Tutorial 6 Solutions

Question 1: Here Fi are the terms of the Fibonacci series. F(0)=0;F(1)=1; All the Fibonacci numbers are calculated as F(n)=F(n-1)+F(n-2).Using this information prove the following

Fibonacci sums: Prove that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ for all $n \in \mathbb{N}$.

Solution-The given equation is taken to be (i).

Base case: When n = 1, the left side of (i) is F1 = 1, and the right side is F3 − 1 = 2 − 1 = 1, so both sides are equal and (i) is true for n = 1.

Induction step: Let k ∈ N be given and suppose (i) is true for n = k. Then

$$\begin{aligned} \sum F_i(i=1 \text{ to } i= k+1) &= \sum F_i(i=1 \text{ to } i= k) + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+3} - 1 \end{aligned}$$

Thus, (i) holds for n = k + 1, and the proof of the induction step is complete. Conclusion: By the principle of induction, it follows that (i) is true for all n ∈ N.

Question 2: Let λ denote the empty string. Let A be any finite nonempty set.
A palindrome over A can be defined as a string that reads the same forward and backwards. Such as 'mom' and 'dad' are palindromes in English Language.
We define a set S as follows:
1. $\lambda \in S$
2. $\forall a \in A$, $a \in S$
3. $\forall a \in A$ and $\forall x \in S$, $axa \in S$
4. All the elements in S must be generated using the above rules.
Prove by structural induction that the set S equals the set of all palindromes over A.

Solution-

Structural induction

P(n): All strings in S of length n are palindromes

• *Inductive Basis:*

By the definitions of S and palindrome, $\lambda \in S$ and it is a palindrome. Likewise, all strings of length 1 are palindromes, and by (2) above, they are in S. Thus, P(0), P(1) is true **[The Basis Holds.]**

• *Inductive Hypothesis:*

P(n) is true.

- *Inductive Step:*

Our task is to prove:

$P(n+1)$ is true.

Consider arbitrary $s \in S$ of length $n+1$. By the definition of S, s is the string aqa, so that $aqa \in S$, where $q \in S$ and q is of length $n-1$. By the strong induction hypothesis, q is a palindrome, and hence aqa is also a palindrome.

[The Inductive Step Holds.]

Question 3-The postage stamp problem: Determine which postage amounts can be created using the stamps of 3 and 7 cents. In other words, determine the exact set of positive integers n that can be written in the form n = 3x+ 7y with x and y nonnegative integers. (Hint: Check the first few values of n directly, then use strong induction to show that, from a certain point n0 onwards, all numbers n have such representation.)

Solution-

We can see that the positive numbers $n < 15$ that have a representation $n = 3x+7y$ with $x, y \in N\cup\{0\}$) are exactly 3, 6, 7, 9, 10, 12, 13, 14. We now use strong induction to show that from 12 onwards every integer has a representation in the above form. In other words, we will prove that the following statement holds for all $n \geq 12$:

(P(n)) : n has a representation (∗) $n = 3x + 7y$ with $x, y \in N \cup \{0\}$

For all positive integers n>=12, we will show that P(n) is true.

Base case: For n = 12, 13, 14, the representations $12 = 3 \cdot 4$, $13 = 3 \cdot 2 + 7$ and $14 = 7 \cdot 2$ show that P(n) is true.

Induction step: Let $k \geq 14$ be given and suppose P(k’) is true for all k’ with k’ = 12, 13, . . . , k, i.e., suppose that all such k’ have a representation in the form (∗). We seek to show that k + 1 also has a representation of this form.

Write $k + 1 = 3 + k\,0$, so that $k' = k - 2$. Note that $k' \leq k$ and also $k' \geq 12$ since we assumed $k \geq 14$. Thus, we can apply the strong induction hypothesis to k' and obtain a representation

$k' = 3x + 7y$, where $x, y \in N \cup \{0\}$.

Adding 3 to both sides of this representation, we get $k + 1 = k' + 3 = 3x + 7y + 3 = 3(x + 1) + 7y$, which is a representation of the desired form for $k + 1$. Hence $P(k + 1)$ is true, and the proof of the induction step is complete.

Conclusion: By the strong induction principle, it follows that $P(n)$ is true for all $n \geq 12$.

Question 4-Consider the following inductive proof and explain the fallacy in it. For all positive integers n, if a and b are positive integers such that max{a, b} = n, then a = b. Proof: By induction on n. The result holds for n = 1, i.e., if max{a, b} = 1, then a = b = 1. Suppose it holds for n, i.e., if max{a, b} = n, then a = b. Now suppose max{a, b} = n + 1. Case 1: a − 1 ≥ b − 1. Then a ≥ b. Hence a = max{a, b} = n + 1. Thus a − 1 = n and max{a − 1, b − 1} = n. By induction hypothesis, a − 1 = b − 1. Hence a = b. Case 2: b − 1 ≥ a − 1. Same argument.

Solution

Fallacy:

In the proof we used the inductive hypothesis to conclude

$\max \{a - 1, b - 1\} = n \dashrightarrow a - 1 = b - 1$.

However, we can only use the inductive hypothesis if $a - 1$ and $b - 1$ are positive integers.

This does not have to be the case as the example $b=1$ shows.