Single Period Market Models

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1 Probability modelling and probability tools: Review 1

1.1 σ-algebras and random variables

The mathematical modelling of any experiment, for which outcome(s) involve randomness, starts with a given triplet $(\Omega, \mathcal{F}, P)$. Ω is the set of all scenarios (the set of all states the world) related to the experiment under consideration, $\mathcal{F}$ is a σ-algebra, and P is a probability measure. **Details, examples and interpretations are given in class, and below is a summary table**.

	Ω general	Ω finite
σ-algebra	collection $\mathcal{F}$ of subsets of Ω s.t. a) $\Omega \in \mathcal{F}$ b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ c) $A_1, A_2, \ldots \in \mathcal{F}$ $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$	one can take $\mathcal{F}$ = power set of Ω = all subsets of Ω $\exists\ B_1, \ldots, B_m$ i) (B_i) disjoint ii) $\bigcup_{i=1}^{m} B_i = \Omega$ iii) every $A \in \mathcal{F}$ is a union of some B_i
probab. measure	$P : \mathcal{F} \to [0, 1]$ s.t. a) $P[\Omega] = 1$ b) $P[\bigcup_{i=1}^{\infty} A_i]$ $= \sum_{i=1}^{\infty} P[A_i]$ for disjoint (A_i)	P is determined by the values $P[\omega_i]$, $\omega_i \in \Omega$.
filtration	σ-algebras $(\mathcal{F}_t)_{t=0,1,\ldots,T}$ with $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$ $\ldots \subseteq \mathcal{F}_T \subseteq \mathcal{F}$	characterized by a sequence $(B_{t,i})$ of increasing partitions of Ω
random variable	map $X : \Omega \to \mathbb{R}$ s.t. $\{\omega \vert X(\omega) \le a\} \in \mathcal{F}$ for any $a \in \mathbb{R}$	if $\mathcal{F}$ = power set, every map is a random variable

1.2 Conditional expectations

Again details, examples and interpretations are given in class.

- **Conditional probability:**

$$P[B|A] = \frac{P[A \cap B]}{P[A]} \quad \text{for } A, B \in \mathcal{F} \text{ with } P[A] > 0;$$

the probability that B happens given we already know that A happens.

- **Conditional expectation with respect to events:** For $A \in \mathcal{F}$ with $P[A] > 0$ and a random variable X, we define the conditional expectation $E[X|A]$ by

$$E[X|A] = E_Q[X],$$

where $E_Q[.]$ is the expectation under the probability Q given by $Q(B) := P[B|A]$ for any event B.

- **Conditional expectation with respect to sub-σ-algebra:** Let $\mathcal{A} \subseteq \mathcal{F}$ be a sub-σ-algebra, that is generated by the sequence of events $(B_i)_{i=1,...n}$ that constitutes a partition of Ω (i.e. $\mathcal{A} := \sigma(B_1, ..., B_n)$, $\bigcup_{i=1}^{n} B_i = \Omega, \quad B_i \cap B_j = \emptyset \quad i \neq j$).
For any random variable X, the conditional expectation $E[X|\mathcal{A}]$ is given by

$$E[X|\mathcal{A}] = \sum_{i=1}^{n} E[X|B_i] I_{B_i},$$

- **Proposition:** Let X be a random variable and $\mathcal{A}$ a sub-σ-algebra. Then

$$Y = E[X|\mathcal{A}] \iff \begin{cases} \text{(i) } Y \text{ is } \mathcal{A}\text{-measurable,} \\ \text{(ii) } E[YI_A] = E[XI_A] \ \forall A \in \mathcal{A}. \end{cases}$$

- **Properties of the conditional expectation:**

  1. **Tower property:**
  $$E[E[X|\mathcal{A}]|\mathcal{B}] = E[X|\mathcal{B}]$$
  for $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{F}$ sub-σ-algebras.
  2. **Linearity:**
  $$E[X_1Y_1 + X_2Y_2|\mathcal{A}] = X_1E[Y_1|\mathcal{A}] + X_2E[Y_2|\mathcal{A}]$$
  for random variables X_1, X_2, Y_1, Y_2 and a sub-σ-algebra $\mathcal{A} \subseteq \mathcal{F}$ with X_1, X_2 $\mathcal{A}$-measurable. In particular, when X is $\mathcal{A}$-measurable, we get $E[X|\mathcal{A}] = X$.
  3. **Independence:**
  $$E[X|\mathcal{A}] = E[X]$$
  if X is independent from $\mathcal{A}$ (i.e. the case where $P[(X \leq a) \cap A] = P[X \leq a]P[A]$ for all $a \in \mathbb{R}$ and $A \in \mathcal{A}$) or when the sub-σ-algebra is equal to the trivial one $\mathcal{F}_0 = \{\Omega, \emptyset\}$.

- **Filtration and stochastic processes:** Details to be given in class.

2 One Period Mathematical/Economic/Financial Model

2.1 Uncertainty Modelling:

- The set Ω of the states of the world is finite:
$$\Omega = \{\omega_1, \ldots, \omega_K\} \quad \text{for } K < \infty$$
- Probability measure P on Ω with $P(\omega) > 0$ for all $\omega \in \Omega$
- Filtration=Flow of information about the market:
$$(\mathcal{F}_t)_{t=0,1} \qquad \mathcal{F}_0 \subset \mathcal{F}_1.$$
Very often we assume that $\mathcal{F}_0$ is trivial, i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

2.2 Underlying Assets:

- The trading dates are the initial date $t = 0$ and the terminal date $t = 1$;
- *Bank account process* (non-risky asset) B: $B_0 = 1$, B_1 random variable with $B_1 \geq 1 \Longrightarrow$ interest rate $r = B_1 - 1 \geq 0$
- *N Stocks* (Risky assets) with price processes given by S: $S_t = (S_1(t), \ldots, S_N(t))$, $t = 0, 1$;
$$\begin{aligned} S_n(0) &= \text{positive, deterministic price of the } n^{th} \text{ security} \\ S_n(1) &= \text{nonnegative, random price of the } n^{th} \text{ security} \end{aligned}$$

2.3 Trading Strategies and Value Processes:

- *Trading strategy* $H = (H_0, \ldots, H_N) \in \mathbb{R}^{N+1}$;
$$\begin{aligned} H_0 &= \text{dollars invested in the bank account} \\ H_n &= \text{units of the } n^{th} \text{ security held by the investor, } n = 1, ..., N. \end{aligned}$$
- *Value process* $V := V^H$
$$\begin{aligned} V_t &= \text{value of the portfolio at time } t \\ &= H_0 B_t + \sum_{n=1}^{N} H_n S_n(t), \qquad t = 0, 1 \end{aligned}$$
- *Gain* $G := G^H$: describes profit ($G > 0$) or loss ($G < 0$) between times 0 and 1:
$$G := V_1 - V_0.$$
Exercise: Prove that
$$\begin{aligned} G &= H_0(B_1 - B_0) + \sum_{n=1}^{N} H_n(S_n(1) - S_n(0)) \\ &= H_0 r + \sum_{n=1}^{N} H_n \Delta S_n, \end{aligned}$$
where $\Delta S_n = S_n(1) - S_n(0)$.

2.4 Discounted prices:

- Normalize prices such that B is constant; the bank account is then called the *numéraire*
- Discounted price process $S^\star$

$$S^\star(t) = (S_1^\star(t), \dots, S_N^\star(t)), \quad \text{where} \quad S_n^\star(t) := \frac{S_n(t)}{B_t}$$

- Discounted value and gains processes:

$$V_t^\star := \frac{V_t}{B_t} = H_0 + \sum_{n=1}^{N} H_n S_n^\star(t), \qquad G^\star := V_1^\star - V_0^\star.$$

Exercise: Prove that

$$G^\star = \sum_{n=1}^{N} H_n(S_n^\star(1) - S_n^\star(0)) = \sum_{n=1}^{N} H_n \Delta S_n^\star.$$

3 Arbitrage and risk neutral probability measures

Definition 1. *1) An* arbitrage opportunity *is a trading strategy H satisfying*

$$V_0 = 0, \quad V_1(\omega) \geq 0 \ \forall \ \omega \in \Omega, \quad \textit{and } E[V_1] > 0.$$

2) The law of one price holds *if for any two trading strategies $\hat{H}$ and $\tilde{H}$, the following holds.*

$$\hat{V}_1(\omega) = \tilde{V}_1(\omega) \ \forall \ \omega \in \Omega \ \textit{implies that } \hat{V}_0 = \tilde{V}_0.$$

3) A probability measure $Q : \Omega \to [0, 1]$ is a risk neutral probability measure *if $Q(\omega) > 0$ for all $\omega \in \Omega$ and*

$$S_n^\star(0) = \sum_{\omega \in \Omega} Q(\omega) S_n^\star(1)(\omega) = E_Q[S_n^\star(1)], \ n = 1, \dots, N.$$

What is the economic/financial meaning of the above?

- Economic point of view: reasonable to study models that are free from arbitrage
- If an arbitrage opportunity were to exist
  $\Longrightarrow$ everybody would use this trading strategy
  $\Longrightarrow$ prices of the securities would be affected so
  that the arbitrage opportunity would vanish!

Proposition 2. *If there are no arbitrage opportunities, the law of one price holds. The converse, however, is not necessarily true.*

It is always NOT easy to check directly whether a model is arbitrage-free or not . The following is very helpful in this direction!

Theorem 3. *The following assertions are equivalent.*
(a) *There are no arbitrage opportunities.*
(b) *There exists a risk neutral probability measure.*

Exercise: Prove that assertion (b) of the above theorem implies assertion (a).

Example 1 (Exercise 1.4 of the textbook): Suppose that there are three scenarios, $\omega_1, \omega_2, \omega_3$, $r = 0$ and two stocks with the following price processes

scenario	$S_1(0)$	$S_2(0)$	$S_1(1)$	$S_2(1)$
ω_1	4	7	8	10
ω_2	4	7	6	8
ω_3	4	7	3	4

Solve the following questions:
(i) Prove that the model has the law of one price
(ii) Provide an arbitrage opportunity for the model.

Example 2: Prove that a market model has arbitrage if there exists a trading strategy satisfying

$$V_0 < 0 \quad \text{and} \quad V_1(\omega) \geq 0, \quad \forall\ \omega \in \Omega.$$

Precisely, we define the following

Definition 4. *A trading strategy, $H = (H_0, H_1, ..., H_N)$, is called a dominating strategy if there exists another strategy $H' = (H'_0, H'_1, ..., H'_N)$ such that*

$$V_0^H = V_0^{H'} \quad \text{and} \quad V_1^H(\omega) > V_1^{H'}(\omega), \quad \forall\ \ \omega \in \Omega.$$

It is easy to prove the following

Lemma 5. *A market admits a dominating strategy if and only if there exists a strategy $H = (H_0, H_1, ..., H_N)$ satisfying*

$$V_0^H = 0 \quad \text{and} \quad V_1^H(\omega) > 0 \quad \forall\ \omega \in \Omega.$$

In fact, it is enough to use the definition of dominating strategy, put $\bar{H} := H - H'$, and then easily check that this strategy $\bar{H}$ satisfies the conditions given by the lemma.

Theorem 6. *Consider the following assertions.*
(a) *The market admits a dominating strategy.*
(b) *There exists a strategy H satisfying*

$$V_0^H < 0 \quad \text{and} \quad V_1^H(\omega) \geq 0, \quad \forall\ \omega \in \Omega.$$

(c) *The market has arbitrages.*
Then, we have (a) $\Longleftrightarrow$(b) $\Longrightarrow$ (c) .

The proof is discussed in class together with examples made up some times on the spot.

Example 3 : Suppose that there are three scenarios, $\omega_1, \omega_2, \omega_3$, $r \geq 0$ deterministic and two stocks with the following price processes

scenario	$S_1(0)$	$S_2(0)$	$S_1(1)$	$S_2(1)$
ω_1	4	7	8	10
ω_2	4	7	a	8
ω_3	4	7	3	4,

where a is a positive number. Solve the following questions:
(i) What relation should a and r satisfy in order that the market model is arbitrage free.
(ii) If the relationship found in part a) holds, determine the set of all risk-neutral measures for the model. Is the market model complete nor not?

4 Contingent claims

Definition 7. *1) A* contingent claim *is a random variable X representing a payoff at time 1.*
2) A contingent claim X is attainable *if there exists a trading strategy H such that*

$$X = V_1.$$

Such an H is called replicating portfolio.

What is a fair value for a contingent claim?

Proposition 8. *Suppose that the market has no arbitrage opportunity.*
1) For every trading strategy, one has $V_0 = E_Q[\frac{V_1}{B_1}]$ for any risk neutral probability measure Q.
2) [**Risk neutral valuation principle**] *If a contingent claim X is attainable, then its unique fair value at time 0, p_0 is given by*

$$E_Q[\frac{V_1}{B_1}] = H_0 + \sum_{n=1}^{N} H_n S_n(0),$$

for any replicating portfolio H and for any risk neutral prob. meas. Q.

Examples of Contingent Claims:

- A *European call option* with strike price K on S_1 is a contingent claim with payoff $X = (S_1 - K)^+ = \max\{0, S_1 - K\}$.
- A *European put option* with strike price K on S_1 is a contingent claim with payoff $X = (K - S_1)^+ = \max\{0, K - S_1\}$.

5 Complete and incomplete markets

We set $\mathbb{M} = \{Q \,|\, Q \text{ is a risk neutral probability measure}\}$ and assume $\mathbb{M} \neq \emptyset$ throughout this section.

Proposition 9. *A contingent claim X is attainable if and only if $E_Q[\frac{X}{B_1}]$ is constant for all $Q \in \mathbb{M}$. In that case, $E_Q[\frac{X}{B_1}]$ is the unique fair value.*

Is there a unique fair value for every contingent claim? What is the set of fair values for a contingent claim?

Definition 10. *The market is* complete *if every contingent claim is attainable. Otherwise, the market is* incomplete.

Theorem 11. *The following are equivalent.*
a) The market is complete.
b) The number of states in Ω equals the number of independent vectors in $\{B_1, S_1(1), \ldots, S_N(1)\}$.
c) $\mathbb{M}$ consists of exactly one element.

Proposition 12. *For any contingent claim X, we have*

$$\textit{fair values} = \Big(or \Big[\inf_{Q \in \mathbb{M}} E_Q \Big[\frac{X}{B_1} \Big], \sup_{Q \in \mathbb{M}} E_Q \Big[\frac{X}{B_1} \Big] \Big] or \Big).$$

The interval is one point if and only if X is attainable. Otherwise, the interval is open.

Examples:

- **Example 1:** Consider the model of example 1.4 of the textbook, which is given by

$$r = \frac{1}{9}, \quad S_1(0) = 5, \quad S_1(1) = \begin{cases} 60/9, & \text{for } \omega_1 \\ 60/9, & \text{for } \omega_2 \\ 40/9, & \text{for } \omega_3 \\ 20/9, & \text{for } \omega_4 \end{cases},$$

$$S_2(0) = 10, \quad S_2(1) = \begin{cases} 120/9, & \text{for } \omega_1 \\ 80/9, & \text{for } \omega_2 \\ 80/9, & \text{for } \omega_3 \\ 120/9, & \text{for } \omega_4 \end{cases}$$

1) Describe the set of all risk-neutral probabilities.
2) Is the market complete? Justify?
3) Describe the set of all fair prices for a call option with strike price $K = 10$ on the second stock. What conclusion can you withdraw?
4) Describe the set of all fair prices for a call option with strike price $K = 50/9$ on the first stock. What conclusion can you withdraw?

- **Example 2:** Consider a single-period model with $K = 3$, $N = 1$, interest rate $r \geq 0$, $S_0 = 1$,

$$S_1 = \begin{cases} 1+u & \text{with probability } p_1 \\ 1+m & \text{with probability } p_2 \\ 1+d & \text{with probability } p_3 \end{cases}$$

with numbers $u > m > d \geq -1$ and positive numbers p_1, p_2, p_3 with $p_1 + p_2 + p_3 = 1$.
1) Provide necessary and sufficient conditions in terms of u, m, d and r for the market model to be free from arbitrage opportunities.
Hint: You can use without proving that for given real numbers $a_1, \ldots, a_N, b$, there exist $q_1 > 0, \ldots, q_N > 0$ with $\sum_{n=1}^N q_n = 1$ and $\sum_{n=1}^N q_n a_n = b$ if and only if $\min_n a_n < b < \max_n a_n$.
2) Under the no-arbitrage assumption, describe the set $\mathbb{M}$ of all risk neutral probability measures and prove that the market is incomplete.
3) For a European call option $(S_1 - K)^+$ with strike price $K = 1 + m$, determine the lower and upper bounds on the fair value of the claim.