BITS Pilani Hyderabad Campus
CS F214 Logic in Computer Science,
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Lecture Notes
Lecture 13-14

Theorem: *The sum of the first n natural numbers in equal to* $\frac{n(n+1)}{2}$

Proof: Let $M(n)$: the sum of the first n natural numbers not equal to $\frac{n(n+1)}{2}$.

- Base Case: To show that $M(1)$ is true.

  The sum of first one natural number is 1.
  Furthermore $M(1)$ states that the sum of first natural number equals $\frac{1(1+1)}{2} = 1$.

  $\therefore M(1)$ is true.

- Inductive Step: Suppose $M(n)$ is true. Consider $M(n+1)$: The sum of the first $(n+1)$ natural numbers is $\frac{(n+1)(n+2)}{2}$.

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$$

  By the induction hypothesis, $M(n)$ is true.

$$\sum_{i=1}^{n+1} = \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$\sum_{i=1}^{n+1} = \frac{(n+1)(n+2)}{2}$$

  $\therefore M(n+1)$ is true.

12 Course of Values Induction or Strong Induction

1. **Base Case**: Natural number 1 has the property i.e. we have a proof of $M(1)$.
2. **Inductive Step**: We assume that $M(1) \wedge M(2) \wedge M(3).. \wedge M(n)$ is true and show that $M(n+1)$ is true.

Statements on parse trees are often shown by strong induction on height.
It is also called as structural Induction.

Theorem: *For every well-formed proposition logic formula, the number of left brackets equal the number of right brackets.*

Proof: Course of values induction on the height of the parse tree corresponding to the well-formed formula.
Let $M(n)$: All formulas of height n, have the same number of left and right brackets.

- **Base Case**: We will show that $M(1)$ is true.
  A parse tree of height 1 has only an atom and no bracket. Hence $M(1)$ is true.

- **Inductive Step**: Suppose $M(1), M(2), ...M(n)$ is true, we will show that $M(n+1)$ is true.
  A parse tree of height ≥ 2 has as its root either of $\neg, \vee, \wedge, \rightarrow$.

  Case (1): Suppose the root as $\neg$. Then the sub-tree rooted at $\neg$ is of height n.
  By the induction hypothesis property as true for the formula ϕ corresponding to that of sub-tree. The formula corresponding to the full tree is $(\neg\phi)$, which has equal number of left and right brackets. Since we added one left bracket and one right bracket.

  Case (2): Let the root be $\vee$. If this i so, there exist two well formed formulas ϕ_1 and ϕ_2 so that the present formula is $(\phi_1 \vee \phi_2)$. The parse trees corresponding to the formulas ϕ_1 and ϕ_2 have height less than or equal to n.
  $\therefore$ by the induction hypothesis both these parse trees have equal number of left and right brackets each.
  $\therefore$ $(\phi_1 \vee \phi_2)$ has equal number of left and right brackets. Since this adds one more left (right) bracket to the sum of left brackets of $(\phi_1 \vee \phi_2)$.

  Case (3), (4) correspond to the binary connectives $\wedge, \rightarrow$, for which the argument is similar.

13 Semantics of Propositional Logic

Def 1: *The set of truth values contains two elements - T and F, where T represents 'true' and F corresponds to 'false'.*

Def 2: *The valuation on model of formula ϕ is an assignment of each propositional atom in ϕ to a truth value. A truth table lists all valuations of a formula ϕ.*

Do all valid sequents preserve truth computed by our truth table semantics?

Def: *If for all the valuations in which all of $\phi_1, \phi_2, ..., \phi_n$ evaluate to T and formula χ also evaluates to T, we say that (the semantic entailment relation) $\phi_1, \phi_2, ..., \phi_n \vDash \chi$ holds and we call $\vDash$ as the semantic entailment relation.*

Examples:

1. $p \wedge q \vDash p$ holds.

p	q	$p \wedge q$	p	$p \vee q$
F	F	F	F	F
F	T	F	F	T
T	F	F	T	T
T	T	T	T	T

2. $p \vee q \vDash p$ does not holds.

3. $p \vDash q \vee \neg q$ holds.