**BITS Pilani Hyderabad Campus**
**CS F214 Logic in Computer Science,**
**I Semester 2021-2022**
**Lecture Notes**
**Lecture 1-5**

# 1 Barber Paradox

**There is a lone male barber who lives on an island. The barber shaves all those men who do not shave themselves and only those men.**

*Consider following statement and determine if it is true or false?* Statement: The barber shaves himself.

Such statements are called as propositions. (Formal definition will be discussed in later lectures).

This statement is paradoxical. The barber cannot shave himself as he only shaves those who do not shave themselves. Thus, if he shaves himself he ceases to be the barber (i.e. person who cannot shave). On the other hand, if barber does not shave himself, then he belongs to the group of people who would be shaved by the barber and hence barber should shave himself.

Hence, there is no clear answer to proposition.

# 2 Naive Set Theory

- This has been proposed by *Georg Cantor*.
- A SET is a collection of objects or entities.
- Bertrand Russell proposed Russell's paradox for above definition

## 2.1 Russell's Paradox

Let R be the set of all sets that are not members of themselves.

$$A = \{1, 2, 3, 4\}, \therefore A \in R$$
$$R = \{X | X \notin X\}$$

Is the statement $R \in R$ true or false?

Similar to the Barber Paradox, if we assume that statement is true that R now belongs to itself. Now, R is a set which contains itself and hence should not be included in R. If we assume statement is false that R does not belong to itself. Then, as per the definition R is now a set which does not contain itself and hence should be included in the set R.

Hence, answering this results in paradox. There is no clear answer to the statement.

## 2.2 Zermelo Frankel Set Theory

Ernst Zermelo and Abraham Frankel came up with axiomatic approach to resolve Russell's Paradox.

Example:

1. If the train arrives late and there are no taxis at the station, then John is late for the meeting.
2. John is not late for the meeting.
3. The train did arrive late.

$\therefore$ There were taxis at the station.

Solution: p = Train arrives late. q = There are no taxis at the station. r = John is late for meeting.

$$
\begin{aligned}
&1.((p \wedge q) \rightarrow (r)) \\
&2.\neg r \\
&3.p \\
&4.\neg q \quad \text{(conclusion)}
\end{aligned}
$$

Statement 1 is true when proposition q is false. i.e. when conclusion is true. Hence the conclusion follows.

Example:

1. If it is raining and Jane does not have umbrella then she will get wet.
2. Jane is not wet.
3. It is raining.

$\rightarrow$ Therefore, Jane has her umbrella with her.

| Argument 1 | Corresponding argument 2 |
|---|---|
| Train is late (p) | It is raining |
| There are taxis at the station(q) | Jane has an umbrella |
| John is late (r) | Jane is wet |

# 3 Propositional Logic

## 3.1 Declarative Sentences Or Propositions

Sentences that are in principle, either true or false (but not both).

Examples:

1. This is a class on logic.

2. The sun orbits the earth.
3. The sum of 2 and 2 is 5.
4. Every even natural number greater than 2 is the sum of 2 prime numbers. (Goldbach's conjecture. We are not sure whether this statement is true or false.)

## 3.2 Non-Declarative Sentences

1. Pay Attention!
2. Would you accompany me to class?
3. May fortune come your way.

Non-Declarative Sentences are not propositions.

## 3.3 Twin Prime Conjecture

- For all the numbers $n$, there exists an $n_0 > n$, such that $n_0$ and $n_0 + 2$ are prime numbers.
- Twin prime conjecture has not been proved yet.

## 3.4 Bounded Gap Theorem

- For all natural numbers $n$, there exists a constant $c$, for all natural numbers $n_0 > n$, so that there are two prime numbers in interval $[n_0, n_0 + c]$
- This has been proved by Yitang Zhang.

## 3.5 Truth Tables

Given these atomic propositions p,q,r,.. or p1,p2,p3.. we can combine them in compositional ways to form other propositions using logical connections.

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

Table 1: Negation ($\neg$)

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

Table 2: Conjuction ($\wedge$)

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

Table 3: Disjunction ($\vee$)

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

Table 4: Conditional (Implication)

p $\leftrightarrow$ q $\implies$ (p$\to$q) $\wedge$ (q$\to$p)

# 4 Rules for Natural Deduction

## 4.1 And Introduction

- Given premises $\phi$, $\psi$, we can conclude $\phi \wedge \psi$.

$$\frac{\phi, \psi}{\phi \wedge \psi}\wedge_i$$

## 4.2 And Elimination

- Given premises $\phi \wedge \psi$, we can conclude $\phi$.

$$\frac{\phi \wedge \psi}{\phi}\wedge_{e_1}$$

$$\frac{\phi \wedge \psi}{\psi}\wedge_{e2}$$

Example Prove that the following statement is valid $p$, $p \wedge q$, $r \vdash q \wedge r$

| | |
|---|---|
| $1.p \wedge q$ | premise |
| $2.r$ | premise |
| $3.q$ | $e_2$ 1 |
| $4.q \wedge r$ | $\wedge_i 3, 2$ |

## 4.3 Double Negation

**Double Negation Elimination**

$$\frac{\neg\neg\phi}{\phi}\neg\neg e$$

**Double Negation Introduction**

$$\frac{\neg\neg\phi}{\phi}\neg\neg i$$

## 4.4 Eliminating Implication (Modus Ponens)

$$\frac{\phi, \phi \to \psi}{\psi} \to e$$

Example: $p$ : It rained. $q$ : The street is wet. $p \to q$: If it rained then the street is wet

## 4.5 Modus Tollens

$$\frac{\phi \to \psi, \neg\psi}{\neg\phi} \to MT$$

e.g. $p \to q$: If it rained then the street is wet. (premise) $\neg q$: The street is not wet. (premise) $\therefore$ it did not rain ($\neg p$) (conclusion)

Prove the following statement $p \to (q \to r), p, \neg r \vdash \neg q$ .

| | |
|---|---|
| $1.p \to (q \to r)$ | premise |
| $2.p$ | premise |
| $3.\neg r$ | premise |
| $4.q \to r$ | $\to$e 2,1 |
| $5.\neg q$ | MT 4,3 |

By using Modus Tollens Elimination,

$$\frac{p, p \to (q \to r)}{q \to r}$$

By using Modus Tollens,

$$\frac{q \to r, \neg r}{\neg q}$$

### 4.6 Implies Introduction

- We wish to build implications that do not already appear as premises (or parts theorem) in our proof.

Example Prove if following condition is valid: $p \to q \vdash \neg q \to \neg p$

| | |
|---|---|
| 1.$p \to q$ | premise |
| 2.$\neg q$ | assumption |
| 3.$\neg p$ | Modus Tollens 1,2 |
| 4.$\neg q \to \neg p$ | [$\to$i 2,3] |

2,3 $\implies$ Scope of the assumption.
* Implies introduction is formulated as :

$$\frac{\phi, \psi}{\phi \to \psi} \to i$$