CS F222: Discrete Structures for Computer Science

Tutorial - 4 (Set Theory and Functions)

1. Prove or give a counter example for the following.

   (a) Let A and B be two sets such that $2^A \subseteq 2^B$. Then $A \subseteq B$. (2^X is the power set of X.)

   (b) Let A, B, and C be sets such that $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

   (c) Let A, B, and C be sets such that $A \in B$ and $B \in C$ then $A \in C$.

   (d) Let A, B, and C be sets such that $A \subset B$ and $A \subset C$ then $A \subset B \cap C$.

   Solution: **(a)** Recall that 2^A is the set of all subsets of A, including A itself. The condition tells us that every subset of A is also a subset of B, and in particular A itself is a subset of B. So $A \subseteq B$.

   (b) Consider any element $a \in A$. Since $A \subseteq B$, every element of A is also an element of B, so $a \in B$. By the same reasoning, $a \in C$ since $B \subseteq C$. Thus every element of A is an element of C, so $A \subseteq C$.

   (c) Let $A = \{x\}$, $B = \{x, \{x\}\}$, and $C = \{\{x, \{x\}\}, y\}$. These sets satisfy $A \in B$ and $B \in C$, but $A \notin C$.

   (d) Note that $\subset$ denotes the *proper subset*. Hence the statement will not hold to true for the case $A = B \cap C$.

2. Prove the following qualities for sets

   (a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

   (b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (Home work)

   Proof of (a). We show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

   To show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$, let $x \in \overline{A \cup B}$. Thus, $x \notin A \cup B$, which implies $x \notin A$ and $x \notin B$. This implies, $x \in \overline{A}$ and $x \in \overline{B}$. Hence, , $x \in \overline{A} \cap \overline{B}$. Hence, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

   To show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, let $x \in \overline{A} \cap \overline{B}$. Thus, $x \notin A$ and $x \notin B$, which implies $x \notin A \cup B$. Hence, $x \in \overline{A \cup B}$. Therefore, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

   Proof of (b) is similar.

3. Prove that $S = (S \cap T) \cup (S - T)$ for all sets S and T.

   Solution: We will show both of the following: $S \subseteq (S \cap T) \cup (S - T)$ and $(S \cap T) \cup (S - T)S$.

   To prove first case, consider any element $x \in S$. Either $x \in T$ or $x \notin T$.

   - If $x \in T$, then $x \in S \cap T$, and thus also $x \in (S \cap T) \cup (S - T)$.
   - If $x \notin T$, then $x \in (S - T)$, and thus again $x \in (S \cap T) \cup (S - T)$.

   To prove the latter part, consider any $x \in (S \cap T) \cup (S - T)$. Either $x \in S \cap T$ or $x \in S - T$

   - If $x \in S \cap T$, then $x \in S$.
   - If $x \in S - T$, then also $x \in S$

4. Answer the following questions on Cartesian product of sets.

   (a) Let $A = \{a, b, c\}$ and $\mathcal{P}(A)$ be the power set of A. Find the set $\mathcal{P}(A) \times A$.

   (b) Find a set A such that $A \subseteq A \times A$.

   (c) Let A, B, C and D be sets such that $A \subseteq C$ and $B \subseteq D$, show that $A \times B \subseteq C \times D$.

(d) Let A, B, C and D be sets such that $A \times B \subseteq C \times D$. Prove or disprove that $A \subseteq C$ and $B \subseteq D$.

Solution: (a) Here, $A = \{a, b\}$ and $\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.
Thus, $\mathcal{P}(A) \times A = \{(\phi, a), (\phi, b), (\{a\}, a), (\{a\}, b), (\{b\}, a), (\{b\}, b), (\{a, b\}, a), (\{a, b\}, b)\}$

(b) $A = \phi$. Note that $A \times A = \phi$ when $A = \phi$.

(c) We know that $A \subseteq C$ and $B \subseteq D$. To show that $A \times B \subseteq C \times D$, let $(a, b) \in A \times B$ where $a \in A$ and $B \in B$. Since $A \subseteq C$ and $B \subseteq D$, $a \in C$ and $b \in D$. Thus, $(a, b) \in C \times D$. Hence, $A \times B \subseteq C \times D$.

(d) Not true. Let $A = \{a\}$, $B = \emptyset$, $C = \{c\}$, and $D = \{d\}$. In this case, $A \times B = \phi$, hence $A \times B \subseteq C \times D$. But, A is not a subset of C.

5. Let A be a set with n elements. Prove that there are 2^{n^2} binary relations on A by using mathematical induction. Also, compute that number of ternary relations on A and prove the correctness by mathematical induction.

Solution:

We prove this by induction on number of elements in A i.e., n.

Base Case: If $n = 0$ then, number of relations is $2^0 = 1$ (Empty set). If $n = 1$ then, number of binary relations is $2^1 = 2$ In particular, if $A = \{x\}$ then, $A \times A = \{(x, x)\}$).

Hypothesis: Assume that the statement is true for all sets with at most k elements where $k \geq 1$.

Induction Step: Let A be the set with $k+1$ elements, $k \geq 1$. Let $A = \{x_1, x_2, ..., x_k, x_{k+1}\}$. From hypothesis, we know that for k elements, number of binary relations are 2^{k^2}. For $(k+1)$th element, we have the following $2k+1$ binary elements: $(x_1, x_{k+1}), (x_2, x_{k+1}), ..., (x_k, x_{k+1}), (x_{k+1}, x_1), (x_{k+1}, x_2), ..., (x_{k+1}, x_k), (x_{k+1}, x_{k+1}$
Therefore, number of binary relations for the set $A = 2^{k^2} \times 2^{2k+1} = 2^{k^2+2k+1} = 2^{(k+1)^2}$

The number of ternary relations on A are 2^{n^3}.

6. Determine whether each of the following function is a bijection from $\mathbf{R}$ to $\mathbf{R}$.

   (a) $f(x) = 2x + 1$

   (b) $f(x) = -3x^2 + 7$

   (c) $f(x) = (x^2 + 1)/(x^2 + 2)$

   (d) $f(x) = (x + 1)/(x + 2)$

Solution:

(a) Yes. *One-to-one:* Let $f(x_1) = f(x_2)$, which implies $2x_1 + 1 = 2x_2 + 1$. Thus, $x_1 = x_2$.

Onto function. Let y be a real number (in co-domain) such that $f(x) = y$, which gives us $2x + 1 = y$. Thus, $x = (y - 1)/2$ is also a real number. Thus, for any real number y in the co-domain, there exists a pre-image $(y - 1)/2$ in the domain.

(b). No. The function is not one-to-one function. For $x = 1$ and $x = -1$, the function maps to the same number 4 in the co-domain.

(c). No. The function is not one-to-one function. For $x = 1$ and $x = -1$, the function maps to the same number $2/3$ in the co-domain.

(d) No. The function is not define at $x = -2$.