# Math 112
# LQuiz 13

2022–03–24 (R)

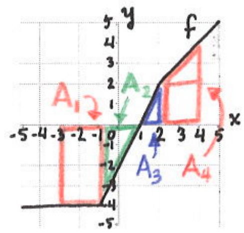
Your name: ______________________________

## Exercise

(4 pt) Let $f : \mathbf{R} \to \mathbf{R}$ be the piecewise function whose rule of assignment is

$$f(x) = \begin{cases} -4 & \text{if } x \leqslant -1 \\ 2x - 2 & \text{if } -1 \leqslant x \leqslant 2 \\ x & \text{if } x \geqslant 2 \end{cases}$$

The function $f$ is graphed below. This exercise explores the signed area under the graph of $f$ from $x = -3$ to $x = 4$.




(a) (1 pt) Use geometry to show that $\int_{-3}^{4} f(x)\,dx = -5$.

**Solution:** Partition the area between the graph of $f$ and the $x$-axis as shown above. We compute the signed area $A_i$ of each piece:[1]

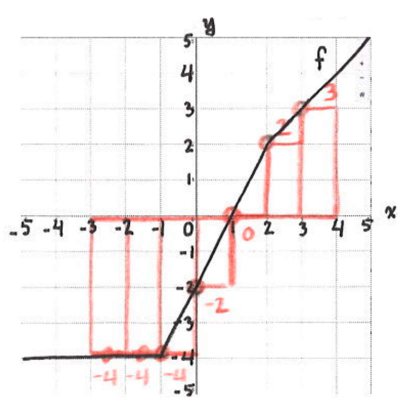
$$A_1 = -(2)(4) = -8 \qquad A_3 = +\frac{1}{2}(1)(2) = 1$$

$$A_2 = -\frac{1}{2}(2)(4) = -4 \qquad A_4 = +\left[(2)(2) + \frac{1}{2}(2)(2)\right] = 6$$

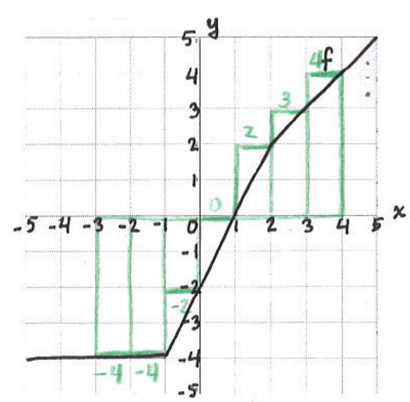
Thus the total area is

$$A = A_1 + A_2 + A_3 + A_4 = -8 + (-4) + 1 + 6 = -5$$

(b) (2 pt) On separate graphs below, draw a lower sum and an upper sum, each with seven intervals of width 1, to estimate $\int_{-3}^{4} f(x)\,dx$. Show that $L \leqslant \int_{-3}^{4} f(x)\,dx \leqslant U$.




Lower sum (L)




Upper sum (U)

[1] Recall that the area is positive if it lies above the $x$-axis (corresponding to positive $y$-values), and negative if it lies below the $x$-axis (corresponding to negative $y$-values).

**Solution:** The lower sum $L$ and upper sum $U$ are sketched above. We compute the value of each by adding the (signed) areas of their rectangles:

$$L = -4 + (-4) + (-4) + (-2) + 0 + 2 + 3 = -9$$
$$U = -4 + (-4) + (-2) + 0 + 2 + 3 + 4 = -1$$

Using our result from part (a), we confirm that

$$L = -9 \qquad \leqslant \qquad \int_{-3}^{4} f(x)\,dx = -5 \qquad \leqslant \qquad U = -1$$

For reference, $f : \mathbf{R} \to \mathbf{R}$ is the piecewise function whose rule of assignment is

$$f(x) = \begin{cases} -4 & \text{if } x \leqslant -1 \\ 2x - 2 & \text{if } -1 \leqslant x \leqslant 2 \\ x & \text{if } x \geqslant 2 \end{cases}$$

(c) (1 pt) Find an antiderivative $F_i(x)$ for each "piece" of $f(x)$. Use these antiderivatives and the fundamental theorem of calculus to compute the value of each integral on the right side of the following equation:

$$\int_{-3}^{4} f(x)\,dx = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx \tag{1}$$

Add your values for the three integrals on the right, and compare the result to part (a).

**Solution:** Let $f_i(x)$ denote the rule of assignment for each "piece" of $f(x)$. That is,

$$f_1(x) = -4 \qquad f_2(x) = 2x - 2 \qquad f_3(x) = x$$

We compute an antiderivative for each of these functions (on their respective intervals):

$$F_1(x) = -4x \qquad F_2(x) = x^2 - 2x \qquad F_3(x) = \frac{1}{2}x^2$$

Using these, and the fundamental theorem of calculus, we compute the value of each integral on the right side of Equation (1):

$$\int_{-3}^{-1} f(x)\,dx = F_1(-1) - F_1(-3) = [(-4)(-1)] - [(-4)(-3)] = 4 - 12 = -8$$
$$\int_{-1}^{2} f(x)\,dx = F_2(2) - F_2(-1) = \left[(2)^2 - 2(2)\right] - \left[(-1)^2 - 2(-1)\right] = 0 - 3 = -3$$
$$\int_{2}^{4} f(x)\,dx = F_3(4) - F_3(2) = \left[\frac{1}{2}(4)^2\right] - \left[\frac{1}{2}(2)^2\right] = 8 - 2 = 6$$

Referring to Equation (1), adding these three results gives

$$\int_{-3}^{4} f(x)\,dx = (-8) + (-3) + 6 = -5$$

This equals the exact value of the definite integral that we computed in part (a), as the fundamental theorem of calculus guarantees.