

Mathematical Optimization

- When you optimize something, you are "making it best".
- 'Best' can vary. If you're a football player, you might want to maximize your running yards and ~~also~~ also minimize your fumbles. Both maximizing and minimizing are types of optimizations.
- A branch of applied mathematics useful in many different fields. Here are a few examples.
 - Manufacturing
 - Production
 - Control
 - Transportation
 - Scheduling
 - Networks
 - Economics
 - Policy making

Objective function $f(x)$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ — variables}$$

Constraints

Equality constraints $g(x) = 0$

Inequality constraints $h(x) \leq 0$

Types of Optimization

- Some problems have constraints and some do not
- There can be ~~be~~ one variable or many
- Variables can be discrete or continuous



- Mathematical optimization works better than traditional "guess and check" methods.

- $\frac{df}{dx} \Big|_{x^*} = 0$, $\frac{d^2f}{dx^2} \Big|_{x^*} > 0 \rightarrow \text{C.R. conditions for minima}$

- $\min_{x_1, x_2} F(x) = F_0 + (x_1 - a)^2 + (x_2 - b)^2$
 $F(x)$ has minimum at $x = (a, b)$

- $\min_{x_1, x_2} F(x) = F_0 + (x_1 - a)^2 + (x_2 - b)^2$

Subj to $G(x) = x_1 - x_2 = 0$

Solution $\rightarrow x_1 = \frac{a+b}{2}, x_2 = \frac{a+b}{2}$

- Alternate way (use of Lagrange multipliers)

~~$\tilde{F}(x) = F_0 + (x_1 - a)^2 + (x_2 - b)^2 + \lambda(x_1 - x_2)$~~

$$\tilde{F}(x) = F_0 + (x_1 - a)^2 + (x_2 - b)^2 + \lambda(x_1 - x_2)$$

λ = Lagrange Multiplier

$$\frac{\partial \tilde{F}}{\partial x_1} = 2(x_1 - a) + \lambda = 0$$

$$\frac{\partial \tilde{F}}{\partial x_2} = 2(x_2 - b) - \lambda = 0$$

$$x_1^* = a - \frac{\lambda}{2}, \quad x_2^* = b + \frac{\lambda}{2}$$

$$\frac{\partial \tilde{F}}{\partial \lambda} = x_1^* - x_2^* = 0$$

$$\lambda^* = a - b$$

$$x_1^* = \frac{a+b}{2}, \quad x_2^* = \frac{a+b}{2}$$

Let $f(x) = f(x_1, x_2, \dots, x_n)$ denote a scalar-valued function of the vector x .

$$x \in \mathbb{R}^n \quad f: \mathbb{R}^n \rightarrow \mathbb{R}$$

Suppose that

i) $f(x)$ is continuous for all x

ii) The gradient vector

$$\frac{\partial f}{\partial x} \stackrel{\Delta}{=} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{is continuous for all } x$$

iii) The second-derivative (or Hessian) matrix

$$\frac{\partial^2 f}{\partial x^2} \stackrel{\Delta}{=} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

is continuous for all x .

→ necessary condition $\left. \nabla f \right|_{x=x^*} = 0$

→ sufficient " $\left. \frac{\partial^2 f}{\partial x^2} \right|_{x^*}$ is positive definite

$$\circ \min F(x)$$

$$\text{subj. to } G_i(x) = 0 \quad \forall i$$

$$\rightarrow \min F(x)$$

$$\text{subj. to } g(x) = 0$$

$$\text{we define } L(x, \lambda) = f(x) + \lambda^T g(x)$$

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

\rightarrow Example:

$$\text{minimize } f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

$$x_1, x_2, x_3$$

$$\text{subj. to } x_3 = x_1 x_2 + 5$$

$$x_1 + x_2 + x_3 = 1$$

Solution:

Define the Lagrangian

$$L(x, \lambda) = x_1^2 + x_2^2 + x_3^2 + \lambda_1 (x_1 x_2 + 5 - x_3)$$

$$+ \lambda_2 (x_1 + x_2 + x_3 - 1)$$

$$\left. \frac{\partial L}{\partial x} \right|_{x^*, \lambda^*} = 0 \quad \text{gives}$$

$$2x_1^* + \lambda_1^* x_2^* + \lambda_2^* = 0$$

$$2x_2^* + \lambda_1^* x_1^* + \lambda_2^* = 0$$

$$2x_3^* - \lambda_1^* + \lambda_2^* = 0$$

$$\left. \frac{\partial L}{\partial \lambda} \right|_{x^*, \lambda^*} = 0 \quad \text{gives}$$

$$x_1^* x_2^* + 5 - x_3^* = 0$$

$$x_1^* + x_2^* + x_3^* - 1 = 0$$

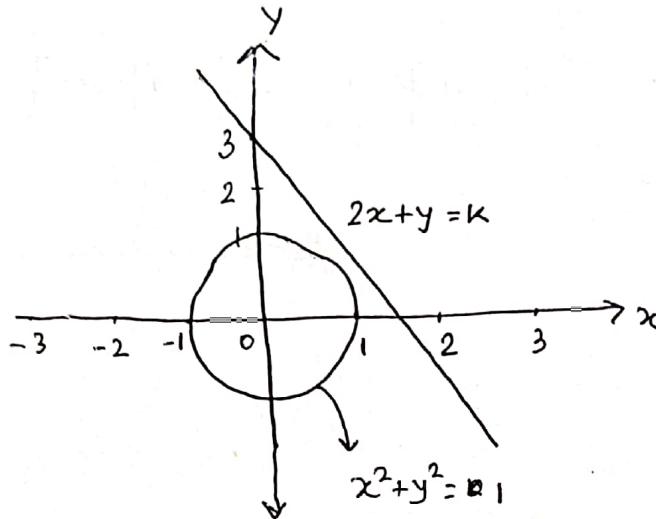
Solving the five equations, we get

$$x^* = \begin{bmatrix} (2, -2, 1) \\ (-2, 2, 1) \end{bmatrix}$$

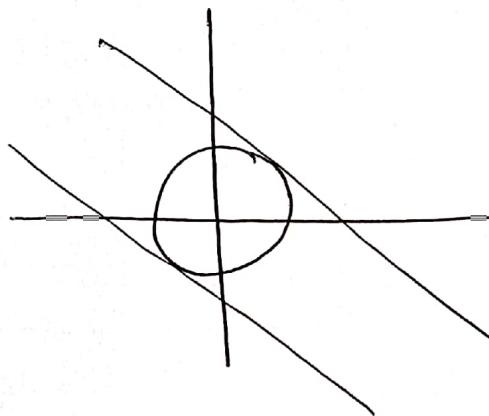
$$\text{max. } f(x, y) = 2x + y$$

x, y

$$\text{Subj. to } x^2 + y^2 = 1$$



Contour line of $f(x, y)$
is the set of all points
where $f(x, y) = K$ for
some constant K .



When the contour lines of two functions f and g
are tangent, their ~~and~~ gradient vectors are parallel.

$$\Rightarrow \nabla f(x_0, y_0) = -\lambda_0 \nabla g(x_0, y_0)$$

$$\nabla f(x_0, y_0) + \lambda_0 \nabla g(x_0, y_0) = 0$$