Motivation for the logistic function in logistic regression.

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In this short note some additional motivation for the logistic function is provided. The idea is that we will posit a general model for the joint distribution of (y, x) and show that under this model the conditional probability $p(y = 1|\mathbf{x})$ has the form of a logistic function.

First we assume that the marginal distribution of y is $y \sim \text{Bernoulli}(\pi)$. Next, we assume that the conditional density of $\mathbf{x}|y = 1$ and $\mathbf{x}|y = 1$ is of the form:

$$p(\mathbf{x}|y = 1) = \exp(\boldsymbol{\theta}_1^\top \mathbf{x} - \kappa(\boldsymbol{\theta}_1))h(\mathbf{x})$$
$$p(\mathbf{x}|y = 0) = \exp(\boldsymbol{\theta}_0^\top \mathbf{x} - \kappa(\boldsymbol{\theta}_0))h(\mathbf{x}),$$

where $\boldsymbol{\theta}_0, \boldsymbol{\theta}_1$ are unknown parameters. Any random variable with this kind of density is said to come from an exponential family. A couple examples of this are:

1. $(p = 1)$ the Poisson distribution with parameter has a density/pmf of the form:

   $$p(x) = \frac{\lambda^x}{x!} \exp(-\lambda) = \exp(\log(\lambda)x - \lambda)\frac{1}{x!}.$$

   This is an exponential family of densities with parameter $\theta = \log(\lambda)$, $\kappa(\theta) = \lambda$ and $h(x) = (x!)^{-1}$.

2. The multivariate Gaussian distribution $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ forms an exponential family. For simplicity let's show this in the case that $\boldsymbol{\Sigma} = \mathbf{I}_p$. The density can be written as

   $$p(\mathbf{x}) = (2\pi)^{-p/2} \exp\left(-\frac{1}{2}\mathbf{x}^\top \mathbf{x} + \mathbf{x}^\top \mu - \frac{1}{2}\mu^\top \mu\right)$$
   $$= (2\pi)^{-p/2} \exp\left(\boldsymbol{\mu}^\top \mathbf{x} - \frac{1}{2}\boldsymbol{\mu}^\top \boldsymbol{\mu}\right)(2\pi)^{-p/2} \exp\left(-\frac{1}{2}\mathbf{x}^\top \mathbf{x}\right),$$

   So $\boldsymbol{\theta} = \boldsymbol{\mu}$, $\kappa(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\mu}^\top \boldsymbol{\mu}$, and $h(\mathbf{x}) = (2\pi)^{-p/2} \exp\left(-\frac{1}{2}\mathbf{x}^\top \mathbf{x}\right)$.

Exponential family distributions are ubiquitous and cover a wide variety of potential conditional distributions for the feature vector $\mathbf{x}$.

Let's return to the joint model of $(y, \mathbf{x})$ and compute $p(y = 1|\mathbf{x})$:

$$p(y = 1|\mathbf{x}) = \frac{p(y = 1, \mathbf{x})}{p(\mathbf{x})} = \frac{p(\mathbf{x}|y = 1)p(y = 1)}{p(\mathbf{x}|y = 1)p(y = 1) + p(\mathbf{x}|y = 0)p(y = 0)}$$
$$= \frac{\exp(\boldsymbol{\theta}_1^\top \mathbf{x} - \kappa(\boldsymbol{\theta}_1))h(\mathbf{x})\pi}{\exp(\boldsymbol{\theta}_1^\top \mathbf{x} - \kappa(\boldsymbol{\theta}_1))h(\mathbf{x})\pi + \exp(\boldsymbol{\theta}_0^\top \mathbf{x} - \kappa(\boldsymbol{\theta}_0))h(\mathbf{x})(1 - \pi)}.$$

Canceling the $h(\mathbf{x})$ terms and dividing the numerator by the denominator we get:

$$p(y = 1|\mathbf{x}) = \frac{1}{1 + \exp\left((\boldsymbol{\theta}_0 - \boldsymbol{\theta}_1)^\top \mathbf{x} + \kappa(\boldsymbol{\theta}_1) - \kappa(\boldsymbol{\theta}_0) + \log(\frac{1-\pi}{\pi})\right)}$$
$$= \sigma\left(((\boldsymbol{\theta}_1 - \boldsymbol{\theta}_0)^\top \mathbf{x} + \kappa(\boldsymbol{\theta}_0) - \kappa(\boldsymbol{\theta}_1) + \log(\frac{\pi}{1-\pi})\right).$$

Consequently, if we call $\boldsymbol{\beta} = \boldsymbol{\theta}_1 - \boldsymbol{\theta}_1$ and $\beta_0 = \kappa(\boldsymbol{\theta}_0) - \kappa(\boldsymbol{\theta}_1) + \log(\frac{\pi}{1-\pi})$, the above expression can be written as $\sigma(\boldsymbol{\beta}^\top \mathbf{x} + \beta_0)$. Thus, the logistic regression model encapsulates any model for the joint distribution of $(y, \mathbf{x})$ where the conditional distribution of $\mathbf{x}$ is from an exponential family. Note that if we were willing to make more assumptions about the form of the exponential family distribution from which $\mathbf{x}$ is derived this would potentially place additional constraints on $\boldsymbol{\beta}$ and β_0.