Math 112
LQuiz 14

2022–03–29 (T)

Your name: ______________________________

Exercise

(4 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the function whose rule of assignment is

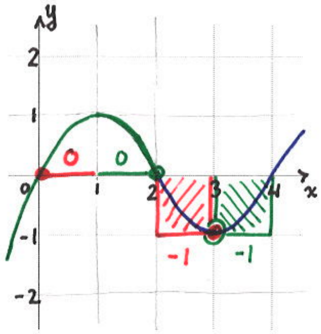
$$f(x) = \begin{cases} 2x - x^2 & \text{if } x \leqslant 2 \\ \sin\left(\frac{\pi}{2}x\right) & \text{if } x \geqslant 2 \end{cases}$$

The function f is graphed below. This exercise explores the signed area under the graph of f from $x = 0$ to $x = 4$.

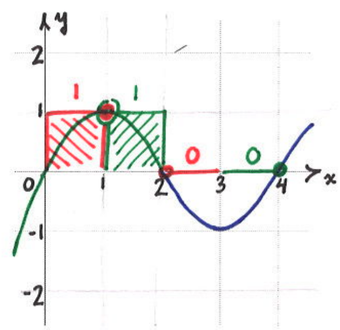
(a) (1 pt) Briefly (!) explain why we can't use finite geometry to find the exact value of $\int_0^4 f(x)\, dx$.

Solution: We cannot use finite geometry to compute the exact signed area between the graph of $f(x)$ and the x-axis, because we cannot partition the graph of $f(x)$ into "nice" geometric shapes for which we know exact area formulas.

(b) (1 pt) On separate graphs below, draw a lower sum and an upper sum, each with four intervals of width 1. Use these to compute a lower and upper estimate for $\int_0^4 f(x)\, dx$.



Lower sum (L)



Upper sum (U)

Solution: The lower and upper sums are sketched above. We compute

$$L = 1(0) + 1(0) + 1(-1) + 1(-1) = -2 \qquad\qquad U = 1(1) + 1(1) + 1(0) + 1(0) = 2$$

(c) (2 pt) Find an antiderivative $F_i(x)$ for each "piece" of $f(x)$. Use these antiderivatives and the fundamental theorem of calculus to compute the integrals on the right side of

$$\int_0^4 f(x)\, dx = \int_0^2 f(x)\, dx + \int_2^4 f(x)\, dx \tag{1}$$

Add your results to determine the integral on the left side. Show that $L \leqslant \int_0^4 f(x)\, dx \leqslant U$..

Solution: By "running the derivative in reverse", and tweaking our original guess as needed, we find the antiderivatives

$$F_1(x) = x^2 - \frac{1}{3}x^3 \qquad\qquad F_2(x) = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)$$

Note that (i) these are not (!) the most general antiderivatives of the corresponding $f_i(x)$—they are particular antiderivatives of $f_i(x)$, where we have chosen the constant $C_i = 0$ in both cases; and (ii) we can quickly check that each $F_i(x)$ is indeed an antiderivative of the corresponding $f_i(x)$, by computing the derivative of the former:

$$F_1'(x) = 2x - \frac{1}{3}3x^2 = 2x - x^2 = f_1(x)$$
$$F_2'(x) = -\frac{2}{\pi}\left[-\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)\right] = \sin\left(\frac{\pi}{2}x\right) = f_2(x)$$

We may use these antiderivatives in the fundamental theorem of calculus to compute the integrals on the right side of Equation (1):

$$\begin{aligned}\int_0^2 f(x)\,dx &= F_1(2) - F_1(0)\\ &= \left[(2)^2 - \frac{1}{3}(2)^3\right] - \left[(0)^2 - \frac{1}{3}(0)^3\right]\\ &= \left[4 - \frac{8}{3}\right] - [0 - 0]\\ &= \frac{4}{3}\end{aligned}$$

$$\begin{aligned}\int_2^4 f(x)\,dx &= F_2(4) - F_2(2)\\ &= \left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}(4)\right)\right] - \left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}(2)\right)\right]\\ &= \left[-\frac{2}{\pi}(1)\right] - \left[-\frac{2}{\pi}(-1)\right]\\ &= -\frac{4}{\pi}\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^4 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx\\ &= \frac{4}{3} - \frac{4}{\pi} \approx 0.0601\end{aligned}$$

We note that

$$L = -2 \qquad \leqslant \qquad \int_0^4 f(x)\,dx \approx 0.0601 \qquad \leqslant \qquad U = 2$$

as required by the theory of lower and upper sums.

Remarks.

1. The total signed area is close to zero, as suggested by the graph of f.

2. Even without a calculator, we can know for sure that $\frac{4}{3} - \frac{4}{\pi}$ is positive, by the following argument (the symbol "⇔" means "if and only if" or "is equivalent to"):

$$3 < \pi \approx 3.1416 \qquad \Leftrightarrow \qquad \frac{1}{3} > \frac{1}{\pi} \qquad \Leftrightarrow \qquad \frac{4}{3} > \frac{4}{\pi} \qquad \Leftrightarrow \qquad \frac{4}{3} - \frac{4}{\pi} > 0$$