(1)

3.6) bosonization

in second quantization:

$$H = \sum_{s=R,L} \sum_q \left(a^\dagger_{sq} \sigma_s U_F q a_{sq} + g_4 \hat{\rho}_{sq} \hat{\rho}_{s-q} + g_2 \hat{\rho}_{sq} \rho_{s,-} \right.$$

In coordinate space:

$$S_0(\psi) = \sum_{s=\pm 1} \int dx \, d\tau \, \psi^\dagger_s(-i s U_F \partial_x + \partial_\tau)\psi_s \text{ ; } \text{her}$$

$$\psi^\dagger_+(q) = \psi^\dagger_{K_F+q} \text{ ; } \psi^\dagger_-(q) = \psi^\dagger_{-K_F+q} \text{ ; } \quad \text{---} \quad |q| << K_F \text{ assum}$$

in the vicinity of fermi points, ψ is like wave pack with $\pm K_F$ velocity. so,

$$\psi(x) = \psi_L \cdot e^{-ikx} + \psi_R \cdot e^{ikx}$$

so,

$$S_0(\psi) = \int d^2x \left(\psi^\dagger (\sigma_0 \partial_{x_0} - i \sigma_3 \partial_{x_1}) \psi \right) =$$

$$= \int d^2x \left(\bar{\psi} (\sigma_1 \partial_{x_0} - \sigma_2 \partial_{x_1}) \psi \right.$$

set $U_F = 1$; $\psi = (\psi_R, \psi_L)^T$; $X = (X_0, X_1) = (\tau, x)$

let $\psi^\dagger \rightarrow \bar{\psi} = \psi^\dagger \sigma_1$

let's introduce

$$\gamma^0 = \sigma_1 ; \gamma^1 = \sigma_2 ; \gamma^5 = -\sigma_3$$

$$S_0(\psi) = \int d^2x \, \bar{\psi} (\partial_\mu \gamma^\mu) \psi$$

remember: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu}$

Symmetries

$$\bar{\Psi} \to e^{-i\varphi_\nu}\Psi$$

$$\Psi \to e^{i\varphi_\nu}\Psi \to \& \Psi + \boxed{(i\delta\varphi_\nu)}\Psi$$

$F_{i,a}$

by noether theorem:

$$j^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)}\Psi = \bar{\Psi}\gamma_\mu\Psi$$ — noether current.

Vector current: under rotation: space-time:

$$x^\mu \to (R \cdot X)^\mu$$

we know that:

$$j^\mu \to (R \cdot j)^\mu$$

j_μ — Vector current

---

components of vector:

$$j^0 = \bar{\Psi}\gamma_0\Psi = \Psi^+(\gamma_0)^2\Psi = \Psi_R^+\Psi_R + \Psi_L^+\Psi_L \equiv \rho$$

density

$$j^1 = \bar{\Psi}\gamma_1\Psi = -\bar{\Psi}\sigma_1\sigma_2\Psi = -i\,\Psi^+\sigma_3\Psi = -i(\Psi_R^+\Psi_R - \Psi_L^+\Psi_L) =$$

$$= -ij$$

current

$$\partial_\mu j^\mu = 0$$

So, $-i\,\partial_\mu j^\mu = i\partial_t\rho + \partial_x j = 0 \to$ conservation of particle current in imaginary time those are in grassman fields. then:

$$\langle\psi| a_R^+ a_R + a_L^+ a_L |\psi\rangle = \psi_L^+ \psi_L + \psi_R^+ \psi_R$$

so, this is really density operator representation. and because continuity equation holds in both representation form of j should be automaticly

atother symmetry:

$$\psi \to e^{i\varphi_a \gamma^5}\psi \quad \text{and} \quad \bar\psi \to \bar\psi e^{i\varphi_a \gamma^5}$$

$$\psi \to \psi + (i\varphi_a)\gamma^5\psi \Rightarrow \boxed{j_a^\mu = i\bar\psi \gamma^\mu \gamma^5 \psi = \varepsilon^{\mu\nu}\gamma^\mu\psi}$$

check that hamiltonian stays the same:

$$\bar\psi e^{i\varphi_a\gamma^5} \partial_\mu \gamma^\mu \left(e^{i\varphi_a\gamma^5}\right)\psi = \bar\psi$$

$e^{i\varphi\gamma^5} = \cos\varphi I + i\sin\varphi \gamma^5$ and $\{\gamma^5, \gamma^\mu\} = 0$

so, this is because $e^{i\varphi_a\gamma^5}\gamma^\mu e^{i\varphi_a\gamma^5} = \gamma^\mu$

let e_2 ∥ perpendicular to the plane.

at least I expicitely check for $\gamma^0 = \sigma_1$ and $\gamma_1 = -\sigma_2$

at first it might be suspicious that both of ~~the~~ signs are positive, but let's check.

$$\psi \to e^{i\varphi_a\gamma^5}\psi = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi}\end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = \begin{pmatrix} e^{-i\varphi}\psi_L \\ e^{i\varphi}\psi_R\end{pmatrix}$$

$$\bar\psi \to \bar\psi e^{i\varphi_a\gamma_5} = \begin{pmatrix}\psi_R^+ & \psi_L^+\end{pmatrix}\begin{pmatrix} e^{-i\varphi_a} & 0 \\ 0 & e^{i\varphi_a}\end{pmatrix} = \begin{pmatrix}\psi_R^+ e^{-i\varphi_a} \\ \psi_L^+ e^{i\varphi_a}\end{pmatrix}$$

(from all the arguments, we can say that so,) so, we can also say that:

$$j_a = e_\epsilon \times j_v$$

where je -- axial ~~vector~~ current;

physicaly first symmetry gives us

$$a: i\partial_0 \rho + \partial_x j = i\left(\frac{\partial n_R}{\partial t} + \frac{\partial n_L}{\partial t}\right) + \frac{\partial n_R}{\partial x} - \frac{\partial n_L}{\partial x} = 0$$

$$V: i\partial_0 j + \partial_x \rho = i\left(\frac{\partial n_R}{\partial x} - \frac{\partial n_L}{\partial x}\right) + \frac{\partial n_R}{\partial t} + \frac{\partial n_L}{\partial t}$$

(if we multiply first on i, it's easy to check that)

$$\left(i\left(\frac{\partial n_R}{\partial t}\right) + \frac{\partial n_R}{\partial x}\right) + \left(i\left(\frac{\partial n_L}{\partial t}\right) - \frac{\partial n_L}{\partial x}\right) = 0$$

if we sum and minus a and v equations we will get continuity for n_R and n_L seperately. so, in classical physics they move indipendently from each other. however quantum fluctuations might destroy this "separation", this phenomena is known as chiral anomaly

Interaction

we should add

$$S_{int}|\psi) = \frac{1}{2}\sum_s \int dx\, d\beta \left(g_2 \hat{\rho}_s \hat{\rho}_{\bar{s}} + g_4 \hat{\rho}_s \hat{\rho}_s\right)$$

$$\hat{\rho}_s = \psi_s^+ \psi_s \rightarrow \psi_s^+ \psi_s \text{ in each } v, a \text{ symmetry}$$

as we checked in the last chapters we should describe model by little excitation.

it conserves the symmetries

okay, so if we take $\hat{\varphi}_{(x)} \to \varphi_i$ and $\theta(x) \to \hat{\theta}_i = \pi \Sigma \rho_i$

like creation operator

36.2. one-dimensional ~~Electron~~ gas (bosonic theory)

~~let~~ as we've already seen in ~~Holstein-Primakoff~~ 2.2.4 transformations, that small excitations should be described by bosons.

so, let's consider bosons:

$$b(x) = \rho_{(x)}^{1/2} e^{i\varphi} ; b(x) = e^{-i\varphi} \rho_{(x)}^{1/2}$$

$\hat{\rho}_{(x)}$ _ density; $\hat{\varphi}$ _ Phase operator.

$[b, b^+] \to [\hat{\rho}, \hat{\varphi}]$ is a canonical transformation, so

$$[\hat{\varphi}_{(x)}, \hat{\rho}_{(y)}] = -i\delta(x-y)$$

let's see why do we require it:

let's consider $\hat{\rho} \to \hat{\rho} + K_F/\pi$ $\to$ to consider fluctuations around Fermi space

$$\hat{b} = \hat{\rho}^{1/2} \exp(i\varphi) ; b^+ = \exp(-i\varphi) \hat{\rho}^{1/2}$$

$$[b, b^+] = [\hat{\rho}^{1/2} e^{i\hat{\varphi}}, e^{-i\hat{\varphi}} \rho^{1/2}] = \hat{\rho} - e^{-i\hat{\varphi}} \hat{\rho} e^{i\hat{\varphi}} =$$

$$= \cancel{\hat{\rho}} - \cancel{\hat{\rho}} + i[\hat{\varphi}, \hat{\rho}] - \dots = 1 \Rightarrow [\varphi, \rho] = -i$$

our goal is to turn from fermionic fields integral to real value integrals.

in the above formalism, $\hat{\varphi}$ is the „momentum conjugate" of $\hat{\rho}$, so we can imagine it as a unit charge translation operator.

$\exp(i\hat{\varphi})$

well, now let's find something that would be valid to exchange for holding fermion statistic

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let's recall Jordan wigner transformation:

1) $c_i^+ = a_i^+ \exp(-\pi i \sum a_j^+ a_j)$

$c_i = \exp(\pi i \sum a_j^+ a_j)\, a_i$

$|\uparrow\rangle = 1 = a^+|0\rangle, \quad |0\rangle = |\downarrow\rangle = a|1\rangle$

problem is that $\{a_i, a_i^+\} = 1$

and $[a_i, a_j^+] = 0$

but $\{c_i^+, c_j^+\} = \delta_{ij}, \quad \{c_i, c_j\} = 0$

and we can consider them as a fermions.

$\boxed{e^A e^B = e^B e^A e^{[A,B]}}$

by same logic we define operator:

$$\hat{\theta}(x) = \pi \int_{-\infty}^{x} dx' \hat{\rho}(x')$$

we can say that

$$[\varphi(x), \theta(y)] = -i\pi\,\theta(x-y)$$

Let's observe:

$$[e^{is\hat{\theta}(x)} e^{i\hat{\varphi}(x)}, e^{is'\hat{\theta}(x')} e^{i\hat{\varphi}(x')}]$$

$$e^{is\hat{\theta}(x)} e^{i\hat{\varphi}(x)} e^{is'\hat{\theta}(x')} e^{i\hat{\varphi}(x')} = \underbrace{e^{is\theta(x)} \cdot e^{is'\theta(x')}}_{A(x)} \cdot \underbrace{e^{i\varphi(x)} e^{-i\pi s'\theta(x'-x)} e^{i\varphi(x')}}_{A(x')}$$

by same logic we would treat second term and we would take $s = s' = 1$;

$$[\quad]_+ \sim e^{-i\pi s'\theta(x'-x)} + e^{-i\pi s(x-x')} = 0$$

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okay, so if we take $\hat{\varphi}(x) \to \varphi_i$ and $\theta(x) \to \hat{\theta}_i = \pi \sum \rho_i$

like creation operator

so, in principle let's consider continuum to discrete case.

our $c_i = \frac{1}{\sqrt{2}} : \exp(is\hat{\theta}_i) \exp(i\hat{\varphi}_i)$

and we can show that $[c_i, c_i^+]_+ = \delta_{ij}$ holds exactly like it's in discrete case:

$$[c_i, c_i^+] = \exp(i\varphi_i - i\varphi_i) = 1$$

when $i \neq j$: $[c_i, c_j^+] = \exp(-i\varphi_i) \cdot \exp\left(\sum_i^{j-1} \rho_i\right) \exp(i\varphi_j) +$

$$+ \exp(i\varphi_j) \exp\left(\sum_i^{j-1} \rho_i\right) \exp(-i\varphi_i) =$$

$$= \left(\exp(-i\varphi_i) \exp(i\varphi_j) - \exp(i\varphi_i) \exp(-i\varphi_j)\right) \cdot \exp(\Sigma) = 0$$

this is because $\{a_i, 1 - 2a_i^+ a_i\} = 0$

we should have some divergent constant, because in reality we are in continous space. let it be Γ;

so, $c(x) = \Gamma \sum e^{i(s)(\pi k_F + \theta(x))} e^{i\varphi(x)}$

right and left going parts;

slowly varying

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A little about spinor transformation

at first, let's observe, that we are working with $\bar{\Psi}$ and not ψ^+, because: $D\bar{\psi} D\psi \to S^+ S\, D\psi^+ D\psi$

and $S^+ S \neq 1$; but if we work with

$$\bar{\psi} = \psi^+ \gamma^0 = \psi^+ \sigma^1$$

$$\bar{\Psi} \to S^{-1} \bar{\Psi} S^{-1}(\Lambda)$$

$$D\bar{\psi} D\psi \to D\bar{\psi}\, D\psi\, S^{-1}(\Lambda)\, S(\Lambda)$$

but let's find S first, it should be such that action stays the same.

$$\bar{\Psi}\, \partial_\mu \gamma^\mu \psi \to \bar{\psi} S^{-1} \gamma^\mu S\, \partial_{\mu'} \bar{\Psi}$$

$$\text{so, } \bar{\Psi}\, \partial_\mu \gamma^\mu \psi \to \bar{\psi} S^{-1} \gamma^\mu S (\Lambda)^{-1\,\nu}_{\mu'}\, \partial_\nu \bar{\psi}$$

$$S^{-1} \gamma^\mu (\Lambda^{-1})^\nu{}_\mu S = \gamma^\nu,$$

$$\text{so, } \quad S^{-1} \gamma^\mu S = \gamma^\nu \Lambda^\mu{}_\nu$$

if we check,

$$S(\theta) = \exp\left(\frac{1}{2}\theta_3\right)$$

satisfy this condition

36.2. bosonic theory

so, if we come back to our definition:

$$c(x) = \Gamma \sum_{s=\pm 1} e^{is(k_F x + \theta(x))} e^{i\varphi(x)}$$

$\psi_s \to e^{i\varphi_\nu}\psi_s$ and $\psi_s = e^{is\varphi_a}\psi_s$ of course

still should be symmetry.

from the $c(x)$, it's easy to see that $a: \theta(x) \to \theta(x) + \varphi_a$; and $V: \varphi(x) + \varphi_v$, it's intuitive because $\exp(i\varphi(x))$ is fermion creation operator and there should be connection with it and $c(x)$. this means that our lagrangian should only consist terms with derivatives.

On the other hand we know that $\rho(x) = \frac{\partial_x \theta}{\pi}$. because of screening we are assuming that $U \sim \rho^2$ because coulomb interaction should be screened. besides, we already know from second-quantization method that $H \sim \rho^2$

but there's problem: if we only take $S = \frac{c}{2}\int d\tau\, dx \cdot (\partial_x \theta)^2$, action wouldn't be lorentz invariant. that's why we should choose:

$$S = \frac{c}{2}\int d\tau\, dx\left((\partial_x\theta)^2 + (\partial_\tau \theta)^2\right)$$

we should understand c coefficient. essentially we should rely on the fact that

$$C(x,\tau) = \langle (\psi_+^\dagger \psi_-)(x,\tau)\, \psi_-^\dagger \psi_+(0,0)\rangle$$

should be same on the both: fermionic and bosonic representation. in the fermionic case: $Z_\pm = \int D\psi \exp(-S_\pm(\psi))$, where $S_\pm(\psi) = \int dx\, d\tau\, \bar\psi(\partial_\tau \mp i\partial_x)\psi$, $\Rightarrow \langle \bar\psi_k \psi_k \rangle_\pm = -\frac{T}{L}\sum_{p,\omega_n} \frac{1}{i\omega_n \pm p} e^{-ipx + i\omega_n \tau}$

$$C(x,t) = T\sum \frac{e^{ikx - i\omega t}}{i\omega_n + p} \int \sum \frac{e^{ikx - i\omega t}}{i\omega_n - p}$$

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so, result is: $\frac{1}{(2\pi)^2}\cdot\frac{1}{x^2+\tau^2}$

for bosonic representation:

$$C(x,\tau) = \Gamma^2 \langle e^{2i(\theta(x,\tau)-\theta(0,0))} \rangle =$$

$$= \Gamma^2 \cdot \left(\frac{a^2}{x^2+\tau^2}\right)^{\frac{c}{\pi}}$$

so, $c = \pi$ and $\Gamma = \frac{1}{2\pi a}$, where $a^{-1} = \frac{E_F}{v_F}$ - cutoff

anyways:

$$S(\theta) = \frac{1}{2\pi}\int d\tau\, dx\left((\partial_\tau\theta)^2 + (\partial_x \theta)^2\right)$$

~~so, $\Pi_\theta = \frac{\partial \mathcal{L}}{\partial \partial_\tau\theta} = \partial_\tau\theta$~~ let's go back to real

$t = -i\tau$ time.

$$S(\theta) = \frac{1}{2\pi}\int dt\, dx\left((\partial_t\theta)^2 - (\partial_x \theta)^2\right)$$

$$\Pi_\theta = \frac{\partial \mathcal{L}}{\partial(\partial_t\theta)} = \frac{1}{2\pi}\cancel{\int dt\, dx}\, \partial_t\theta$$

$\exp(-S) =$

$= \exp(iS_M)$

$$\text{so, } H = \frac{1}{2\pi}\left((\pi\Pi_\theta)^2 + (\partial_x\theta)^2\right)$$

if we derivade $[\theta(y), \varphi(x)] = i\pi\,\theta(y-x)$,

we will take $[\partial_x\varphi, \theta_y] = i\pi\,\delta(y-x)$

on the other hand

$$[\Pi_\theta(x), \theta(y)] = -i\,\delta(x-y)$$

$$\text{so, } \frac{1}{\pi}\partial_t\theta = -\frac{1}{\pi}\partial_x\varphi$$

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so: $S_m(\theta,\varphi) = \frac{1}{2\pi}\int dt\,dx(\Pi_\theta \partial_t\theta - H) = \frac{1}{2\pi}\int dt\,dx\{-2\partial_x\varphi\partial_t\theta - (\partial_x\varphi)^2 - (\partial_x\theta)^2\}$

and if we return in imaginary time, it is:

$$S(\theta,\varphi) = \frac{1}{2\pi}\int dx\,d\tau\left((\partial_x\theta)^2 + (\partial_x\varphi)^2 + 2i\,\partial_\tau\theta\,\partial_x\varphi\right)$$

from here: j^μ if we consider $\varphi \to \varphi + \varphi_r$, we will get

$$j^0 = 0;\; j^1 = -\left(\frac{\partial_x\varphi}{\pi} + i\frac{\partial_\tau\theta}{\pi}\right) = i\left(-i\frac{\partial_x\varphi}{\pi} + \partial_\tau\theta\right)$$

$$\partial_\mu j^\mu = 0 \Rightarrow \partial_x\left(\frac{\partial_\tau\theta}{\pi} - i\frac{\partial_x\varphi}{\pi}\right) = 0$$

so, $\partial_\tau\underbrace{\left(\frac{\partial_x\theta}{\pi}\right)}_{\rho} - i\,\partial_x\frac{\varphi}{\pi} = 0 \to$ from fermionic

$$i\,\partial_\tau\rho + \partial_x j = 0,$$

we can interpret $\frac{\partial_x\varphi}{\pi} = j$;

so, $\begin{cases} n_R + n_L = \frac{\partial_x\theta}{\pi} \\ n_R - n_L = \frac{\partial_x\varphi}{\pi} \end{cases} \Rightarrow \begin{aligned} n_R &= \frac{\partial_x(\theta+\varphi)}{2\pi} \\ n_L &= \frac{\partial_x\theta - \partial_x\varphi}{2\pi} \end{aligned}$

So, we can easily add interaction term.

$$S_{int} = \frac{1}{8\pi^2}\sum_s \int dx\,d\tau\; g_2\cdot\partial_x(\theta+s\varphi)\,\partial_x(\theta-s\varphi) + $$

$$+ g_4\,\partial_x(\theta+s\varphi)\,\partial_x(\theta+s\varphi)\Big) =$$

$$= \frac{1}{4\pi^2}\int dx\,d\tau\left(\cancel{g_2}(\partial_x\theta)^2(g_2+g_4) + (g_4-g_2)(\partial_x\varphi)^2\right)$$

So, $S = \frac{1}{4\pi^2}\int dx\,d\tau\Big((2\pi v_F + g_4 + g_2)(\partial_x\theta)^2 + (2\pi v_F + g_4 - g_2)\cdot$

$$\cdot(\partial_x\varphi)^2 + 2i\,\partial_\tau\theta\,\partial_x\varphi\Big)$$