



# Recurrence Plots of Dynamical Systems

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## Recurrence Plots of Dynamical Systems.

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**Abstract.** – A new graphical tool for measuring the time constancy of dynamical systems is presented and illustrated with typical examples.

In recent years a number of methods have been devised to compute dynamical parameters from time series [1]. Such parameters are the information dimension, entropy, Liapunov exponents, dimension spectrum, etc. In all cases it is assumed that the time series is obtained from an *autonomous* dynamical system, *i.e.* the evolution equations do not contain the time explicitly. It is also assumed that the time series is much longer than the characteristic times of the dynamical system. In the present letter we present a new diagnostic tool which we call *recurrence plot*; this tool tests the above assumptions, and gives useful information also when they are not satisfied. As the examples will show, the information obtained from recurrence plots is often surprising, and not easily obtainable by other methods.

Let  $x(i)$  be the  $i$ -th point on the orbit describing a dynamical system in  $d$ -dimensional space, for  $i = 1, \dots, N$ . The recurrence plot is an array of dots in a  $N \times N$  square, where a dot is placed at  $(i, j)$  whenever  $x(j)$  is sufficiently close to  $x(i)$ . In practice one proceeds as follows to obtain a recurrence plot from a time series  $\{u_i\}$ . First, choosing an embedding dimension  $d$ , one constructs the  $d$ -dimensional orbit of  $\tilde{x}(i)$  by the method of time delays (*i.e.* if the  $u_i$  are scalar,  $\tilde{x}(i) = (u_i, u_{i+1}, \dots, u_{i+d-1})$ ). Next, one chooses  $r(i)$  such that the ball of radius  $r(i)$  centred at  $\tilde{x}(i)$  in  $\mathbf{R}^d$  contains a reasonable number of other points  $\tilde{x}(j)$  of the orbit (in the examples below, we have chosen a sequence of increasing radii and stopped when we found at least 10 neighbours). In our computation we have allowed the radius  $r(i)$  to depend on the point  $\tilde{x}(i)$ , and  $r(i)$  has been selected by a routine used in our algorithm for the determination of the Liapunov exponents—see ref. [2]. Finally, one plots a dot at each point  $(i, j)$  for which  $\tilde{x}(j)$  is in the ball of radius  $r(i)$  centred at  $\tilde{x}(i)$ . We call this picture a *recurrence plot*.

Note that the  $i, j$  are in fact times: therefore a recurrence plot describes natural (but subtle) *time correlation* information. Recurrence plots tend to be fairly symmetric with respect to the diagonal  $i = j$  because if  $\tilde{x}(i)$  is close to  $\tilde{x}(j)$ , then  $\tilde{x}(j)$  is close to  $\tilde{x}(i)$ . There is, however, no complete symmetry because we do not require  $r(i) = r(j)$ .

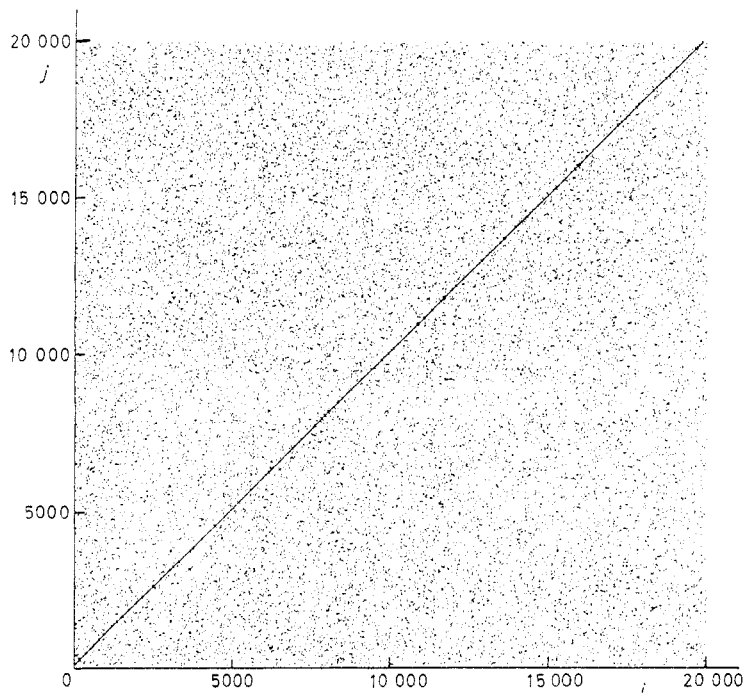


Fig. 1. - A scatter plot for 20 000 iterates of the Henon map  $(x, y) \rightarrow (1 - 1.4x^2 + y, 0.3x)$ . The embedding dimension is 8. In this figure, and in figs. 2 and 4, the diagonal is added.

We shall now analyse some recurrence plots obtained from experimental or computer-generated time series. We distinguish—somewhat arbitrarily—between large-scale *typology* and small-scale *texture*.

For an autonomous dynamical system, if all the characteristic times are short compared to the length of the time series, the typology of the recurrence plot is *homogeneous*. This means that the overall pattern is uniformly grey although at small scale nontrivial texture may be visible (see below). An example of a homogeneous recurrence plot is provided by the Henon map, as exhibited in fig. 1. Another typology is provided by the time evolution with *drift*, *i.e.* by dynamical systems which are not quite homogeneous, but contain adiabatically (slowly) varying parameters [3]. To get an example we started with a time series obtained from the Lorenz system, and added a term varying linearly with time (from 0 to 10 percent of the amplitude of the original signal). The corresponding recurrence plot is our fig. 2. The rich structure of this plot is due to the Lorenz system itself and not to the drift; the main effect of the drift is to make the plot somewhat paler away from the diagonal and darker near it (progressive decorrelation at large time intervals). Since the human eye/brain is not very good at seeing small variations of overall darkness, a histogram of darkness as a function of  $i - j$  is given in fig. 3; this shows unquestionably the existence of a drift. Our next recurrence plot, fig. 4, shows what might be called a *periodic* typology; it is obtained from an experimental time series communicated by Ciliberto and described in ref. [2]. This striking picture shows that—after a transient—the system goes into slow oscillations superimposed on the chaotic motion which is otherwise known to exist (2 positive characteristic exponents have been found in ref. [2]). These slow oscillations (which are not quite periodic) are not easily recognizable on the original time series (and certainly have gone unrecognized in the original analysis).

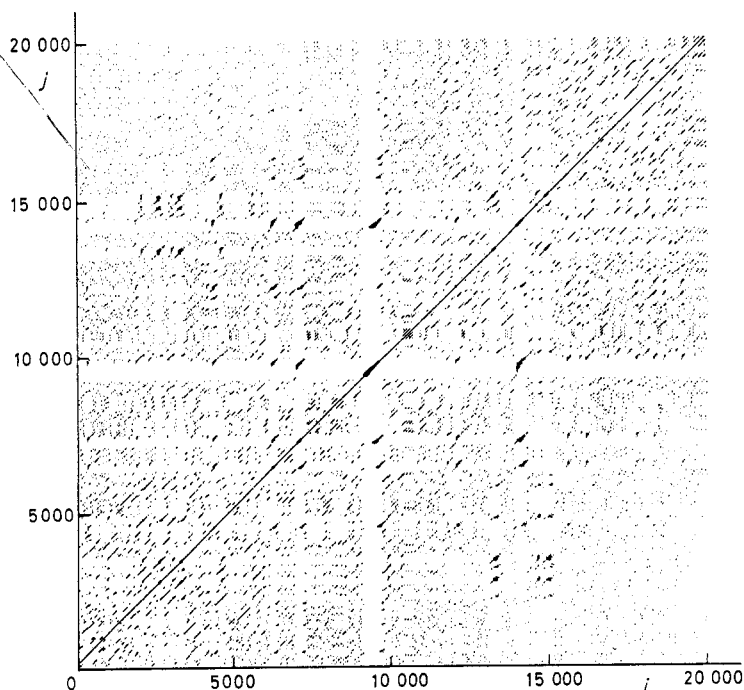


Fig. 2. – The Lorenz system with 10% drift for a total time of 200. The embedding dimension is 3, using the three coordinates as 3 channels of data. A total of  $20\,000 \times 3$  data points is used.

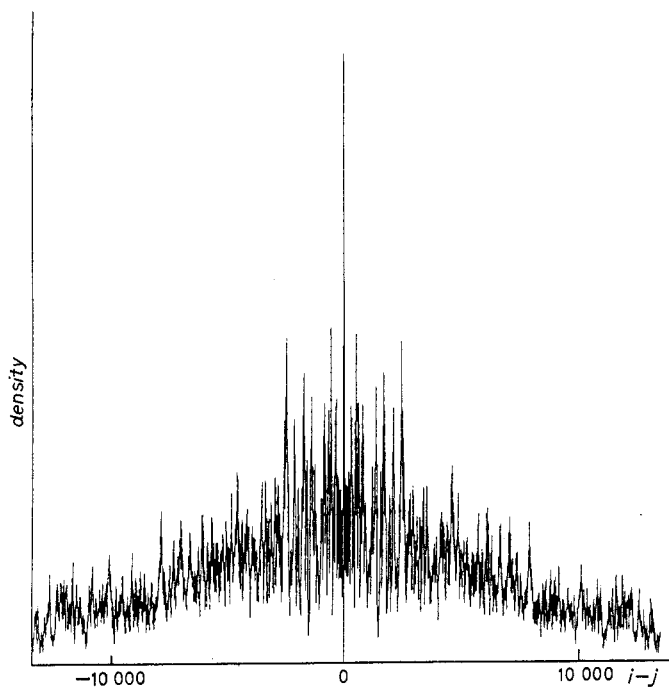


Fig. 3. – The density of distances of black points, as a function of  $i-j$ , averaged over 1% of the time, for the middle 2/3 of fig. 2.

Small-scale *texture* is visible in fig. 2 in the form of *short lines* parallel to the diagonal of the recurrence plot. Such lines would also be visible in fig. 1 and 4 at a larger magnification: they correspond to sequences  $(i, j)$ ,  $(i+1, j+1)$ , ...,  $(i+k, j+k)$  such that the piece of trajectory  $x(j)$ ,  $x(j+1)$ , ...,  $x(j+k)$ , is close to  $x(i)$ ,  $x(i+1)$ , ...,  $x(i+k)$ . The length of the lines is thus related to the inverse of the largest positive Liapunov exponent. If the  $x(i)$  were randomly chosen rather than coming from a dynamical system, there would be no such lines. Another type of texture, the *checkerboard* texture, is expected for the Lorenz system, corresponding to the fact that  $x(i)$  moves on a spiral sometimes around one, sometimes

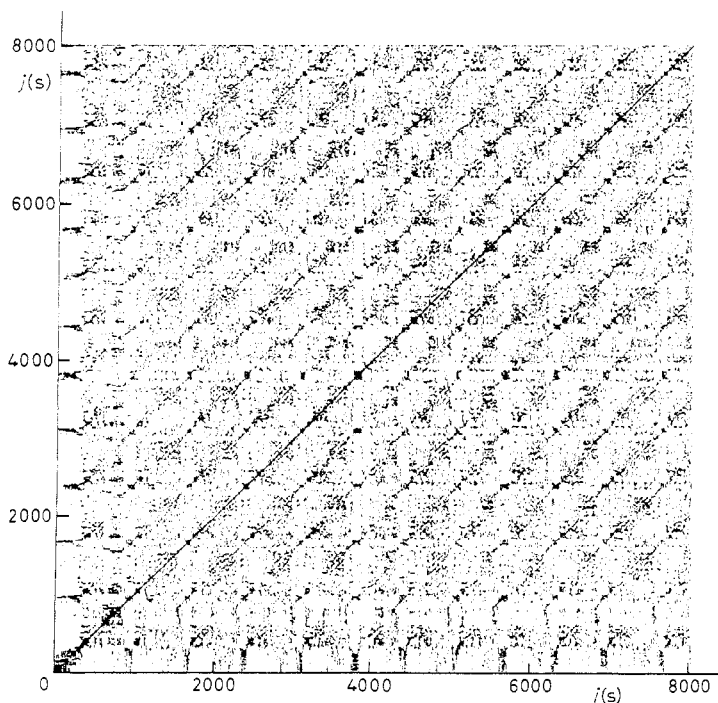


Fig. 4. – The experiment analysed in [2]. A total of 40 000 data points has been used, the time is 8000 s. The embedding dimension is 9.

around the other of the two symmetric fixed points of the system. The checkerboard structure, however, is only barely visible in fig. 2, but a careful analysis shows that the diagonal black spots fall into two mutually exclusive groups having again black spots on the intersection of the horizontal and vertical lines. In fact, the striking events visible in fig. 2 are those when  $x(i)$  spirals for a long time around one of the fixed points. The pattern thus created extends to large scales, and can no longer quite be called a texture.

To conclude, we wish to stress that the recurrence plots are rather easily obtained aids for the diagnosis of dynamical systems. They display important and easily interpretable information about time scales which are otherwise rather inaccessible.

\* \* \*

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