Problem Set 4

Assigned Oct 26 2023, Due Nov 11 2023

Problem 1 (15 points): The goal of this problem is to take descriptions of a few biological systems and develop Markov model representations of those systems. For each description, formally identify a state set then draw a graph showing the transition set. You do not need to identify probabilities for the transitions and can omit self-transitions from your graphs.

a. Assume we are interested in describing the location of a drug molecule given to a patient. We assume it enters the digestive tract (D), can move from there to the bloodstream (B), from bloodstream to kidney (K) or fat (F), and from fat to bloodstream. You can also assume it may stay in its current location on any step.

b. Now suppose we need to simultaneously track two identical copies of the same molecule. Each of the two obeys the same rules as in part a and on any step of the model at most one protein can undergo a state change.

c. Now suppose that we are instead interested in tracking a drug that exists in many copies but we only want to know which subset of the locations have at least one copy of the drug in them, not how many copies each has. Assume there are enough copies that once the drug enters a location, there is always some of the drug present in that location. Assume each individual drug molecule behaves according to the same rules as in part a.

Problem 2 (15 points): Let us assume we have a consumer resource model with the following rules:

$$\frac{dn_1}{dt} = \theta - acn_1n_2$$
$$\frac{dn_2}{dt} = \epsilon acn_1n_2 - \gamma n_2$$

Resources n_1 enter the system at constant rate and are constantly eaten by a consumer species n_2. For example resources could be larvae that are predated. Assume that 1000 new larvae per unit time ($\theta = 1000$), that the per capita contact rate is $c = 0.01$ per unit time, the consumption upon detection is $a = 1$, that one larva is the energetic equivalent of $\epsilon = 0.0005$ and per capita death rate of predators is $\gamma = 0.001$.

a) Draw a vector-field plot for this model.

b) Determine the null clines (the directions in which each of the two variables is constant in time) and add them to the plot.

c) Describe what happens to the predators and prey over time and any potential limitations of the vector field or phase plane plot in this case.

Problem 3 (20 points): Metastasis of malignant tumors model. Metastasis is a process by which cancer cells spread throughout the body. Sometimes cancer cells move via the bloodstream and become lodged in the capillaries of different organs. Some of these cells then move across the capillary wall, where they initiate new tumors. We will construct a model for the dynamics of the number of cancer cells lodged in the capillaries of an organ, C, and the number of cancer cells that have actually invaded the organ, I. Supposes that cells are lost from the capillaries by dislodgment or death at a per capita rate γ_1 and that they invade the organ from the capillaries at a per capita rate β. Once in the organ, they die at per capita rate γ_2 and the cancer cells here replicate at rate ρ. All these parameters are assumed positive.

Draw the events possible just like we did for SIR models and then write out the rules governing C and I in equation form, as well as vector form.

The question of interest is: Will a new tumor be able to take hold and grow or will it disappear? To this end find the equilibria of this system, study their stability and describe what you can conclude.

Problem 4 (30 points): This is a programming problem in which we will apply a variant of common kind of model of disease propagation. Suppose we have been attacked by zombies. We would like to know if our society is likely to survive the attack. We will investigate that question by using a SIR (or Susceptible, Infected, Recovered model) model. In this version of an SIR model, we will assume that we have a population of N people. Initially m of these people are infected with the disease (zombie-ism) and $N - m$ are healthy.

Each infected person is assumed to have contact with λ_1 people per unit time. As a result, the rate at which new people are infected will depend on the number of people currently infected and the fraction of the contacts that are with susceptible people, $\lambda_1 I \frac{S}{N}$, where S and I are the numbers of susceptible and infected people.

The R in an SIR model classically refers to people who recover from the disease and become immune, although in this case we will assume that no one recovers from being a zombie but zombies sometimes die, e.g., by wandering off a cliff. We will initially assume that the rate at which a zombie dies is λ_2, so the overall rate of zombie death in the population is $\lambda_2 I$.

a. Pose this system as a continuous time Markov model. Describe a generic state, list all transitions from that state, and provide their rates.

b. Provide pseudocode for a simulator for this model. You can assume the simulation stops when $I = 0$.

c. Suppose we want to consider how we might change the odds of our survival if we learn to fight zombies effectively. We assume that if living people learn to fight zombies, then we can introduce a second rate of zombie death to be proportional to S. We will then have azombie death of the form $\frac{\lambda_3 IS}{S+I+R}$ from living people killing zombies in addition to the natural zombie death rate $\lambda_2 I$. Modify your pseudocode to implement this variant of the problem.

d. Create code implementing your two versions of the model that takes as input λ_1, λ_2, λ_3, N, m, and one additional parameter t, a number of trials. Your simulators should run the model until $I = 0$ for t independent trials. At the end, it should output the fraction of trials in which everyone ended up infected ($R = N, S = 0$ when the simulator terminates).

e. We would like to ask if there are conditions under which it matters whether we fight back against the zombies. Try running each simulator for $N = 10,000$, $m = 100$, $\lambda_1 = 1.5$, $\lambda_3 = 1$, and $t = 100$, but with varying λ_2: 0.001, 0.01, 0.1, 1, 10, 100. Provide a plot of your results for each of the two assumptions, fighting back or not fighting back.

f. Would you conclude that fighting back against the zombies always helpful, never helpful, or sometimes helpful? Can you draw any conclusions about why fighting back does or does not help?

***Problem 5 (20 points):** Simulate the stochastic model of logistic growth and compare the trajectories with the deterministic model. Discuss what you observe. To set up the stochastic model, start with a population of ten individuals and draw the number of individuals at generation (t+1) from a Poisson distribution with mean $(1 + r(1 - n(t)/K))n(t)$ where $r = 0.2$, $K = 100$, for 100 generations.

Now simulate and plot trajectories for the the stochastic model with r=2.4 and compare with the deterministic logistic model with r=2.7. What do you observe? What do you conclude in terms of our ability to infer the rule or type of process from observed trajectory/data?

*Recall that the starred problems are only for the 02-712 students.