Math 357
Long quiz 04C

2024–04–10 (W)

Your name: ______________________________

Let $\mathbf{Q}$ denote the field of rational numbers; given a prime $p \in \mathbf{Z}_{>0}$, let $\mathbf{F}_p \cong \mathbf{Z}/(p)$, a finite field with p elements; and let t be an indeterminate. For each of the quotient rings below, characterize its algebraic structure as "field", "integral domain but not field", or "ring but not integral domain". Justify your characterization.

$$\begin{aligned} R_1 &= \mathbf{F}_5[t]/(t^3+2) \\ R_2 &= \mathbf{Q}[t]/(t^4-8t^2+20) \\ R_3 &= \mathbf{Q}[t]/(t^3-11t^2+57t+12) \end{aligned}$$

Solution: We analyze each ring in turn.

R_1 : Ring but not integral domain. $t=2$ is a zero of f_1, as a function $\mathbf{F}_5 \to \mathbf{F}_5$. Equivalently, $t-2$ is a factor of t^3-2 in $\mathbf{F}_5[t]$.

R_2 : Field. View f_2 as a function $\mathbf{R} \to \mathbf{R}$. We compute

$$f_2' = 4t^3 - 16t = 4t(t+2)(t-2)$$

Because $\lim_{t\to\pm\infty} f_2(t) = +\infty$, the values of f_2 at the "outer" horizontal tangents, at $t=\pm 2$, are the candidate global minima of f_2. We compute

$$f_2(\pm 2) = 16 - 32 + 20 = 4 > 0$$

We conclude that f_2 has no zeros in $\mathbf{R}$, and hence in $\mathbf{Q}$. Therefore f_2 is irreducible in $\mathbf{Q}[t]$, so the principal ideal (f_2) is maximal, so $R_2 = \mathbf{Q}[t]/(f_2)$ is a field.

R_3 : We present two approaches.

Approach 1: Consider f_3 as a function $\mathbf{R} \to \mathbf{R}$. We compute

$$f_3'(t) = 3t^2 - 22t + 57$$

which has discriminant

$$(-22)^2 - 4(3)(57) = 484 - 684 = -200 < 0$$

It follows that the function f_3' has no zeros in $\mathbf{R}$, and therefore $f_3'(t) > 0$ for all $t \in \mathbf{R}$. That is, the function f_3 is monotonically increasing, and therefore has exactly one zero in $\mathbf{R}$. We compute

$$f_3(-1) = -57 < 0 \qquad\qquad f_3(0) = 12 > 0$$

so by the intermediate value theorem, this unique real zero lies in the interval $(-1,0)$. The rational zeros test applied to the polynomial f_3 implies that any rational zero of f_3 is an integer. There are no integers in the (open) interval $(-1,0)$, so we conclude that f_3 has no zero in $\mathbf{Q}$. Because f_3 is degree 3, this is equivalent to f_3 being irreducible in $\mathbf{Q}[t]$.

Approach 2: Consider $f_3(t+2)$:

$$f_3(t+2) = t^3 - 5t^2 + 25t + 90$$

This is irreducible in $\mathbf{Z}[t]$ by the Eisenstein–Schönemann criterion, and therefore irreducible in $\mathbf{Q}[t]$ by Gauß's lemma. Irreducibility of $f_3(t+2)$ in $\mathbf{Q}[t]$ is equivalent to irreducibility of $f_3(t)$ in $\mathbf{Q}[t]$. Thus f_3 is irreducible in $\mathbf{Q}[t]$

Because f_3 is irreducible in $\mathbf{Q}[t]$, the principal ideal (f_3) is maximal in $\mathbf{Q}[t]$, so $R_3 = \mathbf{Q}[t]/(f_3)$ is a field.