BITS Pilani Hyderabad Campus
CS F214 Logic in Computer Science,
I Semester 2021-2022
Lecture Notes
Lecture 18

14 Completeness of Propositional Logic

Whenever $\phi_1, .., \phi_n \vDash \psi(1)$ holds then there exists a natural deduction proof for the sequent $\phi_1, .., \phi_n \vdash \psi$.

Proof Sketch

1. Assuming (1), we show that $\vDash \phi_1 \to (\phi_2 \to (..(\phi_n \to \psi))$ holds.
2. We show that $\vdash \phi_1 \to (\phi_2 \to (..(\phi_n \to \psi))$ is valid.
3. $\phi_1, .., \phi_n \vdash \psi$ is valid.

Please refer the textbook for the proof.

14.1 Corollary[Soundness and Completeness of Propositional Logic]

Let $\phi_1, ..., \phi_n$ and ψ be formulae of propositional logic. Then, $\phi_1, ..\phi_n \vdash \psi$ is valid.

14.2 Semantic Equivalence

Let ϕ and ψ be formulas in propositional logic. We say that, ϕ an ψ are semantically equivalent iff $\psi \vDash \psi$ holds and $\psi \vDash \phi$ holds as well.
We write $\phi \equiv \psi$
We call ϕ valid iff $\vDash$ holds. Semantic equivalence is identical to provable equivalence.
e.g. $p \to \equiv \neg q \to \neg p$
$p \to q \equiv \neg p \vee q$

We want to transform formulaes into forms in which validity checks are easy. $\phi \to \psi \equiv \neg\phi \vee \psi$.

Definition: *A literal L is either an atom or negation of atom.*

A formula C in Conjunctive Normal Form (CNF) if it is a conjunction of the

clauses where each clause is a disjunction of literals.
e.g.

$$(1)\ (p \vee q) \wedge (\neg p \vee r)(2)(\neg(q \vee \neg p) \vee r) \wedge (p \vee q)$$

(2) is not in CNF form as it has negation of clause.

Definition of CNF in Backus Norm Form (BNF)

Literal	$L ::= p\|\neg p$
Clause	$D ::\neq L\|L \vee D$
CNF Formula	$C ::= D\|D \wedge C$

Observations:

1. A CNF is a conjunction of clauses $C_1, C_2, ..., C_n$
   i.e. $C \equiv C_1 \wedge C_2 \wedge ... \wedge C_n$.
   For C to be true it must be the case that each one of $C_1, C_2, ..., C_n$ are true.
   Suppose C_i is not a valid formula then C is not valid. Now there may be a single clause featuring all n atoms.