CHAPTER 2 — PREDICATE LOGIC

Express things like
- "for all"
- "there exists"

Satisfiability: Does there exist a valuation for which the formula evaluates to true

Validity: Is the formula true for all valuations?

PREDICATES
OR PROPOSITIONAL FUNCTIONS

$S(\text{Rohit}) = \text{True}$
$S(\text{Venkat}) = \text{False}$

Unary predicate

$$S(x): \quad x \text{ is a student}$$
$$I(x): \quad x \text{ is an instructor}$$
$$Y(x, y): \quad x \text{ is younger than } y.$$

binary predicate

Every student x is younger than some instructor y.'

Quantifiers: ① Universal Quantifier
"For all" $\forall$

② Existential quantifier: "There exists"
$\exists$

"Everybody is a student"
$\forall x \, S(x)$

Every student x is younger than some instructor y.'

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$$\forall x\,(S(x) \to (\exists y\,(I(y) \wedge Y(x,y)))).$$

For every x, if x is a student, then there is some y which is an instructor such that x is younger than y.

2 $\forall x\,(S(x) \to (\exists y\,(I(y) \to Y(x,y))))$

$F(x,y)$: y is mother of x

Not all birds can fly.

For that we choose the predicates B and F which have one argument expressing

$$\begin{aligned} B(x) &: \quad x \text{ is a bird} \\ F(x) &: \quad x \text{ can fly.} \end{aligned}$$

$\forall x \, \exists y \, F(x,y)$

$\neq \exists y \, \forall x \, F(x,y)$

The sentence 'Not all birds can fly' can now be coded as

$$\neg(\forall x \, (B(x) \to F(x)))$$

saying: 'It is not the case that all things which are birds can fly.' Alternatively, we could code this as

$$\exists x \, (B(x) \land \neg F(x))$$

younger than

Every child is younger than its mother.

Using predicates, we could express this sentence as

$$\forall x \forall y \, (C(x) \wedge M(y,x) \to Y(x,y))$$

$C(x)$: x is a child

$M(y,x)$: y is x's mother

$Y(x,y)$: x is younger than y

→ However, every child has a unique mother.

$m(x)$: "returns" mother of x

$\forall x \, [C(x) \to Y(x, m(x))]$

Define 2 kinds of things

- "Objects" - variables, constants, functions such as $m(x)$ which we will call <u>terms</u>

- <u>Predicates</u>

Definition 2.1 Terms are defined as follows.

- Any variable is a term.
- If $c \in \mathcal{F}$ is a nullary function, then c is a term.
- If $t_1, t_2, \ldots, t_n$ are terms and $f \in \mathcal{F}$ has arity $n > 0$, then $f(t_1, t_2, \ldots, t_n)$ is a term.
- Nothing else is a term.

In Backus Naur form we may write

$$t ::= x \mid c \mid f(t, \ldots, t)$$

n-ary

where x ranges over a set of variables var, c over nullary function symbols in $\mathcal{F}$, and f over those elements of $\mathcal{F}$ with arity $n > 0$.

S(Rohit) = true

Definition 2.3 We define the set of formulas over $(\mathcal{F}, \mathcal{P})$ inductively, using the already defined set of terms over $\mathcal{F}$:

- If $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, and if $t_1, t_2, \ldots, t_n$ are terms over $\mathcal{F}$, then $P(t_1, t_2, \ldots, t_n)$ is a formula.
- If ϕ is a formula, then so is $(\neg\phi)$.
- If ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $(\phi \to \psi)$.
- If ϕ is a formula and x is a variable, then $(\forall x\, \phi)$ and $(\exists x\, \phi)$ are formulas.
- Nothing else is a formula.

Note how the arguments given to predicates are always terms. This can also be seen in the Backus Naur form (BNF) for predicate logic:

$$\phi ::= P(t_1, t_2, \ldots, t_n) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid (\forall x\, \phi) \mid (\exists x\, \phi) \tag{2.2}$$

where $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, t_i are terms over $\mathcal{F}$ and x is a variable. Recall that each occurrence of ϕ on the right-hand side of the $::=$ stands for any formula already constructed by these rules. (What role could predicate symbols of arity 0 play?)

Convention 2.4 For convenience, we retain the usual binding priorities agreed upon in Convention 1.3 and add that $\forall y$ and $\exists y$ bind like $\neg$. Thus, the order is:

- $\neg$, $\forall y$ and $\exists y$ bind most tightly;
- then $\vee$ and $\wedge$;
- then $\rightarrow$, which is right-associative.

$$\forall x \left[(P(x) \rightarrow Q(x)) \wedge (S(x,y)) \right]$$