BITS Pilani Hyderabad Campus
CS F214 Logic in Computer Science,
I Semester 2021-2022
Lecture Notes
Lecture 11-16

10 Well Formed Formulas

10.1 What is allowable formula ϕ?

- Formulas are string over the alphabet $\{p, q, r..\} \cup \{p_1, p_2, ...\} \cup \{\neg, \wedge, \vee, \rightarrow, (,)\}$.
- But not all strings are admissible.
  e.g $(\neg)() \wedge pqr \rightarrow$.

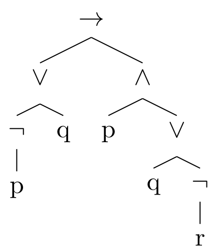
Definition : The well formed formulas of propositional logic are those which we obtain by using the construction rules below and only those by applying them finitely many times.

1. atom: Every propositional atom $p, q, r, ..$ or $p_1, p_2, p_3, ..$ is a well formed formula.
2. $\neg$: If ϕ is a well formed formula then so is $\neg\phi$.
3. $\wedge$: If ϕ and ψ are well formed formulas so is $\phi \wedge \psi$.
4. $\vee$: If ϕ and ψ are well formed formulas so is $\phi \vee \psi$.
5. $\rightarrow$: If ϕ and ψ are well formed formulas so is $\phi \rightarrow \psi$.

10.2 How do we show that a formula is well formed?

Construct in a top down manner, a parse tree where its leaves are atoms. e.g.

$$(((\neg p) \vee q) \rightarrow (p \wedge (q \vee (\neg r))))$$



- Parse trees of well formed formulas are either an atom as root or the root contains $\neg, \wedge, \vee$ or $\rightarrow$.

- In case of $\neg$, there is only one sub-tree coming out of the root. In case of $\vee, \wedge, \rightarrow$ these are forms.
- A Sub-formula of a formula corresponds to a sub-tree of the parse tree.
  - You can obtain the formula back by using In-order traversal of the parse tree.
  - In-order traversal can be obtained by recursively printing left sub-tree, printing root, printing right sub-tree.
  - Pre-order traversal can be obtained by recursively printing root, printing left sub-tree, printing right sub-tree.
  - Post-order traversal can be obtained by recursively printing left sub-tree, printing right sub-tree, printing root.

10.3 Height of a Parse Tree:

Definition: Given a well-formed formula /phi, its height is defined to be 1 plus length of the longest path in the parse tree starting from the root.

Simple example of different arithmetic notations is as follows:

- x+y: Infix Notation
- xy+: Postfix Notation
- +xy: Prefix Notation

11 Mathematical Induction

- C.F. Gauss derived the formula of $\sum_{n=1}^{n} = \frac{n(n+1)}{2}$ as a child prodigy.
- One of the ways to prove this is by the use of mathematical induction.

Suppose, we wish to show $M(n), M \in N$.

1. **Base Case**: Natural number 1 has the property i.e. we have a proof of M(1).
2. **Inductive Hypothesis**: We assume that M(n) is true.
3. **Induction Step**: Prove that if $M(n)$ is true then $M(n+1)$ is true.

Theorem: *The sum of the first n natural numbers in equal to* $\frac{n(n+1)}{2}$

Proof: Let $M(n)$: the sum of the first n natural numbers not equal to $\frac{n(n+1)}{2}$.

- Base Case: To show that $M(1)$ is true.

  The sum of first one natural number is 1.
  Furthermore $M(1)$ states that the sum of first natural number equals $\frac{1(1+1)}{2} = 1$.

  $\therefore M(1)$ is true.

- Inductive Step: Suppose $M(n)$ is true. Consider $M(n+1)$: The sum of the first $(n+1)$ natural numbers is $\frac{(n+1)(n+2)}{2}$.

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$$

By the induction hypothesis, $M(n)$ is true.

$$\sum_{i=1}^{n+1} = \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$\sum_{i=1}^{n+1} = \frac{(n+1)(n+2)}{2}$$

$\therefore M(n+1)$ is true.

12 Course of Values Induction or Strong Induction

1. **Base Case**: Natural number 1 has the property i.e. we have a proof of $M(1)$.

2. **Inductive Step**: We assume that $M(1) \wedge M(2) \wedge M(3).. \wedge M(n)$ is true and show that $M(n+1)$ is true.

Statements on parse trees are often shown by strong induction on height.
It is also called as structural Induction.

Theorem: *For every well-formed proposition logic formula, the number of left brackets equal the number of right brackets.*

Proof: Course of values induction on the height og the parse tree corresponding to the well-formed formula.

Let $M(n)$: All formulas of height n, have the same number of left and right brackets.

- **Base Case**: We will show that $M(1)$ is true.
  A parse tree of height 1 has only an atom and no bracket. Hence $M(1)$ is true.

- **Inductive Step**: Suppose $M(1), M(2), ...M(n)$ is true, we will show that $M(n+1)$ is true.
  A parse tree of height ≥ 2 has as its root either of $\neg, \vee, \wedge, \rightarrow$. Suppose the root as $\neg$. Then the sub-tree rooted at $\neg$ is of height n.
  By the induction hypothesis property as true for the formula ϕ corresponding to that of sub-tree. The formula corresponding to the full tree is $(\neg\phi)$, which has equal number of left and right brackets. Since we added one left bracket and one right bracket.

  Case (2): Let the root be $\vee$. If this i so, there exist two well formed formulas ϕ_1 and ϕ_2 so that the present formula is $(\phi_1 \vee \phi_2)$. The parse trees corresponding to the formulas ϕ_1 and ϕ_2 have height less than or equal to n.
  $\therefore$ by the induction hypothesis both these parse trees have equal number of left and right brackets each.
  $\therefore$ $(\phi_1 \vee \phi_2)$ has equal number of left and right brackets. Since this adds one more left (right) bracket to the sum of left brackets of $(\phi_1 \vee \phi_2)$.

  Case (3), (4) correspond to the binary connectives $\wedge, \rightarrow$, for which the argument is similar.

13 Semantics of Propositional Logic

Def 1: *The set of truth values contains two elements - T and F, where T represents 'true' and F corresponds to 'false'.*

Def 2: *The valuation on model of formula ϕ is an assignment of each propositional atom in ϕ to a truth value. A truth table lists all valuations of a formula ϕ.*

Do all valid sequents preserve truth computed by our truth table semantics?

Def: *If for all the valuations in which all of $\phi_1, \phi_2, ..., \phi_n$ evaluate to T and formula χ also evaluates to T, we say that (the semantic entailment relation) $\phi_1, \phi_2, ..., \phi_n \vDash \chi$ holds and we call $\vDash$ as the semantic entailment relation.*

Examples:

1. $p \wedge q \vDash p$ holds.

p	q	$p \wedge q$	p	$p \vee q$
F	F	F	F	F
F	T	F	F	T
T	F	F	T	T
T	T	T	T	T

2. $p \vee q \vDash p$ does not holds.

3. $p \vDash q \vee \neg q$ holds.

13.1 Soundness of Propositional Logic

Theorem: *Let $\phi_1, \phi_2, ..., \phi_n$ and ψ be propositional logic formulas.*
If $\phi_1, \phi_2, ..., \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, ..., \phi_n \vDash \psi$ holds.

Proof: Since $\phi_1, \phi_2, ..., \phi_n \vdash \psi$ is valid, we know that there is a proof of ψ from the premises $\phi_1, \phi_2, ..., \phi_n$.

We will do course of values induction on the length of this proof (the numbering lines in it).

$M(R)$: For all the sequents $\phi_1, \phi_2, ..., \phi_n \vdash \psi, n \geq 0$, which have a proof of length R. It is the case that $\phi_1, \phi_2, ..., \phi_n \vDash \psi$ holds.

We will show $M(k)$ is true for $k \in N$ by course of values induction.
Example: Consider $p \wedge q \to r \vdash p \to q \to r$.

Solution:

1.$p \wedge q \to r$ premise
2.p assumption
3.q assumption
4.$p \wedge q$ $\wedge$,2,3
5.r $\to$e 4,1
6.$q \to r$ $\to$i 3-5
7.$p \to (q \to r)$ $\to$i 2-6

1. **Base Case**: We wish to show M(1) is true.This is an example of sequent with one line proof.

$$1.\phi \quad \text{premise}$$

Above is a proof that shows $\phi \vdash \phi$.
$\phi \vDash \phi$ hols because whenever ϕ is true, ϕ is true. $\vdash \phi \vee \neg\phi$

$$1.\phi \vee \neg\phi \quad \text{LEM}$$

$\vDash \phi \vee \neg\phi$ For every valuation for ϕ is true, $\neg\phi$ is false. Therefore, $\phi \vee \neg\phi$ is true. Similarly, for every valuation for ϕ is false, $\neg\phi$ is true.
$\therefore$ for every valuation $\phi \vee \neg\phi$ is true.
$\therefore \vDash \phi \vee \neg\phi$ holds.

2. **Inductive Step**: We assume that shortest proof of the sequent $\phi_1, \phi_2, ..., \phi_n \vdash \psi$ is of length K. We assert the inductive hypothesis of all sequents that have a proof of length < K.

   A proof has the structure

$$\begin{array}{rl} 1.\ \phi_1 & \text{premise} \\ & ..... \\ n.\ \phi_n & \text{premise} \\ K.\ \psi & \text{justification} \end{array}$$

   Two issues the proof needs to deal with.

   (a) What happens in between. (We hope this will be solved by induction hypothesis).
   (b) What is the last rule?(Proof needs to consider all such cases).

   Cases (corresponding to which was the last rule applied)

   (a) Consider $\wedge i$ to be the last rule applied.
   ψ has to be of the form $\psi_1 \wedge \psi_2$ citing lines K_1 and K_2 respectively.
   K_1 and K_2 ¡ K.
   Lines 1-K_1 constitute a proof of the sequent. $\phi_1, \phi_2, ..., \phi_n \vdash \psi_1$.
   Likewise, Lines 1-K_2 constitute a proof of the sequent. $\phi_1, \phi_2, ..., \phi_n \vdash \psi_2$.

   Induction hypothesis is that $M(1), .., M(k-1)$ is true. By the induction hypothesis $\phi_1, \phi_2, ..., \phi_n \vDash \psi_1$ holds and $\phi_1, \phi_2, ..., \phi_n \vDash \psi_2$ holds.

$$1.\phi_1, \phi_2, ..., \phi_n \vDash \psi_1 2.\phi_1, \phi_2, ..., \phi_n \vDash \psi_2$$