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Chapter 1

Group Theory

1.1 Lie Theory

1.1.1 Lie Groups

Lie theory begins with the observation of the existence of Lie groups, which are groups that are also manifolds. Just as finite groups represent discrete symmetries continuous symmetries naturally correspond to a class of groups which are themselves continuous (and thus infinite). These are Lie groups, and since they correspond to continuous symmetries there is a natural notion of a continuum of group elements with a clear sense of which elements are close to each other, even which are infinitesimally close. From this observation it's clear that we may view our group as a manifold where each element is a point and nearby elements are nearby points and so on. Moreover we can expect to find coordinates for this manifold by finding parameters by which our group elements can be smoothly parameterized. As usual the number of such coordinates required to span the space defined the dimensionality of the manifold and thus of the group. Now before we go any further I turn to examples to build intuition.

Example 1. SO(2) & U(1)

SO(2) is the group of 2D rotations as such every element of SO(2) may be expressed by a rotation matrix of the form:

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \forall \qquad g \in SO(2)$$

As such each element may be labeled by a particular angle $\theta \in [0, 2\pi]$. As such we can imagine SO(2) as a unit circle, a 1D manifold. This result is even more obvious if you remember that SO(2) is isomorphic to U(1), the group of unit complex numbers which is naturally a circle in the complex plane.

Example 2. SO(3) & SU(2)

SO(3) being the group of 3D rotations, which are specified not by 1 but 3 angles. 2 to specify the axis and one two specify the magnitude of the rotation. That is there exists a mapping:

$$g \in SO(3) \qquad \Leftrightarrow \qquad (\theta, \phi, \psi) \in \{[0, \pi], [0, 2\pi), [0, 2\pi)\}$$

Now naively we might presume this means SO(3) is a 3-sphere in analogy to SO(2) being a circle but thats not quite the case (as can be seen by the ranges of the angles not being quite right for a 3-sphere). In fact SO(3) turns out ot be a rather topologically complex manifold, as is seen most clearly by studying the related group SU(2). SU(2) formally is the group of 2x2 unitary complex matrices but it's much easier to think about once you recognize it's isomorphic to $U(1, \mathbb{H})$, the group of unit quaternions, which you are encouraged to prove as an exercise. Since quaternions form a 4D space the group of unit quaternions is then naturally a 3-sphere. With this in mind we are only one step away from understanding the manifold of SO(3). The last key insight we need is that SU(2) is a double cover of SO(3), that is to say every element of SU(2), and thus every unit quaternion corresponds to a rotation in 3D space, but at a 2-1 ratio, there are in fact two unit quaternions for every 3D rotation. This is a fact well known to those who work in computer graphics where quaternions not rotation matrices are the preferred tools to rotate objects. With this in mind we not see SO(3) is something like a half of a 3-sphere which has been reconnected to itself in a topologically nontrivial way after its other half was removed. Note this means in principle rotating by 2π can pick up a sort of topological charge, since mapping onto unit quaternions we see a rotation of 2π just goes halfway around the 3-sphere and is thus a nontrivial path. Alternatively a rotation of 4π completely circumnavigates SU(2) rendering it a trivial path since SU(2) is topologically simple. Via it's mapping to SO(3) we then know rotations by 4π are topologically trivial in SO(3) as well. So when you rotate through 2π in SO(3) your path ends up in a strange sort of knot that can only be undone by going back or wrapping around again. It is this topological oddity which allows for the existence of fermions, quantum systems which flip sign based on this topological charge and thus only truly reset after rotations of 4π. It turns out that remarkably the situation doesn't change in higher dimensions, SO(N) always retains this topological oddity and so you always have fermions and bosons. Lower dimensions are a different story though: think about the topology of S0(2) and see if you can determine how many classes of particle spins you'd expect in a 2D universe (hint: google anyons).

Ok so perhaps now you're thinking "cool some groups are manifolds, so what?", well if you ask a mathematician what a manifold is the answer you're most likely to get is "something you can do calculus on" so it's worth our time to ponder how we might do calculus on a Lie group. It should be clear that since we can map our manifold with coordinates we can do integrals over it, the natural question though is what should be the integration measure? For some groups like SO(2) and SU(2) where we have a clear picture of what the manifold looks like it may feel like there are natural coordinates. After all integrating around a circle with $\int d\theta$ is far more sensible than $\int \frac{d\theta}{\tan\theta}$, but why? If we probe our sensibilities we'll find the intuition here is that a good integration measure should a lot the same volume to every region of the manifold. Note this is not to say we should always use a trivial measure, after-all for a euclidean sphere the measure which grants equal area to all regions is $\int \sin\theta d\theta d\phi$. It seems we must take the time to work out the criterion of a sensible measure. The so called Haar measure is the usual choice and is defined via either left of right invariance. That is to say for the left haar measure

$$\int_G d\mu f(g) = \int_G d\mu f(hg) \qquad \forall \qquad h \in G$$

and the analogous statement for the right haar measure. This is the natural "every region has the same volume" measure because it means that integrating a functions over the whole group manifold is the same even if you translate it across the manifold via application of a group element to it's argument. Now at this point I've kept the left and right haar measures distinct for mathematical formality but the reality is for groups relevant to physicists they're nearly always the same so going forward I'll just say Haar measure to reference a measure which is both left and right invariant. Should the distinction ever be important I'll explicitly clarify it. Now this condition of invariance does not quite uniquely define the Haar measure, it defines it up to a constant which we define via normalization:

$$\int_G d\mu = 1$$

These two criterion, along with some mathematically technical ones about the measure being finite and well behaved, uniquely specify the haar measure for any group. And indeed, the haar measure for SO(2) and SU(2) is exactly the intuitive integration measured for a circle and 3-sphere respectively, albeit normalized.

Now the other half of calculus, notably differentiation does not need much explaining. You have a manifold mapped with coordinates, if I gave you a function of those coordinates you could differentiate it in the usual way if you like.

1.1.2 Lie Algebras

Now one particularly useful thing to do with a Lie group is construct it's tangent plane at the identity. Doing this is straightforward but remarkably fruitful. We start by simply writing an element of the group near the identity as:

$$g = \mathbb{1} + i\epsilon^i \hat{T}_i$$

Where $\hat{T}_i$ are the so-called generators of the lie algebra. They are a set of unit vectors which span the tangent plane. It is worth noting these generators, while having many of the same properties of group elements are not themselves group elements (typically). As you should expect of any tangent plane it's dimensionality, and thus the number of generators will be the same as the dimensionality of the underlying manifold and thus the group.

Example 3. SO(2)

As stated for $g \in SO(2)$ we can write it in terms of an angle, letting this angle be infinitesimal we have:

$$g = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix} + \mathcal{O}(\epsilon^2) = \mathbb{1} + i\epsilon \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \mathcal{O}(\epsilon^2)$$

So as expected of a 1D group we have one generator of SO(2) which is:

$$\hat{T} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Now you may wonder why we have inserted the factor of i in our expansion and thus definition of the generators, after all it seemed rather unnatural in our example just now. The answer come from remembering we'd eventually like to do quantum mechanics and this factor of i makes life much better for us when we do as explained in more detail in the aside below. For now you may just take it as a convention.

Aside 1. Hermiticity of Generators in Quantum Mechanics

Since we know we typically want to be doing quantum mechanics and we know symmetries of quantum systems are represented by unitary operators it is reasonable to limit our attention ot only groups which can be represented as unitary operators. As such we expect $g^\dagger = g^{-1}$. Now within the Lie algebra the hermitian conjugate is easily calculable:

$$g^\dagger = \left(1 + i\epsilon^i \hat{T}_i\right)^\dagger = 1 - i\epsilon^i \hat{T}_i^\dagger$$

Additionally the inverse is straightforwardly calculable since we know if we transform infinitesimally in the ϵ^i direction we should undo it by transforming in the $-\epsilon^i$ direction so:

$$g^{-1} = 1 + i(-\epsilon^i)\hat{T}_i = 1 - i\epsilon^i \hat{T}_i$$

The unitarity condition is thus reframed as:

$$\hat{T}_i^\dagger = \hat{T}_i$$

That is to say the generators are all hermitian. Now we can see the utility of that factor of i because had we left it our we'd be concluding all generators are anti-hermitian which is much less useful.

Now as it is the Lie Algebra of a group, which may be considered a vector space in it's own right, usually denoted $\mathfrak{g}$ for a Lie group G, but it's not yet obvious how this construction is helpful or interesting. To see this we must ask ourselves how they are connected.

Let us suppose my Lie group has some coordinates x^i the I can take the element $g(x^i) \in G$, which recall corresponds to some sort of transformation/symmetry operation and represent it as a composition of smaller transformations:

$$g(x^i) = g\left(\frac{x^i}{N}\right) \ldots \text{n times} \ldots g\left(\frac{x^i}{N}\right) = g^N\left(\frac{x^i}{N}\right)$$

We could even then go so far as to define it as the limit of the composition of infinitely many infinitesimal transformations, and here is where we see the connection to the Lie algebra which coincides with the group infinitely close to the identity:

$$g(x^i) = \lim_{N\to\infty} g^N\left(\frac{x^i}{N}\right) = \lim_{N\to\infty}\left(1 + \frac{x^i}{N}\hat{T}_i\right)^N = \exp\left(x^i \hat{T}_i\right)$$

And her we see the remarkable utility of the Lie Algebra, every element of the Lie group may be expressed as the exponentiation of an element of the Lie Algebra:

$$\forall\, g \in G \qquad \exists \qquad x \in \mathfrak{g} \qquad \text{such that} \qquad g = \exp(x)$$

Funnily enough you sort of "already knew this" with respect to one, possibly two groups. After all you knew every element of U(1), ie every unit complex number could be written as $e^{i\theta}$. Now from our Lie algebra perspective you see this is because the tangent line of the circle of unit complex numbers at the identity is pointing in the i direction so factoring our the i as usual in our definition of generators we see the generator of U(1) is just 1 as we could have read off from Euler's formula above. The other group you may have already known this for is the group of unit quaternions which have a similarly straightforward Euler formula. The fact that we've now extended this sort of representation to all Lie groups and algebras is particularly powerful. Now working out the generators of SO(3) and SU(2) are straightforward exercises so I won't do them as examples, in any event a derivation of SU(2)'s generators can be found in any quantum mechanics book as they are the pauli matrices.

Additionally all Lie algebras come with a natural inner product due to the fact that the basis vectors of this space (aka the generators) are themselves operators and you can always take the trace of operators. In particular the linearity of traces so we have:

$$\mathrm{Tr}(xy) = \mathrm{Tr}\left(x^i \hat{T}_i y^j \hat{T}_j\right) = x^i y^j\, \mathrm{Tr}\left(\hat{T}_i \hat{T}_j\right) \qquad \forall \qquad x, y \in \mathfrak{g}$$

Additionally since the cyclic property of traces ensures $\mathrm{Tr}\left(\hat{T}_i\hat{T}_j\right)$ is symmetric we are free to treat is as the metric of our Lie Algebra which defines the inner product. The existence of such an inner product is of course very useful and also motivates us to find orthonormal generators. In reality while we always choose an orthogonal set of generators we often do not choose a normalized ones (for as far as I can tell historical reasons). So I will in general presume them to take the normalization:

$$\mathrm{Tr}\left(\hat{T}_i\hat{T}_j\right) = c\delta_{ij}$$

Now I should note all this time I've been treating Lie algebras like just vector spaces, but of course to be an algebra there must also be a vector x vector multiplication. The canonical choice for Lie algebras is the commutator, as it serves as an operation between vectors that spits out a vector in all the necessary ways to qualify as an algebra. It's also particularly useful as the commutator of the generators determines the curvature of your Lie Group (This connection will not be clear to those who have not studied differential geometry, to those who have simply recall that the Riemann curvature tensor can be thought of as a commutator of vectors). Now formally a Lie algebras multiplication is just an abstract Lie bracket which is anti-symmetric and under which the algebra closes, however we usually use anti-commutator where possible. For example if the Lie Algebra of SO(3) or SU(2) which are 3D you may define the Lie Bracket to be a cross product. Now in general since the Lie Bracket is a linear operation which is closed we can write:

$$[x, y] = x^i y^j \left[\hat{T}_i, \hat{T}_j\right] = ix^i y^j f_{ij}{}^k \hat{T}_k \qquad \forall \qquad x, y \in \mathfrak{g}$$

Where we have defined the structure constants f_{ijk} through the relation:

$$\left[\hat{T}_i, \hat{T}_j\right] = if_{ij}{}^k \hat{T}_k$$

While it's obvious that the structure constants are antisymmetric on the first two indices owing to the Lie Bracket it's also provable that it's in fact totally antisymmetric on all indices. To prove this merely expand the trace $\mathrm{Tr}\left([\hat{T}_i, \hat{T}_j]\hat{T}_k\right)$ and abuse the cyclic property of the trace.

With this we can now compute the inner product or Lie bracket of any two elements of the Lie algebra easily enough.

Depending on the group there may also be a completeness relation of some kind defined as products of the generators however at the level of an abstract lie algebra products of the generators are not defined and when they are the result differs from group to group.

One last comment is about integration over the group via the lie algebra, naturally if we wish to do this our Haar measure becomes:

$$d\mu \to \prod_i dx_i J(x^i)$$

where the jacobians $J(x^i)$ can in principle be as nasty as they like. It has however been my experience that, likely due to homogeneity of the haar measure, the taylor series of $J(x^i)$ tend to start at a fairly high order allowing you to get away with ignoring them at low orders in perturbation theory which as a physicists is most of what you do anyway. It's also worth noting that while the Lie algebra is explicitly the tangent plane at the identity since the Haar measure is translation invariant you are free to expand around whatever point you like for the purposes of perturbation theory and the like.