Analytic Structure of Correlation Functions

Motivating and Defining the Real Time Correlation Functions

We are interested in correlations between observables so we shall treat all operators in the heisenberg picture. In euclidean time the structure of the correlation function is obvious:

$$C^{\tau}(\tau_1 - \tau_2) = -\langle \mathcal{T}_\tau X_1(\tau_1) X_2(\tau_2) \rangle = \begin{Bmatrix} -\langle X_1(\tau_1) X_2(\tau_2) \rangle & \tau_1 \geq \tau_2 \\ -\zeta_X \langle X_2(\tau_2) X_1(\tau_1) \rangle & \tau_1 < \tau_2 \end{Bmatrix}$$

Where the sign is conventional and $\mathcal{T}_\tau$ is the euclidean time ordering operator. $\zeta_X = \pm 1$ based on the statistics of our operator X.

Now an obvious generalization to real time would be:

$$C^{t}(t_1 - t_2) = -i \langle \mathcal{T}_t X_1(t_1) X_2(t_2) \rangle$$

where the factor of i is conventional. It turns out however this is not always the most useful object. To see why let us compute the evolution of X in the presence of a linear perturbation F' such that the hamiltonian is $H = H_0 + F'(t)X'$

$$X(t) = \langle U^{-1} X U \rangle = \langle (U^{-1} U_0)(U_0^{-1} X U_0)(U_0^{-1} U) \rangle = \langle U_{F'}^{-1} X_0 U_{F'} \rangle$$

Where all time evolution operators are taken to evolve from $-\infty$ to t. We now see that the operator $U_{F'} = U_0^{-1} U$ controls the deviation of $X(t)$ from its unperturbed value $X_0(t)$ so it's worthwhile to understand $U_{F'}$. Starting from it's definition:

$$d_t U_{F'} = d_t(U_0^{-1} U) = (it\hat{H}_0) U_0^{-1} U + (-it\hat{H}_0 - itF'X') U_0^{-1} U = -itF'X'U^F$$

The solution to this DE is a simple exponential, imposing the boundary condition that $F' = 0 \,\&\, U_F = \mathbb{1}$ in the far past we can write the solution explicitly:

$$U_F(t) = \mathcal{T}_t \exp\left(-i \int_{-\infty}^{t} dt' F'(t') X'(t')\right) \approx \left(\mathbb{1} - i \int_{-\infty}^{t} dt' F'(t') X'(t')\right)$$

Where this expansion is justified because we only care about small perturbations. Substituting this result into the evolution equation for $X(t)$ we see:

$$
\begin{aligned}
X(t) &= \langle U_{F'}^{-1} X_0 U_{F'} \rangle \\
&\approx \langle \left(\mathbb{1} + i \int_{-\infty}^{t} dt' F'(t') X'(t') \right) X_0(t) \left(\mathbb{1} - i \int_{-\infty}^{t} dt' F'(t') X'(t') \right) \rangle \\
&= \langle X_0(t) + \left(i \int_{-\infty}^{t} dt' F'(t') X'(t') \right) X_0(t) + X_0(t) \left(-i \int_{-\infty}^{t} dt' F'(t') X'(t') \right) + \mathcal{O}(F^2) \rangle \\
&= \langle X_0(t) + \cancel{X_0(t) \left(i \int_{-\infty}^{t} dt' F'(t') X'(t') \right)} - \left(i \int_{-\infty}^{t} dt' F'(t') \left[X_0(t), X'(t') \right] \right) + \cancel{X_0(t) \left(-i \int_{-\infty}^{t} dt' F'(t') X'(t') \right)} + \mathcal{O}(F^2) \rangle \\
&= \langle X_0(t) - i \left(\int_{-\infty}^{t} dt' F'(t') \left[X_0(t), X'(t') \right] \right) + \mathcal{O}(F^2) \rangle \\
&\approx \langle X_0(t) \rangle - i \int_{-\infty}^{\infty} dt' \theta(t - t') F'(t') \langle [X_0(t), X'(t')] \rangle \\
&= \langle X_0(t) \rangle - i \int_{-\infty}^{\infty} dt' \theta(t - t') F'(t') \langle \left[X_{(}t), X'(t') \right] \rangle + \mathcal{O}(F^2) \\
&\approx \langle \hat{X}(t) \rangle + \int_{-\infty}^{\infty} dt' C^{+}(t - t') F'(t')
\end{aligned}
$$

So we see the appropriate greens function for real time is actually the so called retarded correlation functions:

$$
C^{+}(t_1 - t_2) = -i\theta(t_1 - t_2) \langle [X_1(t_1), X_2(t_2)]_{\zeta_X} \rangle
$$

In direct contrast we can also define the advanced corelation function which we could have similarly derived by the above procedure if we had instead insisted the perturbation vanish in the far future rather than far past (A situation which is mathematically just as valid but physically far more dubious):

$$
C^{-}(t_1 - t_2) = +i\theta(t_2 - t_1) \langle [X_1(t_1), X_2(t_2)]_{\zeta_X} \rangle
$$

Lehmann representation

Now since we typically compute $C^{\tau}(\tau_1 - \tau_2)$ from our euclidean path integral yet we need $C^{+}(t_1 - t_2)$ we should investigate how all these are related. First we will decompose it into a hamiltonian eigenbasis $|\Psi_\alpha\rangle$ (letting $t_1 = t$ and $t_2 = 0$):

$$
\begin{aligned}
C^{t}(t) &= -i \langle \mathcal{T}_t X_1 X_2 \rangle \\
&= -\frac{i}{\mathcal{Z}} \operatorname{Tr} \left[\mathcal{T}_t X_1(t) X_2(0) \exp\left(-\beta(H_0 - \mu N)\right) \right] \\
&= -\frac{i}{\mathcal{Z}} \Big(\theta(t) \operatorname{Tr} \left[e^{it(H_0 - \mu N)} X_1 e^{-it(H_0 - \mu N)} X_2 \exp\left(-\beta(H_0 - \mu N)\right) \right] \\
&+ \zeta_X \theta(-t) \operatorname{Tr} \left[e^{it(H_0 - \mu N)} X_2 e^{-it(H_0 - \mu N)} X_1 \exp\left(-\beta(H_0 - \mu N)\right) \right] \Big) \\
&= -\frac{i}{\mathcal{Z}} \Big(\theta(t) \sum_{\alpha\beta} \left[e^{it(E_\alpha - \mu N_\alpha)} X_{1,\alpha\beta} e^{-it(E_\beta - \mu N_\beta)} X_{2\beta\alpha} \exp\left(-\beta(E_\alpha - \mu N_\alpha)\right) \right] \\
&+ \zeta_X \sum_{\gamma\delta} \theta(-t) \left[e^{it(E_\gamma - \mu N_\gamma)} X_{2,\gamma\delta} e^{-it(E_\delta - \mu N_\delta)} X_{1,\delta\gamma} \exp\left(-\beta(E_\gamma - \mu N_\gamma)\right) \right] \Big) \\
&= -\frac{i}{\mathcal{Z}} \sum_{\alpha\beta} X_{1,\alpha\beta} X_{2\beta\alpha} e^{it(E_\alpha - \mu N_\alpha - E_\beta + \mu N_\beta)} \left[\theta(t) \exp\left(-\beta(E_\alpha - \mu N_\alpha)\right) + \zeta_X \theta(-t) \exp\left(-\beta(E_\beta - \mu N_\beta)\right) \right] \\
&= -\frac{i}{\mathcal{Z}} \sum_{\alpha\beta} X_{1,\alpha\beta} X_{2\beta\alpha} e^{it\Xi_{\alpha\beta}} \left[\theta(t) e^{-\beta\Xi_\alpha} + \zeta_X \theta(-t) e^{-\beta\Xi_\beta} \right]
\end{aligned}
$$

Where for convenience we've defined $\Xi_\alpha = E_\alpha - \mu N_\alpha$ and $\Xi_{\alpha\beta} = \Xi_\alpha - \Xi_\beta$

Moving into the frequency domain we write:

$$
\begin{aligned}
C^t(\omega) &= \lim_{\eta\to 0}\int dt C^t(t)e^{i\omega t-\eta|t|} \\
&= \lim_{\eta\to 0} -\frac{i}{\mathcal{Z}}\sum_{\alpha\beta} X_{1,\alpha\beta}X_{2\beta\alpha}\int dt e^{i(\omega+\Xi_{\alpha\beta})t-\eta|t|}\left[\theta(t)e^{-\beta\Xi_\alpha}+\zeta_X\theta(-t)e^{-\beta\Xi_\beta}\right] \\
&= \lim_{\eta\to 0} -\frac{i}{\mathcal{Z}}\sum_{\alpha\beta} X_{1,\alpha\beta}X_{2\beta\alpha}\left[-\frac{e^{-\beta\Xi_\alpha}}{i\omega+i\Xi_{\alpha\beta}-\eta}+\zeta_X\frac{e^{-\beta\Xi_\beta}}{i\omega+i\Xi_{\alpha\beta}+\eta}\right] \\
&= \lim_{\eta\to 0} \frac{1}{\mathcal{Z}}\sum_{\alpha\beta} X_{1,\alpha\beta}X_{2\beta\alpha}\left[\frac{e^{-\beta\Xi_\alpha}}{\omega+\Xi_{\alpha\beta}+i\eta}-\zeta_X\frac{e^{-\beta\Xi_\beta}}{\omega+\Xi_{\alpha\beta}-i\eta}\right]
\end{aligned}
$$

Note here the factor of η was motivated by the need for the fourier transformation to converge however we could also motivate it in reverse by starting with:

$$
C^t(\omega) = \frac{1}{\mathcal{Z}}\sum_{\alpha\beta} X_{1,\alpha\beta}X_{2\beta\alpha}\left[\frac{e^{-\beta\Xi_\alpha}}{\omega+\Xi_{\alpha\beta}}-\zeta_X\frac{e^{-\beta\Xi_\beta}}{\omega+\Xi_{\alpha\beta}}\right] = \frac{1}{\mathcal{Z}}\sum_{\alpha\beta}\frac{X_{1,\alpha\beta}X_{2\beta\alpha}}{\omega+\Xi_{\alpha\beta}}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\beta\Xi_\beta}\right]
$$

Then noticing it has poles on the real line at $\omega = -\Xi_{\alpha\beta}$ so to perform the contour integral back to real space we need to push the poles up or down off the real line, take the integral then take the limit as they move back to the real line. There are of course 4 ways we could go about this adding a $\pm i\eta$ to each denominator. We've already seen $+-$ which pushes one up and one down corresponds to $C^t(\omega)$. We note if we'd pushed both up then the contour integral vanishes if we close in the lower half plane and vice versa for pushing them down and the upper half plane. We also notice looking at the fourier transform:

$$
C^t(t) = \int đ\omega C^t(\omega)e^{-i\omega t}
$$

That for convergence we must close in the upper half plane when $t < 0$ and the lower half plane for $t > 0$ so we see then that pushing both down means our corelation function necessarily vanishes when $t < 0$, it must then be the retarded correlation function. Similarly pushing both up must be the advanced correlation function. So the respective denominators are:

$$
\begin{Bmatrix} C^t(\omega) \\ C^+(\omega) \\ C^-(\omega) \end{Bmatrix} \sim \begin{Bmatrix} \omega+\Xi_{\alpha\beta}+i\eta & \omega+\Xi_{\alpha\beta}-i\eta \\ \omega+\Xi_{\alpha\beta}+i\eta & \omega+\Xi_{\alpha\beta}+i\eta \\ \omega+\Xi_{\alpha\beta}-i\eta & \omega+\Xi_{\alpha\beta}-i\eta \end{Bmatrix}
$$

If this simply argument is not convincing it's straightforward to work it our explicitly as we did above.

Key Observations: Since the fourier integral has exponential suppression of large imaginary values built in the response may always be calculated by a contour integral in at least one half plane where the arc vanishes. This means for the integral to be non-vanishing there must be poles. Additionally since the response is fully real, the greens function must be fully real so the poles often land on the real line. Thus to perform calculations we must push them off the real line to take contours. So then we see generically no matter the pole structure pushing all of them down (up) will give us a correlation functions vanishing in the past (future), since there is only one way to push all down (up) this must be the unique retarded (advanced) correlation function. Should the poles naturally lie off the real line this means the dynamics contain sufficient damping that the boundary conditions in the arbitrary past and future are of no consequence so all the retarded/advanced correlation functions coincide. Another key feature is since the greens function must be real valued the only imaginary pieces will be our artificial pole pushing terms. Now since pushing all poles up and pushing all poles down obviously requires imaginary pole pushing terms of opposite signs we can immediately conclude:

$$
C^+(\omega) = \left[C^-(\omega)\right]^*
$$

Additionally it should now be obvious that in fact all possible configurations of pole pushing have the same real part (so long as the integral converged) so:

$$
\Re C^t(\omega) = \Re C^+(\omega) = \Re C^-(\omega)
$$

The relationship between the imaginary components is less obvious but it can be shown to be:

$$
\Im C^t(\omega) = \pm\Im C^\pm(\omega)\times\begin{Bmatrix} \coth(\beta\omega/2), & \textit{bosons} \\ \tanh(\beta\omega/2), & \textit{fermions} \end{Bmatrix}
$$

From this it's made manifest that all the correlation functions contain the same information, they merely solve the same problem for different boundary conditions.

Returning to Imaginary Time

Performing the exact same calculation as above we compute:

$$C^{\tau}(\tau) = -\frac{1}{\mathcal{Z}}\sum_{\alpha\beta} X_{1\alpha\beta}X_{2\beta\alpha}e^{\tau\Xi_{\alpha\beta}}\left[\theta(\tau)e^{-\beta\Xi_\alpha} + \zeta_X\theta(-\tau)e^{-\beta\Xi_\beta}\right]$$

Now recall the imaginary time correlation functions are periodic, this can be worked out explicitly or seen simply from:

$$\begin{aligned}
C^{\tau}(\tau+\beta) &= -\left\langle \mathcal{T}_\tau X_1(\tau+\beta)X_2(0)\right\rangle \\
&= -\frac{1}{Z}\,\mathrm{Tr}\left[\mathcal{T}_\tau e^{-\beta\Xi}e^{\beta\Xi}X_1(\tau)e^{-\beta\Xi}X_2(0)\right] \\
&= -\frac{1}{Z}\,\mathrm{Tr}\left[\mathcal{T}_\tau X_1(\tau)e^{-\beta\Xi}X_2(0)\right] \\
&= -\frac{1}{Z}\,\mathrm{Tr}\left[\mathcal{T}_\tau e^{-\beta\Xi}X_2(0)X_1(\tau)\right] \\
&= -\zeta_X\frac{1}{Z}\,\mathrm{Tr}\left[\mathcal{T}_\tau e^{-\beta\Xi}X_1(\tau)X_2(0)\right] \\
&= \zeta_X C^{\tau}(\tau)
\end{aligned}$$

Or if you like you may explicitly show it from our formula above. Now with this in mind we can fourier decompose in terms of the discrete Matsubara frequencies over the region $0 < \tau < \beta$:

$$\begin{aligned}
C^{\tau}(\omega_n) &= \int_0^\beta d\tau C^{\tau}(\tau)e^{i\omega_n\tau} \\
&= \int_0^\beta d\tau - \frac{1}{\mathcal{Z}}\sum_{\alpha\beta} X_{1\alpha\beta}X_{2\beta\alpha}e^{\tau\Xi_{\alpha\beta}}\left[\theta(\tau)e^{-\beta\Xi_\alpha}\right]e^{i\omega_n\tau} \\
&= -\frac{1}{\mathcal{Z}}\sum_{\alpha\beta} X_{1\alpha\beta}X_{2\beta\alpha}e^{-\beta\Xi_\alpha}\int_0^\beta d\tau e^{(i\omega_n+\Xi_{\alpha\beta})\tau} \\
&= -\frac{1}{\mathcal{Z}}\sum_{\alpha\beta} X_{1\alpha\beta}X_{2\beta\alpha}e^{-\beta\Xi_\alpha}\left[\frac{\zeta_X e^{\Xi_{\alpha\beta}\tau}}{i\omega_n+\Xi_{\alpha\beta}} - \frac{1}{i\omega_n+\Xi_{\alpha\beta}}\right] \\
&= \frac{1}{\mathcal{Z}}\sum_{\alpha\beta}\frac{X_{1\alpha\beta}X_{2\beta\alpha}}{i\omega_n+\Xi_{\alpha\beta}}\left[e^{-\beta\Xi_\alpha} - \zeta_X e^{-\Xi_\beta\tau}\right]
\end{aligned}$$

At this point we realize we've derived the exact same expression as we did for the real time case with $\omega \to i\omega_n$. We see then that we can defined:

$$C(z) = \frac{1}{\mathcal{Z}}\sum_{\alpha\beta}\frac{X_{1\alpha\beta}X_{2\beta\alpha}}{z+\Xi_{\alpha\beta}}\left[e^{-\beta\Xi_\alpha} - \zeta_X e^{-\Xi_\beta\tau}\right]$$

With the knowledge:

$$C^{\tau}(\omega_n) = C(i\omega_n) \qquad \& \qquad C^{+}(\tau)(\omega) = \lim_{z\to\omega^+} C(z) \qquad \& \qquad C^{-}(\tau)(\omega) = \lim_{z\to\omega^-} C(z)$$

where I use the $\pm$ in the limits to mean approaching from above or below the real axis. This means should be wish to compute the retarded (advanced) corelation functions it would suffice to compute the imaginary time correlation function at every positive (negative) Matsubara frequency then find an analytic continuation over the upper (lower) half plane and approach the real axis.

0.1 Spectral Density

Recall the corelation function of a free particle is simply:

$$C_0(z) = \frac{1}{z - H + \mu} = \frac{1}{z-\omega}$$

Where ω is the "natural" frequency of the system. Now looking back it seems our correlation function always has free particle like simple poles at some frequencies $\Xi_{\alpha\beta}$ so it's worth seeming if we can sensibly isolate the pole structure from the rest of the correlation function:

$$\begin{aligned}
C(z) &= \frac{1}{\mathcal{Z}}\sum_{\alpha\beta}\frac{X_{1\alpha\beta}X_{2\beta\alpha}}{z+\Xi_{\alpha\beta}}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\Xi_\beta\tau}\right]\\
&= \left[\frac{1}{\mathcal{Z}}\sum_{\alpha\beta}X_{1\alpha\beta}X_{2\beta\alpha}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\Xi_\beta\tau}\right]\right]\frac{1}{z+\Xi_{\alpha\beta}}\\
&= \int đ\omega\left[2\pi\delta(\omega+\Xi_{\alpha\beta})\frac{1}{\mathcal{Z}}\sum_{\alpha\beta}X_{1\alpha\beta}X_{2\beta\alpha}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\Xi_\beta\tau}\right]\right]\frac{1}{z-\omega}\\
&= \int đ\omega\frac{A(\omega)}{z-\omega}\\
&= \int đ\omega C_0(z,\omega)A(\omega)
\end{aligned}$$

Where $C_0(z,\omega)$ is the greens function of a free particle with frequency ω and we have now defined the spectral density:

$$A(\omega)=\delta(\omega+\Xi_{\alpha\beta})\frac{2\pi}{\mathcal{Z}}\sum_{\alpha\beta}X_{1\alpha\beta}X_{2\beta\alpha}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\beta\Xi_\beta}\right]$$

This means we can always think of the correlation function in terms of a free particle correlation of various energies. Should $A(\omega)$ turn out to contain a singularity we will understand this to mean the free particle state of definite particle number is still an exact eigenstate of the hamiltonian, ie that we are looking at a free theory. In an interacting theory we know the hamiltonian and number operator do not commute so we should expect to see the singularities blurred our as the energy eigenstates of our interacting theory will mixtures of single particle states.

One can derive another particularly useful relation:

$$\begin{aligned}
-2\Im C^+ &= -2\Im\left[\lim_{\eta\to 0}\frac{1}{\mathcal{Z}}\sum_{\alpha\beta}\frac{X_{1,\alpha\beta}X_{2\beta\alpha}}{\omega+\Xi_{\alpha\beta}+i\eta}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\beta\Xi_\beta}\right]\right]\\
&= -\frac{2}{\mathcal{Z}}\sum_{\alpha\beta}X_{1,\alpha\beta}X_{2\beta\alpha}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\beta\Xi_\beta}\right]\Im\left[\lim_{\eta\to 0}\frac{1}{\omega+\Xi_{\alpha\beta}+i\eta}\right]\\
&= -\frac{2}{\mathcal{Z}}\sum_{\alpha\beta}X_{1,\alpha\beta}X_{2\beta\alpha}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\beta\Xi_\beta}\right]\Im\left[-i\pi\delta(\omega+\Xi_{\alpha\beta})+\frac{1}{\omega+\Xi_{\alpha\beta}}\right]\\
&= \frac{2\pi}{\mathcal{Z}}\sum_{\alpha\beta}X_{1,\alpha\beta}X_{2\beta\alpha}\left[e^{-\beta\Xi_\alpha}-\zeta_X e^{-\beta\Xi_\beta}\right]\delta(\omega+\Xi_{\alpha\beta})\\
&= A(\omega)
\end{aligned}$$

Where we've used the indentity $\lim_{\delta\to 0}\frac{1}{x+i\delta}=-i\pi\delta(x)+\frac{1}{x}$. This relation makes it easy for use to derive the spectral density from the corelation function so we can explore it's properties in known scenarios.

Example: Unstable particles A key feature of unstable particles is an (negative) imaginary contribution to the self energy (if it is positive you spontaneously gain particles not lose them), additionally if particles are unstable single particle states are clearly not eigenstates of the hamiltonian so something interesting must happen ot the spectral density. Note the poles do not lie on the real line so all correlation functions coincide, this is because the particles are unstable so the boundary conditions in arbitrarily distant times have no meaning. We take our self energy to be of the form $\Sigma - i\Gamma$ so the Greens function is:

$$C(z)=\frac{1}{z-\omega-\Sigma+i\Gamma}$$

$$
\begin{aligned}
\Rightarrow A(\omega) &= -2\Im\left[\frac{1}{1-\omega-\Sigma+i\Gamma}\right] \\
&= -2\Im\left[\frac{1-\omega-\Sigma-i\Gamma}{(1-\omega-\Sigma)^2+\Gamma^2}\right] \\
&= \frac{2\Gamma}{(1-\omega-\Sigma)^2+\Gamma^2}
\end{aligned}
$$

So we see the singularity of free stable particles has been smeared out into a lorentzian where the real part of the self energy shifts the peak and the imaginary part dictates the width (It is for this reason Γ is often called "decay width").

For weakly interacting theories it is typical to see this broadening of the singularities, for strongly interacting theories there is no need for the spectral density to even remain peaked.

One last useful observation: Consider the corelation function between fields themselves:

$$
C^+(t) = -i\theta(t)\left\langle \left[c(t), c^\dagger(0)\right]_{\zeta_c}\right\rangle = \lim_{\eta\to 0}\frac{1}{\mathcal{Z}}\sum_{\alpha\beta}\frac{c_{\alpha\beta}c^\dagger_{\beta\alpha}}{\omega+\Xi_{\alpha\beta}+i\eta}\left[e^{-\beta\Xi_\alpha}-\zeta_c e^{-\beta\Xi_\beta}\right]
$$

$$
\begin{aligned}
\Rightarrow \int đ\omega A(\omega) &= \int đ\omega \frac{2\pi}{\mathcal{Z}}\sum_{\alpha\beta} c_{\alpha\beta}c^\dagger_{\beta\alpha}\left[e^{-\beta\Xi_\alpha}-\zeta_c e^{-\beta\Xi_\beta}\right]\delta(\omega+\Xi_{\alpha\beta}) \\
&= \left[\langle cc^\dagger\rangle - \zeta_c\langle c^\dagger c\rangle\right] \\
&= \left\langle \left[c, c^\dagger\right]_{\zeta_c}\right\rangle \\
&= 1
\end{aligned}
$$

So we have a normalization for the canonical spectral density: $\int đ\omega A(\omega) = 1$ Another useful fact is you can see when the spectral density is expressed as a sum every term is non-negative. This is obvious for fermions and slightly less obvious for bosons but note for $\omega > 0$ we have $\Xi_{\alpha\beta} = \Xi_\alpha - \Xi_\beta < 0$ so with $\beta > 0$ we have $e^{-\beta\Xi_\alpha} - e^{\beta\Xi_\beta} > 0$ and indeed every term is positive. So for fermions, and bosons of positive energy and non-negative temperature the spectral density is a positive definite and normalized distribution over the free particle greens functions which allows us to interpret it as a sort of probability distribution. Indeed for our unstable particle case it does genuinely describe the probability distribution for the energy of the decay event. Really we should think of the spectral density as like a probability distribution describing the energy which may be absorbed by the environment (in the event of a decay or some such). It becomes negative for bosons of negative temperature precisely because in negative temperature systems the endowment tends to emit more than it absorbs that is you expect processes to result in spontaneous formation of new excitations (ie the gain in a laser).