Math 112
Exam 02

2022–03–10 (R)

Your name: ______________________________

Instructions

Number of exercises	: 5
Permitted time	: 75 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- Work hard, do your best, and have fun!

Exercise	Total	(a)	(b)	(c)	(d)	(e)
1	/10	/2	/2	/2	/2	/2
2	/16	/4	/4	/4	/4	
3	/12	/4	/4	/4		
4	/10	/2	/4	/4		
5	/18	/4	/2	/4	/4	/4
Total	/66					

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

For parts (a)–(b), let $f(x)$ be a function, and let $F_1(x)$ and $F_2(x)$ be antiderivatives of $f(x)$.

(a) (2 pt) The function $2F_1(x)$ is an antiderivative of $2f(x)$.

true false

Solution: True. By definition of antiderivative, the hypothesis that $F_1(x)$ is an antiderivative of $f(x)$ means that $F_1'(x) = f(x)$. Therefore, by linearity of the derivative,

$$(2F_1(x))' = 2F_1'(x) = 2f(x)$$

This equation says that $2F_1(x)$ is an antiderivative of $2f(x)$.

(b) (2 pt) The function $F_1(x) + F_2(x)$ is an antiderivative of $2f(x)$.

true false

Solution: True. By definition of antiderivative, $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$. Therefore, by linearity of the derivative,

$$(F_1(x) + F_2(x))' = F_1'(x) + F_2'(x) = f(x) + f(x) = 2f(x)$$

This equation says that $F_1(x) + F_2(x)$ is an antiderivative of $2f(x)$.
For parts (c)–(d), let $f(x)$ and $g(x)$ be functions such that

$$\int_0^2 f(x)\,dx = 1 \qquad \int_0^2 g(x)\,dx = -4$$

(c) (2 pt) $\displaystyle\int_0^2 \left[3f(x) + \frac{1}{2}g(x)\right]\,dx = 1$

true false

Solution: True. The definite integral is linear. Therefore

$$\int_0^2 \left[3f(x) + \frac{1}{2}g(x)\right]\,dx = 3\int_0^2 f(x)\,dx + \frac{1}{2}\int_0^2 g(x)\,dx = 3(1) + \frac{1}{2}(-4) = 1$$

(d) (2 pt) $\displaystyle\int_0^1 g(x)\,dx + \int_1^2 g(x)\,dx = -2$

true false

Solution: False. Note that

$$\int_0^1 g(x)\ dx + \int_1^2 g(x)\ dx = \int_0^2 g(x)\ dx = -4$$

(e) (2 pt) If the definite integral of a function is zero, then the function must be the zero function.

true false

Solution: False. For example, consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = \sin x$. This is not the zero function, and we can check that $\int_0^{2\pi} \sin x\ dx = 0$. The positive and negative areas exactly cancel.

Exercise 2

(16 pt) Evaluate each of the following limits. Briefly but clearly justify your work.

(a) (4 pt) Use the Taylor series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \tag{1}$$

where the terms in "$\dots$" all involve x to the power 4 or higher, to evaluate

$$\lim_{x \to 0} \frac{x^2 + x - \ln(1+x)}{x^2}$$

Solution: Substituting the Taylor series (1) into the limit, the simplifying, we get

$$\begin{aligned}
\lim_{x \to 0} \frac{x^2 + x - \ln(1+x)}{x^2} &= \lim_{x \to 0} \frac{x^2 + x - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots\right)}{x^2} \\
&= \lim_{x \to 0} \frac{\frac{3}{2}x^2 - \frac{1}{3}x^3 + \dots}{x^2} \\
&= \lim_{x \to 0} \frac{\frac{3}{2} - \frac{1}{3}x + \dots}{1} \\
&= \frac{3}{2}
\end{aligned}$$

where in evaluating the final limit we use the fact that, after canceling the factor x^2 in the third equality, the terms in "$\dots$" all involve x to the power 2 or higher.

(b) (4 pt) Use l'Hôpital's rule to evaluate

$$\lim_{x \to 0} \frac{x^2 + x - \ln(1+x)}{x^2}$$

(Note that this is the same limit as in part (a).)

Solution: At each step, we (i) check that direct evaluation (abbreviated "D.E." below) of the limit gives $\frac{0}{0}$, so l'Hôpital's rule applies; and (ii) apply l'Hôpital's rule (differentiate numerator and denominator). The first result from direct evaluation that is not an indeterminate form is the limit.

$$\begin{aligned}
&\lim_{x \to 0} \frac{x^2 + x - \ln(1+x)}{x^2} \xrightarrow{\text{D.E.}} \frac{0^2 + 0 - \ln(1)}{0^2} = \frac{0}{0} \\
&= \lim_{x \to 0} \frac{2x + 1 - \frac{1}{1+x}}{2x} \xrightarrow{\text{D.E.}} \frac{0 + 1 - 1}{0} = \frac{0}{0} \\
&= \lim_{x \to 0} \frac{2 + \frac{1}{(1+x)^2}}{2} = \frac{2+1}{2} = \frac{3}{2}
\end{aligned}$$

(c) (4 pt) $\lim_{x\to 2} \dfrac{1-\sqrt{5-2x}}{x-2}$

Solution: Direct evaluation of the limit gives the indeterminate form $\frac{0}{0}$. Thus l'Hôpital's rule applies, and we may use it to find the limit. However, let us instead multiply numerator and denominator by the conjugate of the numerator (equivalent to multiplying by 1):

$$\begin{aligned}
\lim_{x\to 2} \frac{1-\sqrt{5-2x}}{x-2} &= \lim_{x\to 2} \frac{1-\sqrt{5-2x}}{x-2} \cdot \frac{1+\sqrt{5-2x}}{1+\sqrt{5-2x}} \\
&= \lim_{x\to 2} \frac{1-(\sqrt{5-2x})^2}{(x-2)(1+\sqrt{5-2x})} \\
&= \lim_{x\to 2} \frac{2x-4}{(x-2)(1+\sqrt{5-2x})}
\end{aligned}$$

Factoring the numerator and canceling the common factor $x-2$ allows us to compute the limit by direct evaluation:

$$\begin{aligned}
&= \lim_{x\to 2} \frac{2(x-2)}{(x-2)(1+\sqrt{5-2x})} \\
&= \lim_{x\to 2} \frac{2}{1+\sqrt{5-2x}} \\
&= \frac{2}{1+\sqrt{1}} = 1
\end{aligned}$$

(d) (4 pt) $\lim_{x\to +\infty} x^3 e^{-x}$

Solution: Direct evaluation of the limit gives $+\infty \cdot 0$, an indeterminate form. It is not an indeterminate form to which l'Hôpital's rule applies. We massage the given function into an equivalent form by moving the exponential factor to the denominator:

$$\lim_{x\to +\infty} x^3 e^{-x} = \lim_{x\to +\infty} \frac{x^3}{e^x}$$

Direct evaluation of this limit gives $\frac{+\infty}{+\infty}$, an indeterminate form to which l'Hôpital's rule applies. Applying it iteratively (three times total), we get

$$\begin{aligned}
&= \lim_{x\to +\infty} \frac{3x^2}{e^x} \xrightarrow{\text{D.E.}} \frac{+\infty}{+\infty} \\
&= \lim_{x\to +\infty} \frac{6x}{e^x} \xrightarrow{\text{D.E.}} \frac{+\infty}{+\infty} \\
&= \lim_{x\to +\infty} \frac{6}{e^x} \xrightarrow{\text{D.E.}} \frac{1}{+\infty} = 0
\end{aligned}$$

(Recall that a limit of $\frac{1}{+\infty}$ means that we're making the denominator larger and larger—arbitrarily large—while the numerator stays constant, so the fraction as a whole becomes smaller and smaller—arbitrarily small, that is, close to 0.) Thus the limit is 0.

Exercise 3

(12 pt) Evaluate the indefinite integrals. (That is, find the most-general antiderivative $F(x)$ of the integrand $f(x)$ in the following integrals $\int f(x)\,dx$.)

(a) (4 pt) $\displaystyle\int 4x^3 - 3x^2 + 1\,dx$

Solution: For each indefinite integral in this exercise, we (1) use linearity of the integral to "break" the integral into separate terms and factor out constants, (2) guess and check to find an antiderivative for each smaller integral, and (3) include "+C" to get the most-general antiderivative, where C is an arbitrary real number (that is, a constant). We check our result by verifying that differentiating our most-general antiderivative (what we've been calling $F(x)$) gives the original integrand (the function in the original integral, which we've been calling $f(x)$).

We compute

$$\begin{aligned}\int 4x^3 - 3x^2 + 1\,dx &= 4\int x^3\,dx - 3\int x^2\,dx + \int 1\,dx\\ &= 4\left[\frac{1}{4}x^4\right] - 3\left[\frac{1}{3}x^3\right] + x + C\\ &= x^4 - x^3 + x + C\end{aligned}$$

We check

$$\left(x^4 - x^3 + x + C\right)' = 4x^3 - 3x^2 + 1 = f(x)$$

(b) (4 pt) $\displaystyle\int 6e^{3x} - 25\cos(5x)\,dx$

Solution: We compute

$$\begin{aligned}\int 6e^{3x} - 25\cos(5x)\,dx &= 6\int e^{3x}\,dx - 25\int \cos(5x)\,dx\\ &= 6\left[\frac{1}{3}e^{3x}\right] - 25\left[\frac{1}{5}\sin(5x)\right] + C\\ &= 2e^{3x} - 5\sin(5x) + C\end{aligned}$$

We check

$$\left(2e^{3x} - 5\sin(5x) + C\right)' = 2e^{3x}\cdot 3 - 5\cos(5x)\cdot 5 = 6e^{3x} - 25\cos(5x) = f(x)$$

(c) (4 pt) $\displaystyle\int \frac{1 - x\sin x}{x}\,dx$

Solution: We compute

$$\begin{aligned}\int \frac{1 - x\sin x}{x}\,dx &= \int \frac{1}{x} - \sin x\,dx\\ &= \int \frac{1}{x}\,dx - \int \sin x\,dx\\ &= \ln x - (-\cos x) + C\\ &= \ln x + \cos x + C\end{aligned}$$

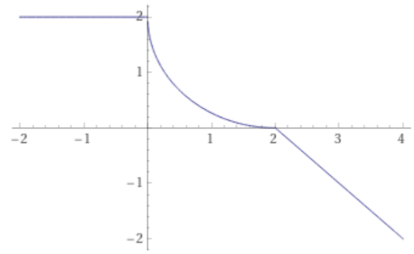
(More precisely, $\int \frac{1}{x}\, dx = \ln|x| + C$. We'll ignore the subtlety about the absolute value for now.) We check

$$(\ln x + \cos x + C)' = \frac{1}{x} - \sin x = \frac{1}{x} - \frac{x \sin x}{x} = \frac{1 - x \sin x}{x} = f(x)$$

Exercise 4

(10 pt) Consider the piecewise function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 2 & \text{if } x \leqslant 0 \\ 2 - \sqrt{4x - x^2} & \text{if } -2 \leqslant x \leqslant 0 \\ 2 - x & \text{if } x \geqslant 2 \end{cases}$$



A graph of f is shown at right.

(a) (2 pt) Consider the rule of assignment for f on the interval $[0, 2]$, namely,

$$y = 2 - \sqrt{4x - x^2}$$

Use algebra to massage this equation into the form $(x - a)^2 + (y - b)^2 = r^2$, thus showing that the graph of f on the interval $[0, 2]$ is part of a circle of radius $r = 2$.

Solution: First, we need to get rid of the square root, which we'll do by squaring. So let's get the square root by itself

$$y - 2 = -\sqrt{4x - x^2}$$

and then square both sides

$$(y - 2)^2 = \left(-\sqrt{4x - x^2}\right)^2 = 4x - x^2$$

To put the right side in the form $(x - a)^2$, we complete the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = -(x^2 - 4x + 4) + 4 = -(x - 2)^2 + 4$$

We substitute this into the previous expression, then move the x and y expressions to the same side, obtaining

$$(x - 2)^2 + (y - 2)^2 = 4 = 2^2$$

This is the equation of a circle with center $(a, b) = (2, 2)$ and radius $r = 2$.

(b) (4 pt) Use geometry to compute the exact value of $\int_{-2}^{4} f(x)\, dx$. You may leave your answer in terms of π; if you prefer, you may use that $\pi \approx 3.14$. (Hint: Use the conclusion from part (a) to help find the area on the interval $[0, 2]$. The area of a full circle of radius r is πr^2. Remember, area is signed (+/−)!)

Solution: We break the definite integral on the interval $[-2, 4]$ into three pieces, based on the geometry of the graph of $f(x)$:

$$\int_{-2}^{4} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx \tag{2}$$

Let's call these areas $A_{[-2,0]}$, $A_{[0,2]}$, and $A_{[2,4]}$, respectively.

On the interval $[-2, 0]$, the area $A_{[-2,0]}$ between the graph of $f(x)$ and the x-axis is a rectangle (in fact, square) of base 2 and height 2. This area is positive because it lies above the x-axis. So

$$A_{[-2,0]} = (2)(2) = 4$$

On the interval $[0, 2]$, the area $A_{[0,2]}$ between the graph of $f(x)$ and the x-axis is a square minus a quarter circle, where the sides of the square and the radius of the circle are 2. This area is positive because it lies above the x-axis. So

$$A_{[0,2]} = (2)(2) - \frac{1}{4}\pi(2)^2 = 4 - \pi$$

On the interval $[2, 4]$, the area $A_{[2,4]}$ between the graph of $f(x)$ and the x-axis is a triangle (in fact, a right triangle) of base 2 and height 2. This area is negative because it lies below the x-axis. So

$$A_{[2,4]} = -\frac{1}{2}(2)(2) = -2$$

Substituting these results into (2), we conclude

$$\begin{aligned}\int_{-2}^{4} f(x)\,dx &= A_{[-2,0]} + A_{[0,2]} + A_{[2,4]} \\ &= [4] + [4 - \pi] + [-2] \\ &= 6 - \pi\end{aligned}$$

(c) (4 pt) Is the definite integral $\int_0^4 f(x)\,dx$ positive, negative, or zero? Justify. (Hint: You need not give an exact value, though you may. It is enough to justify your answer using the geometry of the graph of $f(x)$.)

Solution: The area left over when we subtract the quarter circle from its enclosing square—what we called $A_{[0,2]}$ in part (b)—fits inside the right triangle of base 2 and height 2—what we called $A_{[2,4]}$—leaving a gap of area along the hypotenuse of the triangle. Because $A_{[2,4]}$ is negative area and $A_{[0,2]}$ is positive area, the overlapping parts cancel, and the gap of area that remains (which belongs to $A_{[2,4]}$) is negative. Thus we expect $\int_0^4 f(x)\,dx$ to be negative.

Note that, if we use our values from part (b) to compute the exact value of this definite integral, we get

$$\int_0^4 f(x)\,dx = A_{[0,2]} + A_{[2,4]} = [4 - \pi] + [-2] = 2 - \pi < 0$$

(Note that $2 - \pi = -(\pi - 2) \approx -1.1416$.)

Exercise 5

(18 pt) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = 4x^3 - 12x + 8 \tag{3}$$

(a) (4 pt) Find an antiderivative $F(x)$ of $f(x)$. Verify that it is indeed an antiderivative.

Solution: We compute

$$\begin{aligned}\int 4x^3 - 12x + 8 \, dx &= 4\int x^3 \, dx - 12\int x \, dx + 8\int 1 \, dx \\ &= 4\left[\frac{1}{4}x^4\right] - 12\left[\frac{1}{2}x^2\right] + 8x + C \\ &= x^4 - 6x^2 + 8x + C\end{aligned}$$

We check

$$\left(x^4 - 6x^2 + 8x + C\right)' = 4x^3 - 12x + 8 = f(x)$$

We're asked to find an antiderivative, not necessarily the most-general antiderivative, of $f(x)$. So, if desired, we may set C equal to a particularly convenient value, for example, $C = 0$. In this case, we get a particular antiderivative

$$F(x) = x^4 - 6x^2 + 8x$$

We'll use this as $F(x)$ in the remainder of this exercise.

(b) (2 pt) Using your antiderivative $F(x)$ from part (a), show that

$$\int_{-1}^{2} f(x) \, dx = F(2) - F(-1) = 21$$

Take the first equality as given. (It is the fundamental theorem of calculus.) Just show that $F(2) - F(-1) = 21$.

Solution: We compute

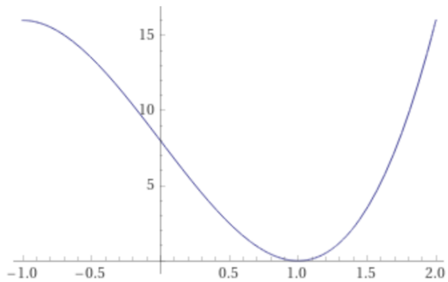
$$\begin{aligned}F(2) &= 2^4 - 6 \cdot 2^2 + 8 \cdot 2 = 16 - 24 + 16 = 8 \\ F(-1) &= (-1)^4 - 6 \cdot (-1)^2 + 8 \cdot (-1) = 1 - 6 - 8 = -13\end{aligned}$$

Thus

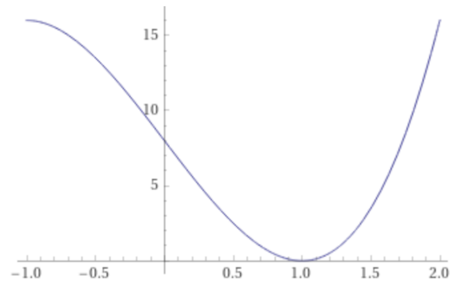
$$F(2) - F(-1) = 8 - (-13) = 21$$

as we were asked to show.

(c) (4 pt) On the graphs of f below, draw a lower- and upper-sum approximation to the definite integral $\int_{-1}^{2} f(x)\, dx$. Partition the interval $[-1, 2]$ into three subintervals, each of width 1.



Lower sum



Upper sum

Solution:

(d) (4 pt) Compute the upper- and lower-sum approximations you sketched in part (c). (Use the rule of assignment for $f(x)$, given in Equation (3) at the start of this exercise, to help find the heights of your rectangles.) Show that these approximations bound your value of the definite integral computed in part (b).

Solution: For the lower-sum approximation, we compute

$$\begin{aligned} L_3 &= 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(1) \\ &= 1 \cdot 8 + 1 \cdot 0 + 1 \cdot 0 \\ &= 8 \end{aligned}$$

For the upper-sum approximation, we compute

$$\begin{aligned} U_3 &= 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(2) \\ &= 1 \cdot (-4 - (-12) + 8) + 1 \cdot 8 + 1 \cdot (32 - 24 + 8) \\ &= 16 + 8 + 16 \\ &= 40 \end{aligned}$$

We verify that

$$L_3 = 8 \leqslant \int_{-1}^{2} f(x)\, dx = 21 \leqslant U_3 = 40$$

(e) (4 pt) Your friend computes lower- and upper-sum approximations for $\int_{-1}^{2} f(x)\, dx$ with 300 subintervals, each of width $\frac{1}{100}$. Your friend reports the results are $L_{300} \approx -1.2723$ and $U_{300} \approx 20.8603$. Explain to your friend why neither result can be correct. (Be kind—your friend just did a lot of work!)

Solution: The lower sum cannot be correct, because the function $f(x)$ that we are integrating is nonnegative (that is, greater than or equal to 0) everywhere on the interval $[-1, 2]$ over which we are integrating. Thus the smallest possible value of any lower-sum approximation is 0.

The upper sum cannot be correct, because the actual area under the graph of $f(x)$ on the interval $[-1, 2]$ is given by $\int_{-1}^{2} f(x)\, dx$, which we showed in part (b) equals 21; and any upper-sum approximation must be greater than or equal to the actual area.