CS F222: Discrete Structures for Computer Science

Tutorial - 5 (Countable Sets and Relations)

1. If $A_1, A_2, A_3 \dots,$ is a collection of countably many countably infinite sets, then $\bigcup_{i=1}^{\infty} A_i$ is also a countable set.

   Solution: Since each A_i is countable set, there exists an ordering of elements in A_i such that every element in A_i appears in the ordering. Let $a_{i1}, a_{i2}, a_{i3}, \dots,$ be the ordering of elements in A_i. To show that $\bigcup_{i=1}^{\infty} A_i$ is countable, it is sufficient to give an ordering of elements in $\bigcup_{i=1}^{\infty} A_i$ such that the order contains each element of the set. The ordering of elements in $\bigcup_{i=1}^{\infty} A_i$ is $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, \dots,$. The elements a_{ij} are listed in non-decreasing order of $i + j$ such that if $i + j = r + s$ then a_{ij} is listed before $a_{r,s}$ if and only if $i < r$. Thus $\bigcup_{i=1}^{\infty} A_i$ is countable.

2. Show that the set of all finite binary sequences/string is countable.

   Solution: We note that there are finite number of binary strings of finite length i.e., there are exactly 2^n binary strings of length n. We use the above result, the countable union of countable sets is countable. Let S_k be the set of binary strings of exactly length k. Thus, S_k is a finite set and hence countable. Further, Further, $S_0, S_1, S_3, S_4, \dots,$ is a countable collection of countable sets. Hence, $\bigcup_{k=0}^{\infty} S_k$ is countable. Further, $\bigcup_{k=0}^{\infty} S_k$ is the set of finite binary sequences. Therefore, the set of all finite binary sequences is countable.

3. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable. ($\mathbf{Z}^+$ is the set of positive integers.)

   Solution: Recall that $\mathbf{Z}^+ \times \mathbf{Z}^+ = \{(i, j) \mid i \in \mathbf{Z}^+ \text{ and } j \in \mathbf{Z}^+\}$. We use the above result, the countable union of countable sets is countable. Let $S_k = \{(i, j) | i + j = k\}$ be a subset of $\mathbf{Z}^+ \times \mathbf{Z}^+$. Note that for each $k = 1, 2, \dots,,$ the set S_k is finite and hence countable. Further, $S_2, S_3, S_4, \dots,$ is a countable collection of countable sets. Hence, $\bigcup_{k=2}^{\infty} S_k$ is countable i.e., $\mathbf{Z}^+ \times \mathbf{Z}^+ = \bigcup_{k=2}^{\infty} S_k$ is countable.

4. The set S of all infinite binary sequences/strings is uncountable.

   Solution: The proof is by contradiction. Suppose that S is countable. Since S is an infinite set, there is an infinite sequence $b_1, b_2, b_3, \dots,$ that contains every element of S (at least once). We will obtain a contradiction by exhibiting an element t of S that, by its definition, can not be b_i for any positive integer i. For each positive integer i, let b_i be the infinite binary sequence $b_{i1}b_{i2}b_{i3}\dots$. The infinite binary sequence $t = t_1t_2t_3\dots$ is defined by $t_i = 1 - b_{ii}$, for $i = 1, 2, \dots$. We claim that t can not be b_i for any positive integer i. For every positive integer i, the i-th element of the sequence t is different from b_{ii}, the i-th element of b_i. Therefore there cannot be an infinite sequence $b_1, b_2, b_3, \dots$ that contains every element of S. Hence, S is uncountable

5. The power set of any infinite set is uncountable.

   Solution: Let $S = \{e_1, e_2, e_3, \dots,\}$ be an infinite set and $\mathcal{P}(S)$ be the power set of S. We now show that $\mathcal{P}(S)$ is uncountable. For the sake of contradiction, assume that $\mathcal{P}(S)$ is countable. Thus, the exists an ordering of subsets S (members of $\mathcal{S}$) such that the ordering contains each subset of S. Let $S_1, S_2, S_3, \dots,$ be the ordering elements (subsets of S) in $\mathcal{P}(S)$. We now encode each S_i to an infinite binary string $b_i = b_{i1}b_{i2}b_{i3}\dots\dots$ where $b_{ij} = 1$ if $e_j \in S_i$, otherwise $b_{ij} = 0$. We now show that there exists a set $T \in \mathcal{P}(S)$ such that $T \neq S_i$ for all i. Let $t = t_1t_2t_3\dots$ be the infinite binary string such that $t_i = 1$ if $b_{ii} = 0$ and $t_i = 0$ if $b_{ii} = 1$. We claim that the i-th bit of t (i.e., t_i) is differ with i-th bit of b_i (i.e., b_{ii}). Now define $T = \{e_i | t_i = 1\}$. Thus, T is not the same as any S_i for all i. This is a contradiction to the fact that there exists an ordering of elements of a countable set that includes all the elements of the set. Hence, $\mathcal{P}(S)$ is not countable.

6. Consider the following relations on $\{1, 2, 3, 4\}$ $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

   $R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,

$R_6 = \{(3,4)\}$.

Which of the above relations are reflexive, symmetric, antisymmetric, and transitive.

Solution:

Reflexive: R_3 and R_5.

Symmetric: R_2 and R_3.

Antisymmetric: R_4, R_5, and R_6.

Transitive: R_4, R_5, and R_6.

7. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a,b) \in R$ if and only if

   (a) everyone who has visited Web page a has also visited Web page b.

   (b) there are no common links found on both Web page a and Web page b.

   (c) there is at least one common link on Web page a and Web page b.

   (d) there is a Web page that includes links to both Web page a and Web page b.

   Solution:

   (a): Reflexive and Transitive
   (b): Symmetric
   (c): Symmetric
   (d): Symmetric