Note 8: Comparison Theorems in Riemannian Geometry and Applications

In this Note, we first introduce some applications of Rauch Comparison Theorem. Then we will give another two Comparison Theorems, Hessian Comparison Theorem and Toponogov Comparison Theorem, and also some applications of these two theorems.

Applications of Rauch Comparison Theorem

Recall the Rauch Comparison Theorem in Note 7. By Hopf-Rinow Theorem, if M is a complete Riemannian manifold with non-positive sectional curvature, $\exp_p$ is defined on the whole T_pM. Then for any $p \in M$, $\exp_p : T_pM \to M$ is a distance-expanding map. Furthermore, there is an interesting corollary, which can be found in [1], Corollary 3, p.119.

Corollary 1

Let M be a simply connected complete Riemannian manifold with non-positive sectional curvature, and let ABC be a geodesic triangle in M (i.e. all sides of the triangle are minimal geodesics), with corresponding angles A, B, C, and corresponding side a, a, c, then
(1) $a^2 + b^2 - 2ab\cos C \leq c^2$,
(2) $A + B + C \leq \pi$.

Proof. Let x be the vertex of angle C, and refer to Figure 8.2 in the tangent space T_xM. Draw radial line segments $\overline{Op}$ and $\overline{Oq}$ with lengths a and b, respectively, such that $\exp_x$ maps $\overline{Op}$ and $\overline{Oq}$ to the sides of the geodesic triangle with lengths a and b. Let ζ be the $\exp_x^{-1}$ image of side c. Since $\exp_x$ is a distance-expanding map, we have $L(\zeta) \leq c$, so the length of the line segment $\overline{pq}$ in T_xM is $L(\overline{pq}) \leq L(\zeta) \leq c$. The law of cosines for the triangle Opq in T_xM is given by:

$$a^2 + b^2 - 2ab\cos C = L(\overline{pq})^2,$$

which proves (1). In $\mathbb{R}^2$, construct a triangle with boundary lengths a, b, and c. This is possible because the sides of a geodesic triangle are always the shortest geodesic paths, and the sum of the lengths of any two sides must be greater than the length of the third side. Let A', B', C' denote the angles of this triangle in $\mathbb{R}^2$. From (1), we know that $C \leq C'$. Similarly, we can deduce $A \leq A'$ and $B \leq B'$. Thus, (2) is proven. $\square$

This corollary is a special case of Toponogov Comparison Theorem. The proof of the Toponogov theorem essentially relies on the Rauch Comparison theorem, and the method is similar to the proof of this corollary. However, it typically involves conjugate points, making the situation more complex.

Theorem 1

Let M and $\tilde{M}$ be m-dimensional Riemannian manifolds, and let K and $\tilde{K}$ be the sectional curvatures, respectively. Consider a point p in M and $\tilde{p}$ in $\tilde{M}$. Let $\varphi : T_pM \to T_{\tilde{p}}\tilde{M}$ be an isometric linear isomorphism. Assume that W and $\tilde{W} = \varphi(W)$ are neighborhoods of the origin in T_pM and $T_{\tilde{p}}\tilde{M}$, respectively, such that the exponential mappings

$$\exp_p : W \to \exp_p(W) \subset M \quad \text{and} \quad \exp_{\tilde{p}} : \tilde{W} \to \exp_{\tilde{p}}(\tilde{W}) \subset \tilde{M}$$

are both diffeomorphisms. If for any corresponding points

$$q \in \exp_p(W) \quad \text{and} \quad \tilde{q} = \exp_{\tilde{p}} \circ \varphi \circ (\exp_q)^{-1}(q) \in \exp_{\tilde{p}}(\tilde{W}),$$

and for any two-dimensional sections $\sigma \subset T_qM$ and $\tilde{\sigma} \subset T_{\tilde{q}}\tilde{M}$, the following inequality holds:

$$K(\sigma) \geq \tilde{K}(\tilde{\sigma}),$$

then, for any smooth curve $\beta(u)$ in W with $0 \leq u \leq 1$, the following inequality holds:

$$L(\exp_p \circ \beta) \leq L(\exp_{\tilde{p}} \circ (\varphi \circ \beta)).$$

Proof. Without loss of generality, we can assume that β is nonzero everywhere (i.e., β does not pass through the origin). Consider the geodesic variation:

$$\alpha(t, u) = \exp_p(t\beta(u)), \quad 0 \leq t \leq 1, \quad 0 \leq u \leq 1.$$

Let

$$V(t, u) = \alpha_{*(t,u)}\left(\frac{\partial}{\partial u}\right) = \left(\exp_p\right)_{*t\beta(u)}\left(t\beta'(u)\right).$$

Then:

$$V(1, u) = \left(\exp_p\right)_{*\beta(u)}\left(\beta'(u)\right),$$

$$L\left(\exp_p \circ \beta\right) = \int_0^1 |V(1, u)| \mathrm{d}u.$$

Similarly, we have:

$$L\left(\exp_{\tilde{p}} \circ \varphi \circ \beta\right) = \int_0^1 |\tilde{V}(1, u)| \mathrm{d}u,$$

where:

$$\tilde{V}(1, u) = \frac{\mathrm{d}}{\mathrm{d}u}\left(\exp_{\tilde{p}}(\varphi \circ \beta(u))\right) = \left(\exp_{\tilde{p}}\right)_{*\varphi\circ\beta(u)}\left(\varphi\left(\beta'(u)\right)\right).$$

For any fixed $u \in (0, 1)$, consider the vector fields $V(t, u)$ and $\tilde{V}(t, u)$ along the geodesics γ_u and $\tilde{\gamma}_u$, respectively, where

$$\gamma_u(t) = \exp_p(t\beta(u)) \quad \text{and} \quad \tilde{\gamma}_u(t) = \exp_{\tilde{p}}(t\varphi \circ \beta(u)).$$

According to the process of defining Jacobi fields, $V(t,u)$ and $\tilde{V}(t,u)$ are Jacobi fields along γ_u and $\tilde{\gamma}_u$, respectively. By definition:

$$
\begin{aligned}
&V(0,u) = \tilde{V}(0,u) = 0, \\
&\frac{\nabla V}{\mathrm{d}t}(0,u) = \beta'(u), \quad \frac{\nabla \tilde{V}}{\mathrm{d}t}(0,u) = \varphi\left(\beta'(u)\right), \\
&\gamma_u'(0) = \beta(u), \quad \tilde{\gamma}_u'(0) = \varphi(\beta(u)).
\end{aligned}
$$

Based on the assumption that, for any $X \in T_{\gamma_u(t)}M$ and $\tilde{X} \in T_{\tilde{\gamma}_u(t)}\tilde{M}$:

$$
K\left(\gamma_u', X\right) \geq \tilde{K}\left(\tilde{\gamma}_u', \tilde{X}\right),
$$

we can apply Rauch Comparison Theorem, which implies:

$$
|V(t,u)| \leq |\tilde{V}(t,u)|.
$$

In particular:

$$
|V(1,u)| \leq |\tilde{V}(1,u)|,
$$

and thus:

$$
L\left(\exp_p \circ \beta\right) \leq L\left(\exp_{\tilde{p}} \circ \varphi \circ \beta\right).
$$

□

This corollary plays an essential role in the proof of the following Toponogov's Theorem and that is how the theorem relies on the Rauch Comparison theorem.

Hessian Comparison Theorem and its application

We will introduce a naturally defined distance function on the manifold as well as Gauss's Lemma expressed using the distance function.

To avoid introducing the concepts of cut points and cut locus, we introduce another specialized definition: a geodesic is called "stably shortest" if all geodesics nearby it are also shortest. Specifically, let

$$
\begin{aligned}
\widetilde{\gamma} : [0,a] &\to T_{\gamma(0)}M \\
t &\mapsto t\dot{\gamma}(0)
\end{aligned}
$$

be a radial line segment with $\gamma(0) = x$. We call $\gamma = \exp_x \widetilde{\gamma}$ "stably shortest" if there exists a sector neighborhood $\mathcal{U}$ of $\widetilde{\gamma}$ in T_xM such that:

$$
\mathcal{U} = \{tp \mid t \in [0,a], |p - \dot{\gamma}(0)| < \text{ some constant } \}.
$$

All radial line segments in $\mathcal{U}$ are mapped to the shortest geodesics in M starting from x by $\exp_x$. From the continuity of the cut locus (see [1], p.158, Lemma 5), one can show that γ is stably shortest if and only if it does not contain any cut points along γ away from $\gamma(0)$.

For any given point $x \in M$, we can define a distance function $\rho : M \to \mathbb{R}$ as follows: for any $y \in M$, $\rho(y)$ is defined as the distance between x and y, denoted as $d(x,y)$. It is evident that the function ρ is continuous, but it may not necessarily be differentiable. If $\gamma : [0,a] \to M$ is a normal (with $|\gamma'| \equiv 1$) stably shortest geodesic, then it has a neighborhood $\exp_x \mathcal{U}$ (as described above) such that $\exp_x : \mathcal{U} \to \exp_x \mathcal{U}$ is a diffeomorphism, and

$$\rho\left(\exp_x z\right) = |z|, \quad \forall z \in \mathcal{U}.$$

Thus, ρ is smooth on $\exp_x \mathcal{U} - \{x\}$. By Gauss's Lemma in Note 6, the gradient of the function ρ is $\frac{\partial}{\partial \rho}$, i.e. for any tangent vector X in $\exp_x \mathcal{U} - \{x\}$,

$$X(\rho) = \left\langle X, \frac{\partial}{\partial \rho} \right\rangle.$$

Given $f \in C^2(M)$, the second-order covariant derivative of f, i.e. $\nabla^2 f = \nabla df$, is called the Hessian of f. By definition, for any smooth vector fields X, Y on M, we have:

$$\begin{aligned} \nabla^2 f(X,Y) &= \nabla df(X,Y) \\ &= Y(Xf) - (\nabla_X Y)f. \end{aligned}$$

As $\nabla^2 f$ is a $(0,2)$-order symmetric tensor field, it induces a symmetric bilinear functional on T_pM for each point p in M. The Laplacian of f, denoted as Δf, is equal to the trace of $\nabla^2(f)$, i.e.,

$$\Delta f = \operatorname{tr}(\nabla^2(f)) = \operatorname{tr}(\nabla^2 f).$$

If we choose a local coordinate system (x^i) around a point $p \in M$, then by Proposition 3 in Note 6, we have:

$$\begin{aligned} \nabla^2 f\left(\frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^j}(p)\right) &= \frac{\partial^2 f}{\partial x^i \partial x^j}(p), \\ \Delta f(p) &= \sum_i \frac{\partial^2 f}{\left(\partial x^i\right)^2}(p). \end{aligned}$$

From this, we can see that the value of $\nabla^2 f(X,Y)$ at point p depends only on the values of the vectors X and Y at point p and is independent of their behavior near p.

Lemma 1

Let $\gamma : [0,a] \to M$ be normal stably shortest geodesic and J is a normal Jacobi field along γ with $J(0) = 0$. Then

$$\nabla^2 \rho\left(J\left(a\right), J\left(a\right)\right) = \left\langle J'\left(a\right), J\left(a\right)\right\rangle.$$

Proof. By Proposition 1 in Note 6, we know that Jacobi fields can be represented as transversal vector fields to a one-parameter family of geodesics. Take ε small enough so that for any $u \in [0,\varepsilon]$, $uJ'(0) \in \mathcal{U} \subset T_{\gamma(0)}M$. Define $\alpha : [0,a] \times [0,\varepsilon] \to M$, given by

$$\alpha(t,u) = \exp_{\gamma(0)}\left(t\left(\gamma'(0) + uJ'(0)\right)\right).$$

Let $T = \alpha_*\left(\frac{\partial}{\partial t}\right)$ and $V = \alpha_*\left(\frac{\partial}{\partial u}\right)$, and we have:

$$V(\gamma(t)) = J(t) \quad \text{and} \quad T(\gamma(t)) = \frac{\partial}{\partial \rho}\Big|_{\gamma(t)}.$$

Now, let's compute at $\gamma(a)$:

$$\begin{aligned}
\nabla^2\rho(J(a),J(a)) &= VV\rho - (\nabla_V V)\,\rho \\
&= V\left\langle V, \frac{\partial}{\partial\rho}\right\rangle - \left\langle \nabla_V V, \frac{\partial}{\partial\rho}\right\rangle \\
&= V\langle V,T\rangle - \langle \nabla_V V, T\rangle \\
&= \langle \nabla_V V, T\rangle + \langle V, \nabla_V T\rangle - \langle \nabla_V V, T\rangle \\
&= \langle \nabla_V T, V\rangle = \langle \nabla_T V, V\rangle \\
&= \langle \dot{J}(a), J(a)\rangle.
\end{aligned}$$

In the above calculation, we used Gauss's Lemma and $[V,T]=0$. Additionally, for simplicity, we did not explicitly indicate that the functions should be evaluated at $\gamma(a)$. $\square$

Let $M,\widetilde{M}$ be n-dimensional , $\gamma:[0,a]\to M, \widetilde{\gamma}:[0,a]\to \widetilde{M}$ are normal geodesics and $x=\gamma(0), \widetilde{x}=\widetilde{\gamma}(0)$. Suppose $\rho,\widetilde{\rho}$ be distance functions on $M,\widetilde{M}$, associated with $x,\widetilde{x}$, respectively.

Theorem 2: Hessian Comparison Theorem

Suppose

(1) γ and $\widetilde{\gamma}$ are stably shortest;

(2) For all t and all $x\in T_{\gamma(t)}(M), \tilde{x}\in T_{\tilde{\gamma}(t)}(\tilde{M})$, we have

$$\tilde{K}\left(\tilde{x},\tilde{\gamma}'(t)\right) \geq K\left(x,\gamma'(t)\right).$$

Then $\rho,\widetilde{\rho}$ are smooth in some neighborhoods of $\gamma,\widetilde{\gamma}$. Moreover, for all $t\in[0,a]$ and $X\in T_{\gamma(t)}M, \tilde{X}\in T_{\widetilde{\gamma}(t)}\widetilde{M}$, if $|X|=|\widetilde{X}|, \langle X,\dot{\gamma}(t)\rangle = \langle \widetilde{X}, \dot{\widetilde{\gamma}}(t)\rangle$, then we have

$$\nabla^2\widetilde{\rho}(\widetilde{X},\widetilde{X}) \leq \nabla^2\rho(X,X).$$

We simply denote the above by $\nabla^2\widetilde{\rho}\prec\nabla^2\rho$ along $\gamma,\widetilde{\gamma}$.

Proof. X can be uniquely decomposed as follows:

$$X = \langle X,\gamma'(t)\rangle\gamma'(t) + Y,$$

where $\langle Y,\gamma'(t)\rangle = 0$. Due to the following:

$$\begin{aligned}
\nabla^2\rho(\gamma'(t),\gamma'(t)) &= \dot{\gamma}(t)\dot{\gamma}(t)\rho - \left(\nabla_{\gamma'(t)}\gamma'(t)\right)\rho = 0, \\
\nabla^2\rho(\gamma'(t),Y) &= \gamma'(t)Y\rho - \left(\nabla_{\gamma'(t)}Y\right)\rho \\
&= \gamma'(t)\left\langle Y, \frac{\partial}{\partial\rho}\right\rangle - \left\langle \nabla_{\gamma'(t)}Y, \frac{\partial}{\partial\rho}\right\rangle \\
&= \left\langle Y, \nabla_{\gamma'(t)}\frac{\partial}{\partial\rho}\right\rangle = 0,
\end{aligned}$$

we have:

$$\begin{aligned}\nabla^2\rho(X,X) &= \langle X,\gamma'(t)\rangle^2\nabla^2\rho(\gamma'(t),\gamma'(t))\\ &\quad + 2\langle X,\gamma'(t)\rangle\nabla^2\rho(\gamma'(t),Y) + \nabla^2\rho(Y,Y)\\ &= \nabla^2\rho(Y,Y).\end{aligned}$$

Similarly, we have:

$$\tilde{X} = \langle \tilde{X}, \dot{\tilde{\gamma}}(t)\rangle\dot{\tilde{\gamma}}(t) + \widetilde{Y}, \quad \text{and} \quad \nabla^2\widetilde{\rho}(\tilde{X},\widetilde{X}) = \nabla^2\widetilde{\rho}(\tilde{Y},\widetilde{Y}).$$

Therefore, to prove $\nabla^2\widetilde{\rho}(\tilde{X},\widetilde{X}) \le \nabla^2\rho(X,X)$, it is equivalent to proving:

$$\nabla^2\widetilde{\rho}(\tilde{Y},\tilde{Y}) \le \nabla^2\rho(Y,Y).$$

At this point:

$$\begin{aligned}\langle Y,\gamma'(t)\rangle &= 0 = \langle \widetilde{Y},\tilde{\gamma}'(t)\rangle,\\ |Y|^2 &= |X - \langle X,\gamma'(t)\rangle\gamma'(t)|^2\\ &= |X|^2 - \langle X,\gamma'(t)\rangle^2\\ &= |\widetilde{X}|^2 - \langle \widetilde{X},\widetilde{\gamma}'(t)\rangle^2 = |\widetilde{Y}|^2.\end{aligned}$$

Since γ and $\widetilde{\gamma}$ are stably shortest geodesics, there are no conjugate points. We can separately construct normal Jacobi fields J and $\widetilde{J}$ along γ and $\widetilde{\gamma}$, such that:

$$J(t) = Y, \quad \widetilde{J}(t) = \widetilde{Y}.$$

By Lemma 1, we only need to prove

$$\langle \tilde{J}'(a), \tilde{J}(a)\rangle \le \langle J'(a), J(a)\rangle. \tag{1}$$

In the proof of Rauch Comparison Theorem (general case), replace the definitions

$$W_{t_1}(t) = \frac{J(t)}{|J(t_1)|} \quad \text{and} \quad \tilde{W}_{t_1}(t) = \frac{\tilde{J}(t)}{\left|\tilde{J}(t_1)\right|}.$$

by

$$W_{t_1}(t) = J(t) \quad \text{and} \quad \tilde{W}_{t_1}(t) = \tilde{J}(t).$$

Then followed the same procedure, one can prove inequality (1). □

Now we give some corollaries which are under the same prerequisites as Theorem 2.

Corollary 2

If $f : [0,+\infty) \to \mathbf{R}$ is a C^∞ increasing function (i.e., $f' \ge 0$), then along γ and $\widetilde{\gamma}$, we have:

$$\nabla^2 f(\widetilde{\rho}) \prec \nabla^2 f(\rho).$$

Proof. Since

$$\begin{aligned}\nabla^2 f(\rho)(X,X) &= XXf(\rho) - (\nabla_X X)\, f(\rho)\\ &= f'(\rho)\left(XX\rho - (\nabla_X X)\,\rho\right) + f''(\rho)(X\rho)^2\\ &= f'(\rho)\nabla^2\rho(X,X) + f''(\rho)\langle X, \dot\gamma(t)\rangle^2,\end{aligned}$$

the conclusion is evident. □

Since Δ represents the trace of ∇^2, we can also compare the Laplacian of distance functions.

Corollary 3

For any $t \in (0,a]$, we have

$$\Delta\widetilde{\rho}(\widetilde{\gamma}(t)) \leq \Delta\rho(\gamma(t)).$$

Toponogov's Theorem

In this section we state Toponogov's Theorem[2], which is a powerful global generalization of the Rauch Theorem. There will be two equivalent statements, and it will be convenient to prove them simultaneously. All indices below are to be taken modulo 3.

Definition 1

A geodesic triangle in the Riemannian manifold M is a set of three geodesic segments parameterized by arc length $(\gamma_1, \gamma_2, \gamma_3)$ of lengths l_1, l_2, l_3 such that $\gamma_i\left(l_i\right) = \gamma_{i+1}(0)$ and $l_i + l_{i+1} \geq l_{i+2}$. Set

$$\alpha_i = \angle\left(-\gamma'_{i+1}\left(l_{i+1}\right), \gamma'_{i+2}(0)\right),$$

the angle between $-\gamma'_{i+1}\left(l_{i+1}\right)$ and $\gamma'_{i+2}(0), 0 \leq \alpha_i \leq \pi$.

We shall specify a geodesic triangle by giving its sides $(\gamma_1, \gamma_2, \gamma_3)$.

Let M be a complete manifold with $K_M \geq H$, i.e. for all plane sections $\sigma \in M$, $K(\sigma) \geq H$.

Theorem 3: Toponogov's Theorem A

Let $(\gamma_1, \gamma_2, \gamma_3)$ determine a geodesic triangle in M. Suppose γ_1, γ_3 are minimal and if $H > 0$, suppose $L\left[\gamma_2\right] \leq \frac{\pi}{\sqrt{H}}$. Then in M^H, the simply connected 2-dimensional space of constant curvature H, there exists a geodesic triangle $(\bar\gamma_1, \bar\gamma_2, \bar\gamma_3)$ such that $L\left[\gamma_i\right] = L\left[\bar\gamma_i\right]$ and $\bar\alpha_1 \leq \alpha_1, \bar\alpha_3 \leq \alpha_3$. Except in case $H > 0$ and $L\left[\gamma_i\right] = \frac{\pi}{\sqrt{H}}$ for some i, the triangle in M^H is uniquely determined.

Theorem 4: Toponogov's Theorem B

Let γ_1, γ_2 be geodesic segments in M such that $\gamma_1(l_1) = \gamma_2(0)$ and $\angle(-\gamma_1'(l_1), \gamma_2'(0)) = \alpha$. We call such a configuration a hinge l and denote it by $(\gamma_1, \gamma_2, \alpha)$. Let γ_1 be minimal, and if $H > 0$,

$$L[\gamma_2] \leq \frac{\pi}{\sqrt{H}}$$

Let $\bar{\gamma}_1, \bar{\gamma}_2 \subset M^H$ be such that $\gamma_1(l_1) = \gamma_2(0), L[\gamma_i] = L[\bar{\gamma}_i] = l_i$ and $\angle(-\bar{\gamma}_1'(l_1), \bar{\gamma}_2'(0)) = \alpha$. Then

$$\rho(\gamma_1(0), \gamma_2(l_2)) \leq \rho(\bar{\gamma}_1(0), \bar{\gamma}_2(l_2))$$

For the proof is too long but the necessary tools to prove this theorem are all given by us so far, we shall not present the proof here.

Reference

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[2] Jeff Cheeger and D. G. Ebin. *Comparison Theorems in Riemannian Geometry*. AMS Chelsea Publishing. AMS Chelsea Publishing, Providence, R.I, 2008.