Math 357
Short quiz 11

2024–03–20 (W)

Your name: ______________________

(a) Define the minimal polynomial of a field element. Briefly but clearly introduce any hypotheses and auxiliary objects that you use in your definition.

Solution: Let $K : K_0$ be a field extension, and let $\alpha \in K$ be algebraic over K_0. The **minimal polynomial** of α over K_0 is the (unique) monic, irreducible polynomial $p \in K_0[t]$ such that $p(\alpha) = 0$.[1] We will denote this minimal polynomial by m_{α,K_0}.

Note that the hypothesis that α be algebraic over K_0 is important. If α is transcendental (not algebraic) over K_0, then by definition there is no polynomial $p \in K_0[t]$ such that $p(\alpha) = 0$, let alone a monic, irreducible one.

(b) Let $\alpha = \sqrt{-\sqrt{2}} \in \mathbf{C}$. State the minimal polynomial of α over $\mathbf{Q}$, over $\mathbf{R}$, and over $\mathbf{C}$.

Solution: We may manipulate the defining equation for α in an attempt to get a "polynomial" relation involving powers of α and coefficients in the desired base field. If we are successful, then we may replace α with t to get a polynomial with α as a zero. (We still must certify that the polynomial is monic and irreducible.) Playing this game, we get candidates

$$\begin{array}{rclcrcl} \alpha = \sqrt{-\sqrt{2}} & \Leftrightarrow & \alpha - \sqrt{-\sqrt{2}} = 0 & \rightsquigarrow & m_{\alpha,\mathbf{C}} = t - \sqrt{-\sqrt{2}} \\ \alpha^2 = -\sqrt{2} & \Leftrightarrow & \alpha^2 + \sqrt{2} = 0 & \rightsquigarrow & m_{\alpha,\mathbf{R}} = t^2 + \sqrt{2} \\ \alpha^4 = 2 & \Leftrightarrow & \alpha^4 - 2 = 0 & \rightsquigarrow & m_{\alpha,\mathbf{Q}} = t^4 - 2 \end{array}$$

All these polynomials are monic. As for irreducibility, note that

1. $m_{\alpha,\mathbf{Q}}$ is irreducible in $\mathbf{Q}[t]$ by the Eisenstein–Schönemann criterion with $p = 2$ and Gauß's lemma.[2]

2. $m_{\alpha,\mathbf{R}}$ is irreducible in $\mathbf{R}[t]$ because, if it factored, it would do so into two degree-1 factors, which would imply that $\alpha = \sqrt{-2}$ is in $\mathbf{R}$, a contradiction.[3]

3. $m_{\alpha,\mathbf{C}}$ is irreducible in $\mathbf{C}[t]$ because all degree-1 polynomials over a field are irreducible.[4]

[1] We may view the ring (field) operations $+$ and $\times$ used to evaluate p at α as being those of K.

[2] See DF3e, Proposition 9.13, p 309 and Proposition 9.5, p 303, respectively.

[3] See DF3e, Proposition 9.10, p 308.

[4] Note that the hypothesis that the ring of coefficients is a field is important. For example, the degree-1 polynomial $2t - 2$ is reducible in $\mathbf{Z}[t]$ (why?).