# CS F222: Discrete Structures for Computer Science

## Tutorial - 5 (Countable Sets and Relations)

1. If $A_1, A_2, A_3 \ldots,$ is a collection of countably many countably infinite sets, $\bigcup_{i=1}^{\infty} A_i$ is also a countable set.
2. Show that the set of all finite binary sequences/string is countable.
3. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable. ($\mathbf{Z}^+$ is the set of positive integers. )
4. The set $S$ of all infinite binary sequences/strings is uncountable.
5. The power set of any infinite set is uncountable.
6. Consider the following relations on $\{1, 2, 3, 4\}$

   $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

   $R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

   $R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

   $R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

   $R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

   $R_6 = \{(3, 4)\}$.

   Which of the above relations are reflexive, symmetric, antisymmetric, and transitive.
7. Determine whether the relation $R$ on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
   (a) everyone who has visited Web page $a$ has also visited Web page $b$.
   (b) there are no common links found on both Web page $a$ and Web page $b$.
   (c) there is at least one common link on Web page $a$ and Web page $b$.
   (d) there is a Web page that includes links to both Web page $a$ and Web page $b$.

$f : \bigcup_{i=1}^{\infty} A_i \to Z^+$

$A_1, A_2$

Tut_5 (1)

21 September 2021 14:01

$Z^+ =$ $A_1, A_2, A_3, \ldots$

$A_i = \{a_{i1}\ a_{i2}\ \ldots\ \}$

$\bigcup_{i=1}^{k} A_i$, $\quad a_{11}\ a_{21}\ a_{31}\ \cdots\ a_{k1}, a$

$A_1 \quad A_2, \ldots \quad A_k$

Tut_5 (1)

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4. The set $S$ of all infinite binary sequences/strings is uncountable.
5. The power set of any infinite set is uncountable.
6. Consider the following relations on $\{1, 2, 3, 4\}$
   $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$, $\notin (3,3)$
   $R_2 = \{(1,1), (1,2), (2,1)\}$,
   $R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$,
   $R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$,
   $R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$,
   $R_6 = \{(3,4)\}$.
   Which of the above relations are reflexive, symmetric, antisymmetric, and transitive.
7. Determine whether the relation $R$ on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
   (a) everyone who has visited Web page $a$ has also visited Web page $b$.
   (b) there are no common links found on both Web page $a$ and Web page $b$.
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$(4,1)\ (1,2) \Rightarrow (4,2) \notin R_1$

1

Reflexive : $R_3$ & $R_5$
Symmetric : $R_2$ and $R_3$
Antisymmetric : $R_4$, $R_5$ and $R_6$
Transitive : $R_4$, $R_5$ and $R_6$.

$(a,b) \in R,\ (b,a) \in R \Rightarrow a = b$

Q1. Each set $A_i$ is countable set, there exists an ordering of elements in $A_i$ such that each element of $A_i$ appears in this ordering.

$f : A \to Z^+$ : $f$ is one-one and onto

$A = \{a_1, a_2, \ldots\ \}$
$Z^+ = \{1, 2, \ldots\ \}$

Consider this $A_i$, let $a_{i1}, a_{i2}, \ldots$ be this ordering of elements in $A_i$

From each $A_i$, I will collect all elements sum of indices is equal to $j$, where $j = 2, 3, 4, \ldots$

when $j=2$, $j=3$, $j=4$
$a_{11}$, $\quad a_{12}, a_{21}$ $\quad a_{13}, a_{22}, a_{31}\ \cdots$

within each such collection (for every $a_{ij}, a_{rs}$, $i+j = r+s$) the elements $a_{ij}$ are listed such that $a_{ij}$ is before $a_{rs}$ if $i < r$.

The elements in $\bigcup_{i=1}^{\infty} A_i$ are ordered as $a_{11}\ a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, \ldots$ This is not lexicographic order

Therefore, $\bigcup_{i=1}^{\infty} A_i$ is countable.

Q2. S.T. the set of all finite binary strings is countable.

fix the length of string to be $n$.

The set $S_0$ contains all strings of length 0.
" $S_1$ " 1,
⋮
" $S_k$ " $k$. $\quad |S_k| = 2^k$

The ordering
$S_0, S_1, S_2, \ldots$

$\bigcup_{k=0}^{\infty} S_k$ is the set of all strings and is countable.

Q3. S.T. $Z^+ \times Z^+$ is countable ($Z^+$ is the set of all positive integers)

$Z^+ \times Z^+ = \{ (i,j) \mid i \in Z^+ \text{ and } j \in Z^+ \}$,

To see the above result, define $S_k = \{ (i,j) \mid i+j = k \} \subseteq Z^+ \times Z^+$.

For each $k = 1, 2, \ldots$, the set $S_k$ is finite and hence countable.

clearly, we have the ordering below,
$S_2, S_3, S_4, \ldots,$

Therefore, $\bigcup_{k=0}^{\infty} S_k$ is countable. $\quad Z^+ \times Z^+$ is countable.

S.T. $Z^+ \times Z^+$ is countable by showing that the polynomial function $f : Z^+ \times Z^+ \to Z^+$ with $f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$, is one-to-one and onto.

$f(1,1) = 1$ $\quad f(2,1) = 3$ $\quad f(3,1) = 6$ $\quad f(4,1) = 10$ $\quad f(5,1) = 15$
$f(1,2) = 2$ $\quad f(2,2) = 5$ $\quad f(3,2) = 9$ $\quad f(4,2) = 14$
$f(1,3) = 4$ $\quad f(2,3) = 8$ $\quad f(3,3) = 13$
$f(1,4) = 7$ $\quad f(2,4) = 12$
$f(1,5) = 11$

The function takes on successive values as $m+n$ increases.

when $m+n = 2$, $f(m,n) = 1$, $\quad$ when $m+n = 3$, $f(m,n) = 2$ or $3$

The range of values the fn takes on a fixed value of $m+n$, say $m+n = x$, is $(x-2)(x-1)$ , , through. $(x-2)(x-1)$ - -

$m+n=2,\ f(1,1)=1,$ when $m+n=3,\ f(\ldots)=2$ or $3$

The range of values the fn takes on a fixed value of $m+n$, say $m+n=x$, is $\frac{(x-2)(x-1)}{2}+1$ through $\frac{(x-2)(x-1)}{2}+(x-1)$

we need to show that the range of values for $x+1$ picks up precisely when the ranges of values for $x$ left off i.e.,

$$f(x-1,1)+1 = f(1,x)$$

$$= \boxed{\underbrace{\frac{(x-2)(x-1)}{2}+(x-1)}_{f(x-1,1)}}+1 = \frac{x^2-x+2}{2} = \frac{(x-1)x}{2}+1 = f(1,x)$$

$\therefore$ $f$ is one-one and onto. and hence $Z^+ \times Z^+$ is countable.

**Q4** S.T. the set $S$ of all infinite binary strings is uncountable.

The proof is contradiction. Suppose that $S$ is countable. Since $S$ is an infinite set, there is an infinite sequence $b_1, b_2, b_3, \ldots$, that contain every element of $S$.

Here, each of $b_i$ is of infinite length.

$b_i = b_{i1}, b_{i2}, b_{i3}, \ldots,$

$b_1 = b_{11}\ b_{12}\ b_{13}\ \ldots$
$b_2 = b_{21}\ b_{22}\ b_{23}\ \ldots$
$b_3 \quad b_{31}\ b_{32}\ b_{33}\ \ldots$
$\vdots$

Consider the following binary string of infinite length.
$t = t_1 t_2 t_3 \ldots,$ where $t_i = 1-b_{ii}$ for $i=1,2,\ldots$

claim: $t \neq b_i$ listed in the above ordering.
because $i$-bit of $t$ differs with the $i$-bit of $b_i$.

The string $t$ is not appearing in the above ordering.
which is contradiction.

$\therefore$ $S$ is uncountable.

Cantor's Diagonalization method

$0.\ d_{11}\ d_{12}\ \ldots$
$0.\ d_{21}\ d_{22}\ \ldots$
$\vdots$

infinitely

$1 \ldots 2 \ldots$

Q7. (a) Reflexive and transitive
(b) Symmetric
(c) Symmetric
(d) Symmetric