Math 357
Long quiz 02A

2024–02–05 (M)

Your name: ________________________________

(a) Consider the polynomial ring $\mathbf{Z}[t_1, t_2, t_3]$ ($\mathbf{Z}$ denotes the integers). For each of the following polynomials, state its (total) degree and its number of (nonzero) homogeneous components.

$$f = t_1^3 + t_1 t_2 t_3 + 2t_2^2 t_3^2 - t_2 t_3^3 + t_3^2$$
$$g = (t_1^2 + t_2^2 + t_3^2)^3 - (t_1^3 + t_2^3 + t_3^3)^2$$

Hint: Think before you compute.

Solution: Let "# h.c." denote "number of homogeneous components". We have

	deg	# h.c.
f	4	3
g	6	1

Note that to determine the degree and number of homogeneous components of the polynomial g, we don't need to expand the powers and group like terms. It suffices to note that (i) all terms in the expanded form have (total) degree 6, and (ii) not all terms can cancel. For example, the $t_1^2 t_2^2 t_3^2$ term from the first expansion has nonzero coefficient (why?); and no term from the second expansion has the same multidegree, $(2, 2, 2)$ (why?).

(b) By popular demand, you are explaining ideals to a group of your friends. One of them exclaims, "Ah! So the ideal of all polynomials whose terms all have even (total) degree is an analog, in polynomial rings, to the ideal of even integers in $\mathbf{Z}$." Respond.

Solution: "I would not trade your enthusiasm for all the prime ideals in $\mathbf{Z}$ (of which there are infinitely many)," I begin. "Aber achtung: Are we sure that the *set* of all polynomials whose terms all have even (total) degree is an *ideal*?" After murmuring and shared scribbles passed among my friends, one retorts, "The set contains the zero polynomial and is closed under addition and multiplication." "I agree," I agree, "but is that all we demand of ideals?" Further murmuring and shared scribbles produces a moan: "Oh no—the set is not strongly closed by multiplication of all ring elements." "For example?" I prod. "Multiply any polynomial whose terms all have even degree by any linear monomial," my friend observes, "and let the monomial's coefficient be 1, to keep things simple." My smile expresses my agreement and my satisfaction.