Tutorial 11 Solutions

. Try to write down a sentence of predicate logic which intuitively holds in a model iff the model has (respectively)

(a) exactly three distinct elements

(b) at most three distinct elements

Solution

a) $\exists x \exists y \exists z \forall w[(\neg(x = y)) \wedge (\neg(y = z)) \wedge (\neg(x = z)) \wedge ((w = x) \vee (w = y) \vee (w = z))]$

b) $\forall w \forall x \forall y \forall z[(w=x) \vee (w=y) \vee (w=z) \vee (x=y) \vee (x=z) \vee (y=z)]$

. Let P be a predicate with two arguments. Find a model which satisfies the sentence $\forall x\, \neg P(x, x)$; also find one which doesn't.

Solution

- Model M1: Let A={p,q,r}. P^{M1} ={(p,q),(q,p),(p,r),(r,p),(q,r),(r,q)}, this satisfies $\forall x \neg$P(x,x)
- Model M2: Let A={p,q,r}. P^{M2} ={(p,p),(p,q),(q,p),(p,r),(r,p),(q,r)}, this does not satisfy $\forall x \neg$P(x,x)

Consider the three sentences

$$\phi_1 \stackrel{\text{def}}{=} \forall x\, P(x,x)$$
$$\phi_2 \stackrel{\text{def}}{=} \forall x\, \forall y\, (P(x,y) \rightarrow P(y,x))$$
$$\phi_3 \stackrel{\text{def}}{=} \forall x\, \forall y\, \forall z\, ((P(x,y) \wedge P(y,z) \rightarrow P(x,z)))$$

which express that the binary predicate P is reflexive, symmetric and transitive, respectively. Show that none of these sentences is semantically entailed by the other ones by choosing for each pair of sentences above a model which satisfies these two, but not the third sentence – essentially, you are asked to find three binary relations, each satisfying just two of these properties.

Solution

- Model M1: Let A={p,q,r}; P^{M1}={(p,p),(q,q),(r,r),(p,q),(q,p),(p,r),(r,p)}, M1 $\models$ ϕ_1(reflexive), M1 $\models$ ϕ_2(symmetric) but not ϕ_3(transitive) (since P^{M1} has (q,p) and (p,r) but not (q,r))
- Model M2: Let A={p,q,r}; P^{M2}={(p,q),(q,p),(p,p),(q,q)}, M2 $\models$ ϕ_2(symmetric), M2 $\models$ ϕ_3(transitive) but not ϕ_1(reflexive) (since (r,r) is not in P^{M2})
- Model M3: Let A={p,q,r}; P^{M3} ={(p,p),(q,q),(r,r),(p,q)}, M3 $\models$ ϕ_1(reflexive), M3 $\models$ ϕ_3(transitive) but not ϕ_2(symmetric) (since (p,q) is in P^{M3} but not (q,p))

Given a set of states and a state transition relation, two nodes u and v are said to be even-reachable, if there is a path from u to v whose length is an even number. (The length of a path is the number of edges in the path; e.g. if there is an edge from u to v, there is said to be a path of length 1 from u to v.). Write down -- in Existential Second-order logic -- a formula that specifies even-unreachability, namely the fact that there is no even-length path from u to v. (You do not need to explain why the formula is correct.)

Solution

Here is one Existential Second-order logic formula for even unreachability:
$\exists P\ \forall x \forall y \forall z\ (C1 \land C2 \land C3 \land C4 \land C5)$
where
C1: $E(x, x)$
C2: $E(x, y) \land R(y, z) \rightarrow P(x, z)$
C3: $E(u, v) \rightarrow \bot$
C4: $P(x, y) \land R(y, z) \rightarrow E(x, z)$
C5: $R(x, y) \rightarrow P(x, y)$.
where $R(x,y)$ denotes an edge from x to y.