Lab Assignment 2

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Question 1: Optimizing parameters

1

Finding a0, a1, a2 that minimizes squared error between the selected function and interpolate function using optim()

```
parabolic_inter = function(a,x){
  a0 = a[1]
  a1 = a[2]
  a2 = a[3]
  res = a0 + a1*x + a2*(x^2)
  return(res)
}
sum_square_error = function(a,x,func){
  x0 = x[1]
  x1 = x[2]
  x2 = x[3]
  res = sum((func(x0)-parabolic_inter(a,x0))^2,(func(x1)-parabolic_inter(a,x1))^2,
            (func(x2)-parabolic_inter(a,x2))^2)
  return(res)
}
opt = function(x,func){
  a = c(0,0,0)
  res = optim(a,sum_square_error,x = x,func = func)
  res = res$par
  return(res)
}
```

2

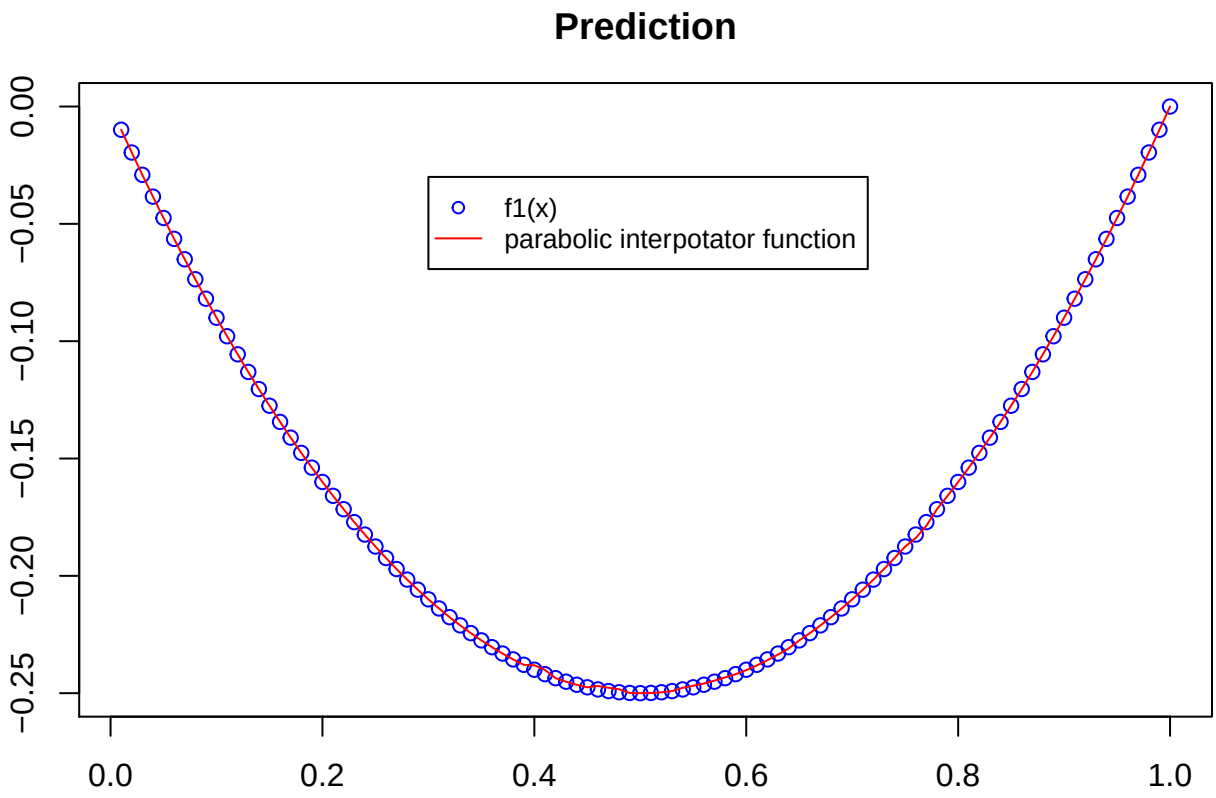
Approximate function

```
approx = function(n,func){
 interval = 1/n
 midpoint = 1/(2*n)
 res = data.frame()
 for(i in 1:n){
   end_point = i/n
```

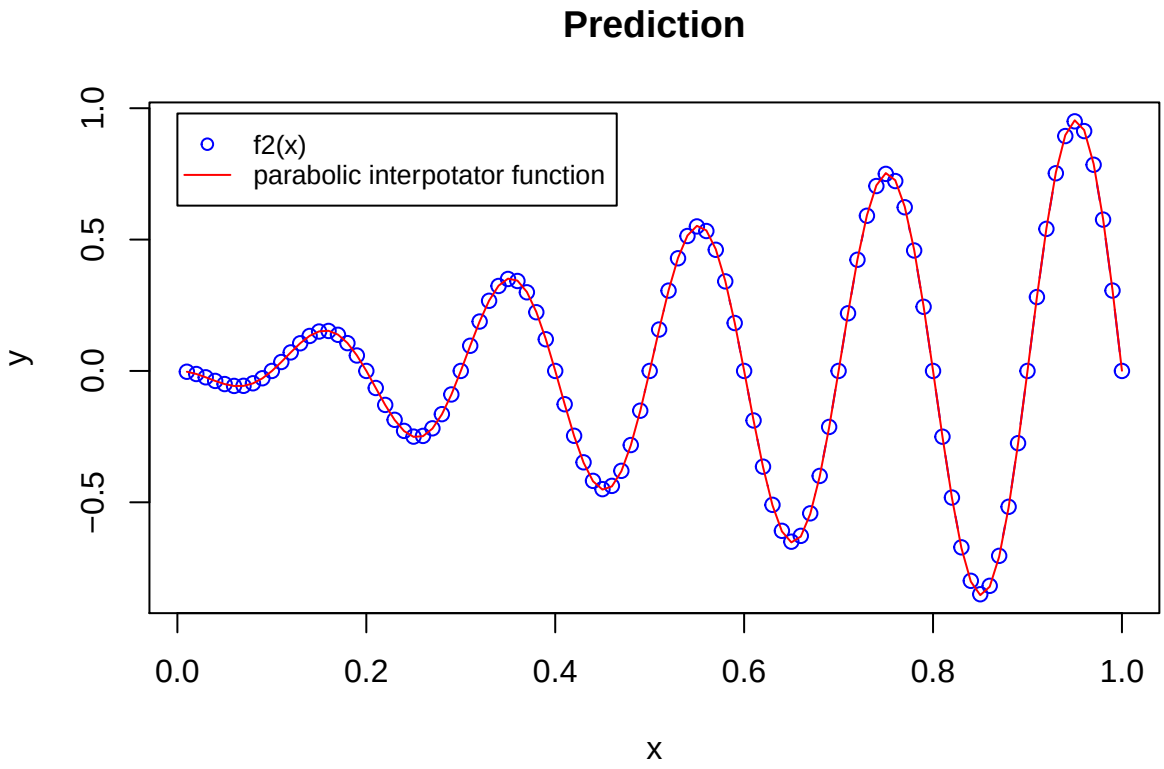
```
   x = c((end_point - interval),(end_point - midpoint),end_point)
   res = append(res,as.data.frame(opt(x,func)))
 }
 return(res)
}
```

3

Comparing the result from f1(x) and parabolic interpolator function for 100 subintervals.



Comparing the result from f2(x) and piecewise parabolic interpolator function for 100 subintervals.



Observing the above two plots , we can say that the prediction from piecewise parabolic interpolater was fair as the predictions has very low errors.

Question 2: Maximizing likelihood

1. Load the data to R environment.

```
load("data.RData")
```

2. A sample from normal distribution with some parameters μ, σ.

The probability density function of normal distribution is:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We have the following n = 100 i.i.d observations:

$$x_1, x_2, ..., x_n$$

The likelihood function:

$$f(x_1, x_2, ..., x_n|\sigma, \mu) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = (\frac{1}{\sigma\sqrt{2\pi}})^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

The log likelihood function:

$$log(f(x_1, x_2, ..., x_n|\sigma, \mu)) = log((\frac{1}{\sigma\sqrt{2\pi}})^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2})$$

$$= nlog\frac{1}{\sigma\sqrt{2\pi}} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -\frac{n}{2}log(2\pi) - nlog\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Let's call

$$log(f(x_1, x_2, ..., x_n|\sigma, \mu))$$

as L and derivation of the log-likelihood by μ by setting partial derivatives to zero

$$\frac{\partial L}{\partial \mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \mid \mu = 0$$

Solve this equation, we get

$$\frac{1}{2\sigma^2}\sum_{i=1}^{n}(2\hat{\mu} - 2x_i) = 0$$

Because σ^2 should be larger than 0, Mean of the sample

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Similarly, derivation of the log-likelihood by σ

$$\frac{\partial L}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n}(x_i-\mu)^2\sigma^{-3} = 0$$

Variance of the sample

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\hat{\mu})^2}{n}$$

Standard Deviation of the sample

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\hat{\mu})^2}{n}}$$

```
mu_hat =  sum(data)/length(data)
# formula for sigma
sigma_hat = sqrt(sum((data - mu_hat)^2)/length(data))
cat(paste("Mean of data", mu_hat))
```

```
Mean of data 1.27552760103638
```

```
cat(paste("Standard Deviation of data", sigma_hat))
```

```
Standard Deviation of data 2.00597647210603
```

3. Minus log–likelihood function with initial parameters μ = 0, = 1.

```r
minusloglikenormal = function(theta, x=data) {
  mu<-theta[1]                    # mean of normal distribution
  sigma2<-theta[2]                # variance of normal distribution
  n = length(x)
  a1 = (n/2)*log(2*pi)-(n/2)*log(sigma2)
  a2 = (sum((x - mu)^2)) / (2 * sigma2^2)
  ans = a1+a2
  return(ans)
}
minusloglikenormal(c(0,1),data)
```

```
[1] 374.4395
```

Gradient method

```r
gradient = function(theta, x = data) {
  mu<-theta[1]
  sigma2<-theta[2]
  n = length(x)
  dmu = sum(mu - x)/sigma2^2
  dsig = (n/sigma2) - sum((x - mu)^2)/sigma2^3
  gr = c(dmu,dsig)
  return(gr)
}
gradient(c(0,1),data)
```

```
[1] -127.5528 -465.0912
```

```r
# Conjugate Gradient method without gradient method:
opt_cg <- optim(par = c(0, 1), fn = minusloglikenormal, method = "CG")

# Conjugate Gradient method without gradient method:
opt_cg_gr <- optim(par = c(0, 1), fn = minusloglikenormal, gr = gradient,  method = "CG")

# BGFS method without gradient method:
opt_bfgs <- optim(par = c(0, 1), fn = minusloglikenormal, method = "BFGS")

# BGFS method with gradient method:
opt_bfgs_gr <- optim(par = c(0, 1), fn = minusloglikenormal, gr = gradient, method = "BFGS")
```

It is more convenient to maximize the log of the likelihood function. Because the logarithm is monotonically increasing function of its argument, maximization of the log of a function is equivalent to maximization of the function itself. Taking the log not only simplifies the subsequent mathematical analysis, but it also helps numerically because the product of a large number of small probabilities can easily underflow the numerical precision of the computer, and this is resolved by computing instead the sum of the log probabilities.

4. Algorithms converge

```r
cg1 = c(opt_cg$convergence, opt_cg$par, opt_cg$value, opt_cg$counts[1],
        opt_cg$counts[2])
cg2 = c(opt_cg_gr$convergence, opt_cg_gr$par, opt_cg_gr$value,
        opt_cg_gr$counts[1], opt_cg_gr$counts[2])
```

```
bfgs1 = c(opt_bfgs$convergence, opt_bfgs$par, opt_bfgs$value,
          opt_bfgs$counts[1], opt_bfgs$counts[2])
bfgs2 = c(opt_bfgs_gr$convergence, opt_bfgs_gr$par, opt_bfgs_gr$value,
          opt_bfgs_gr$counts[1], opt_bfgs_gr$counts[2])
df <- as.data.frame(matrix(c(cg1, cg2, bfgs1, bfgs2), nrow=6))
rownames(df) <- c("Convergence", "Mean", "Standard Deviation", "minusloglikelihood",
                  "function evaluation", "gradient evaluation")
colnames(df) <- c("CG", "BFGS", "CG with gradient", "BFGS with gradient")
knitr::kable(x = df, caption = "Comparison table of the different algorithms")
```

Table 1: Comparison table of the different algorithms

	CG	BFGS	CG with gradient	BFGS with gradient
Convergence	1.0000	0.0000	0.0000	0.0000
Mean	122.8803	127.5528	-17266.3035	127.5528
Standard Deviation	628.7519	466.0912	54456.2258	466.0912
minusloglikelihood	-228.4387	-211.6542	-448.3364	-211.6542
function evaluation	201.0000	64.0000	17.0000	40.0000
gradient evaluation	101.0000	5.0000	16.0000	2.0000

Appendix

```
knitr::opts_chunk$set(echo = TRUE, comment = NA)
parabolic_inter = function(a,x){
  a0 = a[1]
  a1 = a[2]
  a2 = a[3]
  res = a0 + a1*x + a2*(x^2)
  return(res)
}
sum_square_error = function(a,x,func){
  x0 = x[1]
  x1 = x[2]
  x2 = x[3]
  res = sum((func(x0)-parabolic_inter(a,x0))^2,(func(x1)-parabolic_inter(a,x1))^2,
            (func(x2)-parabolic_inter(a,x2))^2)
  return(res)
}
opt = function(x,func){
  a = c(0,0,0)
  res = optim(a,sum_square_error,x = x,func = func)
  res = res$par
  return(res)
}
approx = function(n,func){
 interval = 1/n
 midpoint = 1/(2*n)
 res = data.frame()
 for(i in 1:n){
```

```
   end_point = i/n
   x = c((end_point - interval),(end_point - midpoint),end_point)
   res = append(res,as.data.frame(opt(x,func)))
 }
 return(res)
}
f1 = function(x){
  res = -x *(1-x)
  return(res)
}
f2 = function(x){
  res = -x * sin(10*pi*x)
  return(res)
}
x = c()
for(i in 1:100){x[i] = i/100}

approx1 = approx(100,f1)
interpolate_result1 = c()
for(i in 1:length(approx1)){
  a = as.vector(approx1[[i]])
  interpolate_result1 = append(interpolate_result1,parabolic_inter(a,x[i]))
}
par(mar = c(3,3,3,0))
plot(x,f1(x),xlab = "x",ylab = "y",col = "blue", type = "b")
lines(x,interpolate_result1,col = "red")
title("Prediction")
legend(0.3, -0.03, legend=c("f1(x)", "parabolic interpotator function"),
       col=c("blue", "red"), pch=c(1,NA),lty=c(0,1), cex=0.8)

approx2 = approx(100,f2)
interpolate_result2 = c()
for(i in 1:length(x)){
  a = as.vector(approx2[[i]])
  interpolate_result2 = append(interpolate_result2,parabolic_inter(a,x[i]))
}
plot(x,f2(x),xlab = "x",ylab = "y",col = "blue", type = "b")
lines(x,interpolate_result2,col = "red")
title("Prediction")
legend(0, 0.98, legend=c("f2(x)", "parabolic interpotator function"),
       col=c("blue", "red"), pch=c(1,NA),lty=c(0,1), cex=0.8)
load("data.RData")
mu_hat =  sum(data)/length(data)
# formula for sigma
sigma_hat = sqrt(sum((data - mu_hat)^2)/length(data))
cat(paste("Mean of data", mu_hat))
cat(paste("Standard Deviation of data", sigma_hat))
minusloglikenormal = function(theta, x=data) {
  mu<-theta[1]                    # mean of normal distribution
  sigma2<-theta[2]                # variance of normal distribution
  n = length(x)
  a1 = (n/2)*log(2*pi)-(n/2)*log(sigma2)
  a2 = (sum((x - mu)^2)) / (2 * sigma2^2)
```

```r
  ans = a1+a2
  return(ans)
}
minusloglikenormal(c(0,1),data)
gradient = function(theta, x = data) {
  mu<-theta[1]
  sigma2<-theta[2]
  n = length(x)
  dmu = sum(mu - x)/sigma2^2
  dsig = (n/sigma2) - sum((x - mu)^2)/sigma2^3
  gr = c(dmu,dsig)
  return(gr)
}
gradient(c(0,1),data)
# Conjugate Gradient method without gradient method:
opt_cg <- optim(par = c(0, 1), fn = minusloglikenormal, method = "CG")

# Conjugate Gradient method without gradient method:
opt_cg_gr <- optim(par = c(0, 1), fn = minusloglikenormal, gr = gradient,  method = "CG")

# BGFS method without gradient method:
opt_bfgs <- optim(par = c(0, 1), fn = minusloglikenormal, method = "BFGS")

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cg1 = c(opt_cg$convergence, opt_cg$par, opt_cg$value, opt_cg$counts[1],
        opt_cg$counts[2])
cg2 = c(opt_cg_gr$convergence, opt_cg_gr$par, opt_cg_gr$value,
        opt_cg_gr$counts[1], opt_cg_gr$counts[2])
bfgs1 = c(opt_bfgs$convergence, opt_bfgs$par, opt_bfgs$value,
          opt_bfgs$counts[1], opt_bfgs$counts[2])
bfgs2 = c(opt_bfgs_gr$convergence, opt_bfgs_gr$par, opt_bfgs_gr$value,
          opt_bfgs_gr$counts[1], opt_bfgs_gr$counts[2])
df <- as.data.frame(matrix(c(cg1, cg2, bfgs1, bfgs2), nrow=6))
rownames(df) <- c("Convergence", "Mean", "Standard Deviation", "minusloglikelihood",
                  "function evaluation", "gradient evaluation")
colnames(df) <- c("CG", "BFGS", "CG with gradient", "BFGS with gradient")
knitr::kable(x = df, caption = "Comparison table of the different algorithms")
```