Math 357
Long quiz 04

2024–02–14 (W)

Your name: ______________________________

Let $\mathbf{Q}$ denote the field of rational numbers; given a prime $p \in \mathbf{Z}_{>0}$, let $\mathbf{F}_p \cong \mathbf{Z}/(p)$ denote the finite field with p elements; and let t be an indeterminate. For each of the quotient rings below, characterize its algebraic structure as "field", "integral domain but not field", or "ring but not integral domain". Justify your characterization.

$$R_1 = \mathbf{F}_5[t]/(t^3 - t^2 + 2t - 1) \qquad R_2 = \mathbf{Q}[t]/(t^3 - t^2 + 2t - 1) \qquad R_3 = \mathbf{Q}[t]/(t^6 - 300)$$

Solution: We analyze each quotient ring in turn.

R_1 : Ring but not integral domain. Let $f_1 = t^3 - t^2 + 2t - 1 \in \mathbf{F}_5[t]$. Because $\deg f_1 = 3$, f_1 is reducible if and only if it has a linear factor $t - \alpha$ for some $\alpha \in \mathbf{F}_5$, which we have shown is equivalent to the statement that α is a zero of the function $f_1 : \mathbf{F}_5 \to \mathbf{F}_5$. Evaluating the function f_1 at the five elements of $\mathbf{F}_5$, we find that $f_1(-1) = 0$.[1] Thus f_1 is reducible; in fact,

$$f_1 = (t+1)(t^2 - 2t - 1)$$

It follows that $t - 1$ and $t^2 - 2t - 1$ are zero divisors in R_1. Hence R_1 is a (commutative) ring that is not an integral domain.

R_2 : Field. Let $f_2 = t^3 - t^2 + 2t - 1 \in \mathbf{Z}[t]$. We may be tempted to apply the reduction homomorphism corresponding to the proper ideal $(5) \lhd \mathbf{Z}$ to f_2. This gives the polynomial $f_1 \in \mathbf{F}_5[t]$, which we just showed is reducible. However, this does not imply that f_2 is reducible! Indeed, if we apply the reduction homomorphism corresponding to the proper ideal $(2) \lhd \mathbf{Z}$,[2] then we get

$$\overline{f}_2 = t^3 + t^2 + 1$$

Because $\deg \overline{f}_2 = 3$, $\overline{f}_2$ is reducible if and only if it has a linear factor, which is equivalent to the function $\overline{f}_2$ having a zero in $\mathbf{F}_2$. It is straightforward to check that the function $\overline{f}_2$ has no zero in $\mathbf{F}_2$, so $\overline{f}_2$ is irreducible, and thus $f_2 \in \mathbf{Z}[t]$ is irreducible. Finally, Gauß's lemma implies that $f_2 \in \mathbf{Q}[t]$ is irreducible. Hence $(f_2) \trianglelefteq \mathbf{Q}[t]$ is maximal, so R_2 is a field.

R_3 : Field. Let $f_3 = t^6 - 300 \in \mathbf{Z}[t]$. Note that $300 = 2^2 \cdot 3 \cdot 5^2$. In particular, 3 divides all coefficients of f_3 except the leading coefficient, and 3^2 does not divide the constant term. Thus we may apply the Eisenstein–Schönemann criterion to f_3 using the prime ideal $(3) \lhd \mathbf{Z}$ to conclude that f_3 is irreducible in $\mathbf{Q}[t]$. Hence $(f_3) \trianglelefteq \mathbf{Q}[t]$ is maximal, so R_3 is a field.

[1] One can check that -1 is the only zero of f_1 in $\mathbf{F}_5$, and that it has multiplicity 1.

[2] An analogous argument works with other proper ideals of $\mathbf{Z}$; for example, the principal ideals generated by one of $3, 4, 6, 8, 9, 10, 12, 13$.