identification allows us to view the given isomorphism as an equality $(R[t])^\times = R^\times$. We prove set containment in both directions.[1]

($(R[t])^\times \supseteq R^\times$) Immediate. (Why?)

($(R[t])^\times \subseteq R^\times$) We have seen that if R is an integral domain, then for all $p, q \in R[t]$,

$$\deg(pq) = \deg p + \deg q \tag{1}$$

Let $p \in (R[t])^\times$. By definition of unit, there exists a $q \in R[t]$ such that

$$pq = 1$$

Applying the degree function to this equation and using (1), we have

$$0 = \deg 1 = \deg(pq) = \deg p + \deg q \tag{2}$$

Because $p, q \in (R[t])^\times$, it follows that $p, q \neq 0$, and therefore $\deg p, \deg q \geqslant 0$. Hence (2) implies that $\deg p, \deg q = 0$; that is, p and q are constant polynomials; that is, $p, q \in R$. Thus $(R[t])^\times \subseteq R^\times$, as desired.

(b) Now let R be a commutative ring with a $1 \neq 0$. Give an example to show that the isomorphism in part (a) can fail.

Solution: Let $R = \mathbf{Z}/2\mathbf{Z}$, and let $f = 2t + 1 \in R[t]$. We compute

$$f^2 = (2t+1)^2 = 4t^2 + 4t + 1 \equiv 1$$

because $4 \equiv 0$ in the ring $\mathbf{Z}/4\mathbf{Z}$ of coefficients. Thus $f \in (R[t])^\times$. However, f does not correspond to a unit in R, whose two units $1, 3$ correspond to the constant polynomials (with the same constant terms) in $R[t]$.

In fact, for each $n \in \mathbf{Z}_{\geqslant 0}$, we may define

$$f_n = 2t^n + 1$$

An analogous computation to the one above shows that $f_n \in (R[t])^\times$. Thus $(R[t])^\times$ has infinitely many elements, whereas $R^\times = \{1, 3\}$ has only two. Thus they cannot even be isomorphic as sets.

[1]The isomorphism as groups follows from the fact that multiplication on $R[t]$ is defined, in part, using the multiplication on R.