Math 212
Quiz 34

W 30 Nov 2016

Your name: ____________________

Exercise

(5 pt) For each of the following integrals, name a theorem that allows you to write an equivalent expression (e.g., integral), write it, and evaluate it. *Guiding light:* Integrating a derivative over a region is related to evaluating the original function on the boundary of that region.

(a) (2.5 pt) Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ be given by

$$\mathbf{F}(x, y, z) = (xy, 2z, 3y)\,,$$

and let $C \subseteq \mathbf{R}^3$ be the curve of intersection of the plane $x+z=5$ and the cylinder $x^2+y^2=9$, oriented counterclockwise when viewed from above. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 9\pi$.

Solution: Stokes's theorem: C is a (piecewise-) smooth simple closed curve, and the component functions of $\mathbf{F}$ are continuously differentiable, so taking S to be any piecewise-smooth oriented surface with boundary C permits us to apply Stokes's theorem, rewriting the given line integral as a surface integral. A convenient choice of surface S is the region of the plane $x+z=5$ bounded by C, equipped with upward-pointing unit normal vectors (to agree with the given orientation on C). Let $D \subseteq \mathbf{R}^2$ be the disc in the xy-plane bounded by the given cylinder $x^2+y^2=9$:

$$D = \left\{(x, y) \in \mathbf{R}^2 \,|\, x^2 + y^2 \leqslant 9\right\} = \{(r, \theta) \,|\, r \in [0, 3], \theta \in [0, 2\pi]\}\,.$$

Then a parametrization of this surface S is given by

$$\begin{aligned} \mathbf{r} : D &\to \mathbf{R}^3 \\ (x, y) &\mapsto (x, y, 5 - x)\,. \end{aligned}$$

Note that

$$\mathbf{r}_x = (1, 0, -1)\,, \qquad \mathbf{r}_y = (0, 1, 0)\,,$$

so the induced normal vector at each point of S is

$$\mathbf{r}_x \times \mathbf{r}_y = (1, 0, 1)\,,$$

upward-pointing, as required. Stokes's theorem gives

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA \\
&= \iint_D (1, 0, -x) \cdot (1, 0, 1)\, dA = \iint_D (1 - x)\, dA \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} (1 - r\cos\theta) r\, dr\, d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \left[\frac{1}{2} r^2 - \frac{1}{3} r^3 \cos\theta\right]_{r=0}^{r=3} d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \left(\frac{9}{2} - 9\cos\theta\right) d\theta \\
&= \left[\frac{9}{2}\theta - 9\sin\theta\right]_{\theta=0}^{\theta=2\pi} \\
&= 9\pi.
\end{aligned}$$

(b) (2.5 pt) Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ be given by

$$\mathbf{F}(x, y, z) = \left(x^4, -x^3 z^2, 4xy^2 z\right),$$

and let $S \subseteq \mathbf{R}^3$ be the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$, where S is equipped with its outward-pointing unit normal vectors. Show that $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{3}$.

Solution: Gauss's (divergence) theorem: Let $E \subseteq \mathbf{R}^3$ denote the solid bounded by S. Note that E is a simple region, S has positive (i.e. outward-pointing) orientation (by hypothesis), and the component functions of $\mathbf{F}$ are continuously differentiable. Thus Gauss's theorem applies, and we can rewrite the given surface integral as a triple integral. Using cylindrical coordinates to compute the resulting triple integral, we find

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV &= \iiint_E \left(4x^3 + 0 + 4xy^2\right) dV = 4 \iiint_E x\left(x^2 + y^2\right) dV \\
&= 4 \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=2+r\cos\theta} (r\cos\theta) r^2 \, r \, dz \, dr \, d\theta \\
&= 4 \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r^4 \cos\theta\right) \int_{z=0}^{z=2+r\cos\theta} dz \, dr \, d\theta \\
&= 4 \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(2r^4 \cos\theta + r^5 \cos^2\theta\right) dr \, d\theta \\
&= 4 \int_{\theta=0}^{\theta=2\pi} \left[\frac{2}{5} r^5 \cos\theta + \frac{1}{6} r^6 \cos^2\theta\right]_{r=0}^{r=1} d\theta \\
&= 4 \int_{\theta=0}^{\theta=2\pi} \left(\frac{2}{5}\cos\theta + \frac{1}{12}\left(1 + \cos(2\theta)\right)\right) d\theta \\
&= 4 \left[\frac{2}{5}\sin\theta + \frac{1}{12}\theta + \frac{1}{24}\sin(2\theta)\right]_{\theta=0}^{\theta=2\pi} \\
&= \frac{2\pi}{3}.
\end{aligned}$$