Chapter 1

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Solutions of Non-linear Equations :-

In science and Engineering, mathematical models for different problems can be formulated into the equations of the form,

$$f(x)=0 \quad \text{--- (1)}$$

where, x and $f(x)$ may be real, complex. The solution process often involves findings the values of x that would satisfy $f(x)=0$. These values are called roots of the equation. The equation (1) may belongs to one of the following types of equations.

1. Algebric equation
2. Polynomial equation
3. Transcendental equation

Non-linear Function :

Any function of one variable which does not graph as a straight line in two dimensions or any function of two variables which does not graph as a plane in three dimensions can be said to be non linear.

Consider the function :

$y=f(x)$, $f(x)$ is said to be non linear if the response of the dependent variable y is not direct or exact proportion to the changes in the independent variable x.

For example : $y=x^2+1$

Linear Equation:-

Consider the function,

$y = f(x)$, $f(x)$ is a linear function if the dependent variable y changes in direct propotion to the change in independent variable x.

eg: $y = 2x+3$

Types of Equations:-

1) **Algebric Equation**: An equation of the form $y = f(x)$ is said to be algebric if it can be expressed as:

$$f_n y_n + f_{n-1} y_{n-1} + \cdots + f_1 y_1 + f_0 = 0,$$

where, f_i is an i^{th} order polynomial in x (say).

eg:- $2x+4y-19=0$, $x^3 - xy - 4y^3 = 0$

In general form it can also be represented as $f(x,y)=0$

2) **Polynomial Equations**:- Polynomial equations are simple class of algebric equation that are represented as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

This is called n^{th} degree polynomial and has n roots. The roots may be

i) real and distinct

ii) real and repeated

iii) complex

3) Transcendental Equation:-

A non-algebric equation is called a transcendental equation. These include the trigonometric, exponential and logarithmic functions.

eg: i) $2\sin x - x = 0$,

ii) $e^x \sin x - \frac{1}{2}x = 0$

iii) $\log x^2 - 1 = 0$

Root: The value 'α' of x which satisfies $f(x)=0$, is called root of $f(x)=0$

Geometrically, a root of $f(x)=0$ is that value of x where the graph of $y=f(x)$ intersects the x-axis.

The process of finding the roots of an equation is known as the solution of that equation.

Q. Solve the equation:

$$2x^3 + x^2 - 13x + 6 = 0$$

⇒ Sol^n/

By inspection we can find $x=2$, which satisfies the given equation.

∴ 2 is its root, i.e. $x-2$ is a factor of

$x-2 \mid 2x^3 + x^2 - 13x + 6 \mid$

Dividing this polynomial by $x-2$, we get the quotient $2x^2+5x-3$ and remainder 0.

$x-2 \mid 2x^3 + x^2 - 13x + 6 \mid$

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The quotient is in the form of ax^2+bx+c
where, $a=2$, $b=5$, $c=-3$
So,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25+24}}{4}$$

$$= \frac{-5 \pm 7}{4}$$

Taking +ve

$$x = \frac{-5+7}{4}$$

$$= 1/2$$

Taking -ve sign

$$x = \frac{-5-7}{4}$$

$$= -3$$

Errors: The deviation from the actual value to the observed value is termed as error.

Types of errors:

1. Inherent errors: Error which are already present in the statement of a problem before its solution, are called inherent error. Such errors arise either due to the given data being approximate or due to the limitations of mathematical tables, calculators, or the digital computer.

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2) Numerical Errors:

Numerical errors (also known as procedural errors) are introduced during the process of implementation of numerical method. They are classified into two classes as:

a) Roundoff Errors: This type of errors arise from the process of rounding off the numbers during the computation. Such errors are unavoidable in most of the calculations due to the limitations of the computing aids. Roundoff errors can however be reduced:-

i) By changing the calculation procedure so as to avoid subtraction of nearly equal number or division by small number.

ii) By retaining at least one more significant figure at each step than that given in the data and rounding off at the last step.

b) Truncation Errors:- These errors are caused by using approximate results or replacing an infinite process by a finite one. If we are using a decimal computers having a fixed word length of 4 digits, rounding off of 13.658 gives 13.66 whereas truncation gives 13.65

Example :

If $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \infty = x$ (say)

is replaced by

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = x'$; then the truncation error is $(x - x')$

Absolute, Relative and Percentage error

If x is the true value of a equation x' is its approximate value, then $|x - x'|$ is called absolute (E_a) error.

The relative error is given as:

$$E_r = \left| \frac{x - x'}{x} \right|$$

and the percentage error is

$$E_p = 100 \left| \frac{x - x'}{x} \right| \%$$

Taylors Theorem:

The Taylor series of a real valued function $f(x)$ that is infinitely differentiable at a rear number 'a' is:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

which can be written in the more compact form as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where, $n!$ denotes the factorial of n and $f^{(n)}(a)$ denotes the n^{th} derivative of 'f' evaluted at the point a. The derivative of order zero of 'f' is defined to be f itself and $(x-x)^0$ and $0!$ both are defined to be 1. when $a=0$ the series is also called Maclaurin series.

Q. Derive the Maclaurin series of $\sin(x)$

⇒ ~~sol~~ $\sin$:

$f(x) = \sin x$

$f'(x) = \cos x$ $\quad$ $f''(x) = -\sin x$

$f'''(x) = -\cos x$ $\quad$ $f''''(x) = \sin x$

$f'''''(x) = \cos x$

$f(0) = 0$ $\quad$ $f'(0) = 1$

$f''(0) = 0$ $\quad$ $f'''(0) = -1$

$f''''(0) = 0$ $\quad$ $f'''''(0) = 1$

Maclaurin series is :-

~~$f(x) = x - \frac{x^3}{3!} +$~~

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f''''(0)x^4}{4!} + \dots$$

$$0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

$$\Rightarrow x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

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Methods of Solution:

There are number of methods to find the roots of non-linear eqⁿ as:-

1) Direct analytical method
2) Trial and error method
3) Graphical method
4) Iterative method

1. **Direct Analytical Method:-**

Equations are solved by using derived formulas. We just need to supply the value of input parameters in the eqⁿ.

eg:-

Solution of quadratic equation $ax^2+bx+c=0$ is given by,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Q. Solve the equation $2x^2+5x+2=0$

$a=2, b=5, c=2$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25-4\times2\times2}}{2\times2}$$

$$= \frac{-5 \pm \sqrt{25-16}}{4}$$

$$= \frac{-5 \pm 3}{4}$$

Taking +ve

$$\frac{-5+3}{4} = \frac{-2}{4} = -\frac{1}{2}$$

Taking -ve

$$\frac{-5-3}{4} = \frac{-8}{4} = -2$$

2. Trial and Error Method:-

In this method, we make a series of guess for x and we evaluate the function $f(x)$ at that x, if it is close to zero, it is one of the approximate root of the given non linear equation, otherwise make another guess for x and repeat the same process again until the x for which $f(x)=0$ is found.

Example:

Solve the eqn $f(x) = 2x^2 + x - 6 = 0$

⇒ Soln:

Step 1: Guess $x = 0 \Rightarrow f(x) = -6$
Step 2: Guess $x = 1 \Rightarrow f(x) = -3$
Step 3: Guess $x = 2 \Rightarrow f(x) = 4$
Step 4: Guess $x = -2 \Rightarrow f(x) = 0$
Step 5: Guess $x = 3/2 \Rightarrow f(x) = 0$

3. Iterative Method:-

There are number of iterative methods that have been tried and use successfully in various problem. All these methods, typically generate a sequence of estimators of the solution, which is expected to converge to the true solution.

They are categorized into two classes:-

i) Bracketing methods (Interpolation method)
a. Bisection method (half interval)
b. False position method

ii) Open-end Method (Extrapolation Method)
a. Newton-Raphson method
b. Secant Method
c. Muller's Method
d. Fixed point Method
e. Bairstow's Method

Bisection Method :-

The bisection method is one of the simplest and most reliable iterative methods for the solutions of non-linear equation. It is also known as half-interval method, relies on the fact that if $f(x)$ is continuous in the interval $a < x < b$, and $f(a)$ and $f(b)$ are of opposite signs ie

$f(a) * f(b) < 0$,

then, there is at least one real root in the interval between a and b.

Let $x_1 = a$ and $x_2 = b$, let us also define another point x_0 to the mid-point betn a and b, ie

$$x_0 = \frac{a+b}{2}$$

Now, there exists the following three conditions :-

1. If $f(x_0) = 0$, we have a root at x_0.
2. If $f(x_0) \cdot f(x_1) < 0$, there exist a root betⁿ x_0 and x_1.
3. If $f(x_0) f(x_2) < 0$, there is a root betⁿ x_0 and x_2.

Pictorically, it ~~cat~~ can be represented as:-

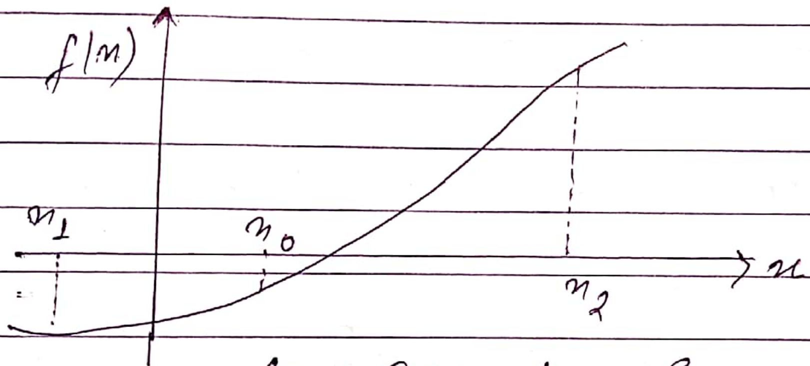


fig:- Illustration of Bisection Method.

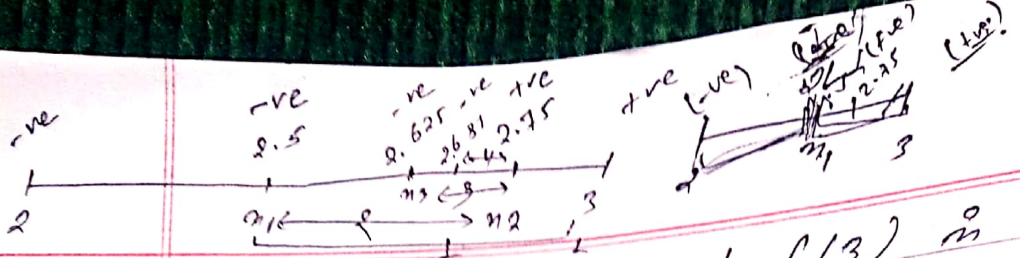
From figure $f(x_0)$ and $f(x_1)$ are of opposite sign, a root lies betⁿ x_0 and x_1, we can further divide this subinterval into two halves to locate a new subinterval containing the root. This process can be repeated until the interval containing the root.

Example: Find the root of the equation $x^3 - 4x - 9 = 0$, using the bisection method within the range $2 < x < 3$ correct upto three decimal places.

⇒ Solⁿ:- Let $f(x) = x^3 - 4x - 9$.

$f(2) = 8 - 8 - 9 = -9$ i.e. -ve

$f(3) = 27 - 12 - 9 = 6$ i.e. +ve



Since $f(2)$ is -ve and $f(3)$ is +ve, so root lies between 2 and 3.

$\therefore$ First approximation to the root is

$x_1 = \frac{2+3}{2} = 2.5$

$f(x_1) = f(2.5) = (2.5)^3 - 4 \times 2.5 - 9$
$= -3.375$ (ie -ve)

$\therefore$ The root lies betn $x_1 = 2.5$ and 3

Second approximation to the root is

$x_2 = \frac{2.5+3}{2} = 2.75$

$f(x_2) = f(2.75) = (2.75)^3 - 4 \times 2.75 - 9$
$= 0.797$ ie +ve

$\therefore$ The root lies betn $x_1 = 2.5$ and $x_2 = 2.75$

Third approximation to the root is

$x_3 = \frac{2.5 + 2.75}{2} = 2.625$

$f(x_3) = (2.625)^3 - 4 \times 2.625 - 9 = -1.412$ ie -ve

$\therefore$ The root lies betn $x_2 = 2.75$ and $x_3 = 2.625$

Fourth approximation to the root is

$x_4 = \frac{2.75 + 2.625}{2} = 2.6875$.

$f(x_4) = (2.6875)^3 - 4 \times 2.6875 - 9 = -0.339$ ie -ve

$\therefore$ The root lies betn $x_4 = 2.6875$ and $x_2 = 2.75$

2 +ve, 2.5 -ve, 2.625 -ve, 2.6875 -ve, 2.7031 -ve, 2.7050 -ve, 2.7060 +ve, 2.7066 +ve, 2.7071 +ve, 2.7109 +ve, 2.7185 +ve, 2.75 +ve, 3 +ve

Fifth approximation to the root is:

$$x_5 = \frac{2.6875 + 2.75}{2} = 2.7185$$

$f(x_5) = (2.7185)^3 - 4 \times 2.7185 - 9 = 0.2209$ ie +ve

∴ The root lies betn x_4 and x_5.

Sixth approximation to the root is:-

$$x_6 = \frac{2.6875 + 2.7185}{2} = 2.7031$$

$f(x_6) = (2.7031)^3 - 4 \times 2.7031 - 9 = -0.0610$ ie -ve

∴ The root lies betn $x_6 = 2.7031$ and $x_5 = 2.7185$

Seventh approximation to the root is:-

$$x_7 = \frac{2.7031 + 2.7185}{2} = 2.7109$$

$f(x_7) = (2.7109)^3 - 4 \times (2.7109) - 9$

$= 0.076$ ie +ve

∴ The root lies betn $x_6 = 2.7031$ and $x_7 = 2.7109$

Eighth approximation to the root is:-

$$x_8 = \frac{2.7031 + 2.7109}{2} = 2.7071$$

$f(x_8) = (2.7071)^3 - 4 \times 2.7071 - 9 = 9.04 \times 10^{-3}$ ie +ve

∴ The root lies betn $x_8 = 2.7071$ and $x_6 = 2.7031$

Ninth approximation to the root is:-

$$x_9 = \frac{2.7031 + 2.7071}{2}$$

$= 2.7050$

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$ $\qquad x = -\frac{a_{n-1}}{a_n}$

$$|x_{max}| = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

$f(x_9) = -0.026$ i.e. -ve

$\therefore$ The root lies betn $x_8 = 2.7071$ and $x_9 = 2.7050$

Tenth approximation to the root :-

$$x_{10} = \frac{2.7050 + 2.7071}{2} = 2.70605$$

$f(x_{10}) = -8.5 \times 10^{-3}$ i.e -ve

$\therefore$ The root lies betn $x_{10} = 2.7060$ & $x_8 = 2.7071$

Eleventh approximation to the root is:-

$$x_{11} = \frac{2.70605 + 2.7071}{2} = 2.70655$$

$f(x_{11}) = 2.69 \times 10^{-4}$ i.e +ve

$\therefore$ The approximate root is 2.706

Algorithm :-

1. Start
2. Decide initial values for x_1 and x_2 and stopping criteria E (error)
3. Calculate $f_1 = f(x_1)$ and $f_2 = f(x_2)$.
4. If $f_1 * f_2 > 0$ and x_1 and x_2 does not contain any root and goto step 8. otherwise continue.
5. Calculate $x_0 = (x_1 + x_2)/2$ and compute $f_0 = f(x_0)$
6. If $f_0 * f_2 < 0$ then set $x_1 = x_0$ otherwise set $x_2 = x_0$ set $f_2 = f_0$

$f_1 * f_0 < 0$ set $x_2 = x_0$ otherwise set $x_1 = x_0$ set $f_1 = f_0$

$|a-b| < 0.0005$ or $(a+b)/2 < 0.0005$

7. If absolute value of $(x_2 - x_1)/x_2$ is less than error E, then

root $= (x_1 + x_2)/2$

print the value of root.

goto step 8.

else

goto step 5

8. Stop

Q. Find the root of equation $x^2 - 4x - 10 = 0$ using bisection method within interval $-2 < x < -1$

Soln:

Let, $f(x) = x^2 - 4x - 10$

$f(-2) = (-2)^2 - 4 \times -2 - 10$

$= 2$ ie +ve

$f(-1) = (-1)^2 - 4 \times -1 - 10$

$= -5$ ie -ve

So, root lies betn -2 and -1.

$\therefore x_1 = \dfrac{-2-1}{2} = -1.5$

$f(x_1) = -1.75$ ie -ve

The root lies betn -1.5 and -2.

$x_2 = \dfrac{-1.5-2}{2} = -1.75$

$f(x_2) = 0.0625$ ie +ve

$\therefore$ The root lies betn -1.5 and -1.75

$\therefore x_3 = \dfrac{-1.5-1.75}{2}$

$f(x_3) = -0.85937$ i.e. $-ve$

$\therefore$ Root lies betⁿ -1.75 and -1.625

$$x_4 = \frac{-1.75 + 1.625}{2} = -1.6875$$

$f(x_4) = -0.402$ i.e. $-ve$

$\therefore$ Root lies betⁿ -1.75 and -1.6875

$$x_5 = \frac{-1.75 - 1.6875}{2} = -1.71875$$

$f(x_5) = -1.017$ i.e. $-ve$

$\therefore$ Root lies betⁿ -1.758 to -1.71875.

$$x_6 = \frac{-1.758 - 1.7187}{2} = -1.734$$

$x_7 = -1.742$

$x_8 = -1.738$

$x_9 = -1.740$

$x_{10} = -1.741$

$x_{11} = -1.742$ //

Secant Method:

Secant method, like bisection method, uses two initial estimates but does not require that they must bracket the root.

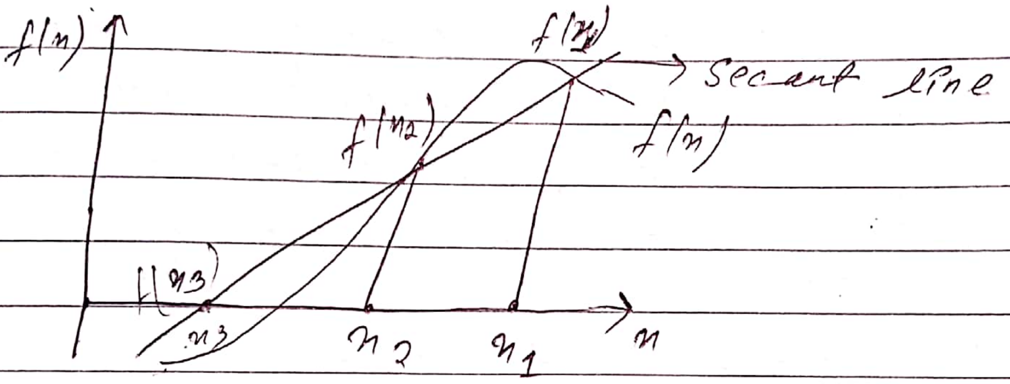


Fig:- Graphical representation of secant method.

Let x_1 and x_2 as starting values they do not bracket the root. slope of the secant line passing through x_1 and x_2 is given by:

$$\frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

i.e. $f(x_1)(x_2 - x_3) = f(x_2)(x_1 - x_3)$

$\Rightarrow x_3[f(x_2) - f(x_1)] = f(x_2)x_1 - f(x_1)x_2$

$$\Rightarrow x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

By adding and subtracting $f(x_2)x_2$ to the numerator and rearranging the terms we get,

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

This formula is known as secant formula.

General form for i^{th} root,

$$x_i = x_{i-1} - \frac{f(x_{i-1})(x_{i-1} - x_{i-2})}{f(x_{i-1}) - f(x_{i-2})}$$

The approximate value of root can be defined by repeating this procedure by replacing x_1 and x_2 by x_2 and x_3 respectively. The next approximate value is given as:

$$x_4 = x_3 - \frac{f(x_3)(x_3 - x_2)}{f(x_3) - f(x_2)}$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

H.W. C prog of Bisection

```
#include <stdio.h>
#include<conio.h>
#define eqn x*x*x-2*x-5

float f(float x)
{
    float ans;
    ans=eqn;
    return(ans);
}

main()
```

Q. Use the Secant method to find the root of the equation $x^2-4x-10=0$ with initial estimates of $x_1=4$ and $x_2=2$, calculate upto 6th iterations.

⇒ Soln:

Let given eqn is $x^2-4x-10=f(x)$

$f(x)=x^2-4x-10$

$x_1=4$, $x_2=2$

Now,

$f(x_1)=4^2-4\times4-10 = -10$

$f(x_2)=2^2-4\times2-10 = -14$

∴ The first approximation to the root is: $x_3=x_2-\frac{f(x_2)(x_2-x_1)}{f(x_2)-f(x_1)}$

$=2-\frac{-14\times-2}{-14+10}$

$=9$

$f(x_3)=9^2-4\times9-10=35$

∴ The second approximation to the root is

$x_4=x_3-\frac{f(x_3)(x_3-x_2)}{f(x_3)-f(x_2)}$

$=9-\frac{35\times(9-2)}{35+14}$

$=4$

$f(x_4)=4^2-4\times4-10=-10$

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∴ The third approximation of root

$$x_5 = \cancel{f(x)} \cdot x_4 - \frac{f(x_4)(x_4 - x_3)}{f(x_4) - f(x_3)}$$

$$= 4 - \frac{-10 \times (4-9)}{-10 - 35}$$

$$= 5.1111$$

$$f(x_5) = (5.1111)^2 - 4 \times 5.1111 - 10 = -4.3210$$

∴ The fourth approximation to the root is

$$x_6 = x_5 - \frac{f(x_5)(x_5 - x_4)}{f(x_5) - f(x_4)}$$

$$= 5.1111 - \frac{-4.3210 \times (5.1111 - 4)}{-4.3210 + 10}$$

$$= 5.9565$$

$$f(x_6) = 1.6538$$

∴ The fifth approximation to the root is

$$x_7 = x_6 - \frac{f(x_6)(x_6 - x_5)}{f(x_6) - f(x_5)}$$

$$= 5.9565 - \frac{1.6538 \times (5.9565 - 5.1111)}{1.6538 + 4.3210}$$

$$= 5.7224$$

$$f(x_7) = -0.1437$$

∴ The sixth approximation to the root is

$$x_8 = x_7 - \frac{f(x_7)(x_7 - x_6)}{f(x_7) - f(x_6)}$$

$$= 5.7224 - \frac{-0.1437 \times (5.7224 - 5.9565)}{-0.1437 - 1.6538}$$

$$= 5.7411$$

$$\therefore f(x_8) = -4.17079 \times 10^{-3}$$

Algorithm for secant method:

1. Start
2. Decide two initial points x_1 and x_2, accuracy level E.
3. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$.
4. Compute $x_3 = x_2 - \frac{f_2(x_2 - x_1)}{f_2 - f_1}$.
5. Test for accuracy of x_3 if

$$\left| \frac{x_3 - x_2}{x_3} \right| > E, \text{ then}$$

set $x_1 = x_2$ and $f_1 = f_2$

set $x_2 = x_3$ and $f_2 = f(x_3)$

goto step 4

otherwise,

set root $= x_3$

print the value of root. goto step 6

6. Stop

Q. Find the root of the equation $xe^x = \cos x$ using the secant method. Correct to four decimal places. Take initial estimates are $x_1 = 0$ and $x_2 = 1$.

⇒ Sol^n:

The given eq^n is $f(x) = xe^x - \cos x$

$f(x_1) = 0.e^0 - \cos 0 = 0 - 1 = -1$

$f(x_2) = 1.e^1 - \cos 1 = 2.71828 - 0.99984$

$= 1.71844$

First approximation to the root is:

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 1 - \frac{1.71844\,(1-0)}{1.71844 + 1}$$

$$= 0.36785$$

$$f(x_3) = 0.36785\, e^{(0.36785)} - \cos(0.36785)$$

$$= -0.46857$$

$\therefore$ Second approximation to the root is

$$x_4 = x_3 - \frac{f(x_3)(x_3 - x_2)}{f(x_3) - f(x_2)}$$

$$= 0.36785 - \frac{-0.46857\,(0.36785 - 1)}{-0.46857 - 1.71844}$$

$$= 0.50328$$

$$f(x_4) = -0.16746$$

$\therefore$ Third approximation to the root is

$$x_5 = x_4 - \frac{f(x_4)(x_4 - x_3)}{f(x_4) - f(x_3)}$$

$$= 0.50328 - \frac{-0.16746\,(0.50328 - 0.36785)}{-0.16746 + 0.46857}$$

$$= 0.57859$$

$$f(x_5) = 0.03197$$

$\therefore$ Fourth approximation to the root is:

$$x_6 = x_5 - \frac{f(x_5)(x_5 - x_4)}{f(x_5) - f(x_4)}$$

$$= 0.57859 - \frac{0.03197\,(0.57859 - 0.50328)}{0.03197 + 0.16746} = 0.56651$$

$f(x_6) = -0.0017$

$\therefore$ fifth approximation to the root is

$$x_7 = 0.56651 - \frac{-0.0017\,(0.56651 - 0.57859)}{-0.0017 - 0.03197}$$

$$= 0.56711$$

$\therefore f(x_7) = -0.000043$

$\therefore$ sixth approximation to the root is

$$x_8 = x_7 - \frac{f(x_7)\,(x_7 - x_6)}{f(x_7) - f(x_6)}$$

$$= 0.56711 - \frac{(-0.000043)(0.56711 - 0.56651)}{-0.000043 + 0.0017}$$

$$= 0.56712$$

$\therefore f(x_8) = -0.000053$

$\therefore$ The approximation root of eqn $xe^x = \cos x$

is 0.56712

Newton-Raphson Method:

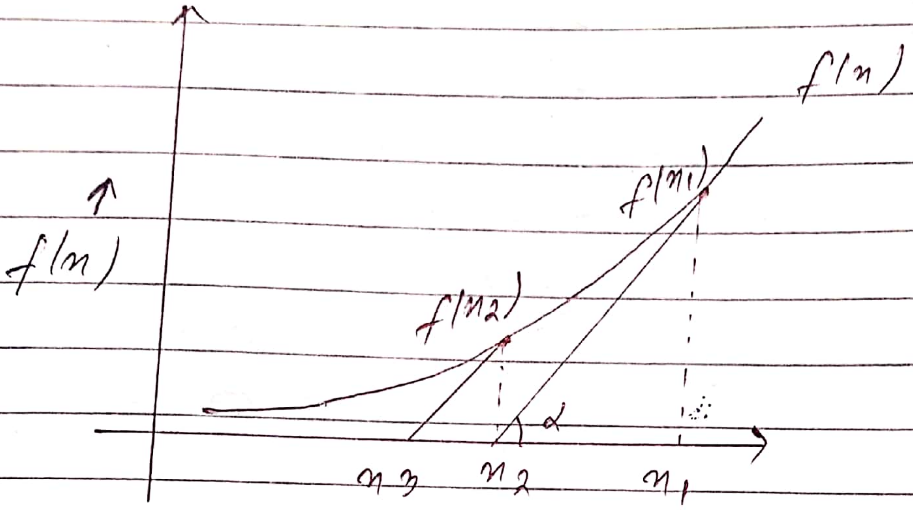

fig:- Newton-Raphson Method

Consider a graph $f(x)$, let us assume that x_1 is an approximate root of $f(x)=0$ Now, draw a tangent at the curve $f(x)$ at $x=x_1$ as show in figure. The point of intersection of this tangent with x-axis gives the second approximation to the root.

Let this point of the intersection be x_2. The slope of tangent is given as,

$$\tan\alpha = \frac{f(x_1)}{x_1-x_2} = f'(x_1)$$

where $f'(x_1)$ is the slope of $f(x)$ at $x=x_1$.

Now, solving for x_2, we get

$$\frac{f(x_1)}{x_1-x_2} = f'(x_1)$$

or, $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The next approximate would be

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

which is known as ~~Netonton~~ Newton iteration formula.

Q. Find the root of the equation $f(x) = x^2 - 3x + 2$ in the vicinity of $x = 0$ using Newton-Raphson method.

⇒ Soln:

The given eqn is: $f(x) = x^2 - 3x + 2$

$f'(x) = 2x - 3$

Initial point $x_0 = 0$

Now, $f(x_0) = 0 - 3 \times 0 + 2 = 2$

$f'(x_0) = -3$

∴ The first approximation to the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{2}{-3}$$

$$\doteq 0.6666$$

$f(x_1) = 0.4445$

$f'(x_1) = -1.6668$

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∴ Second approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6666 - \frac{0.4445}{+1.6668}$$

$$= 0.9332$$

$f(x_2) = \cancel{0.2221}\ 0.0712$

$f'(x_2) = -1.1336$

Third approximation :-

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.9332 - \frac{0.0712}{-1.1336}$$

$= \cancel{1.4061}\ 0.9960$

$f(x_3) = 0.0040$

$f'(x_3) = -1.008$

∴ fourth approximation :-

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.9960 + \frac{0.0040}{1.008}$$

$$= 0.9999$$

$f(x_4) = 0.0001$

$f'(x_4) = -1.0003$

Fifth approximation

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$= 0.9999 + \frac{0.0001}{1.0002}$$

$$= 0.9999$$

$f(x_5) = 0.0001$

$f'(x_5) = -1.0002$

$\therefore$ The approximation root of the eq^n is 0.9999.

Algorithm:-

1. Start
2. Assign an initial value to x say x_0, define error E.
3. Evalute $f(x_0)$ and $f'(x_0)$
4. Find the improved estimate of x_0

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

5. Check the accuracy of the latest estimate compare relative error to a predefined error value E.

If $\left|\frac{x_1 - x_0}{x_1}\right| \le E$ stop,

otherwise continue.

6. Replace x_0 by x_1 and repeat step 4 and step 5.

1Q. Solve $\cos x = 2x$ using Newton-Raphson method, correct upto 5 decimal places with initial guess $x_0 = 0.5$

2Q. Find the root of the equation by Newton-Raphson's method $3x = \cos x + 1$ with initial guess $x_0 = 0.6$ correct upto 4 decimal places.

1) Soln:

$f(x) = \cos x - 2x$

$f'(x) = -\sin x - 2$

Initial point $x_0 = 0.5$

Now,

$f(x_0) = \cos 0.5 - 2\times 0.5 = -0.000038$

$f'(x_0) = -\sin 0.5 - 2 = -2.00872$

$\therefore$ The first approximation to the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{0.000038}{2.00872}$$

$$= 0.49998$$

$f(x_1) = 0.0000019$

$f'(x_1) = -2.00872$

$\therefore$ Second approximation to the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.49998 + \frac{0.0000019}{2.00872}$$

$$= 0.49998$$

$f(x_2) = 0.0000019$

$f(x_1) = -2.00872$

$\therefore$ The approximated root is 0.49998

$f(x_3) = -0.00013$
$f'(x_3) = 3.01163$
$\therefore$ The approximated root is 0.6666

Derivation of Newton-Raphson method using Taylor series expansion.

$\Rightarrow$ Let x_0 be an approximate root of the equation $f(x)=0$. If $x_1 = x_0 + h$ be the exact root then $f(x_1)=0$

Expanding $f(x_0+h)$ by Taylor series:

$$f(x_0) + h\frac{f'(x_0)}{1!} + h^2\frac{f''(x_0)}{2!} + \dots$$

Since, h is small neglecting h^2 and higher power of h and equilize with zero.

we get,

$$f(x_0) + h f'(x_0) = 0$$

$$h = \frac{-f(x_0)}{f'(x_0)}$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

II Fixed Point Method :-

Any function of the form $f(x)=0$ can be manipulated, such that x is on the left hand side of the equation as,

$$x = \phi(x) \quad \text{——} \quad (2)$$

Equation (1) & (2) are equivalent and therefore, a root of eq^n (1) is also root of eq^n (2). The root of the eq^n (2) is given the point of intersection of the curve $y = x$ and $y = \phi(x)$. This point of interection is known as fixed point of $\phi(x)$.

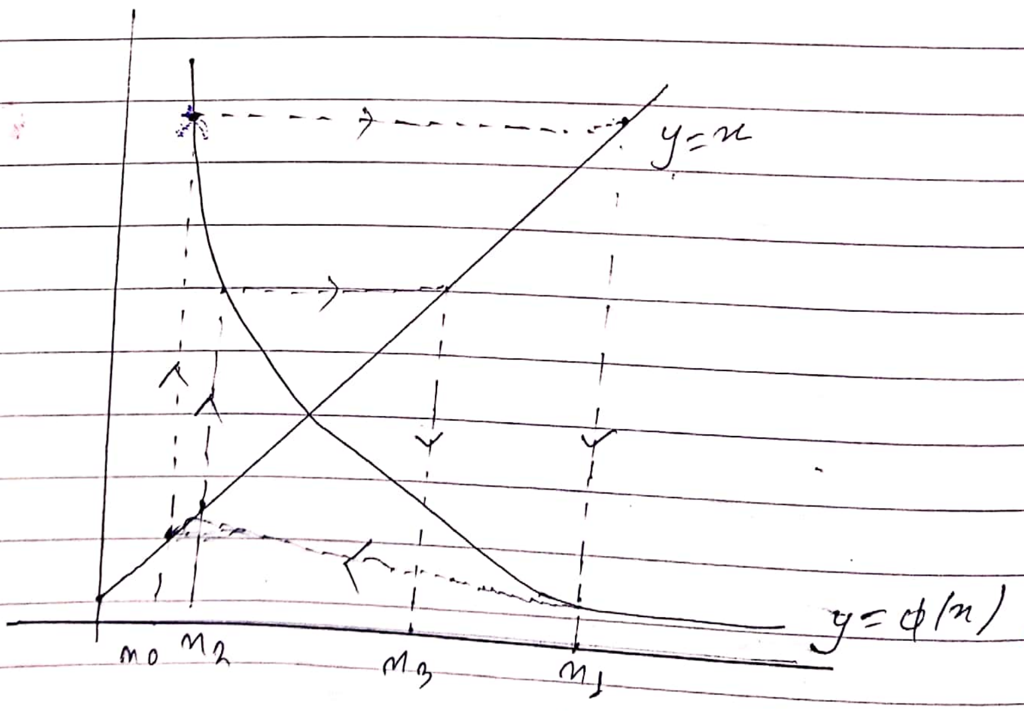

fig:- Geometrical interpretation of fixed point method

The equation $x = \phi(x)$ is known as the fixed point equation. It provides a convenient form for predecting the value of x as function of x. If x_0 is the initial guess to a root, then the next approximation is given by,

$$x_1 = \phi(x_0)$$

Further approximation is given by,

$$x_2 = \phi(x_1)$$

This iteration process can be expressed in general form as

$x_{i+1} = \phi(x_i)$, which is called fixed point iteration formula.

8. Find the root of the equation $\cos x = 3x - 1$, correct upto three decimal places using the fixed point method. Take initial guess as $x_0 = 0$

⇒ soln:

The given eqn : $f(x) = \cos x - 3x + 1$

$$3x = \cos x + 1$$

$$x = \frac{\cos x + 1}{3} \quad \text{ie } \phi(x)$$

Initial guess : $x_0 = 0$

∴ First approximation to the root:

$$x_1 = \phi(x_0) = \frac{\cos 0 + 1}{3}$$

$$= 0.666$$

Newton-Raphson method for system of non-linear equation.

The first order Taylor series is given as:

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) \quad \text{--- (1)}$$

is used to derive the Newton's iteration formula.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{--- (2)}$$

For simplicity, let us again consider two equation of non-linear system.

$f(x,y) = 0$ and $g(x,y) = 0$

First order Taylor series of these eqn can be written as:

$$f(x_{i+1}, y_{i+1}) = f(x_i, y_i) + (x_{i+1} - x_i)\left|\frac{\partial f_i}{\partial x}\right| + (y_{i+1} - y_i)\left|\frac{\partial f_i}{\partial y}\right| \quad \text{--- (3)}$$

$$g(x_{i+1}, y_{i+1}) = g(x_i, y_i) + (x_{i+1} - x_i)\left|\frac{\partial g_i}{\partial x}\right| + (y_{i+1} - y_i)\left|\frac{\partial g_i}{\partial y}\right| \quad \text{--- (4)}$$

If the root estimates are x_{i+1} and y_{i+1} then,

$f(x_{i+1}, y_{i+1}) = g(x_{i+1}, y_{i+1}) = 0$

Now,

Substituting this in eqn (3) & (4) respectively, we get the following two linear eqn:

$$\Delta x f_1 + \Delta y f_2 + f = 0 \quad \text{—— (5)}$$
$$\Delta x g_1 + \Delta y g_2 + g = 0 \quad \text{—— (6)}$$

where, we denote,

$$\Delta x = x_{i+1} - x_i$$
$$\Delta y = y_{i+1} - y_i$$

$$f_1 = \left(\frac{\partial f_i}{\partial x}\right), \quad f_2 = \left|\frac{\partial f_i}{\partial y}\right|, \quad g_1 = \left|\frac{\partial g_i}{\partial x}\right|, \quad g_2 = \left|\frac{\partial g_i}{\partial y}\right|$$

$$f = f(x_i, y_i), \quad g = g(x_i, y_i)$$

Solving for x and y we get,

$$\Delta x = -\left(\frac{f \cdot g_2 - g \cdot f_2}{f_1 g_2 - f_2 g_1}\right) = -\frac{D_x}{D} \quad \text{—— (7)}$$

$$\Delta y = -\left(\frac{g \cdot f_1 - f \cdot g_1}{f_1 g_2 - f_2 g_1}\right) = -\frac{D_y}{D} \quad \text{—— (8)}$$

where,

$$D = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} = f_1 g_2 - g_1 f_2,$$ is called Jacobian matrix

from eq^n (7) & (8) we can establish the following relations.

$$x_{i+1} = x_i - \frac{D_x}{D} \quad \text{—— (9)}$$

$$y_{i+1} = y_i - \frac{D_y}{D} \quad \text{—— (10)}$$

eq^n (9) and (10) are called the two-eq^n Newton's formula.

Example: Determine the root of the eq^n

$x^2 + xy = 6$

$x^2 - y^2 = 3$ using Newton-Raphson method with Initial guess $x_0 = 1$ and $y_0 = 1$

$\Rightarrow$ Sol^n:

Given eq^n are: $f(x,y) = x^2 + xy - 6$

$g(x,y) = x^2 - y^2 - 3$

Initial case: $x_0 = 1$, $y_0 = 1$

~~$f(x_0, y_0)$~~

$f_1 = \frac{\partial f}{\partial x} = 2x + y$, $g_1 = \frac{\partial g}{\partial x} = 2x$

$f_2 = \frac{\partial f}{\partial y} = x$ $g_2 = \frac{\partial g}{\partial y} = -2y$

~~first:~~

We have $x_0 = 1$, and $y_0 = 1$

~~$\therefore D = f_1 \; f_2$~~ Iteration 1:

~~$g_1 \; g_2$~~ $f_1 = 2 + 1 = 3$ $f_2 = 1$

$g_1 = 2$ $g_2 = -2$

$$\therefore D = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$$

$= -6 - 2$

$= -8$

The value of functions at x_0 and y_0:

$f = 1 + 1 - 6 = -4$

$g = 1 - 1 - 3 = -3$

Algorithm

1. Start
2. Define the functions f and g.
3. Define the Jacobian elements f_1, f_2, g_1 and g_2.
4. Decide starting points x_0 and y_0 and error tolerance E.
5. Evaluate f, g, f_1, f_2, g_1, g_2 at (x_0, y_0).
   Compute D_x, D_y and D.
   $x_1 = x_0 - D_x / D$
   $y_1 = y_0 - D_y / D$
6. Test for accuracy
   If $|x_1 - x_0| < E$ and $|y_1 - y_0| < E$, then
   Soln obtained;
   goto step 8
7. Otherwise, set $x_0 = x_1$
   $y_0 = y_1$
   goto step 5
8. print the root
9. Stop.

1. Start
2. $f(x_0) = 0$, x_0
3. $f(x) = 0$ into $x = g(x)$
4. $x_1 = g(x_0)$
5. Error,
   $|x_1 - x_0| > E$
   $x_0 = x_1$ goto 4
   otherwise
   print root
6. Stop.

1 Start
2 $f(x) = 0$, x_0
3 convert $f(x) = 0$ into form $x = g(x)$
4 Initial guess x_0
   do ~~5~~. 4. $x_1 = g(x_0)$
   $x_{i+1} = g(x_i)$

~~5~~. Error $|x_{i+1} - x_i| > E$ $|x_1 - x_0| > E$
   $x_{i+1} = x_i$ $x_0 = x_1$ goto step 4
   otherwise root $= x_1$
6. Stop.

Synthatic Division :-

Synthatic division is a method of performing polynomial division, with less writing and fewer calculations. It is mostly used for division by binomials of the form $(x-a)$. The most useful aspects of synthatic division is that, it allows to calculate without writing variables and uses fewer calculations.

Synthatic division is performed as follows:

Let $p(x)$ be a polynomial of degree n. If we divide $p(x)$ by $(x-a)$ we get another polynomial $q(x)$ of degree $n-1$.

Assume,

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

and $q(x) = b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_0$

$\Rightarrow p(x) = (x-a).\ q(x)$ [$\because q(x)$ = deflated polynomial]

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = (x-a)(b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_0)$

Comparing the coefficient on both sides we get,

$b_{n-1} = a_n$ // [$a_n + a * b_n^0$]

$b_{n-2} = a_{n-1} + a * b_{n-1}$

$\vdots$

$b_0 = a_1 + a * b_1$

Thus,

$\boxed{b_{i-1} = a_i + a * b_i^0, \text{ where } i = n, n-1, \dots 1 \text{ and } b_n = 0}$

This process is also known as deflation process.

Example:

The polynomial equation $p(x) = x^3 - 7x^2 + 15x - 9$ has root at $x = 3$. Find the quotient polynomial $q(x)$ such that $p(x) = (x-3)\,q(x)$

⇒ Soln:

The given eqn is:

$$P(x) = x^3 - 7x^2 + 15x - 9 = 0$$

we have,

$$a_3 = 1, \quad a_2 = -7, \quad a_1 = 15, \quad a_0 = -9$$

Now, $b_n = 0$ i.e. $b_3 = 0$ [general method $b_2 = a_3 + a * b_3$]

$$b_{n-1} = b_2 = a_3 = 1$$

$$b_{n-2} = b_1 = a_2 + a * b_2$$
$$= -7 + 3 \times 1$$
$$= -4$$

$$b_{\cancel{n-3}} = b_0 = a_1 + a * b_1$$
$$= 15 + 3 * (-4)$$
$$= 3$$

$$q(x) \Rightarrow b_2 x^2 + b_1 x + b_0 = 0$$
$$\Rightarrow x^2 - 4x + 3 = 0$$

∴ The quotient polynomial $q(x)$ is $x^2 - 4x + 3 = 0$

Algorithm:

1. Start
2. Read coefficient of dividend polynomial say $a_n, a_{n-1} \ldots a_0$
3. Read magnitude of division polynomial of the form $x-a$.
4. Set $b_n = 0$.
5. Repeat till $n > 0$
   $b_{n-1} = a_n + b_n * a$
6. Remainder $(R) = a_0 + b_0 * a$
7. Display quotient polynomial & remainder
8. Stop

1 Q. Divide $2x^5 - 3x^4 + 4x^3 - 5x^2 + 6x - 9$ by $x^2 - x + 2$ Synthetically.

2. Using Newton-Raphson method, find a root of the equation $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 3, correct upto 4 decimal places.

1 ⇒ Solⁿ:

$P(x) \Rightarrow 2x^5 - 3x^4 + 4x^3 - 5x^2 + 6x - 9 = 0$ we get,

$a_5 = 2$, $a_4 = -3$, $a_3 = 4$, $a_2 = -5$, $a_1 = 6$, $a_0 = -9$

Now, $b_n = 0$ i.e. $b_5 = 0$

Then: $x^2 - x - 2 = 0$

$\Rightarrow x^2 - 2x + x - 2 = 0$

$\Rightarrow x(x-2) + 1(x-2) = 0$

$\Rightarrow (x+1)(x-2) = 0$

Either, or

$x + 1 = 0$ $\quad$ $x - 2 = 0$

$x = -1$ $\quad$ $x = 2$

$\therefore x = -1, 2$

The root is 2, $b_5 = 0$, $a = 2$

$b_{n-1} = b_{5-1} = b_4 = a_5 + a * b_5$

$= 2 + 2 * 0$

$= 2$

$b_{n-1} = b_3 = a_4 + a * b_4$

$= -3 + 2 \times 2$

$= -3 + 4$

$= 1$

$b_2 = a_3 + a * b_3$
$= 4 + 2 \times 1$
$= 6$
$b_1 = a_2 + a * b_2$
$= -5 + 2 * 6$
$= -5 + 12$
$= 7$
$b_0 = a_1 + a * b_1$
$= 6 + 2 * 7$
$= 6 + 14$
$= 20$

$q(x) \Rightarrow b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0 = 0$
$\Rightarrow 2x^4 + x^3 + 6x^2 + 7x + 20 = 0$

$\therefore$ The quotient polynomial $q(x)$ is
$2x^4 + x^3 + 6x^2 + 7x + 20 = 0$.

2 ⇒ Soln:

$f(x) = x^4 + x^3 - 7x^2 - x + 5 = 0$
$f'(x) = 4x^3 + 3x^2 - 14x - 1 = 0$

Initial guess: $x_0 = 2$

Then, $f(x_0) = 2^4 + 2^3 - 7 \times 2^2 - 2 + 5 = -1$
$f'(x_0) = 15$

$\therefore$ 1st approximation to the root is:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2 - \frac{1}{15}$$
$$= 2.0666$$

$f(x_1) = (2.0666)^4 + (2.0666)^3 - 7\times(2.0666)^2 - 2.0666 + 5$
$= 0.10478$

$f'(x_1) = 18.18752$

$\therefore$ Second approximation to the root :-

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.0666 - \frac{0.10478}{18.18752}$$

$$= 2.06089$$

$f(x_2) = 0.00067$

$f'(x_2) = 17.90194$

$\therefore$ Third approximation to the root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.0608$$

$\therefore$ The approximated root is 2.0608

Remainder Theorem:-

Remainder theorem states that "if the ~~progra~~ polynomial $p(x)$ is divided by $x-c$, then remainder is $f(c)$". This means that we can apply synthetic division and the last number on the ~~left~~ right, which is the remainder, will tell us what the functional value of c is. Thus, the remainder theorem is useful for evaluating polynomials at a given value of x.

Example:

Let $f(x) = 3x^3 + x^2 + x - 5$, use the remainder theorem to find $f(-2)$.

$\Rightarrow$ ~~Soln:~~

We have, $c = -2$

$f(x) = 3x^3 + x^2 + x - 5$

Now,

$$\begin{array}{r|rrrr} -2 & 3 & 1 & 1 & -5 \\ & \downarrow & -6 & 10 & -22 \\ \hline & 3 & -5 & 11 & \boxed{-27} \end{array} \rightarrow \text{Remainder}$$

$q(x) = 3x^2 - 5x + 11$

$\therefore f(-2) = -27$

[If remainder is zero then divided is root]

I Horner's Method for polynomial Evaluation:-

Horner's method is a way to evaluate a polynomial. The method splits the polynomial into its individual terms, and solve them incrementally. This method seperates the lowest degree term in the polynomial from the rest, by grouping any terms with a higher degree into one unit, with degree 1, thus making any polynomial $k*x+c$, where k is the group of all terms with degree higher than one and c is the term with degree 0. When the terms are grouped into k, their degree decrease by one as follows,

$$x^4+8x^3+6x^2+4x+3$$
$$= k*x+3, \text{ where } k=(x^3+8x^2+6x+4)$$

This method can be applied recursively inside the k to simplify the polynomial that results after grouping, by repeating these steps, until there are no terms with degree higher than 1 and will yield an equation where there is no need to calculate any term of x^n. After applying this method to above polynomial we end up with,

$x^4 + 8x^3 + 6x^2 + 4x + 3$

$= (x^3 + 8x^2 + 6x + 4)x + 3$

$= ((x^2 + 8x + 6)x + 4)x + 3$

$= (((x + 8)x + 6)x + 4)x + 3$

$= ((((1)x + 8)x + 6)x + 4)x + 3$

Generalizing this we can derive recursive relation for polynomial evaluation as

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$

where, $a_0, a_1, \ldots a_n$ are real numbers, we wish to evaluate a polynomial at a specific value say at x_0.

To accomplish this we need to define a given polynomial in terms of nested multiplication as below:

$p(x) = (a_0 + x(a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}))$

$p(x) = (a_0 + x(a_1 + x(a_2 + a_3 x + \cdots + a_n x^{n-2})))$

$p(x) = (a_0 + x(a_1 + x(a_2 + x(a_3 + x(a_4 + \cdots + (a_{n-1} + x(a_n)))))))$

Now, we can define a new sequence of constant as

$b_n = a_n$

$b_{n-1} = a_{n-1} + b_n x_0$

Then b_0 is the value of $p(x_0)$.

Example:

Evaluate the polynomial

$p(x) = 3x^3 - 4x^2 + 5x - 6$ at $x_0 = 2$

$\Rightarrow$ Solⁿ:

Given polynomial eqⁿ

$p(x) = 3x^3 - 4x^2 + 5x - 6$

Now,

$a_3 = 3$, $a_2 = -4$, $a_1 = 5$, $a_0 = -6$, $x_0 = 2$

Then, $b_n = a_n$

$b_3 = a_3 = 3$

$b_2 = a_2 + b_3 x_0$ $[b_{n-1} = a_{n-1} + b_n x_0]$

$= -4 + 3 \times 2$

$= 2$

$b_1 = a_1 + b_2 x_0$

$= 5 + 2 \times 2 = 9$

$b_0 = a_0 + b_1 x_0$

$= -6 + 9 \times 2 = 12$

Hence, the functional value of given polynomial $p(x)$ at $x_0 = 2$ is 12.

Convergence say

Algorithm:-

1. Start.
2. Read degree of polynomial say n.
3. Read the value at which polynomial to be evaluated, x.
4. Initially set $b_n = a_n$.
5. while $(n \geq 0)$
   $b_{n-1} = a_{n-1} + b_n * x$
6. Print the value of b_0, which is the value of polynomial at x.
7. Stop.