Math 357
Long quiz 02

2024–01–29 (M)

Your name: ______________________________

Consider $\mathbf{R}[x,y]$, the polynomial ring in two indeterminates x, y whose ring of coefficients is the field $\mathbf{R}$ of real numbers. Let $f, g \in \mathbf{R}[x,y]$ be the polynomials

$$f(x,y) = x^2y - xy - xy^3 \qquad\qquad g(x,y) = x^2 - xy - 2y^2$$

(a) For each polynomial, state its (total) degree and its number of homogeneous components.

Solution: By definition, the (total) degree of a nonzero polynomial is the maximum of the (total) degrees of its nonzero monomial terms. Let "# h.c." denote "number of homogeneous components". We have

	deg	# h.c.
f	4	3
g	2	1

In particular, each nonzero monomial term of f is a different homogenous component, because each term has a different (total) degree; and g has a single nonzero homogeneous component, and is therefore homogeneous.

(b) Consider the following statement: "If a polynomial is homogeneous, then the zeros of the induced function are well defined on lines through the origin." Use the polynomials f and g to explain this statement. *Hint:* What is $\{\lambda(x_0, y_0) \mid \lambda \in \mathbf{R}\}$?

Solution: As noted above, the polynomial g is homogeneous, whereas f is not. We observe that $(2,1)$ is a zero of g:

$$g(2,1) = 4 - 2 - 2 = 0$$

Let $\ell_{(2,1)}$ denote the line in $\mathbf{R}^2$ passing through the origin and the point $(2,1)$. Then

$$\ell_{(2,1)} = \{\lambda(2,1) \mid \lambda \in \mathbf{R}\}$$

In particular, for every point $P \in \ell_{(2,1)}$, there exists a $\lambda \in \mathbf{R}$ such that $P = \lambda(2,1)$. Evaluating g at P, we find

$$g(P) = g(2\lambda, \lambda) = 4\lambda^2 - 2\lambda^2 - 2\lambda^2 = \lambda^2(4 - 2 - 2) = 0$$

Because $P \in \ell_{(2,1)}$ was arbitrary, we conclude that the polynomial g evaluates to 0 on the line $\ell_{(2,1)}$. That is, if a homogeneous polynomial evaluates to zero at a nonzero point P, then it evaluates to zero at any point on the line through P and the origin.

(c) Make a conjecture.

Solution: We observe that $(2,1)$ is also a zero of the polynomial f:

$$f(2,1) = 4 - 2 - 2 = 0$$

The point $(-2,-1) = -1(2,1)$ is on the line through $(2,1)$ and the origin. We compute

$$f(-1(2,1)) = f(-2,-1) = -4 - 2 - 2 = -8 \neq 0$$

Perhaps we could strengthen the statement in part (b)? Conjecture: Let R be an integral domain, let $n \in \mathbf{Z}_{>0}$, and let t_i be indeterminates. A polynomial $f \in R[t_1, \ldots, t_n]$ is homogeneous if and only if the zeros of the induced function $f : R^n \to R$ are well defined on lines through the origin. (Do we need the ring R of coefficients to be an integral domain? to be a field? to be infinite?)

One might also consider fractions of homogeneous polynomials. Conjecture: Let R, n, and t_i be as above, and let $f, g \in R[t_1, \ldots, t_n]$ be nonzero homogeneous polynomials of the same degree. Let $Z(g)$ denote the set of zeros of g:

$$Z(g) = \{a \in R^n \mid g(a) = 0\}$$

Then f/g is well defined as a function $R^n - Z(g) \to R$.