Math 211
Exam 04

R 08 Aug 2019

Your name : ____________________

Start time : ____________ End time : ____________

Honor pledge :

Exam instructions

Number of exercises : 9
Permitted time : 3 hours
Permitted resources : None

Remarks:

- Manage your time deliberately.
- If the statement of an exercise is unclear to you, briefly (one sentence) write your understanding of the exercise, then proceed.
- You have worked hard and learned a lot these past five weeks. Do your best, finish strong, and have fun!

Notation:

- $\mathbf{R}$ denotes the real numbers.
- $\mathbf{C}$ denotes the complex numbers.
- For a complex number $z \in \mathbf{C}$, $\overline{z}$ denotes the complex conjugate of z.

Exercise	Total	(a)	(b)	(c)	(d)	(e)	(f)
1	/10						X
2	/6						
3	/8	/2	/3	/3	X	X	X
4	/8	/1	/2	/2	/1	/2	X
5	/17	/4	/2	/2	/9	X	X
6	/20	/2	/2	/5	/1	/4	/6
7	/14	/4	/2	/2	/4	/2	X
8	/10	X	X	X	X	X	X
9	/7	X	X	X	X	X	X
Total	/100						

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary (though brief justification may help check your intuition).

(a) (1 pt) Let $y_1(t)$ be a solution to the nonhomogeneous 3rd-order linear ODE

$$y^{(3)} - (\sin t)y'' + 3y = t,$$

and let $y_2(t)$ be a solution to the nonhomogeneous 3rd-order linear ODE

$$y^{(3)} - (\sin t)y'' + 3y = 2t.$$

Then $2y_1 - y_2$ is a solution to the homogeneous 3rd-order linear ODE

$$y^{(3)} - (\sin t)y'' + 3y = 0.$$

true false

(b) (1 pt) Let $\mathbf{A}$ be an $n \times m$ matrix. For any $\mathbf{b} \in \mathbf{R}^n$, the set of solutions to the matrix equation $\mathbf{Ax} = \mathbf{b}$ is a vector space over $\mathbf{R}$.

true false

(c) (1 pt) Let $f_1(t) = t^2$ and $f_2(t) = t\,|t|$ be two functions $\mathbf{R} \to \mathbf{R}$. Their wronskian $W[f_1, f_2](t)$ is the zero function (i.e. identically 0). *Hint:* The derivative $f_2'(t)$ is defined. Think graphically.

true false

(d) (1 pt) $f_1(t) = t^2$ and $f_2(t) = t\,|t|$ are linearly dependent. *Hint:* Think graphically.

true false

(e) (1 pt) Let $\mathbf{A}$ be a 2×2 matrix with entries in $\mathbf{R}$, and let $\lambda_+, \lambda_- \in \mathbf{C}$ be complex, nonreal eigenvalues of $\mathbf{A}$ that are complex conjugates. If $\mathbf{v} \in \mathbf{C}^2$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue λ_+, then the complex conjugate vector $\overline{\mathbf{v}}$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue λ_-.

true false

(f) (1 pt) For a 1st-order system of ODEs that satisfies the (existence and) uniqueness statement of Picard's theorem, trajectories in the phase plane cannot cross. *Hint:* What information do these trajectories capture, and how?

true false

(g) (1 pt) There exist $n \times n$ matrices $\mathbf{A}$ such that the n columns of the matrix exponential function $e^{\mathbf{A}t}$ are linearly dependent.

true false

(h) (1 pt) Let $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where all matrices are $n \times n$, and $\mathbf{P}$ is invertible. Then for every nonnegative integer j, $\mathbf{A}^j = \mathbf{P}\mathbf{D}^j\mathbf{P}^{-1}$.

true false

(i) (1 pt) Every $n \times n$ matrix with entries in $\mathbf{R}$ can be diagonalized.

true false

(j) (1 pt) Every ODE can be solved, i.e. we can always find a closed-form solution (e.g., an explicit equation for $y(t)$).

true false

Exercise 2

(6 pt) Match each of the homogeneous 1st-order 2×2 linear systems of ODEs with its corresponding phase plane in Figures 1 and 2 (on the last page of this exam). N.B. In each ODE, $\mathbf{x} = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ is a 2×1 matrix of scalar-valued functions.

(a) $\mathbf{x}' = \frac{1}{4}\begin{bmatrix} 10 & -1 \\ -4 & 10 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\mathbf{x}$

(b) $\mathbf{x}' = \frac{1}{4}\begin{bmatrix} -10 & 1 \\ 4 & -10 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\mathbf{x}$

(c) $\mathbf{x}' = \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\mathbf{x}$

(d) $\mathbf{x}' = \begin{bmatrix} -1 & -\frac{1}{2} \\ -2 & -1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\mathbf{x}$

(e) $\mathbf{x}' = \begin{bmatrix} 2 & -2 \\ -8 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\mathbf{x}$

(f) $\mathbf{x}' = \begin{bmatrix} 2 & 2 \\ 8 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\mathbf{x}$

(g) $\mathbf{x}' = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}^{-1}$

(h) $\mathbf{x}' = \frac{1}{8}\begin{bmatrix} -2 & 65 \\ -4 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1-8i & 1+8i \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}\begin{bmatrix} 1-8i & 1+8i \\ 2 & 2 \end{bmatrix}^{-1}$

(i) $\mathbf{x}' = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}^{-1}$

(j) $\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}^{-1}$

(k) $\mathbf{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x}$

(l) $\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\mathbf{x}$

Exercise 3

(8 pt) Consider the homogeneous 1st-order nonlinear ODE

$$e^{t}y' - (1 - e^{2t})y^{\frac{1}{3}} = 0. \tag{1}$$

(a) (2 pt) Show that the ODE (1) has exactly one equilibrium solution. Justify completely.

(b) (3 pt) Find all solutions to the initial value problem given by the ODE (1) and the initial condition $y(0) = 0$. Justify briefly why this is all solutions.

(c) (3 pt) Briefly explain how the equilibrium in part (a) relates to our work in part (b). How do the above results relate to the existence and uniqueness statements of Picard's theorem?

Exercise 4

(8 pt) Consider the linear map T given by

$$T : \mathbf{R}^4 \to \mathbf{R}^3$$
$$\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 & + & 2x_2 & - & x_3 & + & 4x_4 \\ 2x_1 & + & 4x_2 & - & 2x_3 & + & 3x_4 \\ -x_1 & - & x_2 & + & 2x_3 & + & 4x_4 \end{bmatrix}.$$

(a) (1 pt) Write the matrix **A** corresponding to the linear map T. *Hint:* Double-check your matrix before you continue!

(b) (2 pt) Apply the row reduction algorithm to find the reduced row echelon form of **A**.

(c) (2 pt) State a basis for the image $\mathrm{im}(T)$ and a basis for the kernel $\ker(T)$.

(d) (1 pt) Confirm the rank–nullity theorem holds for the linear map T.

(e) (2 pt) For part (c), your friend writes that

$$\mathrm{basis}(\mathrm{im}(T)) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

and gets her problem marked wrong. Argue why your friend deserves full points.

Exercise 5

(17 pt) Consider the following 1st-order nonlinear system of ODEs:

$$x_1' = 2x_1 - x_1^2 - x_1 x_2$$
$$x_2' = x_2^2 - x_1 x_2.$$

(a) (4 pt) State the x_1- and x_2-nullclines, and show that each consists of two lines. Plot them in the (x_1, x_2)-plane.

(b) (2 pt) Find the three equilibrium points of this system. State them explicitly.

(c) (2 pt) Write the jacobian matrix for this system, at the general point (x_1, x_2).

(d) (9 pt) For each equilibrium point, write the corresponding linearized system, and classify its stability.

Exercise 6

(20 pt) Consider the following homogeneous 2nd-order linear ODE IVP:

$$y'' - 3y' + 2y = 0, \qquad y(0) = 2, \qquad y'(0) = 7. \tag{2}$$

(a) (2 pt) Translate this 2nd-order linear ODE into a 1st-order linear system of ODEs. Also translate the initial conditions into an initial condition matrix (of order 2×1).

(b) (2 pt) Write the characteristic polynomial $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$ associated to the coefficient matrix $\mathbf{A}$ of our linear system from part (a). How does this characteristic polynomial relate to the original ODE (2)?

(c) (5 pt) Solve the 1st-order linear system IVP from part (a). *Hint:* Remember to apply the initial conditions!

(d) (1 pt) State the solution to the original 2nd-order ODE IVP (2). *Hint:* If you have done part (c), then you do not need to do any math in this step.

(e) (4 pt) Let y be a "suitable" function (i.e. a function of exponential order α, for some $\alpha \in \mathbf{R}$). Using the definition of the laplace transform,

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\, dt,$$

show that

$$\mathcal{L}\left\{y'\right\} = s\mathcal{L}\{y\} - y(0) \qquad \text{and} \qquad \mathcal{L}\left\{y''\right\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0).$$

Hint: After computing $\mathcal{L}\{y'\}$, use recursion to deduce $\mathcal{L}\{y''\}$.

(f) (6 pt) Use the laplace transform to solve the IVP (2), thus verifying your solution in part (d). Part (e), and the following transform–inverse-transform pairs, may be useful:

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \qquad s > 0; \qquad\qquad \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \qquad s > a;$$

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}, \qquad s > a;$$

$$\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2+b^2}, \quad s > a; \qquad \mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2}, \quad s > a.$$

Exercise 7

(14 pt) Consider the homogeneous 1st-order 2×2 linear system of ODEs

$$\mathbf{x}' = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \mathbf{x}, \tag{3}$$

where $\mathbf{x} = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$. Let $\mathbf{A}$ denote the 2×2 coefficient matrix.

(a) (4 pt) Find the eigenvalues of $\mathbf{A}$, and specify their corresponding eigenspaces.

(b) (2 pt) Write the general solution to the ODE (3).

(c) (2 pt) Use our results from part (a) to diagonalize the matrix $\mathbf{A}$, i.e. to write it in the form

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

where $\mathbf{D}$ is a diagonal matrix. Specify $\mathbf{P}$ explicitly. (You may write $\mathbf{P}^{-1}$ as the 2×2 matrix $\mathbf{P}$ followed by the superscript -1 inverse symbol.)

(d) (4 pt) Compute the matrix exponential function $e^{t\mathbf{A}}$. Leave your answer in the form of a product of three matrices, the middle matrix a function of t. *Hint:* Recall that, by definition,

$$e^{t\mathbf{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n.$$

Use part (c) to make this computable.

(e) (2 pt) Let $\mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$ be a 2×1 matrix of constants $c_1, c_2 \in \mathbf{R}$. Using our result from part (d), compute the product

$$e^{t\mathbf{A}}\mathbf{P}\mathbf{c}.$$

How does this compare with our answer in part (b)? Comment briefly. *Hint:* Note that for any 2×1 matrix $\mathbf{c} \in \mathbf{R}^2$, $\mathbf{P}\mathbf{c}$ is again a 2×1 matrix in $\mathbf{R}^2$. Also note that, by construction, $\mathbf{P}$ is invertible.

Exercise 8

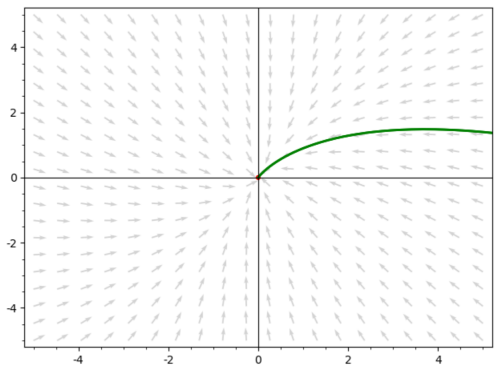
(10 pt) Find the general real solution to the homogeneous 1st-order linear system of ODEs

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}.$$

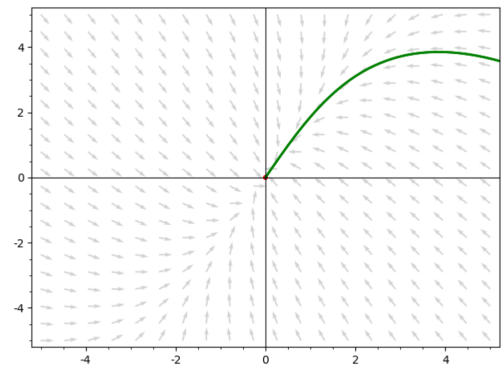
Exercise 9

(7 pt) Find the general solution to the nonhomogeneous 1st-order 2×2 linear system of ODEs

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -6e^{3t} \\ 8e^{3t} \end{bmatrix}.$$



(1)



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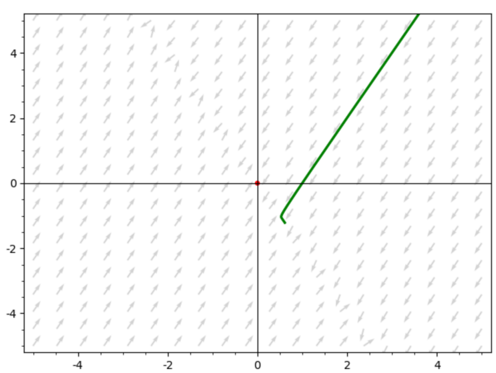

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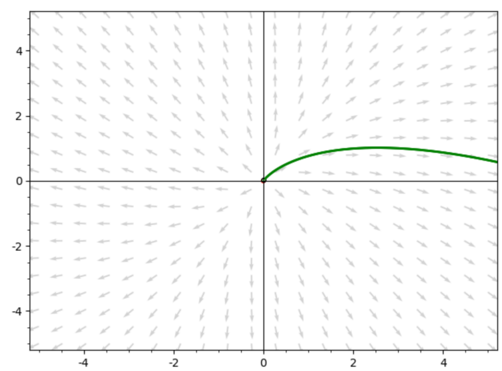

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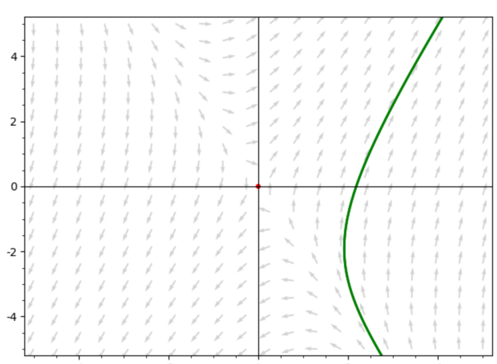

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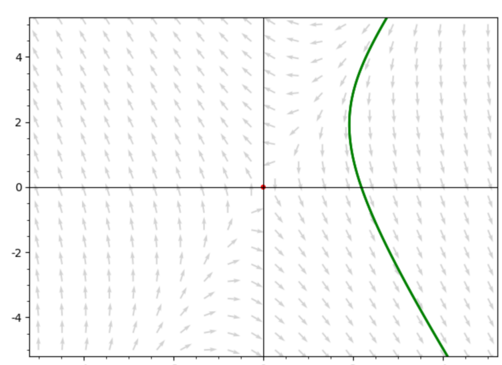
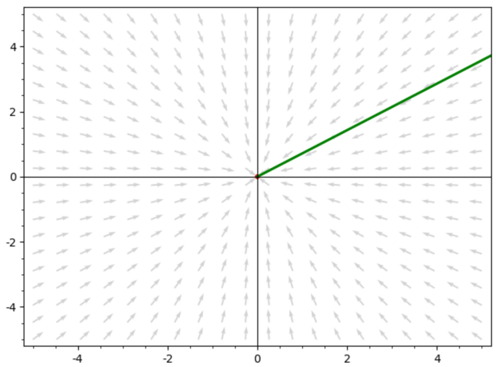


(6)

Figure 1: Phase planes for Exercise 2, in the (x_1, x_2) plane.



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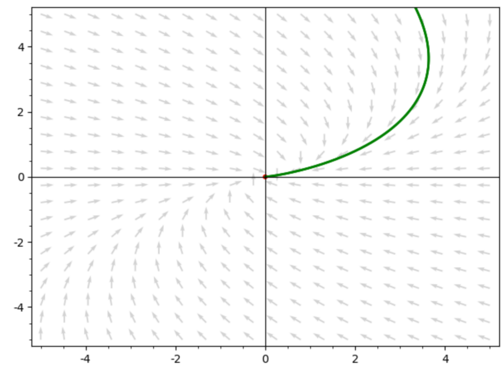

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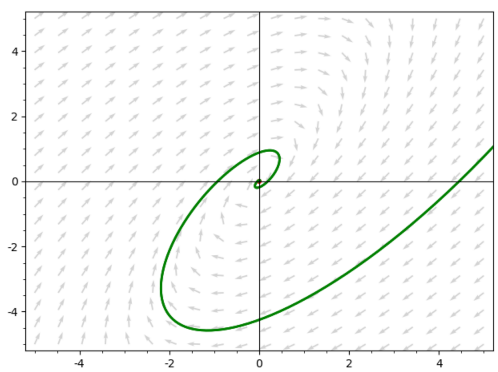

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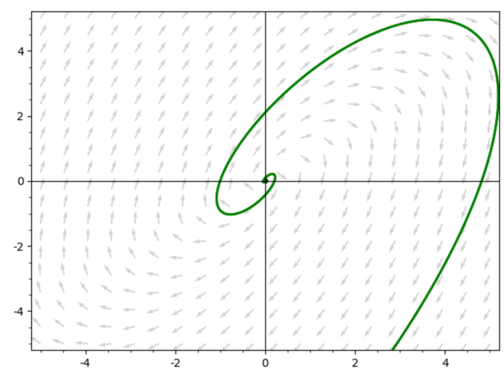

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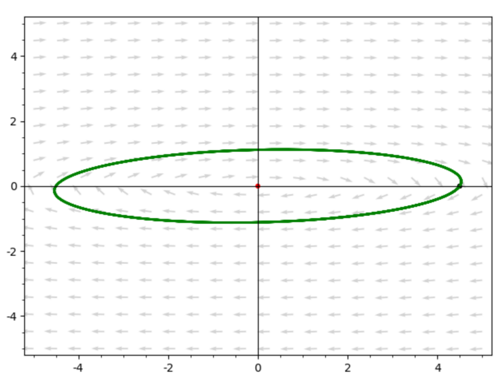

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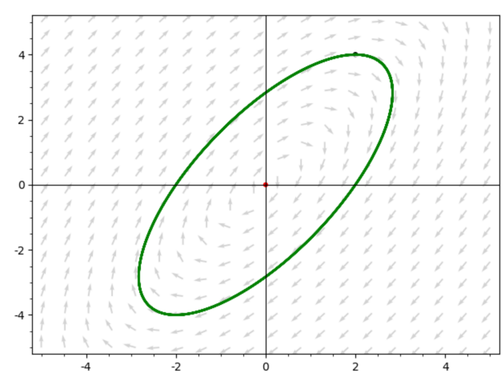

(12)

Figure 2: Phase planes for Exercise 2, in the (x_1, x_2) plane.