Karush - Kuhn - Tucker Conditions:

$$\min_{X} f(X)$$

X is a $n \times 1$ vector on real number

$X \in \mathbb{R}^n$

Subj. to $l_j(X) = 0$, $j = 1, \dots, r$

$h_i(X) \leq 0$, $i = 1, \dots, m$

Optimality Conditions

We define the Lagrangian as

$$L(x, \lambda, \mu) = \cancel{f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}}$$

$$= f(x) + \sum_{j=1}^{r} \lambda_j l_j(x) + \sum_{i=1}^{m} \mu_i h_i(x)$$

The necessary conditions of optimality are called KKT conditions. They are as follows:

- $$\frac{\partial L}{\partial x_k} = \frac{\partial f(x)}{\partial x_k} + \sum_{j=1}^{r} \lambda_j \frac{\partial l_j(x)}{\partial x_k} + \sum_{i=1}^{m} \mu_i \frac{\partial h_i(x)}{\partial x_k} = 0$$

  $k \in (1, 2, \dots, n)$

- $l_j(x) = 0$, $h_i(x) \leq 0$ for all i, j
- $\mu_i h_i(x) = 0$ for all i
- $\mu_i \geq 0$ for all i

Example 1

Minimize $f = x_1^2 + 2x_2^2 + 3x_3^2$

$g_1 = x_1 - x_2 - 2x_3 \leq 12$

$g_2 = x_1 + 2x_2 - 3x_3 \leq 8$

Using KKT conditions

Solution

$$L = (x_1^2 + 2x_2^2 + 3x_3^2) + \mu_1 (x_1 - x_2 - 2x_3 - 12) + \mu_2 (x_1 + 2x_2 - 3x_3 - 8)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 + \mu_1 + \mu_2 = 0 \qquad \text{— (1)}$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - \mu_1 + 2\mu_2 = 0 \qquad \text{— (2)}$$

$$\frac{\partial L}{\partial x_3} = 6x_3 - 2\mu_1 - 3\mu_2 = 0 \qquad \text{— (3)}$$

$$\mu_1(x_1 - x_2 - 2x_3 - 12) = 0 \qquad \text{— (4)}$$

$$\mu_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \qquad \text{— (5)}$$

$$x_1 - x_2 - 2x_3 - 12 \leq 0 \qquad \text{— (6)}$$

$$x_1 + 2x_2 - 3x_3 - 8 \leq 0 \qquad \text{— (7)}$$

$$\mu_1 \geq 0, \quad \mu_2 \geq 0 \qquad \text{— (8)}$$

Case I: $\mu_1 = 0$

putting $\mu_1 = 0$ in (1), (2), (3)

$$x_1 = x_2 = -\frac{\mu_2}{2}, \quad x_3 = \frac{\mu_2}{2}$$

Lets write eq (5),

$$\mu_2\left(-\frac{\mu_2}{2} - \frac{2\mu_2}{2} - 3\cdot\frac{\mu_2}{2} - 8\right) = 0$$

$$-3\mu_2^2 - 8\mu_2 = 0$$

$$\therefore \mu_2 = 0 \text{ or } -\frac{8}{3}$$

$$\Rightarrow \mu_2 = 0 \longrightarrow x^* = [0, 0, 0]$$

Case II: $x_1 - x_2 - 2x_3 - 12 = 0$

or, $$-\frac{(\mu_1+\mu_2)}{2} - \frac{(\mu_1 - 2\mu_2)}{4} - \frac{2\mu_1 + 3\mu_2}{3} - 12 = 0$$

or, $17\cancel{\lambda_1}^{\,\mu_1} + 12\cancel{\lambda_2}^{\,\mu_2} = -144$ — inconsistent with $\mu_1 \geq 0, \mu_2 \geq 0$

$\therefore$ Solution is $[0\ 0\ 0]^T$