CS F222: Discrete Structures for Computer Science

Tutorial - 2 (Proof Methods)

1. Prove that if n is an integer and n^2 is odd, then n is odd. First try to use the direct proof and if you cannot conclude the proof, then use proof of contraposition.

   Solution:
   Direct proof: Suppose that n is an integer and n^2 is odd. From the definition of an odd integer, there exists an integer k such that $n^2 = 2k+1$, this implies $n = \pm\sqrt{2k+1}$ which is not terribly useful.

   Proof by contraposition: Let n is not odd. Because every integer is odd or even, this means that n is even. Thus, there exists an integer k such that $n = 2k$. This implies, $n^2 = 4k^2 = 2(2k^2)$, which implies that n^2 is also even. We have proved that if n is an integer and n^2 is odd, then n is odd.

2. Prove that for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

   Solution: We use proof of contraposition. Let $p = x + y \geq 100$ and $q = x \geq 50$ or $y \geq 50$. Instead of showing $p \rightarrow q$, we show that $\neg q \rightarrow \neg p$.

   Let $\neg q$ is true i.e, it is not the case that $x \geq 50$ or $y \geq 50$. This implies, both x and y are less than 50 i.e., $x < 50$ and $y < 50$. Thus, $x + y < 50 + 50 = 100$. Therefore, $\neg p$ is true.

3. If p is a prime number grater than 3, then p^2 has the form $12k+1$, where k is an integer.

   Solution: (Direct Proof) Let p be a prime number greater than 3. Hence, p is a odd number. Thus, p^2 is odd (You can prove that if n is an odd integer then n^2 is odd.) note that $p^2 - 1 = (p+1)(p-1)$, thus $p^2 - 1$ is divisible by 4. We now show that $p^2 - 1$ is also divisible by 3. We know that one of any three consecutive integer is divisible by 3. Thus, one among $p-1, p, p+1$ is divisible by three. Since p is greater than 3, it must be the case that either $p-1$ or $p+1$ is divisible by 3. Thus, $p^2 - 1$ is divisible by 3. Now, $p^2 - 1$ is divisible by both 3 and 4, and $gcd(3, 4) = 1$, thus $p^2 - 1$ is also divisible by 12. Therefore, $p^2 - 1 = 12k$ for some integer. This gives us, $p^2 = 12k + 1$.

4. Show that $\sqrt{3}$ is irrational.

   Solution: Assume that $\sqrt{3}$ is not an irrational number, thus $\sqrt{3}$ is rational. Let $\sqrt{3} = a/b$ where a and $b \neq 0$ are two integers such that $gcd(a, b) = 1$. We have, $a^2 = 3b^2$. Here, a^2 is a multiple of 3 and hence, a is also multiple of 3 (If a is not a multiple of 3, then a is of the form either $3k+1$ or $3k+2$. Thus, a^2 is of the form either $9k^2 + 6k + 1$ or $9k^2 + 12k + 4$, which is a contradiction to our assumption that a^2 is a multiple of 3.)

   So, assume that $a = 3k$, for some kN. Thus, $a^2 = 3b^2$ this implies $9k^2 = 3b^2 \implies 3k^2 = b^2$, thus b^2 is also a multiple of 3 and hence, b is also multiple of three. In this case $gcd(a, b) \geq 3$, which is a contradiction.

5. Show that $\log_2 3$ is irrational.

   Solution: The proof proceeds by contradiction. Assume that $log_2 3$ is rational. By definition, there exist two numbers p, $q \in Z$, with $q \neq 0$, such that $log_2 3 = \frac{p}{q}$, which means that $2^{\frac{p}{q}} = 3$. Thus, $2p = 3q$.

   We can assume that $p, q > 0$. (Indeed, if $p/q > 0$ then we can just work with $|p|$ and $|q|$, and if $p/q \leq 0$ we reach a contradiction of the form $3 = 2^{p/q} \leq 2^0 = 1$.) Now, any positive integer power of 2 is even, because it has 2 as a divisor, so 2^p, is even. On the other hand, a positive integer power of 3 is odd, as we've seen previously. We have reached a contradiction.

6. Show that the sum of a rational number and an irrational number is an irrational number.

   Solution: The proof proceeds by contradiction. Assume that the sum of a rational number and an irrational number results in a rational number. Let us denote the irrational number by x and the other two rational numbers as a/b and c/d respectively, where a, b, c, d are integers and $b, d \neq 0$. Now, from our assumption

   $$a/b + x = c/d$$

   $$\text{or, } x = c/d - a/b$$

   $$\text{or, } x = \frac{cb-ad}{bd}$$

   which is rational. But this contradicts our assumption that x is irrational. This completes the proof.

7. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (Hint: Use a proof of cases,$x = y$, $x > y$ and $x < y$.)

   Solution: We use proof by cases, by considering the three cases (i)$x = y$ (ii) $x > y$ and (iii) $x < y$.

   Case (i). Let $x = y$. Then, $\min(x, y) = \max(x, y) = x$ (and $\min(x, y) = \max(x, y) = y$). Further, $x + y = 2x$ (resp. $x + y = 2y$). Thus, $\max(x, y) + \min(x, y) = x + y$.

   Case (ii). Suppose, $x > y$. In this case, $\min(x, y) = y$ and $\max(x, y) = x$. Thus, $\max(x, y) + \min(x, y) = x + y$.

   Case (iii). Suppose, $x < y$. In this case, $\min(x, y) = x$ and $\max(x, y) = y$. Thus, $\max(x, y) + \min(x, y) = x + y$.

8. Let $f(x) = x^3 + x - 5$ be a function. Show that there exists a real number c such that $f'(c) = 7$.

   Solution: We give a constructive existence proof. The derivative of f is $f'(x) = 3x^2+1$. Our goal is to find a positive number c such that $f'(c) = 7$.

   Thus, $3c^2 + 1 = 7 \implies c^2 = 2 \implies c = \pm\sqrt{2}$. Then, $c = \sqrt{2}$.