Note 6: Comparison theorems on Riemannian geometry

Jacobi field and conjugate points

In the Euclidean plane, geodesics originating from the same point may move farther apart, or figuratively speaking, they ”diverge”. However, on a unit sphere, any two geodesics originating from the same point will intersect at the antipodal point, or in other words, they ”converge”.

To study this ”divergence” or ”convergence” on a Riemannian manifold, the usual approach is to embed the geodesics into a family of geodesics, by considering a ”variation of geodesics” with a fixed starting point, and analyzing the corresponding variation vector field which is known as a Jacobi field.

We now determine the differential equation satisfied by $V\big|_{\alpha(t,0)}$. By the induced covariant derivatives (see [1], pp.149-150, Example 8.2), we have

$$\nabla_T V - \nabla_V T - \alpha_*\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\right) = 0.$$

But $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$, so $\nabla_T V = \nabla_V T$. Therefore $\nabla_T \nabla_T V = \nabla_T \nabla_V T$. Since $\nabla_T T = 0$, we may write

$$\nabla_T \nabla_T V = \nabla_T \nabla_V T - \nabla_V \nabla_T T$$

Using the definition of the curvature tensor and the fact that

$$[T, V] = \nabla_T V - \nabla_V T = \alpha_*\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\right) = 0$$

we get the Jacobi equation $\nabla_T \nabla_T V = R(T, V)T$.

Definition 1: Jacobi fields

A vector field J, along a geodesic γ with tangent vector T satisfying the equation

$$\nabla_T \nabla_T J = R(T, J)T$$

is called a Jacobi field.

Let $\{E_i(t)\}$ be orthogonal and parallel along $\alpha(t, 0)$. Then the Jacobi equation may be written as the linear second-order system of ordinary differential equations

$$\begin{bmatrix} \langle J, E_1 \rangle \\ \vdots \\ \langle J, E_n \rangle \end{bmatrix}'' = \left[\langle R\left(T, E_i\right) T, E_j \rangle\right]_{ij} \begin{bmatrix} \langle J, E_1 \rangle \\ \vdots \\ \langle J, E_n \rangle \end{bmatrix}.$$

From the theory of ordinary differential equations it follows that the space of solutions of this system is $2n$-dimensional and that there exists a unique solution with prescribed initial value and first derivative. This is equivalent to prescribing $J(0)$ and $J'(0) = \nabla_T J|_{t=0}$. Also, we know that the zeros of a Jacobi field must be discrete provided it is not constantly zero. Otherwise, there will be a limit point of zeros at which J and J' are both 0, which leads to $J \equiv 0$.

Notice that since $\nabla_T T = 0$, we have

$$\langle J, T \rangle'' = \langle J'', T \rangle = \langle R(T, J)T, T \rangle = 0.$$

Therefore any Jacobi field J may be written uniquely as

$$J = J^{\perp} + (at + b)T$$

where $\left\langle J^{\perp}, T \right\rangle \equiv 0$. This shows that those Jacobi fields, called normal Jacobi fields, perpendicular to T are what we really care about. One can calculate the coefficients:

$$a = \langle J'(0), T(0) \rangle \quad \text{and} \quad b = \langle J(0), T(0) \rangle.$$

Thus J is a normal Jacobi field if and only if

$$J(0) \perp T(0) \quad \text{and} \quad J'(0) \perp T(0).$$

Finally, if J is a Jacobi field, then J comes from a variation of geodesics.

Proposition 1

Let M be a Riemannian manifold, $\gamma : [0, 1] \to M$ a geodesic, and J a Jacobi field along γ. Then there exists a geodesic variation $\alpha(t, s)$, where $\alpha(t, 0) = \gamma(t)$, such that

$$J(t) = V \big|_{\alpha(t,0)}.$$

Proof. Choose a curve $\lambda(s), s \in (-\varepsilon, \varepsilon)$ in M such that $\lambda(0) = \gamma(0), \lambda'(0) = J(0)$. Along λ choose a vector field $W(s)$ with $W(0) = \gamma'(0), \nabla_{\frac{\partial}{\partial s}} W(0) = J'(0)$. Define $\alpha(s, t) = \exp_{\lambda(s)} tW(s)$ and verify that $V(0, 0) = \lambda'(0) = J(0)$ and

$$\nabla_{\frac{\partial}{\partial t}} V(0, 0) = \nabla_{\frac{\partial}{\partial s}} T(0, 0) = \nabla_{\frac{\partial}{\partial s}} W(0) = J'(0).$$

□

Corollary 1

Let $\gamma : [0, a] \to M$ be a geodesic. Then a Jacobi field J along γ with $J(0) = 0$ is given by

$$J(t) = \exp_{t\gamma'(0)}\left(tJ'(0)\right), \quad t \in [0, a].$$

The Gauss lemma tells us that the exponential mapping preserves the orthogonality with radial geodesics, and the length along the radial geodesic is preserved. Meanwhile, the change in length of the tangent vector orthogonal to the radial geodesic under the exponential mapping can be attributed to the computation of the magnitude of the Jacobi field. Now we are going to relate the rate of spreading of the geodesics that start from $p \in M$ with the curvature at p.

Proposition 2

Let $p \in M$ and $\gamma : [0, a] \to M$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$. Let $w \in T_v(T_pM)$ and let J be a Jacobi field along γ given by

$$J(t) = \exp_{tv}(tw), \quad 0 \le t \le a$$

Then the Taylor expansion of $|J(t)|^2$ about $t = 0$ is given by

$$|J(t)|^2 = |w|t^2 + \frac{1}{3}\langle R(v, w)v, w\rangle t^4 + o(t^4).$$

Proof. Since $J(0) = 0, J'(0) = w$, let

$$f(t) = \langle J(t), J(t)\rangle = |J(t)|^2$$

and we have:

$$f'(t) = 2\left\langle J'(t), J(t)\right\rangle, \quad f''(t) = 2\left\langle J''(t), J(t)\right\rangle + 2\left\langle J'(t), J'(t)\right\rangle.$$

Then for the first three coefficients:

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2|w|^2.$$

Take covariant derivative of $f''(t)$ and we have

$$\begin{aligned} f'''(t) &= 6\left\langle J''(t), J'(t)\right\rangle + 2\left\langle J'''(t), J(t)\right\rangle, \\ f^{(4)}(t) &= 8\left\langle J'''(t), J'(t)\right\rangle + 6\left\langle J''(t), J''(t)\right\rangle + 2\left\langle J^{(4)}(t), J(t)\right\rangle. \end{aligned}$$

By Jacobi equation $J''(t) = R\left(\gamma', J\right)\gamma'$,

$$f'''(0) = 6\left\langle R\left(\gamma', J\right)\gamma', J'\right\rangle\big|_{t=0} = 0, \quad f^{(4)}(0) = 8\left\langle J'''(t), J'(t)\right\rangle\big|_{t=0},$$

and

$$\begin{aligned} &\left\langle J'''(t), J'(t)\right\rangle = \left\langle D_{\gamma'}\left(R\left(\gamma', J\right)\gamma'\right), J'\right\rangle \\ &= \gamma'\left(\left\langle R\left(\gamma', J\right)\gamma', J'\right\rangle\right) - \left\langle R\left(\gamma', J\right)\gamma', J''\right\rangle \\ &= \gamma'\left(\left\langle R\left(\gamma', J'\right)\gamma', J\right\rangle\right) - \left\langle R\left(\gamma', J\right)\gamma', J''\right\rangle \\ &= \left\langle D_{\gamma'}\left(R\left(\gamma', J'\right)\gamma'\right), J\right\rangle + \left\langle R\left(\gamma', J'\right)\gamma', J'\right\rangle - \left\langle R\left(\gamma', J\right)\gamma', J''\right\rangle. \end{aligned}$$

Therefore, $f^{(4)}(0) = 8\langle R(v, w)v, w\rangle$. From Taylor expansion, we have

$$\begin{aligned} f(t) &= f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \frac{t^4}{4!}f^{(4)}(0) + o\left(t^4\right) \\ &= |w|^2t^2 + \frac{1}{3}\langle R(v, w)v, w\rangle t^4 + o\left(t^4\right). \end{aligned}$$

□

Corollary 2

If $\gamma : [0, \ell] \to M$ is parametrized by arc length, (i.e. $|v| = 1$). Letting $w \in T_v(T_pM)$ with $|w| = 1$ satisfying $\langle w, v\rangle = 0$, and $J(t) = \exp_{tv}(tw)$, then we have

$$|J(t)|^2 = t^2 - \frac{1}{3}K(v, w)t^4 + o(t^4),$$

where σ is the plane generated by v and w. Moreover,

$$|J(t)| = t - \frac{1}{6}K(v, w)t^3 + o(t^3). \tag{1}$$

Proof. Since $v \perp w, |v| = |w| = 1$, then $\langle R(v, w)v, w\rangle = -K(v, w)$. Therefore,

$$|J(t)| = t\sqrt{1 - \frac{1}{3}K(v, w)t^2 + o\left(t^2\right)}.$$

Together with expansion

$$\sqrt{1 - x} = 1 - \frac{1}{2}x + o(x),$$

one can prove the corollary. □

We now recall the introduction at the beginning of this subsection. The expression (1) essentially contains the relation between geodesics and curvature. Indeed, considering the geodesic variation

$$\alpha(t, s) = \exp tv(s), \quad t \in [0, \delta], \quad s \in (-\varepsilon, \varepsilon),$$

where$v(s)$ is a curve in T_pM with $|v(s)| = 1, v(0) = v, v'(0) = w$, we see that the rays $t \mapsto tv(s), t \in [0, \delta]$, that start from the origin 0 of T_pM, deviate from the ray $t \to tv(0)$ with the velocity

$$\left|\left(\frac{\partial}{\partial s}tv(s)\right)(0)\right| = |tw| = t$$

On the other hand, (1) tells us that the geodesics $t \mapsto \exp_p(tv(s))$ deviate from the geodesic $\gamma(t) = \exp_p tv(0)$ with a velocity that differs from t by a term of the third order in t, given by $-\frac{1}{6}K(v, w)t^3$. This tells us that, locally, the geodesics spread apart less than the rays in T_pM, if $K(v, w) > 0$, and that they spread apart more than the rays in T_pM, if $K(v, w) < 0$. Actually, for t small, the value $K(v, w)t^3$ furnishes an approximation for the extent of this spread with an error of order t^3.

Now we are going to turn to the relationship between the singularities of the exponential map and Jacobi fields.

Definition 2: Conjugate points

Let $\gamma : [0, a] \to M$ be a geodesic. The point $\gamma\left(t_0\right)$ is said to be conjugate to $\gamma(0)$ along $\gamma, t_0 \in (0, a]$, if there exists a Jacobi field J along γ, not identically zero, with $J(0) = 0 = J\left(t_0\right)$.

Observe that if $\gamma(t_0)$ is conjugate to $\gamma(0)$, then $\gamma(0)$ is conjugate to $\gamma(t_0)$

The following proposition relates conjugate points with the singularities of the exponential map.

Proposition 3

Let $\gamma : [0, a] \to M$ be a geodesic and put $\gamma(0) = p$. The point $q = \gamma(t_0), t_0 \in (0, a]$, is conjugate to p along γ if and only if $v_0 = t_0\gamma'(0)$ is a critical point of $\exp_p$.

Proof. The point $q = \gamma(t_0)$ is a conjugate point of p along γ if and only if there exists a non-zero Jacobi field J along γ with $J(0) = J(t_0) = 0$. Let $v = \gamma'(0)$ and $w = J'(0)$. From Corollary 1,

$$J(t) = \exp_{*tv}(tw), t \in [0, a].$$

Observe that J is non-zero if and only if $w \neq 0$. Therefore, $q = \gamma(t_0)$ is conjugate to p if and only if

$$0 = J(t_0) = \exp_{*t_0 v}(t_0 w), \quad w \neq 0,$$

that is, if and only if, $t_0 v$ is a critical point of $\exp_p$. The first assertion is therefore proved.

□

Rauch Comparison Theorem

With enough preparations, we state Rauch comparison theorem and will show some examples of application. Then we will give a brief proof as $n = 2$ given that the motivation of the proof in this case is most clear. For general situations, we need introduce Morse index forms to complete the proof which will be discussed in the sequel subsection.

Let M and $\widetilde{M}$ be two n-dimensional Riemannian manifolds with $p \in M$ and $\widetilde{p} \in \widetilde{M}$. Let $\varphi : T_pM \to T_{\widetilde{p}}\widetilde{M}$ be a linear isometry. Let $x \in T_pM$ and $\widetilde{x} = \varphi(x)$. Consider the two geodesics $\gamma : [0, 1] \to M$ and $\widetilde{\gamma} : [0, 1] \to \widetilde{M}$ determined by $\gamma(t) = \exp_p tx$ and $\widetilde{\gamma}(t) = \exp_{\widetilde{p}} t\widetilde{x}$, respectively. Let $X \in T_x(T_pM)$, and let $\widetilde{X} = \varphi_*(X) \equiv \varphi(X)$, where $\widetilde{X} \in T_{\widetilde{x}}(T_{\widetilde{p}}M)$.

Theorem 1: Rauch comparison theorem

Assume that

(1) $\tilde{\gamma}$ does not have conjugate points on $(0, a]$,

(2) for all t and all $x \in T_{\gamma(t)}(M), \tilde{x} \in T_{\tilde{\gamma}(t)}(\tilde{M})$, we have

$$\tilde{K}(\tilde{x}, \tilde{\gamma}'(t)) \geq K(x, \gamma'(t)).$$

Then

$$|(\exp_{\widetilde{p}})_* \widetilde{X}| \leq |(\exp_p)_* X|.$$

If M is a complete Riemannian manifold with non-positive sectional curvature, $\exp_p$ is defined on all of T_pM by Hopf-Rinow Theorem. Then for any $p \in M$, $\exp_p : T_pM \to M$ is a distance-expanding map, i.e.

$$\left|(\exp_p)_* X\right| \geq |X|, \quad \forall X \in T_p M$$

Furthermore, there is an interesting corollary (a special case of Toponogov Comparison Theorem) as follows:

Corollary 3

Let M be a simply connected complete Riemannian manifold with non-positive sectional curvature, and let ABC be a geodesic triangle in M (i.e. all sides of the triangle are minimal geodesics), with corresponding angles A, B, C, and corresponding side a, a, c, then
(1) $a^2 + b^2 - 2ab\cos C \leqslant c^2$,
(2) $A + B + C \leqslant \pi$.

The proof of above corollary together with some explanation of how Rauch Comparison Theorem applies to sphere theorem can be found in [2], Chapter 8.

Since $(\exp_p)_*$ is always isometric in the radial direction, without loss of generality, we can assume that $X \perp x, \widetilde{X} \perp \widetilde{x}$ in $T_p M, T_{\widetilde{p}}\widetilde{M}$, respectively, and describe $(\exp_p)_* X$ via Jacobi fields. By these observations, we can restate the Rauch comparison theorem with slight generalization of $\widetilde{M}$ as follows:

Theorem 2: Rauch comparison theorem

Let $\gamma : [0, a] \to M^n$ and $\tilde{\gamma} : [0, a] \to \tilde{M}^{n+k}$, $k \geq 0$, be geodesics with the same velocity (i.e. $|\gamma'(t)| = |\tilde{\gamma}'(t)|$). Let J and $\tilde{J}$ be normal Jacobi fields along γ and $\tilde{\gamma}$, respectively, with $J(0) = \tilde{J}(0) = 0$ such that

$$|J'(0)| = \left|\tilde{J}'(0)\right|.$$

Assume that
(1) $\tilde{\gamma}$ does not have conjugate points on $(0, a]$,
(2) for all t and all $x \in T_{\gamma(t)}(M), \tilde{x} \in T_{\tilde{\gamma}(t)}(\tilde{M})$, we have

$$\tilde{K}\left(\tilde{x}, \tilde{\gamma}'(t)\right) \geq K\left(x, \gamma'(t)\right).$$

Then

$$|\tilde{J}| \leq |J|$$

Proof of Theorem 2 as $n = 2$. First, we know that $J, \widetilde{J}$ are normal Jacobi fields. If we let $W(t), \widetilde{W}(t)$ be the unit vector fields parallel along $\gamma, \widetilde{\gamma}$, respectively, such that $W \perp \dot{\gamma}, \widetilde{W} \perp \dot{\widetilde{\gamma}}$, then we can write

$$J(t) = f(t)W(t), \quad \widetilde{J}(t) = \tilde{f}(t)\widetilde{W}(t).$$

Then the theorem becomes: Assuming $\tilde{f}, f$ are functions on $[0,a]$ satisfying the following equations:

$$\begin{cases} f'' + Kf = 0, \\ f(0) = 0, \quad f'(0) = b > 0; \end{cases} \qquad \begin{cases} \tilde{f}'' + \tilde{K}\tilde{f} = 0, \\ \tilde{f}(0) = 0, \quad \tilde{f}'(0) = b > 0; \end{cases}$$

where $\tilde{K}, K$ are functions on $(0,a]$ and

$$\tilde{K}(t) \geq K(t), \quad \forall t \in [0,a],$$

then for any $\tilde{f} > 0$ on $[0,a]$, we have $f \geq \tilde{f}$.

To prove this, we only need to let

$$\Phi(t) = (f f' - \tilde{f}' f)(t).$$

It is known that when $\tilde{f}, f \geq 0$, we always have $\Phi'(t) \geq 0$. In fact, $f > 0$ on $(0,a]$, which will be illustrated later. Therefore, since $\Phi(0) = 0$, we have

$$\Phi(t) \geq 0, \quad \forall t \in [0,a].$$

Then rewrite $\Phi \geq 0$ as

$$\frac{f'}{f} \geq \frac{\tilde{f}'}{\tilde{f}} \tag{2}$$

and integrate it yields

$$\ln \frac{f(t)}{f(\epsilon)} \geq \ln \frac{\tilde{f}(t)}{\tilde{f}(\epsilon)}$$

where $0 < \epsilon < t \leq a$. Therefore,

$$\frac{f(t)}{\tilde{f}(t)} \geq \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\tilde{f}(\epsilon)} = \frac{b}{b} = 1. \tag{3}$$

Thus we have $f(t) \geq \tilde{f}(t), \forall t \in (0,a]$.

Now we prove that $f > 0$ on $(0,a]$. Otherwise, there exists $r \in (0,a)$ such that $f(r) = 0$ and for any $t \in (0,r)$, $f(t) > 0$ (notice that $f'(0) > 0$). Thus, $\tilde{f}, f > 0$ on $(0,r)$. Taking $t \to r$ in (3), we have

$$0 = f(r) \geq \tilde{f}(r) > 0,$$

which leads to a contradiction. □

For general situations, let

$$f(t) = \langle J(t), J(t) \rangle, \quad \tilde{f}(t) = \langle \widetilde{J}(t), \widetilde{J}(t) \rangle.$$

Motivated by above proof, if $f(a) \geq \tilde{f}(a)$ holds, it is necessary to prove:

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{\tilde{f}(\epsilon)} = 1 \quad \text{and} \quad \frac{f'}{f} \geq \frac{\tilde{f}'}{\tilde{f}}, \text{ on } (0,a] \text{ on which } \tilde{f}, f > 0.$$

The former can be obtained by L'Hospital's rule:

$$\begin{aligned}\lim_{t\to 0}\frac{f(t)}{\tilde{f}(t)} &= \lim_{t\to 0}\frac{\langle J'(t), J(t)\rangle}{\left\langle \tilde{J}'(t), \tilde{J}(t)\right\rangle} \\ &= \lim_{t\to 0}\frac{\left\langle J'(t), \frac{J(t)-J(0)}{t}\right\rangle}{\left\langle \tilde{J}'(t), \frac{\tilde{J}(t)-\tilde{J}(0)}{t}\right\rangle} \\ &= \frac{|J'(0)|^2}{\left|\tilde{J}'(0)\right|^2} = 1.\end{aligned}$$

To prove $\tilde{f}, f > 0$ on $(0,a]$, let's assume that $\tilde{J}$ is a non-zero Jacobi field; otherwise, the conclusion is trivial. Since there are no conjugate points to $\tilde{\gamma}(0)$ along $\tilde{\gamma}$, the function $|\tilde{J}(t)|$ has no zeros other than $t=0$, i.e. $\tilde{f} > 0$ on $(0,a]$. Also, one can prove $f > 0$ on $(0,a]$ in the same way as the final part of the proof for the case $n=2$, provided the latter inequality holds true. However, to verify the inequality, one needs the minimal property of Jacobi fields which is known as the basic index lemma.

Morse index form and proof of Rauch comparison theorem

To see how the Morse index form is motivated, we shall introduce "path space" and one can see more details in [3], Part III.

The set of all piecewise smooth paths from p to q in M will be denoted by $\Omega(M;p,q)$, or briefly by $\Omega(M)$ or Ω. We will think of Ω as being something like an "infinite dimensional manifold", for example, a topological space in which each point has a neighborhood homeomorphic to an open set in a Banach space. To start the analogy we make the following definition.

By the tangent space of Ω at a path w will be meant the vector space consisting of all piecewise smooth vector fields W along ω for which vanishes at endpoints. The notation $T_\omega\Omega$ will be used for this vector space.

If F is a real valued function on Ω it is natural to ask what

$$F_{*\omega} : T_\omega\Omega \to T_{F(\omega)}\mathbb{R}$$

the induced map on the tangent space, should mean. When F is a function which is smooth in the usual sense, on a Riemannian manifold M. Given $X \in T_pM$, choose a smooth path $u \to \alpha(u)$ in M, which is defined for $-\varepsilon < u < \varepsilon$, so that

$$\alpha(0) = p, \quad \frac{d\alpha}{du}(0) = X.$$

Then $F_*(X)$ is equal to

$$\left.\frac{d(F(\alpha(u))}{du}\right|_{u=0},$$

multiplied by the basis vector $\left(\frac{d}{dt}\right)_{F(p)} \in T_{F(p)}\mathbb{R}$.

A variation of ω denoted by $\bar{\alpha}(u) = \alpha(u,t)$ for $u \in (-\varepsilon, \varepsilon)$ may be considered as a "smooth path" in Ω. And more generally if $(-\varepsilon, \varepsilon)$ in the variation is replaced by a neighborhood of $0 \in \mathbb{R}^n$, then it is called an n-parameter variation of ω. By analogy with the definition given above, if F is a real valued function on Ω, we attempt to define $F_{*\omega}$ as follows. Given $W \in T_\omega\Omega$ choose a variation $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$ with

$$\bar{\alpha}(0) = \omega, \frac{d\bar{\alpha}}{du}(0) = W$$

and set $F_{*\omega}(W)$ equal to $\left.\frac{d(F(\bar{\alpha}(u))}{du}\right|_{u=0}$ multiplied by the tangent vector $\left(\frac{d}{dt}\right)_{F(\omega)}$. Of course without hypothesis on F there is no guarantee that this derivative will exist, or will be independent of the choice of $\bar{\alpha}$. We will not investigate what conditions F must satisfy but we have indicated how F_* might be defined only to motivate the following.

Definition 3

A path ω is a critical path for a function $F : \Omega \to \mathbb{R}$ if and only if

$$\left.\frac{d(F(\alpha(u))}{du}\right|_{u=0},$$

for every variation of ω.

We denote the length of the piecewise smooth curve $\gamma : [a,b] \to M$ by $L(\gamma)$. By definition,

$$L(\gamma) = \int_a^b |\gamma'(t)| \mathrm{d}t$$

It follows from the chain rule that $L(\gamma)$ does not depend on a particular choice of parameterization. Then the length $L : \Omega \to \mathbb{R}$ is a real valued function on Ω. By first variation of arc length ([1], Section 3.3), we know that the path ω is a critical point for the function L if and only if ω is a geodesic.

Continuing with the analogy developed in the previous, we now wish to define a bilinear functional

$$L_{**\gamma} : T_\gamma\Omega \times T_\gamma\Omega \to \mathbb{R}$$

when γ is a critical point of the function L, i.e. a geodesic. This bilinear functional will be called the Hessian of L at γ.

If f is a real valued function on a Riemannian manifold M with critical point p, then the Hessian

$$\nabla^2 f : T_pM \times T_pM \to \mathbb{R}$$

is a $(0,2)$-tensor can be defined as $\nabla^2 f = \nabla(df)$. Equivalently, given $X_1, X_2 \in TM_p$ choose a smooth map $\alpha : U \to M$ where U is a neighborhood of $(0,0)$ in $\mathbb{R}^2$ such that

$$\alpha(0,0) = p, \frac{\partial\alpha}{\partial u_1}(0,0) = X_1, \frac{\partial\alpha}{\partial u_2}(0,0) = X_2.$$

Then

$$\nabla^2 f(X_1, X_2) = \left.\frac{\partial^2 f(\alpha(u_1, u_2))}{\partial u_1 \partial u_2}\right|_{(0,0)}$$

This suggests defining L_{**} as follows. Given vector fields $W_1, W_2 \in T_\gamma \Omega$ choose a 2-parameter variation $\alpha : U \times [a, b] \to M$, such that

$$\alpha(0,0,t) = \gamma(t), \quad \frac{\partial \alpha}{\partial u_1}(0,0,t) = W_1(t), \quad \frac{\partial \alpha}{\partial u_2}(0,0,t) = W_2(t).$$

Then the Hessian $L_{**}(W_1, W_2)$ will be defined to be the second partial derivative

$$\left.\frac{\partial^2 L(\bar{\alpha}(u_1, u_2))}{\partial u_1 \partial u_2}\right|_{(0,0)}$$

where $\bar{\alpha}(u_1, u_2) \in \Omega$ denotes the path $\bar{\alpha}(u_1, u_2)(t) = \alpha(u_1, u_2, t)$. By second variation formula theorem ([4], pp.16-18), $L_{**\gamma}(W_1, W_2)$ is well defined symmetric and bilinear function and depends only on W_1, W_2 to γ. This Hessian is a Morse index form and more generally, we write the index form as:

$$I_\gamma(V, W) = \int_a^b \langle \nabla_T V, \nabla_T W \rangle + \langle R(W, T)V, T \rangle \mathrm{d}t,$$

where V and W are smooth vectors fields along γ. For cases of piecewise smooth ones, see [4], pp.16-18 as well. Sometimes, we simply write $I(V, W)$ or $I_a^b(V, W)$.

In a certain sense, Jacobi fields minimize the index form.

Lemma 1: Basic index lemma

Let γ be a geodesic in M from p to q containing no points conjugate to p. Let W be a piecewise smooth vector field on γ and V the unique Jacobi field such that $V(p) = W(p) = 0$ and $V(q) = W(q)$. Then

$$I(V, V) \leq I(W, W),$$

and equality holds only if $V = W$.

We are now in the position to prove Rauch Comparison Theorem.

Proof of Theorem 2. 首先考虑 $J \perp \gamma', \tilde{J} \perp \tilde{\gamma}'$ 的情形. 此时,

$$J(0) = \tilde{J}(0) = 0, \quad |J'(0)| = \left|\tilde{J}'(0)\right|.$$

要证明的结论是

$$\frac{|J(t)|^2}{|\tilde{J}(t)|^2} \geq 1, \quad \forall t \in (0, l].$$

利用 L'Hospital 法则和 (5.1) 式得到

$$
\begin{aligned}
\lim_{t\to 0}\frac{|J(t)|^2}{|\tilde{J}(t)|^2} &= \lim_{t\to 0}\frac{\langle J'(t), J(t)\rangle}{\left\langle \tilde{J}'(t), \tilde{J}(t)\right\rangle} \\
&= \lim_{t\to 0}\frac{\langle J''(t), J(t)\rangle + \langle J'(t), J'(t)\rangle}{\left\langle \tilde{J}''(t), \tilde{J}(t)\right\rangle + \left\langle \tilde{J}'(t), \tilde{J}'(t)\right\rangle} \\
&= \frac{|J'(0)|^2}{\left|\tilde{J}'(0)\right|^2} = 1.
\end{aligned}
$$

因此, 只需要证明 $\frac{|J(t)|^2}{|\tilde{J}(t)|^2}$ 是 t 的增函数, 即对于任意的 $t>0$ 有

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|J(t)|^2}{|\tilde{J}(t)|^2}\right) \geq 0.
$$

又因为在 $J(t)\neq 0$ 的点处,

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|J(t)|^2}{|\tilde{J}(t)|^2}\right) &= \frac{\langle J', J\rangle}{\langle \tilde{J}, \tilde{J}\rangle} - \frac{\langle J, J\rangle \left\langle \tilde{J}', \tilde{J}\right\rangle}{\langle \tilde{J}, \tilde{J}\rangle^2} \\
&= \frac{\langle J, J\rangle}{\langle \tilde{J}, \tilde{J}\rangle}\left(\frac{\langle J', J\rangle}{\langle J, J\rangle} - \frac{\left\langle \tilde{J}', \tilde{J}\right\rangle}{\langle \tilde{J}, \tilde{J}\rangle}\right).
\end{aligned}
$$

所以, 只需要证明 J 在 $(0,l]$ 上处处不为零并且

$$
\frac{\langle J', J\rangle}{\langle J, J\rangle} \geq \frac{\left\langle \tilde{J}', \tilde{J}\right\rangle}{\langle \tilde{J}, \tilde{J}\rangle}, \quad t\in(0,l].
$$

假定 $r\in(0,l]$, 使得 Jacobi 场 J 在 $(0,r)$ 上无零点. 首先要证明 (5.4) 式在区间 $(0,r)$ 上成立. 对于任意取定的 $t_1\in(0,r)$, 由于 □

Remark

Rauch Comparison Theorem still holds when $J(0), \tilde{J}(0)\neq 0$ or moreover, J and $\tilde{J}$ are not normal Jacobi fields. In that case, J and $\tilde{J}$ should satisfy:

$$
\begin{aligned}
\langle J(0), \gamma'(0)\rangle &= \left\langle \tilde{J}(0), \tilde{\gamma}'(0)\right\rangle, \\
\angle\left(J'(0), \gamma'(0)\right) &= \angle\left(\tilde{J}'(0), \tilde{\gamma}'(0)\right).
\end{aligned}
$$

For more details see [1], Theorem 5.1, p. 342.

Reference

[1] 陈维桓 and 李兴校. 黎曼几何引论. 北京大学出版社, Beijing, di 1 ban edition, 2002.

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[4] Jeff Cheeger and D. G. Ebin. *Comparison Theorems in Riemannian Geometry*. AMS Chelsea Publishing. AMS Chelsea Publishing, Providence, R.I, 2008.