Tut_8

12 October 2021 14:02

Tut_8

$a_n = c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \cdots + c_0 a_0 + F(n)$

CS F222: Discrete Structures for Computer Science

Tutorial - 8 (Recurrence relations)

1. Find a recurrence relation for the number of ways to climb $n$ stairs if the person climbing the stairs can take one stair or two stairs at a time. What are the initial conditions?
2. Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds. Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in $n$ microseconds. What are the initial conditions?
3. Find a recurrence relation for $C_n$, the number of ways to parenthesize the product of $n+1$ numbers, $x_0 \times x_1 \times x_2 \times \cdots \times x_n$ to specify the order of multiplication. What are the initial conditions?
4. Solve the recurrence relations using iterative method.
   (a) $a_n = a_{n-1} + 2n + 3$ with $a_0 = 4$
   (b) $f(n) = 5f(n/2) + 3$ with $f(1) = 7$
5. Solve the following recurrence relation
   (a) $a_n = 2a_{n-1} + 3^n$ for all $n \geq 2$ with $a_1 = 5$.
   (b) $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ for all $n \geq 3$ with $a_0 = -2, a_1 = 0$, and $a_2 = 5$
   (c) $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$ for all $n \geq 2$ with $a_0 = 1$ and $a_1 = 0$.

$a_n = 2a_{n-1}$

$r - 2 = 0$

$r = 2$

$a_n^{(h)} = \alpha 2^n$

$a_n^{(P)} = P_0 3^n$

$P_0 3^n = 2P_0 3^{n-1} + 3^n$

$P_0 = 3$

$a_n = a_n^{(h)} + a_n^{(P)}$

$= \alpha 2^n + 3^{n+1}$

$a_n = a_{n-1} + 2n + 3, \quad a_0 = 4$

$a_n = ( \qquad ) + 2n + 3$

$= (a_{n-2} + 2(n-1) + 3) + 2n + 3$

$= a_{n-2} + 2((n-1) + n) + 2 \cdot 3$

$= a_{n-3} + 2(n-2) + 3 + 2((n-1) + n) + 2 \cdot 3$

$= a_{n-3} + 2((n-2) + (n-1) + n) + 3 \cdot 3$

Q1. Let $a_n$ be the number of steps needed to climb $n$ stairs.

$a_{n-1}, \quad a_{n-2}$

$$\boxed{a_n = a_{n-1} + a_{n-2}, \text{ for } n \geq 2 \qquad a_0 = 1, \quad a_1 = 1}$$

Q2. $T_n \rightarrow$ # of different messages that can be sent in $n$ microseconds

$$\boxed{T_0 = 1 \qquad T_1 = 1 \qquad T_n = T_{n-1} + T_{n-2} \text{ for } n \geq 2}$$

Q3. $C_n \rightarrow$ the number of ways of parenthesizing the product

$x_0 \times x_1 \times x_2 \times \cdots \times x_n$.

$x_0 \times x_1 \times x_2 \times x_3$ $\qquad C_3 = 5$

$((x_0 \cdot x_1) \cdot x_2) \cdot x_3$

$(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$

$(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$

$x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$

$x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$ $\qquad C_0 = C_1 = 1$

Let that dot be lying b/w $x_k$ and $x_{k+1}$

$x_0 \; x_1 \; x_2 \; \cdots \; x_k \cdot x_{k+1} \; \cdots \; x_n$

$C_n = C_0 \cdot C_{n-1} + C_1 \cdot C_{n-2} + \cdots + C_{n-1} C_0$

$= \sum_{i=1}^{n-1} C_i C_{n-i-1}$

$C_n = \binom{2n}{n} / (n+1)$

(a)

Q4. Find a recurrence relation for the number of ternary strings of length $n$ that do not contain two consecutive 0's or two consecutive 1's.

(b) what are the initial conditions?

(c) How many ternary strings of length six do not contain two consecutive 0s or two consecutive 1s?

Symbols: 0, 1, 2

$a_n \rightarrow$ # of those strings of length $n$

1, 2222 2021,

2 ___ $n-1$

02 ___ $n-2$

12 ___ $n-2$

012 ___ $n-3$

102 ___

0102 ___ $n-4$

$a_n = a_{n-1} + 2a_{n-2} + \cancel{2a_{n-3}} + \cancel{2a_{n-4}} + \cdots + 2a_0 + 2$ — ①

$a_{n-1} = a_{n-2} + \cancel{2a_{n-3}} + \cancel{2a_{n-4}} + \cdots + 2a_0 + 2$ — ②

① − ②

$1\,0\,2\,\underline{\qquad\qquad}$

$0\,1\,0\,2\,\underbrace{\qquad\qquad}_{n-4}$

$1\,0\,1\,2\,\underbrace{\qquad\qquad}_{n-4}$

$\vdots$

$0\,1\,0\,1\,0\cdots 2\,\underbrace{\qquad\qquad}_{0}$

$1\,0\,1\,0\cdots 2\,\underbrace{\qquad\qquad}_{0}$

$\underbrace{0\,1\,0\,1\,0\cdots\cdots 1\,0}_{n}$

$\underbrace{1\,0\,1\,0\cdots\cdots 1\,0}_{n}$

① − ②

$a_n - a_{n-1} = a_{n-1} + a_{n-2}$

$$\boxed{\begin{array}{c} a_n = 2a_{n-1} + a_{n-2} \\ \forall\ n \geq 2 \\ a_0 = 1 \\ a_1 = 3 \end{array}}$$

(c) $a_2 = a_1 + 2a_0 + 2 = 3 + 2\cdot 1 + 2 = 7$

$a_3 = a_2 + 2a_1 + 2a_0 + 2 = 7 + 2\cdot 3 + 2\cdot 1 + 2 = 17$

$\vdots$

$a_6 = a_5 + 2a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 2$

$= 99 + 2\cdot 41 + 2\cdot 17 + 2\cdot 7 + 2\cdot 3 + 2\cdot 1 + 2$

$= \underline{239}$

$C_n = \binom{2n}{n} / (n+1)$

Q5. (a) Find a recurrence relation for the number of ternary strings of length $n$ that do not contain consecutive symbols that are the same.

(b) what are the initial conditions?

$a_n \rightarrow$ # those strings

$\frac{a_n}{3}$ of these must start with each of the symbols 0, 1 and 2

$0\;\overset{1\text{ or }2}{\underbrace{\qquad\qquad}_{n-1}}$

we append 0, 1 or 2 to only $2/3\ a_{n-1}$ strings

$a_n = 3\cdot\frac{2}{3}a_{n-1}$

$$\boxed{\begin{array}{l} a_n = 2a_{n-1} \quad \text{for every } n \geq 2 \\ a_1 = 3, \quad a_0 = 1 \end{array}}$$

$\underline{a_n = a_{n-1} + 2n + 3}, \quad a_0 = 4$

$a_n = (\qquad\qquad) + 2n + 3$

$= (a_{n-2} + 2(n-1) + 3) + 2n + 3$

$= a_{n-2} + 2((n-1) + n) + 2\cdot 3$

$= a_{n-3} + 2(n-2) + 3 + 2((n-1) + n) + 2\cdot 3$

$= a_{n-3} + 2((n-2) + (n-1) + n) + 3\cdot 3$

$\vdots$

$= a_{n-i} + 2((n-i+1) + (n-i) + \cdots + n) + i\cdot 3$

let $n = i$

$= a_0 + 2(1 + 2 + \cdots + n) + n\cdot 3$

$= n^2 + 4n + 4$