Theorem 2.25 (Löwenheim-Skolem Theorem) Let ψ be a sentence of predicate logic such for any natural number $n \geq 1$ there is a model of ψ with at least n elements. Then ψ has a model with infinitely many elements.

PROOF: The formula $\phi_n \stackrel{\text{def}}{=} \exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)$ specifies that there are at least n elements. Consider the set of sentences $\Gamma \stackrel{\text{def}}{=} \{\psi\} \cup \{\phi_n \mid n \geq 1\}$ and let Δ be any of its finite subsets. Let $k \geq 1$ be such that $n \leq k$ for all n with $\phi_n \in \Delta$. Since the latter set is finite, such a k has to exist. By assumption, $\{\psi, \phi_k\}$ is satisfiable; but $\phi_k \to \phi_n$ is valid for all $n \leq k$ (why?). Therefore, Δ is satisfiable as well. The compactness theorem then implies that Γ is satisfiable by some model $\mathcal{M}$; in particular, $\mathcal{M} \vDash \psi$ holds. Since $\mathcal{M}$ satisfies ϕ_n for all $n \geq 1$, it cannot have finitely many elements. $\Box$

If $\mathcal{M}$ has finitely many elements say q of them, then ϕ_{q+1} should not be satisfiable in $\mathcal{M}$. But it is.

Theorem 2.26 Reachability is not expressible in predicate logic: there is no predicate-logic formula ϕ with u and v as its only free variables and R as its only predicate symbol (of arity 2) such that ϕ holds in directed graphs iff there is a path in that graph from the node associated to u to the node associated to v.

PROOF: Suppose there is a formula ϕ expressing the existence of a path from the node associated to u to the node associated to v. Let c and c' be constants. Let ϕ_n be the formula expressing that there is a path of length n from c to c': we define ϕ_0 as $c = c'$, ϕ_1 as $R(c, c')$ and, for $n > 1$,

$$\phi_n \stackrel{\text{def}}{=} \exists x_1 \ldots \exists x_{n-1}(R(c, x_1) \wedge R(x_1, x_2) \wedge \cdots \wedge R(x_{n-1}, c')).$$

Let $\Delta = \{\neg\phi_i \mid i \geq 0\} \cup \{\phi[c/u][c'/v]\}$. All formulas in Δ are sentences and Δ is unsatisfiable, since the 'conjunction' of all sentences in Δ says that there is no path of length 0, no path of length 1, etc. from the node denoted by c to the node denoted by c', but there is a finite path from c to c' as $\phi[c/u][c'/v]$ is true.

However, every finite subset of Δ is satisfiable since there are paths of any finite length. Therefore, by the Compactness Theorem, Δ itself is satisfiable. This is a contradiction. Therefore, there cannot be such a formula ϕ. $\square$

EXISTENTIAL SECOND-ORDER LOGIC

$$\exists P \, \phi$$

$$\exists P \, \forall x \forall y \forall z \, (C_1 \wedge C_2 \wedge C_3 \wedge C_4)$$

where each C_i is a Horn clause[4]

$$\begin{aligned}
C_1 &\stackrel{\text{def}}{=} P(x, x) \\
C_2 &\stackrel{\text{def}}{=} P(x, y) \wedge P(y, z) \longrightarrow P(x, z) \\
C_3 &\stackrel{\text{def}}{=} P(u, v) \rightarrow \bot \\
C_4 &\stackrel{\text{def}}{=} R(x, y) \rightarrow P(x, y).
\end{aligned}$$

Given a model $\mathcal{M}$ with interpretations for all function and predicate symbols of ϕ in (2.11), *except* P, let $\mathcal{M}_T$ be that same model augmented with an interpretation $T \subseteq A \times A$ of P, i.e. $P^{\mathcal{M}_T} = T$. For any look-up table l, the semantics of $\exists P\, \phi$ is then

$$\mathcal{M} \vDash_l \exists P\, \phi \qquad \text{iff} \qquad \text{for some } T \subseteq A \times A,\ \mathcal{M}_T \vDash_l \phi. \tag{2.13}$$

UNIVERSAL SECOND-ORDER LOGIC

$$\forall P\, \exists x \exists y \exists z\, (\neg C_1 \vee \neg C_2 \vee \neg C_3 \vee \neg C_4) \qquad (2.14)$$

Theorem 2.28 Let $\mathcal{M} = (A, R^{\mathcal{M}})$ be any model. Then the formula in (2.14) holds under look-up table l in $\mathcal{M}$ iff $l(v)$ is R-reachable from $l(u)$ in $\mathcal{M}$.

Reachability is expressible in Universal second-order Logic.