Math 357
Exam 01

2024–03–01 (F)

Your name: ______________________________

Honor pledge:

Instructions

1. In the space above, please legibly write your name and the Rice Honor Pledge, then sign.
2. Full time for this exam is exactly 50 minutes. No resources are allowed.
3. Your reasoning—correctness and clarity—is more important than your "answer".
4. If you think there is ambiguity or error in an exercise, then briefly (!) write your understanding of the exercise and any additional hypotheses you are making, then proceed.

This exam is an imperfect measure of my understanding at a particular point in time. It is not a measure of who I am or who I will be.

Exercise	Total	(a)	(b)	(c)	(d)
1	/4	/4	/4	/4	
2	/4				
3	/4	/4	/4	/4	/4
4	/4				
5	/4	/4	/4		
Total	/20				

Exercise 1

(4 pt) Let R be a commutative ring with a multiplicative identity $1 \neq 0$. An element $a \in R$ is **nilpotent** if there exists an $n \in \mathbf{Z}_{>0}$ such that $a^n = 0$.

(a) Let $a \in R$ be nonzero. Prove that if a is nilpotent, then a is a zero divisor.

(b) Give an example to show that the converse of (a) is false.

(c) Let $a \in R$ be nilpotent. Prove that $1 - a$ is a unit.

Exercise 2

(4 pt) Let R be an integral domain. Prove that if R is a euclidean domain, then R is a principal ideal domain. (Heart points: Give—without proof—an example of a principal ideal domain that is not a euclidean domain.)

Exercise 3

(4 pt) For each of the following polynomials, state whether it is reducible or irreducible in the indicated polynomial ring. Justify your assertions.

$$f_1 = t^2 + 2 \in \mathbf{F}_7[t] \qquad f_3 = 6t^4 + 24t^3 + 18t + 81 \in \mathbf{Q}[t]$$
$$f_2 = 4t^3 + 9t^2 + 7t - 12 \in \mathbf{Z}[t] \qquad f_4 = t^4 - 42t^2 + 30t + 12 \in \mathbf{Q}[t]$$

Exercise 4

(4 pt) Let F be a field, let $\alpha \in F$, and let t be an indeterminate. We may give $F[t]$ the structure of a ring, an $F[t]$-module, or a $\mathbf{Z}$-module. For each map below, state whether it is a ring homomorphism, an $F[t]$-module homomorphism, or a $\mathbf{Z}$-module homomorphism. Justify your assertions. *Hint:* A given map may satisfy several or none of these conditions. The characterization may depend on the value of α.

$$\begin{aligned} \varphi : F[t] &\to F[t] \\ f(t) &\mapsto \alpha f(t) \end{aligned} \qquad\qquad \begin{aligned} \psi : F[t] &\to F[t] \\ f(t) &\mapsto f(\alpha) \end{aligned}$$

Exercise 5

(4 pt) Let F be a field, let G be a finite group, and let V be a unital FG-module.

(a) We have seen that V affords a representation $\rho : G \to GL(V)$. Given $g \in G$, define $\rho(g)$.

(b) Prove that for each $g \in G$, $\rho(g) \in GL(V)$.