Contents

0.1 Infinite Order Expansions

0.1.1 Set Up

To make any progress with infinite order expansions you must first identify a parameter, which in some limit, causes one "small" class of diagrams to dominate over all others. The idea being that in this limit one can neglect all except this class of diagrams and this class is small enough that is can be sensibly summed as a convergent series. The following section will be devoted to an example case. Consider ϕ^4 theory where our field is in factor a N component object and out lagrangian is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi^a + \frac{1}{2}m^2\phi_a\phi^a + \frac{g}{4N}\phi_a\phi^a\phi_b\phi^b$$

Where the metric on the space of ϕ^a's components is euclidean so ϕ_a and ϕ^a have the same components, it's just notational convenience.

Here N will be our control parameter, where we note that every interaction vertex will carry a power of N^{-1} and every "loop" which causes us to sum over all component of ϕ will produce a propagator of order N. We hope then that we can sensibly order our diagrams by powers of N and the leading class of diagrams, which well approximates the entire series in this limit, is small enough it can be sensibly summed.

Before we we do any of that though we need our usual ingredients, the propagator and vertex factor. The (momentum space) propagator can be clearly seen to simply be:

$$\Delta_0^{ab}(p) = \frac{\delta^{ab}}{p_\mu p^\mu + m^2}$$

The vertex factor is a bit more interesting, performing the usual calculation from the path integral to first order:

$$\begin{aligned}
Z(J) &= \int \mathcal{D}\phi \exp\left(-\int d^dx \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi^a + \frac{1}{2}m^2\phi_a\phi^a + \frac{g}{4N}\phi_a\phi^a\phi_b\phi^b + J_a\phi^a\right) \\
&= \exp\left(-\frac{g}{4N}\int d^dy \frac{\delta}{\delta J^c}\frac{\delta}{\delta J^c}\frac{\delta}{\delta J^d}\frac{\delta}{\delta J^d}\right)\int \mathcal{D}\phi \exp\left(-\int d^dx \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi^a + \frac{1}{2}m^2\phi_a\phi^a + J_a\phi^a\right) \\
&\approx \left(1 - \frac{g}{4N}\int d^dy \frac{\delta}{\delta J^c}\frac{\delta}{\delta J^c}\frac{\delta}{\delta J^d}\frac{\delta}{\delta J^d}\right)\int \mathcal{D}\phi \exp\left(-\int d^dx \frac{1}{2}\phi_a\left(-\partial^2 + m^2\right)\phi^a + J_a\phi^a\right) \\
&= \left(1 - \frac{g}{4N}\int d^dy \frac{\delta}{\delta J^c}\frac{\delta}{\delta J^c}\frac{\delta}{\delta J^d}\frac{\delta}{\delta J^d}\right)\exp\left(\int d^dx' d^dx' J(x)_a \Delta_0^{ab}(x-x')J(x')_b\right) \\
&= \left(1 + \ldots \mathcal{O}(J^{<4})\cdots - \frac{g}{4N}\int d^dy d^dx J^a(x)\Delta_0^{ac}(x-y)J^b(x)\Delta_0^{bc}(x-y)J^e(x)\Delta_0^{ed}(x-y)J^f(x)\Delta_0^{fd}(x-y)\right)Z_0(J)
\end{aligned}$$

Where I've neglected terms of order less than J^4 since we know our goal is the vertex factor and thus the 4-point correlation function. We also now have the cumulant generating function:

$$W(J) = \ln Z(J) = \mathcal{O}(J^{<4})\cdots - \frac{g}{4N}\int d^dy d^dx J^a(x)\Delta_0^{ac}(x-y)J^b(x)\Delta_0^{bc}(x-y)J^e(x)\Delta_0^{ed}(x-y)J^f(x)\Delta_0^{fd}(x-y)\cdots + \mathcal{O}(J^{>4})$$

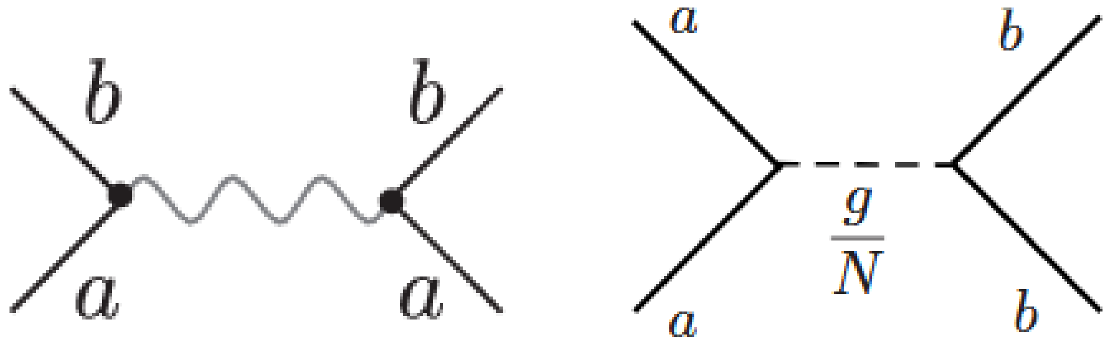
And so we compute:

$$\begin{aligned}\langle\phi^a(x_1)\phi^b(x_2)\phi^c(x_3)\phi^d(x_4)\rangle &= \frac{\delta}{\delta J^a(x_1)}\frac{\delta}{\delta J^b(x_2)}\frac{\delta}{\delta J^c(x_3)}\frac{\delta}{\delta J^d(x_4)}F(J)\\ &= -\frac{g}{4N}\int d^dy\Delta_0^{ai}(x_1-y)\Delta_0^{bj}(x_2-y)\Delta_0^{cj}(x_3-y)\Delta_0^{dj}(x_4-y)+(a\leftrightarrow b\leftrightarrow c\leftrightarrow d)\\ &= -\frac{g}{4N}\delta^{ab}\delta^{cd}\int d^dy\Delta_0^{aa}(x_1-y)\Delta_0^{bb}(x_2-y)\Delta_0^{cc}(x_3-y)\Delta_0^{dd}(x_4-y)+(a\leftrightarrow b\leftrightarrow c\leftrightarrow d)\end{aligned}$$

Where in the last line we have used the face that $\Delta_0^{ab}\propto\delta^{ab}$ and the part at the end in parenthesis indicates all other orders of $a, b, c, \&d$ (bringing their respective positions with them). Now we see that because of the delta functions these are not 4 independent ingoing/outgoing lines, rather they are two pairs which must have matching indices, as such there are not the usual 4! identical diagrams here, instead each different choice of pairing is a topologically different diagram. Once we've chosen such a pairing we can of course swap each pair without changing anything leading to a symmetry factor of just 4 (now we see why we use $\frac{1}{4}$ rather than the usual $\frac{1}{4!}$) in our lagrangian. From here it's clear the vertex factor is:

$$\Gamma^{abcd}=-\frac{g}{N}\delta^{ab}\delta^{cd}$$

Due to this pairing of indices at the vertex if one does not wish to label every line with an index the convenient thing to do is separate out the vertex into a line with a pair of lines connecting at each endpoint. Note this should not be taken to indicate any mediating particle, it's merely notational convenience so it's always clear at a glance which pairs share the index. There are two common conventions for this:



Which convention you use doesn't matter it just changes your diagrams a bit (ie it determines if rainbows or tadpoles will be our leading contribution). I will proceed with the right hand diagrams convention because it allows me to say the indices are conserved at the new effective 3 point vertices and that loops involve sums over a free index.

0.1.2 Self Energy

The next interesting feature to define is the self energy operator Σ. The self energy is simply put the the sum of all one particle irreducible the ways the field can scatter off itself, it is a sort of special effective vertex with only one line in a and one line out. The idea being the field goes in, scatters of itself in any number of ways then goes out and the self energy is the sum over all such processes. We restrict the self energy to only the 1 particle irreducible scatterings because any other self scattering event must be representable as the sum of two 1 particle irreducible self scattering events. As such with our new special self energy "vertex" defined we know the exact propagator is the sum over all possible self energy scattering events, ie:

$$\Delta=\Delta_0+\Delta_0\Sigma\Delta_0+\Delta_0\Sigma\Delta_0\Sigma\Delta_0+\cdots=\Delta_0+\Delta_0\Sigma\Delta$$

$$\Rightarrow(1-\Delta_0\Sigma)\Delta=\Delta_0\qquad\Rightarrow\qquad\Delta=(1-\Delta_0\Sigma)^{-1}\Delta_0=(1-\Delta_0\Sigma)^{-1}(\Delta_0^{-1})^{-1}=(\Delta_0^{-1}-\Sigma)^{-1}$$

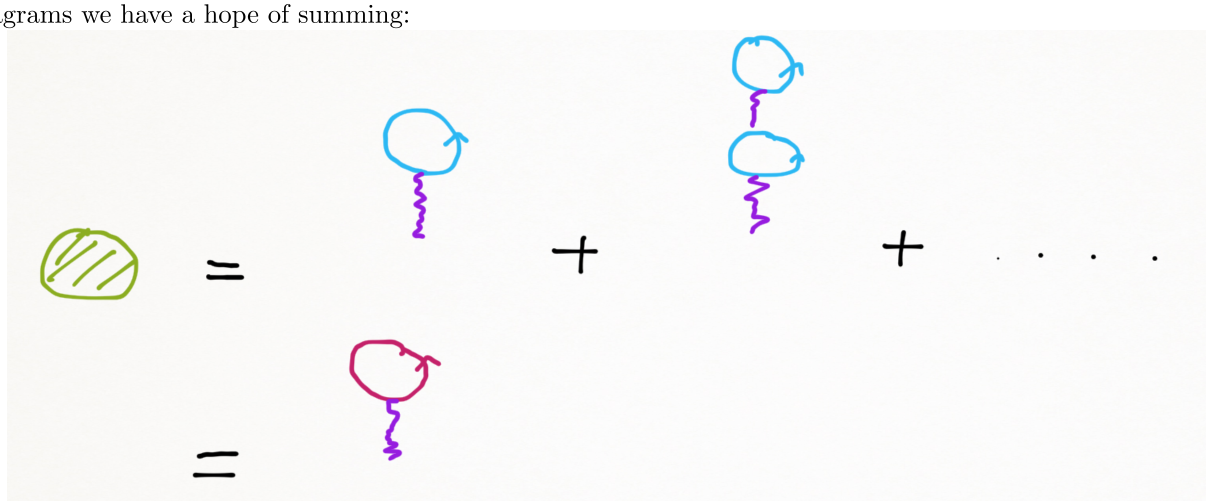
And of course we know the inverse of the free propagator so we arrive at the formal expression of the exact propagator:

$$\Delta^{ab}(p)=\left(p_\mu p^\mu\delta^{ab}+m^2\delta^{ab}-\Sigma^{ab}(p)\right)^{-1}$$

0.1.3 The Large N Limit

If we want to compute the exact propagator in the limit of $N \gg 1$ the first thing we must do is identify our leading order in N. Obviously the 0th order (in g) diagrams are of order unity in N so it's at least that high. Now clearly everytime we add an interaction line our order in N descends by 1, and every time we add a loop our order in N ascends by 1. However the catch here is of course that you cannot have a loop connected to your propagator without an interaction line so adding at best leaves your order in N unchanged. From this it's clear that you can never construct a diagram of higher order in N that unity which is nice because it means we have a special class of diagram which remains finite in the limit of $N \to \infty$ while all others die. This is exactly what we wanted.

Now lets focus on the self energy. Recall the only operation we're allowed if we want stay at leading order is adding loops attached by an interaction line (so called tadpoles) and we must remain 1 particle irreducible so we must always add each tadpole to the head of an existing tadpole. As such the only diagrams that contribute to the self energy in the large N limit are towers of tadpoles. Note this is exactly the sort of thing we were hoping for, a countable infinite class of diagrams we have a hope of summing:



Where the red line indicates the exact propagator. From here we can write down the equation:

$$\Sigma^{ab}(p) = -\frac{g}{N}\delta^{ab}\int^{\Lambda} đ^d p' \Delta_c{}^c(p')$$

Note as we might have guessed earlier by the momentum independence of the vertex factor we now see explicitly there is no remaining momentum on the right hand side which has not been integrated over and it is proportional to a delta function so it must be the case that $\Sigma^{ab}(p) = \Sigma\delta^{ab}$ is actually momentum independent. Using our identity from the previous section we can then write:

$$\Sigma = -\frac{g}{N}\int^{\Lambda} đ^d p \left(p_\mu p^\mu + m^2 - \Sigma\right)^{-1}\delta^a_a = -g\int^{\Lambda} đ^d p \left(p_\mu p^\mu + m^2 - \Sigma\right)^{-1}$$

This integral is of course computable:

$$\begin{aligned}
\Sigma &= -g\int^{\Lambda} đ^d p \left(p^2 + m^2 - \Sigma\right)^{-1} \\
&= -gS_{d-1}\int^{\Lambda} p đ p \left(p^2 + m^2 - \Sigma\right)^{-1} \\
&= -g\frac{S_{d-1}}{2(2\pi)^d}\int_0^{\Lambda^2} đ(p^2)\left(p^2 + m^2 - \Sigma\right)^{-1} \\
&= -g\frac{S_{d-1}}{2(2\pi)^d}\ln\left(\frac{\Lambda^2 + m^2 - \Sigma}{m^2 - \Sigma}\right)
\end{aligned}$$

This equation is not analytically solvable but is easily in reach of numerical and graphical methods. One thing that is worth noting (check for yourself on Desmos) that a solution always exists and is always negative. This tells us (as we might have expected) the interaction adds an effective energy to the mass:

$$m^2_{eff} = m^2 - \Sigma > m^2$$

Additionally as we recall the mass in the lagrangian also sets the correlation inverse length $m^{-1} \sim \xi$so this also suggests the interaction decreases correlation length increasing disorder in the system.

0.1.4 The Vertex (Let's do it all again!)

We might notice that if we wish to compute the exact vertex of our theory we should start, in analogy with self energy, by defining an operator which consists of all 2-particle irreducible diagrams which contribute to the vertex. Then by exactly the same logic we can write the exact vertex as:

$$\Gamma^{aa'bb'}(p,p',q) = \Gamma_0^{aa'bb'}(p,p',q) + \int đ^d p'' \Gamma_0^{aa'cc'}(p,p'',q)\Delta_{cd}(p'')\Delta_{c'd'}(p''+q)\Gamma^{dd'bb'}(p',p'',q)$$

Where we note here we've cast the exact vertex in terms of the exact propagator. Since we know the bare vertex and propagator are both proportional to delta functions we will ansatz:

$$\Gamma^{aa'bb'} = \Gamma^{ab}\delta^{aa'}\delta^{bb'}$$

We will also ansatz the exact vertex is independent of incoming momenta, as the exact vertex is. The viability of this assumption will be tested by our ability to find solutions to the equations we're about to derive. We then write:

$$\begin{aligned}
\Gamma^{ab}(q) &= \Gamma_0^{ab}(q) + \int đ^d p'' \Gamma_0^{ac}(q)\Delta_{cd}(p'')\Delta_{cd}(p''+q)\Gamma^{db}(q) \\
&= \Gamma_0^{ab}(q) + \Gamma_0^{ac}(q)\left(\int đ^d p'' \Delta_{cd}(p'')\Delta_{cd}(p''+q)\right)\Gamma^{db}(q) \\
&= \Gamma_0^{ab}(q) + \Gamma_0^{ac}(q)P(q)N\Gamma^{cb}(q)
\end{aligned}$$

Where we've defined $P(q)\delta_{cd}\delta_{cd} = P(q)N\delta_{cd} = \int đ^d p''\Delta_{cd}(p'')\Delta_{cd}(p''+q)$ as the analogue of the self energy for the vertex. Now that we have it in this form we just cite the algebra above to write:

$$\Gamma^{ab}(q) = \left[(\Gamma_0^{ab})^{-1}(q) - NP(q)\right]^{-1} = -\frac{g}{N}\frac{1}{1+gP(q)}$$

Which is a complete formal solution since $P(q)$ is a function of the exactly propagator which we know from our discussion above.

One interesting thing to note here is that unlike the exact propagator, the exact vertex vanishes in the $N \to \infty$ limit since the leading order of the interacting diagrams contains at least one more interaction line than loops such that it can connect to all the external lines. As such unlike with the propagator if we want our theory to be interesting we must take N to be very large such that taking only the leading order is a suitably good approximation but we cannot complete the limit making our approximation exact because the exact vertex factor will be exactly 0.