Math 211
Quiz 15

F 26 Jul 2019

Your name : ______________________________

Exercise

(5 pt) Solve the initial value problem given by the homogeneous 2nd-order ODE

$$y'' + 6y' + 8y = 0 \tag{1}$$

and the initial conditions

$$y(0) = 3, \qquad\qquad y'(0) = -10.$$

Your final answer should be an explicit equation for $y(t)$.[1]

Solution: Applying the change of variables

$$x_0 = y, \qquad\qquad x_1 = y',$$

we get the corresponding homogeneous 1st-order linear system

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

The characteristic polynomial of the (constant) coefficient matrix, call it A, is

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -8 & -6-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 8.$$

Note that this is the same characteristic polynomial we obtain by replacing each derivative $y^{(n)}$ with λ^n in the given ODE (1). The eigenvalues are the zeros of this characteristic polynomial:

$$0 \underset{\text{set}}{=} p(\lambda) = (\lambda + 4)(\lambda + 2) \qquad \Leftrightarrow \qquad \lambda = -4, -2.$$

Because all the eigenvalues are distinct, the general solution y to (1) is obtained by putting these eigenvalues as coefficients inside exponentials, and taking the linear combination:

$$y(t) = c_1 e^{-4t} + c_2 e^{-2t}.$$

To find the solution to the IVP, we compute y',

$$y'(t) = -4c_1 e^{-4t} - 2c_2 e^{-2t},$$

and apply the initial conditions (the degree of the ODE (1) is 2, so 2 initial conditions are required):

$$\begin{aligned} 3 &= y(0) = c_1 + c_2 \\ -10 &= y'(0) = -4c_1 - 2c_2. \end{aligned}$$

[1] *Hint:* We have two approaches to find the general solution $y(t)$ to (1): (1) translate the given higher-order linear ODE into the corresponding 1st-order linear system (what do the initial conditions become in this case?), and (2) jump directly to the characteristic equation. The general solution involves two parameters, call them c_1, c_2. Compute $y(t)$, take its derivative to get $y'(t)$ (which will also involve c_1, c_2), then apply the initial conditions to get a system of two linear equations in the two unknowns c_1, c_2. Now solve this system, e.g., using our techniques from linear algebra.

This is a linear system of equations in the unknowns c_1, c_2, which we can solve in several ways. Here we'll form the augmented matrix associated to the system and row reduce:

$$\left[\begin{array}{cc:c} 1 & 1 & 3 \\ -4 & -2 & -10 \end{array}\right] \xrightarrow{R_2 = R_2 + 4R_1} \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & 2 & 2 \end{array}\right] \xrightarrow{R_2 = \frac{1}{2}R_2} \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right].$$

Reading of the solution, $c_1 = 2$ and $c_2 = 1$. Thus the solution to the IVP is

$$y(t) = 2e^{-4t} + e^{-2t}.$$

One can check that this $y(t)$ indeed satisfies both the ODE and the initial conditions, as required.