Note 8: Comparison Theorems in Riemannian Geometry and Applications

In this Note, we first introduce some applications of Rauch Comparison Theorem. Then we will give another two Comparison Theorems, Hessian Comparison Theorem and Toponogov Comparison Theorem, and also some applications of these two theorems.

Applications of Rauch Comparison Theorem

Recall the Rauch Comparison Theorem in Note 7. By Hopf-Rinow Theorem, if M is a complete Riemannian manifold with non-positive sectional curvature, $\exp_p$ is defined on the whole T_pM. Then for any $p \in M$, $\exp_p : T_pM \to M$ is a distance-expanding map. Furthermore, there is an interesting corollary, which can be found in [1], Corollary 3, p.119.

Corollary 1

Let M be a simply connected complete Riemannian manifold with non-positive sectional curvature, and let ABC be a geodesic triangle in M (i.e. all sides of the triangle are minimal geodesics), with corresponding angles A, B, C, and corresponding side a, a, c, then

(1) $a^2 + b^2 - 2ab\cos C \le c^2$,

(2) $A + B + C \le \pi$.

Proof. Let x be the vertex of angle C, and refer to Figure 8.2 in the tangent space T_xM. Draw radial line segments $\overline{Op}$ and $\overline{Oq}$ with lengths a and b, respectively, such that $\exp_x$ maps $\overline{Op}$ and $\overline{Oq}$ to the sides of the geodesic triangle with lengths a and b. Let ζ be the $\exp_x^{-1}$ image of side c. Since $\exp_x$ is a distance-expanding map, we have $L(\zeta) \le c$, so the length of the line segment $\overline{pq}$ in T_xM is $L(\overline{pq}) \le L(\zeta) \le c$. The law of cosines for the triangle Opq in T_xM is given by:

$$a^2 + b^2 - 2ab\cos C = L(\overline{pq})^2,$$

which proves (1). In $\mathbb{R}^2$, construct a triangle with boundary lengths a, b, and c. This is possible because the sides of a geodesic triangle are always the shortest geodesic paths, and the sum of the lengths of any two sides must be greater than the length of the third side. Let A', B', C' denote the angles of this triangle in $\mathbb{R}^2$. From (1), we know that $C \le C'$. Similarly, we can deduce $A \le A'$ and $B \le B'$. Thus, (2) is proven. $\square$

This corollary is a special case of Toponogov Comparison Theorem. The proof of the Toponogov theorem essentially relies on the Rauch comparison theorem, and the method is similar to the proof of this corollary. However, it typically involves conjugate points, making the situation more complex.

Theorem 1

Let M and $\tilde{M}$ be m-dimensional Riemannian manifolds, and let K and $\tilde{K}$ be the sectional curvatures, respectively. Consider a point p in M and $\tilde{p}$ in $\tilde{M}$. Let $\varphi : T_pM \to T_{\tilde{p}}\tilde{M}$ be an isometric linear isomorphism. Assume that W and $\tilde{W} = \varphi(W)$ are neighborhoods of the origin in T_pM and $T_{\tilde{p}}\tilde{M}$, respectively, such that the exponential mappings

$$\exp_p : W \to \exp_p(W) \subset M \quad \text{and} \quad \exp_{\tilde{p}} : \tilde{W} \to \exp_{\tilde{p}}(\tilde{W}) \subset \tilde{M}$$

are both diffeomorphisms. If for any corresponding points

$$q \in \exp_p(W) \quad \text{and} \quad \tilde{q} = \exp_{\tilde{p}} \circ \varphi \circ (\exp_q)^{-1}(q) \in \exp_{\tilde{p}}(\tilde{W}),$$

and for any two-dimensional sections $\sigma \subset T_qM$ and $\tilde{\sigma} \subset T_{\tilde{q}}\tilde{M}$, the following inequality holds:

$$K(\sigma) \geq \tilde{K}(\tilde{\sigma}),$$

then, for any smooth curve $\beta(u)$ in W with $0 \leq u \leq 1$, the following inequality holds:

$$L(\exp_p \circ \beta) \leq L(\exp_{\tilde{p}} \circ (\varphi \circ \beta)).$$

Proof. Without loss of generality, we can assume that β is nonzero everywhere (i.e., β does not pass through the origin). Consider the geodesic variation:

$$\alpha(t, u) = \exp_p(t\beta(u)), \quad 0 \leq t \leq 1, \quad 0 \leq u \leq 1.$$

Let

$$V(t, u) = \alpha_{*(t,u)}\left(\frac{\partial}{\partial u}\right) = \left(\exp_p\right)_{*t\beta(u)}(t\beta'(u)).$$

Then:

$$V(1, u) = \left(\exp_p\right)_{*\beta(u)}(\beta'(u)),$$

$$L\left(\exp_p \circ \beta\right) = \int_0^1 |V(1, u)|\mathrm{d}u.$$

Similarly, we have:

$$L\left(\exp_{\tilde{p}} \circ \varphi \circ \beta\right) = \int_0^1 |\tilde{V}(1, u)|\mathrm{d}u,$$

where:

$$\tilde{V}(1, u) = \frac{\mathrm{d}}{\mathrm{d}u}\left(\exp_{\tilde{p}}(\varphi \circ \beta(u))\right) = \left(\exp_{\tilde{p}}\right)_{*\varphi \circ \beta(u)}(\varphi(\beta'(u))).$$

For any fixed $u \in (0, 1)$, consider the vector fields $V(t, u)$ and $\tilde{V}(t, u)$ along the geodesics γ_u and $\tilde{\gamma}_u$, respectively, where

$$\gamma_u(t) = \exp_p(t\beta(u)) \quad \text{and} \quad \tilde{\gamma}_u(t) = \exp_{\tilde{p}}(t\varphi \circ \beta(u)).$$

According to the process of defining Jacobi fields, $V(t, u)$ and $\tilde{V}(t, u)$ are Jacobi fields along γ_u and $\tilde{\gamma}_u$, respectively. By definition:

$$V(0, u) = \tilde{V}(0, u) = 0,$$
$$\frac{\mathrm{D}V}{\mathrm{d}t}(0, u) = \beta'(u), \quad \frac{\mathrm{D}\tilde{V}}{\mathrm{d}t}(0, u) = \varphi\left(\beta'(u)\right),$$
$$\gamma_u'(0) = \beta(u), \quad \tilde{\gamma}_u'(0) = \varphi(\beta(u)).$$

Based on the assumption that, for any $X \in T_{\gamma_u(t)}M$ and $\tilde{X} \in T_{\tilde{\gamma}_u(t)}\tilde{M}$:

$$K\left(\gamma_u', X\right) \geq \tilde{K}\left(\tilde{\gamma}_u', \tilde{X}\right),$$

we can apply Rauch Comparison Theorem, which implies:

$$|V(t, u)| \leq |\tilde{V}(t, u)|.$$

In particular:

$$|V(1, u)| \leq |\tilde{V}(1, u)|,$$

and thus:

$$L\left(\exp_p \circ \beta\right) \leq L\left(\exp_{\tilde{p}} \circ \varphi \circ \beta\right).$$

$\square$

Reference

[1] 伍鸿熙, 沈纯理, and 虞言林. 黎曼几何初步. 高等教育出版社, Bei jing, 2014.