Math 212
Quiz 06

F 02 Sep 2016

Your name: ______________________________

Exercise

(5 pt) Find an equation for the plane in $\mathbf{R}^3$ passing through the points

$$(2,-2,0), \qquad (2,0,2), \qquad (0,-2,2).$$

Hint: If you have trouble determining a normal vector for this plane, think whether you can use the given points to "create" two nonparallel vectors that lie in the plane. Then think whether you can use these to determine a normal vector for the plane.

Solution: Geometrically, a plane is determined by (i) a point in the plane and (ii) a normal vector to the plane. We're given a point (in fact, three points) in the plane, so we just need to determine a normal vector, and then translate the geometry into algebra.

A systematic way to form a normal vector from three (noncolinear) points A, B, C is to form the vectors $\vec{AB}, \vec{AC}$, then compute their cross product. Doing so, we have

$$\vec{AB} = (2-2, 0-(-2), 2-0) = (0,2,2), \qquad \vec{AC} = (0-2, -2-(-2), 2-0) = (-2,0,2),$$

so a normal vector $\mathbf{n}$ to the plane determined by the three points is

$$\begin{aligned}
\mathbf{n} &= \vec{AB} \times \vec{AC} \\
&= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 2 \\ -2 & 0 & 2 \end{pmatrix} \\
&= (-1)^{1+1} \det \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \mathbf{i} + (-1)^{1+2} \det \begin{pmatrix} 0 & 2 \\ -2 & 2 \end{pmatrix} \mathbf{j} + (-1)^{1+3} \det \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{k} \\
&= 4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} = (4,-4,4).
\end{aligned}$$

Because the given three points are relatively "nice", you may also be able to "see" a normal vector to the plane they define: Find a corner of a room (maybe an empty room, if you're self-conscious), view the three lines emanating from that corner as the positive x-, negative y-, and positive z-axes (appropriately assigned), then put your feet at the point $(2,-2,0)$ (on the floor), your left hand at the point $(2,0,2)$ (on the left wall), and your right hand at the point $(0,-2,2)$ (on the right wall). If you now hold your body in a planar "plank" (great ab workout!), and then think about it (great mental-math workout!), you'll "see" that a normal vector to your body is given by $(1,-1,1)$. Note that this vector is a scalar multiple of the normal vector $\mathbf{n}$ that we found above.

Either way, we now have a normal vector for our plane. Choosing any of the given three points, say $(2,0,2)$, we can form an equation for the plane (see the general discussion below):

$$0 = \mathbf{n} \cdot (x-2, y-0, z-2) = (4,-4,4) \cdot (x-2, y-0, z-2) = (4x-8) + (-4y) + (4z-8).$$

If we wish, we can isolate the x, y, z terms, move the constant terms to the other side, and divide through by a common factor of 4 to obtain

$$x - y + z = 4.$$

General discussion: Remember that a plane is determined by (i) a point in the plane and (ii) a normal vector to the plane. In particular, for a plane in $\mathbf{R}^3$, let $P_0 = (x_0, y_0, z_0)$ be a point in the plane, and let $\mathbf{n} = (n_1, n_2, n_3)$ be a normal vector to the plane. Given any point $P = (x, y, z)$ in the plane, we can form the vector from P_0 to P:

$$\vec{P_0P} = (x - x_0, y - y_0, z - z_0).$$

This vector $\vec{P_0P}$ "lives in" the plane. (More precisely, if we move $\vec{P_0P}$ so that its initial point lies in the plane, then its terminal point — and hence the whole vector — lies in the plane.) Because $\mathbf{n}$ is normal (i.e. perpendicular) to the plane, we must have

$$\mathbf{n} \cdot \vec{P_0P} = 0.$$

Writing out what this equation means in terms of components, we find

$$0 = \mathbf{n} \cdot \vec{P_0P} = (n_1, n_2, n_3) \cdot (x - x_0, y - y_0, z - z_0) = n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0).$$

Note that this is a scalar equation, and it is linear in the variables x, y, z. If desired, we can combine the scalar terms into a single term

$$d = n_1x_0 + n_2y_0 + n_3z_0$$

and write the equation of the plane as

$$n_1x + n_2y + n_3z = d.$$

This is the general form of an equation of a plane in $\mathbf{R}^3$. (Can you generalize this to the general form of an equation of a plane in $\mathbf{R}^n$?) This gives us another way to solve the above exercise: We're given three points (x, y, z) that solve this equation, and we need to find the four values n_1, n_2, n_3, d. This gives us three equations (one for each point) in four unknowns.