Tutorial 7

Prove that the following sequent is not valid:

$$\neg p \vee (q \rightarrow p) \vdash (\neg p \wedge q).$$

Solution :

For the valuation with p = T and q = T,

Premise :

p	q	¬p	q→p	¬p∨(q→p)
T	T	F	T	T

Conclusion :

p	q	¬p	¬p∧q
T	T	F	F

Since the premise evaluates to T and the conclusion evaluates to F for the same valuation, the sequent is not valid.

Construct a formula in CNF based on each of the following truth table:

p	q	r	ϕ_2
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

Solution :

The CNF formula equivalent to the given truth table is

$(\neg p \lor \neg q \lor r) \land (\neg p \lor q \lor \neg r) \land (\neg p \lor q \lor r) \land (p \lor \neg q \lor r) \land (p \lor q \lor r)$

(Can also be verified by constructing the truth table of the CNF formula and comparing with the given truth table)

Apply the algorithm discussed in class and determine if the Horn formula below is satisfiable. Go through the steps as outlined in the algorithm.

$$(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (\top \rightarrow r) \wedge (\top \rightarrow q) \wedge (u \rightarrow s) \wedge (\top \rightarrow u)$$

Solution :

The marking algorithm first marks r, q and u through the clauses of the form $\top \rightarrow \cdot$. Then it marks p through the clause $r \rightarrow p$ and s through the clause $u \rightarrow s$. The w never gets marked and so $p \wedge q \wedge w$ can never trigger a marking for $\bot$. Thus the algorithm determines that the formula is satisfiable

Apply the algorithm discussed in class and determine if the Horn formula below is satisfiable. Go through the steps as outlined in the algorithm.

$$(\top \rightarrow q) \wedge (\top \rightarrow s) \wedge (w \rightarrow \bot) \wedge (p \wedge q \wedge s \rightarrow v) \wedge (v \rightarrow s) \wedge (\top \rightarrow r) \wedge (r \rightarrow p)$$

Solution :

Applying the marking algorithm to the given horn formula, it marks q, s and r in the first iteration of the while loop. In the second iteration p gets marked and in the third iteration an inconsistency is found ($p \wedge q \wedge s \rightarrow \bot$) is a subformula of the original formula and p,q,s are all marked. Thus the given Horn formula is not satisfiable.

Consider an arrangement of a finite set of line segments on the plane (assume that no two line segments overlap). Such an arrangement divides the plane into regions. Prove using induction that you can colour these regions using two colours such that two regions sharing a common boundary are of different colours.

Solution :

For simplicity, let us assume that the 2 colours we want to use are red and blue.

We want to prove inductively that:
$\forall$n (The plane is divided by n lines) $\Rightarrow$ (the regions can be coloured using blue and red such that no two adjacent regions have the same colour)

Note that two regions are adjacent if and only if they share an edge.

Base case :
The plane is divided by 0 lines =>we can colour the only region blue=> we can colour it using blue and red such that no two adjacent regions have the same colour

Inductive Hypothesis:
(The plane is divided by n lines) $\Rightarrow$ (the regions can be coloured using blue and red such that

no two adjacent regions have the same colour)

Solution (Continued) :

Induction:

Given any n+1 lines dividing the plane, we can remove a line in the plane to make a total of n lines in the plane. Therefore the arrangement of n+1 lines can be formed as such from an arrangement of n lines by adding the desired line to make an arrangement of n+1 lines in the plane. Let us start with our initial n lines. By the Induction Hypothesis, we can color the regions such that no two adjacent regions have the same colour. Colour them to achieve this. No two adjacent regions have the same colour if and only if all edges obey predicate P:
P(s): At any point on edge s, the color to that point's right is different from the color to that point's left.
Now, add the desired line, let's call it line λ. Invert all the colors on the right of that line (blue becomes red and red becomes blue). We know that prior to the inversion all such edges obeyed predicate P. After the inversion, each edge to the right of λ had the colours on both its right and its left inverted, which implies that the colours remained different! (Blue-Red became Red-Blue). Edges to the left of λ were not affected, and still obey predicate P.

Hence, given the induction hypothesis, we have proved that an arbitrary arrangement of n+1 lines can be colored such that no two adjacent regions have the same color: (The plane is

divided by n+1 lines) ⇒ (the regions can be coloured using blue and red such that no two adjacent regions have the same colour)

Conclusion:

Given the basis case and the induction, we conclude that:
∀n (The plane is divided by n lines) ⇒ (the regions can be coloured using blue and red such that no two adjacent regions have the same colour)