BITS Pilani Hyderabad Campus
CS F214 Logic in Computer Science,
I Semester 2021-2022
Lecture Notes
Lecture 15-16

13.1 Soundness of Propositional Logic

Theorem: *Let $\phi_1, \phi_2, ..., \phi_n$ and ψ be propositional logic formulas.*
If $\phi_1, \phi_2, ..., \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, ..., \phi_n \vDash \psi$ holds.

Proof: Since $\phi_1, \phi_2, ..., \phi_n \vdash \psi$ is valid, we know that there is a proof of ψ from the premises $\phi_1, \phi_2, ..., \phi_n$.

We will do course of values induction on the length of this proof (the numbering lines in it).

$M(R)$: For all the sequents $\phi_1, \phi_2, ..., \phi_n \vdash \psi, n \geq 0$, which have a proof of length R. It is the case that $\phi_1, \phi_2, ..., \phi_n \vDash \psi$ holds.

We will show $M(k)$ is true for $k \in N$ by course of values induction.
Example: Consider $p \wedge q \rightarrow r \vdash p \rightarrow q \rightarrow r$.

Solution:

1.$p \wedge q \rightarrow r$ premise
2.p assumption
3.q assumption
4.$p \wedge q$ $\wedge$,2,3
5.r $\rightarrow$e 4,1
6.$q \rightarrow r$ $\rightarrow$i 3-5
7.$p \rightarrow (q \rightarrow r)$ $\rightarrow$i 2-6

1. **Base Case**: We wish to show M(1) is true.This is an example of sequent with one line proof.

   1.ϕ premise

   Above is a proof that shows $\phi \vdash \phi$.
   $\phi \vDash \phi$ hols because whenever ϕ is true, ϕ is true. $\vdash \phi \vee \neg\phi$

   1.$\phi \vee \neg\phi$ LEM

$\vDash \phi \vee \neg\phi$ For every valuation for ϕ is true, $\neg\phi$ is false. Therefore, $\phi \vee \neg\phi$ is true. Similarly, for every valuation for ϕ is false, $\neg\phi$ is true.
$\therefore$ for every valuation $\phi \vee \neg\phi$ is true.
$\therefore \vDash \phi \vee \neg\phi$ holds.

2. **Inductive Step**: We assume that shortest proof of the sequent $\phi_1, \phi_2, ..., \phi_n \vdash \psi$ is of length K. We assert the inductive hypothesis of all sequents that have a proof of length $<$ K.

A proof has the structure

$$\begin{array}{rr} 1.\ \phi_1 & \text{premise} \\ & \text{.....} \\ n.\ \phi_n & \text{premise} \\ K.\ \psi & \text{justification} \end{array}$$

Two issues the proof needs to deal with.

(a) What happens in between. (We hope this will be solved by induction hypothesis).

(b) What is the last rule?(Proof needs to consider all such cases).

Cases (corresponding to which was the last rule applied)

(a) Consider $\wedge i$ to be the last rule applied.
ψ has to be of the form $\psi_1 \wedge \psi_2$ citing lines K_1 and K_2 respectively.
K_1 and K_2 ¡ K.
Lines 1-K_1 constitute a proof of the sequent. $\phi_1, \phi_2, ..., \phi_n \vdash \psi_1$. Likewise, Lines 1-$K_2$ constitute a proof of the sequent. $\phi_1, \phi_2, ..., \phi_n \vdash \psi_2$.

Induction hypothesis is that $M(1), .., M(k-1)$ is true. By the induction hypothesis $\phi_1, \phi_2, ..., \phi_n \vDash \psi_1$ holds and $\phi_1, \phi_2, ..., \phi_n \vDash \psi_2$ holds.

$$1.\phi_1, \phi_2, ..., \phi_n \vDash \psi_1 2.\phi_1, \phi_2, ..., \phi_n \vDash \psi_2$$