Example 2.5 Consider translating the sentence

Every son of my father is my brother.

into predicate logic. As before, the design choice is whether we represent 'father' as a predicate or as a function symbol.

1. As a predicate. We choose a constant m for 'me' or 'I,' so m is a term, and we choose further $\{S, F, B\}$ as the set of predicates with meanings

$$\begin{aligned} S(x, y):&\quad x \text{ is a son of } y \\ F(x, y):&\quad x \text{ is the father of } y \\ B(x, y):&\quad x \text{ is a brother of } y. \end{aligned}$$

   Then the symbolic encoding of the sentence above is

$$\forall x\, \forall y\, (F(x, m) \wedge S(y, x) \to B(y, m)) \tag{2.3}$$

   saying: 'For all x and all y, if x is a father of m and if y is a son of x, then y is a brother of m.'

2. As a function. We keep m, S and B as above and write f for the function which, given an argument, returns the corresponding father. Note that this works only because fathers are unique and always defined, so f really is a function as opposed to a mere relation.
The symbolic encoding of the sentence above is now

$$\forall x\,(S(x, f(m)) \to B(x, m)) \tag{2.4}$$

meaning: 'For all x, if x is a son of the father of m, then x is a brother of m;' it is less complex because it involves only one quantifier.

Domain-specific knowledge:

$B(m,m)$? Not well defined.

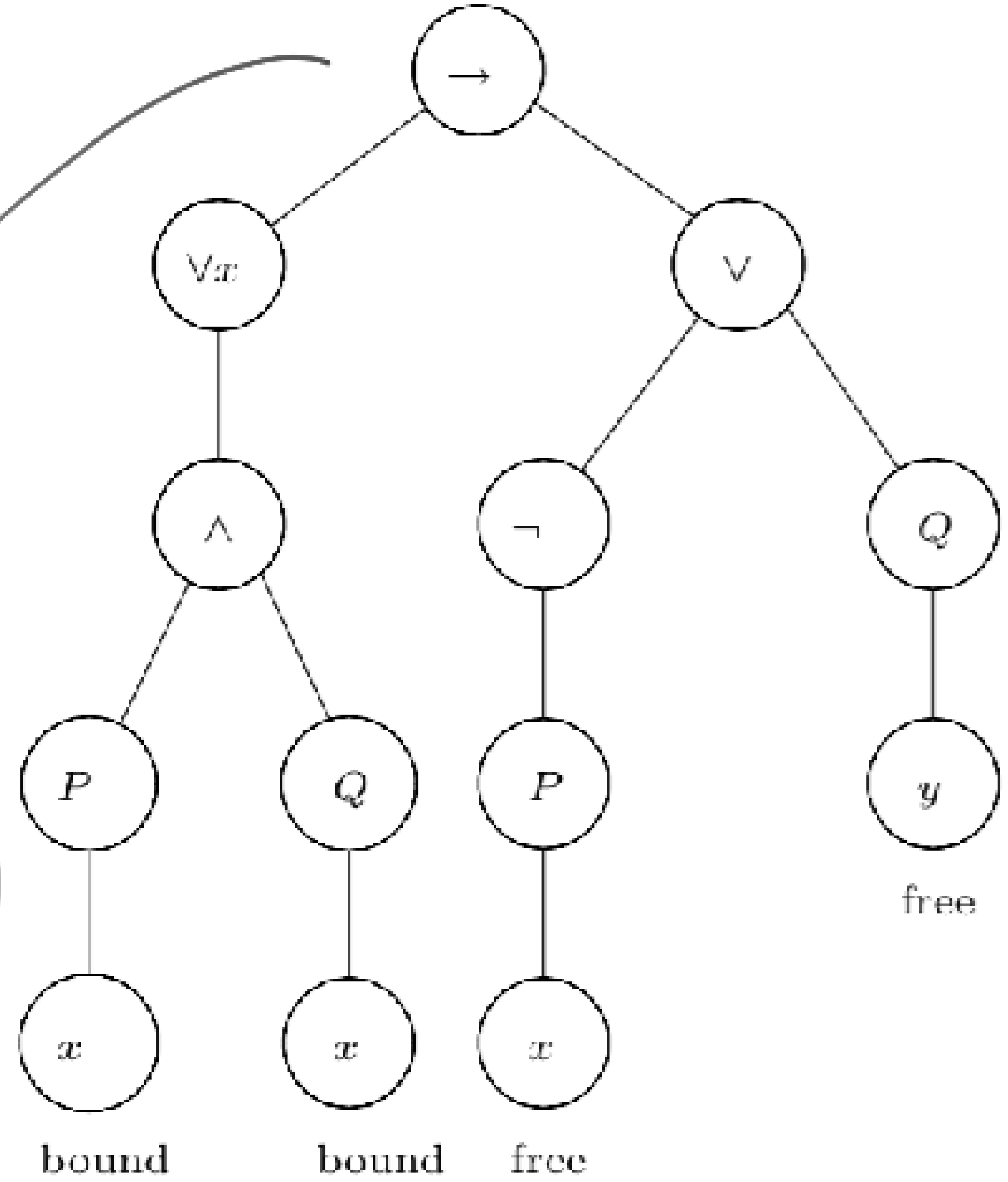
FREE & BOUND VARIABLES

$\forall x [P(x) \wedge (\exists x Q(x)$

Definition 2.6 Let ϕ be a formula in predicate logic. An occurrence of x in ϕ is free in ϕ if it is a leaf node in the parse tree of ϕ such that there is no path upwards from that node x to a node $\forall x$ or $\exists x$. Otherwise, that occurrence of x is called bound. For $\forall x\, \phi$, or $\exists x\, \phi$, we say that ϕ – minus any of ϕ's subformulas $\exists x\, \psi$, or $\forall x\, \psi$ – is the scope of $\forall x$, respectively $\exists x$.

$\forall x\, \underline{P(x, y)}$

$[\forall x\, (P(x) \wedge Q(x))] \to [\neg P(x) \vee Q(y)]$

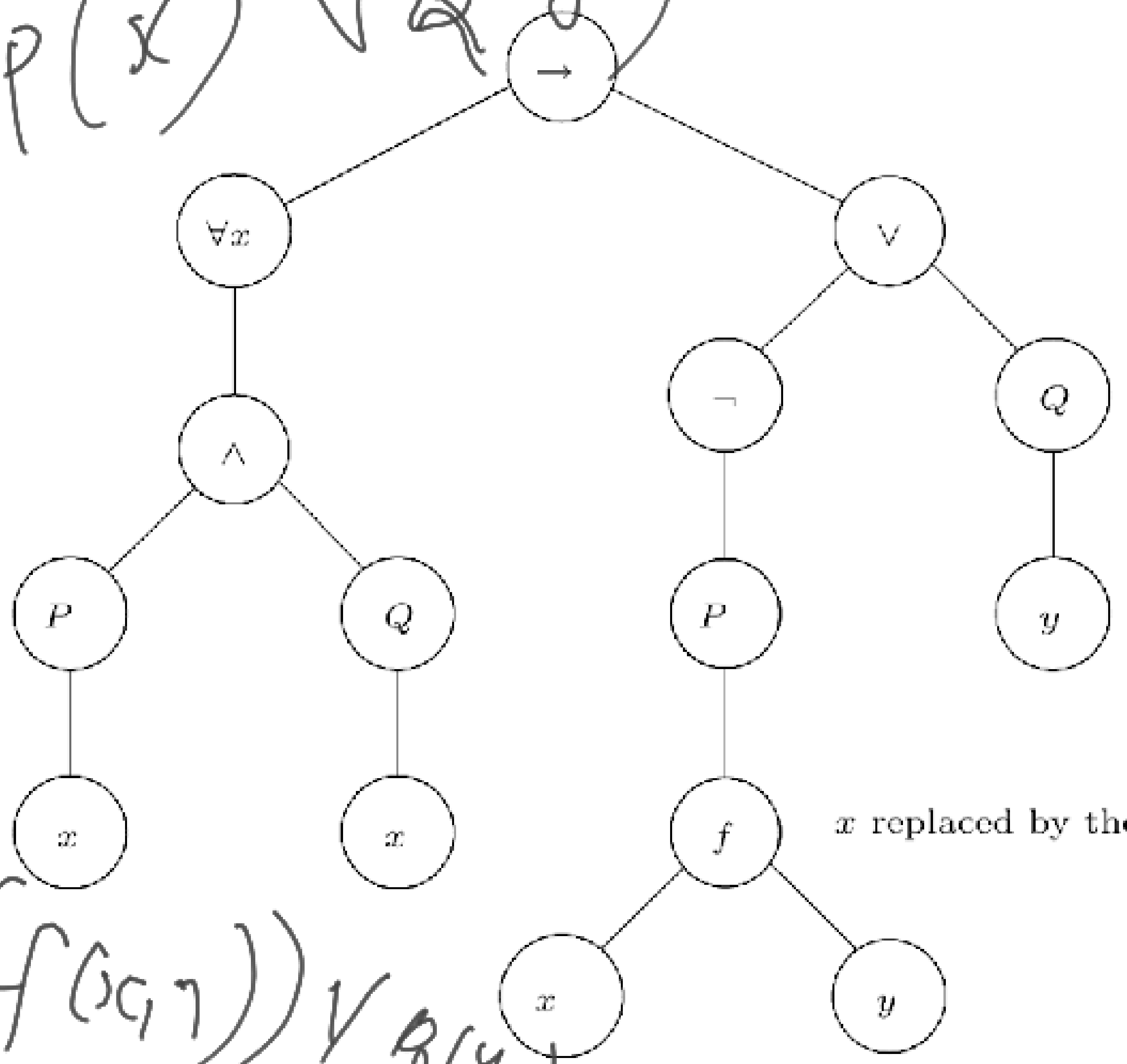


SUBSTITUTION

Definition 2.7 Given a variable x, a term t and a formula ϕ we define $\phi[t/x]$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t.

$\phi(x)$

$\equiv [\forall x(P(x) \wedge Q(x))] \to \neg P(x) \vee Q(y)$



x replaced by the term $f(x, y)$

$\phi[f(x,y)/x]$

$\equiv [\forall x(P(x) \wedge Q(x))] \to \neg P(f(x,y)) \vee Q(y)$

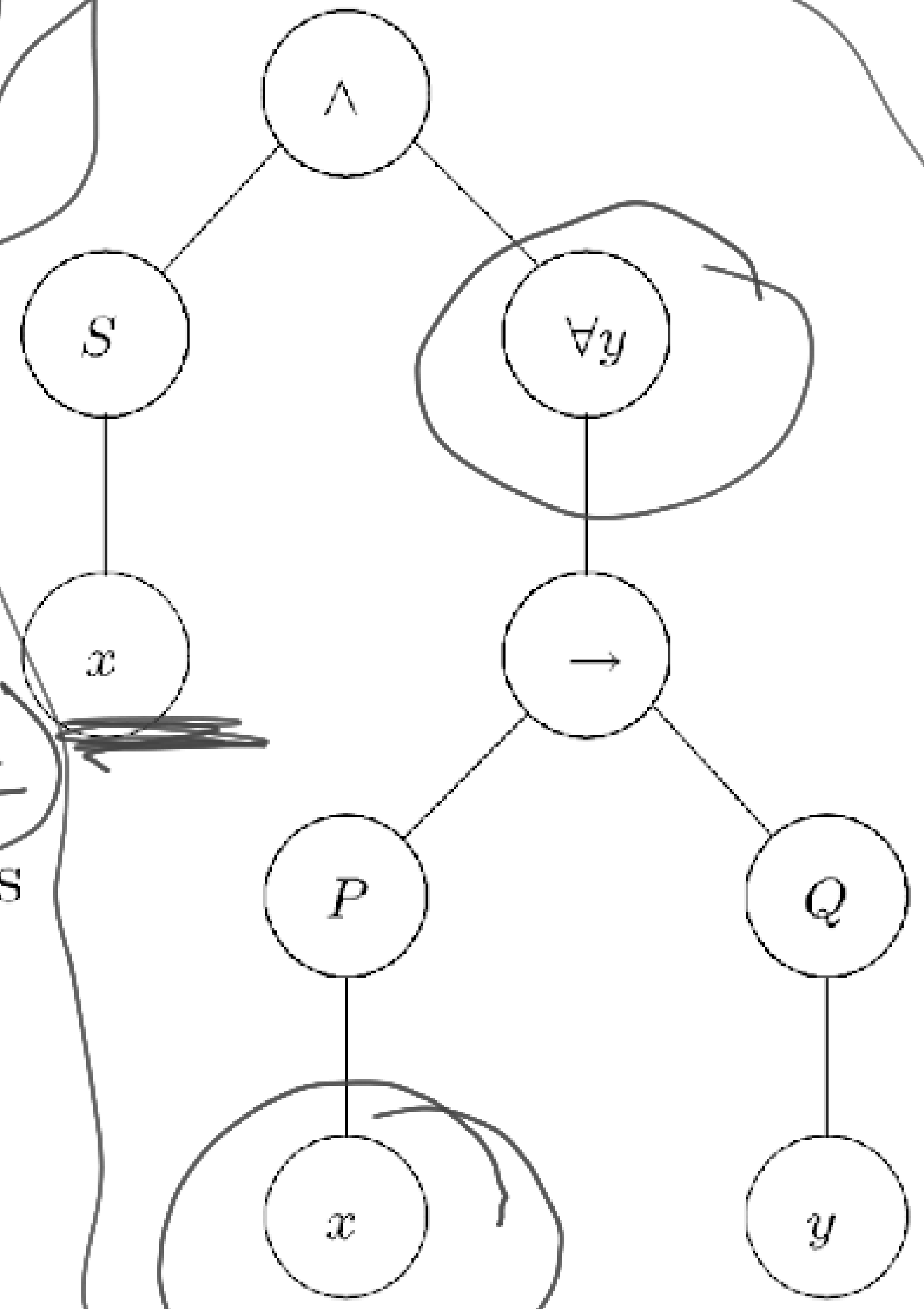
Definition 2.8 Given a term t, a variable x and a formula ϕ, we say that t is free for x in ϕ if no free x leaf in ϕ occurs in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

ϕ: $S(x) \wedge \forall y [P(x) \to Q(y)]$

suppose $f(y,y)$ is a term

$\phi[f(y,y)/x]$

the term $\cancel{f(y,y)}$ $f(z,z)$ is ~~not~~ free for x in this formula



Important to avoid undesirable capture of variables.

PROOF THEORY OF PREDICATE LOGIC ~ NATURAL DEDUCTION RULES

PROOF RULES FOR EQUALITY

EQUALS INTRODUCTION

$$\frac{}{t = t} =\mathrm{i}$$

May only be invoked if t is a term !

EQUALS ELIMINATION

$$\frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} = \text{e.}$$

Convention 2.10 Throughout this section, when we write a substitution in the form $\phi[t/x]$, we implicitly assume that t is free for x in ϕ; for, as we saw in the last section, a substitution doesn't make sense otherwise.

$\phi(y) : (y > 1) \to (y > 0)$

We obtain proof

$t_1 \quad t_2$

$\dfrac{\phi(t_1/y)}{\phi(t_2/y)}$

1	$(x+1) = (1+x)$	premise
2	$(x+1>1) \to (x+1>0)$	premise
3	$(1+x>1) \to (1+x>0)$	$=$e 1, 2

$\phi(x+1/y)$

establishing the validity of the sequent

$$x+1 = 1+x, (x+1>1) \to (x+1>0) \;\vdash\; (1+x)>1 \to (1+x)>0.$$

$$t_1 = t_2 \vdash t_2 = t_1 \tag{2.6}$$

$$t_1 = t_2,\ t_2 = t_3 \vdash t_1 = t_3. \tag{2.7}$$

t_2

A proof for (2.6) is:

$\phi(x)$ is $x = t_1$

1	$t_1 = t_2$	premise
2	$t_1 = t_1$	=i
3	$t_2 = t_1$	=e 1, 2

$\phi(t_1/x)$

$\phi(t_2/x)$

where ϕ is $x = t_1$. A proof for (2.7) is:

1	$t_2 = t_3$	premise
2	$t_1 = t_2$	premise
3	$t_1 = t_3$	=e 1, 2

$\phi(t_2/x)$

$\phi(t_3/x)$

where ϕ is $t_1 = x$, so in line 2 we have $\phi[t_2/x]$ and in line 3 we obtain $\phi[t_3/x]$, as given by the rule =e applied to lines 1 and 2. Notice how we applied the scheme =e with several different instantiations.

∴ equality is reflexive, symmetric & transitive

Universal Quantifier Elimination

$$\frac{\forall x\, \phi}{\phi[t/x]}\ \forall x\,\mathrm{e}.$$

Universal Quantifier Introduction

x_0 is a "fresh" variable.

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad\phi[x_0/x] \end{array}}}{\forall x\, \phi}\ \forall x\,\text{i.}$$

proof of the sequent $\forall x\,(P(x) \to Q(x)),\ \forall x\, P(x)\ \vdash\ \forall x\, Q(x)$:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\forall x\, P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall x\,$e 1
4		$P(x_0)$	$\forall x\,$e 2
5		$Q(x_0)$	$\to$e 3, 4
6		$\forall x\, Q(x)$	$\forall x\,$i 3–5

$$P(t),\ \forall x\,(P(x) \to \neg Q(x))\ \vdash\ \neg Q(t):$$

1	$P(t)$	premise
2	$\forall x\,(P(x) \to \neg Q(x))$	premise
3	$P(t) \to \neg Q(t)$	$\forall x$ e 2
4	$\neg Q(t)$	$\to$e 3, 1