Math 357
Long quiz 03

2024–01–31 (W)

Your name: ______________________________

Let R be a commutative ring, and let t be an indeterminate. Consider the polynomial ring $R[t]$.

(a) Define the degree function, deg, on $R[t]$.

Solution: Given any nonzero $f \in R[t]$, we may write f in the form

$$f = a_m t^m + \ldots + a_0 \tag{1}$$

where $m \in \mathbf{Z}_{\geqslant 0}$, each $a_i \in R$, and $a_m \neq 0$. Define

$$\begin{aligned} \deg : R[t] &\to \mathbf{Z}_{\geqslant 0} \cup \{-\infty\} \\ f &\mapsto \begin{cases} m & \text{if } f \neq 0, f \text{ as in (1)} \\ -\infty & \text{if } f = 0 \end{cases} \end{aligned}$$

Some mathematicians prefer to define $\deg : R[t] - \{0\} \to \mathbf{Z}_{\geqslant 0}$, leaving the degree of the zero polynomial undefined.

(b) Let $p, q \in R[t]$. Prove that if R is an integral domain, then

$$\deg pq = \deg p + \deg q$$

Give an example to show that this equation can fail if R is not an integral domain.

Solution: For the (counter)example, consider the ring $\mathbf{Z}/(4)$ (also written $\mathbf{Z}/4\mathbf{Z}$), which is not an integral domain, and the polynomials $p = q = 2t + 1$ in $(\mathbf{Z}/(4))[t]$. Then

$$pq = (2t+1)(2t+1) = 4t^2 + 4t + 1 \equiv 0t^2 + 0t + 1 = 1$$

so

$$\deg pq = 0 \neq 2 = 1 + 1 = \deg p + \deg q$$

Now suppose R is an integral domain. Case 1: Either p or q is the zero polynomial. In this case, $pq = 0$. With the convention that for all $n \in \mathbf{Z} \cup \{-\infty\}$, $-\infty + n = -\infty = n + -\infty$, we get

$$\deg pq = -\infty = \deg p + \deg q$$

Case 2: Neither p nor q is the zero polynomial. Denote their leading terms by

$$\mathrm{LT}(p) = a_m t^m \qquad\qquad \mathrm{LT}(q) = b_n t^n$$

By hypothesis, $p \neq 0$ and $q \neq 0$; hence $a_m \neq 0$ and $b_n \neq 0$, and $\deg p = m$ and $\deg q = n$. Because R is an integral domain, it follows that $a_m b_n \neq 0$. Hence, by definition of multiplication in $R[t]$,

$$\mathrm{LT}(pq) = a_m b_n t^{m+n}$$

so

$$\deg pq = m + n = \deg p + \deg q$$

as desired.