Tutorial 5 Solutions

Let $*$ be a new logical connective such that $p * q$ does not hold iff p and q are either both false or both true.

(a) Write down the truth table for $p * q$.

(b) Write down the truth table for $(p * p) * (q * q)$.

a)

p	q	p*q
T	T	F
T	F	T
F	T	T
F	F	F

b)

p	q	p*p	q*q	(p*p)*(q*q)
T	T	F	F	F
T	F	F	F	F
F	T	F	F	F
F	F	F	F	F

Prove that

$$(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + \cdots + (2 \cdot n - 1) = n^2$$

by mathematical induction on $n \geq 1$.

- Let P(n) denote the statement (2.1-1)+(2.2-1)+......(2.n-1)=n^2
- Base Case: P(1) is true because 2.1-1=2-1=1=1^2
- Inductive Step: Let us assume that for a natural number n>1 P(n) holds.(inductive hypothesis)

  Then LHS of P(n+1)= (2.1-1)+(2.2-1)+......(2.n-1)+(2.(n+1)-1)= n^2 +(2.(n+1)-1)

  = n^2 +2n+2-1

  = n^2 +2n+1= $(n+1)^2$ =RHS of P(n+1)

  Thus P(n+1) holds

Therefore by the principle of mathematical induction P(n) holds for all natural numbers n>=1.

Consider the following inductive proof to show all horses are the same color. Proof: We will induct on the number of horses. Base case will be one horse which is of course of one color. So base case holds. Now for inductive step, we show that if it is true for any group of n horses, that all have the same color, then it is true for any group of (n + 1) horses. Given any set of (n + 1) horses, if you exclude the last horse, you get a set of n horses. By inductive step these n horses all have the same color. But by excluding the first horse in the pack (n + 1) horses, you can conclude that the last n horses also have the same color by inductive step. Therefore all (n + 1)horses have the same color. Definitely something is not correct with this proof because all horses are not of same color. What is the mistake?

Ans. Consider the inductive step when N+1=2 i.e when there is a set of 2 horses, say {h1, h2}. Excluding the last horse gives just {h1} and by the inductive hypothesis the statement holds for this set. Excluding the first horse gives just {h2} and again the statement holds for this set. However, after combining the sets i.e the set {h1, h2} need not be of the same color because the first and second sets share no elements. Thus, the problem is in the inductive step. If it were possible to prove that any group of two horses has the same color we could add this to the basis to fix the proof, however this is not always true.

2. Define a set, S, of strings of a's and b's recursively as follows:

   (a) $\epsilon \in S$, where ϵ is the empty string,

   (b) if $x \in S$, then $axb \in S$,

   (c) if $x \in S$, then $bxa \in S$,

   (d) if $x, y \in S$, then $xy \in S$.

   Here xy indicate the concatenation of the strings x and y, namely, xy is the string that starts with the sequence of a's and b's in x followed by the a's and b's in y.

   Prove using induction that any string of S will have equal number of a's and b's.

- Let P(n) be the predicate that if any string s is in S formed using the repeated application of (b), (c) or (d) 'n' number of times, then s has the same number of a's and b's.
- Base Case : The string formed by using the rules (b), (c) or (d) zero times is formed using (a) i.e an empty string which has equal number of a's and b's (zero a's and zero b's). Thus P(0) is true.
- Inductive Step:

Assume P(0), P(2), P(3), ... , P(n) is true.(Inductive Hypothesis)

To prove: P(n+1) is true.

- <u>Case 1: (n+1)th rule applied is (b)</u>

The string s will be of the form s = a**x**b, where x ∈ S

String **x** was formed by the application of either of the rules (b), (c) or (d) for ‘n’ number of times. Since P(n) is true (inductive hypothesis), **x** has equal number of a’s and b’s. Consequently the number of a’s and b’s in s are also equal because s is formed by appending a single a and a single b to **x**. Therefore P(n+1) is true.

- <u>Case 2: (n+1)th rule applied is (c)</u>

The string s will be of the form s = b**x**a, where x ∈ S

String **x** was formed by the application of either of the rules (b), (c) or (d) for ‘n’ number of times. Since P(n) is true (inductive hypothesis), **x** has equal number of a’s and b’s. Consequently the number of a’s and b’s in s are also equal because s is formed by appending a single b and a single a to **x**. Therefore P(n+1) is true.

- <u>Case 3: (n+1)th rule applied is (d)</u>

The string s is of the form s = **xy**, where **x**,**y** ∈ S

Strings **x** and **y** were formed by application of rules (b), (c) or (d) less than ‘n’ number of times. Thus by the inductive hypothesis, string **x** will have equal number of a’s and b’s (say nx times) and string **y** will have equal number of a’s and b’s (say ny times).

Thus the number of a’s in s=nx+ny

number of b’s in s=nx+ny=number of a’s. Therefore P(n+1) is true.

<u>From Case 1,2 and 3 P(n+1) is true.</u>

Since P(n+1) is true whenever P(1), P(2), ... , P(n) is true, by principles of strong induction,

P(n+1) is true for all n ∈ N.

For any parse tree T of a propositional logic formula ϕ, prove using structural induction that $|nodes(T)| \leq 2^{h(T)+1} - 1$ where $h(T)$ is the height of the parse tree.

- <u>Base Case: Φ is a propositional atom</u>

Thus h(T) = 1. Hence |nodes(T)| = 1 <= 2^{1+1}-1

Hence base case is true.

- <u>Inductive step: Consider a parse tree T</u>

Let Tl be its left subtree and Tr be its right subtree.

→**|nodes(T)| = |nodes(Tl)|+|nodes(Tr)|+1** and **h(T)=max(H(Tl),H(Tr))+1**

Using structural induction on each of the sub-tree

|nodes(Tl)|<=$2^{h(Tl)+1}$-1

|nodes(Tr)|<=$2^{h(Tr)+1}$-1

→|nodes(T)| = |nodes(Tl)|+|nodes(Tr)|+1

<=$2^{h(Tl)+1}$-1+ $2^{h(Tr)+1}$-1+1

<=2($2^{\max(h(Tl),h(Tr))+1}$)-1

=2($2^{h(T)}$)-1= $2^{h(T)+1}$-1

Hence proved