Math 211
Quiz 23Z

W 07 Aug 2019

Your name : ____

Exercise

(5 pt) Solve the following nonhomogeneous 2nd-order linear initial value problem:

$$y'' + y' + y = e^{-t}, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Hint: Recall that, from the definition of the lapace transform,

$$\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\} - y(0).$$

Applying this result to y'', we get

$$\mathcal{L}\{y''\}(s) = s\mathcal{L}\{y'\} - y'(0) = s^2\mathcal{L}\{y\} - sy(0) - y'(0).$$

The following transform–inverse-transform pairs may be useful:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a},$$

Solution: Applying the laplace transform to both sides of the ODE, and using the fact that the laplace transform is linear, we get

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{y'' + y' + y\} = \mathcal{L}\{e^{-t}\}. \tag{1}$$

Denote

$$Y(s) = \mathcal{L}\{y\}.$$

From the definition of the laplace transform, and using the given initial conditions, we compute

$$\begin{aligned}\mathcal{L}\{y'\} &= s\mathcal{L}\{y\} - y(0) = sY(s) - 0 = sY(s)\\ \mathcal{L}\{y''\} &= s^2\mathcal{L}\{y'\} - sf(0) - f'(0) = s^2Y(s) - 1.\end{aligned}$$

Substituting these results into (1), we get

$$\left(s^2Y(s) - 1\right) + sY(s) + Y(s) = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}.$$

Solving this equation for $Y(s)$, we get

$$Y(s) = \frac{\frac{1}{s+1} + 1}{s^2 + s + 1} = \frac{\frac{s+2}{s+1}}{s^2 + s + 1} = \frac{s+2}{(s+1)(s^2+s+1)}.$$

Setting up the partial fraction decomposition, we have

$$\frac{s+2}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}.$$

Clearing denominators, we get

$$s + 2 = (s^2 + s + 1)A + (s+1)(Bs + C).$$

Now we plug in (at least) three values for t to get a system of equations we can use to solve for A, B, C:

$$\begin{aligned} s = -1 &: 1 = A \\ s = 0 &: 2 = A + C = 1 + C \Rightarrow C = 1 \\ s = 1 &: 3 = 3A + 2(B + C) = 3 + 2B + 2 \Rightarrow B = -1. \end{aligned}$$

Thus

$$\begin{aligned} Y(s) &= \frac{s+2}{(s+1)(s^2+s+1)} \\ &= \frac{1}{s+1} + \frac{-s+1}{s^2+s+1} \\ &= \frac{1}{s+1} - \frac{s-\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} \\ &= \frac{1}{s+1} - \frac{s+\frac{1}{2}-1}{(s+\frac{1}{2})^2+\frac{3}{4}} \\ &= \frac{1}{s+1} - \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{4}{3}\frac{\frac{3}{4}}{(s+\frac{1}{2})^2+\frac{3}{4}}, \end{aligned}$$

where

- in the third equality, in the second term, we have (i) factored out a -1 from the numerator and (ii) completed the square in the denominator;
- in the fourth equality, we have rewritten the numerator so we get the same $s + a$ term as appears in the completed square in the denominator (to set us up for the inverse laplace transform); and
- in the fifth equality, we have explicitly decomposed the second term into two terms, and rewritten the numerator of our second term (factoring out a $\frac{4}{3}$) so that all terms have a form that appear in our dictionary of transform–inverse-transform pairs.

Applying the inverse laplace transform to both sides, and using the fact that it is linear, we get

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\} \\ &= e^{-t} - e \end{aligned}$$