Math 112

Mock Exam 2

Grader's Example

Exercise 1

(10 pt) True/False. For each of the following statements, circle whether it is true or false. No justification is necessary.

(a) (2 pt) If direct evaluation of a limit gives an indeterminate form, then we can always apply l'Hôpital's rule, even though other methods may be faster.

true false

can't use l'Hôpital's rule for $\infty - \infty$

(b) (2 pt) Let $f(x)$ be a function, and let $F(x)$ be an antiderivative of $f(x)$. Then $(F(x))^2$ is an antiderivative of $(f(x))^2$.

true false

let $f(x)=1$ $F(x)=x$ $\frac{d}{dx}(F(x))^2 = \frac{d}{dx}x^2 = 2x \neq 1 = (f(x))^2$

(c) (2 pt) Let $f(x)$ be a function, and let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then the function $F(x) - G(x)$ is always a constant function.

true false

F(x) and G(x) should only differ by a constant

For parts (d)–(e), let f and g be functions such that

$$\int_{-1}^{3} f(x)\,dx = -2 \qquad \int_{-1}^{3} g(x)\,dx = 4$$

(d) (2 pt) $\int_{-1}^{3} [f(x) + g(x)]\,dx = 2$

true false

$$\int_{-1}^{3} [f(x) + g(x)]\,dx = \int_{-1}^{3} f(x)\,dx + \int_{-1}^{3} g(x)\,dx$$

(e) (2 pt) $\int_{-1}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx = -2$

true false

$$\int_{-1}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx = \int_{-1}^{3} f(x)\,dx$$

Exercise 2

(20 pt) Evaluate each of the following limits. Briefly but clearly justify your work.

(a) (4 pt) $\lim_{x\to 0} \frac{x+\cos x}{-1+\sin x} = \frac{1}{-1} = \boxed{-1}$

direct evaluation

(b) (4 pt) $\lim_{x\to -\infty} \frac{6x^3 - x^2 + 5x + 5}{2x^3 + 2x}$

$$\begin{array}{r} 3 \\ 2x^3 + 2x \overline{\big) 6x^3 - x^2 + 5x + 5} \\ -6x^3 \quad\quad - 6x \\ \hline -x^2 - x + 5 \end{array}$$

$= \lim_{x\to -\infty} 3 + \frac{-x^2 - x + 5}{2x^3 + 2x}$

denominator goes to -inf faster

$= \boxed{3}$

(c) (4 pt) Use the Taylor series

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots$$

to evaluate

$$\lim_{x\to 0} \frac{x^5}{\sin(x) - x + \frac{1}{6}x^3}$$

$$= \lim_{x\to 0} \frac{x^5}{\cancel{x} - \cancel{\frac{1}{6}x^3} + \frac{1}{120}x^5 + O(x^7) - \cancel{x} + \cancel{\frac{1}{6}x^3}}$$

$$= \lim_{x\to 0} \frac{1}{\frac{1}{120} + O(x^2)} = \frac{1}{\frac{1}{120}} = \boxed{120}$$

(d) (4 pt) Use l'Hôpital's rule to evaluate

$$\lim_{x\to 0} \frac{x^5}{\sin(x) - x + \frac{1}{6}x^3}$$

(Note that this is the same limit as in part (c).)

$$\lim_{x\to 0} \frac{x^5}{\sin(x) - x + \frac{1}{6}x^3} \quad \begin{matrix} -0 \\ -0 \end{matrix}$$

$$\text{L.H.} \lim_{x\to 0} \frac{5x^4}{\cos(x) - 1 + \frac{1}{2}x^2} \quad \begin{matrix} -0 \\ -0 \end{matrix}$$

$$\text{L.H.} \lim_{x\to 0} \frac{20x^3}{-\sin(x) + x} \quad \begin{matrix} -0 \\ -0 \end{matrix}$$

$$\text{L.H.} \lim_{x\to 0} \frac{60x^2}{-\cos(x) + 1} \quad \begin{matrix} -0 \\ -0 \end{matrix}$$

$$\text{L.H} \lim_{x\to 0} \frac{120x}{\sin(x)} \quad \begin{matrix} -0 \\ -0 \end{matrix}$$

$$\text{L.H.} \lim_{x\to 0} \frac{120}{\cos(x)} = \frac{120}{1} = \boxed{120}$$

(e) (4 pt) $\lim_{x\to 0^{+}} (1+x)^{\frac{1}{x}}$

$$\lim_{x\to 0^+} e^{\ln\left([1+x]^{\frac{1}{x}}\right)} = \lim_{x\to 0^+} e^{\frac{1}{x}\ln(1+x)}$$

$$= \lim_{x\to 0^+} e^{\frac{\ln(1+x)}{x}} \quad \begin{matrix} -0 \\ -0 \end{matrix}$$

$$\text{L.H.} = \lim_{x\to 0^+} e^{\frac{\frac{1}{1+x}}{1}} = e^1 = e$$

Exercise 3

(16 pt) Evaluate the indefinite integrals. (That is, find the most-general antiderivative $F(x)$ of the integrand $f(x)$ in the following integrals $\int f(x)\,dx$.)

(a) (4 pt) $\int 4x^3 - 2x + 1\,dx$

$= \boxed{x^4 - x^2 + x + C}$

(b) (4 pt) $\int e^{2x} - e^{-x}\,dx$

$= \boxed{\frac{1}{2}e^{2x} + e^{-x} + C}$

(c) (4 pt) $\int \frac{x^2 - 1}{\sqrt{x}}\,dx = \int x^{3/2} - x^{-1/2}\,dx$

$= \boxed{\frac{2}{5}x^{5/2} - 2x^{1/2} + C}$

(d) (4 pt) $\int (x^2 - 1)(4x + 3)\,dx = \int 4x^3 + 3x^2 - 4x - 3\,dx$

$= \boxed{x^4 + x^3 - 2x^2 - 3x + C}$

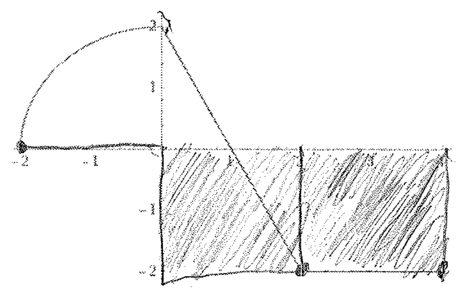
Exercise 4

(16 pt) Consider the piecewise function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant -2 \\ \sqrt{4-x^2} & \text{if } -2 \leqslant x \leqslant 0 \\ 2-2x & \text{if } 0 \leqslant x \leqslant 2 \\ -2 & \text{if } x \geqslant 2 \end{cases}$$

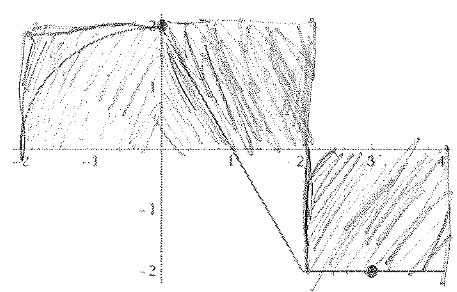
Graphs of f are included in parts (a) and (b).

(a) (4 pt) Draw and compute a lower- and upper-sum estimate (call them L_3 and U_3, respectively) for $\int_{-2}^{4} f(x)\,dx$ by partitioning $[-2, 4]$ into three subintervals, each of width 2.



Lower sum (L_3)

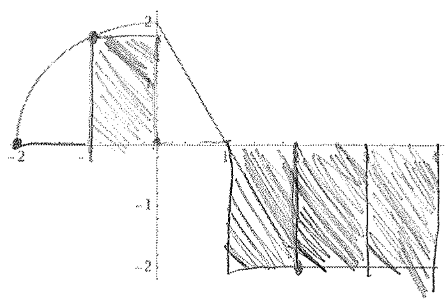
$0 \cdot 2 + (-2) \cdot 2 + (-2) \cdot 2 = \boxed{-8}$



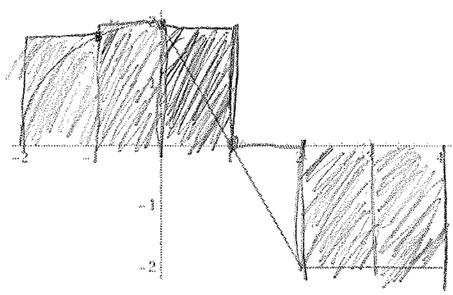
Upper sum (U_3)

$2 \cdot 2 + 2 \cdot 2 + (-2) \cdot 2 = 4$

(b) (4 pt) Draw and compute a lower- and upper-sum estimate (call them L_6 and U_6, respectively) for $\int_{-2}^{4} f(x)\ dx$ by partitioning $[-2, 4]$ into six subintervals, each of width 1. (Note: $f(-1) = \sqrt{3} \approx 1.73$.)



Lower sum (L_6)



Upper sum (U_6)

$\sqrt{3} - 2 - 2 - 2 = \sqrt{3} - 6$
≈ -4.27

$\sqrt{3} + 2 + 2 - 2 - 2 = \sqrt{3}$
≈ 1.73

(c) (4 pt) Use geometry to compute the exact value of $\int_{-2}^{4} f(x)\ dx$.

$[-2, 0]$ semi-circle of radius 2, positive: $\frac{1}{4}\pi(2)^2 = \pi$

$[0, 1]$ triangle w/ base = 1, height = 2, positive: $\frac{1}{2}(1)(2) = 1$

$[1, 2]$ triangle w/ base = 1, height = 2, negative: $-\frac{1}{2}(1)(2) = -1$

$[2, 4]$ square with side = 2, negative: $2^2 \cdot (-1) = -4$

$\pi + 1 - 1 - 4 = \pi - 4 \approx -0.86$

(d) (4 pt) Order all your results, from parts (a)–(c), in increasing order. Make a conjecture about where lower- and upper-sum estimates L_{12} and U_{12}, with twelve subintervals, each of width $\frac{1}{2}$, would go in your order.

$L_3 \leq L_6 \leq L_{12} \leq \int_{-2}^{4} f(x)\,dx \leq U_{12} \leq U_6 \leq U_3$

Exercise 5

(18 pt) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = e^x + \pi\cos(\pi x) + 2x - 1$$

(a) (4 pt) Find an antiderivative $F(x)$ of $f(x)$. Verify that it is indeed an antiderivative.

$$F(x) = \int e^x + \pi\cos(\pi x) + 2x - 1$$
$$= \boxed{e^x + \sin(\pi x) + x^2 - x + C}$$

verify: $[F(x)]' = \frac{d}{dx}\left[e^x + \sin(\pi x) + x^2 - x + C\right] = e^x + \pi\cos(\pi x) + 2x - 1$

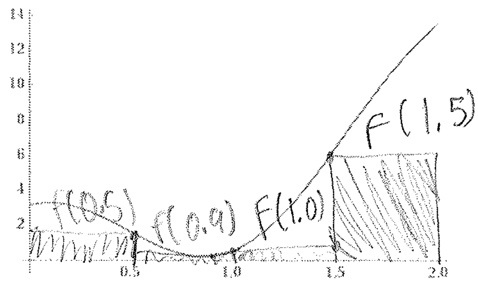
(b) (4 pt) Using your antiderivative $F(x)$ from part (a), show that $\int_0^2 f(x)\,dx = e^2 + 1$ (approximately 8.3890).

$$\int_0^2 f(x)\,dx = F(2) - F(0)$$
$$= e^2 + \cancel{\sin(2\pi)} + (2)^2 - 2 - e^0 \cancel{- \sin(0)} \cancel{- (0)^2} \cancel{+ 0}$$
$$= e^2 + 4 - 2 - 1$$
$$= \boxed{e^2 + 1}$$

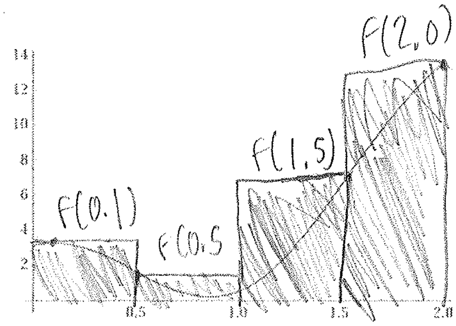
(c) (2 pt) Find the average value of $f(x)$ on the interval $[0, 2]$. (You have already done almost all the work!)

$$\frac{1}{2-0}\int_0^2 f(x)\,dx = \boxed{\frac{1}{2}(e^2 + 1)}$$

(d) (4 pt) On the graphs of f below, draw a lower- and upper-sum approximation to the definite integral $\int_0^2 f(x)\,dx$. Partition the interval $[0, 2]$ into four subintervals, each of width $\frac{1}{2}$.



Lower sum



Upper sum

(e) (4 pt) Using the values f(x) below, compute the upper- and lower-sum approximations you sketched in part (d). Show that these approximations bound your value of the definite integral in part (b).

- $f(0.0) \approx 3.14$
- $f(0.1) \approx 3.29$ (a local maximum)
- $f(0.5) \approx 1.65$
- $f(0.9) \approx 0.24$ (a local minimum)
- $f(1.0) \approx 0.58$
- $f(1.5) \approx 6.48$
- $f(2.0) \approx 13.53$

$$L_4 = \frac{1}{2}\left(f(0.5) + f(0.9) + f(1.0) + f(1.5)\right)$$
$$= \frac{1}{2}(1.65 + 0.24 + 0.58 + 6.48) = 4.475$$

$$U_4 = \frac{1}{2}\left(f(0.1) + f(0.5) + f(1.5) + f(2.0)\right)$$
$$= \frac{1}{2}(3.29 + 1.65 + 6.48 + 13.53) = 12.475$$

$$L_4 \leq \int_0^2 f(x)\,dx \leq U_4$$