QUANTIFIER EQUIVALENCES

✓ $\forall x\, \phi \wedge \forall x\, \psi \dashv\vdash \forall x\, (\phi \wedge \psi)$

$\cancel{\exists x\, \phi \wedge \exists x\, \psi \dashv\vdash \exists x\, [\phi \wedge \psi]}$

$\neg\forall x\, P(x) \vdash \exists x\, \neg P(x)$

1	$\neg\forall x\, P(x)$	premise
2	$\neg\exists x\, \neg P(x)$	assumption
3	x_0	
4	$\neg P(x_0)$	assumption
5	$\exists x \neg P(x)$	$\exists x$ i 4
6	$\bot$	$\neg$e 5, 2
7	$P(x_0)$	PBC 4–6
8	$\forall x\, P(x)$	$\forall x$ i 3–7
9	$\bot$	$\neg$e 8, 1
10	$\exists x\, \neg P(x)$	PBC 2–9

$$\neg\forall x\,\phi \vdash \exists x\,\neg\phi$$

1		$\neg\forall x\,\phi$	premise
2		$\neg\exists x\,\neg\phi$	assumption
3	x_0		
4		$\neg\phi[x_0/x]$	assumption
5		$\exists x\neg\phi$	$\exists x$ i 4
6		$\bot$	$\neg$e 5, 2
7		$\phi[x_0/x]$	PBC 4–6
8		$\forall x\,\phi$	$\forall x$ i 3–7
9		$\bot$	$\neg$e 8, 1
10		$\exists x\,\neg\phi$	PBC 2–9

RAA assumption

RAA – Reductio ad absurdum

$$\exists x\, \neg\phi \vdash \neg\forall x\, \phi$$

1	$\exists x\, \neg\phi$	~~assumption~~ premise
2	$\forall x\, \phi$	assumption
3	x_0	
4	$\neg\phi[x_0/x]$	assumption
5	$\phi[x_0/x]$	$\forall x$ e 2
6	$\bot$	$\neg$e 5, 4
7	$\bot$	$\exists x$ e 1, 3–6
8	$\neg\forall x\, \phi$	$\neg$i 2–7

$\therefore$ we have $\neg\forall x\, \phi \dashv\vdash \exists x\, \neg\phi$

$$\forall x\, \phi \wedge \psi \vdash \forall x\, (\phi \wedge \psi)$$

1	$(\forall x\, \phi) \wedge \psi$	premise
2	$\forall x\, \phi$	$\wedge e_1$ 1
3	ψ	$\wedge e_2$ 1
4	x_0	
5	$\phi[x_0/x]$	$\forall x\, e$ 2
6	$\phi[x_0/x] \wedge \psi$	$\wedge i$ 5, 3
7	$(\phi \wedge \psi)[x_0/x]$	identical to 6, since x not free in ψ
8	$\forall x\, (\phi \wedge \psi)$	$\forall x\, i$ 4–7

1	$\forall x\,(\phi \wedge \psi)$	premise
2	x_0	
3	$(\phi \wedge \psi)[x_0/x]$	$\forall x$ e 1
4	$\phi[x_0/x] \wedge \psi$	identical to 3, since x not free in ψ
5	ψ	$\wedge$e$_2$ 3
6	$\phi[x_0/x]$	$\wedge$e$_1$ 3
7	$\forall x\,\phi$	$\forall x$ i 2–6
8	$(\forall x\,\phi) \wedge \psi$	$\wedge$i 7, 5

Not rigorously correct.

RIGOROUS WAY

11. $\forall x \phi \wedge \psi$ $\wedge i$ 7, 10

1	$\forall x\,(\phi \wedge \psi)$	premise
2	x_0	
3	$(\phi \wedge \psi)[x_0/x]$	$\forall x$ e 1
4	$\phi[x_0/x] \wedge \psi$	identical to 3, since x not free in ψ
5	ψ	$\wedge e_2$ 3
6	$\phi[x_0/x]$	$\wedge e_1$ 3
7	$\forall x\,\phi$	$\forall x$ i 2–6
~~8~~	$\cancel{(\forall x\,\phi) \wedge \psi}$	~~$\wedge$i 7, 5~~
8	$(\phi \wedge \psi)(x_1/x)$	$\forall x$ e 1
9	$\phi(x_1/x) \wedge \psi$	
10	ψ	$\wedge e_2$ 9

$$(\exists x\,\phi) \lor (\exists x\,\psi) \vdash \exists x\,(\phi \lor \psi)$$

1		$(\exists x\,\phi) \lor (\exists x\,\psi)$			premise
2		$\exists x\,\phi$		$\exists x\,\psi$	assumpt.
3	x_0	$\phi[x_0/x]$	x_0	$\psi[x_0/x]$	assumpt.
4		$\phi[x_0/x] \lor \psi[x_0/x]$		$\phi[x_0/x] \lor \psi[x_0/x]$	$\lor$i 3
5		$(\phi \lor \psi)[x_0/x]$		$(\phi \lor \psi)[x_0/x]$	identical
6		$\exists x\,(\phi \lor \psi)$		$\exists x\,(\phi \lor \psi)$	$\exists x$ i 5
7		$\exists x\,(\phi \lor \psi)$		$\exists x\,(\phi \lor \psi)$	$\exists x$ e 2, 3–6
8		$\exists x\,(\phi \lor \psi)$			$\lor$e 1, 2–7

1		$\exists x\,(\phi \vee \psi)$		premise
2	x_0	$(\phi \vee \psi)[x_0/x]$		assumption
3		$\phi[x_0/x] \vee \psi[x_0/x]$		identical
4		$\phi[x_0/x]$	$\psi[x_0/x]$	assumption
5		$\exists x\,\phi$	$\exists x\,\psi$	$\exists x$ i 4
6		$\exists x\,\phi \vee \exists x\,\psi$	$\exists x\,\phi \vee \exists x\,\psi$	$\vee$i 5
7		$\exists x\,\phi \vee \exists x\,\psi$		$\vee$e 3, 4–6
8		$\exists x\,\phi \vee \exists x\,\psi$		$\exists x$ e 1, 2–7

$$\therefore\ \exists x\,(\phi \vee \psi) \dashv\vdash \exists x\,\phi \vee \exists x\,\psi$$

1		$\exists x\, \exists y\, \phi$	premise
2	x_0	$(\exists y\, \phi)[x_0/x]$	assumption
3		$\exists y\, (\phi[x_0/x])$	identical, since x, y different variables
4	y_0	$\phi[x_0/x][y_0/y]$	assumption
5		$\phi[y_0/y][x_0/x]$	identical, since x, y, x_0, y_0 different variables
6		$\exists x\, \phi[y_0/y]$	$\cancel{\forall}\exists x$ i 5
7		$\exists y\, \exists x\, \phi$	$\cancel{\forall}\exists y$ i 6
8		$\exists y\, \exists x\, \phi$	$\exists y$ e3, 4–7
9		$\exists y\, \exists x\, \phi$	$\exists x$ e1, 2–8

Theorem 2.13 Let ϕ and ψ be formulas of predicate logic. Then we have the following equivalences:

1. (a) $\neg\forall x\,\phi \dashv\vdash \exists x\,\neg\phi$
   (b) $\neg\exists x\,\phi \dashv\vdash \forall x\,\neg\phi$.
2. Assuming that x is not free in ψ:
   (a) $\forall x\,\phi \wedge \psi \dashv\vdash \forall x\,(\phi \wedge \psi)$[3]
   (b) $\forall x\,\phi \vee \psi \dashv\vdash \forall x\,(\phi \vee \psi)$
   (c) $\exists x\,\phi \wedge \psi \dashv\vdash \exists x\,(\phi \wedge \psi)$
   (d) $\exists x\,\phi \vee \psi \dashv\vdash \exists x\,(\phi \vee \psi)$
   (e) $\forall x\,(\psi \to \phi) \dashv\vdash \psi \to \forall x\,\phi$
   (f) $\exists x\,(\phi \to \psi) \dashv\vdash \forall x\,\phi \to \psi$
   (g) $\forall x\,(\phi \to \psi) \dashv\vdash \exists x\,\phi \to \psi$
   (h) $\exists x\,(\psi \to \phi) \dashv\vdash \psi \to \exists x\,\phi$.
3. (a) $\forall x\,\phi \wedge \forall x\,\psi \dashv\vdash \forall x\,(\phi \wedge \psi)$
   (b) $\exists x\,\phi \vee \exists x\,\psi \dashv\vdash \exists x\,(\phi \vee \psi)$.
4. (a) $\forall x\,\forall y\,\phi \dashv\vdash \forall y\,\forall x\,\phi$
   (b) $\exists x\,\exists y\,\phi \dashv\vdash \exists y\,\exists x\,\phi$.

[3] Remember that $\forall x\,\phi \wedge \psi$ is implicitly bracketed as $(\forall x\,\phi) \wedge \psi$, by virtue of the binding priorities.