The construction of the consistent self-gravitating galaxy model with specific rotation curve

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Background

- How to set the initial conditions for the galaxy simulation

  > Simulate for some time to achieve a dynamic equilibrium

  > Virialize initially

$$\boxed{\sigma_R^{\ 2}} \quad \sigma_\theta^{\ 2} \quad \overline{\upsilon_R} \quad \overline{\upsilon_\theta}$$

Binney & Tremaine (1987)

Applied by Hernquist(1993) , Springel (2005), etc.

$$\sigma_R \leftarrow Q \equiv \frac{\sigma_R \kappa}{3.36 G\Sigma} \qquad \sigma_\theta = \sigma_R \frac{\kappa}{2\Omega} \qquad \overline{\upsilon_\theta}^2 - \upsilon_c^{\ 2} = \sigma_R^{\ 2}\left(1 - \frac{\kappa^2}{4\Omega^2} - 2\frac{R}{R_d}\right)$$

~~Epicyclic approximation~~

Main idea

- Obtain a consistent distribution function

$$f\left(\vec{\upsilon},\vec{x}\right) \longrightarrow \sigma_R{}^2 \quad \sigma_\theta{}^2 \quad \overline{\upsilon_R} \quad \overline{\upsilon_\theta}$$

$$f\left(E,L_z\right) \longleftarrow \text{Jeans Theorem}$$

$$\frac{df}{dt}=\frac{\partial f}{\partial E}\frac{dE}{dt}+\frac{\partial f}{\partial L_z}\frac{dL_z}{dt}=0$$

Main idea

- A typical profile of distribution function (Entropy maximized)

$$f\left(E,L_z\right)=P\left(L_z\right)\exp\left[\frac{-\left(E-E_c\right)}{\sigma^2}\right]$$

$$\sigma \longleftarrow \quad Q\equiv\frac{\sigma\kappa}{3.36G\Sigma}=1$$

$$\iint P\left(L_z\right)\exp\left[\frac{-\left(E-E_c\right)}{\sigma^2}\right]dp_R dL_z=R\cdot\Sigma\left(R\right)$$

$$\downarrow$$

$$P\left(L_z\right) \longleftarrow \quad \int_0^{R_{h,0}\left(R\right)} K_0\left(R,R_h\right)R_h S\left(R_h\right)dR_h=R\cdot\Sigma\left(R\right)$$

$$R_h^{\ 2}\Omega\left(R_h\right)=L_z$$

Main idea

- Volterra-type integral equation

$$\int_0^{R_{h,0}(R)} \underbrace{K_0\left(R,R_h\right)}_{\text{Weight Function}} \cdot R_h \cdot S\left(R_h\right) dR_h = R \cdot \Sigma\left(R\right)$$

Same Dimension

Main methods

- Discretize the integration into several short intervals, within which apply some kind of integral formula

$$\int \cdots \longrightarrow \sum \cdots$$

- Expand the unknown function into a series of orthogonal polynomials

$$F(x) \longrightarrow \{A_j\}_{j=0,1,2,\ldots}$$

Method I

- The Gauss-Legendre integral formula

$$\int_{-1}^{1} f(x)dx \approx I_*(f) = f\left(1/\sqrt{3}\right) + f\left(-1/\sqrt{3}\right)$$

- Discretize the integration into several short intervals

$$\int_0^{R_{h,0}(R)} K_0(R,R_h)\cdot R_h \cdot S(R_h)dR_h = \sum_{j=0}^{N(R)} \int_{R_h^{(j)}}^{R_h^{(j+1)}} K_0(R,R_h)\cdot R_h \cdot S(R_h)dR_h \qquad R_h^{(j)} = jh,\ j \in [0,N]$$

- In each interval, apply the G-L integral formula

$$\int_{R_h^{(j)}}^{R_h^{(j+1)}} K_0(R,R_h)\cdot R_h \cdot S(R_h)dR_h$$
$$= \left[K_0\left(R,R_{h,-}{}^{(j)}\right)\cdot f\left(R_{h,-}{}^{(j)}\right) + K_0\left(R,R_{h,+}{}^{(j)}\right)\cdot f\left(R_{h,+}{}^{(j)}\right)\right]\cdot\left(R_h^{(j+1)} - R_h^{(j)}\right)/2$$

$$R_{h,-}{}^{(j)} = \left(\left(-1/\sqrt{3}+1\right)/2 + j\right)h, \quad R_{h,+}{}^{(j)} = \left(\left(1/\sqrt{3}+1\right)/2 + j\right)h$$

Method I

- Input a series of data points

$$R \in \left\{ R_i \right\}_{i=0,1,\cdots N}$$

- The integration turns into a matrix multiplication

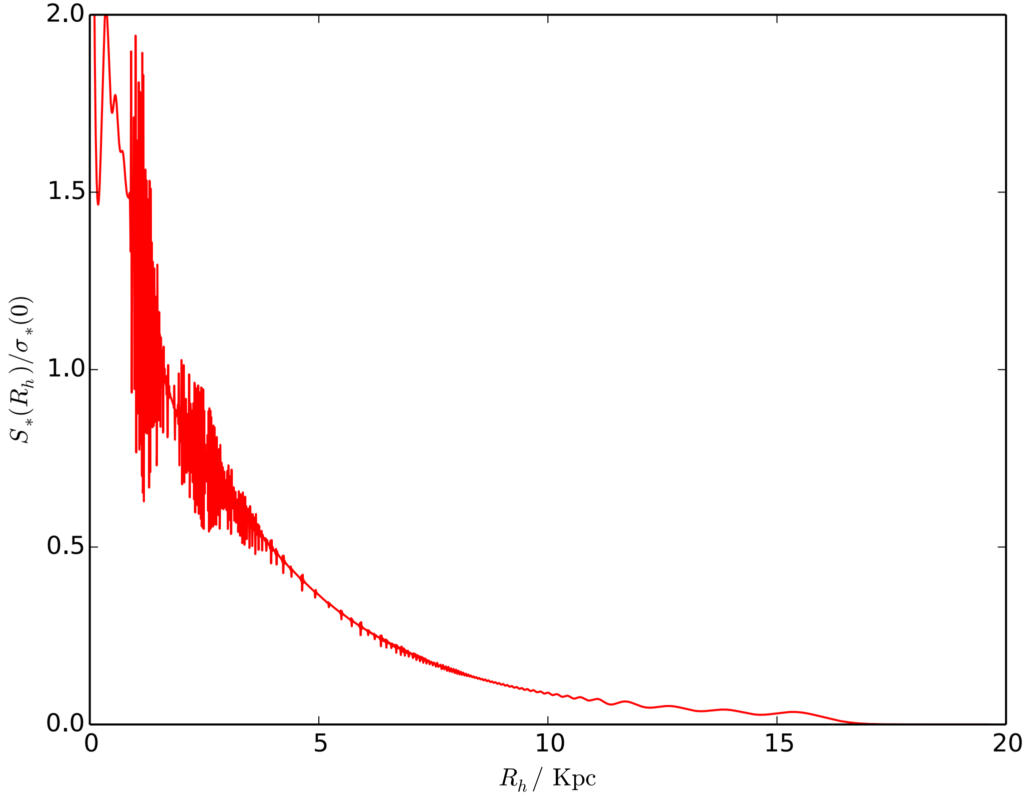
$$U F_* = G_*$$

- Operate the matrix inversion

$$F_* = U^{-1} G_*$$

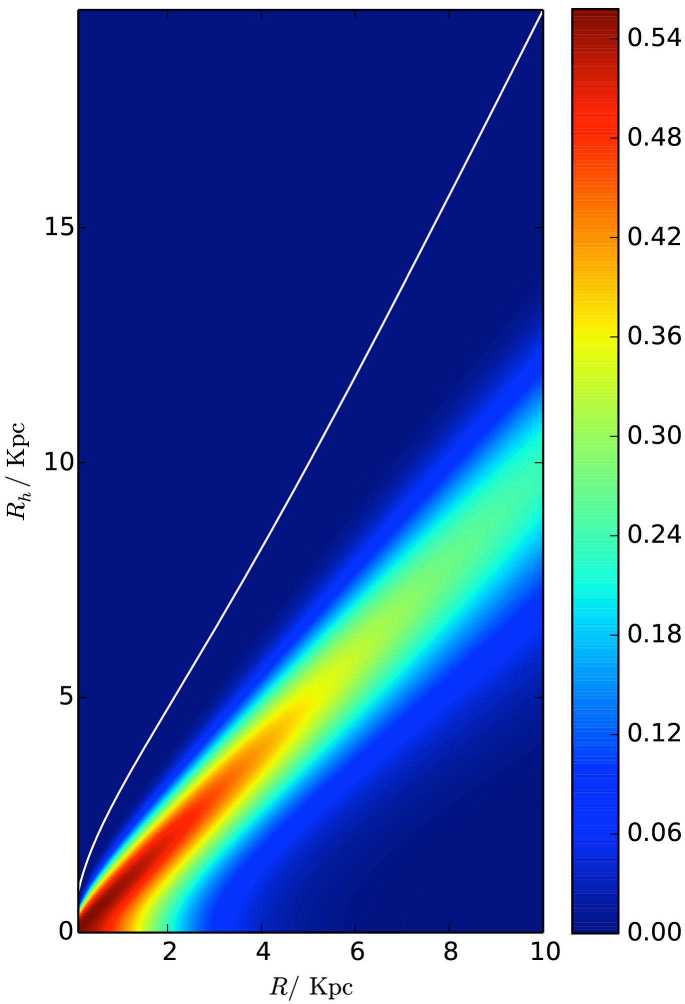
Result I

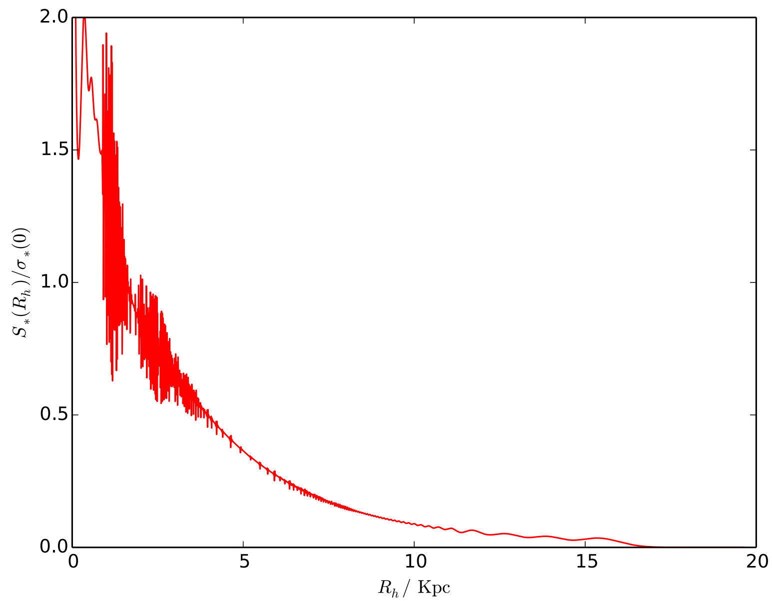
- $S(R_h)$

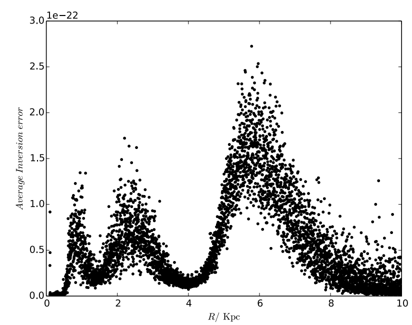


Problem I

- Matrix inversion error







Solution I

- Modify the weight function $K_0(R, R_h)$

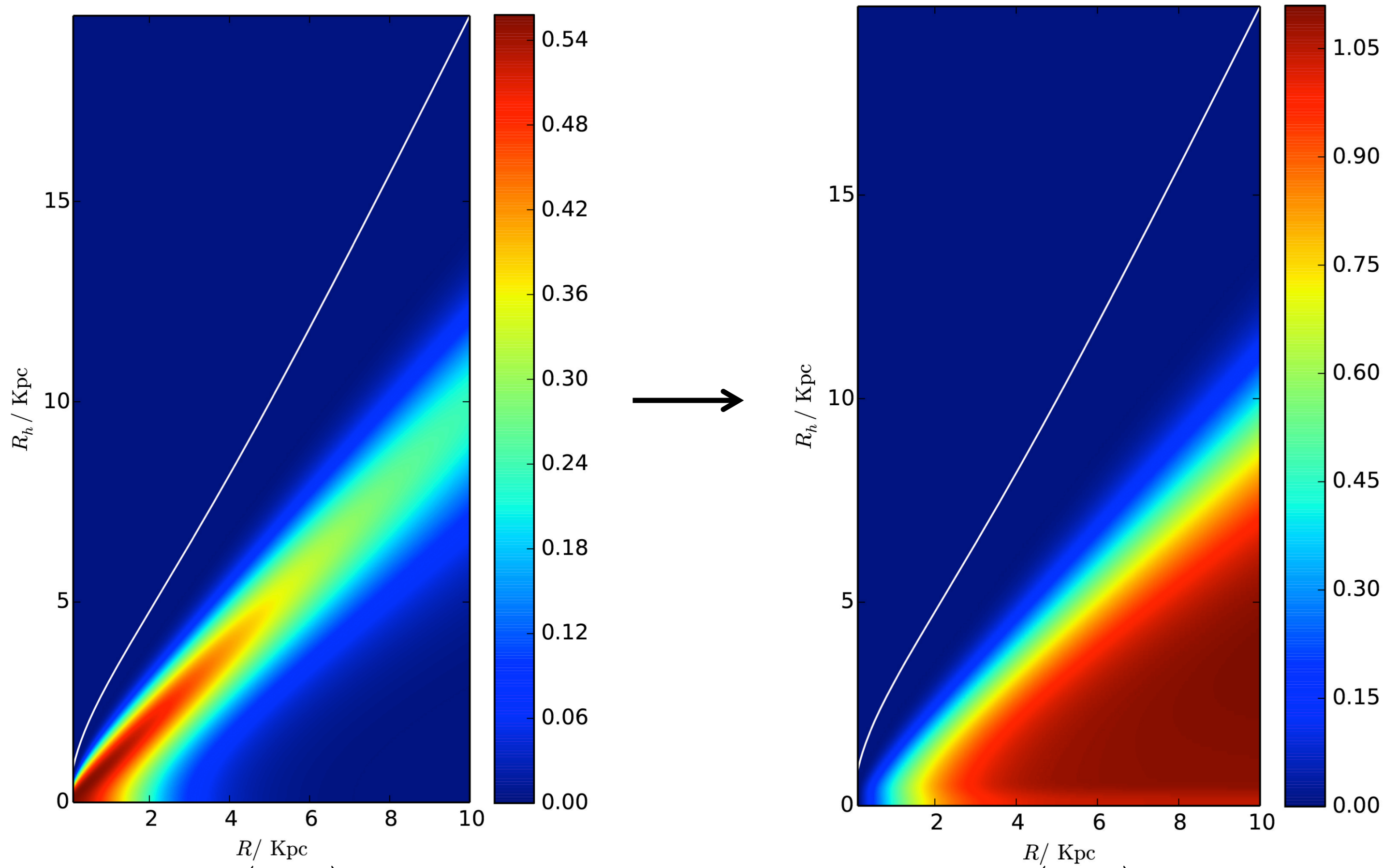
$$\int_0^R dR \int_0^{R_{h,0}(R)} dR_h \, K_0(R, R_h) \cdot R_h \cdot S(R_h) = \int_0^R dR \, R \, \Sigma(R)$$

$$\int_0^{R_{h,0}(R)} dR_h \, K_1(R, R_h) \cdot R_h \cdot S(R_h) = \int_0^R dR \, R \, \Sigma(R)$$

$$K_0(R, R_h) \longrightarrow K_1(R, R_h)$$

Solution I

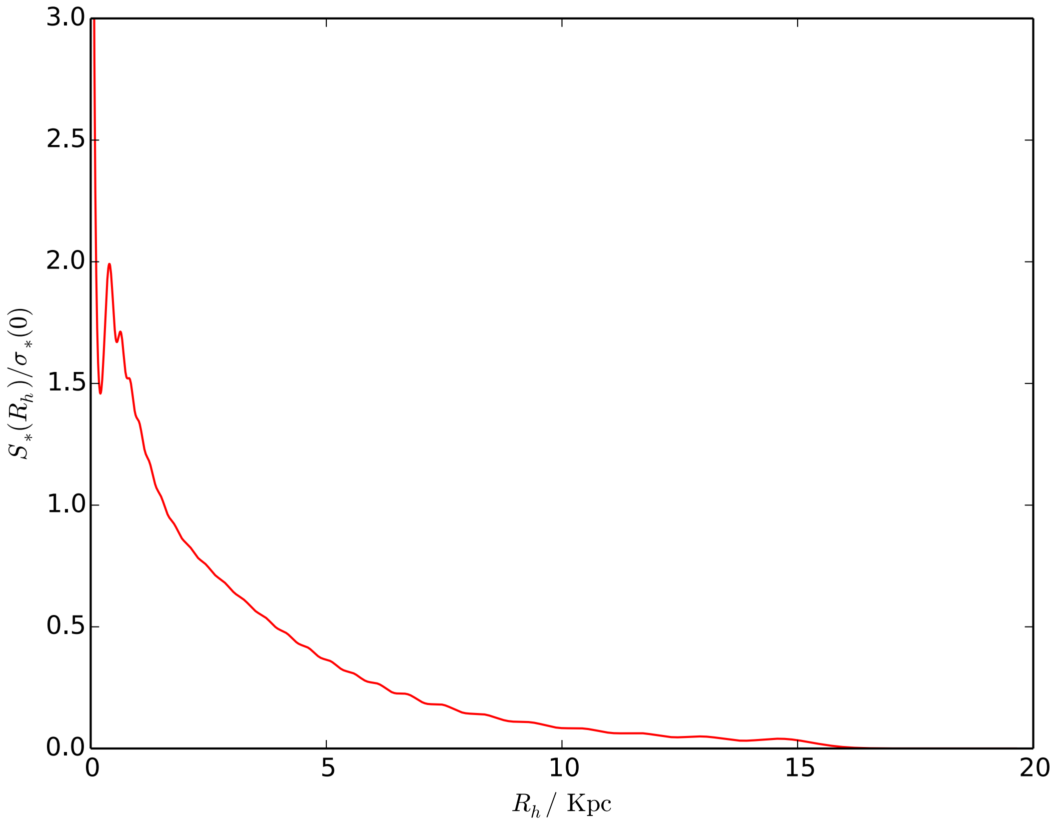
- Modified weight function $K_1\left(R,R_h\right)$



$K_0\left(R,R_h\right)$ $K_1\left(R,R_h\right)$

Result I.I

- $S(R_h)$



Method II

- The Legendre polyminials

$$P_j(x) = \frac{1}{2^j j!} \frac{d^j}{dx^j}\left[\left(x^2 - 1\right)^j\right]$$

- Expand the unknown function

$$S(R_h) = \sum_{j=0}^{N} A_j P_j(x)$$

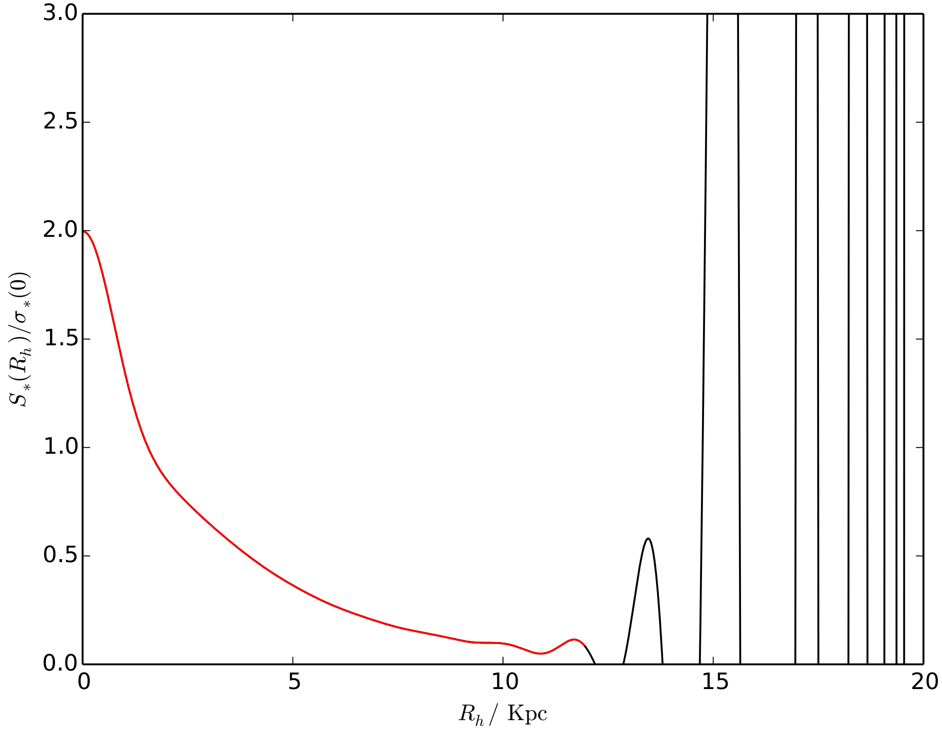
- Calculate the integration for each basis and turn the integral equation into a series of linear equations

$$\sum_{j=0}^{N} A_j \int_0^{R_{h,0}(R)} K_0(R, R_h) \cdot R_h \cdot P_j\left[x(R_h)\right] dR_h = R \cdot \Sigma(R)$$

- Apply the least square method to determine each coefficient

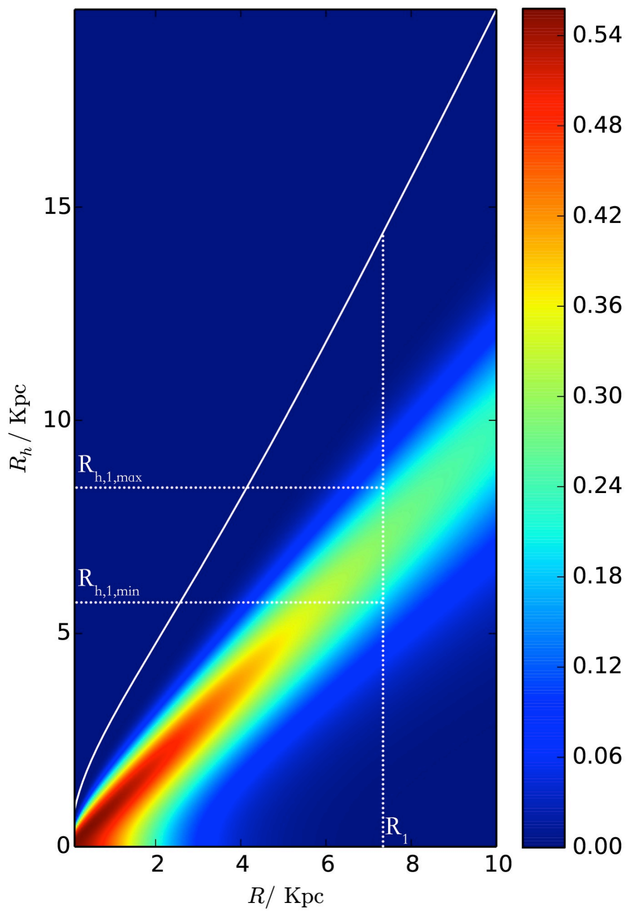
Result II

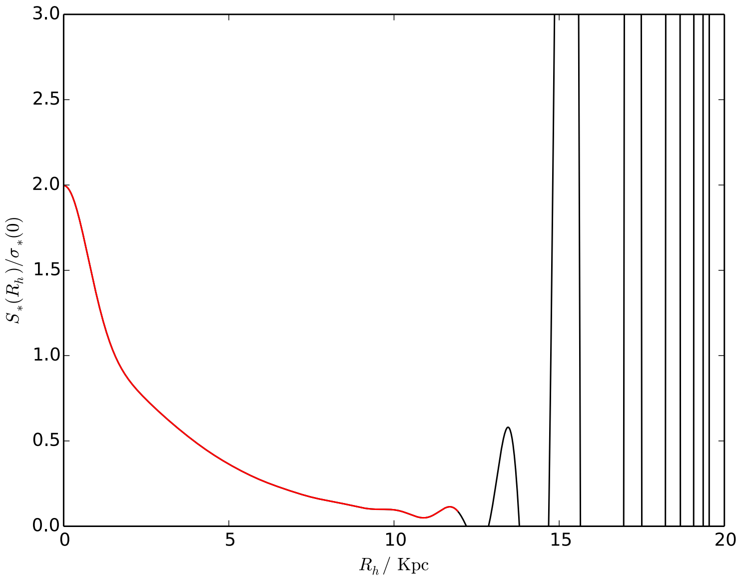
- $S\left(R_h\right)$



Problem II

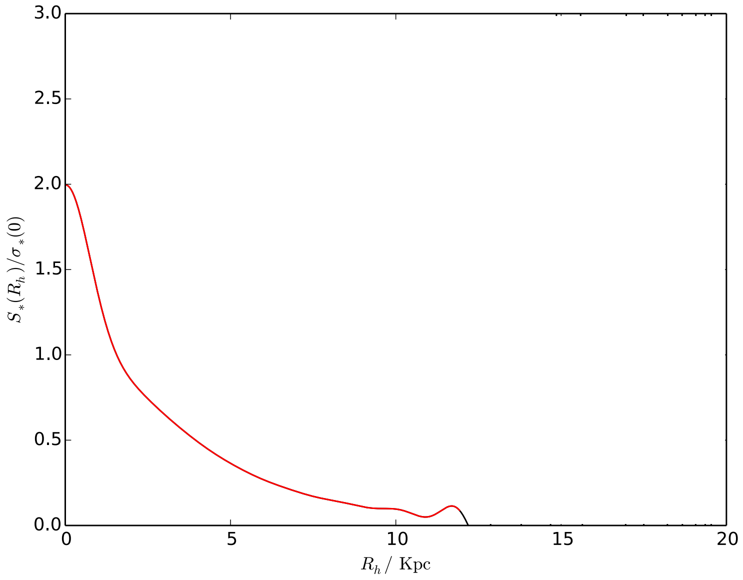
- The singularity of weight function $K_0\left(R,R_h\right)$





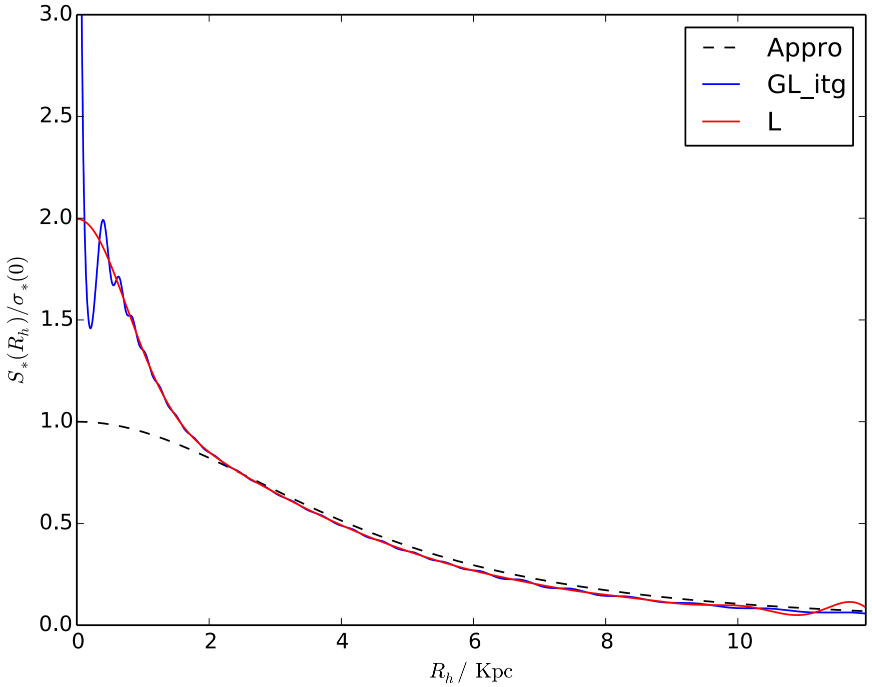
Solution II

- Cut it off!



Comparison

- Shu's appromixation $S\left(R_h\right) \approx \Sigma\left(R_h\right)$
- My derivation: $S\left(R_h\right) \approx 0.92\,\Sigma\left(R_h\right)$



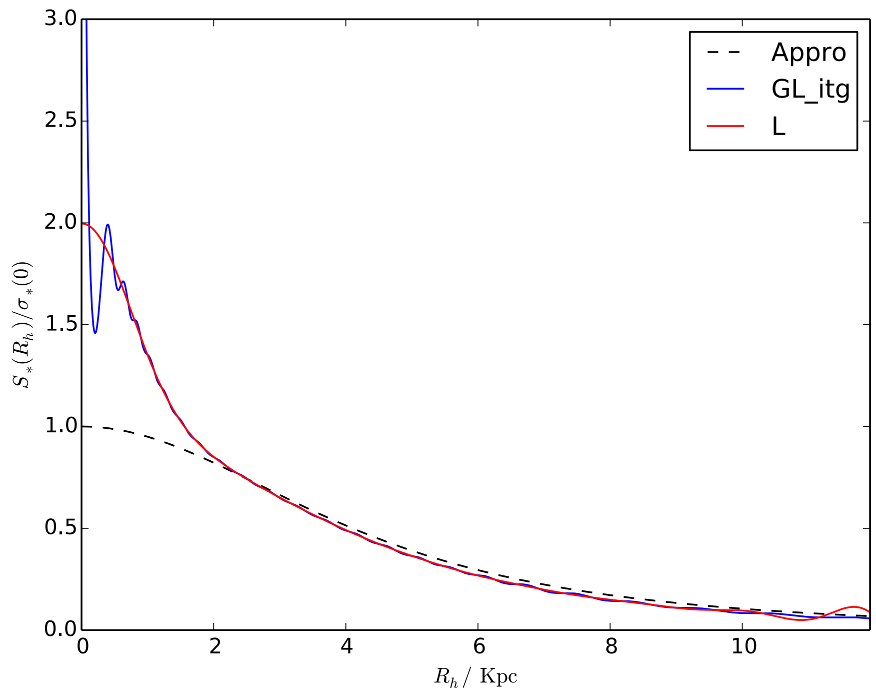
Problem

- $S(0) = ?$
- My derivation: $S(0) \approx \frac{\pi^{3/2} x_T}{2\chi_s} \Sigma(0) = 2.0001\Sigma(0)$

$$\chi_s = \int_0^1 s \exp\left[-\frac{s^4}{(\pi x_T)^2}\right] erf\left[\frac{(1-s^4)^{1/2}}{\pi x_T}\right] ds$$

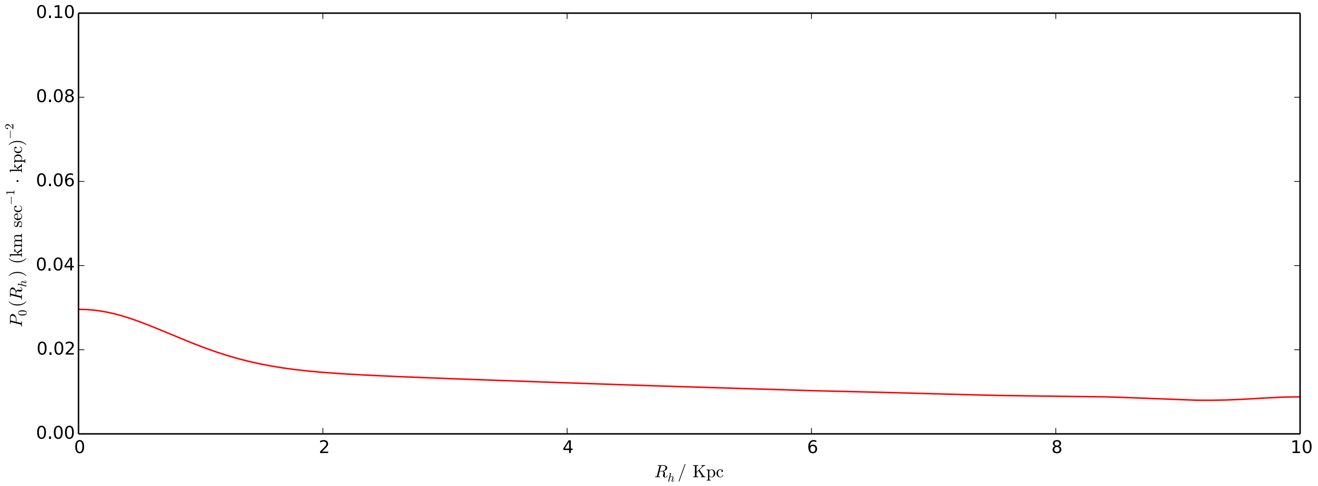
$$x_T = 0.08507$$

Constant from Toomre's criterion



Other result

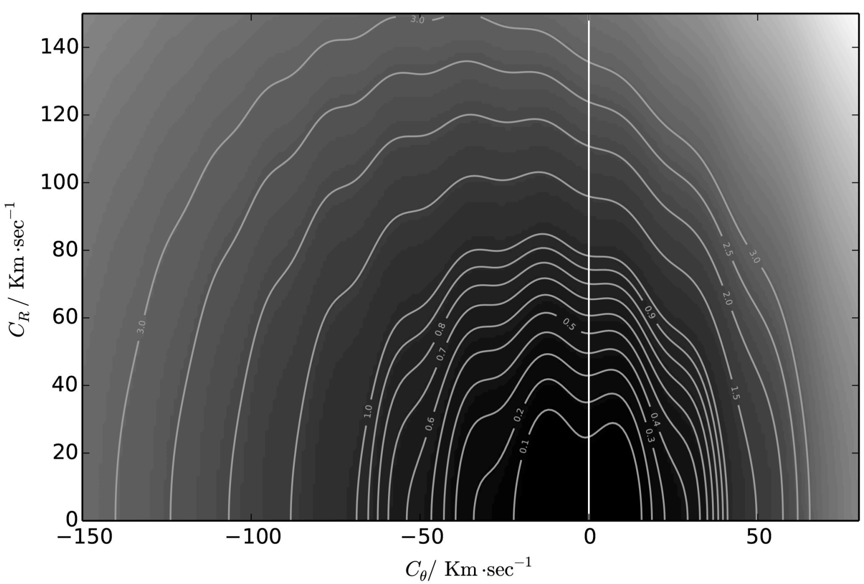
- $P_0\left(R_h\right)$

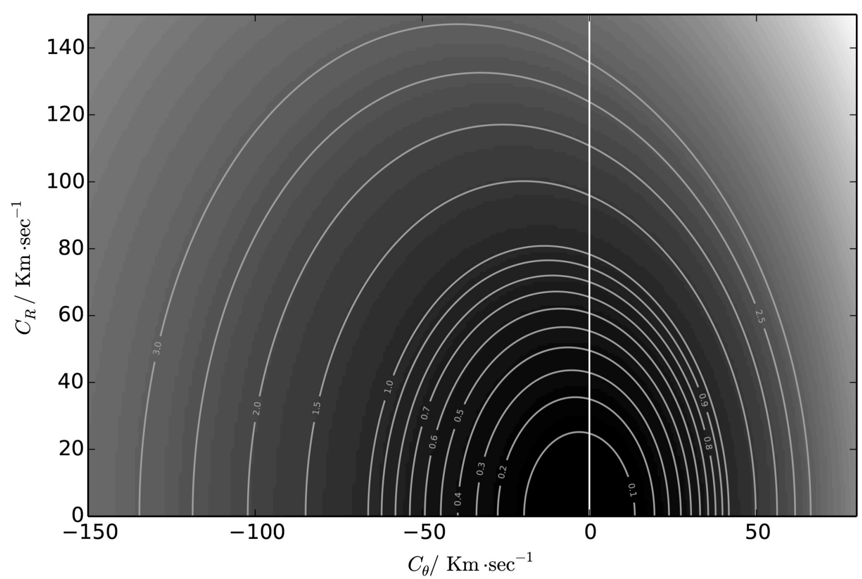


Other result

- Distribution function

$$f\left(E, L_z\right) \longrightarrow f\left(R, C_\theta, C_R\right)$$





G-L

$R = 10\,kpc$

Appro

Thanks for your time