Non-Linear Sigma Models

Often when a symmetry is spontaneously broken you are left with goldstone modes which are fields taking values within the symmetry group. This is because why the symmetry has broken and thus the system has "chosen" an specific vacuum this vacuum can still in principle shift sliding along the manifold which defines the minima of the free energy. Fluctuations of the vacuums state can if large enough reinstate the broken symmetry. Theories of this variety are called non-linear sigma models and the nature of non-abelian group multiplication induces nonlinearities. We can imagine a general action for such a goldstone mode as:

$$S[g] = \frac{1}{\lambda}\int d^d x \,\mathrm{Tr}\left(\nabla g \nabla g^{-1}\right)$$

Where the relevant functional measure for computing the partition function is $\int Dg = \prod_i \int d\mu_i$ where $d\mu$ is the haar measure over the entire group manifold.

Now as usual we immediately transition to the Lie Algebra representation for ease:

$$g = \exp(W) = \exp(i\pi^a T_a) \qquad \text{with normalization} \qquad \mathrm{Tr}(T_a T_b) = c\delta_{ab}$$

$$\begin{aligned}
\Rightarrow S[g] &= \frac{1}{\lambda}\int d^d x \,\mathrm{Tr}\left(\nabla\left[1 + W + W^2/2 + \mathcal{O}(W^3)\right]\nabla\left[1 - W + W^2/2\right] + \mathcal{O}(W^3)\right) \\
&= \frac{1}{\lambda}\int d^d x \,\mathrm{Tr}\left(\left[\nabla W + W\nabla W/2 + \nabla W W/2 + \mathcal{O}(W^3)\right]\left[-\nabla W + W\nabla W/2 + \nabla W W/2 + \mathcal{O}(W^3)\right]\right) \\
&= \frac{1}{\lambda}\int d^d x \,\mathrm{Tr}\left(\left[-\left(\nabla W \nabla W\right) + \mathcal{O}(W^3)\right]\right) \\
&= -\frac{1}{\lambda}\int d^d x \,\mathrm{Tr}\left(\nabla W \nabla W\right) + \mathcal{O}(W^3) \\
&= -\frac{1}{\lambda}\int d^d x \nabla\pi^a \nabla\pi^b \,\mathrm{Tr}\left(T_a T_b\right) + \mathcal{O}(\pi^3) \\
&= -\frac{c}{\lambda}\int d^d x \nabla\pi^a \nabla\pi_a + \mathcal{O}(\pi^3)
\end{aligned}$$

Now we've arrived at a familiar looking free field action. Performing the usual fourier expansion:

$$\begin{aligned}
S[g] &= -\frac{c}{\lambda}\int d^d x \nabla\left(\int \text{đ}p \pi^a_p e^{-ip_\mu x^\mu}\right)\nabla\left(\int \text{đ}p' \pi^a_{p'} e^{-ip'_\mu x^\mu}\right) + \mathcal{O}(\pi^3) \\
&= -\frac{c}{\lambda}\int \text{đ}p \int \text{đ}p' \int d^d x \left(-ip_\mu \pi^a_p e^{-ip_\mu x^\mu}\right)\left(-ip'^\mu \pi^a_{p'} e^{-ip'_\mu x^\mu}\right) + \mathcal{O}(\pi^3) \\
&= \frac{c}{\lambda}\int \text{đ}p \int \text{đ}p' p^2 \pi^a_p \pi^a_{p'} \int d^d x e^{-i(p+p')_\mu x^\mu} + \mathcal{O}(\pi^3) \\
&= \frac{c}{\lambda}\int \text{đ}p \int \text{đ}p' p^2 \pi^a_p \pi^a_{p'} (2\pi)^d \delta^d(p+p') + \mathcal{O}(\pi^3) \\
&= \frac{1}{2}\int \text{đ}p \pi^a_p \left(\frac{2cp^2}{\lambda}\right)\pi^a_{-p} + \mathcal{O}(\pi^3) \\
&= \frac{1}{2}\int \text{đ}p \pi^a_p \Pi^{-1}_p \pi^a_{-p} + \mathcal{O}(\pi^3)
\end{aligned}$$

Where we've defined the propagator: $\Pi_p = \frac{\lambda}{2cp^2}$.

Now we'd like to do a little momentum space renormalization so lets separate our fields in $g = (g_s)(g_f)$ slow and fast components and expand the action appropriately:

$$
\begin{aligned}
S[g] &= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left(\nabla(g_s g_f)\nabla(g_s g_f)^{-1}\right) \\
&= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left(\nabla(g_s g_f)\nabla(g_f^{-1} g_s^{-1})\right) \\
&= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left((g_s\nabla g_f + \nabla g_s g_f)(g_f^{-1}\nabla g_s^{-1} + \nabla g_f^{-1} g_s^{-1})\right) \\
&= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left(g_s\nabla g_f g_f^{-1}\nabla g_s^{-1} + \nabla g_s g_f g_f^{-1}\nabla g_s^{-1} + g_s\nabla g_f\nabla g_f^{-1} g_s^{-1} + \nabla g_s g_f\nabla g_f^{-1} g_s^{-1}\right) \\
&= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left(\nabla g_s^{-1} g_s\nabla g_f g_f^{-1} + \nabla g_s\nabla g_s^{-1} + \nabla g_f\nabla g_f^{-1} + g_s^{-1}\nabla g_s g_f\nabla g_f^{-1}\right) \\
&= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left(\nabla g_s\nabla g_s^{-1}\right) + \mathrm{Tr}\left(\nabla g_f\nabla g_f^{-1}\right) + \mathrm{Tr}\left((g_s^{1}\nabla g_s)^{-1}(g_f\nabla g_f^{-1})^{-1} + g_s^{-1}\nabla g_s g_f\nabla g_f^{-1}\right) \\
&= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left(\nabla g_s\nabla g_s^{-1}\right) + \mathrm{Tr}\left(\nabla g_f\nabla g_f^{-1}\right) + \mathrm{Tr}\left(g_f\nabla g_f^{-1} g_s^{-1}\nabla g_s + g_s^{-1}\nabla g_s g_f\nabla g_f^{-1}\right) \\
&= \frac{1}{\lambda}\int d^dx \,\mathrm{Tr}\left(\nabla g_s\nabla g_s^{-1}\right) + \mathrm{Tr}\left(\nabla g_f\nabla g_f^{-1}\right) + 2\,\mathrm{Tr}\left(g_s^{-1}\nabla g_s g_f\nabla g_f^{-1}\right) \\
&= S[g_s] + S[g_f] + S[g_s, g_f]
\end{aligned}
$$

So we see the coupling actions is:

$$
S[g_s, g_f] = \frac{2}{\lambda}\int d^dx \left(g_s^{-1}\nabla g_s g_f\nabla g_f^{-1}\right)
$$

Now we of course want to integrate over the fast degrees of freedom so we expand in the lie algebra $g_f = \exp(W) = 1 + W + W^2/2 + \mathcal{O}(W^3)$. For simplicity we'll also denote $g_s^{-1}\nabla_\mu g_s = \Phi_\mu$:

$$
\begin{aligned}
S[g_s, g_f] &= \frac{2}{\lambda}\int d^dx \,\mathrm{Tr}\left(\Phi_\mu(1 + W + W^2/2 + \mathcal{O}(W^3))\nabla^\mu(1 - W + W^2/2 + \mathcal{O}(W^3))\right) \\
&= \frac{2}{\lambda}\int d^dx \,\mathrm{Tr}\left(\Phi_\mu(1 + W + W^2/2 + \mathcal{O}(W^3))(-\nabla^\mu W + W\nabla^\mu W/2 + \nabla W W/w + \mathcal{O}(W^3))\right) \\
&= \frac{2}{\lambda}\int d^dx \,\mathrm{Tr}\left(-\Phi_\mu\nabla^\mu W - \Phi_\mu W\nabla^\mu W + \Phi_\mu W\nabla^\mu W/2 + \Phi_\mu\nabla^\mu W W/2 + \mathcal{O}(W^3)\right) \\
&= \frac{2}{\lambda}\int d^dx \,\mathrm{Tr}\left(-\Phi_\mu\nabla^\mu W - \Phi_\mu W\nabla^\mu W/2 + \Phi_\mu\nabla^\mu W W/2 + \mathcal{O}(W^3)\right) \\
&= \frac{2}{\lambda}\int d^dx - \mathrm{Tr}\left(\Phi_\mu\nabla^\mu W\right) + \frac{1}{2}\mathrm{Tr}\left(\Phi_\mu\left[\nabla^\mu W, W\right]\right) + \mathcal{O}(W^3)
\end{aligned}
$$

Now here we've stopped at quadratic order because we are only interesting in the 1 loop correction to the propagator and higher powers necessarily involve more loops. As another note the first order effect must be exactly 0 since to first order there is no 1 loop diagram as the 3 point vertex from the first order term involves only 1 fast mode particle so you need 2 vertices to close the loop and thus it's a second order effect. As such we turn our attention in entirety to the quadratic term performing the usual fourier analysis:

$$\begin{aligned}
S[g_s, g_f] &\approx \frac{1}{\lambda}\int d^d x\,\mathrm{Tr}\left(\Phi_\mu\left[\nabla^\mu W, W\right]\right) \\
&= \frac{1}{\lambda}\int d^d x \int đqđpđp'\,\mathrm{Tr}\left(\Phi_{\mu,q}e^{-iq_\mu x^\mu}\left[\nabla^\mu W_p e^{-ip_\mu x^\mu}, W_{p'}e^{-ip'_\mu x^\mu}\right]\right) \\
&= \frac{1}{\lambda}\int d^d x \int đqđpđp'\,\mathrm{Tr}\left(\Phi_{\mu,q}e^{-iq_\mu x^\mu}\left[ip^\mu W_p e^{-ip_\mu x^\mu}, W_{p'}e^{-ip'_\mu x^\mu}\right]\right) \\
&= -\frac{i}{\lambda}\int d^d x \int đqđpđp'\,p^\mu\,\mathrm{Tr}\left(\Phi_{\mu,q}\left[W_p, W_{p'}\right]\right)e^{-i(q+p+p')_\mu x^\mu} \\
&= -\frac{i}{\lambda}\int đqđpđp'\,p^\mu\,\mathrm{Tr}\left(\Phi_{\mu,q}\left[W_p, W_{p'}\right]\right)(2\pi)^d\delta(q+p+p') \\
&= -\frac{i}{\lambda}\int đqđp\,p^\mu\,\mathrm{Tr}\left(\Phi_{\mu,q}\left[W_p, W_{-p-q}\right]\right) \\
&\approx -\frac{i}{\lambda}\int đqđp\,p^\mu\,\mathrm{Tr}\left(\Phi_{\mu,q}\left[W_p, W_{-p}\right]\right)
\end{aligned}$$

Where in the last step we've made the approximation that the momentum of the slow field is much less than the fast field such that $p + q \approx p$. Here Atland seems to assume the W's anticommute for reasons entirely opaque to me and declares the result:

$$S_2[g_s, g_f] \approx -\frac{2i}{\lambda}\int đqđp\,p^\mu\,\mathrm{Tr}\left(\Phi_{\mu,q}W_p W_{-p}\right)$$

To integrate W out of the partition function we expand the couping order by order:

$$\begin{aligned}
Z &= \int_0^\Lambda \mathcal{D}g\exp(-S[g]) \\
&= \int_0^\Lambda \mathcal{D}g\exp(-S[g_s] - S[g_s, W] - S[W]) \\
&= \int_0^\Lambda \mathcal{D}g\exp(-S[g_s])\left[1 - S[g_s, W] + \frac{1}{2}S^2[g_s, W] + \mathcal{O}(S^3[g_s, W])\right]\exp(-S[W]) \\
&= \int_0^{\Lambda/b} \mathcal{D}g\exp(-S[g_s])\left[1 - \langle S[g_s, W]\rangle_W + \frac{1}{2}\langle S^2[g_s, W]\rangle_W + \mathcal{O}(S^3[g_s, W])\right] \\
&= \int_0^{\Lambda/b} \mathcal{D}g\exp\left(-S[g_s] + \ln\left(1 - \langle S[g_s, W]\rangle_W + \frac{1}{2}\langle S^2[g_s, W]\rangle_W + \mathcal{O}(S^3[g_s, W])\right)\right) \\
&= \int_0^{\Lambda/b} \mathcal{D}g\exp\left(-S_{eff}[g]\right)
\end{aligned}$$

So we have the RG step:

$$S[g] \to S[g_s] - \ln\left(1 - \langle S[g_s, W]\rangle_W + \frac{1}{2}\langle S^2[g_s, W]\rangle_W + \mathcal{O}(S^3[g_s, W])\right) \approx S[g_s] - \frac{1}{2}\langle S^2[g_s, W]\rangle_W$$

Where in particular we are safe in throwing away all higher orders because each power of the coupling action has a derivative of the slow field so any term with more than two such derivatives is irrelevant. Additionally Atland here throws away the first order term for reasons not clear to me. So the relevant term to compute is:

$$\langle S^2[g_s, W]\rangle_W = -\frac{4}{\lambda^2}\int đqđq'đpđp'p^\mu p'^\nu\left\langle \mathrm{Tr}(\Phi_{\mu,q}W_p, W_{-p})\,\mathrm{Tr}(\Phi_{\nu,q'}W_{p'}, W_{-p'})\right\rangle$$

Now we need Wicks theorem, there are a few relevant variants:

$$\begin{aligned}
\langle \mathrm{Tr}(AW_p)\,\mathrm{Tr}(A'W_{p'})\rangle &= -\,\langle \pi^a_p \pi^{a'}_{p'}\rangle\,\mathrm{Tr}(AT_a)\,\mathrm{Tr}(A'T_{a'}) \\
&= -\delta_{p,-p'}\delta^{aa'}\Pi_p\,\mathrm{Tr}(AT_a)\,\mathrm{Tr}(A'T_{a'}) \\
&= -\delta_{p,-p'}\Pi_p A_{ji}A'_{lk}T^{ij}_a T^{a,kl} \\
&= -\delta_{p,-p'}\Pi_p A_{ji}A'_{lk}\,(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}) && \Leftarrow \mathrm{O(N)} \\
&= -\delta_{p,-p'}\Pi_p\left(A_{ji}A'_{ij}-A_{ji}A'_{ji}\right) \\
&= -\delta_{p,-p'}\Pi_p\left(\mathrm{Tr}(AA')-\mathrm{Tr}(AA'^T)\right)
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{Tr}(AW_pA'W_{p'})\rangle &= -\,\langle \pi^a_p \pi^{a'}_{p'}\rangle\,\mathrm{Tr}(AT_aA'T_{a'}) \\
&= -\delta_{p,-p'}\delta^{aa'}\Pi_p\,\mathrm{Tr}(AT_aA'T_{a'}) \\
&= -\delta_{p,-p'}\Pi_p A_{li}A'_{jk}T^{ij}_a T^{a,kl} \\
&= -\delta_{p,-p'}\Pi_p A_{li}A'_{jk}\,(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}) && \Leftarrow \mathrm{O(N)} \\
&= -\delta_{p,-p'}\Pi_p\left(A_{ii}A'_{jj}-A_{ji}A'_{ji}\right) \\
&= -\delta_{p,-p'}\Pi_p\left(\mathrm{Tr}(A)\,\mathrm{Tr}(A')-\mathrm{Tr}(AA'^T)\right)
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{Tr}(AW_pW_{p'}A')\rangle &= -\,\langle \pi^a_p \pi^{a'}_{p'}\rangle\,\mathrm{Tr}(AT_aT_{a'}A') \\
&= -\delta_{p,-p'}\delta^{aa'}\Pi_p\,\mathrm{Tr}(AT_aT_{a'}A') \\
&= -\delta_{p,-p'}\Pi_p A_{li}A'_{kl}T^{ij}_a T^{a,jk} \\
&= -\delta_{p,-p'}\Pi_p A_{li}A'_{kl}\,(\delta_{ik}\delta_{jj}-\delta_{ij}\delta_{jk}) && \Leftarrow \mathrm{O(N)} \\
&= -\delta_{p,-p'}\Pi_p A_{li}A'_{kl}\,(N\delta_{ik}-\delta_{ik}) \\
&= -\delta_{p,-p'}\Pi_p(N-1)A_{li}A'_{il} \\
&= -\delta_{p,-p'}\Pi_p(N-1)\,\mathrm{Tr}(AA')
\end{aligned}$$

Where at the indicated rows we have invoked a completeness relation unique to O(N) so this derivation is no longer general. Now let us specifically wonder about, we must sum over all possible pairs which will be quite long so lets do it in parts, first the (1-4)(2-3) pairing where keep in mind W's are hermitian (symmetric) and the A's are valued in the lie algebra and thus traceless:

$$\begin{aligned}
\langle \mathrm{Tr}(A_{q,p}W_{p+q}W_{-p})\,\mathrm{Tr}(A'_{q',p'}W_{p'+q'}W_{-p'})\rangle &> -\delta_{p,-p'}\Pi_p\left(\mathrm{Tr}(A_{q,p}W_{p+q}A'_{q',p'}W_{p'+q'})-\mathrm{Tr}(A_{q,p}W_{p+q}W_{p'+q'}A'^T_{q',p'})\right) \\
&= \delta_{q,-q'}\Pi_p\Pi_{p+q}\left(\cancel{\mathrm{Tr}(A_{q,p})\,\mathrm{Tr}(A'_{q',-p})}-\mathrm{Tr}(A_{q,p}A'^T_{q',-p})-(N-1)\,\mathrm{Tr}(A_{q,p}A'^T_{q',-p})\right) \\
&= \Pi_p\Pi_{p+q}\left(-N\,\mathrm{Tr}(A_{q,p}A'^T_{-q,-p})\right) \\
&= N\Pi_p\Pi_{p+q}\,\mathrm{Tr}(-A_{q,p}A'^T_{-q,-p})
\end{aligned}$$

Now let us do (1-3)(2-4):

$$\begin{aligned}
\langle \mathrm{Tr}(A_{q,p}W_{p+q}W_{-p})\,\mathrm{Tr}(A'_{q',p'}W_{p'+q'}W_{-p'})\rangle &= \langle \mathrm{Tr}(A_{q,p}W_{p+q}W_{-p})\,\mathrm{Tr}(W_{-p'}A'_{q',p'}W_{p'+q'})\rangle \\
&> -\delta_{p,-p'-q'}\Pi_p\left(\mathrm{Tr}(A_{q,p}W_{p+q}W_{-p'}A'_{q',p'})-\mathrm{Tr}(A_{q,p}W_{p+q}A'^T_{q',p'}W_{-p'})\right) \\
&= \delta_{p',p+q}\Pi_p\Pi_{p+q}\Big((N-1)\,\mathrm{Tr}(A_{q,p}A'_{-p-p',p'})-\cancel{\mathrm{Tr}(A_{q,p})\,\mathrm{Tr}(A'^T_{-p-p',p'})}+\mathrm{Tr}(A_{q,p}A'_{-p-p',p}
\end{aligned}$$

$$= N\Pi_p\Pi_{p+q}\,\mathrm{Tr}(A_{q,p}A'_{-q,p+q})$$

Putting these together we finally have:

$$\langle \mathrm{Tr}(A_{q,p}W_{p+q}W_{-p})\,\mathrm{Tr}(A'_{q',p'}W_{p'+q'}W_{-p'})\rangle = N\Pi_p\Pi_{p+q}\,\mathrm{Tr}(A_{q,p}A'_{-q,p+q}-A_{q,p}A'^T_{-q,-p})$$

Now for reasons I cannot deduce Atland claims the result is:

$$\langle \mathrm{Tr}(A_{q,p}W_{p+q}W_{-p})\,\mathrm{Tr}(A'_{q',p'}W_{p'+q'}W_{-p'})\rangle = (N-2)\Pi_p\Pi_{p+q}\,\mathrm{Tr}(A_{q,p}A'_{-q,p+q} - A_{q,p}A'_{-q,-p})$$

Now we can use this result directly to evaluate our adjustment to the action:

$$\begin{aligned}
\langle S^2[g_s,W]\rangle_W &= -\frac{4}{\lambda^2}\int đqđq'đpđp'p^\mu p'^\nu \,\langle \mathrm{Tr}(\Phi_{\mu,q}W_p,W_{-p})\,\mathrm{Tr}(\Phi_{\nu,q'}W_{p'},W_{-p'})\rangle \\
&= -\frac{4}{\lambda^2}\int đqđq'đpđp'(N-2)p^\mu p'^\nu \Pi_p^2\left[\delta_{p',p+q}\delta_{p,-p'-q'} - \delta_{q,-q'}\delta_{p,-p'}\right]\mathrm{Tr}\left(\Phi_{\mu,q}\Phi_{\nu,-q}\right) \\
&= \frac{4}{\lambda^2}\int đqđp(N-2)p^\mu p^\nu\frac{\lambda^2}{4p^4}\,\mathrm{Tr}\left(\Phi_{\mu,q}\Phi_{\nu,-q}\right) \\
&= (N-2)\int đqđp\frac{p^\mu p^\nu}{p^4}\,\mathrm{Tr}\left(\Phi_{\mu,q}\Phi_{\mu,-q}\right) \\
&= (N-2)\int đqđq'đp\frac{p^\mu p^\nu}{p^4}\,\mathrm{Tr}\left(\Phi_{\mu,q}\Phi_{\nu,q'}\right)(2\pi)\delta^d(q+q') \\
&= (N-2)\int đqđq'đp\int d^dx\frac{p^\mu p^\nu}{p^4}\,\mathrm{Tr}\left(\Phi_{\mu,q}\Phi_{\nu,q'}\right)e^{-i(q+q')_\mu x^\mu} \\
&= (N-2)\int đp\frac{p^\mu p^\nu}{p^4}\int d^dx\,\mathrm{Tr}\left(\Phi_\mu\Phi_\nu\right) \\
&= (N-2)\int đp\frac{p^\mu p^\nu}{p^4}\int d^dx\,\mathrm{Tr}\left(g_s^{-1}\nabla_\mu g_s g_s^{-1}\nabla_\nu g_s\right)
\end{aligned}$$

Noting the integrand is odd when $\mu \neq \nu$ and thus vanishes we can set $\mu = \nu$:

$$\begin{aligned}
\langle S^2[g_s,W]\rangle_W &= \frac{2(N-2)}{d}\int đp\frac{1}{p^2}\int d^dx\,\mathrm{Tr}\left(\nabla_\mu g_s\nabla^\mu g_s^{-1}\right) \\
&= \frac{N-2}{d(2\pi)^d}\int\frac{dp}{p^2}\lambda S[g] \\
&= \frac{(N-2)\Omega_{d-1}}{d(2\pi)^d}\int_{\Lambda/b}^{\Lambda} dp p^{d-3}\lambda S[g] \\
&= \frac{(N-2)\Omega_{d-1}\left(\Lambda^{d-2} - (\Lambda/b)^{d-2}\right)}{d(2\pi)^d}\lambda S[g] \\
&\approx \frac{(N-2)\ln(b)}{2\pi}\lambda S[g] \qquad\qquad d = 2+\epsilon
\end{aligned}$$

So at last we have:

$$S[g] \to \left(1 - \frac{(N-2)\ln(b)}{4\pi}\lambda\right)S[g_s] \to \left(1 - \frac{(N-2)\ln(b)}{4\pi}\lambda\right)b^\epsilon S[g]$$

Naturally we then absorb this factor into the coupling so:

$$\frac{1}{\lambda_r} = \left(1 - \frac{(N-2)\ln(b)}{4\pi}\lambda\right)b^\epsilon \Rightarrow \lambda_r = \lambda\left(1 - \frac{(N-2)\ln(b)}{4\pi}\lambda\right)^{-1}b^{-\epsilon}$$

Taking both $\lambda, \epsilon \ll 1$ we can write:

$$\lambda_r = \lambda + \left(\frac{(N-2)\ln(b)}{4\pi}\lambda - \epsilon\ln(b)\right)$$

From here we just read off the beta function:

$$\frac{d\lambda}{d\ln b} = -\epsilon\lambda + \frac{(N-2)\lambda^2}{4\pi} + HOT$$

So we see a transition at d=2 where above d=2 there is a fixed point at:

$$\lambda_c = \frac{4\pi\epsilon}{(N-2)}$$

Above this fixed point lambda flows to higher values and the system becomes strongly coupled, in this phase fluctuations of the goldstone modes become extremely energetically cheap and the symmetry is restored by thermal fluctuations. Below this fixed point lambda flows to lower values and the system decouples, goldstone mode excitations become prohibitively energetically expensive and the symmetry remains broken. Below d=2 no such fixed point exists so we see that O(N) theories of this kind develop a phase transition in dimensions greater than 2.