Math 212
HalfExam 02

N 30 Oct 2016

Your name: ______________________________

Exam instructions

Number of exercises	: 11
Permitted time	: 120 minutes
Permitted resources	: None

Instructor's note:

- Hints to each exercise are provided on the final page. Work through the entire exam prior to looking at the hints, then use the hints to help you with the challenging exercises.
- Manage your time deliberately.
- An effort has been made to make the exercises as clear and unambiguous as possible. If the statement of an exercise seems unclear, briefly (one sentence) write your understanding of the exercise, then proceed.

Exercise	Total
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
11	/10
Total	/110

Exercise 1

Let $D = \{(x, y, z) \in \mathbf{R}^3 \,|\, z \neq 0\}$, and let $f : D \to \mathbf{R}$ be given by

$$f(x, y, z) = \frac{x + y}{z}.$$

Find the maximum rate of change of f at the point $(1, 1, -1)$ and the direction in which it occurs.

Exercise 2

The ideal gas law states that

$$PV = nRT, \tag{1}$$

where P is pressure (in kPa), V is volume (in L), n is number of moles, $R = 8.3144$ J/(mol K), and T is temperature (in K). (You may ignore units; they are chosen so that they work out.[1])

The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s, and its temperature is increasing at a rate of 0.15K/s. Find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

[1] For those of us who mind our units, note that

$$1 \text{ J} = 1 \text{ Pa m}^3 = 1000 \text{ Pa L} = 1 \text{ kPa L}.$$

Exercise 3

Let $D \subseteq \mathbf{R}^2$ be the closed disc

$$D = \left\{(x, y) \in \mathbf{R}^2 \,|\, x^2 + y^2 \leqslant 2\right\},$$

and let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function given by

$$f(x, y) = x^2 + y^2 - x - y + 1.$$

Does f achieve a global minimum and maximum on D? Explain. If yes, find the corresponding values.

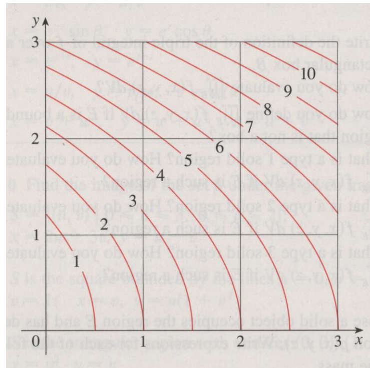

Figure 1: Level sets for the function $f : R \to \mathbf{R}$ in Exercise 4.

Exercise 4

Let $R = [0, 3] \times [0, 3] \subseteq \mathbf{R}^2$, and let $f : R \to \mathbf{R}$ have the level sets shown in Figure 1. Use a Riemann sum with nine terms to estimate the value of the double integral

$$\iint_R f(x, y)\, dA.$$

Take the sample points of each subregion to be the upper-right corner of the subregion.

Exercise 5

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuous function. Rewrite the following sum of iterated integrals as a single iterated integral, by changing the order of integration:

$$\int_{-4}^{0} \int_{0}^{4+x} f(x,y)\,dy\,dx + \int_{0}^{4} \int_{0}^{4-x} f(x,y)\,dy\,dx.$$

Exercise 6

Calculate the value of the iterated integral

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\, dx\, dy.$$

Exercise 7

Let $T \subseteq \mathbf{R}^3$ be the solid tetrahedron with vertices

$$P_0 = (0,0,0), \qquad P_1 = \left(\frac{1}{3}, 0, 0\right), \qquad P_2 = (0,1,0), \qquad P_3 = (0,0,1).$$

Find the value of the triple integral

$$\iiint_T xy \, dV.$$

Exercise 8

Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$.

Exercise 9

Evaluate the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy.$$

Exercise 10

Let X, Y be random variables with probability density function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x, y) = \begin{cases} C(x+y) & \text{if } 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant C.

(b) Find $\text{Prob}(X \leqslant 2, Y \geqslant 1)$.

(c) Find $\text{Prob}(X + Y \leqslant 1)$.

Exercise 11

Let $R \subseteq \mathbf{R}^2$ be the square with vertices

$$(0,2), \qquad (1,1), \qquad (2,2), \qquad (1,3).$$

Evaluate the double integral

$$\iint_R \frac{x-y}{x+y}\,dA.$$

Hints

The following numbers correspond to the exercise numbers above.

1. Consider ∇f.
2. This exercise involves rates of change, i.e. derivatives, each with respect to time. View each variable as a function of time, and use the chain rule to relate the various rates of change.
3. Appeal to the EVT. Parametrize the boundary of D using polar coordinates.
4. Use the definition of a Riemann sum.
5. Sketch the region(s) of integration.
6. Justify why you can change the order of integration.
7. Sketch the region of integration.
8. Choose a convenient coordinate system, based on the region of integration.
9. Choose a convenient coordinate system, based on the region of integration.
10. Use the definition of a probability density function.
11. Apply a change of variables. It may be easier to first write the new variables u, v as functions of the given variables x, y, then invert to find x, y as functions of u, v. Remember the Jacobian determinant.