Math 211
Exam 01

W 17 Jul 2019

Your name : ______________________

Start time : ____________ End time : ____________

Honor pledge :

Exam instructions

Number of exercises	: 6
Permitted time	: 90 minutes
Permitted resources	: None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- You are well-trained. Do your best!

Exercise	Total	(a)	(b)	(c)	(d)	(e)	(f)
1	/15	/3	/3	/3	/3	/3	X
2	/15	/2.5	/2.5	/2.5	/2.5	/2.5	/2.5
3	/20	/10	/10	X	X	X	X
4	/24	/10	/10	/4	X	X	X
5	/16	/8	/8	X	X	X	X
6	/10	X	X	X	X	X	X
Total	/100						

Exercise 1

(15 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) Let y_1 and y_2 be solutions to the first-order ODE

$$ty' + e^t y = t. \tag{1}$$

Then any linear combination $a_1 y_1 + a_2 y_2$ is also a solution to (1).

true false

Solution: False. The ODE is nonhomogeneous, so the superposition principle does not apply. In fact, letting $y = a_1 y_1 + a_2 y_2$, we can compute

$$ty' + e^t y = a_1 y_1' + a_2 y_2' + a_1 e^t y_1 + a_2 e^t y_2 = a_1 \left(y_1' + e^t y_1\right) + a_2 \left(y_2' + e^t y_2\right) = (a_1 + a_2)\, t,$$

which equals t as required by the ODE if and only if $a_1 + a_2 = 1$.

(b) The function $y(t) = e^{4t}$ solves the ODE $y'' - 3y' - 4y = 0$.

true false

Solution: True. We can plug the proposed solution into the ODE and check that the equation is true.

(c) The first-order ODE $y' = \frac{t}{t^2+1} y^2 + \cos(ty)$ has a solution for any initial condition $y(t_0) = y_0$.

true false

Solution: True. The function y' (i.e. the right side of the ODE) is continuous at all (t, y), so by Picard's theorem, there exists a solution (possibly several) through any (t_0, y_0).

(d) Let $y_1(t)$ be a solution to the IVP $y' = e^{ty} + t, y(0) = 1$, and let $y_2(t)$ be a solution to the IVP $y' = e^{ty} + 2t, y(0) = 1$. Then for any $t \in \mathbf{R}_{>0}$ (i.e. for any $t > 0$), $y_1(t) < y_2(t)$.

true false

Solution: True. The first term in the two IVPs, namely e^{ty}, is the same; the second term is t and $2t$. For $t > 0$, $t < 2t$. Thus for $t > 0$, for any point (t, y), the value of y' given by the first IVP is strictly less than the value of y' given by the second IVP. That is, starting at $(t_0, y_0) = (0, 1)$, and letting t increase, the slope of the solution y_1 is always less than the slope of y_2, i.e. the solution y_1 grows less quickly than the solution y_2, i.e. for all $t > 0$, $y_1(t) < y_2(t)$.

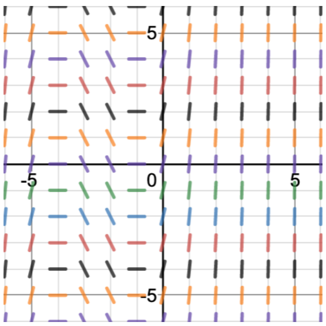
(e) Every ODE has a closed-form solution (e.g., an explicit equation for $y(t)$).

true false

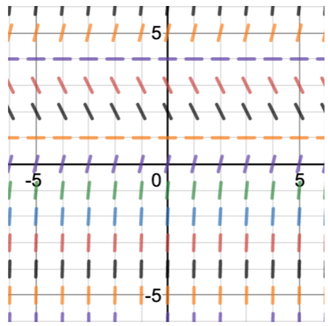
Solution: False.

Exercise 2

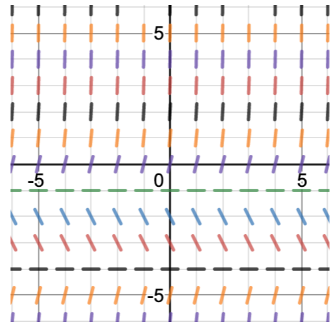
(15 pt) Matching. Write the number of each slope field next to its corresponding ODE.



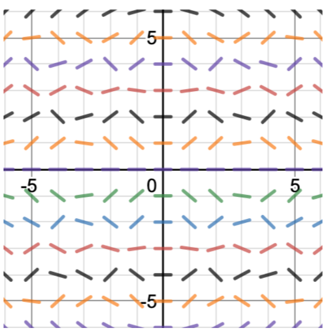
(1)



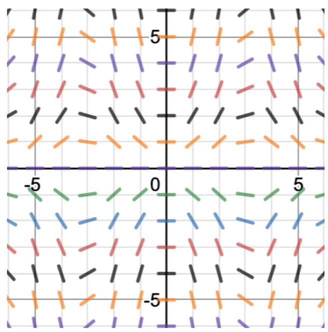
(2)



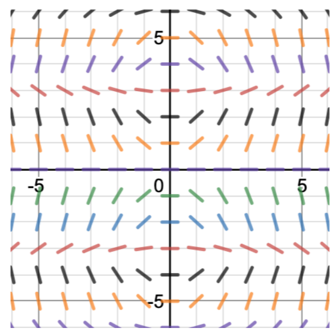
(3)



(4)



(5)



(6)

__(2)__ (a) $\frac{dy}{dt} = y^2 - 5y + 4$

__(3)__ (b) $\frac{dy}{dt} = y^2 + 5y + 4$

__(1)__ (c) $\frac{dy}{dt} = t^2 + 5t + 4$

__(6)__ (d) $\frac{dy}{dt} = t \sin y$

__(5)__ (e) $\frac{dy}{dt} = y \sin t$

__(4)__ (f) $\frac{dy}{dt} = \sin(ty)$

Exercise 3

(20 pt) For each of the following first-order ODEs, find the general solution.

(a) $\dfrac{dy}{dt} = \dfrac{2t+1}{4y^3+4y}$

Solution: This first-order nonhomogeneous nonlinear ODE is separable. Separating variables and integrating, we get

$$\begin{aligned} 4\int \left(y^3+y\right) dy &= \int (2t+1)\, dt \\ y^4+2y^2 &= t^2+t+c_1, \end{aligned}$$

for $c \in \mathbf{R}$. Completing the square on the left, we get

$$y^4+2y^2 = \left(y^4+2y^2+1\right)-1 = \left(y^2+1\right)^2-1.$$

Solving for y, and letting $c = c_1+1$, we conclude that

$$y(t) = \pm\sqrt{-1+\sqrt{t^2+t+c}}.$$

Note that the inner square root originally has a $\pm$ factor in front, but to take the second (i.e. outer) square root, we must choose $+$, so that what we're taking the square root of is nonnegative.

(b) $(t^2+9)y'+ty = 0$

Solution: This first-order homogeneous linear ODE is also separable. Separating variables and integrating, we get

$$\begin{aligned} y' &= -\frac{t}{t^2+9}y \\ \int y^{-1}\, dy &= -\int \frac{t}{t^2+9}\, dt \\ \ln|y| &= -\frac{1}{2}\ln\left|t^2+9\right|+c_1 = \ln\left|t^2+9\right|^{-\frac{1}{2}}+c_1, \end{aligned}$$

where $c_1 \in \mathbf{R}$. Note that $\left|t^2+9\right| > 0$ for all $t \in \mathbf{R}$, so we may drop the absolute value on the right. Exponentiating both sides gives

$$|y| = \left(t^2+9\right)^{-\frac{1}{2}} e^{c_1} = c_2\left(t^2+9\right)^{-\frac{1}{2}},$$

where $c_2 \in \mathbf{R}_{>0}$. This is an equation for $|y|$, so y can be either positive or negative, which we can capture by replacing $c_2 \in \mathbf{R}_{>0}$ with $c_3 \in \mathbf{R}\setminus\{0\}$. However, we can check that $y(t) \equiv 0$, i.e. the zero function, is also a solution to the original ODE, so in fact c_3 can be any real number, call it c. We conclude that the general solution is

$$y(t) = c\left(t^2+9\right)^{-\frac{1}{2}},$$

for any $c \in \mathbf{R}$.

Exercise 4

(24 pt) Consider the following first-order nonhomogeneous linear ODE:

$$y' - 4y = te^{6t}. \tag{2}$$

(a) (10 pt) Find the general solution to the corresponding homogeneous ODE.

Solution: The corresponding homogeneous ODE is

$$y' - 4y = 0.$$

This is ODE is separable:

$$\begin{aligned} y^{-1}\,dy &= 4\,dt \\ \ln|y| &= 4t + c_1 \\ y_h &= c_2 e^{4t}, \end{aligned}$$

where $c_2 \in \mathbf{R}$.

(b) (10 pt) Find the general solution to the nonhomogeneous ODE (2).

Solution: Using variation of parameters, we guess a particular solution of the form

$$y_p = ve^{4t},$$

where $v(t)$ is an unknown function of t. Plugging this into the original ODE (5), we get

$$te^{6t} = y_p' - 4y_p = \left(v'e^{4t} + 4ve^{4t}\right) - 4\left(ve^{4t}\right) = v'e^{4t}.$$

Solving for v', we get

$$v' = e^{-4t}\left(te^{6t}\right) = te^{2t}.$$

Integrating both sides with respect to t — using integration by parts on the right, with

$$u = t, \qquad dv = e^{2t}\,dt,$$

and thus

$$du = dt, \qquad v = \frac{1}{2}e^{2t}$$

— we find

$$v = \int te^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{2}\int e^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + c_0,$$

where $c_0 \in \mathbf{R}$. Because we're looking for a (i.e. any) particular solution, we may choose c_0 to be any value we like. Let's make our lives easy and choose $c_0 = 0$. Then

$$y_p = ve^{4t} = \frac{1}{2}te^{6t} - \frac{1}{4}e^{6t} = \frac{1}{4}(2t-1)\,e^{6t}.$$

By the nonhomogeneous principle, the general solution y to the ODE (5) is

$$y(t) = y_p + y_h = \frac{1}{4}(2t-1)\,e^{6t} + c_2 e^{4t}. \tag{3}$$

(c) (4 pt) Find the particular solution of the solution family you gave in part (b) that satisfies the initial condition $y(0) = 1$.

Solution: Applying the initial condition to our general solution (3), we find

$$1 \underset{\text{set}}{=} y(0) = \frac{1}{4}(0-1) + c_2 \qquad \Leftrightarrow \qquad c_2 = \frac{5}{4}.$$

Thus the particular solution to the IVP with ODE (2) and initial condition $y(0) = 1$ is

$$y(t) = \frac{1}{4}(2t-1)\,e^{6t} + \frac{5}{4}e^{4t}.$$

Exercise 5

(16 pt) Consider the one-parameter family of first-order nonlinear ODEs

$$\frac{dy}{dt} = y^3 + \alpha y, \tag{4}$$

where the parameter α is allowed to take any value in $\mathbf{R}$.

(a) (8 pt) Show that for $\alpha \geqslant 0$, the ODE (4) has a unique equilibrium, and that it is unstable.

Solution: Recall that an equilibrium solution is a solution $y(t)$ that does not change with t; i.e. the function y is constant; i.e. $y' = 0$. For a first-order separable ODE $y' = f(t)g(y)$, equilibrium solutions can be found by setting $g(y) = 0$ and solving for y.

The right side of the ODE (4) factors as

$$\frac{dy}{dt} = y(y^2 + \alpha).$$

We see that the constant function $y = 0$ is always an equilibrium solution. The second factor $y^2 + \alpha$ has...

- ...no real roots when $\alpha > 0$.
- ...a repeated root of algebraic multiplicity 2 at $y = 0$ when $\alpha = 0$.
- ...two distinct real roots of algebraic multiplicity 1 at $y = \pm\sqrt{-\alpha}$ when $\alpha < 0$.

Thus when $\alpha \geqslant 0$, the ODE (4) has a unique equilibrium solution, $y \equiv 0$. Moreover, in this case, if $y \neq 0$, then $y^2 + \alpha > 0$, so the sign of y' matches the sign of y. That is, $y' < 0$ if $y < 0$, and $y' > 0$ if $y > 0$. Thus in this case, $y \equiv 0$ is an unstable equilibrium.

(b) (8 pt) Show that for $\alpha < 0$, the ODE (4) has three equilibria, at 0 and $\pm\sqrt{-\alpha}$. Classify the stability of each equilibrium.

Solution: In part (a) we found that when $\alpha < 0$, the ODE (4) has three distinct equilibrium solutions, $y \equiv 0$ and $y \equiv \pm\sqrt{-\alpha}$. By analyzing the sign of each factor (e.g., sign lines), or by evaluating y' at intermediate points, we find that

- $y \equiv -\sqrt{-\alpha}$ is unstable.
- $y \equiv 0$ is stable.
- $y \equiv \sqrt{-\alpha}$ is unstable.

In particular, notice that when α passes from negative to positive, the stability of the equilibrium solution $y \equiv 0$ changes.

Exercise 6

(10 pt) In Class 5 (Friday 12 July) we argued that the first-order nonhomogeneous linear ODE

$$y' - y = t \tag{5}$$

has the general solution

$$y(t) = -t - 1 + ce^t, \tag{6}$$

where $c \in \mathbf{R}$. The question was posed: How do we know (6) captures *all* the solutions to (5)?

We gave one argument:

> The solutions to the corresponding homogeneous ODE form a one-dimensional vector space over $\mathbf{R}$ (as we will see), and the nonhomogeneous principle guarantees that any solution to (5) has the form $y = y_p + y_h$.

Give another argument, using Picard's theorem about existence and uniqueness of solutions. *Hint:* Consider any point (t_0, y_0). Is there a solution to (5) of the form (6) that passes through (t_0, y_0)? It may help to sketch some solutions ce^t to the corresponding homogeneous ODE.

Solution: Allowing c to be any real number, we see that the solution curves $y_h(t) = ce^t$ to the corresponding homogeneous ODE fill the (t, y) plane. That is, through any point (t, y), we can find such a curve. Graphically, adding the particular solution $y_p(t) = -t - 1$ of the nonhomogeneous ODE (5) to these solutions to the homogeneous ODE corresponds to distorting this family of curves so that the line $-t-1$ plays the role of $y = 0$. The resulting general solutions $y(t) = y_p + y_h$, given by (6), still fill the plane.

We can check that the ODE (5) satisfies the hypotheses of both the existence and uniqueness part of Picard's theorem:

- $y' = t + y$ is continuous everywhere, guaranteeing existence of a solution given any initial condition.
- $\frac{\partial}{\partial y} y' = 1$ is continuous everywhere, guaranteeing uniqueness of these solutions.

The conclusion of our qualitative, graphical analysis above illustrates the existence part of Picard's theorem: Through any point (t, y), we could find the graph of a curve given by (6). The uniqueness part of Picard's theorem says there can be no other solutions. Thus, the family of general solutions (6) indeed captures all solutions to (5).