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UCS654
Predictive Analysis using Statistics

Assignment : Parameter Estimations

1.) Let (X1, X2, ...) be a random sample of size n taken from a Normal Population with parameters : mean = θ1 and variance = θ2. Find the Maximum Likelihood Estimates of these two parameters.

→ In finding the estimators, write the pdf as a function of θ1 and θ2.

$$f(x_i ; \theta_1, \theta_2) = \frac{1}{\sqrt{\theta_2}\sqrt{2\pi}} \exp\left[-\frac{(x_i - \theta_1)^2}{2\cdot\theta_2}\right]$$

for

$$-\infty < \theta_1 < \infty$$

$$-\infty < \theta_2 < \infty$$

that makes the likelihood function as :

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i ; \theta_1, \theta_2) = \theta_2^{-n/2}(2\pi)^{-n/2} \exp\left[\frac{-1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2\right]$$

and

therefore the log of the likelihood function:

$$\log L(\theta 1, \theta 2) = -\frac{n}{2}\log \theta 2 - \frac{n}{2}\log(2\pi) - \frac{\Sigma(x_i - \theta 1)^2}{2\theta 2}$$

Upon taking the partial derivative of the log likelihood with respect to $\theta 1$, and setting to 0, we have:

$$\frac{\partial \log L(\theta 1, \theta 2)}{\partial \theta 1} = \frac{-\cancel{2}\Sigma(x_i - \theta 1)(-1)}{\cancel{2}\theta 2}$$

$$\frac{\partial \log L(\theta 1, \theta 2)}{\partial \theta 1} = \frac{\Sigma(x_i - \theta 1)}{\theta 2} \overset{set}{\equiv} 0$$

Now, multiplying through by $\theta 2$ and distributing the summation, we get;

$$\Sigma x_i - n\theta 1 = 0$$

On solving,

$$\hat{\theta 1} = \frac{\Sigma x_i}{n} = \bar{x}.$$

Now for $\theta 2$

Taking the partial derivatives of the log likelihood with respect to $\theta 2$, and setting to 0, we get:

$$\frac{\partial \log L(\theta 1, \theta 2)}{\partial \theta 2} = \frac{-n}{2\theta 2} + \frac{\Sigma(x_i - \theta 1)^2}{2\theta 2^2} \overset{set}{\equiv} 0$$

Multiplying through by $2\theta 2^2$

$$\frac{\partial \log L(\theta 1, \theta 2)}{\partial \theta 2} = \left[\frac{-n}{2\theta 2} + \frac{\Sigma (x_i - \theta 1)^2}{2\theta 2^2} \text{ set } 0 \right] \times 2\theta 2^2$$

we get :

$$-n\theta 2 + \Sigma (x_i - \theta 1)^2 = 0$$

On solving

$$\hat{\theta 2} = \frac{\Sigma (x_i - \bar{x})^2}{n}$$

Thus,

Maximum Likelihood Estimate for mean = θ_1 is :

$$\hat{\theta}_1 = \frac{\Sigma x_i}{n} = \bar{x}.$$

Maximum Likelihood Estimate for variance = θ_2 is :

$$\hat{\theta}_2 = \frac{\Sigma (x_i - \bar{x})^2}{n}.$$

2.) Let $X_1, X_2, \ldots, X_n$ be a random sample from $B(m, \theta)$ distribution, where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compute value of θ using the M.L.E.

→ Let $X_1, X_2, \ldots, X_n$ be a random sample from $B(m, \theta)$ and we have observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

The maximum likelihood estimate of θ is the value that maximizes the likelihood function

$$L(x_1, x_2, \ldots, x_n; \theta).$$

Remember that when we have a random sample, we can obtain the joint PMF and PDF by multiplying the marginal (individual) PMFs and PDFs.

Thus,

$$L(x_1, x_2, \ldots, x_n; \theta) = P_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n; \theta)$$

$$= \prod_{i=1}^{n} P_{X_i}(x_i; \theta)$$

$$= \prod_{i=1}^{n} \binom{m}{x_i} \theta^{x_i} (1-\theta)^{m-x_i}$$

$$= \left[\prod_{i=1}^{n} \binom{m}{x_i}\right] \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{mn - \sum_{i=1}^{n} x_i}$$

Note that the first term does not depend on θ

so we can write $L(x_1, x_2, \ldots x_n ; \theta)$ as

$$L(x_1, x_2, \ldots, x_n ; \theta) = c\ \theta^s (1-\theta)^{mn-s}$$

where,

c does not depend on θ, and

$s = \sum_{k=1}^{n} x_i$

By differentiating and setting the derivative to 0 we obtain

$$\hat{\theta}_{ML} = \frac{1}{mn} \sum_{k=1}^{n} x_i .$$

Thus, M.L.E. can be written as

$$\hat{\Theta}_{ML} = \frac{1}{mn} \sum_{k=1}^{n} X_i .$$