

CS F222: Discrete Structures for Computer Science

Tutorial - 5 (Countable Sets and Relations)

1. If $A_1, A_2, A_3 \dots$, is a collection of countably many countably infinite sets, $\bigcup_{i=1}^{\infty} A_i$ is also a countable set.
2. Show that the set of all finite binary sequences/string is countable.
3. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable. (\mathbf{Z}^+ is the set of positive integers.)
4. The set S of all infinite binary sequences/strings is uncountable.
5. The power set of any infinite set is uncountable.

6. Consider the following relations on $\{1, 2, 3, 4\}$

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R_6 = \{(3, 4)\}.$$

Which of the above relations are reflexive, symmetric, antisymmetric, and transitive.

7. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
 - (a) everyone who has visited Web page a has also visited Web page b .
 - (b) there are no common links found on both Web page a and Web page b .
 - (c) there is at least one common link on Web page a and Web page b .
 - (d) there is a Web page that includes links to both Web page a and Web page b .

$$A_1, A_2$$
$$Z^+ = \frac{A_1, A_2, A_3, \dots}{A_i = \langle a_{i1}, a_{i2}, \dots \rangle}$$
$$\begin{matrix} & a_{11} & a_{21} & a_{31} & \dots & a_{k1} & a \\ \cup A_i & & & & & & \\ i=1 & \underline{A_1} & A_2 & \dots & A_k & & \end{matrix}$$

Tutorial - 5 (Countable Sets and Relations)

- $(4,1) \rightarrow (4,2)$ $\neq 2$

Which of the above relations are reflexive, symmetric, antisymmetric, and transitive?

7. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive.

- and/or transitive, where $(a, b) \in R$ if and only if

- (b) there are no common links found on both Web page a and Web page b .

- (c) there is at least one common link on Web page a and Web page b .

- (d) there is a Web page that includes links to both Web page a and Web page b .

- $$1 = 2 = 4 = 8 = 16 = 32 = 64 = 128 = 256 = 512 = 1024 = 2048 = 4096 = 8192 = 16384 = 32768 = 65536 = 131072 = 262144 = 524288 = 1048576 = 2097152 = 4194304 = 8388608 = 16777216 = 33554432 = 67108864 = 134217728 = 268435456 = 536870912 = 1073741824 = 2147483648 = 4294967296 = 8589934592 = 17179869184 = 34359738368 = 68719476736 = 137438953472 = 274877906944 = 549755813888 = 1099511627776 = 2199023255552 = 4398046511104 = 8796093022208 = 17592186044416 = 35184372088832 = 70368744177664 = 140737488355328 = 281474976710656 = 562949953421312 = 1125899906842624 = 2251799813685248 = 4503599627370496 = 9007199254740992 = 18014398509481984 = 36028797018963968 = 72057594037927936 = 144115188075855872 = 288230376151711744 = 576460752303423488 = 1152921504606846976 = 2305843009213693952 = 4611686018427387904 = 9223372036854775808 = 18446744073709551616 = 36893488147419103232 = 73786976294838206464 = 147573952589676412928 = 295147905179352825856 = 590295810358705651712 = 1180591620717411303424 = 2361183241434822606848 = 4722366482869645213696 = 9444732965739290427392 = 18889465931478580854784 = 37778931862957161709568 = 75557863725914323419136 = 151115727451828646838272 = 302231454903657293676544 = 604462909807314587353088 = 1208925819614629174706176 = 2417851639229258349412352 = 4835703278458516698824704 = 9671406556917033397649408 = 19342813113834066795298816 = 38685626227668133590597632 = 77371252455336267181195264 = 154742504910672534362390528 = 309485009821345068724781056 = 618970019642690137449562112 = 1237940039285380274899124224 = 2475880078570760549798248448 = 4951760157141521099596496896 = 9903520314283042199192993792 = 19807040628566084398385987584 = 39614081257132168796771975168 = 79228162514264337593543950336 = 158456325028528675187087900672 = 316912650057057350374175801344 = 633825300114114700748351602688 = 1267650600228229401496703205376 = 2535301200456458802993406410752 = 5070602400912917605986812821504 = 10141204801825835211973625643008 = 20282409603651670423947251286016 = 40564819207303340847894502572032 = 81129638414606681695789005144064 = 162259276829213363391578010288128 = 324518553658426726783156020576256 = 649037107316853453566312041152512 = 1298074214633706907132624082305024 = 2596148429267413814265248164610048 = 5192296858534827628530496329220096 = 10384593717069655257060992658440192 = 20769187434139310514121985316880384 = 41538374868278621028243970633760768 = 83076749736557242056487941267521536 = 166153499473114484112975882535043072 = 332306998946228968225951765070086144 = 664613997892457936451903530140172288 = 1329227995784915872903807060280344576 = 2658455991569831745807614120560689152 = 5316911983139663491615228241121378304 = 10633823966279326983230456482242756608 = 21267647932558653966460912964485513216 = 42535295865117307932921825928971026432 = 85070591730234615865843651857942052864 = 170141183460469231731687303715884105728 = 340282366920938463463374607431768211456 = 680564733841876926926749214863536422912 = 1361129467683753853853498429727072845824 = 2722258935367507707706996859454145691648 = 5444517870735015415413993718908291383296 = 10889035741470030830827987437816582766592 = 21778071482940061661655974875633165533184 = 43556142965880123323311949751266331066368 = 87112285931760246646623899502532662132736 = 174224571863520493293247799005065324265472 = 348449143727040986586495598010130648530944 = 696898287454081973172991196020261297061888 = 1393796574908163946345982392040522594123776 = 2787593149816327892691964784081045188247552 = 5575186299632655785383929568162090376495104 = 11150372599265311570767859136324180752990208 = 22300745198530623141535718272648361505980416 = 44601490397061246283071436545296723011960832 = 89202980794122492566142873090593446023921664 = 178405961588244985132285746181186892047843328 = 356811923176489970264571492362373784095686656 = 713623846352979940529142984724747568191373312 = 1427247692705959881058285969449495136382746624 = 2854495385411919762116571938898990272765493248 = 5708990770823839524233143877797980545530986496 = 11417981541647679048466287755595961091061972992 = 22835963083295358096932575511191922182123945984 = 45671926166590716193865151022383844364247891968 = 91343852333181432387730302044767688728495783936 = 1826877046$$

$$(a,b) \in R, (b,a) \in R$$
$$\Rightarrow \underline{a = b}$$

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$$f: A \rightarrow \mathbb{Z}^+ ; f \text{ is one-one and onto}$$
$$A = \{a_1, a_2, \dots\}$$

$$Z^+ = \{1, 2, \dots\}$$

Consider this A_i , let a_{i1}, a_{i2}, \dots be the ordering of elements in A_i

From each A_i , I will collect all elements
sum of indices is equal to j , where $j=2, 3, 4, \dots$

when $j=2$, $j=3$, $j=4$
 a_{11} , a_{12}, a_{21} , a_{13}, a_{22}, a_{31} ...

within each such collection (for every a_{ij}, a_{rs} , $i+j = r+s$)

the elements a_{ij} are listed such that a_{ij} is before a_{rs} if $i < r$.

The elements in $\bigcup_{i=1}^{\infty} A_i$ are ordered as $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{24}, \dots$. This is not lexicographic order.

Therefore, $\bigcup_{i=1}^{\infty} A_i$ is countable.

(Q2) - S.T. the set of all finite binary strings is countable.

fix the length of string to be n .

The set S_0 contains all strings of length 0.

$$s_1, \dots, s_n$$
$$S_K \quad K$$

The ordering

$$S_0, S_1, S_2, \dots$$

$\cup_{k=0}^{\infty} U_{S_k}$ is the set of all strings and is countable.

Q3. s.t. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable (\mathbb{Z}^+ is the set of all positive integers)

$$Z^+ \times Z^+ = \{(i, j) \mid i \in Z^+ \text{ and } j \in Z^+\}$$

To use the above result, define $S_K = \{(i, j) \mid i+j=K\} \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$.

For each $k=1, 2, \dots$, the set S_k is finite and hence countable.

clearly, we have the ordering below,

$$S_2, S_3, S_4, \dots,$$

Therefore, $\bigcup_{k=0}^{\infty} S_k$ is countable. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

S.T. $\mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is countable by showing that the polynomial function $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ with $f(m, n) = \frac{(m+n-2)(m+n-1)}{2} + m$, is one-to-one and onto.

$f(1,1) = 1$ $f(2,1) = 3$ $f(3,1) = 6$ $f(4,1) = 10$ $f(5,1) = 15$
 $f(1,2) = 2$ $f(2,2) = 5$ $f(3,2) = 9$ $f(4,2) = 14$
 $f(1,3) = 4$ $f(2,3) = 8$ $f(3,3) = 13$
 $f(1,4) = 7$ $f(2,4) = 12$
 $f(1,5) = 11$

The function takes on successive values as $m+n$ increases.

when $m+n=2$, $f(m,n)=1$, when $m+n \leq 3$, $f(m,n)=2$ or 3

The range of values the \ln takes on a fixed value of $m+n$, say $m+n=x$.

ü $(x-2)(x-1)$ through $(x-2)(x-1)$

... $m+n=2$, $m+n=1$, with $m+n=0$, $m+n=-1$ or -2 .

The range of values the fn takes on a fixed value of $m+n$, say $m+n=x$,

is $\frac{(x-2)(x-1)}{2} + 1$ through $\frac{(x-2)(x-1)}{2} + (x-1)$

we need to show that the range of values for $x+1$ picks up precisely when the range of values for x left off i.e.,

$$\begin{aligned} f(x-1,1)+1 &= f(1,x) \\ &= \frac{(x-2)(x-1)}{2} + (x-1) + 1 = \frac{x^2-x+2}{2} = \frac{(x-1)x}{2} + 1 = f(1,x) \end{aligned}$$

$\therefore f$ is one-one and onto. and hence $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Q4 S.T. the set of all infinite binary strings is uncountable.

The proof is contradiction. Suppose that S is countable. Since S is an infinite set, there is an infinite sequence b_1, b_2, b_3, \dots that contain every element of S .

Here, each of b_i is of infinite length.

$$b_i = b_{i1}, b_{i2}, b_{i3}, \dots$$

$$\left\{ \begin{array}{l} b_1 = b_{11} \ b_{12} \ b_{13} \ \dots \\ b_2 = b_{21} \ b_{22} \ b_{23} \ \dots \\ b_3 = b_{31} \ b_{32} \ b_{33} \ \dots \\ \vdots \end{array} \right.$$

Consider the following binary string of infinite length.

$$t = t_1 t_2 t_3 \dots, \text{ where } t_i = 1 - b_{ii} \text{ for } i=1, 2, \dots$$

Claim: $t \notin b_i$ listed in the above ordering.

because i -bit of t differs with the i -bit of b_i .

The string t is not appearing in the above ordering, which is contradiction.

$\therefore S$ is uncountable.

Cantor's Diagonalization method

$$\begin{array}{ccccccc} 0. & d_{11} & d_{12} & \dots & \dots & \dots & \dots \\ 0. & d_{21} & d_{22} & \dots & \dots & \dots & \dots \\ & \vdots & \vdots & & & & \end{array}$$

$$\underbrace{1 \dots 2 \dots}_{\text{infinite}}$$

87. (a) Reflexive and Transitive
(b) Symmetric
(c) Symmetric
(d) Symmetric