Math 357
Expositional homework 05

Assigned: 2024–03–18 (M)
Due:

The goal of this homework is to deepen our understanding of representation theory by working with the theory and applications of two results: Maschke's theorem and a version of Schur's lemma.

Corollaries of Maschke's theorem. Let G be a finite group, let F be a field such that $\operatorname{char} F \nmid \#G$, and let V be a finitely generated FG-module (equivalently, a finite-dimensional F-vector space[1]). Prove the following statements.

(a) V is completely reducible.[2]

(b) Let $\rho : G \to GL(V)$ be a representation of G on V—if you like, the representation afforded by the FG-module V. Then there exists a basis $\mathcal{B}$ of V such that, simultaneously (!) for all $g \in G$, the matrix of $\rho(g)$ with respect to $\mathcal{B}$ is block diagonal.[3]

Schur's lemma. Let G be a group; let F be a field; and for $i \in \{1, 2\}$, let V_i be an F-vector space, and let $\rho_i : G \to GL(V_i)$ be a representation of G on V_i. A G**-homomorphism** from ρ_1 to ρ_2 is an F-linear map $\varphi : V_1 \to V_2$ that intertwines the two representations; that is, for all $g \in G$,

$$\varphi \circ \rho_1(g) = \rho_2(g) \circ \varphi$$

as maps $V_1 \to V_2$. A G**-isomorphism** is an invertible G-homomorphism.

Analogous to the notation we developed for modules, let $\operatorname{Hom}_G(\rho_1, \rho_2)$ denote the set of G-homomorphisms from ρ_1 to ρ_2, and let $\operatorname{End}_G(\rho_1)$ denote $\operatorname{Hom}_G(\rho_1, \rho_1)$.

Consider the following version of Schur's lemma.

Lemma 1 (Schur). *Let G be a group, let V be a $\mathbf{C}$-vector space, let $\rho : G \to GL(V)$ be an irreducible representation of G, and let $\varphi \in \operatorname{End}_G(\rho)$. Then φ is a scalar multiple of the identity map on V. That is, there exists a $\lambda \in \mathbf{C}$ such that for all $v \in V$, $\varphi(v) = \lambda v$.*

(c) Prove Schur's lemma. *Hint:* Make sense of the following proof outline:

1. $E_\lambda = \{v \in V \mid \varphi(v) = \lambda v\} \neq \{0_V\}$

[1] Convince yourself of this equivalence! For a concise explanation, see DF3e, p 851.
[2] *Hint:* See DF3e, p 851.
[3] *Hint:* Use Exercise (a). See also DF3e, p 851.

2. E_λ is G-invariant
3. $E_\lambda = V$

(d) For $i \in \{1, 2\}$, let V_i be a $\mathbf{C}$-vector space, and let $\rho_i : G \to GL(V_i)$ be a representation of G on V_i; and let $\varphi \in \mathrm{Hom}_G(\rho_1, \rho_2)$. Prove that if $V_1 \not\cong V_2$, then φ is the zero map; and if $V_1 \cong V_2$ and φ is not the zero map, then φ is a G-isomorphism.

Applications of Schur's lemma. Let G be an abelian group, let V be a nonzero $\mathbf{C}$-vector space, and let $\rho : G \to GL(V)$ be an irreducible representation.

(e) Prove that $\deg \rho = 1$.[4]

(f) Let $n \in \mathbf{Z}_{>0}$, and let $G = \langle g \mid g^n = 1 \rangle$ be the cyclic group of order n with generator g. Use Exercise (e) to show that ρ has the form

$$\begin{aligned} \rho : G &\to C^\times \\ g^j &\mapsto e^{\frac{2jk\pi i}{n}} \end{aligned}$$

for a fixed $k \in \{0, \dots, n-1\}$.

[4] *Hint:* In the setting of Schur's lemma, consider $\varphi = \rho(g)$ for some $g \in G$.