MSCA 37016 - HOMEWORK 4

1] The quadratic characteristic polynomial $f(\lambda)$ of matrix A is as follows:

$$f(\lambda) = \det(A)$$

$$= \begin{bmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix}$$

$$= (5-\lambda)(5-\lambda) - 1(1)$$

$$= \lambda^2 - 10\lambda + 25 - 1$$

Ans. $\boxed{f(\lambda) = \lambda^2 - 10\lambda + 24}$

2] $f(\lambda) = \lambda^2 - 10\lambda + 24$

$$= \lambda(\lambda-6) - 4(\lambda-6)$$

$$= (\lambda-6)(\lambda-4)$$

$\lambda - 6 = 0$ $\qquad\qquad$ $\lambda - 4 = 0$

$\lambda = 6$ $\qquad\qquad$ $\lambda = 4$

Ans. $\boxed{\therefore \lambda_1 = 6 \qquad \lambda_2 = 4}$

3] The matrix A is invertible as it does not have an eigenvalue equal to 0 as seen in answer 2.

4] For $\lambda_1 = 6$, the eigen vector is:

$$\begin{bmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} = \begin{bmatrix} 5-6 & 1 \\ 1 & 5-6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

The reduced row echelon form of this is:

$$\begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The null space of this matrix is:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - v_2 = 0$$

$$v_1 = v_2$$

$$\therefore \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

The unit vector $\hat{v}_1 = \dfrac{\vec{v_1}}{\|\vec{v_1}\|} = \dfrac{1}{\sqrt{1+1}} \vec{v} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

For $\lambda_2 = 4$, the eigen vector is:

$$\begin{bmatrix} 5-4 & 1 \\ 1 & 5-4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The reduced row echelon form of this is:

$$R_2 = R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The null space of this is:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0$$

$$v_1 = -v_2$$

$$\therefore \vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

The unit vector $\hat{v_2} = \dfrac{\vec{v_2}}{\|\vec{v_2}\|} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Ans. unit vector $\hat{v}_1$ for eigenspace $E_{\lambda_1} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix}$

unit vector $\hat{v}_2$ for eigenspace $E_{\lambda_2} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -0.7071 \\ 0.7071 \end{pmatrix}$

5] To verify: $A = VDV^T$

$A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$; $V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$; $D = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$; $V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$VDV^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$VDV^T = \begin{bmatrix} 6/\sqrt{2} & -4/\sqrt{2} \\ 6/\sqrt{2} & 4/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$VDV^T = \begin{bmatrix} 6/2 + 4/2 & 6/2 - 4/2 \\ 6/2 - 4/2 & 6/2 - 4/2 \end{bmatrix}$$

$$\boxed{VDV^T = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}}$$

$$\therefore A = VDV^T = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

Hence spectral theorem was proved.

6] Verify: $tr(A) = \lambda_1 + \lambda_2$

$tr(A) = 5 + 5$

$\boxed{tr(A) = 10}$

$\lambda_1 + \lambda_2 = 6 + 4$

$\boxed{\lambda_1 + \lambda_2 = 10}$

$\therefore$ Hence proved $tr(A) = \lambda_1 + \lambda_2 = 10$

7] Verify: $det(A) = \lambda_1 \lambda_2$

$det(A) = 5(5) - 1(1) = 24$

$\lambda_1 \lambda_2 = 6(4) = 24$

$\therefore$ Hence proved $det(A) = \lambda_1 \lambda_2 = 24$

8] $f(\lambda) = \lambda^2 - 10\lambda + 24$

$f(0) = 0 - 0 + 24 = 24$

$\det(A) = 5(5) - 1(1) = 24$

As seen from above, $f(0) = \det(A) = 24$

Ans. Therefore, the relationship between $f(0)$ and $\det(A)$ is of equality.