BITS Pilani Hyderabad Campus CS F214 Logic in Computer Science, I Semester 2021-2022 Lecture Notes Lecture 6-10

Prove: $\neg q \rightarrow \neg p \vdash p \rightarrow q$ Solution:

$$\begin{array}{ccc}
1. \neg q \rightarrow \neg p & \text{premise} \\
2.p & \text{assumption} \\
3. \neg \neg p & \neg \neg i \ 2 \\
4. \neg \neg q & \text{MT } 3,1 \\
5.q & \neg \neg e \ 4
\end{array}$$

$$\begin{array}{ccc}
6.p \rightarrow q & \rightarrow \text{i } 2-5
\end{array}$$

Consider,

1.p assumption
$$2.p \to p \to i 1-1$$

Above is a poof of the statement $\vdash (p \rightarrow p)$

Def: A logical formula ϕ with a valid sequent $\vdash \phi$ is called a **Theorem**. Remarks: Any sequent $\phi \vdash \psi$ is equivalent to $\vdash \phi \rightarrow \psi$.

Proof of $\vdash \phi \rightarrow \psi$:

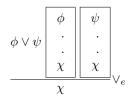
Remark(b): $\phi_1, \phi_2, ..., \phi_n \vdash \psi$ is equivalent to $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (...(.. \rightarrow (\phi_n \rightarrow \psi)))$

5 Rules for Disjunction

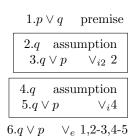
5.1 OR-Introduction

$$\frac{\phi}{\phi \vee \psi} \vee_{i1}$$
$$\frac{\phi}{\psi \vee \phi} \vee_{i2}$$

5.2 OR-Elimination



Prove: $p \lor q \vdash q \lor p$



Prove: $q \to r \vdash p \lor q \to p \lor r$

$$\begin{array}{c|c} 1.q \rightarrow r & \text{premise} \\ \hline \\ 2.p \lor q & \text{assumption} \\ \hline \\ 3.p & \text{assumption} \\ 4.p \lor r & \lor_i \ 3 \\ \hline \\ 5.q & \text{assumption} \\ 6.r & \rightarrow e \ 5.1 \\ 7.p \lor r & \lor_{i2} \ 6 \\ \hline \\ 8.p \lor r & \lor_e \ 2,3\text{-}4,5\text{-}7 \\ \hline \\ .(p \lor q) \rightarrow (p \lor r) & \rightarrow \text{i} \ 2\text{-}8 \end{array}$$

5.3 "Copy" Rule

$$\begin{array}{c|c} \bot p \to (q \to p) \\ \hline \\ 1.p & \text{assumption} \\ \hline \\ 2.q & \text{assumption} \\ 3.p & \textbf{copy 1} \\ \hline \\ 4.q \to p & \to i \ 2\text{-}3 \\ \hline \\ 5.p \to (q \to p) & \to i \ 1\text{-}4 \\ \hline \end{array}$$

6 Rules for Negation

6.1 Contradiction

- Contradictions are expressions of the form $\phi \land \neg \phi$ or $\neg \phi \land \phi$ where ϕ is any proposition.
- ullet represents a contradiction.
- Any proposition can be derived from contradiction.

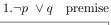
6.2 Bottom Elimination

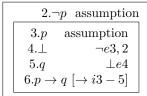
$$\frac{\perp}{\psi} \perp e$$

6.3 Not Elimination

$$\frac{\phi \neg \phi}{\bot} \neg e$$

Example: Prove $\neg p \lor q \vdash p \to q$ is valid.





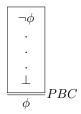
$$11.p \to q \lor 1, 2-6, 7-10$$

6.4 Negation Introduction and Proof by Contradiction

6.4.1 Negation Introduction



6.4.2 Proof By Contradiction (Reductio Ad Absurdum)



Example: Prove the following sequent is valid.

$$p \to q, p \to \neg q \bot \neg p$$

Solution:

$$1.p \rightarrow q \qquad \text{premise}$$

$$2.p \rightarrow \neg q \qquad \text{premise}$$

$$3.p \quad \text{assumption(3)}$$

$$4.q \qquad \rightarrow e \quad 3.1$$

$$5.\neg q \qquad \rightarrow e \quad 3.2$$

$$6.\bot \qquad \neg e \quad 4.5$$

$$7.\neg p \qquad \neg i \quad 3-6$$

Example: Prove $p \to (q \to r), p, \neg r \vdash \neg q$ is valid, without using MT. Solution:

$$\begin{array}{ccc} 1.p \rightarrow (q \rightarrow r) & \text{premise} \\ 2.p & \text{premise} \\ 3. \neg r & \text{premise} \\ \hline \\ 4.q & \text{assumption} \\ 5.q \rightarrow r & \rightarrow e \ 2.1 \\ 6.r & \rightarrow e \ 4.5 \\ 7. \bot & \neg e \ 3.6 \\ \hline \\ 8. \neg q & \neg_i \ 4-7 \\ \end{array}$$

7 Derived Rules

MT can be derived from $\rightarrow e, \neg e$ and \neg_i .

$$\begin{array}{ccc}
1.\phi \to \psi & \text{premise} \\
2.\neg \psi & \text{premise}
\end{array}$$

$$\begin{array}{ccc}
3.\neg \phi & \text{assumption}) \\
4.\neg \psi & \to e & 3,1 \\
5.\bot & \neg e & 2,4
\end{array}$$

$$\begin{array}{cccc}
6.\neg \phi & \neg_i & 3-5
\end{array}$$

8 Law of the Excluded Middle (LEM)

- It is also called as Tertium non datur. (There is no third possibility)
- Says that $\phi \lor \neg \phi$ is always true.

$$\vdash \phi \vee \neg \phi$$

$$\begin{array}{|c|c|c|c|}\hline 1.\neg(\phi\vee\neg\phi) & \text{assumption}\\\hline 2.\phi & \text{assumption}\\ 3.\phi\vee\neg\phi & \vee_i \ 2\\ 4.\bot & \neg e \ 3,1\\\hline 5.\neg\phi & \neg_i \ 2-4\\ 6.\phi\vee\neg\phi & \vee_{i2} \ 5\\ 7.\bot & \neg e \ 6\\\hline 8.\phi\vee\neg\phi & \text{PBC } 1-7\\\hline \end{array}$$

Example: Using LEM, show that $p \to q \vdash \neg p \lor q$ as valid.

$$\begin{array}{ccc} 1.p \rightarrow q & \text{premise} \\ 2.p \vee \neg p & \text{LEM} \\ \hline \\ 3.p & \text{assumption} \\ 4.q & \rightarrow e \ 3.1 \\ 5. \neg p \vee q & \vee_{i2} \ 4 \\ \hline \\ 6. \neg p & \text{assumption} \\ 7. \neg p \vee q & \vee_{i} \ 6 \\ \hline \\ 8. \neg p \vee q & \vee_{e} \ 2.3 \text{-} 5.6 \text{-} 7 \\ \hline \end{array}$$

Provable Equivalence 9

- ϕ and ψ are provable equivalent if and only if the sequents $\phi \vdash \psi$ and $\psi \vdash \phi$ are valid.
- It is denoted by $\phi + \psi$.
- Ultimately we could define the $\phi \dashv \vdash \psi$ as $\vdash (\phi \to \psi) \land (\psi \to \phi)$.

$$\neg (p \to q) \vdash p \land \neg q$$

$$1. \neg (p \to q) \quad \text{premise}$$

$$2. \neg (p \land \neg q) \quad \text{assumption}$$

$$3.p \quad \text{assumption}(3)$$

3.p assumption(3)

4.¬q assumption
5.p $\land \neg q \quad \land_i \ 3,4$ 6. $\bot \quad \neg e \ 2,5$ 7.q PBC 4-6

8.p $\rightarrow q \quad \rightarrow i \ 3$ -7
9. $\bot \quad \neg e \ 8,1$

 $10.p \land \neg q$ PBC 2-9