Math 211
Quiz 06

M 15 Jul 2019

Your name : ______________________________

Exercise

(5 pt) Find the general solution to the nonhomogeneous linear ODE[1]

$$y' + y = \frac{1}{1+e^t}. \tag{1}$$

Solution: By the nonhomogeneous principle, any solution y to (1) has the form

$$y = y_p + y_h,$$

where y_p is any (one) particular solution to (1), and y_h is the general solution to the corresponding homogeneous ODE. The general solution to (1) is $y_p + y_h$.

To find y_h, we solve the corresponding homogeneous ODE:

$$y' + y = 0.$$

This ODE is separable. Solving it (and calling our solution y_h), we find

$$\begin{aligned} y^{-1}\,dy &= -dt \\ \ln|y| &= -t + c_1 \\ y_h &= c_2 e^{-t}. \end{aligned}$$

Note that

$$y_h' + y_h = -c_2 e^{-t} + c_2 e^{-t} = 0,$$

so y_h indeed solves the corresponding homogeneous ODE, as claimed.

To find y_p we can use guess-and-check, integrating factors, or variation of parameters. Variation of parameters is the most systematic of these options, so we'll go with it. Variation of parameters tells us that, to find a particular solution to (1), we take the general solution to the corresponding homogeneous ODE, and turn the constant parameter into a function of t, call it $v(t)$:

$$y_p = ve^{-t}. \tag{2}$$

Plugging this function into the original ODE (1), we get

$$\frac{1}{1+e^t} = y_p' + y_p = \left(v'e^{-t} - ve^{-t}\right) + ve^{-t} = v'e^{-t}.$$

Solving this equation for v, we find

$$v' = \frac{e^t}{1+e^t} \qquad \Leftrightarrow \qquad v(t) = \int \frac{e^t}{1+e^t}\,dt = \ln\left|1+e^t\right| + c_3 = \ln\left(1+e^t\right) + c_3,$$

[1] *Hint:* Recall that the nonhomogeneous principle says we can find (i) one particular solution to the nonhomogeneous ODE, (ii) the general solution to the corresponding homogeneous ODE, then combine them to get the general solution to the nonhomogeneous ODE. While we're not obliged to use variation of parameters here, recall that it says that a particular solution to a nonhomogeneous ODE can be found by treating the constant in the general solution to the corresponding homogeneous ODE as a function of t.

for some $c_3 \in \mathbf{R}$. We can drop the absolute value because $1 + e^t > 0$ (in fact, > 1) for all $t \in \mathbf{R}$. Also, because we only need one particular solution, we can choose the constant c_3 to be any value we like. To make our lives easy, let's choose $c_3 = 0$. Then, plugging this v into (2), we have

$$y_p = e^{-t} \ln\left(1 + e^t\right).$$

By the nonhomogeneous principle, the general solution y to (1) is

$$y(t) = y_p + y_h = e^{-t} \ln\left(1 + e^t\right) + ce^{-t}, \tag{3}$$

where $c \in \mathbf{R}$.

We can check that our general solution (3) is indeed a solution by plugging it into (1):

$$\begin{aligned} y' + y &= \left(-e^{-t} \ln\left(1 + e^t\right) + \frac{e^{-t}}{1 + e^t} e^t - ce^{-t}\right) + \left(e^{-t} \ln\left(1 + e^t\right) + ce^{-t}\right) \\ &= \frac{1}{1 + e^t}, \end{aligned}$$

so the original ODE (1) indeed holds, i.e. our $y(t)$ in (3) is a solution.