Math 211
Quiz 21Z

M 05 Aug 2019

Your name : ________________________________

Exercise

(5 pt) Consider the homogeneous 1st-order 2×2 linear system of ODEs

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 4 & 1 \end{bmatrix} \mathbf{x}. \tag{1}$$

(a) (2 pt) Diagonalize the coefficient matrix (i.e. compute the eigenvalues and corresponding eigenvectors of the given matrix $\mathbf{A}$, and write it in the form $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$).

Solution: Denote the coefficient matrix in (1) by

$$\mathbf{A} = \begin{bmatrix} -1 & -4 \\ 4 & 1 \end{bmatrix}.$$

Eigenvalues. We compute

$$\lambda_{\pm} = \pm i\sqrt{15}.$$

Eigenvectors. We compute

$$\mathbf{v}_{+} = \begin{bmatrix} -1 + i\sqrt{15} \\ 4 \end{bmatrix}, \qquad \mathbf{v}_{-} = \begin{bmatrix} -1 - i\sqrt{15} \\ 4 \end{bmatrix}.$$

Any nonzero (complex) scalar multiples of these vectors will also be eigenvectors. Note that once we compute one of the two eigenvectors $\mathbf{v}_{+}, \mathbf{v}_{-}$, we can obtain the other immediately by taking the complex conjugate of the first.

Diagonalization. Using these results, we conclude

$$\begin{bmatrix} -1 & -4 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 + i\sqrt{15} & -1 - i\sqrt{15} \\ 4 & 4 \end{bmatrix} \begin{bmatrix} i\sqrt{15} & 0 \\ 0 & -i\sqrt{15} \end{bmatrix} \begin{bmatrix} -1 + i\sqrt{15} & -1 - i\sqrt{15} \\ 4 & 4 \end{bmatrix}^{-1}. \tag{2}$$

Call the matrices in this equation as follows:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

(b) (2 pt) Use this information to write the general real solution $\mathbf{x}(t)$ to (1). *Hint:* Recall that when the eigenvalues of a homogeneous 2×2 linear system are complex, we can compute one solution $e^{\lambda t} v$, just as we did with real eigenvalues; decompose it into real and imaginary parts; then use these parts as our basis for the solution space. The formulas

$$e^{a+ib} = e^{a} e^{ib}, \qquad e^{ib} = \cos b + i \sin b,$$

will be useful.

Solution: Call columns 1 and 2 of $\mathbf{P}$ $\mathbf{v}_+$ and $\mathbf{v}_-$, respectively. (They are eigenvectors corresponding to the eigenvalues $\lambda_+ = i\sqrt{15}i$ and $\lambda_- = -i\sqrt{15}$, respectively. Hence our choice of subscripts $+$ and $-$.) Our diagonalization (2) gives one basis for the vector space of complex (!) solutions to (1), namely, the vectors

$$\mathbf{x}_+(t) = e^{\lambda_+ t}\mathbf{v}_+ = e^{i\sqrt{15}t}\begin{bmatrix} -1+i\sqrt{15} \\ 4 \end{bmatrix}, \qquad \mathbf{x}_-(t) = e^{\lambda_- t}\mathbf{v}_- = e^{-i\sqrt{15}t}\begin{bmatrix} -1-i\sqrt{15} \\ 4 \end{bmatrix}.$$

To describe the real (!) solutions to (1), we can decompose either one of these complex basis vectors into real and imaginary parts, $\mathbf{x}_{\mathrm{Re}}$ and $\mathbf{x}_{\mathrm{Im}}$, and take these as our basis vectors. Decomposing $\mathbf{x}_+(t)$, using the formulas in the hint, we find

$$\begin{aligned}
\mathbf{x}_+(t) &= e^{i\sqrt{15}t}\begin{bmatrix} -1+i\sqrt{15} \\ 4 \end{bmatrix} = e^{0t}e^{i\sqrt{15}t}\begin{bmatrix} -1+i\sqrt{15} \\ 4 \end{bmatrix} = (\cos(\sqrt{15}t) + i\sin(\sqrt{15}t))\begin{bmatrix} -1+i\sqrt{15} \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\cos(\sqrt{15}t) - \sqrt{15}\sin(\sqrt{15}t)\right) & +i & \left(\sqrt{15}\cos(\sqrt{15}t) - \sin(\sqrt{15}t)\right) \\ 4\cos(\sqrt{15}t) & +i & \sin(\sqrt{15}t) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} -\cos(\sqrt{15}t) - \sqrt{15}\sin(\sqrt{15}t) \\ 4\cos(\sqrt{15}t) \end{bmatrix}}_{\mathbf{x}_{\mathrm{Re}}} + i\underbrace{\begin{bmatrix} \sqrt{15}\cos(\sqrt{15}t) - \sin(\sqrt{15}t) \\ \sin(\sqrt{15}t) \end{bmatrix}}_{\mathbf{x}_{\mathrm{Im}}}.
\end{aligned}$$

(Note that our definition of $\mathbf{x}_{\mathrm{Im}}$ does not include the coefficient i.) The general real solution to (1) is the real linear combination of $\mathbf{x}_{\mathrm{Re}}$ and $\mathbf{x}_{\mathrm{Im}}$, i.e.

$$\begin{aligned}
\mathbf{x}(t) &= c_1\mathbf{x}_{\mathrm{Re}}(t) + c_2\mathbf{x}_{\mathrm{Im}}(t) \\
&= c_1\begin{bmatrix} -\cos(\sqrt{15}t) - \sqrt{15}\sin(\sqrt{15}t) \\ 4\cos(\sqrt{15}t) \end{bmatrix} + c_2\begin{bmatrix} \sqrt{15}\cos(\sqrt{15}t) - \sin(\sqrt{15}t) \\ \sin(\sqrt{15}t) \end{bmatrix},
\end{aligned}$$

where $c_1, c_2 \in \mathbf{R}$.

(c) (1 pt) Quickly sketch the phase plane. Your sketch need not be exact; focus on whether trajectories move toward or away from the origin, and the direction of rotation.

Solution: The real part of both eigenvalues $\lambda_\pm$ is 0; thus trajectories are closed ellipses. To determine the direction of rotation, we can use the original ODE (1) to compute the tangent vectors $\mathbf{x}'$ at a few points (x_1, x_2):

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}, \qquad \mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}.$$

This shows that, at $(x_1, x_2) = (1, 1)$, tangent vectors point left ($x_1' = -5 < 0$) and up ($x_2' = 5 > 0$); at $(x_1, x_2) = (-1, 1)$, tangent vectors point left ($x_1' = -3 < 0$) and down ($x_2' = -3 < 0$). Thus trajectories rotate counterclockwise.

A phase plane for the linear system (1) is shown in Figure 1.

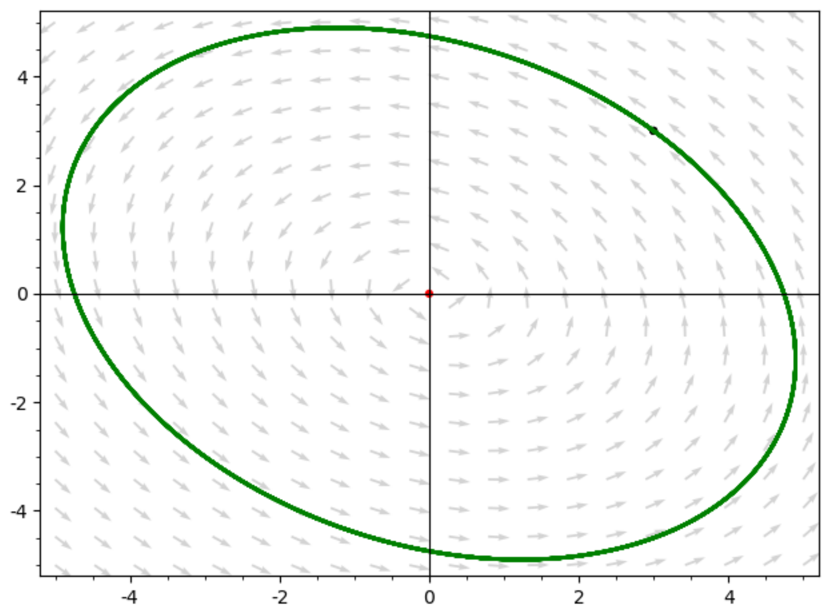

Figure 1: Phase plane, in the (x_1, x_2) plane, for the homogeneous 1st-order linear system (1). Arrows point in the direction of increasing t.