Cost Minimization subject to an output constraint

Let the production function be $z = f(x, y)$
where z = output & x, y — inputs.

→ for any given quantity (output), we need to find the minimum cost needed to produce that

Aim: Minimize cost subject to output constraint

Let prices of x, y be P_x & P_y respectively.

Then Total cost : $TC = P_x x + P_y y$

Objective: $\min\limits_{x,y} \; P_x x + P_y y$

subject to $z = f(x, y)$

Lagrangian:

$$L = P_x x + P_y y + \lambda \left(z - f(x, y)\right)$$

First order conditions:

$$\frac{\partial L}{\partial x} = P_x - \lambda f_x = 0 \quad \text{—— (1)} ; \quad f_x = \frac{\partial f}{\partial x} = MP_x$$

$$\frac{\partial L}{\partial y} = P_y - \lambda f_y = 0 \quad \text{—— (2)} ; \quad f_y = \frac{\partial f}{\partial y} = MP_y$$

$$\frac{\partial L}{\partial \lambda} = z - f(x, y) = 0 \quad \text{—— (3)}$$

From (1) & (2) : $\lambda = \dfrac{P_x}{f_x} = \dfrac{P_y}{f_y}$

$$\Rightarrow \frac{P_x}{P_y} = \frac{f_x}{f_y} = \frac{MP_x}{MP_y}$$

Example:

Consider the production function : $z = x^{0.5} y^{0.5}$

given $P_x = P_y = 1$ & $z = 5$.

Find optimum x & y & total cost.

Objective Minimize total cost subject to output constraint

Production function : $x^{0.5} y^{0.5} = 5$

~~to~~ Total cost : $TC = x + y$ $(P_x = P_y = 1)$

Aim Min $x + y$ subject to $5 = x^{0.5} y^{0.5}$

Lagrangian :

$$L = x + y + \lambda \left(5 - x^{0.5} y^{0.5}\right)$$

F.O.C's :

$$\frac{\partial L}{\partial x} = 1 - \lambda\, 0.5 \frac{y^{0.5}}{x^{0.5}} = 0 \quad \text{———} (1)$$

$$\frac{\partial L}{\partial y} = 1 - \lambda\, 0.5 \frac{x^{0.5}}{y^{0.5}} = 0 \quad \text{———} (2)$$

$$\frac{\partial L}{\partial \lambda} = 5 - x^{0.5} y^{0.5} = 0 \quad \text{———} (3)$$

From (1) & (2) $= \lambda = \frac{1}{0.5} \frac{x^{0.5}}{y^{0.5}} = \frac{1}{0.5} \frac{y^{0.5}}{x^{0.5}}$

$$\Rightarrow x = y$$

Putting this in (3)

$$5 - x^{0.5} x^{0.5} = 0 \quad (\text{as } x = y)$$

$\Rightarrow$ $x = 5$

$\therefore$ $y = 5$

Total cost $= x + y = (5 \times 1) + (5 \times 1) = 10$