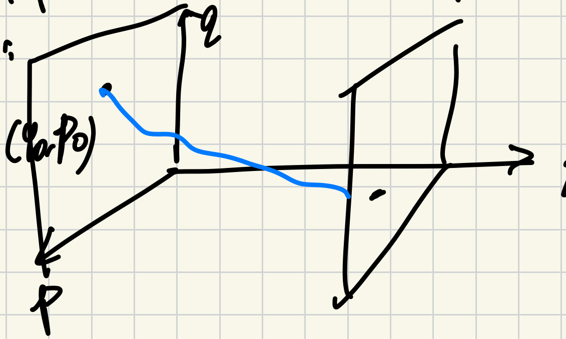
3.4 Construction of the Many-Body Field Integral

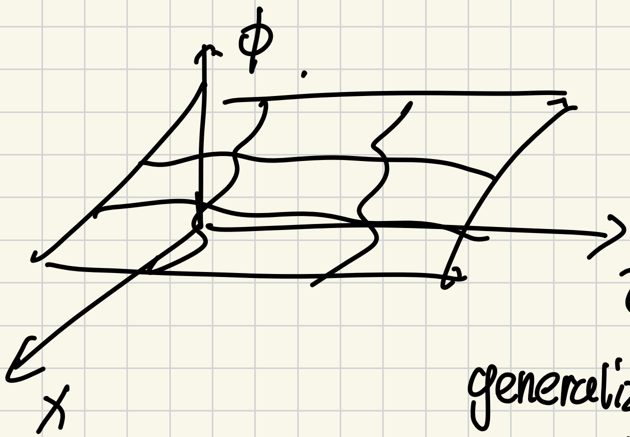
3.4.1 Construction of the field integral

QM:



We consider all path (integral over time) and sum over them.

QFT



$$\int d^4x\, S(\phi)$$

We need to integral over generalized $(d+1)$-dimensional surfaces.

Many particle Hamitonians are conveniently expressed in terms of creation and annihilation operators, an obvious idea would be to search for eigenstates of these operators.

- coherent states (bosons)

$$|\phi\rangle = \sum_{n_1,n_2\cdots} C_{n_1,n_2} \, |n_1, n_2 \cdots\rangle$$

$$|n_1,n_2\rangle = \frac{(a_1^{+})^{n}}{\sqrt{n_1!}} \, \frac{(a_2^{+})^{n}}{\sqrt{n_2!}} \cdots |0\rangle$$

- creation operators a_i^{+} cannot poses eigenstates (minimum number $n_0 \to n_0+1$)

- annihilation operators a_i can poses eigenstates

$0, 1, 2 \cdots\cdots, \infty$

$0, 1, 2 \cdots\cdots \infty$

bosonic coherent states:

$$|\phi\rangle = \exp\Big(\sum_i \phi_i a_i^{+}\Big)|0\rangle$$

- $a_i|\phi\rangle = \phi_i|\phi\rangle$

$$[a_i, \exp(\sum_j \phi_j a_j^\dagger)]$$

$$= [a_i \quad \sum_n \frac{\sum_j (\phi_j a_j^\dagger)^n}{n!}]$$

$$[a_i, (a_j^\dagger)^n] = n (a_j^\dagger)^{n-1} \delta_{ij}$$

$$[a_i, (\sum_j \phi_j a_j^\dagger)^n] = \sum_n \sum_j \frac{(a_j^\dagger)^{n-1}}{(n-1)!} \phi_j^{n-1} \delta_{ij}$$

$$= \phi_i \exp(\sum_i \phi_i a_i^\dagger)$$

$$a_i \exp(\sum_i \phi_i a_i^\dagger)|0\rangle$$

$$= \exp(\sum_i \phi_i a_i^\dagger) \boxed{a_i|0\rangle}^{0} + \phi_i \exp(\sum_i \phi_i a_i^\dagger)|0\rangle$$

$$= \phi_i |\phi\rangle$$

$$\Rightarrow a_i|\phi\rangle = \phi_i |\phi\rangle$$

▷ hermitian conjugate:

$$\langle\phi| a_i^\dagger = \langle\phi| \bar{\phi}_i$$

$$\langle\phi| = \langle 0| \exp(\sum_i \bar{\phi}_i a_i)$$

▷ $a_i^\dagger |\phi\rangle = \partial_{\phi_i} |\phi\rangle$

▷ $\langle\theta|\phi\rangle = \exp\left(\sum_i \bar{\theta}_i \phi_i\right)$

$\langle\theta|\phi\rangle = \langle 0|\exp\left(\sum_i \bar{\theta}_i a_i\right)|\phi\rangle$

$|\psi\rangle$ is the eigenvector of a_i. not orthogonal

$\langle\theta|\phi\rangle = \langle 0|\exp\left(\sum_i \bar{\theta}_i \phi_i\right)|0\rangle = \exp\left(\sum_i \bar{\theta}_i \phi_i\right)$

▷ norm of coherent state:

$\langle\phi|\phi\rangle = \exp\left(\sum_i \bar{\phi}_i \phi_i\right)$

▷ completeness relation

$$\int \prod_i \frac{d\phi_i}{\pi} \exp\left(-\sum_i \bar{\phi}_i \phi_i\right) |\phi\rangle\langle\phi| = \mathbb{1}_F$$

Introduce Schur's lemma:

If an operator commutes with all $\{a_i, a_i^\dagger\}$, it must be proportional to the unit vector.

- The whole Fock space is generated by operators $a_i, a_i^\dagger$

$$|n_1 n_2 \cdots\rangle = \frac{(a_1^\dagger)^{n_1}}{n_1!} \frac{(a_2^\dagger)^{n_2}}{n_2!} \cdots |0\rangle$$

irreducible representation:

- There is no nontrivial subspace that is invariant under the action of all operators in the representation.

$\Rightarrow$ The creation and annihilation operators form an irreducible representation of the Heisenberg algebra on Fock space.

$$a_i \int d\phi\, e^{-\bar\phi\phi} |\phi\rangle\langle\phi| = \int d\phi\, e^{-\bar\phi\phi} \phi_i |\phi\rangle\langle\phi|$$

$\langle\phi| \neq (|\phi\rangle)^\dagger$ in the path integral

We will consider $\phi, \bar\phi$ as independent variables

$$= -\int d\phi\, \partial_{\bar\phi_i}\left(e^{-\bar\phi\phi}\right) |\phi\rangle\langle\phi|$$

$$= -\left(e^{-\bar\phi\phi} |\phi\rangle\langle\phi|\right)_{B.C.}$$

$$+ \int d\phi\, d\bar\phi\, e^{-\bar\phi\phi} |\phi\rangle\left(\partial_{\bar\phi_i} \langle\phi|\right)$$

$\int d\phi\, d\bar\phi$

$-\bar\phi\phi$

$$= \cancel{(\quad)}_{BC} + \int d\phi\, e^{-\bar\phi\phi} |\phi\rangle\langle\phi| a_i$$

$0.$

In complex plane:

$$\int_{\mathbb{C}} \frac{d^2 z}{\pi} e^{-|z|^2} = 1$$

$$\boxed{\int dz\,d\bar{z}\, e^{-|z|^2} = \frac{1}{2}\int dr\,r\,d\theta\, e^{-r^2} = \pi.}$$

$$\Rightarrow \int \prod_i \frac{d\phi_i}{\pi} \exp\left(-\sum_i \bar{\phi}_i \phi_i\right) |\phi\rangle\langle\phi| = 1_{\mathcal{F}}$$

Overcomplete-set of states

$$\langle\theta|\phi\rangle = \exp\left(\sum_i \bar{\theta}_i \phi_i\right)$$

They're not orthogonal.

They span the space. But there are "too many" of them.

- Coherent states (fermions)

Still suppose that annihilation operators are characterized by a set of coherent states such that, for all i.

$$a_i|\eta\rangle = \eta_i|\eta\rangle$$

$$\{a_i, a_j\} = 0 \Rightarrow \underline{\eta_i\eta_j + \eta_j\eta_i = 0}$$

they are not ordinary numbers.

- An algebra is a vector space endowed with a multiple rule $A \times A \to A$

- We can construct an Algebra A tailored to our needs by starting out from a set of elements.
  - Some concept of algebras:

1. $c_0 \cdot I + c_i\eta_i + c_{ij}\eta_i\eta_j \in A$, $c_0, c_i, c_{ij} \in \mathbb{C}$

identity element.

2. $\eta_i\eta_j\eta_x = \eta_i(\eta_j\eta_x)$ associativity

$\eta_i \eta_j = -\eta_j \eta_i$ anticommutative.

$\Rightarrow$ spans a finite-dimensional associative algebra.

all possible index, but some are zero or related by a minus sign.

$$C_0 + \sum_{n=1}^{\infty} \sum_{i_1 \cdots i_n = 1}^{N} C_{i_1 \cdots i_n} \eta_{i_1} \cdots \eta_{i_n}$$

first order $n=1$, $\eta_1, \eta_2 \cdots \eta_N$ $C_0, C_{i_1} \cdots C_{i_n} \in \mathbb{C}$

second order $n=2, \cdots$

finite dimension, the dimension is 2^N.

when $n > N$, it will be repeated

Functions are defined via. Taylor expansion

$$f(\xi_1, \cdots \xi_N) = \sum_{n=0}^{\infty} \sum_{i_1 \cdots i_n = 1}^{K} \frac{1}{n!} \frac{\partial^n f}{\partial \xi_{i_1} \cdots \partial \xi_{i_n}}\bigg|_{\xi=0} \xi_{i_n} \cdots \xi_{i_1}$$

first order

$$f(\eta) = f(0) + f'(0)\eta.$$

Differentiation. (still satisfy anticommutativity)

$$\partial_{\eta_i} \eta_j = \delta_{ij}.$$

$$\partial_{\eta_1}(\eta_1\eta_2) = \eta_2$$

$$\partial_{\eta_1}(\eta_2\eta_1) = -\eta_2$$

$$\partial_{\eta_i}\partial_{\eta_j} = -\partial_{\eta_j}\partial_{\eta_i}.$$

Integration. ($\int d\eta : \mathcal{A} \to \mathbb{C}$)

one variable: $f(\eta) = a + b\eta$

$\int$ and ∂ do the same job

$$\frac{\partial f(\eta)}{\partial \eta} = b$$

$$\int d\eta\, f(\eta) = b$$ (no boundry term)

Many variables:

$$\int d\eta_N \cdots d\eta_1\, \eta_1\eta_2 \cdots \eta_N = 1$$

$\{\eta_i, a_j\} = 0$?

Fermionic coherent states:

$$|\eta\rangle = \exp\left(-\sum_i \eta_i a_i^\dagger\right)|0\rangle$$

$$a_i e^X = e^X a_i - \{a_i, X\} e^X$$

$$\left\{a_i, -\sum_j \eta_j a_j^\dagger\right\} = \eta_i \{a_i, a_i^\dagger\} = \eta_i$$

$$a_i|\eta\rangle = \exp\left(-\sum_i \eta_i a_i^\dagger\right) a_i|0\rangle + \eta_i \exp\left(-\sum_j \eta_j a_j^\dagger\right)|0\rangle$$

$$\Rightarrow a_i|\eta\rangle = \eta_i|\eta\rangle$$

annihilation → still annihilation

adjoint of a fermion coherent state.

$$\langle\eta| = \langle 0|\exp\left(-\sum_i a_i \bar{\eta}_i\right)$$

$$= \langle 0|\exp\left(\sum_i \bar{\eta}_i a_i\right)$$

example.:

one mode: $|\eta\rangle = |0\rangle - \eta|1\rangle$

$$a|\eta\rangle = a|0\rangle - a\eta a^\dagger|0\rangle$$

$$= a|0\rangle + \eta a a^\dagger|0\rangle = \eta|0\rangle$$

$$\eta|\eta\rangle = \eta|0\rangle - \eta^2|1\rangle = \eta|0\rangle$$

$$a|\eta\rangle = \eta|\eta\rangle$$

- Some scane properties

$$\langle\eta|a_i^\dagger = \langle\eta|\bar{\eta}_i$$

$$a_i^\dagger|\eta\rangle = -\partial_{\eta_i}|\eta\rangle$$

$$\langle\eta|\xi\rangle = e^{\bar{\eta}\xi}$$

$$\langle\eta|\eta\rangle = e^{\bar{\eta}\eta}$$

Think it directly example.

$$|\eta\rangle = \eta_i|00\cdots\underset{i}{1}\cdots\rangle$$

$$+ \eta_i\eta_j|\cdots\underset{i}{1}\cdots\underset{j}{1}\cdots\rangle$$

$$+ \cdots$$

$$a_i^\dagger|\eta\rangle = \underset{\text{no } \eta_i}{\eta_j\eta_k}|\cdots\underset{i}{1}\cdots\rangle$$

and over complete:

$$\int d\bar{\eta}d\eta\, e^{-\bar{\eta}\eta}|\eta\rangle\langle\eta| = 1$$

(no π factor, because $\int d\eta\, e^{\bar{\eta}\eta} = 1$)

$$\int d\eta\, e^{-\bar{\eta}\eta} = \int d\eta\,(1-\bar{\eta}\eta)$$
$$= -\int d\bar{\eta}\, d\eta\, \bar{\eta}\eta = 1$$

Grassmann Gaussian integrate.

$$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta} a \eta} = a.$$

$$\int d\bar{\eta}\, d\eta\,(1-\bar{\eta} a \eta) = \int d\bar{\eta}\, d\eta\, \eta a \bar{\eta} = a.$$

Multi-dimensional generalization to matrix and vector.

$$\int d\bar{\phi}\, d\phi\, e^{-\bar{\phi} A \phi} = \det A$$

$$e^{-\bar{\phi} A \phi} = \sum_n \frac{(-\bar{\phi} A \phi)^n}{n!},$$ $d\phi\, d\bar{\phi} = \bar{\phi}_1 \cdots \bar{\phi}_n \phi_1 \cdots \phi_n$ only n term can survive.

$$= \sum_n (-1)^n \underbrace{\sum_{i_1 \cdots i_n} \sum_{j_1 \cdots j_n}}_{n!} \bar{\phi}_{i_1} A_{i_1 j_1} \phi_{j_1} \bar{\phi}_{i_2} A_{i_2 j_2} \phi_{j_2} \cdots$$

arrange it as

$$\eta_n \eta_{n-1} \cdots \eta_1 \bar{\eta}_n \cdots \bar{\eta}_1$$

$$d\bar{\phi}\, d\phi = d\bar{\phi}_1 \cdots d\bar{\phi}_K\, d\phi_1 \cdots d\phi_4$$

only $N=K$, the term can survive.

$$\Rightarrow \int d\bar{\phi} d\phi \sum_{n \in S_N} \mathrm{sgn}(\sigma)\, A_{1\sigma(1)} \cdots A_{N\sigma(N)}$$

$$= \det(A)$$

$$\int d\bar{\phi}\, d\phi\, \exp\left(-\bar{\Phi} A \phi + \bar{\nu}^T \phi + \bar{\phi}^T \nu\right)$$

$$= \det A\, \exp\left(\bar{\nu}^T A^{-1} \nu\right)$$

$$-\bar{\Phi} A \phi + \bar{\nu}^T \phi + \bar{\Phi} \nu = -\left(\bar{\Phi} - \bar{\nu} A^{-1}\right)^T A \left(\phi - A^{-1} \nu\right) + \bar{\nu} A^{-1} \nu$$

Grassman integral can be translated

$$\int d\eta\, f(\eta) = \int d\eta\, f(\eta + \text{const})$$

so $$\int d\bar{\Phi}\, d\phi\, \exp\left(-\bar{\phi}^T A \phi + \bar{\nu} \phi + \bar{\Phi}^T \cdot \nu\right)$$

$$= \det A\, \exp\left(\bar{\nu}^T A^{-1} \nu\right)$$

$$\det A \langle \phi_j \bar{\phi}_i \rangle = \frac{\partial}{\partial \bar{z}_i} \frac{\partial}{\partial z_j} \left[\det A \exp(\bar{z}^T A^{-1} z) \right] \Big|_{\bar{z}=z=0}$$

$$= \det A \cdot A^{-1}_{ji}$$

$$\Rightarrow \langle \phi_j \bar{\phi}_i \rangle = A^{-1}_{ji}$$

$$\Rightarrow \langle \phi_{j_1} \phi_{j_2} \cdots \phi_{j_n} \bar{\phi}_{i_n} \cdots \bar{\phi}_{i_2} \bar{\phi}_{i_1} \rangle$$

$$= \sum_P (\operatorname{sgn} P)\, A^{-1}_{j_1 i_{P_1}} \cdots A^{-1}_{j_n i_{P_n}}$$