CS F214 Logic in Computer Science,
I Semester 2021-2022

Mid-semester Exam Solutions

1. In a DNF, it is straightforward to test for
- Satisfiability
- Unsatisfiability

2. Every satisfiable formula is
- Valid
- Not necessarily valid
- Not valid.

3. To check validity of a CNF formula with n atomic propositions, one needs to evaluate it on how many valuations of the formula
- n
- 2^n
- log n
- 0

4. The negation of every unsatisfiable formula is
- valid
- not necessarily valid
- not valid

5. A contradiction is
- a proposition.
- NOT a proposition.

6. Given $p \to q$, the proposition $\neg q \to \neg p$ is called the
- Converse
- Contrapositive
- Inverse
- Reverse

7. Given $p \rightarrow q$, $q \rightarrow p$ is called its
- Contrapositive
- Converse
- Inverse
- Reverse

8. For a general CNF formula,
- There is an efficient algorithm to determine Satisfiability
- No efficient algorithm is known to determine Satisfiability

9. Every valid sequent has the corresponding semantic entailment relation to hold. This theorem is called
- Soundness
- Completeness
- Soundness & Completeness

10. The negation of a CNF gives us a
- CNF
- DNF
- None of the above

11. If p is false, the proposition $p \rightarrow q$ is
- Always true
- Always false
- Depends on the value of q

12. Every valid formula is
- satisfiable
- not necessarily satisfiable
- Unsatisfiable

13. From a contradiction, one can infer
- No proposition
- Any proposition

Part B1 - 7 mark questions

1.Write a proof of the following lemma.
Lemma. A disjunction of literals L1 ∨ L2 ∨ · · · ∨ Lm is valid if and only if there are1≤i,j≤m such that Li is ¬Lj .

If we have L1 ∨ . . . ∨ Lk ∨ . . . ∨ ¬Lk ∨ . . . ∨ Ln for some 1 <= k <= n then any valuation will make the clause true.
[L1 ∨ . . . ∨ Lk ∨ . . . ∨ ¬Lk ∨ . . . ∨ Ln] = T since [Lk ∨ ¬Lk] = T always.
Then |= L1 ∨ . . . ∨ Lk ∨ . . . ∨ ¬Lk ∨ . . . ∨ Ln holds.

Let, |= L1 ∨ L2 ∨ . . . . . . ∨ Ln hold.

Then L1 ∨ L2 ∨ . . . ∨ Ln is valid for all valuations

Assume no literal L_i is the negation of another literal L_j.

Define a valuation v' such that v'(p) = F if p is the literal L_k and v'(p) = T if ¬p is the literal L_k for some k.

Then valuation of L_k at v' = F for all 1<= k<= n and hence valuation of [L1 ∨ L2 ∨ . . . ∨ Ln] at v' = F which contradicts the fact that |= L1 ∨ L2 ∨ . . . . . . ∨ Ln holds.

Hence proved.

2. A. [3 marks] Prove or disprove the validity of the sequent: p∨(q∧r) ⊢ (p∨q)∧r
B. [4 marks] Prove that the following sequent is valid: p→q, r ⊢ ¬q→(¬p∧r)

SOLUTIONS:

A: The sequent is not valid. Any row of the truth table where the 2 formulas have a different valuation could serve as a counter example.

p	q	r	q^r	pVq	p∨(q∧r)	(p∨q)∧r
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	1	0	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0

1	1	1	1	1	1	1

B.

1. p→q Premise
2. r Premise
----
3. ¬q Assumption
4. ¬p MT(3,1)
5. ¬p ^r ^i(4,2)
----
6. ¬q→(¬p∧r) →i(3-5)

3. Prove that the following sequent is valid: p→q ⊢ (p∨r)→(q∨r)

1. p→q Premise

----

2. p V r Assumption
----
3. p Assumption
4. q ->i(1,3)
5. q V r Vi(4)

----
6. r Assumption
7. q V r Vi(6)
----
8. q V r Ve(3-5,6-7,2)
----
9. (p∨r)→(q∨r) →i(2-8)

4. Prove that the following sequent is valid: p∨(q∨r) ⊢ (p∨q)∨r

1. p∨(q∨r) Premise
----
2. p Assumption
3. p V q Vi(2)
4. (p∨q)∨r Vi(3)

----
5. q V r Assumption
----
6. q Assumption
7. p V q Vi(6)

8. (p V q) V r Vi(7)

----
9. r Assumption
10. (p V q) V r Vi(9)
----
11. (p V q) V r Ve(5,6-8,9-10)
----
12. (p V q) V r Ve(1,2-4,5--11)

5. Using the algorithm discussed in class, determine if the following Horn Formulas are satisfiable. Show your work.
(a) [4 marks] $(p \wedge r \to s) \wedge (T \to p) \wedge (p \to r) \wedge (p \wedge s \to T) \wedge (p \wedge r \wedge s \to \bot)$
(b) [3 marks] $(p \wedge q \to r) \wedge (T \to q) \wedge (T \to r) \wedge (q \wedge r \to p) \wedge (p \wedge r \wedge s \to \bot) \wedge (p \wedge s \to t)$

a) We start with each iteration and marking of clauses having truth value as T as true and their consequent as True.

Iteration1:

$(p \wedge r \to s) \wedge \cancel{(T \to p)} \wedge (p \to r) \wedge (p \wedge s \to T) \wedge (p \wedge r \wedge s \to \bot)$ [p => T]

Iteration 2:

$(p \wedge r \to s) \wedge \cancel{(p \to r)} \wedge (p \wedge s \to T) \wedge (p \wedge r \wedge s \to \bot)$ [p,r => T]

Iteration 3:

$\cancel{(p \wedge r \to s)} \wedge (p \wedge s \to T) \wedge (p \wedge r \wedge s \to \bot)$ [p,r,s => T]

Iteration 4:

$\cancel{(p \wedge s \to T)} \wedge \cancel{(p \wedge r \wedge s \to \bot)}$

In this iteration, $\bot$ gets marked as T, hence it is unsatisfiable

b) We start with each iteration and marking of clauses having truth value as T as true and their consequent as True.

Iteration1:

$(p \wedge q \to r) \wedge \cancel{(T \to q)} \wedge \cancel{(T \to r)} \wedge (q \wedge r \to p) \wedge (p \wedge r \wedge s \to \bot) \wedge (p \wedge s \to t)$ [q,r => T]

Iteration 2 :

$(p \wedge q \to r) \wedge \cancel{(q \wedge r \to p)} \wedge (p \wedge r \wedge s \to \bot) \wedge (p \wedge s \to t)$ [q,r,p => T]

Iteration 3:

$\cancel{(p \wedge q \to r)} \wedge (p \wedge r \wedge s \to \bot) \wedge (p \wedge s \to t)$

Iteration 4:

$(p \wedge r \wedge s \to \bot) \wedge (p \wedge s \to t)$

All the iterations are over and ⊥ is unmarked. Hence satisfiable

6. For the following truth table, write an equivalent formula in Disjunctive Normal Form (DNF) and an equivalent formula in Conjunctive Normal Form (CNF).

p	**q**	**r**	**Ψ**
F	**F**	**F**	**T**
F	**F**	**T**	**T**
F	**T**	**F**	**T**
F	**T**	**T**	**T**
T	**F**	**F**	**T**
T	**F**	**T**	**T**
T	**T**	**F**	**F**
T	**T**	**T**	**T**

DNF

= Row 1 + Row 2 + Row 3 + Row 4 + Row 5 + Row 6 + Row 7

= (¬p ^ ¬q ^ ¬r) V (¬p ^ ¬q ^ r) V (¬p ^ q ^ ¬r) V (¬p ^ q ^ r) V (p ^ ¬q ^ ¬r) V (p ^ ¬q ^ r) V (p ^ q ^ r)

CNF

= ¬(Row 6)

= ¬(p ^ q ^ ¬r)

= ¬p V ¬q V r

Part B2 - 10 mark questions

Q1 Prove that the following sequent is valid: $(p \to q) \to (p \land \neg q \land r),\ q \lor \neg p \vdash \neg q \to (p \land \neg r)$

1. $(p \to q) \to (p \land \neg q \land r)$ premise
2. $q \lor \neg p$ premise

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3. q Assumption

---

4. $\neg q$ Assumption
5. $\bot$ ¬e 3,4
6. $(p \land \neg r)$ ⊥e 5

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7. $\neg q \to (p \land \neg r)$ ->i 4-6

---

8. $\neg p$ Assumption

---

9. $\neg q$ Assumption
10. $\neg p$ Copy 8

---

11. $\neg q \to \neg p$ -> i 9-10

---

12. p Assumption
13. $\neg\neg p$ ¬¬i 12
14. $\neg\neg q$ MT 11,13
15. q ¬¬e 14

---

16. $p \to q$ ->i 12-15
17. $(p \land \neg q \land r)$ -> e 1,16
18. p $\land e_1$ 17
19. $\bot$ ¬e 8,18
20. $\neg q \to (p \land \neg r)$ ⊥e 19

---

21. $\neg q \to (p \land \neg r)$ Ve 2, 3-7, 8-20

22. **Use Mathematical induction to show that n^2 +7n is even, for all positive integers n.**

Let $P(n) = n^2+7n$ is even

Base case:
$n = 1$: $P(1) = 8$ which is even

Induction step:

Let us assume it is true for some m

Therefore

$P(m) = m^2+7m$ = even = $2k$ where k belongs to N

We need to prove it for m+1

$P(m+1)$

$= (m+1)^2+7(m+1)$

$= m^2 + 2m + 1 + 7m + 7$

$= (m^2 + 7m) +2m + 8$

$= 2k + 2(m + 4)$

$= 2k + 2k'$ where $k' = m+4$

$= 2(k+k')$ = even

Therefore it is true for m+1

Therefore it is true for all positive integers n.

Part B3 - 15 mark questions

1. In class, we have seen one class of propositional formulas for which it is straightforward to check the validity and another class for which it is straightforward to check satisfiability. In this question, we will consider a class of propositional formulas for which it is straightforward to do both.

Consider BITS formulas — a subclass of the class of CNF formulas — which have at most one positive literal per clause. Formally, these are formulas that can be generated as an instance of B in the grammar below, which is specified in Backus Naur Form:

P ::= p
N ::= ¬p
M ::= N | M ∨ N
C ::= P | M | M ∨ P
B ::= C | B ∧ C

(a) [1 mark] Outline an efficient* algorithm to determine the validity of a BITS formula.

(b) [14 marks] Describe an efficient* algorithm to determine if a BITS formula is satisfiable. (If you are using any equivalence(s) in your algorithm, you must provide natural deduction proof(s) for them.)

***Here, efficient algorithm means any algorithm that, in the worst case, takes a number of steps that is not at least some exponential function in the number of atomic variables (or atoms) in the formula. e.g. evaluating the propositional formula for all valuations takes exponentially many steps in the number of atomic variables.**

SOLUTION SKETCH:

(a) Since this is still a formula in CNF, the algorithm to check validity of CNF formulas that we discussed in class applies here. For each clause, one needs to check if there is at least one atom p, so that both p and its negation are present in the clause. If every clause has this property, the formula is valid; else it is not.

(b) An important observation here is that any BITS formula can be converted to a Horn formula. This is via DeMorgan's law, $\neg p \vee \neg q \equiv \neg(p \wedge q)$ and the equivalence $\neg p \vee q \equiv p \rightarrow q$. However, 2 special cases need to be dealt with. The clauses L with just negated literals will be written as L $\vee$ $\bot$ and then bottom appears as the consequent of the Horn clause. The clause where there is only one positive literal p and no negative literal, should be written as $\neg \top \vee p$ and then the Horn clause becomes $\top \rightarrow p$. After this, one can apply the algorithm discussed in class to determine whether a Horn formula is satisfiable.

2. In Shakespeare's *Merchant of Venice* Portia had three caskets-gold, silver, and lead-inside one of which was Portia's portrait. The suitor was to choose one of the caskets, and if he was lucky enough (or wise enough) to choose the one with the portrait, then he could claim Portia as his bride. On the lid of each Casket was an inscription to help the suitor choose wisely.

Portia had the following inscriptions put on the Caskets.

GOLD	SILVER	LEAD
THE PORTRAIT IS NOT IN THE SILVER CASKET	THE PORTRAIT IS NOT IN THIS CASKET	THE PORTRAIT IS IN THIS CASKET

Portia explained to the suitors that at least one of the three statements was true and at least one of them was false. Which casket contains the portrait? Put this puzzle in formal propositional logic and solve it using a truth table.

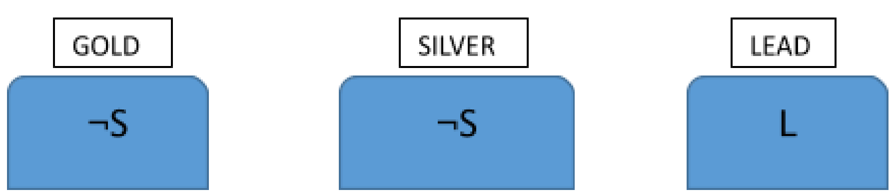
Solution: Let's use the following propositional logic atoms:

G: means portrait is in the GOLD case

S: means portrait is in the SILVER case

L: means portrait is in the LEAD case

So the inscription will now read as



Now, one of the casket contains the portrait. So that can be coded as

$(G \wedge \neg S \wedge \neg L) \vee (\neg G \wedge S \wedge \neg L) \vee (\neg G \wedge \neg S \wedge L)$ --------------(1)

The other fact is that at least one of the three statements is true and at least one of them was false can be formalized as:

$(\neg S \wedge \neg\neg S) \vee (\neg S \wedge \neg L) \vee (\neg\neg S \wedge \neg S) \vee (\neg S \wedge \neg L) \vee (L \wedge \neg\neg S) \vee (L \wedge \neg\neg S)$

which is equivalent to

$(\neg S \wedge \neg L) \vee (L \wedge S)$ ------------------------(2)

So, if we look into the Truth Table:

G	S	L	(1)	(2)
T	T	T	F	T
T	T	F	F	F
T	F	T	F	F
T	**F**	**F**	**T**	**T**
F	T	T	F	T
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Only in the highlighted red row, both (1) and (2) are True. Hence, the portrait is in the gold casket.

NOTE: If you simply argued verbally, without formulating it as a propositional logic formula, as done above, you would have lost about half the points.