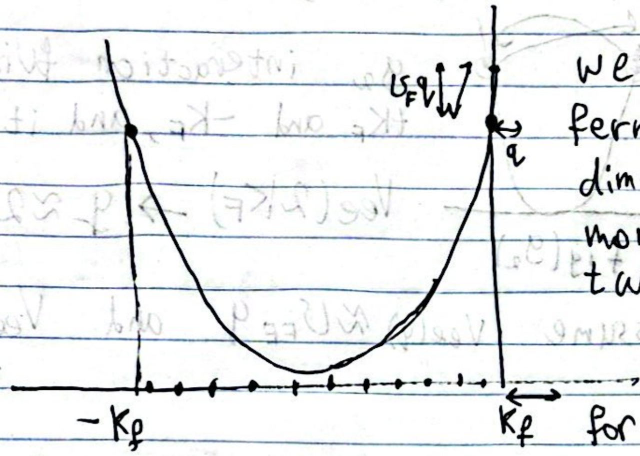
2.2.4.

$\hat{H} \sim \sum_{\mu\nu} a^+_\mu H_{\mu\nu} a_\nu$ - free hamiltonian

usually interaction is quartic.

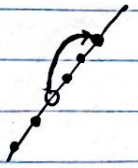
$$\hat{H} = \sum a^+_K \left(\frac{K^2}{2m} - E_F\right) a_K + \frac{1}{2L} \sum V_{(q)} a^+_{K-q} a^+_{K'+q} a_{K'} a_K$$

to optimize energy, electrons push each other in 1D, while they can avoid contact in higher dimensions.



we have only two „isolated" fermi point while in higher dimensions particles have more phase space to two-particle interactions.

We will use notation R/L for $a^+_{+/-K_f+q}$ notation to distinguish Right and Left moving particles



we assume that in the vicinity of fermi point non-interacting Hamiltonian is:

$$\hat{H}_0 \simeq \sum_{S=R,L} \sum a^+_{sq} \sigma_s U_F q a_{sq}$$

$\sigma_s = (+/-)$ for $s = (R/L)$ and for $|q| < \Gamma$, we have cutoff.

let's define:

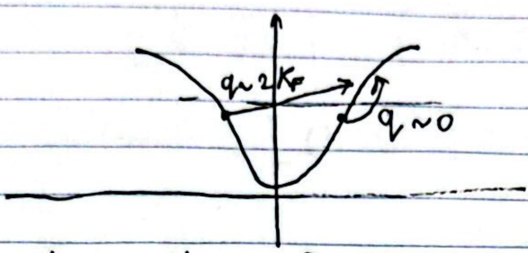
$$\hat{\rho}_{sq} = \sum_k a^+_{s,k+q} a_{sk}$$

when it's acting on ground state, it's excitation of partide from K to $K+q$. it can be interprate as particle-hole super position. It's fourier transform of:

$$\rho_s(x) = \psi^+(x) \psi(x) = \psi^+_L(x) \psi_L(x) + \psi^+_L(x) \psi_R(x) + \psi^+_R(x) \psi_L(x) + \psi^+_R(x) \psi_R(x)$$

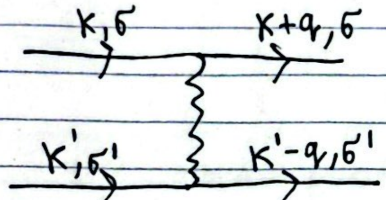
$$\psi(x) \approx \sum e^{ikx} \Psi_k + \sum e^{ikx} C_k$$

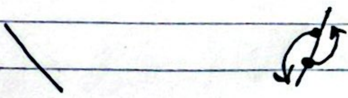
we have two(3) kind of interaction:



we will have three force constants:

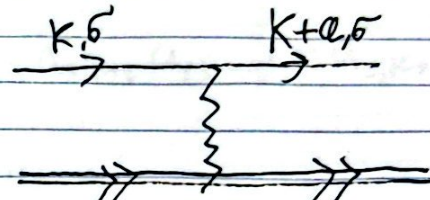
1)





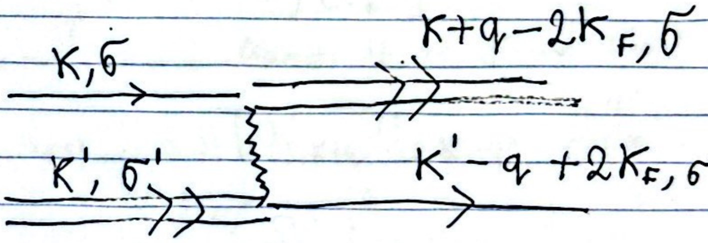
$g_4 = V(q \sim 0)$

2)



$g_2 = V(q \sim 0)$

3)



$g_1(2K_F)$

here one line corresponds K_F, two line $-K_F$

we don't consider g_1 interaction here commutator algebra:

$$[A, BC] = [A, B]C \mp B[A, C]$$

for bosons: $[a^+, a^n] = -n a^{n-1}$; for fermions $a^2=0$

let's check algebra of $\hat{\rho}$ operators.

$$[\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] = \delta_{ss'} \sum_k (a^+_{s,k+q} a_{sk-q'} - a^+_{s,k+q+q'} a_{s,k})$$

we can prove it by

$$\left[\sum a^+_{k+q} a_k, \sum a^+_{k'+q'} a_{k'}\right] =$$

$$= \sum\sum\left([a^+_{k+q} a_k, a^+_{k'+q'}] a_{k'} + a^+_{k'+q'}[a^+_{k+q} a_k, a_{k'}]\right.$$

to make approximation, we should estimate it in ground state.

$$[\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] \approx \delta_{ss'} \sum_k \langle\Omega| a^+_{s,k+q} a_{s,k-q'} - a^+_{s,k+q+q'} a_{sk} |\Omega\rangle$$

$$= \delta_{ss'} \delta_{q,-q'} \sum \langle\Omega| \hat{n}_{s,k+q} - \hat{n}_{sk} |\Omega\rangle$$

but our cutoff might be violated, because of $k \to k+q$ shift changes cutoff. So, this interpretation is invalid.
let's consider that $k<0$ is occupied, $k>0$ is empty and $s=R$
then:

$$\sum_k \langle\Omega| n_{R,k+q} - n_{R,k} |\Omega\rangle = \left(\sum_{-\Gamma \le k \le -q} + \sum_{-q<k\le 0} + \sum_{0<k<\Gamma}\right)$$

$$\langle\Omega|(n_{R,k+q} - n_{Rk})|\Omega\rangle = \sum_{-q<k<0} = -\frac{qL}{2\pi}$$

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for L particles occupation numbers would be opposite, which gives us additional minus. so: (4)

$$[\rho_{sq}, \rho_{s'q'}] = -\delta_{ss'}\delta_{qq'}\sigma_s \frac{qL}{2\pi}$$

let's make ~~same~~ four boson particles:

$$b_q \equiv n_q \hat{\rho}_{L,q} \qquad b_q^+ \equiv n_q \hat{\rho}_{L,-q}$$

$$b_{-q} \equiv n_q \rho_{R,-q} \qquad b_{-q}^+ \equiv n_q \hat{\rho}_{Rq}$$

with $n_q = \left(\frac{2\pi}{Lq}\right)^{1/2}$

then: $$V_{ee} = \frac{1}{2\pi}\sum_{q>0} q \begin{pmatrix} b_q & b_{-q} \end{pmatrix} \begin{pmatrix} g_4 & g_2 \\ g_2 & g_4 \end{pmatrix} \begin{pmatrix} b_q^+ \\ b_q \end{pmatrix}$$

now we need to express Kinetic part by bosons:
let's think like this:
if wee find H' which has same commutations with $\hat{b}$ operators as H_0, then we're done. let's take:

$$H_0' = \frac{2\pi U_F}{L}\sum_{q,s} \rho_{s,q}\rho_{s,-q};$$

$$\frac{2\pi U_F}{L}[\rho_{s,q}\rho_{s,-q}, n_q\rho_{sq}] = n_q\rho_{s,q}[\rho_{s,-q}, \rho_{sq}] =$$

$$= qU_F\sigma_s\rho_{sq}$$

and $$H_0 = \sum_{s,q} q\sigma_s a_{sq}^+ a_{sq}$$

$$[H_0, b_{q'}] = \sum_q q\sigma_s [a_{sq}^+ a_{sq}, \sum a_{s,K+q'}^+ a_{s,K}] =$$

$$= \sigma_s\left[\sum_q q\, a_{sq}^+ a_{sq}, \sum a_{s,K+q'}^+ a_{s,K}\right] =$$

$$= \sum_K \left[\sum_q q\, a_{sq}^+ a_{sq}, a_{s,K+q'}^+\right] a_{s,K} +$$

$$+ a_{s,K+q'}^+\left[\sum_q q\, a_{sq}^+ a_{sq}, a_{sK}\right] =$$

$$= \sum_{K,q}\left(q\, a_{sq}^+ a_{sK}\delta_{K+q',q} + q\, a_{s,K+q'}^+ a_{sq}\delta_{Kq}\right) = \frac{2\pi}{} qU_F\sigma_s\hat{\rho}_{sq}$$

So, total Hamiltonian: (5)

$$\hat{H} = \frac{1}{2\pi} \sum q \begin{pmatrix} b_q & b^+_{-q} \end{pmatrix} \begin{pmatrix} 2\pi v_F + g_4 & g_2 \\ g_2 & 2\pi v_F + g_4 \end{pmatrix} \begin{pmatrix} b^+_q \\ b_{-q} \end{pmatrix}$$

so, maping plan is

$$a \to \hat{\rho} \to b$$

some kind of bosonization.

why ~~do we~~ is it possible make this scheme?

because bosons are like fermions in 1D, they can't pass through in 1D.

let's observe that ~~if~~ if $H = \sum H_{\mu\nu} b^+_\mu b_\nu$ $[\hat{N} = \sum b^+_\lambda b_\lambda, H = \sum b^+_\mu b_\nu H_{\mu\nu}] = 0$, ~~when~~ so number of particles are conserved.

check:

$$[\sum b^+_\mu b_\nu H_{\mu\nu}, \sum b^+_\lambda b_\lambda] =$$

$$= \sum_{\mu,\nu} H_{\mu\nu} [b^+_\mu b_\nu, b^+_\lambda b_\lambda] = \sum H_{\mu\nu} ([b^+_\mu b_\nu, b^+_\lambda] b_\lambda + \ldots$$

$$+ b^+_\lambda [b^+_\mu b_\nu, b_\lambda] = 0$$

in our case number of particles are not conserved. this is logical, because creation of boson is excitation in our case, which is not fixed.

let: $\Psi_q = \begin{pmatrix} b^+_q \\ b_{-q} \end{pmatrix}$; $H = \sum q \Psi^+_q K \Psi_q = \sum q \Psi'^+_q \underbrace{T^+ K_q T}_{\text{diagonal}} \Psi'$

We should ensure that bose algebra stays the same, so: $[\Psi'_{q_i}, \Psi'^+_{q_j}] = [\Psi_{q_i}, \Psi_{q_j}] = (-\sigma_3)_{ij}$

where $\Psi'_{q_i} = T^{-1} \Psi_{q_i}$

we should diagonalize Hamiltonian. remember that ~~such a Hamilto~~ if we transform bose particles by

$$T\Psi = \begin{pmatrix}\cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta\end{pmatrix}\begin{pmatrix} b_q^+ \\ b_{-q}\end{pmatrix} = \begin{pmatrix}\cosh\theta\, b_q^+ + \sinh\theta\, b_{-q} \\ \sinh\theta\, b_q^+ + \cosh\theta\, b_{-q}\end{pmatrix} = \begin{pmatrix} b_q'^{+} \\ b'_{-q}\end{pmatrix}$$

$$-[b_q'^{+}, b'_q] = [\cosh\theta\, b_q^+ + \sinh\theta\, b_{-q}, \cosh\theta\, b_q + \sinh\theta\, b_{-q}^+]$$

$$= -\cosh^2\theta + \sinh^2\theta = -1$$

now lets diagonalize matrix

$2\pi v_F + g_4 \equiv c_1;\ g_2 \equiv c_2$

$$\begin{pmatrix}\cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta\end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_2 & c_1\end{pmatrix}\begin{pmatrix}\cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta\end{pmatrix} =$$

$$= \begin{pmatrix}\cosh\theta c_1 + \sinh\theta c_2 & \cosh\theta c_2 + \sinh\theta c_1 \\ \cosh\theta c_1 + \sinh\theta c_2 & \cosh\theta c_1 + \sinh\theta C_2\end{pmatrix}$$

$$\cdot \begin{pmatrix}\cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta\end{pmatrix} =$$

$$= \begin{vmatrix}\cosh^2\theta c_1 + \sinh\theta\cosh\theta c_2 + \sinh\cosh\theta c_2 + \sinh^2 c_1 & '' \\ \cancel{c_1 - \sinh\theta\cosh\theta} + (\cosh^2\theta + \sinh^2) c_1 + \sinh 2\theta c_2 & ''\end{vmatrix}$$

We can see that $\tanh(2\theta) = -\frac{c_1}{c_2} = -\frac{g_2}{2\pi v_F + g_4}$ to be diagonal; and $\varepsilon = \pm\frac{1}{2\pi}\left((2\pi v_F + g_4)^2 - g_2^2\right)^{1/2}$

So, $\hat{H} = \sum_q \varepsilon_q b_q^{\dagger} b_q$