Modeling and Simulation of Turbulent Flows
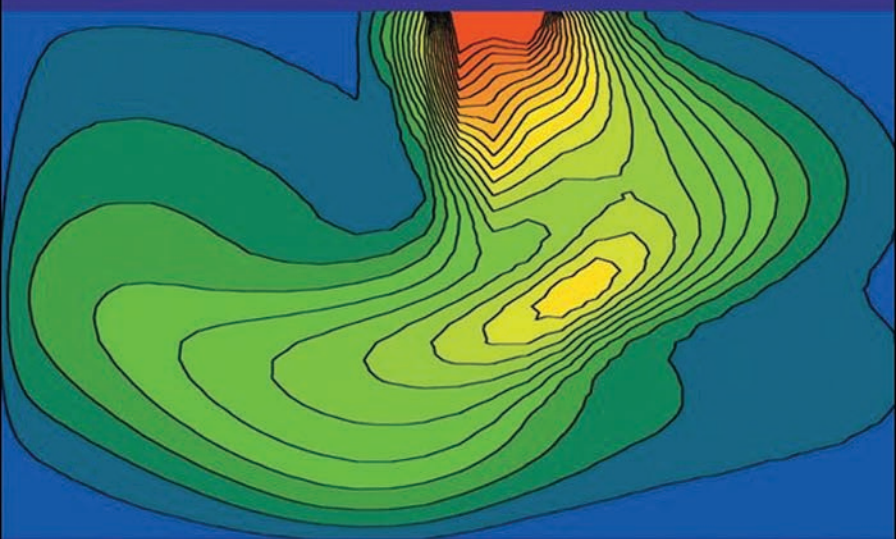
Roland Schiestel

This page intentionally left blank

Modeling and Simulation of Turbulent Flows

This page intentionally left blank

Modeling and Simulation of Turbulent Flows

Roland Schiestel

iSTE   WILEY

Front cover picture (IRPHE): Turbulence kinetic energy levels in the unsteady flow between rotating disks, using Reynolds stress transport modeling (see [RAN 04]).

Table of Contents

Foreword

When, rather more than 10 years ago, Roland Schiestel sent me the manuscript for a new book on turbulent flows that he had written, I was delighted to see that, while rigorous in the development of traditional approaches to turbulence, these were used to serve the main theme of the work, namely the modeling of turbulence in a form suitable for use in CFD solvers. His invitation to write a preface was gladly accepted and the words I wrote then perhaps still merit repeating:

> The fluid mechanics of the world we live in is overwhelmingly dominated by that chaotic, unsteady motion called turbulent flow. Whether it be the flow of air and water in the natural environment or the man-managed interior environment, heat, momentum and mass exchange is brought about by large-scale, irregular eddying motions rather than by molecular diffusion and the design of virtually all types of thermo-fluids equipment: pipes, boilers, compressors, turbines, IC engines, condensers, etc. are variously designed to cope with or exploit the fact that the fluids passing through or around them are in turbulent motion.

> This is such a commonplace observation that the reader may feel it hardly deserves mention. Yet, if – instead of using our eyes to view the world about us – we formed our view of the nature of fluid motion by reading fluid mechanics textbooks, what a different impression would be gained! From such a study we would understand that for a great many problems fluid viscosity is an irrelevancy, in most others the flow remains perfectly laminar while, to handle that rather inconvenient (and apparently unimportant) state called "turbulence" we refer to tediously compiled experimental correlations.

> This distorted view of the relative importance of different strands of engineering fluid mechanics underlines the extent to which academics base the syllabus of their courses on what they know, rather than on what is relevant. That, I suppose, is as inescapable a fact of life as turbulent flow itself.

> At the research level, the computation of turbulent flows has long been a subject receiving greater attention than its scant coverage in textbooks would lead us to expect. Now the rapid growth of software companies marketing commercial CFD packages (coupled with a corresponding growth of users of such software) has helped bring home the need for more – and more systematic – instruction on the internal workings of these black boxes. The aspect of CFD software where questions most often arise and where, through the absence of textbooks, they are least easily handled is on turbulence modeling. There is, manifestly, a need for a comprehensive textbook treatment of engineering turbulence modeling, perhaps particularly one written by an active contributor to the continuing advance of the subject.

The above was the scene, as I perceived it, in the early 1990s when the first edition of Roland Schiestel's book appeared to warm reviews. Over those intervening years, of course, the world of turbulence modeling has moved on, and with the first edition and then an enlarged second edition sold out, the author and publisher have concluded that the time has come for a new edition. The fact that the earlier editions *had* sold out was a good indication that the book was meeting a real need and that the structure and philosophy should remain intact – as it *has*. For example, the book rightly focuses on second-moment closure for it is only at this level that the subject can be developed formally as a branch of mathematical physics (having adopted that starting point, simpler levels of closure naturally emerge as particularly limiting cases that are applicable under increasingly restricted circumstances). Moreover, without recourse to modeling the unknown processes in the second-moment equations, an examination of the exact generation terms explains, at least qualitatively, so many of the paradoxes of turbulent shear flows. For example: why turbulent mixing typically results in twice as much heat flow at *right angles* to the mean temperature gradient as along it; why a secondary strain associated with streamline curvature whose magnitude is only 2% of the primary strain produces a 25% modification in the turbulent shear stress; or why, in orthogonal mode rotation, a relatively weak Coriolis force augments shear stress on the pressure surface by 10% whereas further increase in the rotation rate produces no further augmentation.

Thus, the fabric and style of the very successful original version are retained in this new edition. Among the several new additions, is the inclusion of new approaches to the economical handling of the near-wall region where "wall functions" are normally adopted to escape the crippling cost of a fine-scale resolution of the sublayer and buffer region. The usual log-law based wall functions had such a narrow range of applicability that alternative strategies were sorely needed. These are now included in a new presentation of this material. The number of references has also increased by some 30%, the great majority of which are to works appearing in the last five years. Thus, this new edition continues to serve admirably both those in industrial CFD needing to understand the physical basis of

their software as well as those engaged in or about to start their research in turbulence modeling.

In the foreword to the original edition I had written: "turbulence modeling is still seen by many as a black art founded on bad physics and capable of producing computed flow patterns in accord with measurements only by the arbitrary, case-by-case adjustment of a sackful of empirical constants and other less reputable fudges". That perception is, happily, much less commonly found today. In France, Roland Schiestel's previous editions have been a major contributor to the better-founded appreciation of turbulence modeling by the scientific and industrial communities. May this new edition continue the good work!

Brian E. LAUNDER
Manchester

This page intentionally left blank

Preface

The present book was originally developed for a postgraduate course on Modeling and Simulation of Turbulent Flows taught by the author at Aix-Marseille University for a number of years. A first edition in French appeared in 1993 at Hermes publishers, with new expanded editions in 1998 and 2006. This last French edition was the basis for the present English edition.

Although there exists extensive scientific literature that deals with turbulent flows and their numerical modeling, the information is generally disseminated among numerous papers in specialized international publications. The aim of the present book is to give an introductory, synthetic presentation of numerical modeling and simulation methods for turbulent flows from its basic foundations. It is primarily intended for potential users of numerical models, postgraduate students at university, as well as researchers and practicing engineers interested in the practical calculation of turbulent flows. Some technical details have been marked by a vertical line in the left margin. These may be of interest to some readers but can be skipped if desired. The book gives the physical foundations of the modeling methods and leads the reader to a point where he can implement and make use of the model equations in practical applications with a clear knowledge of the underlying physics and then go deeper into more advanced techniques. Having in mind the actual numerical solution on the computer, some recapitulative tables on numerical methods are grouped together in the last chapter. We shall not claim to or aim at exhaustiveness by any means, considering the numerous works that have been achieved in the field of turbulence modeling, but rather we shall try to follow a rational pathway through the multiform landscape of turbulence modeling, with an emphasis laid upon basic concepts and methods of approach.

In spite of the great variety of experimental studies on the structure of turbulent flows, the fundamental mechanisms in turbulence phenomena still remain

incompletely elucidated and even nowadays many problems remain open, sometimes enigmatic.

However, most fluid flows encountered in the domain of industrial practice are likely to be turbulent and many phenomena, such as heat or mass transfer, are so intimately linked to the fluid motion that their study requires prior calculation of the turbulent flow. Thus, the numerical prediction of turbulent flows is of primary importance for numerous practical applications (industrial, environment, etc.).

The proposed presentation relies on several traditional basic concepts for fluid turbulence phenomenology, and in particular the Kolmogorov theory. The methodology of one point closures is developed all the way to its application to second order moment transport models. Lower order models will then be presented as simplified approaches deduced from second order closures, even if the historical order is the reverse. The impact of spectral theories is also essential, in particular on the notions of spectral equilibrium or on linear and non-linear interactions. These spectral theories, although mainly representative of homogenous turbulence, allow a more refined description of turbulence interaction mechanisms.

Among existing theories and models, preference is given to methods that lead to actual numerical predictions of turbulent flows. In this way, and besides one point and two point statistical closures, the book also addresses large eddy simulation methods that have been developed and increasingly used since the advent of supercomputers. This standpoint has led us to discard detailed presentation of analytical theories and every purely theoretical approach not leading to practical prediction methods.

Compared to the deductive reasoning prevailing in exact sciences, the method of approach used in turbulence modeling may be surprising because of its empiricism. However, the modeling approach is not specific to turbulence, it is also used in many domains in physics. According to M. Dode[1] there are two ways to study natural phenomena: the method of exact science and the model method. The method of exact science such as in thermodynamics, mechanics, optics, electromagnetism, is based on very few fundamental principles, the value of which is considered absolute in the field of the science under consideration. These fundamental principles or postulates have an experimental origin and have been discovered by induction. All scientific laws are then deduced from these first premises by applying mathematics and the rigor of these laws is absolute within the domain in which the postulates are assumed.

---

[1] Translated from M. DODE, *Le deuxième principe*, Sédes Ed., Paris, 1965.

On the other hand, the model method tries to interpret the phenomena and to represent their mechanisms through a picture. This is the case for the atomic theory which proposed a mechanical model for the structure of an atom. In the model method, we try to imagine a mechanism whose details remain hidden and that would be capable of giving an interpretation of observed facts. Once this mechanism is defined, we try to draw all the possible conclusions from it. The general character of the proposed model is then recognized according to the value of the predictions it allows, predictions that exact sciences are not able to provide. The method of exact science and the model method are not conflictive, they are complementary. The further our knowledge advances, the further the physical models move away from direct experimentation and become more and more abstract until they become pure "model equations".

According to this line of thought, a turbulence model is thus composed of "model equations". They describe a phenomenon, which is not actual turbulence but which is sufficiently close to it for representing a useful simplified picture. The accuracy of the model and its ability to represent the properties of a turbulent flow, are directly dependent on our knowledge of the physics of the phenomena that it has been possible to build in the equations.

The model is an almost quantified summary of our present knowledge on fluid turbulence. According to this point of view, mathematical models are perfectible and indeed are improving everyday, enriched by new concepts inspired by experimental results or numerical simulations or progress achieved in theoretical approaches.

The present book gives a prominent position to formalism and equations, thus allowing a rational approach to models in connection with underlying physical concepts and intuition. Beyond the description of existing models, the aim is to show how to develop mathematical models and their elaboration process. In this prospect, equations, which are the very language of science, allow us both to condense the physical concepts into a mathematical scheme and to provide a predictive tool. They call to mind this "bijection" between the physical world and mathematical formalism toward which theories and models are tending.

Recent years have seen remarkable developments in turbulence modeling towards more and more advanced formulae that are the fruit of much research carried on in this domain by various research teams around the world. Effort has been concentrated in particular on the realizability properties allowing us to deal with "extreme" states of the turbulence field (high anisotropies and also compressibility effects, reorganization by rotation, etc.) and also on a more refined description of the underlying physics aiming at a wider universality of description. These efforts towards model development, along with the emergence of new concepts, are currently ongoing, largely boosted by the quick expansion of new

measurement techniques such as lazer velocimetry and especially by the generalized use of direct numerical turbulence simulations that provide numerical databases for calibrating and testing turbulence models. Direct numerical simulations, in spite of their limitations in Reynolds number, allow us not only to consider types of flows or physical parameter values that are impossible to obtain experimentally but also to test the closure hypotheses directly. Thus, the different turbulent phenomena being more easily differentiated and studied separately, this results in a more detailed physical description that goes beyond a mere mimicry of the observed behavior of a turbulent flow. We can say that the development of new techniques of approach including direct numerical simulations of turbulence, far from substituting for modeling, have been on the contrary a catalyst for progression. We can then observe, not only an improvement of turbulence models but also a larger variety of model types (non-linear models, transport equations for new quantities, etc.) arising. Second order closures remain however a preferred reference level of closure allowing both extended potentials in physical description and an efficient numerical solution of the equations. From the user point of view, the methods of approach of turbulent flows are very varied, ranging from one point closure models with a limited number of equations up to advanced transport modeling and large eddy simulations that can be considered as hybrid methods between modeling and simulation. All these methods must be considered as more complementary than competitive and the choice of a particular method will be mainly guided by the type of problem to be solved and by the type of answers that are expected.

Acknowledgements

Introduction

We shall consider, among the numerous approaches of turbulence in fluids, only those which lead to effective prediction methods of turbulent shear flows by numerical calculation, methods which allow the study of real flow situations. In the absence of general predictive theory of turbulent phenomena, practical approaches resort to modeling and simulation methods. These various approaches have different levels of complexity, ranging from statistical modeling based on a hypothesis of turbulence viscosity, then more advanced modeling involving numerous transport equations, up to turbulent large eddy numerical simulations. These various methods are complementary rather than competitive, they correspond to different levels of description, each of them having its own performances and limitations. Thus, among the large variety of turbulence models and possible approaches, the practical user will often be led to make a choice which will be generally dictated by the type of physical problem to be solved and by the type of answers expected (the characteristics of the flow which are looked for). From this perspective, we shall try, in the present introduction, to point out some guiding lines, which will be like an Ariadne's thread through a varied and multiform landscape.

The full description of a random field with probability densities is very complex, therefore the usual methods only consider the first moments on single or several points. The statistical method, using mean value operators, leads to the closure problem of the system equations. This system of equations is closed by introducing modeling hypotheses. Generally, two approaches can be distinguished in the statistical method. They are the one point closures based on a single point moment hierarchy and the two point closures or spectral models based on a two point moment hierarchy or their Fourier transforms which are spectral tensors. The first method has been largely used for the calculation of non-homogenous turbulent shear flows encountered in real situations, the second method is usually limited to homogenous isotropic or anisotropic turbulence.

With the passing years, many turbulence models have been proposed and developed, allowing us to predict in each case the mean velocity distributions, mean temperature or concentration in a turbulent flow. The early empirical theories with very simple formulations are generally based on a turbulence viscosity coefficient or on a mixing length scheme. These types of models and their various improved forms have been widely used and have allowed us, in spite of their theoretical limitations, to reproduce fairly well the measurements obtained by the experimental approach, in particular in the turbulent boundary layers. However, each of these empirical theories has its own formulation which is valid only inside a very limited domain and therefore different schemes must be used for the calculation of confined turbulent flows in pipes and channels, in jets, in wakes and in boundary layers, etc.

Bringing into play more and more elaborate hypotheses describing the behavior of the turbulent fluid has then led to the development of more complex schemes allowing us, in addition to the prediction of mean flow, also to predict the characteristics of the turbulent field. These are the transport equation models, the most popular of which was the energy-dissipation model, the so-called "k-ε" model. The following step, in the direction of increasing complexity, is the development of second order models. The aim of this research on advanced turbulence models composed of an increasing number of transport equations is the development of more universal schemes that would be able to solve a varied group of practical flows in different configurations without having to revise the calibration of numerical constants. We should point out, however, that the domain of application of a particular turbulence model always remains limited and that every progress towards universality is sometimes gained to the detriment of accuracy if we consider a particular flow application. If transport equation models are now widely used in industrial practice, in particular the energy-dissipation model, second order advanced closures still remain under development and progress is being achieved every day.

Transport equation models allow us to deal with the turbulent interaction phenomena more explicitly by assigning a physical interpretation to each unknown statistical correlation to be modeled. They are thus the privileged framework for introducing extra phenomena in interaction with turbulence such as the influence of gravity forces, rotation effects, thermal effects, etc.

The basis of this statistical approach has been codified in the invariant modeling method of J. Lumley. This methodology is presented in the first chapters.

These basic models, with or without transport equations, were generally initially introduced for incompressible fluid and fully developed turbulence. Extensions were then made to account for the effect of the turbulence Reynolds number, in particular close to a wall, often after empirical additions. The methods used for compressible

flows are still very often mere adaptations of models developed for incompressible flows.

Traditional one point closures are based on an implicit hypothesis of single scale description. This hypothesis can be incorrect in turbulent flows with high departures from equilibrium. This is the case when the turbulent flow is subjected to a force varying strongly in space or in time or when energy is fed at particular wavenumbers or frequencies in the spectrum. In these types of flow, multiscale models have been developed in order to account for some spectral information in a simplified way.

One point models require the introduction of hypotheses, often numerous, linking unknown correlations to known quantities. The modeler is guided for this by his physical intuition of the acting turbulent mechanisms, dimensional analysis, fundamental theoretical concepts, comparing experimental results, etc. These hypotheses involve numerical constants that are determined against key experiments that are relatively simple and well known experimentally (grid turbulence, wall boundary layer turbulence, pure strain, homogenous shear, etc.). A turbulence model can then be viewed as a quantified summary of our present knowledge on turbulence that is synthesized as an equation set. The validity of these schemes is subsequently validated or invalidated after confrontation between numerical predictions and experimental results for various turbulent flows. It is thus through its consequences that a turbulence model finds its ultimate justification. The domain of application of the model will be, of course, more or less limited depending on the generality level of the hypotheses that have been retained.

The increase of memory size and speed of scientific computers has more recently allowed the development of simulation methods. If direct numerical solution of the Navier-Stokes equations for turbulent regimes, due to the extent of computational means that are necessary, remains limited to fundamental problems in turbulence at relatively low Reynolds numbers, the numerical simulation of turbulent large eddies now allows us to tackle real situations. This method allows the simulation by a three-dimensional and unsteady calculation of the motion of the larger turbulent eddies whereas fine grained structures of a size smaller than the calculation grid step size and whose characteristics are more universal, are modeled. The aim is thus to simulate realizations of turbulent flows that provide a detailed picture and description of the turbulent structures generated by the calculation and their evolution. In large eddy simulations, these eddies are not exactly real turbulence because of the use of small scale modeling but its statistical properties are expected to be preserved. The various methods of simulation are useful for deeper insights into the study turbulent flows because they allow us to generate large-scale fluctuating fields that can be analyzed using statistical post-processing exactly in the same way as the experimentalist proceeds with laboratory measurements. Numerical simulations can thus provide statistical correlations that would not be attainable by

measurement and thus give the possibility of direct testing of the closure hypotheses. Also, numerical simulations allow us to tackle types of flows or to consider physical parameter values that would not be possible experimentally. The drawback of direct numerical simulations or refined large eddy simulations is their Reynolds number limitations. Another application of simulation is the study of flows in which one point statistical models do not work or are not able to give the desired information, this is the case for complex or pathological flows with unsteady or irregular large-scale evolutions. Numerical simulations may also be useful for the production of databases used to tune simpler models, in particular for turbulent quantities that cannot be measured. We will note, however, that in some sense, numerical simulation methods by calculating more and modeling less, bypass the turbulence problem rather than solving it, the true original turbulence problem remaining statistical in essence.
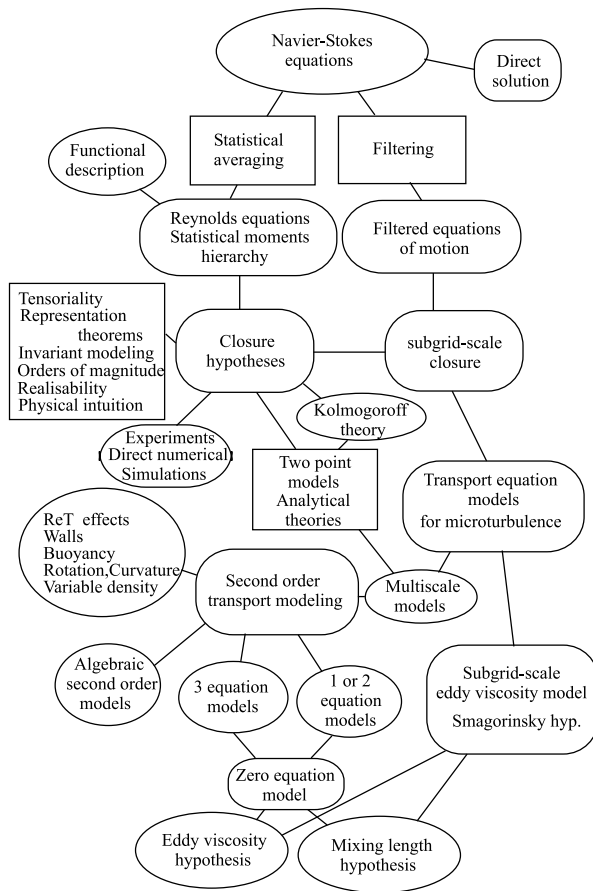
Figure i-1. *Implication diagram*

Chapter 1

Fundamentals in Statistical Modeling:
Basic Physical Concepts

1.1. The nature of turbulence

Turbulence is a property of the flow and not a property of the fluid itself. In flows that are originally laminar, the turbulent regime arises from instabilities that develop as the Reynolds number increases. There is no precise definition of turbulence in fluids, nor does there exist any general theory of turbulence. Turbulence is thus characterized by several observable properties that we shall make clear in the following. If we accept the usually retained hypothesis that the detailed motion of the turbulent fluid is governed by the Navier-Stokes equations, then the fundamental equations of turbulence can be considered as known and thus they can form the basis of any tentative statistical theory to describe the turbulence field.

1.1.1. *Observable properties of turbulent flows*

The chaotic character of turbulent fluctuations appears as a direct consequence of non-linear terms present in the Navier-Stokes equations. These non-linearities are apparent through several important consequences which, considering the absence of any precise definition of turbulence, will serve as characteristic properties.

1.1.1.1. *Irregular signal in space and time*

Physical quantities such as velocity and pressure vary in an apparently random way (cf. Figure 1.1). Let us note that organized or periodic fluctuations are not part

of the turbulent agitation, that is the case for example for pulsed flows in which we have to remove the periodic component in order to get the true turbulent signal.

1.1.1.2. *Rotational flow*

The presence of countless swirling eddies conveys the fact that a turbulent flow is highly rotational. The turbulent motion thus presents strong fluctuations in the curl of velocity. The non-linearies control the interactions between theses eddies of differing size. Let us note, in this connection, that an acoustic field, even random, is not turbulent at all because it is irrotational.

1.1.1.3. *High diffusivity*

A turbulent field diffuses any transportable quantity, such as temperature or a dye, but also momentum, rapidly. In reality, turbulent diffusion is due to convective terms at the fluctuating level. A traced fluid particle marked by a dye is then distorting, branching out and progressively fraying (cf. Figure 1.2). Diffusive shapes are described by fractal dimension ensembles ([VIL 99]).
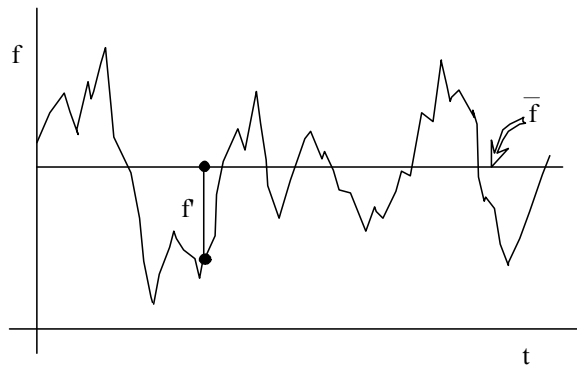
Figure 1.1. *Turbulent signal*

1.1.1.4. *A three-dimensional phenomenon*

Fluctuating turbulent motions are always three-dimensional and unsteady. Let us mention here and now that a so-called "two-dimensional" turbulence also exists and can be found in very special situations. Its mechanisms are very different from those prevailing in three-dimensional turbulence.

1.1.1.5. *Unpredictable character of trajectories*

The detailed properties of a turbulent flow present an extreme sensitivity to initial and boundary conditions. This behavior is apparent if we consider a tiny

deviation in the initial conditions, we then observe that the two flows become rapidly very different from each other if we look at its instantaneous detailed description. This unpredictable character of the detailed fluid particle trajectories on sufficiently long time intervals corresponds to a loss of the memory of initial conditions. This is the unpredictability phenomenon. It explains, for example, the difficulties encountered in long term weather prediction. The unpredictability problem has been studied numerically by Chollet J.P. and Métais O. [CHO 89B].

Some properties of turbulence however remain reproducible, such as statistical properties, mean values and spectral distributions.

The progressive loss of memory of a turbulent flow which forgets, after some elapsed time, the detail of fluctuations in the initial conditions some way justifies the statistical approach to turbulence since, to some extent, the detail of initial conditions can be ignored. In this connection, it is possible to distinguish newly created turbulence which still retain the memory of the conditions in which it was created (much more difficult to study) from fully developed turbulence which has lost the memory of initial conditions (and which can be studied relatively more easily because it is subject to universal laws).

1.1.1.6. *Coexistence of eddies of very different scales*

There exists a whole cascade of eddies of smaller and smaller scales, created by non-linear processes due to inertial terms in the equations of motion.

1.1.1.7. *Dissipation*

Turbulence cannot be sustained by itself, it needs an energy supply. This source of energy can have various origins, the most usual is shear or strain of the mean flow, but the origin can also come from external forces such as Archimedean forces. If turbulence is deprived of any generation process, it decays progressively. Turbulence is dissipative. The mechanism of viscous dissipation of turbulence is related to the existence of strong gradients of the instantaneous velocity field. The instantaneous strain rates indeed become very high inside the smallest eddies and the degradation of the turbulent kinetic energy into heat is thus very strong.

Although there is "turbulence" in plasmas, we shall consider here only the macroscopic approach which supposes that the molecular scales are largely smaller than the scales of the smallest turbulent eddies, which justifies the method of continuous media and the Navier-Stokes equations in particular.

a) t=0

b) t1>0

c) t2>t1
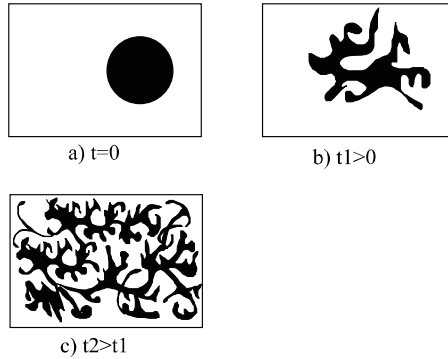
Figure 1.2. *Evolution of a volume marked by a dye under the effect of turbulent diffusion (after Monin and Yaglom, 1971)*

1.1.2. *Traditional point of view on turbulence*

Turbulence is an eddying motion which, at high Reynolds numbers usually reached in practical flows, shows a wide spectrum spreading over a significant range of eddy scales and a corresponding spectrum in frequency. The turbulent motion, always rotational, can be perceived as a muddle of swirling eddies whose rotational curl vectors are directed randomly in space and strongly unsteady.

The largest eddies, which are associated with the low frequency range of the energy spectrum, are determined by the boundary conditions of the flow. Their length scale is comparable to the order of magnitude of the whole domain occupied by the flow itself. The smallest eddies, associated with high frequencies in the spectrum, are determined by viscous forces. The energy spectrum width, i.e. the scale difference between the largest and the smallest eddies, increases with the Reynolds number. Momentum and heat transfer are mainly due to large-scale motions which contribute to statistical correlations $\overline{u_i u_j}$ and $\overline{u_i \gamma}$ where u_i represents the velocity components and γ a transported scalar.

Thus, it is mainly these large eddies that must be considered for determining $\overline{u_i u_j}$ and $\overline{u_i \gamma}$ in turbulence models: the velocity and length scales introduced in the usual turbulence models are basically macroscales.

The large eddies interact with the mean flow (because their characteristic scales have the same order of magnitude), they extract kinetic energy from the mean flow and supply this energy to the large-scale agitations.

Turbulent structures can be considered as swirling vortex elements that are stretching each other. This vortex stretching is one of the most important aspects of turbulent motion. It produces the transfer of energy on smaller and smaller scales until the viscous forces become active and dissipate this energy, this is the energy cascade.

The amount of energy coming from the mean flow and injected into the turbulent motion is determined by the large scale motions, it is only this amount of energy which will be able to cascade to smaller scales and then to be dissipated. Thus, the dissipation rate of kinetic energy is determined by the large-scale motions, although the dissipation is a viscous process that occurs mainly at the level of small eddies. The fluid viscosity does not determine the dissipation rate itself but only the length of the scale at which it happens. The higher the Reynolds number, the smaller the dissipative eddies.

Owing to their interaction with the mean flow, the large eddies are strongly dependent on the boundary conditions of the problem. The mean flow often presents preferred directions that are then imposed on the large-scale turbulent motions. These big eddies may consequently be highly anisotropic. During the cascade process, this directional dependency is however weakened. When the Reynolds number is sufficiently high, then the range of big eddies and the range of small dissipative eddies become clearly separated in the spectrum and this directional dependency is almost completely lost. This is the tendency of fine scale turbulence to local isotropy.

An important contribution to the traditional statistical approach of turbulence was made by the Kolmogorov theory which introduced the concept of universality of locally homogenous and isotropic turbulence. This concept applies to fully developed turbulence at high Reynolds numbers and leads to the (-5/3) slope in the energy spectrum and to scaling laws with (2/3) exponent for spatial correlations. This theory remains an essential basis of the statistical modeling approach, in the sense that it is an implicit hypothesis at the root of one point modeling development. It also represents the traditional phenomenology of turbulence. Intermittency is also a property that can be observed in turbulent flows. We can distinguish two types of intermittencies: the external boundary intermittency which corresponds to the irregular limiting surface at the free boundaries of turbulent flows and the internal intermittency which corresponds to the occurrence of high activity periods interrupted by periods of relative stillness in the dissipative eddies (cf. [*FAV 76], Chapters IV.4 and 5). Turbulence is present everywhere in nature, its effects are sometimes useful or sometimes disastrous depending on individual cases ([*LES 94]).

(a)   (b)
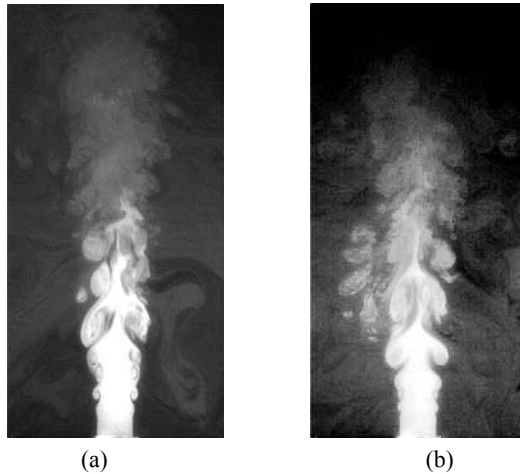
Figure 1.3. *Visualization of a circular jet of air developing in still ambient air, represented in an axial plane using lazer tomography (micro-droplets of oil). Reynolds number at the nozzle exit Re=4935 (a), Re=6148 (b), (Flash photograph IRPHE, thanks to M. Amielh and M-P. Chauve)*

1.1.3. *The origins of turbulence*

The occurrence of turbulence may have various causes. In turbulent shear flows, it is generally due to an increase in the flow Reynolds number $Re = \rho U L / \mu$, which represents the ratio of inertial forces to viscous forces. However, the occurrence of turbulence is also influenced by external forces (Archimedean forces, Coriolis forces, etc.). The framework of linear stability theory (small perturbations away from the reference state with linearization of the equations of motion) allows us to understand the beginning of transition. This theory is based on the Rayleigh equations for free flows and the Orr-Sommerfeld equations for boundary layers. But the mechanisms of transition from laminar to turbulent regime are three-dimensional and very complex, they are not fully clarified anyway at present (cf. Swinney H.L. and Gollub J.P., 1981). One of the acting mechanisms in the onset of turbulence in shear flows is the Kelvin-Helmholtz instability which is a type of vortex sheet instability (cf. [*LES 90]) which is encountered in particular in mixing layers, boundaries of jets or wakes (vortex streets). This instability gives rise to spiral whorls, a precursory of turbulence. Figure 1.3 shows a visualization of a circular jet which images the convective Kelvin-Helmholtz instabilities at the boundaries of the jet after the nozzle exit. Several diameters downstream, the transition to turbulence rapidly takes place, spiral structures break up and the general appearance becomes apparently random. The transition is all the more fast and sudden as the Reynolds number is high.
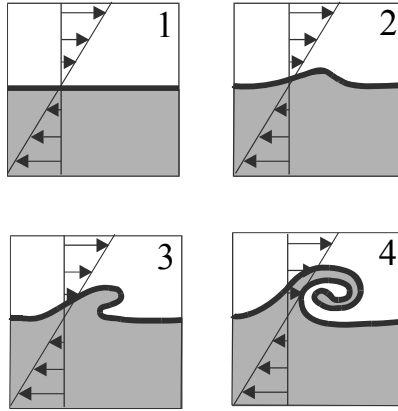
Figure 1.4. *Kelvin-Helmholtz instability*

In a boundary layer, spanwise vortices (perpendicular to the direction of mean flow), resulting from the Kelvin-Helmholtz instability and taking the form of vortex filaments will have the tendency to oscillate away from spanwise direction. Depending on the direction of the deviation (Figure 1.4), the vortex will be entrained at different velocities because of the mean velocity gradient and will develop into "hairpins" and "horseshoes" (Figure 1.5; cf. [*LES 90]). Inside the boundary layer, the transition is initially triggered off by propagating Tollmien-Schlichting waves (which are solutions of the Orr-Sommerfeld equations), and then the turbulence begins to appear in localized regions as "turbulent spots" that are types of concentration zones of hairpins taking on the appearance of a Λ, with the tip of the Λ being directed downstream.
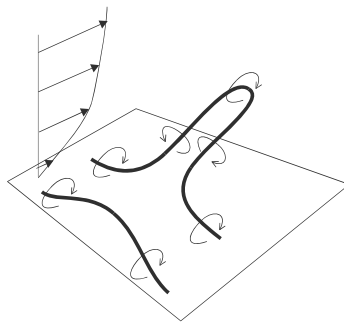
Figure 1.5. *Hairpins in a boundary layer*

In flows submitted to gravity forces (Rayleigh-Bénard instabilities, free convection on a vertical wall), the transition is determined by the Rayleigh number

$\mathscr{R}a = \beta g \Delta T L^3 / \nu\sigma$ (cf. Chapter 15) interpreted as the ratio of the Archimedean thrust to the product of the viscous friction and thermal diffusion. In case the forced convection is combined with natural free convection, it is the Richardson number that will be determinant (cf. Chapter 15).

1.2. The various approaches to turbulence

A general explanatory theory of turbulence phenomena does not exist, but there are numerous partial and incomplete theories. Among these theories, some of them are very rudimentary and very limited but remain nevertheless useful for an industrial approach; some others are more advanced and require more extensive mathematical developments. The approaches to turbulence are thus numerous and various; turbulence is a subject in continuous evolution which is incessantly enriched by new materials. Sometimes, even theories that are apparently extraneous to the studied domain can bring an unexpected perspective and cast new light in spite of their different formalism.

1.2.1. *Direct approach*

Most of the approaches to turbulence assume that the detailed instantaneous motion of the fluid is described by the Navier-Stokes equations. The fluid is then considered a continuum in comparison with the molecular scale. According to this point of view, the equations of turbulence are completely known, and some research works have been directed towards the study of "turbulent solutions" of the Navier-Stokes equations (Agostini L. and Bass J. [AGO 50]; Bass J. [BAS 62]). The study of every turbulent flow could thus be done, in principle, by solving the Navier-Stokes equations directly. In view of predicting a turbulent flow numerically, the question thus arises: why not solve the Navier-Stokes equations? This point of view would consist of making a direct calculation of the turbulent motion for one or several realizations with random initial and boundary conditions and then carrying out a statistical treatment of these solutions. It can be shown however that the number of discretization points necessary to represent the smallest scales of turbulence reaches colossal values.

To fix the ideas, let us try to evaluate quantitatively the necessary computer resources. Turbulence being always three-dimensional and unsteady, the number N^3 of grid points (which dictates the necessary memory space on the computer) in a cubic domain will be proportional to $(\delta x)^{-3}$, δx being the space step, and the calculating time will be proportional to $N^3 / \delta t$ where δt is the time step. In order to describe the smallest scales correctly, δx and δt must be of the order of the Kolmogorov scales.

More precisely:

$$\delta x = \eta / 4 \qquad \eta = \left(v^3 / \varepsilon \right)^{1/4}$$

$$\delta t = \tau / 4 \qquad \tau = \left(v / \varepsilon \right)^{1/2}$$

The order of magnitude of these scales has been divided by four in order to be able to describe (at least very roughly) a sine curve on a full period.

Let $\mathrm{Re}_t = \dfrac{\ell \sqrt{k}}{v}$ be the turbulence Reynolds number, where ℓ is the size of the energetic big eddies $\ell = k^{3/2} / \varepsilon$ and k the kinetic energy of turbulence, noting that $\mathrm{Re}_t = \left(\dfrac{\ell}{\eta} \right)^{4/3}$, we find:

$$N^3 \approx \left(\frac{L}{\delta x} \right)^3 \approx 64 \left(\frac{\ell}{\eta} \right)^3 \approx 64 \, \mathrm{Re}_t^{\,9/4}$$

It has been admitted that the characteristic geometrical size of the flow L had the order of magnitude of ℓ.

Let us consider for example a cross-section in a turbulent pipe flow. Assuming that the order of magnitude of the turbulent Reynolds number usually encountered in practical duct flows is $\mathrm{Re}_t = \mathrm{Re}/10$ where Re is the flow Reynolds number $\mathrm{Re} = \dfrac{UL}{v}$, we find for a moderate Reynolds number Re = 80,000 the value $N^3 \approx 4.10^{10}$ (!). It could easily be shown likewise that the calculating time is proportional to $\mathrm{Re}_t^{\,11/4}$.

This direct approach requires powerful computational means, it can be performed at the present time only for turbulent flows in relatively simple geometries and for low Reynolds numbers. Direct solution of the Navier-Stokes equations in turbulent regimes thus remains dedicated to fundamental research studies on turbulence (Orszag and Patterson [ORS 72]; Orszag and Pao [ORS 74]), and it is not feasible for numerical prediction of turbulent flows in industrial practice, at least, not in the foreseeable future.

There is, however, an intermediate calculation technique between direct solution and full statistical approach. The method consists of calculating the large-scale

eddying motions of the turbulent flow, solving the filtered Navier-Stokes equations on a coarse grid and modeling the turbulent motions whose scale is smaller than the size of the mesh step (subgrid scale models). It is a large eddy simulation (the large eddies are precisely the range of eddies that would be the most difficult to model).

This technique is still costly as regards the amount of numerical calculations (three-dimensional and unsteady) without being always prohibitive (cf. Chapter 19). The approach could be particularly useful for flows in which the big eddies play an important role.

In shear flows, these large eddies marked by the particular geometry of the flow often present some ordered character (coherent structures). However, the existence of coherent structures and their precise definition still pose problems to the specialists.

Direct numerical simulation (tackling the Navier-Stokes equations directly) has been used also for testing, at relatively low Reynolds numbers, the validity of subgrid scale models (Clark R.A., Ferziger J.H. and Reynolds W.C. [CLA 79]; Ferziger J.H. and McMillan O.J. [FRE 77B]) or the one point closure models (Rogallo R.S. [ROG 81A, ROG 81B]).

1.2.2. *Statistical method (statistical equations*, *open system*, *closure problem)*

A turbulent flow solution is always an unsteady complicated solution of the equations of motion, presenting irregular fluctuations in space and time. Faced with the disorderly aspect of turbulent evolutions and this apparent complexity of the phenomenon, the natural and most useful standpoint was to introduce statistical methods. This seemingly randomness of the turbulent evolutions originates in the irregular features in the initial and boundary conditions which are not precisely determined in detail and in which a very small deviation causes a complete change in the detailed structure of the whole flow. The statistical method is not justified by the absence of causality but rather by the ignorance of overabundant causes that are hardly accessible.

The decomposition of an instantaneous characteristic quantity of the turbulent flow into a macroscopic part corresponding to the mean average value and a fluctuating part corresponding to an apparently random part, allows us to develop the statistical treatment of the equations of motion. This treatment, applied to the Navier-Stokes equations which describe the detailed instantaneous motion of the fluid, introduces new unknown terms coming from the convective derivative and which are interpreted as turbulent stresses. In the case of isovolume flows, to which we shall now restrict

ourselves in the first chapters, the two methods, called A and B, introduced in Favre A. *et al.* ([*FAV 76]), lead to the same result (cf. Chapter 17).

Several averaging operators can be used: time averaging, space averaging, stochastic averaging (probability theory) defined from a probability density function, statistical averaging or ensemble averaging (mean value on an ensemble of realizations). It is the last one that we shall use in the following. It is defined as the statistical limit of the arithmetic mean taken over several experimental realizations made in the same general flow conditions. This mean value (Reynolds averaging) denoted $\overline{(.)}$ verifies the so-called "Reynolds rules" (Hinze J.O. [*HIN 75]):

$$\overline{(f + g)} = \overline{f} + \overline{g}$$

$$\overline{(af)} = a.\overline{f} \qquad a, \text{ numerical constant}$$

$$\overline{(\overline{f} \, g)} = \overline{f}.\overline{g}$$

$$\overline{\partial f / \partial \xi} = \partial \overline{f} / \partial \xi \qquad \xi, \text{ space or time variable}$$

In the case of steady turbulent flow, ensemble averages do not depend on time. If the ergodicity conditions are then admitted [*BLA 53] (it is equivalent, on the statistical point of view, to consider repeated analogous experiments or to consider a single experiment indefinitely ongoing), it is possible to show that the time average is equal to the ensemble average. Similarly, in the case of homogenous turbulent flows, it is possible to use the space average.

In the case of periodic unsteady turbulent flows, we also define phase averaging, which is an ensemble average considered for a fixed value of the phase of the periodic motion. This definition leads to a three-term decomposition:

$$f = \underbrace{\langle f \rangle + \widetilde{f}}_{\overline{f}} + f\,'$$

which allows us to distinguish, the constant time mean $\langle f \rangle$, the periodic component $\widetilde{f}(t)$ and the purely turbulent fluctuation $f\,'(t)$, the phase average being represented by $\overline{f}(t) = \langle f \rangle + \widetilde{f}(t)$.

The Navier-Stokes and continuity equations are:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial U_i}{\partial x_j}\right) \tag{1.1}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{1.2}$$

Considering the Reynolds decomposition $U_i = \bar{U}_i + u_i$, $P = \bar{P} + p$ (cf. Figure 1.1) which allows us to distinguish the mean flow from the fluctuating flow, this formalism (by averaging each equation) leads to the Reynolds equations:

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{U}_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\bar{P}}{\rho}\right) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{U}_i}{\partial x_j} - \overline{u_i u_j}\right) \tag{1.3}$$

$$\frac{\partial \bar{U}_i}{\partial x_i} = 0 \tag{1.4}$$

The terms $\overline{u_i u_j}$ give rise to the Reynolds stresses, which come from the non-linearities of the Navier equations and govern the interaction between the mean flow and the fluctuating motion.

System [1.3]–[1.4] involves more unknowns than equations, it is an open system. Taking the mean value of an instantaneous equation implies a loss of information that has to be reintroduced in the form of physical hypotheses: this is the closure problem. Introducing closure hypotheses that express the behavior of the turbulent medium allows us to obtain a number of equations equal to the number of unknowns and thus these equations can be solved numerically. Very schematically, the central problem is thus the problem of connection between the Reynolds stresses $\overline{u_i u_j}$ and the mean field.

The Navier-Stokes equations can also be used to derive new general equations governing the various statistical averages of higher order (which are statistical moments in one or several points and at one or several instants of time). The evolution equations of second order moments (Reynolds stresses) involve triple velocity products (or triple correlations) which come from the non-linear terms. The process can be repeated indefinitely in principle, in order to get a statistical hierarchy of moment equations (Keller L.V. and Friedman A.A. [KEL 24]). However, each new equation introduces an even greater number of unknowns; we have an open hierarchy.

Schematically:

$$\frac{\partial}{\partial t}\underbrace{\overline{u_i u_j \ldots\ldots u_m}}_{<..n\ time..>} = \ldots\left\{\underbrace{\overline{u_i u_j \ldots u_m u_p}}_{<..n+1\ time.>}\right\}\ldots \tag{1.5}$$

In this case also, for achieving the closure of the system, additional hypotheses involving higher order moments, will be necessary.

This method of moments has been up to now, the most investigated and the most fruitful, if we consider the standpoint of the industrial user who wants to carry on the numerical prediction of turbulent flows. This is the method pursued by B.E. Launder and J.L. Lumley in particular. It is inspired by the mechanics of continuous media way of thinking. In the same way as the properties of a continuum are derived from key experiments (tensile test tube, etc.), we can determine in the present case the properties of a "turbulent material".

It is also possible to develop spectral methods using the same point of view.

It is thus necessary to consider homogenous anisotropic turbulence (HAT). The open character of the system is revealed on the spectral tensors which are Fourier transforms of two point correlations.

Spectral closure methods (Jeandel D., Brison J.F. and Mathieu J. [JEA 78]), more refined than one point moment closures, must be amenable to dealing with a turbulent field in which the energy spectrum varies strongly, but the closure hypotheses are all the more delicate to introduce because they require a more detailed knowledge of the acting mechanisms.

Some other approaches make use of probability density functions to describe the turbulence field. The probability density function contains in particular the complete statistical information in one or several points on the turbulent quantity under consideration. The statistical treatment developed on the basis of probability theory introduces a probability density function (or PDF) $\mathcal{P}$ defined by:

$$Probability(\varPhi < a) = \int_0^a \mathcal{P}(\varPhi)d\varPhi$$

The statistical moments of order k are thus defined by (cf. section 4.6.1):

$$a_k = \int_{\mathscr{D}} \varPhi^k \mathcal{P}(\varPhi)d\varPhi \, ,$$

this definition being easily extended to several dimensions.

In the case of weighted averages, we introduce joint probabilities density functions (joint PDF) $\mathcal{P}(\rho, \Phi)$ (Bilger R.W. [BIL 75]) and weighted PDF $\mathcal{P}f(\Phi)$ such that (cf. Chapter 17):

$$\mathcal{P}f(\Phi) = \frac{1}{\rho}\int_0^\infty \rho.\mathcal{P}(\rho, \Phi)d\rho \qquad \text{and} \qquad \widetilde{\Phi} = \int_0^\infty \Phi.\mathcal{P}f(\Phi)d\Phi$$

It is also possible to derive in this case, a hierarchy of equations (which resembles the BBGKY hierarchy of traditional statistical mechanics); this hierarchy is not closed. The use of probability densities proved to be particularly useful in the turbulent combustion domain but it was not much explored otherwise (Lundgren T.S. [LUN 67]; Pope S.B. [POP 85, POP 94A]; Dopazo C. [DOP 94]). In his works, Lundgren introduced in particular a closure hypothesis for the single point distribution function, this closure specifies that the fluctuating pressure term causes the distribution function to tend toward Gaussian distribution law. Yen J.T., [YEN 72], has derived a hierarchy of evolution equations for turbulent correlations directly from the Boltzmann equation.

The Fourier transform, when applied directly to the Navier-Stokes equations, allows us to obtain algebraic relations that are far more manageable. It is within this framework that the analytical theories of turbulence have been developed by Orszag, Kraichnan, Leslie, Lesieur (cf. Lesieur M. [LES 90]).

To summarize, in the usual statistical approaches of turbulence, it is possible to distinguish very schematically two main groups. One of these two groups makes use of two or more point statistics but useful results are limited to homogenous turbulence and in most cases isotropic, the other group relates to one point statistics but can be applied to inhomogenous turbulent shear flows. One point statistical models, in spite of their shortcomings, are particularly useful to the engineer because they allow the numerical prediction of non-homogenous complex flows in various geometries at a moderate calculation cost. The choice of a particular method of approach depends mainly on the type of problem under consideration and the answers that are requested (global mean information or detailed description).

1.2.3. *Moments and cumulants of a random field*

The statistical treatment relies on fundamental concepts of probability theory and statistical methods. For mathematical fundamentals, the interested reader may refer to basic books such as [*BLA 53], [*BAS 67], [*LUM 70], [*MON 71]. In the applications to turbulence, the values of velocity components or those of pressure at different points in the flow field can be considered a set of random variables. The properties of these

random variables are described by their probability density function. If we consider N variables $u_j(A_k)$, $j=1,N$, at N points A_k, the PDF is defined by:

$$\mathcal{P}_{A_1 A_2 ..A_N}(u_1, u_2,...,u_N)du_1 du_2 ....du_N = \text{probability}\left\{u_j < u_j(A_k) < u_j + du_j\right\}_{\substack{j=1,N \\ k=1,N}}$$

Multiple point moments are then obtained from:

$$\mathcal{M}_{i_1 i_2 . i_N} = \overline{u_1^{i_1} u_2^{i_2} ...u_N^{i_N}} =$$
$$\iint ... \int_{\mathbb{R}^N} u_1^{i_1}, u_2^{i_2},...,u_N^{i_N} \mathcal{P}_{A_1 A_2 ..A_N}(u_1, u_2,...,u_N)du_1 du_2 ....du_N \; .$$

Centered moments can be defined as follows:

$$\mathcal{B}_{i_1 i_2 . i_N} = \overline{\left(u_1 - \overline{u_1}\right)^{i_1} \left(u_2 - \overline{u_2}\right)^{i_2} ...\left(u_N - \overline{u_N}\right)^{i_N}} \; .$$

In the particular case of a single random variable (one of the velocity components at one point) the dimensionless ratios:

$$\frac{\mathcal{B}_3}{\mathcal{B}_2^{3/2}} = \mathcal{S} \; , \quad \frac{\mathcal{B}_4}{\mathcal{B}_2^2} = \mathcal{A} \; ,$$

are called skewness factor and flatness factor (or kurtosis) of the random variable u respectively.

The characteristic function φ is the Fourier transform of the probability density function:

$$\varphi_{A_1 A_2 ..A_N}(\zeta_1, \zeta_2,...,\zeta_N) =$$
$$\iint ... \int_{\mathbb{R}^N} \exp\left(i \sum_{k=1}^N \zeta_k u_k\right) \mathcal{P}_{A_1 A_2 ..A_N}(u_1, u_2,...,u_N)du_1 du_2 ....du_N \; .$$

It is generative of statistical moments when developed in MacLaurin series:

$$\varphi_{A_1 A_2 ..A_N}(\zeta_1, \zeta_2,...,\zeta_N) = \sum_{k_1 ...k_N} i^K \frac{\mathcal{M}_{k_1 k_2 . k_N}}{k_1 ! k_2 !...k_N !} \zeta_1^{k_1} \zeta_2^{k_2} ...\zeta_N^{k_N} \; , \quad K = \sum_{j=1}^N k_j$$

The second characteristic function $\psi = Log(\varphi)$ can also be developed in series. Its development involves the cumulants C:

$$\psi_{A_1 A_2 .. A_N}(\zeta_1, \zeta_2, ..., \zeta_N) = \sum_{k_1 ... k_N} i^K \frac{C_{k_1 k_2 .. k_N}}{k_1! k_2! ... k_N!} \zeta_1^{k_1} \zeta_2^{k_2} ... \zeta_N^{k_N}.$$

It is possible to liken the series development of the second characteristic function to the logarithm of the first, and thus to obtain relations between cumulants and moments. The successive cumulants of a Gaussian probability distribution are zero beginning from the third cumulant.

The fourth order centered moment of a velocity component at different points is defined by:

$$\mathcal{B}(A_1, A_2, A_3, A_4) = \overline{\left(u(A_1) - \overline{u(A_1)}\right)\left(u(A_2) - \overline{u(A_2)}\right)\left(u(A_3) - \overline{u(A_3)}\right)\left(u(A_4) - \overline{u(A_4)}\right)},$$

The fourth order cumulant will be:

$$C(A_1, A_2, A_3, A_4) = \mathcal{B}(A_1, A_2, A_3, A_4) - \mathcal{B}(A_1, A_2)\mathcal{B}(A_3, A_4)$$
$$- \mathcal{B}(A_1, A_3)\mathcal{B}(A_2, A_4) - \mathcal{B}(A_1, A_4)\mathcal{B}(A_3, A_2)$$

One of the remarkable properties of the cumulants, such as $C(A_1, A_2, A_3, A_4)$, is that the cumulant goes to zero when the distance of at least two of the four points A_1, A_2, A_3, A_4 becomes infinitely large (thus independent). This property is not verified by the statistical moments centered or not.

1.2.4. *Alternative concepts and new contributions*

There is an abundance of literature. Let us cite a few examples from among them.

– *Turbulence, entropy and thermodynamics.* One of the first works that can be related to this approach is the Malkus theory (Malkus W.V.R. [MAL 56]). This theory is based on the use of an hypothetical principle of maximal viscous dissipation. The stability condition of the mean flow leads to an Orr-Sommerfeld type equation satisfied by the perturbations in the mean values and it is supposed that the smallest scale of the possible turbulent motions is the neutral stability scale, the momentum transfer is then described by a finite series. A variational solution is

given to the stability problem which leads to velocity profiles that are in good agreement with experimental results in the turbulent channel flow between two parallel plates.

Let us also cite the work of Nihoul (Nihoul J.C.J. [NIH 67]) who tackles the problem on the more general point of view of the theory of irreversible phenomena.

The approach by Spiegel E.A., [SPI 72], is based on a description of turbulence using a two fluid model analogous to the Landau equations for superfluids. This point of view suggests conceptual parallels and connections between superfluids and turbulence ([BAR 97]). Spiegel E.A. also introduces the concept of pseudo-thermodynamics of turbulence along with a "turbulence entropy".

The use of extended irreversible thermodynamics [*JOU 01] may also provoke further thoughts on the development of models inspired by non-Newtonian fluids ([HUA 96], [SPE 99], [RAJ 99]). Thermodynamic constraints can also be applied in turbulence modeling ([HUA 01]).

– *Waves and turbulence.* This relates in particular to the undulatory approach of Landahl M.T. and Phillips O.M. (Landahl M.T. [LAN 67]; Phillips O.M. [PHI 67]).

– *Turbulence as a viscoelastic medium.* The study by Crow J., [CRO 68A], [CRO 68B], is the first to introduce an approach to turbulence as a viscoelastic medium. This study resorts to a relation proposed by Townsend. This type of approach clearly shows some of the limitations of simple traditional turbulence models, the Reynolds stresses not being a local property of the fluid, but depending on the whole flow in space and also its past history ([LUM 70]).

– *Turbulence and dynamic systems*. Present works on dynamic systems, strange attractors and on bifurcation theory are a promising way for a renewed explanation of turbulent phenomena also in fluids as in other domains in which chaotic behaviors occur (cf. Bergé P., Pommeau Y., Vidal C.H. [*BER 84]; Mira C. and Gumowski I. [*MIR 80]; in particular). Indeed, the account for non-linearities that take place in the study of dissipative dynamic systems is the origin of a great variety of behaviors in natural phenomena among which turbulence is one of the most beautiful examples. In fact, this chaotic behavior is not inevitably related to the large number of degrees of freedom of the system (cf. Ruelle D. and Takens F. [RUE 71]). In this way, the dynamic system theory allows a generalized view of the turbulence phenomena and chaos reaching beyond the framework of fluid turbulence to encompass all types of systems presenting chaotic behaviors (chemical systems, mechanical systems, biological systems, etc.) and also to explain the different stages towards turbulence according to scenarios of bifurcations of solutions which represent the routes to chaos. The Navier-Stokes equations considered as a dynamic

system (Ruelle D. [RUE 91]) have complicated properties and at the present time the knock-on effects of these studies on the understanding and modeling of turbulence are still limited to the beginnings in turbulence growth. The more philosophical aspects raised by these new concepts and approaches are discussed by Favre A., [FAV 91].

– *Introduction of internal variables*. The turbulent field is usually described with measurable quantities having a direct physical meaning, such as the Reynolds stresses, the viscous dissipation rate, etc. but we can imagine that "hidden" parameters which are not directly accessible to measurements, may however be pertinent. Let us cite first the work of Reynolds W.C. and Kassinos S.C., [REY 94] who introduced new structure tensors (cf. section 8.7) which describe structural properties of eddies in space. Non-equilibrium turbulence is studied by Berezin Y.A. and Trofinov V.M., [BER 95] using a mathematical description involving an asymmetric Reynolds stress tensor, according to a formalism analogous to the one used in helical turbulence.

– *Discrete elements description*. The approach of continuous media is not the only one that can be conceived, let us cite for example the method of lattice-gas automata models which can provide an interesting approach in particular cases (Clavin P. *et al.*, [CLA 88]). Chorin A.J. [CHO 73] has proposed a numerical simulation in which the viscosity is represented by the Brownian motion of "vortex blobs" obtained from a random generator. This type of method could suggest a practical way for studying instability and transition to turbulence.

The method of discrete vortex models introduced by Acton E., [ACT 80] allows us to model the shear layer in a turbulent jet using superimposition of basic "vortex rings" which describe large-scale motions.

1.3. Homogenous and isotropic turbulence (HIT)

1.3.1. *Some fundamental relations*

Turbulent flows encountered in industrial practice are generally non-homogenous and non-isotropic. The local isotropy theory leads us to assume that the fine-grained structures in a non-isotropic turbulence field tends towards isotropy, so this fact shows the usefulness of studying homogenous isotropic turbulence (HIT).

Because of its simplified features, homogenous and isotropic turbulence is relatively more easily tractable by theoretical means and experimental comparisons can be made (decay of grid turbulence). Most experimental studies are performed in a wind tunnel in a mean uniform current with a velocity U which is large compared

to the turbulent fluctuations. Comparisons between theory and experiments will often resort to the Taylor hypothesis (cf. Hinze J.O., [*HIN 75]) which allows us to replace temporal evolution of statistical quantities with the corresponding spatial evolution of these quantities.

We say that turbulence is homogenous and isotropic if its statistical properties are invariant under translation, rotation and reflection of the coordinate system.

Let us consider the most general correlation tensor:

$$B_{ij\ldots p}(\vec{r},\vec{s},\ldots,\vec{q}) = \overline{u_i(M_r)u_j(M_s)\ldots u_p(M_q)} \qquad [1.6]$$

with $\vec{r} = \overrightarrow{OM_r}$, $\vec{s} = \overrightarrow{OM_s}$ and $\vec{q} = \overrightarrow{OM_q}$

It can be shown that in homogenous and isotropic turbulence (Panchev S., [*PAN 71]) the tensor $B_{ij\ldots p}(\vec{r},\vec{s},\ldots,\vec{q})$ can be written as (see also section 3.6):

$$B_{ij\ldots p}(\vec{r},\vec{s},\ldots,\vec{q}) = \sum A(r^2,s^2,r_m s_m,\ldots)r_i r_j \delta_{km}\ldots \qquad [1.7]$$

The terms in the sum involve the indices $i,j\ldots, p$ only for the vector components $\vec{r},\vec{s},\ldots,\vec{q}$ or, by pairs in the Kronecker deltas. Let us examine some consequences of this property (cf. Hinze J.O., [*HIN 75]) on two point correlations.

1.3.1.1. *Pressure correlations*

The two point pressure-velocity correlation $A(\vec{x})$ and $B(\vec{x}+\vec{r})$, can be written as:

$$\Pi_\alpha(\vec{r}) = \overline{p(\vec{x})u_\alpha(\vec{x}+\vec{r})} = A(r^2)r_\alpha \qquad [1.8]$$

Because of the incompressibility relation $\dfrac{\partial u_i}{\partial x_i} = 0$, we deduce that $\Pi_\alpha(\vec{r})$ is zero:

$$\Pi_\alpha(\vec{r}) = 0 \qquad [1.9]$$

It can be shown by considering two distinct independent points $\vec{x}_A = \vec{x}$ and $\vec{x}_B = \vec{x} + \vec{r}$, the vanishing of divergence $\overline{p_A\left(\dfrac{\partial u_\alpha}{\partial x_\alpha}\right)_B} = \overline{p_A\left(\dfrac{\partial u_\alpha}{\partial r_\alpha}\right)} = \dfrac{\partial \Pi_\alpha}{\partial r_\alpha} = 0$

implies after [1.8] that $A + r\dfrac{dA}{dr} = 0$, or $A = Cte/r$ (we have used $\dfrac{\partial r}{\partial r_\alpha} = \dfrac{r_\alpha}{r}$

because $r^2 = r_\alpha r_\alpha$ and $2r\,dr = 2r_\alpha\,dr_\alpha$). The constant Cte must be zero because the correlation cannot become infinite at $r = 0$.

1.3.1.2. *Double velocity correlations*

Double velocity correlations correspond to the Reynolds stresses:

$$R_{\alpha\beta}(\vec{r}) = \overline{u_\alpha(\vec{x})u_\beta(\vec{x}+\vec{r})} = A_1(r^2)r_\alpha r_\beta + B_1(r^2)\delta_{\alpha\beta} \qquad [1.10]$$

If we introduce the two functions $f(r)$ and $g(r)$ (cf. Figure 1.6) defined by:

$$f(r) = \frac{\overline{u_L^A u_L^B}}{u_0^2}, \quad g(r) = \frac{\overline{u_N^A u_N^B}}{u_0^2},$$

with $\vec{r} = \overline{OB} - \overline{OA}$, at $r = 0$, we have $\overline{u_\alpha u_\beta} = u_0^2\delta_{\alpha\beta} = \dfrac{2}{3}k_0\delta_{\alpha\beta}$, so it is possible to write:

$$R_{\alpha\beta}(\vec{r}) = u_0^2\left(\frac{f(r)-g(r)}{r^2}r_\alpha r_\beta + g(r)\delta_{\alpha\beta}\right). \qquad [1.11]$$

The incompressibility condition implies a relationship between f and g that can be written:

$$f + \frac{r}{2}\frac{df}{dr} = g \qquad [1.12]$$

Note: if the invariance is under reflection of coordinate axes, then:

$$R_{\alpha\beta}(\vec{r}) = A_1(r^2)r_\alpha r_\beta + B_1(r^2)\delta_{\alpha\beta} + C_1(r^2)\varepsilon_{\alpha\beta\gamma}r_\gamma$$

This is an extended definition of isotropy, introduced in particular by Lesieur M. ([*LES 90]). The corresponding relations are used in magnetohydrodynamic turbulence and allow us to introduce the helicity of the turbulence field $H = \overline{u_i.\omega_i} = \varepsilon_{ijk}\overline{u_i.\dfrac{\partial u_k}{\partial x_j}}$ (cf. Chapter 2) which is related to coefficient $C_1(r^2)$ in the previous expression.

1.3.1.3. *Triple correlations*

The triple velocity correlations are written:

$$T_{\alpha\beta\gamma}(\vec{r}) = \overline{u_\alpha(\vec{x})u_\beta(\vec{x})u_\gamma(\vec{x}+\vec{r})} =$$
$$= A_2(r^2)r_\alpha r_\beta r_\gamma + B_2(r^2)r_\gamma\delta_{\alpha\beta} + C_2(r^2)(r_\alpha\delta_{\beta\gamma}+r_\beta\delta_{\alpha\gamma})$$

[1.13]

This time we introduce the three functions *k*, *h* and *g* (cf. Figure 1.6) defined by:

$$k(r) = \frac{\overline{u_L^A u_L^A u_L^B}}{u_0^3} \;,\; h(r) = \frac{\overline{u_N^A u_N^A u_L^B}}{u_0^3} \;,\; q(r) = \frac{\overline{u_N^A u_L^A u_N^B}}{u_0^3} \;,$$
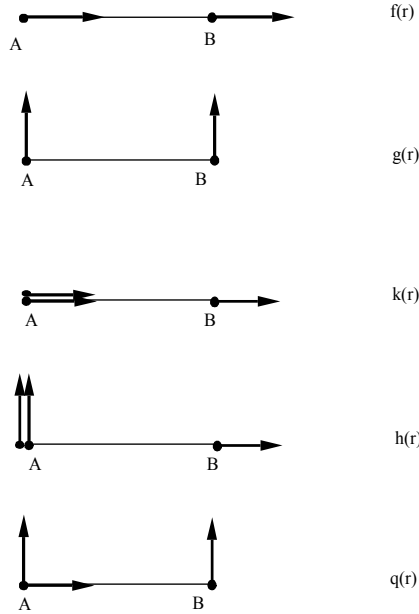
Figure 1.6. *Two point correlations*

Thus,

$$T_{\alpha\beta\gamma}(\vec{r}) = u_0^3\left(\left(k(r)-h(r)-2q(r)\right)\frac{r_\alpha r_\beta r_\gamma}{r^3} + h(r)\frac{r_\gamma\delta_{\alpha\beta}}{r} + q(r)\left(\frac{r_\beta\delta_{\alpha\gamma}}{r} + \frac{r_\alpha\delta_{\beta\gamma}}{r}\right) \right)$$

[1.14]

The incompressibility condition implies the two relationships:

$$q(r) = \frac{1}{4r}\frac{d}{dr}(k.r^2) , \qquad\qquad h = \frac{1}{2}k .$$ [1.15]

1.3.1.4. *Spectral point of view*

The spectral tensor is defined as the Fourier transform of the correlation tensor $R_{\alpha\beta}(\vec{r})$ (cf. also [*HIN 75], [*DEI 98]):

$$\Gamma_{\alpha\beta}(\vec{\kappa}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} R_{\alpha\beta}(\vec{r})e^{-i\vec{\kappa}.\vec{r}}dV(\vec{r})$$ [1.16]

with the reciprocal relation:

$$R_{\alpha\beta}(\vec{r}) = \int_{\mathbb{R}^3}\Gamma_{\alpha\beta}(\vec{\kappa})e^{i\vec{\kappa}.\vec{r}}dV(\vec{\kappa})$$ [1.17]

In the case of homogenous and isotropic turbulence, we can write:

$$\Gamma_{\alpha\beta}(\vec{\kappa}) = C_1(\kappa^2)\kappa_\alpha\kappa_\beta + C_2(\kappa^2)\kappa^2\delta_{\alpha\beta}$$ [1.18]

Accounting for the incompressibility condition which implies $C_1 = -C_2$, we obtain:

$$\Gamma_{\alpha\beta}(\vec{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^2}\Delta_{\alpha\beta}(\vec{\kappa}) \quad \text{with} \quad \Delta_{\alpha\beta}(\vec{\kappa}) = \delta_{\alpha\beta} - \frac{\kappa_\alpha\kappa_\beta}{\kappa^2}$$ [1.19]

This expression involves the three-dimensional spectrum $E(\kappa) = 2\pi\kappa^2\Gamma_{\alpha\alpha}(\vec{\kappa})$ in which $\Gamma_{\alpha\alpha}(\vec{\kappa})$ is dependent only on the modulus κ:

$$E(\kappa) = 2\pi\kappa^2\Gamma_{\alpha\alpha}(\kappa)$$ [1.20]

Equation [1.17] for $r = 0$ provides an expression for the kinetic energy of turbulence:

$$k_0 = \frac{1}{2}\int_{\mathbb{R}^3}\Gamma_{\alpha\alpha}(\vec{\kappa})dV(\vec{\kappa}) = \int_0^\infty E(\kappa)d\kappa$$

Note: the theoretical approach refers to three-dimensional spectral tensors and spectra. On the other hand, experimental studies almost always provide frequency spectra defined as follows:

$$\overline{u_1(\tau)u_1(\tau-t)} = \frac{1}{2}\int_{\mathbb{R}} E_{(11)}(f).e^{2i\pi ft}\,df = \int_0^\infty E_{(11)}(n)\cos(2\pi nt).dn \ .$$

The Taylor hypothesis, in spite of the reservations we can have about its conditions of applicability, allows us to transpose the result into space variables. If the mean flow velocity $\overline{U}$ is constant and the turbulence level low:

$$\xi_1 = \overline{U}.\tau \ , \quad \kappa_1 = \frac{2\pi}{\overline{U}}f \ , \quad E_{(11)}(\kappa_1) = \frac{\overline{U}}{2\pi}E_{(11)}(f) \ .$$

The study of space-time correlations (Favre A. *et al.*, [*FAV 76]) has shown that, strictly speaking, this hypothesis is not true in the general case.

In the wavevector space, the one-dimensional spectra are defined by:

$$E_{(\alpha\beta)}(\kappa_1) = 2\iint \Gamma_{\alpha\beta}(\overline{\kappa})d\kappa_2 d\kappa_3 \ ,$$

and in particular for $\alpha{=}1$, $\beta{=}1$, $E_{(11)}(\kappa_1) = 2\iint \Gamma_{11}(\overline{\kappa})d\kappa_2 d\kappa_3$

Relation [1.17] for $\vec{r} = (r_1,0,0)$ implies:

$$R_{\alpha\beta}(r_1) = \int_{\mathbb{R}^3} \Gamma_{\alpha\beta}(\overline{\kappa})e^{i\kappa_1 r_1}dV(\overline{\kappa}) = \frac{1}{2}\int_{-\infty}^{+\infty} E_{(\alpha\beta)}(\kappa_1)e^{i\kappa_1 r_1}d\kappa_1$$

$$= \int_0^{+\infty} E_{(\alpha\beta)}(\kappa_1)e^{i\kappa_1 r_1}d\kappa_1 = \int_0^{+\infty} E_{(\alpha\beta)}(\kappa_1)\cos(\kappa_1 r_1)d\kappa_1$$

and in particular $R_{11}(r_1) = \int_0^{+\infty} E_{(11)}(\kappa_1)\cos(\kappa_1 r_1)d\kappa_1$.

In isotropic turbulence, the expression $R_{\alpha\alpha}(\vec{r}) = u_0^2\left(f(r)+2g(r)\right)$ deduced from [1.11] allows us, using Fourier transform, to obtain a relationship between three-dimensional spectra and one-dimensional spectra (cf. Hinze J.O., [*HIN 75]):

$$E_{(11)}(\kappa_1) = \int_{\kappa_1}^\infty \frac{E(\kappa)}{\kappa}\left(1-\frac{\kappa_1^2}{\kappa^2}\right)d\kappa \ ,$$

$$E_{(22)}(\kappa_1) = E_{(33)}(\kappa_1) = \frac{1}{2} \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa} \left(1 + \frac{\kappa_1^2}{\kappa^2}\right) d\kappa ,$$

and also the reciprocal relation:

$$E(\kappa_1, t) = \frac{1}{2} \kappa_1^2 \frac{\partial^2 E_{(11)}(\kappa_1, t)}{\partial \kappa_1^2} - \frac{1}{2} \kappa_1 \frac{\partial E_{(11)}(\kappa_1, t)}{\partial \kappa_1} ,$$

with $u_0^2 f(x_1) = \dfrac{1}{2} \int_{\mathbb{R}} E_{(11)}(\kappa_1) e^{i\kappa_1 x_1} dx_1$.

Furthermore, let us cite the useful relations between $E(\kappa)$ and $R_{jj}(r)$:

$$E(\kappa) = \frac{1}{\pi} \int_0^{\infty} R_{jj}(r) \kappa r . \sin(\kappa r) dr , \quad R_{jj}(r) = 2 \int_0^{\infty} E(\kappa) \frac{\sin(\kappa r)}{\kappa r} d\kappa .$$

We will note after [1.11] that $R_{jj}(r) = u_0^2 \left(f + 2g \right) = u_0^2 \left(3f + r df / dr \right)$.

1.3.2. *Dynamics of homogenous isotropic turbulence (HIT)*

1.3.2.1. *Equations in physical space*

The hypothesis of homogenity and isotropy of turbulence made it possible to express the various correlation tensors and spectral tensors with a reduced number of independent functions. We will now examine how to write the equations of turbulence dynamics in the framework of this hypothesis.

The evolution equations of the two point correlations can be derived in the following way.

The equations for the fluctuating velocity are written twice, at two arbitrary distinct points $A(\vec{x})$ and $B(\vec{x} + \vec{r})$. Next, the equation at point A is multiplied by $u_\alpha(B)$ and the equation at point B is multiplied by $u_\beta(A)$, then the right-hand side and left-hand side members of each equation are summed up and the mean value of the resulting equation is taken.

We assume that the mean velocity is uniform and independent of time, and thus taking into account the properties of homogenity and incompressibility, we are led

(cf. Favre A. *et al.* [*FAV 76]; Hinze J.O. [*HIN 75]; Monin A.S. and Yaglom A.M. [*MON 71]) to the following equation :

$$\frac{\partial R_{\alpha\beta}}{\partial t} - \frac{\partial}{\partial r_\gamma}\left(T_{\alpha\gamma\beta} - T_{\gamma\beta\alpha}\right) - \frac{1}{\rho}\left(\frac{\partial \Pi_\beta}{\partial r_\alpha} - \frac{\partial \Pi_\alpha}{\partial r_\beta}\right) - 2v\frac{\partial^2 R_{\alpha\beta}}{\partial r_\gamma \partial r_\gamma} = 0 \qquad [1.21]$$

with $R_{\alpha\beta}$, $T_{\alpha\gamma\beta}$ and Π_α given by definitions [1.8], [1.10] and [1.13].

Considering the particular form of these tensors in the special case of isotropic turbulence, we get $\Pi_\alpha = 0$ while $R_{\alpha\beta}$ and $S_{\alpha\beta} = \frac{\partial}{\partial r_\gamma}(T_{\alpha\gamma\beta} - T_{\gamma\beta\alpha})$ will be functions of u_0^2, f and h.

The resulting equation, obtained after some algebra, is the Karman-Howarth equation:

$$\frac{\partial(u_0^2 f)}{\partial t} + 2u_0^3\left(\frac{\partial h}{\partial r} + 4\frac{h}{r}\right) - 2vu_0^2\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right) = 0 \qquad [1.22]$$

The general structure of this equation is identical to one of the Navier-Stokes equations. The first term represents the time derivative of the double correlation. The second term, space derivative of the triple correlation comes from the non-linear term in the Navier equations. The last term is representative of the effect of molecular viscosity. Of course, the closure problem also pertains to this equation, at this level it consists of estimating the triple correlations according to double correlations.

The behavior of the functions k, h, g and f when $r \to 0$ is worth consideration. We can show, using the isotropy hypothesis and considering the transform of k under a reflection of coordinate axes (Hinze J.O., [*HIN 75]), that the development of k in series about the origin begins with a term of third degree:

$$k = \frac{k'''(0)}{3!}r^3 + \dots.. \qquad [1.23]$$

The result is the same for h and q.

For function f, the development in series can be written:

$$f = 1 - \frac{r^2}{2\lambda^2} \, , \qquad\qquad\qquad [1.24]$$

with λ being the Taylor microscale.

Taking account all of these properties, we deduce from the Karman-Howarth equation the local form of this fundamental equation:

$$\frac{du_0^2}{dt} = -10\nu \frac{u_0^2}{\lambda^2}$$

Introducing the kinetic energy of turbulence $k_0 = \frac{3}{2}u_0^2$ and its dissipation rate $\varepsilon = 10\nu \frac{k_0}{\lambda^2}$, we obtain:

$$\frac{dk_0}{dt} = -\varepsilon \, , \qquad\qquad\qquad [1.25]$$

an equation which is nothing other than the turbulence energy equation, which here describes the decay of turbulence according to k_0 and λ.

1.3.2.2. *Spectral dynamics*

On equation [1.21] for the double correlations:

$$\frac{\partial R_{\alpha\beta}}{\partial t} - S_{\alpha\beta} - 2\nu \frac{\partial^2 R_{\alpha\beta}}{\partial r_\gamma \partial r_\gamma} = 0 \, ,$$

with $S_{\alpha\beta} = \frac{\partial}{\partial r_\gamma}\left(T_{\alpha\gamma\beta} - T_{\gamma\beta\alpha}\right) + \frac{1}{\rho}(\frac{\partial \Pi_\beta}{\partial r_\alpha} - \frac{\partial \Pi_\alpha}{\partial r_\beta})$, we perform a Fourier transform:

$$R_{\alpha\beta} \xrightarrow{\ \mathscr{F}\ } \Gamma_{\alpha\beta} \quad , \qquad\qquad S_{\alpha\beta} \xrightarrow{\ \mathscr{F}\ } F_{\alpha\beta} \, ,$$

which leads to the evolution equation for the spectral tensor of double velocity correlations:

$$\frac{\partial \Gamma_{\alpha\beta}}{\partial t} - F_{\alpha\beta} + 2\nu\kappa^2 \Gamma_{\alpha\beta} = 0 \ . \hspace{2cm} [1.26]$$

Tensorial contraction of this equation gives the equation for the energy spectrum, the so-called Lin equation:

$$\frac{\partial E}{\partial t} = T - 2\nu\kappa^2 E \hspace{3cm} [1.27]$$

with $E = 2\pi\kappa^2 \Gamma_{\alpha\alpha}$ and $T = 2\pi\kappa^2 F_{\alpha\alpha}$.

Integrating on the interval $[0,\kappa]$ in the spectrum gives:

$$\frac{\partial}{\partial t}\int_0^\kappa E.d\kappa' = \int_0^\kappa T.d\kappa' - 2\nu\int_0^\kappa \kappa'^2 E.d\kappa' \hspace{1.5cm} [1.28]$$

The first term of [1.28] represents the derivative of the kinetic energy of turbulence contained in the $[0,\kappa]$ zone corresponding to the largest eddies. The second term represents the energy transfer with the other regions of the spectrum. The last term corresponds to viscous dissipation.

If integration is performed on the whole spectrum, we must obviously recover equation [1.25] because:

$$\int_0^\infty E.d\kappa = k_0 \ , \hspace{1cm} \int_0^\infty T.d\kappa = 0 \ , \hspace{1cm} 2\nu\int_0^\infty \kappa^2 E.d\kappa = \varepsilon \hspace{1cm} [1.29]$$

The shapes of the spectra involved in these equations are sketched in Figure 1.7.
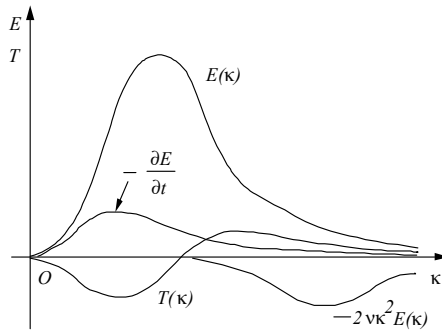
Figure 1.7. *Sketch of the spectral distributions involved in equation [1.27]*

1.3.3. *Two point closures and analytical theories*

Various spectral closures have been proposed in the past (cf. Favre A. *et al.* [*FAV 76]; Hinze J.O. [*HIN 75]; Monin A.S. and Yaglom A.M. [*MON 71]). Most of them rely on the same principle: transfer function T, when κ varies from zero to infinity, grows from negative values to positive values, it represents locally the gain or the loss of energy due to transfer towards other wavenumbers, and the conservation of energy implies $\int_0^\infty T.d\kappa = 0$. It is thus convenient to define a transfer flux function $S(\kappa)$ such that:

$$T(\kappa,t) = -\frac{\partial S}{\partial \kappa}(\kappa,t) \text{ and consequently } S(\kappa,t) = -\int_0^\kappa T(\kappa',t)d\kappa' \qquad [1.30]$$

The contribution of $S(\kappa,t)$ represents the flux of energy from every wavenumber in the $[0,\kappa]$ zone towards wavenumbers in the $[\kappa,+\infty[$ zone.

In integrated equation [1.28]:

$$\frac{\partial}{\partial t}\int_0^\kappa E.d\kappa' + S(\kappa,t) + 2\nu\int_0^\kappa \kappa'^2 E.d\kappa' = 0 \,,$$

the main hypothesis for achieving closure will be of the type:

$$S(\kappa,t) = \mathscr{F}\{E(\kappa,t)\} \,.$$

The simplest closure is the Kovasznay, 1948 hypothesis (cf. Hinze J.O., [*HIN 75]):

$$S(\kappa,t) = \gamma_K E(\kappa,t)^{3/2} \kappa^{5/2} \qquad [1.31]$$

Heisenberg, 1948 (cf. Hinze J.O., [*HIN 75]) proposed to express $S(\kappa)$ as the product of two integrals and assumes:

$$S(\kappa,t) = \nu_H(\kappa,t)\overline{\left(\frac{\partial u}{\partial x}\right)^2}(\kappa,t) \qquad [1.32]$$

where $\nu_H(\kappa,t)$ is an equivalent "viscosity" due to fine-grained turbulence ($[0,+\infty[$ zone of the spectrum).

The expression of this spectral viscosity is given by the integral:

$$\nu_H(\kappa,t) = \int_\kappa^\infty \kappa^{3/2} E(\kappa',t)^{1/2} d\kappa',$$

whereas $\overline{\left(\dfrac{\partial u}{\partial x}\right)^2}(\kappa,t)$ is associated with the $[0,\kappa]$ zone of the spectrum:

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}(\kappa,t) = \int_0^\kappa \kappa'^2 \, E(\kappa',t)d\kappa'.$$

Intuitively, the Heisenberg hypothesis expresses the fact that micro-turbulence acts like a viscosity on the large-scale turbulence.

All these hypotheses give rise to a $\kappa^{-5/3}$ region in the energy spectrum.

There are many other contributions to such theories (Obukhov, Onsager, Corrsin, Von Karman, Pao, (cf. Hinze J.O. [*HIN 75]; Monin A.S. and Yaglom A.M. [*MON 71]).

Still in the framework of two point closures, the analytical theories of turbulence are based on more general statistical hypotheses. They proceed on the Fourier transform of the fluctuating velocities themselves. The Navier-Stokes equations written in Fourier space introduce triadic interactions between wavenumbers verifying $\vec{\kappa}_1 + \vec{\kappa}_2 + \vec{\kappa}_3 = 0$. Closure schemes, thus, operate more directly on the detailed statistical structure of the equations of motion (cf. Leslie D. [*LES 73]; Orszag S.A. [ORS 70]; Lesieur M. [*LES 90]; Mc Comb W.D. [*MCC 96]; Deissler R.C. [*DEI 79]). Among these theories, the most well known are the theories belonging to the family of the EDQNM and DIA approaches. Let us also cite the model by Canuto V.M. *et al.*, [CAN 96] to [CAN 97C].

1.3.3.1. *Quasi-Normal (QN) theory of Proudman and Reid*

The fourth order moments are decomposed into sums of products of the second order moments as with the case of a Gaussian probability distribution, which corresponds to the zero fourth cumulant hypothesis (cf. [*LES 90]). The problem is then closed at the level of third order moments, this can be written symbolically as:

$$\frac{\partial \langle uu \rangle}{\partial t} = \langle uuu \rangle, \quad \frac{\partial \langle uuu \rangle}{\partial t} = \sum \langle uu \rangle \langle uu \rangle,$$

i.e.: $\dfrac{\partial \langle uu \rangle}{\partial t} = \int_0^t \langle uu \rangle \langle uu \rangle (\tau) d\tau$.

In isotropic turbulence, the resulting equation for the energy spectrum is obtained after rather lengthy algebraic developments:

$$\left(\frac{\partial}{\partial t} + 2\nu\kappa^2 \right) E(\kappa,t) = \int_{\vec{\kappa}+\vec{p}+\vec{q}=0} d\vec{p} \int_0^t \frac{\kappa^2}{4\pi p^2 q^2} e^{-\nu(\kappa^2+p^2+q^2)(t-\tau)} d\tau$$

$$\times \begin{bmatrix} 2\kappa^2 a(\kappa,p,q)E(p,\tau)E(q,\tau) - p^2 b(\kappa,p,q)E(\kappa,\tau)E(q,\tau) \\ -q^2 b(\kappa,p,q)E(\kappa,\tau)E(p,\tau) \end{bmatrix} ,$$

where a and b are coefficients determined geometrically:

$$a(\kappa,p,q) = P_{\alpha\beta\gamma}(\vec{\kappa})\Delta_{\beta\delta}(\vec{p})\Delta_{\gamma\rho}(\vec{q})P_{\alpha\delta\rho}(\vec{\kappa})/4\kappa^2 ,$$

$$b(\kappa,p,q) = -P_{\alpha\beta\gamma}(\vec{\kappa})\Delta_{\beta\delta}(\vec{q})P_{\gamma\delta\alpha}(\vec{p})/2\kappa^2 ,$$

$$P_{\alpha\beta\gamma}(\vec{\kappa}) = \kappa_\beta \Delta_{\alpha\gamma}(\vec{\kappa}) + \kappa_\gamma \Delta_{\alpha\beta}(\vec{\kappa}) , \qquad \Delta_{\alpha\beta}(\vec{\kappa}) = \delta_{\alpha\beta} - \kappa_\alpha\kappa_\beta/\kappa^2 .$$

This theory is, however, unacceptable because it implies non-physical effects; negative spectral energies are possible; it also presents too large dynamic memory effects at high Reynolds numbers.

1.3.3.2. *Eddy-Damped Quasi-Normal theory (EDQN)*

The idea is to replace the term $e^{-\nu(\kappa^2+p^2+q^2)(t-\tau)}$ in the *QN* theory by $e^{-\eta_{\kappa,p,q}(t-\tau)}$ where $\eta_{\kappa,p,q}$ accounts for dynamic memory loss due to non-linear effects. Symbolically, this corresponds to:

$$\frac{\partial \langle uu \rangle}{\partial t} = \langle uuu \rangle , \quad \frac{\partial \langle uuu \rangle}{\partial t} = \sum \langle uu \rangle \langle uu \rangle + \langle uuuu \rangle , \quad \langle uuuu \rangle = -\eta_{\kappa,p,q}\langle uuu \rangle ,$$

or: $\dfrac{\partial \langle uu \rangle}{\partial t} = \int_0^t e^{-\eta_{\kappa,p,q}(t-\tau)} \langle uu \rangle \langle uu \rangle (\tau) d\tau$.

Coefficient $\eta_{\kappa,p,q}$ is obtained by:

$$\eta_{\kappa,p,q} = \mu(\kappa,t) + \mu(p,t) + \mu(q,t) ,$$

with $\mu(\kappa,t) = \nu\kappa^2 + \lambda\left[\int_0^{\kappa} p^2 E(p,t)dp\right]^{1/2}$.

Although more satisfactory, this theory does not guarantee realizability.

1.3.3.3. *Eddy-Damped Quasi-Normal Markovianized (EDQNM) theory*

The cure for the problem (markovianization) consists of completely eliminating the long-term memory effects.

The spectral equation from *EDQNM* thus reads ([*LES 90]):

$$\left(\frac{\partial}{\partial t} + 2\nu\kappa^2\right)E(\kappa,t) = \int_{\overline{\kappa}+\overline{p}+\overline{q}=0} d\overline{p}\int_0^t \frac{\kappa^2}{4\pi p^2 q^2} e^{-\eta(t-\tau)} d\tau$$

$$\times\left[\begin{array}{c} 2\kappa^2 a(\kappa,p,q)E(p,\underline{t})E(q,\underline{t}) - p^2 b(\kappa,p,q)E(\kappa,\underline{t})E(q,\underline{t}) \\ - q^2 b(\kappa,p,q)E(\kappa,\underline{t})E(p,\underline{t}) \end{array}\right],$$

where the underlined $\underline{t}$ replace the τ in *QN* theory in the term inside square brackets.

If we assume that $\eta_{\kappa,p,q}$ varies slowly over time (Leith hypothesis, cf. [LES 90]), the integral:

$$\int_0^t e^{-\eta_{\kappa,p,q}(t-\tau)} d\tau = \Theta_{\kappa,p,q}(\tau),$$

can be evaluated simply by:

$$\Theta_{\kappa,p,q}(t) = \frac{1-\exp\{-\eta_{\kappa,p,q}t\}}{\eta_{\kappa,p,q}} = \frac{1-\exp\{-[\mu(\kappa,t)+\mu(p,t)+\mu(q,t)]t\}}{\mu(\kappa,t)+\mu(p,t)+\mu(q,t)},$$

with $\mu(\kappa,t) = \nu\kappa^2 + \lambda\left[\int_0^{\kappa} p^2 E(p,t)dp\right]^{1/2}$, $\lambda = 0.36$

In practice, we use a formulation in the (p,q) plane (modulus of wavevectors), the energy spectrum being obtained by:

$$\frac{\partial E}{\partial t} + 2\nu\kappa^2 E = T(\kappa,t) \text{ (equation [1.27])},$$

$$T(\kappa,t) = \frac{1}{2} \iint_\Delta \Theta_{\kappa,p,q} \frac{\kappa}{pq} b(\kappa,p,q) \times$$

$$\left[\kappa^2 E(p,t)E(q,t) - p^2 E(q,t)E(\kappa,t) \right] dpdq$$

The expression for b given in section 1.3.3.1 can also be written ([*LES 90]) as: $b(\kappa,p,q) = \frac{p}{\kappa}(xy + z^3)$, where x, y, z are the cosines of the interior angles of the triangle $\vec{\kappa}, \vec{p}, \vec{q}$.

1.3.3.4. *Direct Interaction Approximation (DIA) by Kraichnan*

The essential ingredient of the theory is the introduction of the impulsional response function which provides a quantitative measure of dynamic memory effects (cf. Kraichnan R. [KRA 59]; Proudman I. [PRO 61]). This function which can be symbolically represented as $G_{ij}(\kappa,t,t') = \delta u_i / \delta f_j$ describes the velocity variations δu_i produced by an infinitesimal perturbation δf_j applied as an external solenoidal force in the Navier-Stokes equations:

$$\delta u_i = \int_{-\infty}^{t} G_{ij}(\vec{\kappa},t,t')\delta f_j(\vec{\kappa},t')dt' .$$

In isotropic turbulence $G_{ij}(\vec{\kappa},t-t') = G(\kappa,t-t')\varDelta_{ij}(\vec{\kappa})$. Edwards assumes (cf. [*LES 90]) that for a stationary random process:

$$G_{ij}(\vec{\kappa},t,t') \begin{cases} = \varDelta_{ij}(\vec{\kappa})\exp[-\eta(\kappa)(t-t')] & \text{if } t \geq t' \\ = 0 & \text{if } t < t' \end{cases}$$

with $\eta(\kappa) = C^{te}\varepsilon^{1/3}\kappa^{2/3}$ in the inertial zone and $\eta(\kappa) = \nu\kappa^2$ at the limiting boundary of the dissipative zone. In isotropic turbulence, the double velocity correlation is defined by:

$$\overline{\hat{u}_i(\vec{\kappa},t+\tau)\hat{u}_j^*(\vec{\kappa},t)} = \frac{U(\kappa,\tau)}{2}\varDelta_{ij}(\vec{\kappa}), \text{ with } U(\kappa,\tau) = \frac{E(\kappa,\tau)}{2\pi\kappa^2}, \ \tau = t - t',$$

(cf. relation [1.19] and section 3.7.4).

Function $G(\kappa,t)$ can be related to $\eta(\kappa)$ by $\dfrac{1}{\eta(\kappa)} = \int_0^\infty G(\kappa,s)ds$.

The *DIA* equations are complex (cf. Kraichnan R. [KRA 59]; Leslie D.C., [*LES 73]):

$$\left(\frac{\partial}{\partial t} + \nu\kappa^2\right)U(\kappa,t,t') = \frac{1}{2}\kappa^2 \int d\vec{p}\left[\int_0^t a(\kappa,p,q)G(\kappa,t',s)U(p,t,s)U(q,t,s)ds\right.$$

$$\left. -\int_0^t b(\kappa,p,q)U(\kappa,t',s)G(p,t,s)U(q,t,s)ds\right],$$

$$\left(\frac{\partial}{\partial t} + \nu\kappa^2\right)G(\kappa,t,t') = -\frac{1}{2}\kappa^2 \int d\vec{p}.b(\kappa,p,q)\int_0^t G(\kappa,s,t')G(p,t,s)U(q,t,s)ds\ ,$$

where $U(\kappa,t,t') = U(\kappa,t',t)$, $E(\kappa,t) = 2\pi\kappa^2 U(\kappa,t,t)$ and $G(\kappa,t,t) = 1$.

These evolution equations can also be expressed in a form involving integrals in the (p,q) plane using the identity $\iiint_{\vec{p}+\vec{q}=\vec{\kappa}}\mathscr{S}(\kappa,p,q)d\vec{q} \equiv \iint_\Delta \dfrac{2\pi pq}{\kappa}\mathscr{S}(\kappa,p,q)dpdq$.

The main drawback of this theory is the violation of random Galilean invariance. The result is an incompatibility with the Kolmogorov theory. As a consequence, the *DIA* leads to an inertial zone in the energy spectrum satisfying a $-3/2$ power law instead of the $-5/3$ usual power law.

1.3.3.5. *Lagrangian History Direct Interaction Approximation (LHDIA) by Kraichnan*

In this case, the integrals in the *DI* theory are considered in Lagrangian coordinate system (cf. Leslie D.C., [*LES 73]). The resulting equations are statistically invariant under random Galilean transformation and thus are fully compatible with the Kolmogorov theory in the inertial zone.

1.3.3.6. *Extensions*

Let us cite the extensions to the *EDQNM* theory in non-homogenous turbulence developed by Cambon (Cambon C. [CAM 79]; Cambon C. *et al.* [CAM 81]).

1.3.3.7. *Renormalization group (RNG) methods*

Renormalization group theory, a method stemming from other branches of physics (field theory), being transposed to turbulence, has led to the emergence of new turbulence models that will be presented subsequently in the book. The *RNG*

approach assumes universality of the small scales of turbulence and refers to perturbation methods allowing us to represent formally the solutions of the Navier-Stokes equations as a development in powers of the Reynolds number. The technique consists of removing iteratively from the equations of motion slices of wavenumbers (large wavenumbers) in order to get new equations for the remaining modes. A full description of the theory is beyond the scope of the present book, the interested reader will find a basic presentation in the following references: Orszag S.A. *et al.* ([*GAT 96], Chapter 4), Yakhot V. and Orszag S.A., [YAK 86] and Smith L.M. and Reynolds W.C., [SMI 92]. We shall mainly be interested here in the consequences on turbulence models as a spin-off effect from the *RNG* theory (cf. Chapter 8).

1.4. Kolmogorov hypotheses and the local isotropy theory

The local isotropy theory has been developed by Kolmogorov. Following his research work, an important contribution to the understanding of turbulent mechanisms has been made by Von Karman, Lin, Batchelor and Townsend (cf. Hinze J.O., [*HIN 75]; Monin A.S. and Yaglom A.M., [*MON 71]; Batchelor G.K., [*BAT 56]; Townsend A.A., [*TOW 56]). We now recall the basic ideas.

1.4.1. *Phenomenology*

1.4.1.1. *Evolution of eddies, cascade and dissipation mechanisms*

Let us, for example, consider the case of decay of turbulence behind a grid. During the evolution of the turbulence field, the big eddies will generate smaller and smaller eddies by non-linear interactions, thus transferring the energy to the smallest eddies. At the same time, the fluid viscosity will act and the viscous dissipation becomes higher and higher for the small eddies. Thus, the inertial forces are effective in transferring the energy on a longer and longer range of wavenumbers as long as the viscous dissipation does not come into play for damping them at large values of wavenumbers. The equilibrium between these two opposite trends is controlled by the turbulence Reynolds number: the higher the turbulence Reynolds number, i.e. the smaller the molecular viscosity ν, the stronger the efficiency of inertial forces to transfer energy to higher and higher wavenumbers will be.

1.4.1.2. *Independence of eddying motions at remote wavenumbers*

When turbulence is created, only the smaller wavenumbers are excited. The wavenumbers which are involved correspond to the size of the whole mechanical system that create turbulence. The effect of inertial forces being to transfer energy to higher wavenumbers until it reaches the sink due to viscosity, there is thus an entire

spectral zone which obtains energy by transfer only and not directly from the mean flow. We can then suppose that the smaller eddies will be independent from the external conditions which have produced the initial big eddies.

Kraichnan studied the concept of locality of the inertial interactions using analytical theories of turbulence and he has shown that (cf. Lesieur M., [LES 78A]) efficient interactions remain within a decade of wavenumbers.

1.4.1.3. *Permanence of very large eddies*

The very big eddies contain relatively low amounts of energy and moreover their characteristic frequency, of order $u.\kappa$, becomes very small when approaching the origin of the spectrum. This is why we are led to assume the permanent character of very big eddies.

1.4.1.4. *Vortex stretching*

One of the most essential mechanisms responsible for the energy cascade is the vortex stretching phenomenon.

Let us consider the equation for the swirl vector, obtained by taking the curl of the velocity vector in the Navier-Stokes equations.

$$\frac{\partial \omega_i}{\partial t} + U_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial U_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \, ,$$

or equivalently (cf. Tennekes H. and Lumley J.L., [*TEN 72]),

$$\frac{\partial \omega_i}{\partial t} + U_j \frac{\partial \omega_i}{\partial x_j} = \omega_j S_{ij} + \nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \, , \qquad [1.33]$$

with $\qquad \omega_i = \varepsilon_{ijk} \frac{\partial U_k}{\partial x_j} \qquad$ and $\qquad S_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right).$

The first term in the right-hand side represents the source of ω_i due to the action of mean flow. Stretching a vortex filament produces an increase of vorticity, this is a consequence of angular momentum conservation.

Let us consider the schematic example (after Tennekes H. and Lumley J.L., [*TEN 72]), corresponding to a pure strain $S = S_{11} = -S_{22}$ applied in an inviscid fluid.

The evolution of ω_1 and ω_2 is given by (cf. Figure 1.8):

$$\frac{d\omega_1}{dt} = S.\omega_1 \qquad\qquad \frac{d\omega_2}{dt} = -S.\omega_2$$

if at $t = 0$ $\qquad\qquad \omega_1 = \omega_2 = \omega_0$ $\qquad\qquad$ then,

$$\omega_1 = \omega_0.e^{St} \qquad\qquad \omega_2 = \omega_0.e^{-St}$$
$$\omega_1^2 + \omega_2^2 = 2\omega_0^2.\mathrm{Ch}(2S.t)$$

We can see that the total vorticity increases.
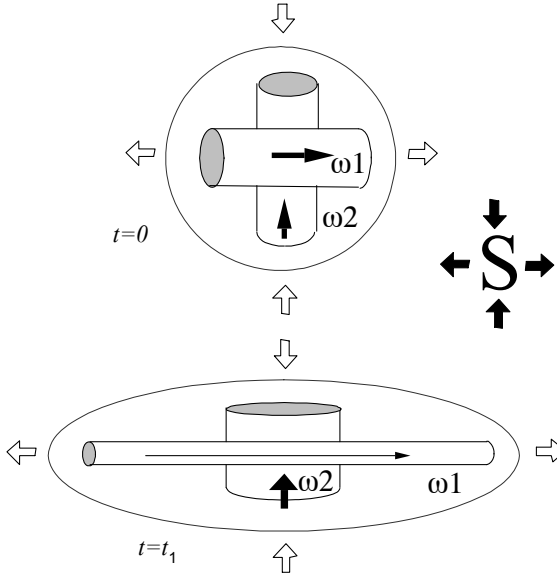
Figure 1.8. *Mechanism of vortex stretching (after Tennekes H. and Lumley J.L., 1972)*

This mechanism is effective for extracting energy from the mean flow. In this case, the most efficient vortices for extracting energy from the mean flow are those who have their principal axes almost aligned with the principal axes of the mean strain field (cf. Figure 1.9).

This same mechanism also acts on smaller scale vortices, transferring energy to higher wavenumbers.
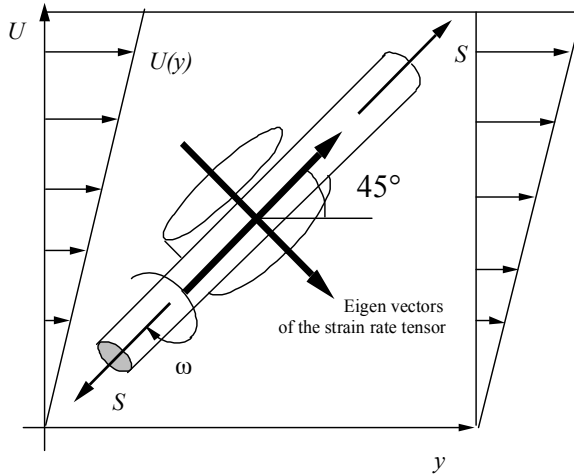
Figure 1.9. *Efficiency of vortex stretching for vortices aligned with the principal axes of mean strain*

1.4.1.5. *Isotropy of microturbulence*

In the cascade process, the small eddies tend to get free from the anisotropy characteristics of the bigger ones and then the microturbulence approaches isotropy. Considering the effect of vortex stretching, an initial stretching in the z-direction enhances motions in the x- and y-directions while reducing the scale. After repeating several times the isotropy is approached (cf. Figure 1.10).

This property of tendency to isotropy of the microturbulence will be used in one point closures, in particular for representing the dissipation term in the Reynolds stress transport equations.

1.4.2. *Local isotropy theory*

We have seen already that the influence of mean flow is lesser and lesser for the small eddies. The motion of the small eddies will thus satisfy universal statistical laws that will depend neither on the mean values nor on the particular geometry of the flow. This constitutes the 1941 Kolmogorov theory ([KOL 41A], [KOL 41B], [KOL 41C]).
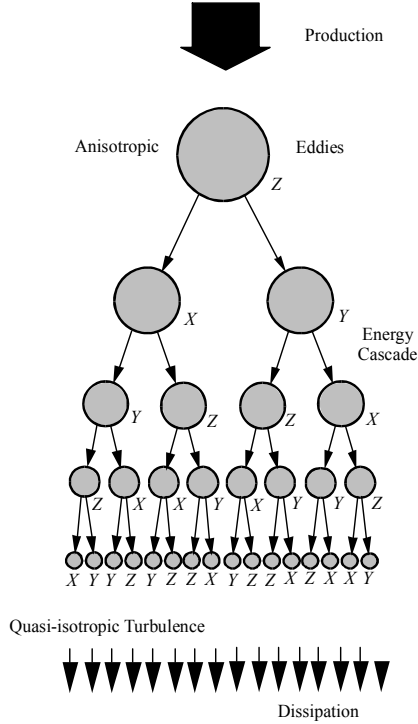
Figure 1.10. *Bradshaw's tree diagram illustrating the tendency to isotropy of microturbulence, schematic description*

As fundamental local characteristics, Kolmogorov introduces the structure functions (longitudinal L and normal N about the separation vector $\vec{r}$ between two points):

$$D_{LL}(r) = \overline{\left[u_L(\vec{x}+\vec{r}) - u_L(\vec{x}) \right]^2}, \quad D_{NN}(r) = \overline{\left[u_N(\vec{x}+\vec{r}) - u_N(\vec{x}) \right]^2},$$

$$D_{LLL}(r) = \overline{\left[u_L(\vec{x}+\vec{r}) - u_L(\vec{x}) \right]^3}. \qquad [1.34]$$

The velocity differences appearing in the structure functions when the two points are sufficiently close to each other are determined by the small eddies, and the big eddies have no influence on these differences. Previous discussions allow us to assume that, when the turbulent Reynolds number is sufficiently high, there exists a domain in the turbulent fluid, which is small compared to the size of the big eddies

and considered during a time interval which is short compared to the characteristic period of turbulence evolution, in which the turbulence field is locally homogenous and isotropic. The concept of local isotropy does not require homogenity and isotropy in the whole flow, but only inside a small domain with characteristic size L during a time interval T.

1.4.2.1. *Kolmogorov first hypothesis*

Inside a domain in which turbulence is locally homogenous and isotropic, the probability laws are dependent on r, ε and ν only. This hypothesis allows us to write in particular:

$$D_{LL}(r) = \vartheta^2 \mathscr{D}_{LL}\left(\frac{r}{\eta}\right), \qquad D_{NN}(r) = \vartheta^2 \mathscr{D}_{NN}\left(\frac{r}{\eta}\right) \qquad [1.35]$$

$\mathscr{D}_{LL}$ and $\mathscr{D}_{NN}$ being universal functions and ϑ, η the Kolmogorov scales:

$$\vartheta = \nu^{1/4} \varepsilon^{1/4}, \qquad \eta = \nu^{3/4} \varepsilon^{-1/4}.$$

1.4.2.2. *Kolmogorov second hypothesis*

This is more restrictive and can be stated as follows:

L is such that, if $\eta \ll r \ll L$, the probability laws relative to $\left[u_\alpha(\vec{x} + \vec{r}) - u_\alpha(\vec{x}) \right]$ depend on ε and r only. Thus:

$$D_{LL}(r) = C.\left(\varepsilon r\right)^{2/3}, \qquad D_{LLL}(r) = C_1.\varepsilon r \qquad [1.36]$$

1.4.2.3. *The Kolmogorov equation*

We can derive the dynamic equation for the locally homogenous and isotropic turbulence from the equation of the fluctuating motion. This is the governing equation of locally homogenous and isotropic turbulence in the same way as the governing equation in *HIT* was the Karman-Howarth equation:

$$6\nu \frac{dD_{LL}}{dr} - D_{LLL} = \frac{4}{5} \varepsilon r \qquad [1.37]$$

Time no longer appears explicitly in equation [1.37] because the hypothesis of locally homogenous and isotropic turbulence includes a steadiness hypothesis.

1.4.2.4. *Spectral point of view*

We now consider the consequences of the Kolmogorov hypotheses on the energy spectra. Indeed, the consequences on the spectra are the most useful.

a) The first hypothesis in spectral terms:

$$E(\kappa) = \vartheta^2 \eta \mathscr{F}(\kappa / K) \tag{1.38}$$

with $K = 1/\eta$, i.e. $E(\kappa) = \varepsilon^{1/4} v^{5/2} \mathscr{F}(\kappa.\varepsilon^{1/4} v^{-3/4})$

b) The second hypothesis in spectral terms:

$$E(\kappa) = C \, \varepsilon^{2/3} \kappa^{-5/3} \tag{1.39}$$

In order to account for the dissipation zone, we can extend the previous formula in the following form $E(\kappa) = C.\varepsilon^{2/3} \kappa^{-5/3} f(\kappa / K)$ in which the function $f(\kappa / K)$ represents the cut-off due to viscosity in such a way that when $\kappa \to 0$, then $f(\kappa / K) \to 1$ and when $\kappa \to \infty$, then $f(\kappa / K) \to 0$.
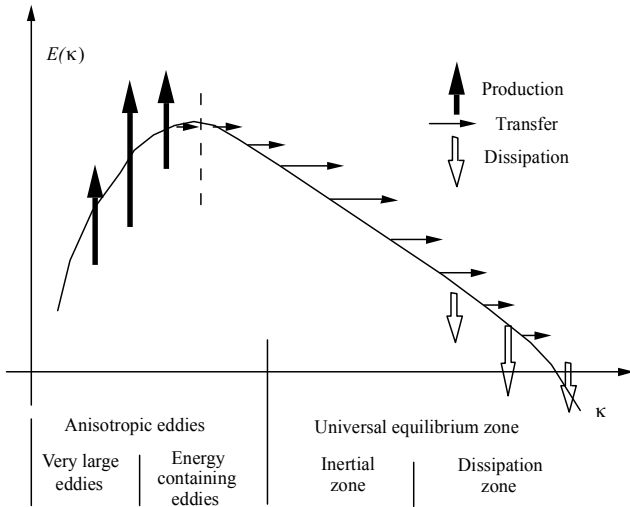
Figure 1.11. *Sketch of the turbulence energy spectrum*

At sufficiently high Reynolds numbers, there exists a range of high wavenumbers in which the turbulence is in statistical equilibrium (uniquely determined by the two parameters ε and v): this is the universal equilibrium range.

When the Reynolds number is sufficiently high, there may exist a region of considerable extent, located inside and at the beginning of the equilibrium range and in which the dissipation is negligible. Inside this zone, the energy transfer by inertial forces is the dominant process and the effect of viscosity vanishes; this is the inertial subrange (cf. Figure 1.11).

The inertial zone is all the more extended as the Reynolds number is higher. Figure 1.12 shows the extension of the (-5/3) law region when the viscosity is diminished, all other things being equal.

The introduction of the energy cascade allows us to ignore the details of small scale turbulence and to characterize the whole energy cascade by ε only, which is the value of the total dissipation rate of turbulent energy by viscosity. This energy dissipation is practically totally independent of the value of viscosity, it is self adaptive in order to balance the action of large-scale eddies. As a consequence, a one point theory will be sufficient to define the connection between the non-universal large-scale part and the universal equilibrium zone of the spectrum, if we assume that this latter part automatically complies with the value of ε.
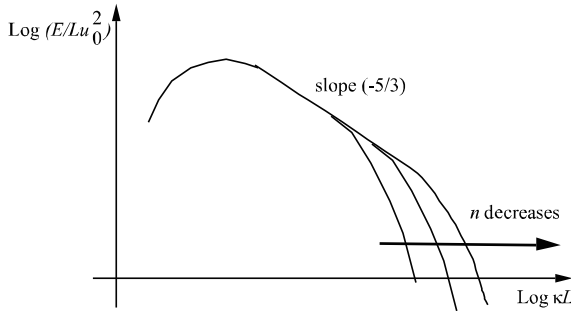
Figure 1.12. *Influence of Reynolds number on the energy spectrum (with normalization on the macroscales L and u_0)*

The local isotropy theory thus allows us to build a bridge between two point modeling and one point modeling.

1.4.2.5. *Departures from the Kolmogorov theory*

Higher order statistics are often studied, these are the normalized moments of the type: $Z_n = \dfrac{\overline{u_{1,1}^{\,n}}}{\left(\overline{u_{1,1}^{\,2}}\right)^{n/2}}$. The usual ones are the skewness coefficient of derivatives

obtained for $n = 3$, and the flatness coefficient obtained for $n = 4$. The Kolmogorov theory implies that these normalized moments would be universal constants, but the measurements show that deviations occur depending on the Reynolds number.

The structure functions of order n, which generalize expressions [1.34], are defined as follows:

$$D_{(n)}(r) = \overline{\left[u_L(\vec{x} + \vec{r}) - u_L(\vec{x}) \right]^n} \, .$$

In the inertial zone, on the basis of dimensional arguments, the Kolmogorov hypotheses would imply an exponent $n/3$:

$$D_{(n)}(r) = \alpha_n \left(\varepsilon r \right)^{n/3} \, .$$

However the experiment [ANS 84] shows that the exponents differ from this value, all the more as n is large. These discrepancies which are also noticeable on the probability density functions, are attributed to the internal intermittency phenomenon. The dissipation is strongly active during a fraction of time which decreases with the scale under consideration. This observation led Kolmogorov to propose a revised form of his theory [KOL 62]. Several models (multifractal models) have also been introduced in order to account for this phenomenon and predict the anomalies in the exponents (cf. Frisch U., [*FRI 95], Chapter 8).

1.5. One point closures

One point theories are based on the development of a hierarchy of one point moment equations and they are mainly useful for studying non-homogenous turbulent shear flows, i.e. those usually encountered in industrial practice. These models are commonly called "RANS" ("Reynolds averaged Navier-Stokes"). The term "URANS" refers to the case of unsteady flows ("Unsteady RANS"). A calculation based on URANS makes use of a statistical model and predicts mean values (statistical means based on ensemble averaging) that are dependent on time. It thus allows us to reveal organized vortices or evolving structures which are *not* turbulent big eddies because they are preserved by ensemble averaging. This must not be confused with the LES method which allows us to simulate the large-scale part of turbulent fluctuations. The two approaches are thus conceptually different.

The Reynolds equation [1.3] corresponds to the first statistical moment. The transport equations for $\overline{u_i u_j}$ are second order moment equations, they are derived from the fluctuating velocity equation using the method already described for the

two point double velocity correlations. In the case of the turbulent transfer of a passive scalar γ the correlations which are involved are the scalar-velocity correlations $\overline{u_i \gamma}$.

1.5.1. *Classification of one point closures*

The classification is usually made according to the number of transport equations. Another type of classification may rely on the different representation methods for the turbulence scales and the Reynolds stresses.

1.5.1.1. *Classification according to the descriptive scales: velocity scale V and length scale L, characteristic of turbulence*

We can distinguish very schematically:

1) Local definitions for *L* and *V*

   – *L*: mixing length ℓ given by an algebraic function (cf. Chapter 12)

   – *V*: velocity scale given by the mean strain rate of the flow field $V = \ell \partial U / \partial y$ (Prandtl, 1925; cf. Hinze J.O., [*HIN 75])

2) Transport equation for *V* only

   – *L*: characteristic length given by an algebraic function

   – *V*: velocity scale obtained from a transport equation (for example a transport equation for turbulence kinetic energy *k* or for the viscosity of turbulence)
(Bradshaw P., Ferriss D.H. and Atwell N.P. [BRA 67]; Donaldson C. du P., [DON 69]; Nee W. and Kovasznay L.S.G. [NEE 69A and B])

3) Transport equations for *L* and *V*

   – *L*: obtained from an evolution equation for $k^\alpha L^\beta$ (for example $\varepsilon = k^{3/2} / L$)

   – *V*: obtained from a transport equation (for example a transport equation for turbulent kinetic energy *k* or for all the Reynolds stress tensor components)
(Rodi W. and Spalding D.B. [ROD 70]; Hirt C.W., [HIR 69]; Spalding D.B. [SPA 69]; Daly B.J. and Harlow F.H. [DAL 70]; Harlow F.H. and Nakayama P.I. [HAR 67, HAR 68A, HAR 68B]; Lumley J.L. [LUM 78A]; Launder B.E., Reece G.J. and Rodi W. [LAU 75C]; Mathieu J. and Jeandel D. [MAT 75])

4) Several transport equations for *L* (cf. Chapter 18)

   – *L*: multiple scales obtained from transport equations

   – *V*: transport equations
(Schiestel R. [SCH 74A, SCH 83, SCH 87A]; Launder B.E. and Schiestel R. [LAU 78]).

1.5.1.2. *Methods for obtaining the Reynolds stresses*

1) Turbulent eddy viscosity hypothesis $v_t = LV = L^2/T$

where *L*, *T* and *V* are respectively the characteristic length scale, time scale and velocity scale of turbulence.

$$R_{12} = \overline{uv} = -v_t \frac{\partial U}{\partial y} \quad \text{or} \quad R_{ij} = \frac{2}{3}k\delta_{ij} - v_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

(Boussinesq, 1877 – Prandtl, 1925 – Kolmogorov, 1942; cf. Launder B.E. and Spalding D.B., (LAU 72]; Nee W. and Kovasznay L.S.G., [NEE 69A and B])

2) Turbulent stress transport equations

$$\frac{dR_{ij}}{dt} = P_{ij} + \Phi_{ij} + D_{ij} - \varepsilon_{ij}$$

(Rotta J.C. [ROT 51A and B]; Hirt C.W., [HIR 69]; Daly B.J. and Harlow F.H., [DAL 70]; Launder B.E., Reece G.J. and Rodi W., [LAU 75C]; Lumley J.L. and Khajeh-Nouri B., [LUM 74]; Fu S., [FU 88]; Craft T.J. *et al.*, [CRA 89]; Launder B.E., [LAU 89]; Speziale C.G. *et al.*, [SPE 91B])

3) Algebraic stress modeling (ASM)

$$R_{ij} = f(R_{hm}, \partial U_h/\partial x_m, L, V, ...)$$

(Rodi W., [ROD 76])

4) Non-linear models and explicit algebraic stress models (EASM)

$$R_{ij} = f(S_{hm}, S_{hp}S_{pm}, \omega_{hm}, \omega_{hp}\omega_{pm}, L, V, ...)$$

(Craft T.J. *et al.*, [CRA 93], Gatski T. *et al.*, [GAT 93])

5) Other methods
   – Restructuring model (Mathieu J. and Jeandel D. [MAT 75])
   – Elliptic relaxation model (Durbin P.A. [DUR 91A, DUR 93])
   – Structure model (Reynolds W.C. *et al.* [REY 94]; Cambon C. *et al.* [CAM 92A]), etc.

1.5.1.3. *Different types of models*

The reader will also find in the references of Launder B.E. and Spalding D.B., 1972; Gibson M.M. *et al.*, [GIB 79B]; Launder B.E. and Spalding D.B., [LAU 74B]; Mellor G.L. and Herring H.J., [MEL 73] a comprehensive overview of traditional turbulence models. Very roughly, we can distinguish on the one hand the first order closures that consist of linking the double correlations (R_{ij} and $F_{\gamma i}$) to the mean values $\overline{U}_i$ and $\overline{\Gamma}$ without introducing any new transport equation (zero equation models, cf. Chapter 12) or by using one or two additional equations (one or two equation models, cf. Chapters 11 and 12), and on the other hand the second order closures in which the Reynolds stresses and the turbulent scalar fluxes are given by closure equations (either as algebraic equations or as transport equations). The main types of turbulence models used in practice are described hereafter.

a) Turbulence eddy viscosity models

We should point out here that the eddy viscosity ν_t is not a property of the fluid but rather a property of turbulent motion itself. It follows that ν_t will have to be chosen differently for every particular flow case. The approximation of turbulent stresses based on velocity gradient hypothesis neglects one of the main aspects of turbulence which is its memory. Indeed, the Reynolds stresses do not conform instantaneously to the mean velocity gradients that are applied on them, and this rules out any pointwise relation between the Reynolds stresses and the mean strain field. The approach of space-time correlations has clearly proved this character (Favre A. *et al.*, [*FAV 76], Chapter 4.3).

Moreover, the gradient hypothesis may fail in asymmetric turbulent flows which exhibit a zone with "negative production" of turbulent energy (in these flows, the turbulent shear stress does not vanish exactly at the point where the maximum mean velocity occurs).

In spite of their theoretical limitations, these simple closure schemes still remain useful for the industrial approach of the engineer. J. Boussinesq, 1877 was the first to represent the effect of turbulence through a turbulence viscosity coefficient named eddy viscosity (Boussinesq model). Initially, the eddy viscosity was assumed to be a constant. Afterwards, several authors introduced variations in the eddy viscosity according to the flow parameters. Let us cite in this respect, the contributions of Prandtl, 1925, Von Karman, 1930, Taylor, 1932, Reichardt, 1942, Prandtl, 1942, Clauser, 1956, Escudier, 1966 (cf. Launder B.E. and Spalding D.B., [*LAU 72]; Hinze J.O., [*HIN 75]).

Memory effect of turbulence is taken into account, at least partially, in turbulence models composed of evolution equations. Hinze J.O., Sonnenberg R.E. and Builtjes P.J.H., [HIN 74] and then Builtjes P.J.H., [BUI 77] have introduced a

memory function modifying the Boussinesq hypothesis, but they have shown that this formulation is equivalent to a relaxation equation that resembles in its structure, the modeled transport equation for the turbulent shear stress.

b) One-equation models

In the case of models using a single transport equation, it is still necessary to specify analytically a characteristic length scale L which will be dependent on the geometry of the specific flow under consideration. This group of models accounts in some ways for a memory effect, but it cannot be easily applied to complex geometries because of the difficulty to choose correctly the length scale L throughout the entire flow. Because of this, they do not reach a much better degree in universality. Let us quote for this type of closure: Bradshaw P., Ferriss D.H. and Atwell N.P., [BRA 67]; Nee W. and Kovasznay L.S.G., [NEE 69A]; Jeandel D., [JEA 72A]; Jeandel D. and Mathieu J., [JEA 73] and also Trusov B.E., [TRU 72] who examines the relation between k and its dissipation rate more precisely. The model by Spalart P.R. and Allmaras S.R. [SPA 94] is, for example, widely used in aerodynamic applications.

c) Two- and three-equation models

In these models, the characteristic length scale L can be determined by a transport equation, and thus these models allow an improved degree of generality and wider universality. These models have been largely used (the k-ε model being the most popular) in various flow configurations and they offer an interesting compromise between universality and accuracy for global predetermination of turbulent shear flows. Let us quote: Rodi W. and Spalding D.B., [ROD 70]; Saffman P.G., [SAF 70]; Wolfshtein M., [WOL 70]; Glushko G.S., [GLU 72]; Ng K.H. and Spalding D.B., [NG 72]; Spalding D.B., [SPA 72]; Saffman P.G., [SAF 74]; Mathieu J. and Jeandel D., [MAT 75]; Ng K.H. and Spalding D.B., [NG 76].

The k-ε model, which will be presented in Chapter 11, has been largely used in practical applications, in this model the turbulent field is described using the two parameters that are the turbulence kinetic energy k and its dissipation rate ε. These two parameters are obtained in the flow field by solving their modeled partial differential transport equations (written here in two dimensions):

$$\frac{\partial k}{\partial t} + U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y} = P - \varepsilon + \frac{\partial}{\partial x}\left(\frac{v_t}{\mathrm{Pr}_k}\frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{v_t}{\mathrm{Pr}_k}\frac{\partial k}{\partial y}\right)$$

$$\frac{\partial \varepsilon}{\partial t} + U\frac{\partial \varepsilon}{\partial x} + V\frac{\partial \varepsilon}{\partial y} = C_{\varepsilon 1}\frac{\varepsilon}{k}P - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x}\left(\frac{v_t}{\mathrm{Pr}_\varepsilon}\frac{\partial \varepsilon}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{v_t}{\mathrm{Pr}_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)$$

In the case where $\partial U / \partial y$ is the major velocity gradient (for example, in boundary layer type flows), the turbulent shear stress and the turbulence production rate P are approached by:

$$R_{12} = -c_\mu \frac{k^2}{\varepsilon} \frac{\partial U}{\partial y} \text{ and } P = c_\mu \frac{k^2}{\varepsilon} \left(\frac{\partial U}{\partial y} \right)^2$$

This class of models has been improved by the introduction of non-linear k-ε models (Speziale C.G., [SPE 87B]; Craft T.J. *et al.*, [CRA 93]) and also the k-ε R.N.G. models (Yakhot V., [YAK 86, YAK 92]). Some recent models introduce a third transport equation for an anisotropy invariant (Craft T.J. *et al.*, [CRA 95B]; Suga K., [SUG 97]), allowing us to sensitize the k-ε model to the anisotropy of the turbulent field.

d) Turbulent Reynolds stress transport models

These models allow us to account more precisely for the anisotropy of the Reynolds stress tensor. They are potentially more general and universal than the previous models but more difficult to develop because the closure hypotheses are more numerous and represent more detailed mechanisms and numerical constants to be determined are also more numerous. The underlying philosophy in the introduction of more and more transport equations remains the search for a more universal model based on a finer and more realistic description of turbulence interaction phenomena. These closures still pose modeling problems which are imperfectly solved and are the subject of current research. Let us cite the contributions of Chou P.Y., [CHO 45A and B]; Rotta J.C., [ROT 51A and B]; Harlow F.H. and Nakayama P.I., [HAR 67, HAR 68B]; Donaldson C. du P., [DON 69]; Harlow F.H. and Hirt C.W., [HAR 69]; Hirt C.W., [HIR 69]; Daly B.J. and Harlow F.H., [DAL 70]; Rotta J.C., [ROT 72]; Launder B.E., Reece G.J. and Rodi W., [LAU 75C]. These models often give rise to simplified versions, such as the k-R_{12}-ε model for two-dimensional thin shear flows (Hanjalic K. and Launder B.E., [HAN 72]). The more recent contributions introduce higher order approximations in the closure hypotheses (Craft T.J. *et al.*, [CRA 89]; Launder B.E., [LAU 89A]; Speziale C.G. *et al.*, [SPE 91B], etc.).

e) Algebraic Reynolds stress models

These can be considered as a simplified expression of the previous Reynolds stress transport models, which can be more easily solved numerically. They are composed of a set of algebraic equations for the individual components of the Reynolds stress tensor coupled with a two-equation transport model for k and ε for example (Rodi W., [ROD 76]; Launder B.E., [LAU 82]). Explicit algebraic models

(Gatski T. *et al.*, [GAT 93]) express the Reynolds stress tensor in a resolved analytical form.

Models consisting of an even larger number of transport equations are far less used in practice (Kolovandine B.A. and Vatutine I.A., [KOL 69, KOL 72]) or they are very specific to some specific type of problems (André J.C. *et al.*, [AND 77]). The more recent contributions of Reynolds W.C. and Kassinos S.C., [REY 94], no longer use directly the transport equations for the Reynolds stresses.

f) Multiscale models

In all the previous approaches there is an underlying hypothesis of spectral equilibrium. Indeed, in these models which are single-scale models, all the turbulent eddies are characterized from a single length scale L. The multiple scale models allow us to get free from any spectral equilibrium hypothesis. Consequently, the number of equations to solve is greater. These models are more recent and their development is still in progress (Schiestel R., [SCH 74A, SCH 83, SCH 87A]; Launder B.E. and Schiestel R., [LAU 78]).

1.5.1.4. *Transport of a passive scalar (temperature)*

As previously, we can distinguish in the model classification:

a) the calculation method for the length and velocity scales (for example: $L_\gamma = L$ and $V_\gamma = V$);

b) the relationships between the turbulent fluxes $\overline{u_i \gamma}$ and the turbulence scales, the mean field Γ, the Reynolds stresses and the mean velocity field U_i (for example: a hypothesis on the turbulent Prandtl number Pr_t or alternatively transport equations for $\overline{u_i \gamma}$).

Considering that the turbulent fluxes of a passive scalar are highly influenced by the stresses and the characteristic scales of the dynamic field, it is impossible to get a good model for the turbulent fluxes of a scalar before having settled a good model for the Reynolds stresses themselves. As for the dynamic turbulence field, various levels of closure have been developed for scalar turbulence up to high levels (Kolovandine B.A., [KOL 68]).

1.5.2. *The development of turbulence models*

Turbulence transport modeling based on evolution equations of one point statistical moments require the introduction of closure hypotheses, often numerous, that link the unknown correlations to the known quantities. To do that, the modeler is guided by his physical intuition of the acting mechanisms, dimensional analysis, tensorial properties, equation compatibility, etc. The main techniques for the development of one point closures, generally combine the formalist approach (cf. section 4.4) which emphasizes the mathematical and functional aspects, the phenomenological approach (cf. Chapter 6 *et seq.*) which mainly relies on physical intuition for modeling turbulent interaction processes and sometimes also the integral approach (cf. section 5.10) which follows from the two point closures.

One of the preliminary options, prior to any turbulence model development is the specification of known quantities chosen for describing the turbulence field and also consequently the closure level which will determine the number of equations in the system. Second order closures correspond to a level of description which offers a good compromise between potentialities for representing turbulent phenomena of practical interest and the possibility of using efficient numerical treatment. Lower order closures remain limited in their field of application, while higher order closures are heavy to carry on and generally difficult to develop. In every process of closure of the moment equations, there is however the underlying idea that higher order moments are relatively less influential than the first moments, a hypothesis that justifies the truncation of the hierarchy of equations.

In spite of difficulties that are fundamental in nature, significant progress has been accomplished in the numerical prediction of turbulent shear flows encountered in industrial practice, aerodynamics and the environment.

The closure hypotheses involve "universal" numerical constants. These constants are determined (normally once and for all) by applying the model to key experiments, that are relatively simple experiments, well known experimentally, such as the decay of grid turbulence, wall turbulence, pure homogenous strain and homogenous shear (cf. Huffman G.D. *et al.*, [HUF 73]). The domain of application of the model will be, of course, more or less limited depending on the degree of generality of the hypotheses introduced.

1.5.3. *The role of turbulence models*

As in every domain in physics, the aim of a turbulence model is the predetermination of the observable behavior of a turbulent flow, using a calculation scheme. The aim of turbulence models is thus practical, the numerical prediction of

turbulent flows, and not really to explain the fundamental mechanisms of the phenomenon. Turbulence models can only give an approximate description and they are only applicable to some classes of flows:

> Although the state is not as statistically universal as we once thought, we can still analyze turbulence using the principles of physics, identifying intrinsic mechanisms (from Lumley J.L. ([*CHA 00], preface)).

A model in which the numerical constants need to be adjusted for each particular flow is in its essence almost nothing more than an interpolation method between experimental data. A good turbulence model must allow extrapolation from the empirical information it contains. It is necessary of course to examine the limits beyond which the extrapolation is no longer valid.

The qualities that will be appreciated in a turbulence model will be: a large domain of application, the accuracy of the predictions it allows, the simplicity in the implementation and computer time efficiency. However, a more universal model is not necessarily a better choice in a very specific flow case. The choice of the model must be made in each problem considered according to the objectives in view.

The choice of a turbulence model among the numerous proposals available in the scientific literature may be often a delicate matter for the final user. This choice is mainly dependent on the expected answers, i.e. on the type of information that we wish to get and also on the variety of flows under consideration, that dictate the level of complexity in modeling and its degree of universality.

One of the main advantages in numerical prediction of turbulent flows is the possibility of easily varying the geometrical, dynamic and thermophysical parameters of the problem which is being treated. We can thus study the influence of these parameters, avoiding repeating numerous experiments that are often long and costly.

1.5.4. *The relationship with experiments*

A turbulence model is generally built from a conceptual scheme stemming from physical intuition, theoretical data and mathematical properties of the equations (invariance, tensoriality, etc.). The experimental information is embodied globally in the models through physical intuition by the synthetic and comparative knowledge of the data coming from measurements of various flow situations.

In fact, experiments are more directly involved at two levels: firstly, for adjusting the closure hypotheses (numerical constants) in the case of fundamental

experiments intended to exhibit a particular phenomenon, which is as "pure" as possible, secondly, it is then involved in testing the models that are already built. More complex turbulent flows encountered in practice may often serve as a test for turbulence models and sometimes suggest empirical corrections that allow us to get round some weaknesses they have been revealed in modeling. However, turbulent flows encountered in practice often present several complex features whose effects interfere, making the interpretation of measurements complicated. Thus, it becomes difficult to take advantage directly of the measurements made in such a particular flow if we have in view modeling improvements of some general significance. This practice, in spite of its undeniable interest, thus remains of limited scope.

From the viewpoint of the development of new models, it seems necessary, after figuring out and separating out the main difficulties, to devise fundamental experiments performed with the aim to study as far as possible a "pure" phenomenon or a more specific mechanism. This approach, very attractive in its principle, still entails, however, some weaknesses: experimental investigations and quantities to be measured are dependent on the prior choice that has been made on the type of mathematical description of the turbulent field in the model (the mathematical formalism and the level of closure of the turbulent field must be chosen in order to offer a fruitful framework able to account for phenomena, to reveal the true acting parameters and to introduce consistently the closure hypotheses); the second difficulty comes from the fact that the turbulent field does not behave like a "turbulent material" and the various basic acting phenomena simply do not add up. Second order closures seems to be the adequate level of closure to study the various turbulent shear flows encountered in industrial practice and the environment.

Turbulent shear flows generally contain swirling large eddies that are strongly dependent on the particular geometry of the flow. From this point of view, each type of turbulent flow always retains some specific characters. The study of the structure of the turbulent field and the interactions between organized eddies is essential to the understanding of acting mechanisms in free turbulent shear flows such as the mixing layer (cf. Cantwell B.J., [CAN 81]; Lesieur M., [LES 90]) and also in turbulent wall flows (cf. Kline S.J. *et al.*, [KLI 67]).

One point statistical models can be applied in principle in the case of unsteady turbulent flows in the mean, such as time periodic pulsed flows. However, the question of the validity of the closure hypotheses can be raised when the frequencies of the unsteady flow interfere with the characteristic frequencies of turbulence itself. Second order models are better suited to account for delays occurring in the evolution of the turbulent field compared to the mean field (Hanjalic K. and Stosik N., [HAN 85]; Kebede W. *et al.*, [KEB 85]).

Any reader whose main interest lies in the practical aspects of modeling from the point of view of the industrial user can directly refer to Chapter 6 dealing with second order closures or Chapter 11 and the following dealing with simplified models. The first chapters giving the fundamentals of modeling can be studied subsequently without disadvantage. The description of the various modeling techniques in turbulence presented in the book do not claim to be exhaustive at all, considering the great variety of existing turbulence models, we have tried especially to bring out the basic methodologies and the main types of models.

1.6. Functional description of turbulence

In order to complete this introductory chapter, we shall give some information on the functional description of turbulence.

The space-time characteristic functional allows a complete statistical description of the turbulent velocity field. It is defined as the Fourier transform of the probability density over the continuum set of the random field (cf. Monin A.S. and Yaglom A.M., [*MON 71]):

$$\Phi\left[\vec{\Theta}(\vec{x},t)\right] = \int_{\Omega} e^{i\left\langle\left\langle\vec{\Theta}\,\vec{u}\right\rangle\right\rangle} d\mu(\vec{u}) = \overline{\exp\left\{i\left\langle\left\langle\vec{\Theta}\,\vec{u}\right\rangle\right\rangle\right\}},$$

with $\left\langle\left\langle\vec{\Theta}.\vec{u}\right\rangle\right\rangle = \int_{\Omega}\Theta_m(\vec{x},t)u_m(\vec{x},t).d\vec{x}dt$.

The characteristic function (cf. section 1.2.3) was defined for a discrete set of N points in the flow. Extension to the whole continuum of the turbulent field is made by replacing the characteristic function by a characteristic functional. A less complete description than that contained in the space-time characteristic functional is given by the spatial characteristic functional:

$$\Phi\left[\vec{\Theta}(\vec{x}),t\right] = \int_{\Omega} e^{i\left\langle\vec{\Theta}.\vec{u}\right\rangle} d\mu(\vec{u}) = \overline{\exp\left\{i\left\langle\vec{\Theta}.\vec{u}\right\rangle\right\}},$$

with $\left\langle\vec{\Theta}\,\vec{u}\right\rangle = \int_{\Omega}\Theta_m(\vec{x})u_m(\vec{x},t).d\vec{x}$.

The measure $d\mu(\vec{u})$ is defined by $d\mu(\vec{u}) = \mathscr{P}(\vec{u})\mathscr{D}(\vec{u})$ where $\mathscr{P}(\vec{u})$ is the probability density of the velocity field, $\mathscr{P}(\vec{u})\mathscr{D}(\vec{u})$ thus represents the probability

of occurrence of a given field $\vec{u}(\vec{x},t)$, $\mathscr{D}(\vec{u})$ is the hypervolume element in the functional space Ω, and $\mathscr{P}(\vec{u})$ satisfies the normalization relation $\int \mathscr{P}(\vec{u})\mathscr{D}(\vec{u}) = 1$.

We can conceive in more practical terms the hypervolume element $\mathscr{D}$ by considering an approximate space of finite dimension in which we would get the following definition:

$$\mathscr{D}(\vec{u}) = C.\prod_{\vec{x}} du_1(\vec{x})du_2(\vec{x})du_3(\vec{x})$$

Functional Volterra derivatives can be defined by:

$$\mathscr{D}_{g_j}\Phi = \frac{\partial \Phi}{\partial g_j(\vec{x})d\vec{x}}(\vec{u})$$

The functional Volterra derivative is dependent on the parameter function $g_j(\vec{x})$ at point $\vec{x}$.

This definition can be interpreted from the mathematical expression of the differential in finite dimension:

$$d\Phi = \sum_i \frac{\partial \Phi}{\partial g_i}dg_i \ ,$$

transposed for a continuum:

$$d\Phi = \int \mathscr{D}_{g_i}\Phi dg_i(\vec{x}).d\vec{x} \ .$$

It is the Fréchet differential at "point" $g_i(\vec{x})$, which is linear with respect to dg_i.

The consequent following relations can be used in practice:

$$d_{g_i}\Phi[h_i] = \frac{d}{d\varepsilon}\Phi[g_i + \varepsilon h_i] \text{ with } h_i(\vec{x}) = dg_i(\vec{x}),$$

$$\mathscr{D}_{g_j(\vec{y})}\Phi = \lim_{\varepsilon \to 0} \frac{\Phi[g_j + \varepsilon\delta(\vec{x} - \vec{y})] - \Phi[g_j]}{\varepsilon},$$

δ being the Dirac distribution at point $\vec{x} = \vec{y}$.

A generalized Taylor series development of Φ in which higher order differentials appear, thus provides all the statistical moments of the probability distribution.

$$\Phi(g_j) = \Phi(g_{0j}) + \int \mathcal{D}_{g_{0j}(\vec{x})} \Phi . h_j(\vec{x}) . d\vec{x} + \frac{1}{2!} \iint \mathcal{D}^2 g_{0j}(\vec{x}), g_{0q}(\vec{y}) \Phi . h_j(\vec{x}) h_q(\vec{y}) . d\vec{x} d\vec{y} + \dots$$

The characteristic function $\Phi[\theta_m(\vec{x}), t]$ satisfies the Hopf equation, consequence of the Navier-Stokes equations:

$$\frac{\partial \Phi}{\partial t} = i \left\langle \tilde{\theta}_m \frac{\partial \mathcal{D}_m \mathcal{D}_\alpha \Phi}{\partial x_\alpha} \right\rangle + v \left\langle \tilde{\theta}_m . \Delta \mathcal{D}_m \Phi \right\rangle .$$

The vector field $\tilde{\theta}_m$ is defined by $\vec{\theta} = \tilde{\vec{\theta}} + \text{grad}\,\Phi$ with $\text{div}\,\tilde{\vec{\theta}} = 0$ and $\tilde{\theta}_m \big|_{\partial \Omega} = 0$ at the boundary.

This splitting allows us to eliminate the pressure functional in the Hopf equation.

In spectral space, we can also define the functional:

$$\Psi \left[\vec{\zeta}(\vec{\kappa}), t \right] = \Phi \left[(2\pi)^{-3} \int e^{-i\vec{\kappa}.\vec{x}} \zeta(\vec{\kappa}) d\vec{\kappa}, t \right],$$

$$\vec{\zeta}(\vec{\kappa}) = \int e^{-i\vec{\kappa}.\vec{x}} \vec{\Theta}(\vec{x}) d\vec{x},$$

$$\Psi \left[\vec{\zeta}, t \right] = \exp \left\{ i \overline{\left\langle \vec{\zeta}^* . \hat{u} \right\rangle} \right\},$$

with $\left\langle \vec{\zeta}^* . \hat{u} \right\rangle \equiv \left\langle \vec{\Theta} . \vec{u} \right\rangle$.

Approximate methods towards the analytical solution of the Hopf equation have been proposed (cf. Monin A.S. and Yaglom A.M., [*MON 71]).

1.7. Turbulent diffusion and Lagrangian description

The diffusive properties of turbulent flows are a matter for the diffusion theory by continuous movements. This theory deals with the movement of particles

entrained by the fluid using Lagrangian description. If $\xi_i(X_j, t/t_0)$ stands for the prior position at the initial instant of time t_0 of the particle which is located in X_j (Lagrangian coordinates) at the instant of time t, the Lagrangian velocity of the particle is given by $U_i\left(\xi_m, t\right) = \partial X_j\left(\xi_m, t\right)/\partial t$ and its trajectory by:

$$X_i = \xi_i + \int_{t_0}^t U_i\left[\xi_j\left(X_m, t/t_0\right), t_1\right]dt_1$$

The momentum equations can be written in terms of Lagrangian coordinates ([*MON 71]) and the statistical treatment yields the Lagrangian correlations. The displacement of the particle and its fluctuation,

$$Y_j(\tau) = X_j(\xi_m, t_0 + \tau) - \xi_j = \int_{t_0}^{t_0+\tau} U_j(\xi_m, t)dt \ ,$$

$$Y'_j(\tau) = Y_j(\tau) - \overline{Y_j(\tau)} = \int_{t_0}^{t_0+\tau} U'_j(\xi_m, t)dt \ ,$$

allow us to obtain the Lagrangian variance and correlation coefficient:

$$D_{ij}(\tau) = \overline{Y'_i Y'_j} = \int_{t_0}^{t_0+\tau}\int_{t_0}^{t_0+\tau} \overline{U'_i(\xi_m, t_1)U'_j(\xi_m, t_2)}dt_1 dt_2 \ ,$$

$$\overline{U'_i(\xi_m, t_1)U'_j(\xi_m, t_2)} = \left(\overline{u'^2_i u'^2_j}\right)^{1/2} R^L_{ij}(t_2 - t_1) \ ,$$

which satisfy the equation:

$$\frac{dD_{ij}(\tau)}{d\tau} = \left(\overline{u'^2_i u'^2_j}\right)^{1/2} \int_0^\tau \left(R^L_{ij}(s) + R^L_{ji}(s)\right)ds \ .$$

For large values of time τ, we can show ([*MON 71]) that the variance of displacements becomes proportional to τ, while for short values of time it is quadratic in τ. In the case of shear flows, the variance is considerably increased.

Even if Lagrangian methods do indeed give a very detailed description of the phenomena in turbulent dispersion, which is useful in specific cases for fine analysis, the practice of turbulence models usually leads to making use of Eulerian approaches for taking into account the turbulent diffusion effects.

1.8. Two-dimensional turbulence

In two-dimensional turbulence, the characteristic functions are dependent on two space variables x and y, in addition to time t. Thus:

$$\vec{U} \begin{vmatrix} u_x \\ u_y \\ u_z \end{vmatrix} \qquad\qquad \vec{v} \begin{vmatrix} u_x \\ u_y \end{vmatrix} \qquad\qquad \text{and} \qquad\qquad w = u_z \ .$$

In order for a turbulent flow to be two-dimensional, it is necessary that an external constraint force it to remain two-dimensional, otherwise instabilities will develop and bring it back to three-dimensionality.

Two-dimensional turbulent flows are encountered in very particular circumstances: atmospheric turbulence, turbulent flows submitted to high rotation rates, etc. (Lesieur M., [*LES 90]).

The Navier-Stokes equations can be written in the case of two-dimensional turbulence:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \ ,$$

$$\frac{\partial v_j}{\partial x_j} = 0 \ ,$$

$$\frac{\partial w}{\partial t} + v_j \frac{\partial w}{\partial x_j} = \nu \frac{\partial w}{\partial x_j \partial x_j} \ .$$

Let us consider the curl of the velocity field:

$$\omega_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$$

We can deduce from the momentum equations:

$$\frac{\partial \omega_i}{\partial t} + v_j \frac{\partial \omega_i}{\partial x_j} = \nu \frac{\partial \omega_i}{\partial x_j \partial x_j}$$

It can be observed that for $v = 0$, the rotational or curl is conserved. Also taking into account the incompressibility relation, the characteristic property of conservation of the enstrophy $\overline{\omega^2}$ thus results:

$$\overline{\omega^2} = C^{te} \quad \text{if} \quad v = 0$$

The two-dimensional Euler equations thus satisfy the conservation of energy:

$$\frac{d}{dt} \int_0^\infty E(\kappa, t) d\kappa = 0 ,$$

and also the conservation of the enstrophy $J = \int_0^\infty \kappa^2 E(\kappa, t) d\kappa$:

$$\frac{d}{dt} \int_0^\infty \kappa^2 E(\kappa, t) d\kappa = 0 .$$

Two-dimensional turbulence is driven by interaction mechanisms that are very different from those acting in the usual three-dimensional turbulence, in particular, there is no energy cascade towards small eddies. In these interaction mechanisms, the inverse transfer is far more significant than the direct transfer (for the enstrophy to be conserved).

The balance of enstrophy J leads to the equation:

$$\frac{dJ}{dt} = 2v \int_0^\infty \kappa^4 E(\kappa, t) d\kappa$$

in which $\int_0^\infty \kappa^4 E(\kappa, t) d\kappa$ represents the "palinstrophy".
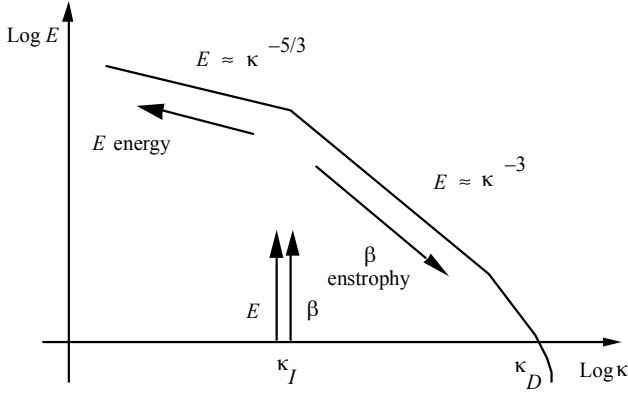
Figure 1.13. *Energy spectrum in two-dimensional turbulence (after Lesieur M., [LES 78A])*

Dimensional reasoning supposing that E is dependent only on enstrophy transfer rate β and wavenumber κ leads to:

$$E(\kappa) \sim \beta^{2/3}\kappa^{-3}$$

If energy and enstrophy are injected about the vicinity of wavenumber κ_I (cf. Figure 1.13.) the enstrophy will cascade towards smaller eddies (large wavenumbers) down to the wavenumber $\kappa_D \sim \left(\beta/\nu^3\right)^{1/6}$, this is the enstrophy cascade, while the energy will go back towards larger eddies (smaller wavenumbers) according to a $\kappa^{-5/3}$ spectrum (cf. Lesieur M., [*LES 90] in particular).

Chapter 2

Turbulence Transport Equations
for an Incompressible Fluid

2.1. General transport equations

The statistical method is based on a hierarchy of moment equations. These equations thus form the basis for the description of the turbulent field and reasoning will be developed by physical interpretation of the terms of these equations.

We shall give in the present chapter the master equations for the first moments which will be used for developing turbulence models in incompressible fluid. All these equations are obtained by taking the statistical mean of instantaneous equations and applying the "Reynolds rules" (cf. Chapter 1).

All these transport equations can be written in the general form:

$$\frac{d\phi}{dt} = \text{Production} + \text{Diffusion} - \text{Dissipation} + \text{Additional Sources} \qquad [2.1]$$

where $\dfrac{d}{dt}$ stands for the material derivative.

Let us recall that the material derivative of a quantity ϕ is defined by:

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + U_j\frac{\partial\phi}{\partial x_j} \ .$$

The material derivative represents the variation rate of the quantity ϕ in the direction of the fluid flow, following the fluid particle. In a change of Galilean frame of reference, the convective term $U_j \partial\phi/\partial x_j$ considered in itself is not conserved, but the material derivative as a whole is conserved.

We shall use, unless otherwise stated, the hypothesis of summation of dummy indices (Einstein summation convention) which implies:

$$A_j B_j = \sum_{j=1,2,3} A_j B_j .$$

The transport equations express the balance of the source/sink of a quantity ϕ inside a control volume.

Equation [2.1] is the local consequence of the integral balance equation applied to a fluid domain Ω entrained by the flow. Thus,

$$\frac{d}{dt}\int_\Omega \phi.dV = \int_\Omega S.dV + \oint_{\partial\Omega} F.dS ,$$

where S represents the volumic sources and F the diffusive flux across the boundary $\partial\Omega$, i.e., also:

$$\int_\Omega \frac{\partial\phi}{\partial t}.dV = \int_\Omega S.dV + \oint_{\partial\Omega} (F - \phi\overline{U}_j.n_j)dS ,$$

where $\phi\overline{U}_j.n_j$ represents the convective flux across the boundary $\partial\Omega$.

In the case of mean equations, it will be more convenient to make use of the mean material derivative:

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \overline{U}_j \frac{\partial\phi}{\partial x_j} ,$$

which is the derivative in the direction of mean flow.

Although it is a misuse of language, we shall continue to call "transport equation" every equation written in the form of [2.1] where ϕ is a pointwise

quantity characteristic of the turbulent fluid (ϕ is no longer necessarily a transportable quantity with conservative properties in the physical sense).

Some of the terms, often encountered, have an immediate physical meaning:

$$\sigma \frac{\partial \phi}{\partial x_j} , \qquad\qquad \text{molecular flux}$$

$$\overline{\phi u_j} , \qquad\qquad \text{turbulent diffusion flux}$$

$$\overline{U}_j \frac{\partial \overline{\phi}}{\partial x_j} , \qquad\qquad \text{convection}$$

$$\sigma \overline{\frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_j}} , \qquad\qquad \text{dissipative sink term}$$

2.1.1. *Transport equations of dynamics*

The Navier-Stokes equations:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial U_i}{\partial x_j}\right) \qquad\qquad [2.2]$$

are a direct consequence of the momentum conservation theorem.

The mass conservation equation or continuity equation can be written for an incompressible fluid:

$$\frac{\partial U_i}{\partial x_i} = 0 . \qquad\qquad [2.3]$$

The divergence of equation [2.2], taking into account [2.3], yields the Poisson equation for pressure:

$$\frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{P}{\rho}\right) = -\frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} .$$

The Reynolds equations (obtained by averaging equation [2.2] and introducing the Reynolds decomposition)

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\overline{P}}{\rho}\right) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \overline{U}_i}{\partial x_j} - \overline{u_i u_j}\right) \tag{2.4}$$

give rise to the double velocity correlations $\overline{u_i u_j}$ which express the influence of the turbulent field on the mean field. They can be interpreted as apparent stresses $\rho \overline{u_i u_j}$ which form the Reynolds stress tensor. This tensor is symmetric and its trace is equal to twice the kinetic energy of turbulence.

The continuity equation of the mean field is obtained by averaging equation [2.3]:

$$\frac{\partial \overline{U}_i}{\partial x_i} = 0 \,. \tag{2.5}$$

The fluctuating velocity equation is obtained by difference [2.2]-[2.4]:

$$\frac{\partial u_i}{\partial t} + \overline{U}_j \frac{\partial u_i}{\partial x_j} = -u_j \frac{\partial \overline{U}_i}{\partial x_j} - u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j} + \overline{u_i u_j}\right). \tag{2.6}$$

The divergence of the velocity fluctuations (equation obtained by difference [2.3]-[2.5]), like the divergence of mean velocity, also vanishes:

$$\frac{\partial u_i}{\partial x_i} = 0 \,. \tag{2.7}$$

The evolution equations of various moments are obtained by statistical treatment:

– Equation for the double velocity correlations $R_{ij} = \overline{u_i u_j}$ (or Reynolds stresses) obtained by $\overline{u_i \times Eq.of\ u_j + u_j \times Eq.of\ u_i}$:

$$\frac{\partial R_{ij}}{\partial t} + \overline{U}_m \frac{\partial R_{ij}}{\partial x_m} = -R_{im}\frac{\partial \overline{U}_j}{\partial x_m} - R_{jm}\frac{\partial \overline{U}_i}{\partial x_m} + \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

$$-\frac{\partial}{\partial x_m}\left(\overline{u_i u_j u_m}\right) - \frac{\partial}{\partial x_m}\left[\overline{\frac{p}{\rho}\left(u_i \delta_{jm} + u_j \delta_{im}\right)}\right] - 2\nu \overline{\frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m}} + \nu \frac{\partial^2 R_{ij}}{\partial x_m \partial x_m} \tag{2.8}$$

This equation shows on its right-hand side several contributions: the first two terms express the action of mean velocity gradients on the turbulent field and are interpreted as the production of the Reynolds stresses (its sign can be negative in some particular cases); the pressure-strain term vanishes by tensorial contraction and thus causes a redistribution of energy among the components of the Reynolds stress tensor; the following terms express the diffusive transport due to velocity fluctuations and pressure fluctuations; then comes a destruction term called viscous dissipation by misuse of language because it represents a sink term due to molecular viscosity; finally comes a viscous diffusion term. Schematically, we will write:

$$\frac{\partial R_{ij}}{\partial t} + \overline{U}_m \frac{\partial R_{ij}}{\partial x_m} = P_{ij} + \Phi_{ij} + D_{ij} - \varepsilon_{ij} + \nu \frac{\partial^2 R_{ij}}{\partial x_m \partial x_m}$$

Modeling of this equation will be considered in Chapter 6.

– Equation for kinetic energy of turbulence $k = \frac{1}{2} R_{jj}$:

$$\frac{\partial k}{\partial t} + \overline{U}_j \frac{\partial k}{\partial x_j} = -R_{jm} \frac{\partial \overline{U}_j}{\partial x_m} - \frac{\partial}{\partial x_m} \left(\overline{u_m \frac{u_j u_j}{2}} \right) - \frac{\partial}{\partial x_j} \left(\overline{\frac{u_j p}{\rho}} \right)$$

$$- \nu \overline{\frac{\partial u_j}{\partial x_m} \frac{\partial u_j}{\partial x_m}} + \nu \frac{\partial^2 k}{\partial x_m \partial x_m} \ . \tag{2.9}$$

This equation, which can be obtained by tensorial contraction of the Reynolds stress equation, expresses that the variation of kinetic energy of turbulence is due to the effect of the different terms in the right-hand side: production of energy by the action of mean velocity gradients on the turbulence, diffusive transport due to velocity and pressure fluctuations, viscous dissipation and finally molecular diffusion. Schematically, we will write:

$$\frac{\partial k}{\partial t} + \overline{U}_m \frac{\partial k}{\partial x_m} = P + D - \varepsilon + \nu \frac{\partial^2 k}{\partial x_m \partial x_m} \ .$$

The equation for k is used in particular in simplified one-equation and two-equation transport models (cf. Chapters 11 and 12).

– Equation for triple correlations $T_{ijk} = \overline{u_i u_j u_k}$ (obtained by $\overline{u_i u_j \times Eq.of \ u_k + u_j u_k \times Eq.of \ u_i + u_i u_k \times Eq.of \ u_j}$):

$$\frac{\partial T_{ijk}}{\partial t} + \overline{U}_m \frac{\partial T_{ijk}}{\partial x_m} = -T_{ijm} \frac{\partial \overline{U}_k}{\partial x_m} - T_{ikm} \frac{\partial \overline{U}_j}{\partial x_m} - T_{jkm} \frac{\partial \overline{U}_i}{\partial x_m}$$

$$-\frac{1}{\rho}\left(\overline{u_i u_j \frac{\partial p}{\partial x_k}} + \overline{u_i u_k \frac{\partial p}{\partial x_j}} + \overline{u_j u_k \frac{\partial p}{\partial x_i}} \right) + \nu \frac{\partial^2 T_{ijk}}{\partial x_m \partial x_m}$$

$$+\left(R_{jk} \frac{\partial R_{im}}{\partial x_m} + R_{ki} \frac{\partial R_{jm}}{\partial x_m} + R_{ij} \frac{\partial R_{km}}{\partial x_m} \right) - \frac{\partial}{\partial x_m}\left(\overline{u_m u_i u_j u_k} \right)$$

$$-2\nu\left(\overline{u_i \frac{\partial u_j}{\partial x_m} \frac{\partial u_k}{\partial x_m}} + \overline{u_j \frac{\partial u_i}{\partial x_m} \frac{\partial u_k}{\partial x_m}} + \overline{u_k \frac{\partial u_i}{\partial x_m} \frac{\partial u_j}{\partial x_m}} \right). \qquad [2.10]$$

Third order closures are seldom used. Nevertheless, the interpretation of source terms in the formal equations of third order moments is particularly useful for physical analysis of the effects of triple correlations (cf. Chapter 6).

– Equation for the dissipation rate of turbulent kinetic energy $\varepsilon = \nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}}$

(obtained by: $2\nu \overline{\frac{\partial u_i}{\partial x_j} \times \frac{\partial}{\partial x_j}(Eq.of\,u_i)}$):

$$\frac{\partial \varepsilon}{\partial t} + \overline{U}_j \frac{\partial \varepsilon}{\partial x_j} = -2\nu \overline{\frac{\partial u_k}{\partial x_j} \frac{\partial u_m}{\partial x_j}} \frac{\partial \overline{U}_m}{\partial x_k} - 2\nu \overline{\frac{\partial u_j}{\partial x_k} \frac{\partial u_j}{\partial x_m}} \frac{\partial \overline{U}_m}{\partial x_k}$$

$$-2\nu \overline{\left(u_k \frac{\partial u_j}{\partial x_m} \right)} \frac{\partial^2 \overline{U}_j}{\partial x_k \partial x_m} - 2\nu \overline{\frac{\partial u_j}{\partial x_m} \frac{\partial u_k}{\partial x_m} \frac{\partial u_j}{\partial x_k}} - \nu \frac{\partial}{\partial x_k}\overline{\left(u_k \left(\frac{\partial u_j}{\partial x_m} \right)^2 \right)}$$

$$-2\frac{\nu}{\rho} \overline{\frac{\partial u_j}{\partial x_m} \frac{\partial^2 p}{\partial x_m \partial x_j}} + \nu \frac{\partial^2 \varepsilon}{\partial x_m \partial x_m} - 2\nu^2 \overline{\left(\frac{\partial}{\partial x_k} \frac{\partial u_j}{\partial x_m} \right)^2}. \qquad [2.11]$$

The equation for dissipation rate of turbulent kinetic energy is more tricky for analyzing and modeling (cf. Chapter 7). It is mainly used for calculating the characteristic length scale of turbulence.

2.1.2. *Transport equations of turbulence associated with a passive scalar*

We are supposing that the turbulent field is transporting a scalar quantity (such as temperature or concentration of a contaminant).

The instantaneous convection-diffusion equation of a passive scalar can be written:

$$\frac{\partial \Gamma}{\partial t} + U_j \frac{\partial \Gamma}{\partial x_j} = \sigma \frac{\partial^2 \Gamma}{\partial x_j \partial x_j}.$$

[2.12]

In the particular case of heat transfer, coefficient σ is the thermal diffusivity of the fluid, related to the thermal conductivity λ and to the specific heat C_p by $\sigma = \lambda / \rho C_p$. The heat flux is $J_i = -\lambda \frac{\partial \Gamma}{\partial x_i}$ while the temperature flux is

$$\mathscr{J}_i = \frac{J_i}{\rho C_p} = -\sigma \frac{\partial \Gamma}{\partial x_i}.$$

The statistical equations can be obtained as previously:

– Transport equation of a passive scalar in the mean (averaging equation [2.12] taking into account the decomposition $\Gamma = \overline{\Gamma} + \gamma$):

$$\frac{\partial \overline{\Gamma}}{\partial t} + \overline{U}_j \frac{\partial \overline{\Gamma}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\sigma \frac{\partial \overline{\Gamma}}{\partial x_j} - \overline{u_j \gamma} \right).$$

[2.13]

– Equation for the scalar fluctuation (obtained by the difference equation [2.12] – equation [2.13]):

$$\frac{\partial \gamma}{\partial t} + \overline{U}_j \frac{\partial \gamma}{\partial x_j} = -u_j \frac{\partial \overline{\Gamma}}{\partial x_j} - u_j \frac{\partial \gamma}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\sigma \frac{\partial \gamma}{\partial x_j} - \overline{u_j \gamma} \right).$$

[2.14]

– Equation for turbulent fluxes of a passive scalar $F_{\gamma i} = \overline{u_i \gamma}$, (obtained by $\overline{u_i \times Eq.of \ \gamma + \gamma \times Eq.of \ u_i}$):

$$\frac{\partial F_{\gamma i}}{\partial t} + \overline{U}_j \frac{\partial F_{\gamma i}}{\partial x_j} = -R_{ij} \frac{\partial \overline{\Gamma}}{\partial x_j} - F_{\gamma j} \frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial}{\partial x_j} \overline{u_i u_j \gamma}$$

$$-\frac{1}{\rho} \overline{\left(\gamma \frac{\partial p}{\partial x_i} \right)} + v\gamma \overline{\frac{\partial^2 u_i}{\partial x_m \partial x_m}} + \sigma u_i \overline{\frac{\partial^2 \gamma}{\partial x_m \partial x_m}}.$$

[2.15]

The equations for the turbulent fluxes of a scalar allow a more refined approach of turbulent diffusion than the one given by gradient diffusion theories. Its closure is studied in Chapter 9.

– Equation for the variance of a passive scalar $q = \dfrac{1}{2}\overline{\gamma^2}$ (obtained by $\overline{\gamma \times Eq.of\ \gamma}$).

The notation q stands for the intensity of the fluctuations of the transported scalar.

$$\frac{\partial q}{\partial t} + \overline{U}_j \frac{\partial q}{\partial x_j} = -F_{\gamma j}\frac{\partial \overline{\Gamma}}{\partial x_j} - \frac{\partial}{\partial x_m}\overline{\left(u_m \frac{\gamma^2}{2} \right)} + \sigma\frac{\partial^2 q}{\partial x_j \partial x_j} - \sigma\overline{\frac{\partial \gamma}{\partial x_j}\frac{\partial \gamma}{\partial x_j}} \qquad [2.16]$$

– Equation for the dissipation rate of the variance of a passive scalar, $\varepsilon_\gamma = \sigma\overline{\dfrac{\partial \gamma}{\partial x_j}\dfrac{\partial \gamma}{\partial x_j}}$ (obtained by $2\sigma\overline{\dfrac{\partial \gamma}{\partial x_j} \times \dfrac{\partial}{\partial x_j}\left(Eq.of\ \gamma \right)}$):

$$\frac{\partial \varepsilon_\gamma}{\partial t} + \overline{U}_j \frac{\partial \varepsilon_\gamma}{\partial x_j} = -2\sigma\overline{\frac{\partial \gamma}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\frac{\partial \overline{\Gamma}}{\partial x_i} - 2\gamma u_i\overline{\frac{\partial \gamma}{\partial x_j}}\frac{\partial^2 \overline{\Gamma}}{\partial x_i \partial x_j}$$

$$-2\sigma\overline{\frac{\partial \gamma}{\partial x_i}\frac{\partial \gamma}{\partial x_j}}\frac{\partial \overline{U}_i}{\partial x_j} - 2\sigma\overline{\frac{\partial \gamma}{\partial x_j}\frac{\partial u_i}{\partial x_j}\frac{\partial \gamma}{\partial x_i}} - 2\sigma\overline{\left(\frac{\partial^2 \gamma}{\partial x_i \partial x_j} \right)^2}$$

$$-\sigma\frac{\partial}{\partial x_i}\overline{\left[u_i \left(\frac{\partial \gamma}{\partial x_j} \right)^2 \right]} + \sigma\frac{\partial^2 \varepsilon_\gamma}{\partial x_i \partial x_i} \ . \qquad [2.17]$$

The physical interpretation of terms in these two last equations is more complex because characteristic scales of the scalar are involved in addition to the purely dynamic scales, and it will be detailed in the chapters devoted to closure models (Chapter 10).

2.1.3. *Equations for quantities related to vorticity*

– Equation for the instantaneous vorticity $\Omega_i = \varepsilon_{ijk}\dfrac{\partial U_k}{\partial x_j}$

This equation is obtained by taking the curl of the Navier-Stokes equations, having noted in particular that the term $\varepsilon_{ijk}\left(\dfrac{P}{\rho}\right)_{,jk}$ vanishes because of the property of antisymmetry of ε_{ijk} (cf. Chapter 3). Moreover, the term $\varepsilon_{ijk}\dfrac{\partial U_q}{\partial x_k}\dfrac{\partial U_j}{\partial x_q}$ can also be written $\varepsilon_{ijk}\dfrac{\partial U_q}{\partial x_k}\left(\dfrac{\partial U_j}{\partial x_q}-\dfrac{\partial U_q}{\partial x_j}\right)$, using the relation $\dfrac{\partial U_j}{\partial x_q}-\dfrac{\partial U_q}{\partial x_j}=-\varepsilon_{jqh}\Omega_h$, we find that the initial term is equal to $-\dfrac{\partial U_i}{\partial x_j}\Omega_j$.

$$\frac{\partial \Omega_i}{\partial t}+U_j\frac{\partial \Omega_i}{\partial x_j}=+\Omega_j\frac{\partial U_i}{\partial x_j}+\frac{\partial}{\partial x_j}\left(v\frac{\partial \Omega_i}{\partial x_j}\right),$$

or
$$\frac{\partial \Omega_i}{\partial t}+U_j\frac{\partial \Omega_i}{\partial x_j}=+\Omega_j S_{ij}+\frac{\partial}{\partial x_j}\left(v\frac{\partial \Omega_i}{\partial x_j}\right).$$

Indeed, the relation $\Omega_i\left(\dfrac{\partial U_i}{\partial x_j}-\dfrac{\partial U_j}{\partial x_i}\right)=-\varepsilon_{ijm}\Omega_i\Omega_m=0$ implies $\Omega_i\dfrac{\partial U_i}{\partial x_j}=\Omega_i S_{ij}$, with $S_{ij}=\dfrac{1}{2}\left(\dfrac{\partial U_i}{\partial x_j}+\dfrac{\partial U_j}{\partial x_i}\right)$. The vorticity is also divergence-free $\dfrac{\partial \Omega_i}{\partial x_i}=0$.

We can notice that in two-dimensional flows, the term $\Omega_j S_{ij}$ vanishes.

The mean equations for vorticity are:

$$\frac{\partial \overline{\Omega}_i}{\partial t}+\overline{U}_j\frac{\partial \overline{\Omega}_i}{\partial x_j}=+\overline{\Omega}_j\frac{\partial \overline{U}_i}{\partial x_j}-\overline{u_j\frac{\partial \omega_i}{\partial x_j}}+\overline{\omega_j\frac{\partial u_i}{\partial x_j}}+\frac{\partial}{\partial x_j}\left(v\frac{\partial \overline{\Omega}_i}{\partial x_j}\right),$$

or
$$\frac{\partial \overline{\Omega}_i}{\partial t}+\overline{U}_j\frac{\partial \overline{\Omega}_i}{\partial x_j}=+\overline{\Omega}_j\overline{S}_{ij}-\overline{u_j\frac{\partial \omega_i}{\partial x_j}}+\overline{\omega_j s_{ij}}+\frac{\partial}{\partial x_j}\left(v\frac{\partial \overline{\Omega}_i}{\partial x_j}\right),$$

and for the fluctuating part:

$$\frac{\partial \omega_i}{\partial t} + \overline{U}_j \frac{\partial \omega_i}{\partial x_j} = -u_j \frac{\partial \overline{\Omega}_i}{\partial x_j} + \omega_j \overline{S}_{ij} + \overline{\Omega}_j s_{ij} - \left(u_j \frac{\partial \omega_i}{\partial x_j} - \overline{u_j \frac{\partial \omega_i}{\partial x_j}} \right)$$

$$+ \left(\omega_j s_{ij} - \overline{\omega_j s_{ij}} \right) + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial \omega_i}{\partial x_j} \right),$$

with $\dfrac{\partial \overline{\Omega}_i}{\partial x_i} = 0$ and $\dfrac{\partial \omega_i}{\partial x_i} = 0$.

– Equation for instantaneous enstrophy:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \Omega_i \Omega_i \right) + U_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} \Omega_i \Omega_i \right) = +\Omega_i \Omega_j S_{ij}$$

$$+\nu \frac{\partial^2}{\partial x_j \partial x_j} \left(\frac{1}{2} \Omega_i \Omega_i \right) - \nu \frac{\partial \Omega_i}{\partial x_j} \frac{\partial \Omega_i}{\partial x_j} .$$

– Equation for enstrophy of turbulence $\dfrac{1}{2} \overline{\omega_i \omega_i}$:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \overline{\omega_i \omega_i} \right) + \overline{U}_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} \overline{\omega_i \omega_i} \right) = -\overline{u_j \omega_i} \frac{\partial \overline{\Omega}_i}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{u_j \left(\frac{1}{2} \omega_i \omega_i \right)} + \overline{\omega_i \omega_j s_{ij}}$$

$$+\overline{\omega_i \omega_j}.\overline{S}_{ij} + \overline{\Omega}_j \overline{\omega_i s_{ij}} + \nu \frac{\partial^2}{\partial x_j \partial x_j} \left(\frac{1}{2} \overline{\omega_i \omega_i} \right) - \nu \overline{\frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}} .$$

We will notice the analogy of this equation with the equation of dissipation rate of turbulence energy.

– Equation for helicity of turbulence $H = \overline{u_i \omega_i}$:

$$\frac{\partial H}{\partial t} + \overline{U}_j \frac{\partial H}{\partial x_j} = -R_{ij} \frac{\partial \overline{\Omega}_i}{\partial x_j} + \left(\overline{u_i \omega_j} - \overline{u_j \omega_i} \right) \frac{\partial \overline{U}_i}{\partial x_j} + \overline{\Omega}_j \overline{u_i s_{ij}}$$

$$+\overline{u_i \omega_j s_{ij}} - \overline{\omega_i \frac{\partial}{\partial x_i} \left(\frac{p}{\rho} \right)} - \frac{\partial}{\partial x_j} \left(\overline{u_j H'} \right) + \nu \frac{\partial^2 H}{\partial x_j \partial x_j} - 2\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}} .$$

2.2. Equations specific to the main types of turbulent flows

The physical knowledge of the properties of traditional turbulent flows and their specific characters, form an essential contribution to the development of turbulence modeling. Numerous experimental studies have been amassed for many years on various types of flows using diverse investigation techniques: measurements by hot wire anemometry or lazer anemometry, supplemented by visualizations, etc. Nowadays, add to that the databases generated by direct numerical simulations of turbulent flows (at low Reynolds numbers). All this knowledge is useful, not only to reach the physical intuition of observed phenomena which is necessary for conceiving turbulence models but also for testing, calibrating and improving already developed models. The interested reader can refer to the vast existing scientific literature (numerous papers in scientific journals and several synthetic books [*HIN 75], [*MON 71], [*BAI 03], [*CHA 00], [*PIQ 99], [*MAT 00]). We believe it is nevertheless useful to make a short visit to the "menagerie" of turbulent flows studied in laboratories, while specifying for each of them the particular equations and the usual simplifying hypotheses in the study of these flows. These relations will be useful in the following in particular for calibrating the numerical constants in the models.

2.2.1. *Free flows: jets*, *wakes*, *mixing layers*

Traditional laboratory free flows are generally represented by jets, wakes behind an obstacle and mixing layers (Figures 2.1 to 2.3); they can be plane or axisymmetric.
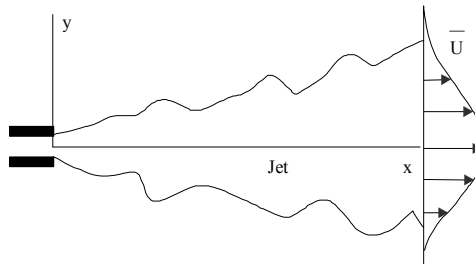
Figure 2.1. *Sketch of a turbulent jet developing into a stagnant atmosphere*

We can also mention the turbulent plumes (Figure 2.4) that are created by buoyancy forces. The external boundaries of these free flows are always irregular, due to boundary intermittency. The external region of a jet located beyond the boundary, although non-turbulent, can be animated by induced irrotational fluctuations.
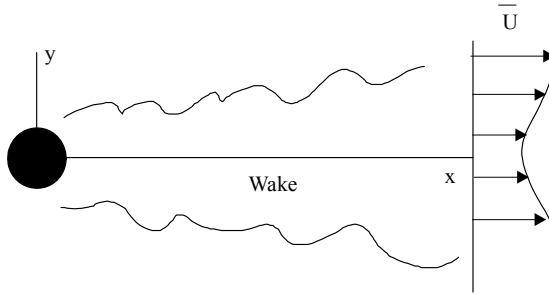
Figure 2.2. *Sketch of a turbulent wake developing behind an obstacle*

We can generally distinguish in these free flows, an initial region, in the immediate vicinity of the obstacle if it is a wake, or just after the nozzle if it is a jet, this region is followed by a transition region and then by the main region. In this main region, it is possible to make the hypotheses of thin shear flow (that are in fact identical to the boundary layer approximations) $\dfrac{\partial}{\partial x} \ll \dfrac{\partial}{\partial y}$. Thus, the momentum equations can be greatly simplified.
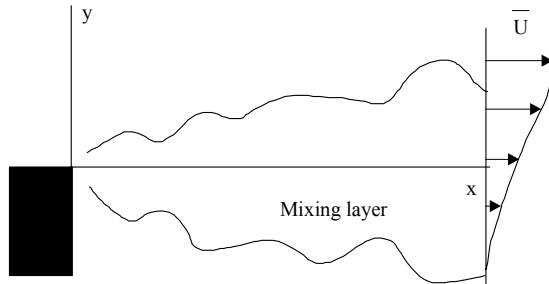
Figure 2.3. *Sketch of a turbulent mixing layer developing downstream of a step*

The main region generally has scale similarity properties, i.e. the solution of the momentum equations can be written in the form $\dfrac{U}{U_e} = \varphi(\eta)$ with $\eta = \eta(x, y) = y / x^{\alpha}$ and where U_e stands for the characteristic mean velocity (axial velocity of the jet, axial velocity defect for the wake, etc.).
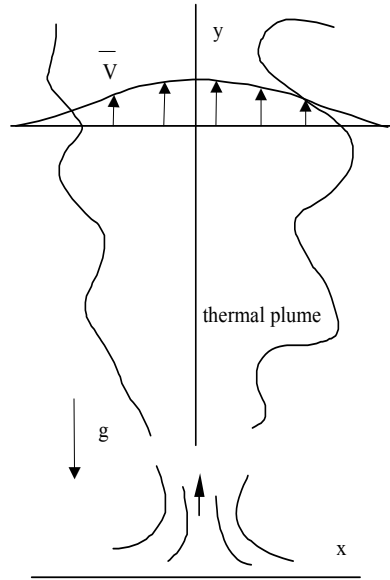
Figure 2.4. *Sketch of a thermal plume*

This is equivalent to a separation in variables $U = U_e(X).\varphi(\eta)$, which further simplify the equations and their numerical solution. Similarity, depending on the type of flow, can be exact or approximate. We can easily show that under these conditions $U_e(x) \approx x^{\beta}$, with analogous properties for all the other mean quantities (Reynolds stresses, etc.).

Plane wake	$\alpha = 1/2$	$\beta = -1/2$
Axisymmetric wake	1/3	-2/3
Plane jet	1	-1/2
Axisymmetric jet	1	-1
Plane plume	1	0
Axisymmetric plume	1	-1/3
Plane mixing layer	1	0

Table 2.1. *Values of α and β for the usual turbulent free flows*

The values of α and β in the usual turbulent free flows are very different from the ones prevailing in laminar flows, in which similarity also exists. The simplification of the transport equations for momentum and turbulence in incompressible plane free flows is made on the basis of thin shear flow approximations. The mean momentum equations thus read:

first momentum equation: $U \dfrac{\partial U}{\partial x} + V \dfrac{\partial U}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} - \dfrac{\partial R_{12}}{\partial y}$,

second momentum equation: $0 = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} - \dfrac{\partial R_{22}}{\partial y}$

$\Rightarrow \quad \dfrac{P}{\rho} + R_{22} = \text{constant}$

For illustrating the equations of the turbulent field, let us take the example of the k-ε model equations which will be developed in the subsequent chapters.

$$U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y} = -R_{12}\frac{\partial U}{\partial y} - \varepsilon + \frac{\partial}{\partial y}\left(\frac{v_t}{\text{Pr}_k}\frac{\partial k}{\partial y}\right)$$

$$U\frac{\partial \varepsilon}{\partial x} + V\frac{\partial \varepsilon}{\partial y} = -C_{\varepsilon 1}\frac{\varepsilon}{k}R_{12}\frac{\partial U}{\partial y} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial y}\left(\frac{v_t}{\text{Pr}_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)$$

$$R_{12} = -c_\mu \frac{k^2}{\varepsilon}\frac{\partial U}{\partial y}$$

For obtaining the transport equations in axisymmetric flows see Chapter 3 for tensor equations in cylindrical coordinates.

2.2.2. Semi-free flows: boundary layer, wall jets

Flows that are limited by walls are more complicated than free flows because the presence of the wall introduces new additional characteristic scales.

Depending on the distance from the wall, we can distinguish in the flat plate boundary layer (Figure 2.5) several regions which are governed by different scales. In the framework of the boundary layer approximations, the first momentum equation can be written:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{\partial R_{12}}{\partial y} + v\frac{\partial^2 U}{\partial y^2}\ ,$$

and the second:

$$\frac{P}{\rho} + R_{22} = \text{constant}$$

Turbulent boundary layer on a flat plate
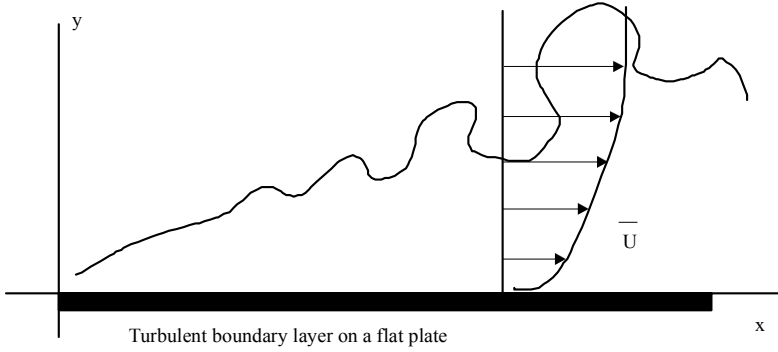
Figure 2.5. *Sketch of the flat plate turbulent boundary layer*

In a boundary layer without pressure gradient, $\partial P / \partial x = 0$.

The equations of the turbulent field will be written as previously but this time using a low Reynolds number version of the turbulence model in order to describe the region very close to the wall influenced by the molecular viscosity.
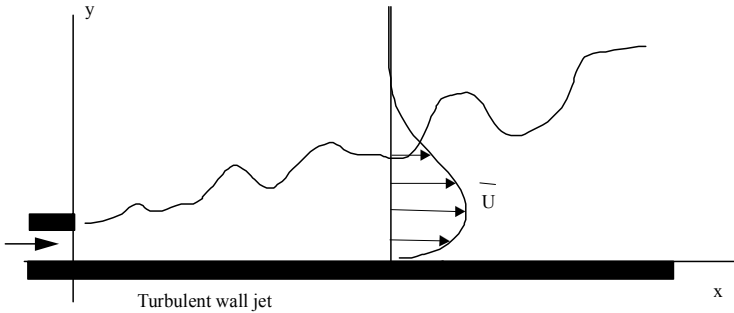
Turbulent wall jet

Figure 2.6. *Sketch of a turbulent wall jet*

The wall jet (Figure 2.6) is similar to a boundary layer near its lower boundary and similar to a free jet near its upper boundary.

In the region very near the wall in the boundary layer we notice that the molecular viscosity is dominant compared to turbulence viscosity and that consequently $\nu \dfrac{\partial U}{\partial y} = \dfrac{\tau_0}{\rho}$, the mean velocity thus varies linearly $U = \dfrac{\tau_0}{\mu} y$; it is the viscous sublayer. The wall friction is given by the wall shear stress $\tau_0 = \mu \left(\dfrac{\partial U}{\partial y} \right)_w$.

On the other hand, when the distance y increases, the stresses related to molecular viscosity become negligible. Dimensional analysis and similarity considerations (Monin and Yaglom, [*MON 71]) lead us to assume $\dfrac{\partial U}{\partial y} = \dfrac{1}{\mathbb{K}} \dfrac{u_*}{y}$ where K is the Karman constant, this relation after integration yields the logarithmic velocity profile (Figure 2.7).

When the distance y further increases, the stresses decrease progressively with the turbulence level and then go to zero on the boundary which moreover is intermittent.

Quantitively, we can say that near the wall, the flow is dependent on the parameters u_* (velocity scale) and v/u_* (length scale), from which we deduce the wall law $\dfrac{U}{u_*} = f\left(\dfrac{yu_*}{v}\right)$ and $-\dfrac{R_{12}}{u_*^2} = g\left(\dfrac{yu_*}{v}\right)$; this is the internal region which is subdivided into the viscous sublayer $U = \dfrac{u_*^2}{v} y$ and the logarithmic layer

$$U^+ = \frac{1}{\mathbb{K}} Log\ Ey^+ , \mathrm{K} = 0.41, E = 9.0, (U^+ = \frac{U}{u_*}, y^+ = \frac{yU^+}{v}).$$

Further from the wall, it is admitted that the velocity defect is dependent on u_*^2 and δ the boundary layer thickness. It is the external region, in which prevails the velocity-defect Von Karman law $\dfrac{U_{ext} - U}{u_*} = F\left(\dfrac{y}{\delta}\right)$, $\dfrac{\tau}{\tau_0} = G\left(\dfrac{y}{\delta}\right)$.

In a (fictitious) boundary layer with infinite thickness and without pressure gradient we would have $-R_{12} + v \dfrac{\partial U}{\partial y} = $ constant at any y.

On the wall $R_{12} = 0$ and then $v\left(\dfrac{\partial U}{\partial y}\right)_0 = $ constant , so that $-R_{12} + v\dfrac{\partial U}{\partial y} = u_*^2$.

Further from the wall, $v\dfrac{\partial U}{\partial y}$ becomes negligible and then $-R_{12} = u_*^2$.

In the framework of gradient closure hypothesis, we would write $-R_{12} = v_t \dfrac{\partial U}{\partial y}$

with $v_t = u_* \mathrm{K} y$ and $\dfrac{\partial U}{\partial y} = \dfrac{u_*}{\mathrm{K} y}$.
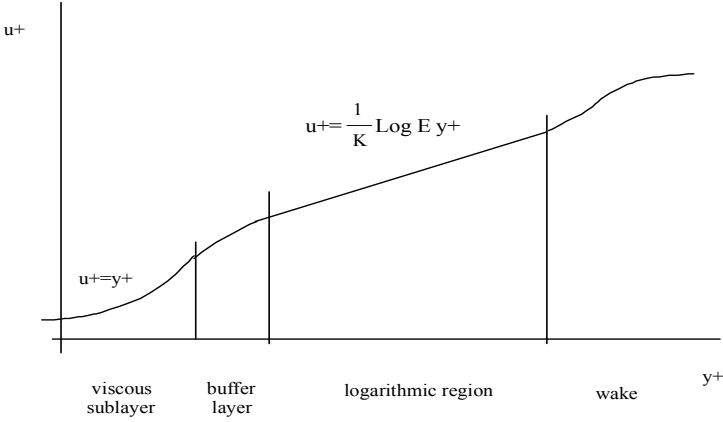
Figure 2.7. *Mean velocity profile in a turbulent boundary layer in logarithmic coordinates*

In the logarithmic layer, the molecular viscosity is negligible, compared to turbulence viscosity. The physical arguments which have led to the universal velocity law and the logarithmic velocity profile also extend to all the one point statistical moments. These moments will be dependent only on y, τ, v and thus can be noted in the form:

$$\underbrace{\overline{u_i u_j .... u_q}}_{n \text{ terms}} = u_*^n f\left(\frac{y u_*}{v} \right)$$

When the distance y is sufficiently large (logarithmic zone), the statistical regime of the turbulent fluctuations is no longer dependent on v and these universal functions go to a constant:

$$\underbrace{\overline{u_i u_j .... u_q}}_{n} = A_{(i, j, ... q)} \, u_*^n$$

in particular $k = C . u_*^2$ and $R_{ij} = C_{(i, j)} . u_*^2$.

So, in the logarithmic layer, we assume that the mean quantities and the turbulent quantities are functions of the two scales u_* and y, and this implies $k = C.u_*^2$ but also $\varepsilon = C'u_*^3 / y$. These laws are recovered from experiments (with some approximation).

Within the viscous sublayer, it will be necessary also to take into account the viscous scale; we can make use, for example, of the three scales u_*, y and ν.

2.2.3. Confined flows: tubes and channels
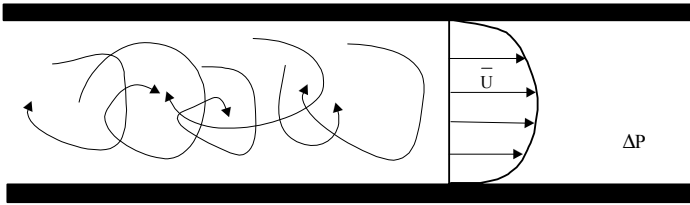
Figure 2.8. *Turbulent flow in a plane channel*

In the case of duct flows (plane channel or circular tube for example), the total stresses can be determined in the fully developed regime ($\partial / \partial x = 0$, except for P) by applying the momentum theorem.

The momentum equation in the case of a plane channel reads:

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{\partial R_{12}}{\partial y} + \nu \frac{\partial^2 U}{\partial y^2}.$$

The mean pressure gradient (which is the driving force) is usually imposed as a constant value $\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} = \Lambda$. After integration, we obtain $\dfrac{\tau}{\tau_0} = 1 - \dfrac{y}{h}$ and $\dfrac{\tau_0}{\rho} = u_*^2 = -h\Lambda$, where h is the channel half-width between the two parallel planes.

The head drop coefficient is given by $\lambda_* = -\dfrac{h\Lambda}{U_m^{\,2}/2} = 2\left(\dfrac{u_*}{U_m}\right)^2$, where U_m is the bulk velocity. Near the wall, we observe, as in the boundary layer, a viscous sublayer and a logarithmic layer.

The results are similar in a circular tube, it is, however, necessary in this case to consider cylindrical coordinates.

2.2.4. *Homogenous turbulent flows*

Homogenous turbulent flows, in the statistical sense, have their properties that do not depend on the particular point considered in space. In principle, they thus occupy an infinite extent of space, but in practice they can be obtained approximately only in a limited portion of space. These flows are simpler in their concept, since the effects of diffusion will be completely canceled.

2.2.4.1. *Example of homogenous isotropic turbulent flow*

The decay of homogenous and isotropic turbulence can be performed experimentally in an approximate way by considering a uniform flow far from any wall boundary. For this, a generating turbulence grid is placed in a uniform current with constant mean velocity U inside a wind tunnel and we observe and measure the evolution of turbulence downstream of the grid. The time evolution is thus replaced by spatial evolution. For example, the equation of energy decay $dk / dt = -\varepsilon$ is then replaced by $U dk / dx = -\varepsilon$. Of course, a small non-homogenity appears in the x direction, but it is very weak and can be neglected.

2.2.4.2. *Examples of homogenous anisotropic turbulent flows*

Homogenity is compatible with the existence of mean velocity gradients only if these gradients are constant [CRA 58], the associated turbulent field then becomes homogenous and anisotropic. The equations for momentum and turbulent field can thus be written:

$$U = \text{constant}, \quad \frac{\partial k}{\partial t} = -R_{ij}\Lambda_{ij} - \varepsilon \quad \text{with} \quad \frac{d\overline{U}_i}{dx_j} = \Lambda_{ij}$$

and $\dfrac{\partial \varepsilon}{\partial t} = -C_{\varepsilon 1}\dfrac{\varepsilon}{k}R_{ij}\Lambda_{ij} - C_{\varepsilon 2}\dfrac{\varepsilon^2}{k}$ for the k-ε model for example.

$$R_{ij} = \frac{2}{3}k\delta_{ij} - c_\mu \frac{k^2}{\varepsilon}\left(\Lambda_{ij} + \Lambda_{ji}\right)$$

Performing experiments on homogenous anisotropic turbulent flows is however not easy since homogenity can be obtained only in a limited region in space. Generally a distorting channel is used, its shape being devised in order to impose the

mean velocity gradients that are required. Hereafter are given the main traditional examples of homogenous anisotropic turbulent flows.

a) Homogenous turbulent shear flow

In this flow, the mean velocity gradient $\dfrac{d\overline{U}_1}{dx_2} = A_{12}$ is constant.

b) Pure plane distortion

The mean velocity field is given by $\left[A_{ij} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{bmatrix}$

The fluid particle is distorted only in directions x_2 and x_3.

c) Axisymmetric contraction

This flow is produced in the central region of a channel whose cross-section is suddenly reduced. The fluid particle is then elongated in direction x_1 and contracted in directions x_2 and x_3, the mean velocity field is given by $\left[A_{ij} \right] = \begin{bmatrix} b & 0 & 0 \\ 0 & -b/2 & 0 \\ 0 & 0 & -b/2 \end{bmatrix}$. Chapter 5 will be devoted to the study of these flows.

Chapter 3

Mathematical Tools

The purpose of this chapter is to sum up some mathematical definitions and basic properties that will be useful in the remainder of the book to elaborate statistical closure models. We shall restrict ourselves to an intuitive short presentation and to some useful operating formulae. The reader interested in mathematical aspects which are not considered here, can refer to the mathematical books cited in the bibliography.

3.1. Tensors

The very synthetic shorthand notation of tensors and their powerful properties will be very useful for the formulation of mathematical models of turbulence. Indeed, all the statistical moments that are involved in the transport equations of the turbulence field are tensors, the closure relations will thus take the form of relations between tensors, and as a consequence important formal properties follow, it is essential to account for them. The tensors are geometrical entities which generalize the traditional concept of vector.

3.1.1. *Affine tensors*

Affine tensors are defined in a vector space (cf. Lichnerowicz A., [*LIC 67]; Choquet-Bruhat Y., [*CHO 68]).

Given a vector space $\mathcal{E}_n$ of dimension n and $\{e_i\}$ a vector basis in $\mathcal{E}_n$, let $\mathcal{E}_n^*$ be the dual space and $\{\theta^i\}$ the dual basis of $\{e_i\}$, a p-covariant tensor attached to $\mathcal{E}_n$ is defined as a p-linear form on $\mathcal{E}_n$ denoted:

$$t(\vec{V}_{(1)}, \vec{V}_{(2)}, \ldots) = v_{(1)}^i v_{(2)}^j \ldots t(\vec{e}_i, \vec{e}_j, \ldots) = v_{(1)}^i v_{(2)}^j \ldots t_{ij} \ldots$$

It can be expressed as a linear combination of particular tensors denoted $\theta^i \otimes \theta^j \otimes \ldots$ which are p-linear forms. When operating on p vectors $\vec{V}_{(h)}$, these forms give the product $v_{(1)}^i v_{(2)}^j \ldots$ of p of their components, and thus

$$t = t_{ij} \ldots \theta^i \otimes \theta^j \ldots \qquad\qquad [3.1]$$

A p-contravariant tensor is a p-linear form on $\mathcal{E}_n^*$, and can be expressed as:

$$t = t^{ij} \ldots \vec{e}_i \otimes \vec{e}_j \ldots \qquad\qquad [3.2]$$

A mixed tensor p times covariant and q times contravariant is a multilinear form on the direct product of $\mathcal{E}_n$, p times by itself and of $\mathcal{E}_n^*$, q times by itself:

$$t = t_{j\ldots}^{i\ldots} \vec{e}_i \otimes \theta^j \ldots \qquad\qquad [3.3]$$

Examples: the tensor $t = t^i \vec{e}_i$ is a vector of $\mathcal{E}_n$, $t = t_i \theta^i$ is a linear form (an element of $\mathcal{E}_n^*$).

Various operations can be performed on tensors, they are in particular:

– the tensorial product: the product $t \otimes u$ of two tensors t and u, respectively p- and q-linear forms, is the $p+q$ linear product form:

$$w = t \otimes u, \qquad w_k^{ij} = t_k^i u^j \qquad\qquad [3.4]$$

– the contraction: this is obtained by summing up a covariant index (subscript) and a contravariant index (superscript):

$$c^i = w_j^{ij} \qquad\qquad\qquad\qquad [3.5]$$

– the contracted product:

$$c^i = a^{ij} b_j \qquad\qquad\qquad\qquad [3.6]$$

As for usual vectors, transformation formulae for the components of a tensor in a change of frame, can be expressed linearly in terms of the elements of the basis transformation matrix [*LIC 67].

The change of basis, from the old basis $\{\vec{e}_i\}$ to the new one $\{\vec{E}_i\}$ is made using the basis change matrix $\left[\alpha_i^j\right]$ according to $\vec{E}_i = \alpha_i^j \vec{e}_j$ and inversely by using the basis change matrix $\left[\beta_i^j\right] = \left[\alpha_i^j\right]^{-1}$ according to $\vec{e}_i = \beta_i^j \vec{E}_j$, these matrices being inverse one to each other $\beta_i^j \alpha_k^i = \delta_k^j$.

The change of frame for any tensor having components $t_{ih\ldots m}^{jk\ldots l}$ in the basis $\{\vec{e}_i\}$ and components $T_{ih\ldots m}^{jk\ldots l}$ in the new basis $\{\vec{E}_i\}$ is made using the formulae:

$$T_{ih\ldots m}^{jk\ldots l} = \alpha_i^p \alpha_h^q \ldots \alpha_m^u \beta_r^j \beta_s^k \ldots \beta_t^l t_{pq\ldots u}^{rs\ldots t}$$

Moreover, these transformation relations are tensoriality criteria. Quantities that are transformed in a change of frame according to the previous relations are proven to form the components of a tensor.

3.1.2. Euclidian tensors

We can define on $\mathcal{E}_n$ the scalar product of two vectors $\vec{V}.\vec{W} = v_i w_j \times \left(\vec{e}_i, \vec{e}_j\right)$ which introduces the metric tensor g_{ij}:

$$g_{ij} = \left(\vec{e}_i, \vec{e}_j\right). \qquad\qquad\qquad\qquad [3.7]$$

We shall denote g^{ij} the tensor defined such that matrix $[g^{ij}]$ associated with g^{ij} is the inverse matrix of $[g_{ij}]$ associated with g_{ij}, thus:

$$g_{im} g^{jm} = \delta_i^j .$$

To the same geometrical entity, it is thus possible to associate various tensors with differing variance modes, by using the contracted product with the metric tensor. For a vector $\vec{V}$ with contravariant components v^i, we can thus define its covariant components by $v_i = g_{ij} v^j$. We shall get the expressions $\vec{V} = v^i \vec{e}_i$ and also $(\vec{V}.\vec{e}_i) = v_i$, since $(\vec{V}.\vec{e}_i) = v^m (\vec{e}_m.\vec{e}_i)$. The same for higher order tensors:

$$t^i_{jk} g_{il} = t_{jkl} , \qquad t^i_{jk} g^{kl} = t^{il}_j , \qquad t^i_{jk} g^{kl} g^{jh} = t^{ilh} ,$$
$$t_{hkj} g^{hi} g^{kl} g^{jh} = t^{ilh} .$$

3.1.3. *Cartesian tensors*

If the basis $\{\vec{e}_i\}$ is orthonormal, the previous relations become:

$$\vec{V}.\vec{W} = v_m w_m \quad \text{and} \quad (\vec{e}_i, \vec{e}_j) = \delta_{ij}$$

Thus, $g_{ij} = \delta_{ij}$, and it is no longer necessary to distinguish the modes of variance because the corresponding components are numerically equal,

$$t^{ik}_j \delta_{im} = t^k_{jm} = t_{kjm} .$$

In practice, all the indices will be placed as subscripts:

$$c_i = a_{ij} b_j$$

3.1.4. *Extensions and pseudo-tensors*

Completely antisymmetric tensors are involved in many cases. They have the antisymmetry property for each pair of indices. A completely antisymmetric tensor $Y^{i_1 i_2 i_3 \cdots i_p}$ of order p consists of C_n^p strict components $Y^{(k_1 k_2 k_3 \cdots k_p)}$ such as

$k_1 < k_2 < ... < k_p$, the indices are denoted enclosed in parentheses meaning that it is a strict component. The corresponding basis is denoted $\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_p$.

If we introduce π, as a permutation $\{i_1 i_2 i_3 ... i_p\}$ of the first p integers, we can define the antisymmetric operator A applied to a tensor T of order p on $(\otimes E_n)^p$ by:

$$AT = \sum_{S_p} sign(\pi).\pi T \; , \; \pi \in S_p \; , \text{ group of permutations of } p \text{ integers.}$$

The components read: $(AT)_{i_1 i_2 i_3 ... i_p} = \delta_{i_1 i_2 i_3 ... i_p}^{k_1 k_2 k_3 ... k_p} T_{k_1 k_2 k_3 ... k_p}$ where $\delta_{i_1 i_2 i_3 ... i_p}^{k_1 k_2 k_3 ... k_p}$ is the generalized Kronecker tensor of order $2p$ defined by:

$$\delta_{i_1 i_2 i_3 .. i_p}^{k_1 k_2 k_3 .. k_p} = 0 \text{ if } i_1 i_2 i_3 ... i_p \text{ is not a permutation of } k_1 k_2 k_3 ... k_p$$
$$= +1 \text{ if } i_1 i_2 i_3 ... i_p \text{ is an even permutation of } k_1 k_2 k_3 ... k_p$$
$$= -1 \text{ if } i_1 i_2 i_3 .. i_p \text{ is an odd permutation of } k_1 k_2 k_3 ... k_p$$

A completely antisymmetric tensor $Y^{i_1 i_2 i_3 ... i_p}$ of order $p \leq n$ can be written:

$$Y^{i_1 i_2 i_3 ... i_p} = \delta_{(k_1 k_2 k_3 ... k_p)}^{i_1 i_2 i_3 .. i_p} Y^{(k_1 k_2 k_3 ... k_p)} \text{ denoted } Y^{i_1 i_2 i_3 .. i_p} = \varepsilon^{i_1 i_2 i_3 .. i_p} Y^{(k_1 k_2 k_3 ... k_p)}$$

where $\varepsilon^{i_1 i_2 i_3 .. i_p}$ is numerically equal to $\delta_{(k_1 k_2 k_3 ... k_p)}^{i_1 i_2 i_3 ... i_p}$, but is no longer a true tensor because of the reduced number of indices.

A completely antisymmetric tensor of order $p = n$ has only a single strict component:

$$Y^{i_1 i_2 i_3 ... i_n} = \delta_{(12 .. n)}^{i_1 i_2 i_3 ... i_n} Y^{(12...n)} \text{ or } Y^{i_1 i_2 i_3 ... i_n} = \varepsilon^{i_1 i_2 i_3 .. i_n} Y^{(12...n)} .$$

The quantity $Y^{(12...n)}$ is not a true scalar, for it is transformed when undergoing a change of basis according to $Y^{(12...n)} = \Delta.Y'^{(12...n)}$ where Δ is the determinant of the basis change matrix, $Y^{(12...n)}$ is called a scalar capacity ([*BRI 60]). In the case

of a covariant tensor the transformation law becomes $Y_{(12...n)} = \dfrac{1}{\Delta}.Y'_{(12.. n)}$, $Y_{(12...n)}$

is thus called a scalar density. The main pseudo-tensors fall into these two types; a tensorial density is obtained by the product of a scalar density by a true tensor; a tensorial capacity is obtained by the product of a scalar capacity by a true tensor.

We define the exterior product of two antisymmetric tensors of orders p and q by:

$$T \wedge U = \frac{1}{p!q!} A(T \otimes U).$$

For the vectors $X = \{x^i\}$ and $Y = \{y^i\}$, it is an antisymmetric tensor of second order $E = \{e_{ij}\}$: $E = X \wedge Y = A(X \otimes Y) = X \otimes Y - Y \otimes X$.

In an Euclidian space, it can be shown that ([LIC67]) the change of basis on $g = det\big[g_{ij}\big]$ is $g' = \Delta^2 g$, or equivalently $\dfrac{1}{\sqrt{g}} = |\Delta|\dfrac{1}{\sqrt{g'}}$. Restricting ourselves to a basis with the same orientation (Δ>0) the quantity $\dfrac{1}{\sqrt{g}}$ is transformed like the strict component of a completely antisymmetric tensor of order n:

$$\eta^{i_1.. i_n} = \varepsilon^{i_1.. i_n}\frac{1}{\sqrt{g}} \quad \text{or} \quad \eta_{i_1...i_n} = \varepsilon_{i_1...i_n}\sqrt{g}.$$

The adjoint $*T$ of a completely antisymmetric tensor T is defined by:

$$(*T)_{i_{q+1}.. i_n} = \frac{1}{q!}\eta_{i_1...i_q i_{q+1}...i_n}T^{i_1.. i_q}$$

In the previous example, the strict components of $E = X \wedge Y$ are in fact the components of the vector product in basic geometry. The adjoint vector of E is

$$V = (*E) = \begin{Bmatrix} x^2 y^3 - x^3 y^2 \\ x^3 y^1 - x^1 y^3 \\ x^1 y^2 - x^2 y^1 \end{Bmatrix} \text{ also denoted } V_i = \varepsilon_{ijk}x^j y^k. \text{ In this expression, } \varepsilon_{ijk} \text{ is a}$$

pseudo-tensor whose components are equal to +1 if $\{ijk\}$ is an even permutation of

$\{123\}$, equal to -1 for an odd permutation and equal to zero in the other cases. This property is also verified by $V_i = \varepsilon_{ijk} x^j y^k$ which is a pseudo-vector (axial vector of the basic geometry).

Some useful identities in Cartesian coordinates are given hereafter:

$$\varepsilon_{ijk} \varepsilon_{lmn} = \delta_{il} \delta_{jm} \delta_{kn} + \delta_{im} \delta_{jn} \delta_{kl} + \delta_{in} \delta_{jl} \delta_{km}$$
$$- (\delta_{il} \delta_{jn} \delta_{km} + \delta_{im} \delta_{jl} \delta_{kn} + \delta_{in} \delta_{jm} \delta_{kl})$$
$$\varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$
$$\varepsilon_{ijk} \varepsilon_{ljk} = 2 \delta_{il}$$

Let us emphasize that, unlike the Kronecker unit tensor δ_i^j which is independent of any change in frame axes (true tensor), ε_{ijk} is a pseudo-tensor, in particular ε_{ijk} changes sign according to the orientation of the frame axes.

We can show also that the volume element $d\tau = dx^1 dx^2 ... dx^n$ is a capacity, strict component of the volume element n-form $dx^1 \wedge dx^2 \wedge ... \wedge dx^n$.

3.2. Euclidian space in curvilinear coordinates, tensor fields

A tensor field is defined by its components at every point M in space, components relative to a local vector frame $\{\vec{e_i}\}$.

The techniques presented in the present section will be mainly useful for writing the tensor equations for the statistical moments in non-Cartesian coordinate systems. Let us consider the following change in coordinates:

$$x^i = f^i(y^1, y^2, ..., y^n), \quad \text{for} \quad i=1, ... \, n$$

giving the rectilinear coordinates x^i, according to y^i, curvilinear coordinates of a point M.

As local vector basis, we choose the natural vector basis defined by:

$$\vec{e_i} = \frac{\partial \overline{M}}{\partial y^i}, \text{ which is also denoted more simply } \vec{e_i} = \partial_i \overline{M} \qquad [3.8]$$

The dual basis will then be $\theta^i \equiv dy^i$.

The metric tensor defines the infinitesimal length element by:

$$ds^2 = g_{ij} dy^i dy^j \quad \text{with} \ \ g_{ij} = \left(\vec{e}_i, \vec{e}_j \right).$$

3.2.1. *Derivatives of the vectors of the natural basis*

The derivative and the differential of the basis vectors can be represented in the local basis itself by the relations:

$$\frac{\partial \vec{e}_i}{\partial y^k} = \Gamma_{ki}^{j} \vec{e}_j \qquad , \qquad d\vec{e}_i = \Gamma_{ki}^{j} \vec{e}_j dy^k \quad ,$$

$$\frac{\partial \theta^i}{\partial y^k} = -\Gamma_{kp}^{j} \theta^p \quad \text{and} \quad d\theta^j = -\Gamma_{kp}^{j} \theta^p dy^k$$

It can be demonstrated that (cf. Lichnerovicz A., [*LIC 67]):

$$\Gamma_{ik}^{j} = \frac{1}{2} g^{jl} \left(\frac{\partial g_{lk}}{\partial y^i} + \frac{\partial g_{il}}{\partial y^k} + \frac{\partial g_{ki}}{\partial y^l} \right) \tag{3.9}$$

It must be noted that the Christoffel symbols Γ_{ik}^{j} are not tensors.

3.2.2. *Absolute differential and covariant derivative*

The usual derivative of the component of a tensor, such as $\partial t_{ij} / \partial y_m$, is no longer a tensor, since the basis vectors are dependent on point M under consideration, it is also necessary to take into account the derivatives of the basis vectors. In order to simplify the presentation, the formalism is given for the particular case of a second order tensor, but the method is the same in the general case.

Consider the tensor $t = t_j^i \vec{e}_i \otimes \theta^j$, its differential can be written:

$$dt = dt_j^i . \vec{e}_i \otimes \theta^j + t_j^i . d\vec{e}_i \otimes \theta^j + t_j^i . \vec{e}_i \otimes \theta^j$$

$$dt = \left(dt^i_j + t^s_j \Gamma^i_{ls} dy^l - t^i_s \Gamma^s_{lj} dy^l \right).\vec{e}_i \otimes \theta^j \quad 1$$

$$dt = \left(\nabla t \right)^i_j .\vec{e}_i \otimes \theta^j \quad \text{and} \quad \left(\nabla t \right)^i_j = t^i_{j,m} .\vec{e}_i \otimes \theta^m$$

$$\underbrace{}_{A.D.} \qquad \underbrace{\phantom{t^i_{j,m}}}_{C.D.}$$

(A.D. = absolute differential and C.D. = covariant derivative)

implying $dt = t^i_{j,m} \theta^m \otimes \vec{e}_i \otimes \theta^j$ and

$$t^i_{j,m} = \partial_m t^i_j + \Gamma^i_{ms} t^s_j - \Gamma^s_{mj} t^i_s , \qquad [3.10]$$

the partial derivative $\dfrac{\partial t^i_j}{\partial y^m}$ being denoted more simply $\partial_m t^i_j$.

The covariant derivative $t^i_{j,m}$ is a tensor whereas the partial derivative $\partial_m t^i_j$ is not. We can show that $g_{ij,m} \equiv 0$, $\forall m$.

In the particular case of Cartesian tensors $y^m = x^m$ and the covariant derivative thus become identified with the usual partial derivative.

$$t^i_{j,m} = t_{ij,m} = \partial_m t_{ij} \qquad [3.11]$$

3.2.3. *Practical derivation of the equations in curvilinear coordinates*

On a practical point of view, we can simply proceed as follows:

1) From the usual equation in Cartesian coordinates, restore the general tensor notation with correct modes of variance, replacing partial derivatives by covariant derivatives and using the metric tensor when necessary.

2) Calculate g_{ij} and Γ^i_{jk} .

3) Express the covariant derivatives.

4) Introduce the physical components of the tensors on the normed local basis[2] using scaling factors $h_i = \sqrt{g_{ii}}$.

---

[1] We have used the same notation for the coordinate forms dy^i and for $dy^i = dy^i(\vec{V})$, which results from its application to a vector $\vec{V}$.

[2] The natural basis is not normed.

5) Substitute in the relations such obtained the values of g_{ij} and Γ^i_{jk} calculated in point 2).

3.2.4. *Usual operators*

The previous relations can be used to introduce easily all the usual operators in analytical geometry.

The gradient of a tensor t^{pq}_{ij} is defined by $t^{pq}_{ij,m}$, the divergence of a contravariant vector v^j by $v^j{}_{,j}$, the divergence of a covariant vector v_j by $g^{jp}v_{p,j}$, the Laplacian of f by $g^{jm}f_{,jm}$. In the particular case of Cartesian tensors, we can write $v_{j,j}$ for the divergence, and $f_{,mm}$ for the Laplacian.

The curl of a vector v_j is $curl_{ij}\vec{V} = v_{i,j} - v_{j,i} = \partial_j v_i - \partial_i v_j$, it is a second order tensor. The rotational vector or curl in basic geometry denoted $curl\,\vec{V}$ or $rot\,\vec{V}$ is in fact the adjoint vector of $curl_{ij}\vec{V}$ defined by $R^m = \frac{1}{2}\varepsilon^{ijm}\left(\partial_i v_j - \partial_j v_i\right) = \varepsilon^{ijm}\partial_i v_j$ which is a pseudo-tensor (tensor density) also called "axial vector" in basic geometry. We can deduce from the identities relating the products of $\overset{\equiv}{\varepsilon}$ and $\overset{\equiv}{\delta}$ (cf. section 3.1.4.) that $curl_{ij}\vec{V} = -\varepsilon_{ijm}R^m$.

3.3. Orthogonal curvilinear coordinates

We shall consider here the example of two-dimensional problems but the method can be applied in the same way to three-dimensional problems.

Let y_j be the rectilinear coordinates of a point M in the plane referring to the orthonormal basis $\{\vec{E}_i\}$. A curvilinear coordinate system y_j is defined by the transformation relations:

$$\begin{cases} Y_1 = f_1(y_1, y_2) \\ Y_2 = f_2(y_1, y_2) \end{cases} \tag{3.12}$$

The coordinate curves are given by the curves $y_1 = C^{te}$ and $y_2 = C^{te}$. The two tangent vectors at a point M to these curves yield a local basis that can be chosen orthonormal (cf. Figure 3.1):

$$\vec{e}_i = \frac{\partial_i \overrightarrow{M}}{\left\| \partial_i \overrightarrow{M} \right\|} \qquad (\partial_i \overrightarrow{M} \text{ denotes } \frac{\partial \overrightarrow{M}}{\partial y_i}) \qquad\qquad [3.13]$$

(which is thus different from the natural basis in M).

We are interested in curvilinear orthogonal coordinate systems, i.e. we shall assume:

$$\vec{e}_i . \vec{e}_j = \delta_{ij} \qquad\qquad [3.14]$$

$\left\{ \vec{e}_i \right\}$ is thus an orthonormal local basis.
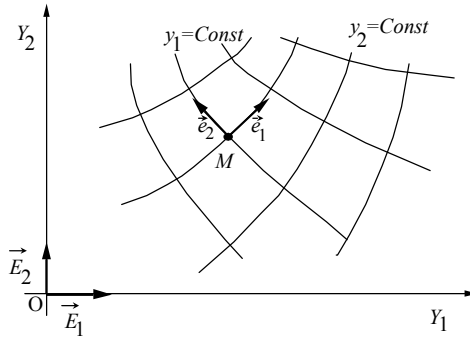
Figure 3.1. *Orthogonal curvilinear coordinates. Orthonormal local basis*

Given an element vector:

$$d\overrightarrow{M} = \sum_j dY_j . \vec{E}_j \; ,$$

the length element is given by the ds^2 in space:

$$ds^2 = \left(\overrightarrow{dM} \cdot \overrightarrow{dM} \right) = dY_1^2 + dY_2^2 \; ,$$
$$ds^2 = \sum_{i,j} \partial_i \overrightarrow{M} . \partial_j \overrightarrow{M} \, dy_i dy_j \; . \qquad\qquad [3.15]$$

After [3.13] and [3.14] we obtain:

$$\partial_i \overline{M}.\partial_j \overline{M} = \sum_p \frac{\partial f_p}{\partial y_i} \frac{\partial f_p}{\partial y_j} = 0, \quad \text{if} \quad i \neq j$$

from which there results:

$$ds^2 = \sum_i \left\| \partial_i \overline{M} \right\| dy_i^2,$$

We shall denote:

$$h_i^2 = \left\| \partial_i \overline{M} \right\|,$$

and consequently:

$$ds^2 = h_1^2 dy_1^2 + h_2^2 dy_2^2. \qquad [3.16]$$

Using the notation $d\xi(i) = h_i dy_i$ (notation by Pope S.B., [POP 78B]) we thus get $ds^2 = dY_1^2 + dY_2^2 = d\xi(1)^2 + d\xi(2)^2$.

Given a vector $\vec{V} = \sum_j V_j \vec{E}_j = \sum_j v_j \vec{e}_j$ and its covariant derivative in a normed basis:

$$\frac{\partial \vec{V}}{\partial \xi(i)} = \sum_j v_{j,i} \vec{e}_j = \sum_j \left(\frac{\partial v_j}{\partial \xi(i)} \vec{e}_j + v_j \frac{\partial \vec{e}_j}{\partial \xi(i)} \right)$$

the derivative of the basis vector is:

$$\frac{\partial \vec{e}_j}{\partial \xi(i)} = \sum_m \gamma_{mij} \vec{e}_m, \qquad [3.17]$$

and using [3.14] and [3.17] the expression of the corresponding Christoffel symbol is inferred to be:

$$\gamma_{pij} = \frac{\partial \vec{e}_j}{\partial \xi(i)} \vec{e}_p, \qquad [3.18]$$

after [3.14] and [3.18] we obtain the relation:

$$\gamma_{pij} + \gamma_{jip} = 0 \quad \text{or} \quad \gamma_{pij} = -\gamma_{jip}$$

and then if $p = j$, we find $\gamma_{jij} = 0$.

We now consider the identity obtained by reversing the order of derivatives applied to vector $\overrightarrow{OM}$:

$$\frac{\partial^2 \overrightarrow{M}}{\partial y_i \partial y_j} = \frac{\partial^2 \overrightarrow{M}}{\partial y_j \partial y_i} \quad \text{or} \quad \frac{\partial (h_j \vec{e}_j)}{\partial y_i} = \frac{\partial (h_i \vec{e}_i)}{\partial y_j} .$$

Now making the scalar product of each side by $\vec{e}_\alpha$, we obtain:

$$\frac{\partial h_j}{\partial y_i} \delta_{j\alpha} + h_j h_i \gamma_{\alpha ij} = \frac{\partial h_i}{\partial y_j} \delta_{i\alpha} + h_i h_j \gamma_{\alpha ji} ,$$

if α is different from i and j:

$$\gamma_{\alpha ij} = \gamma_{\alpha ji} \quad (i \neq \alpha \text{ and } j \neq \alpha),$$

if moreover i is different from j, then:

$$\gamma_{\alpha ij} = 0 \quad (i \neq \alpha, j \neq \alpha \text{ and } i \neq j),$$

since indeed $\gamma_{\alpha ij} = \gamma_{\alpha ji} = -\gamma_{ij\alpha} = -\gamma_{i\alpha j} = \gamma_{j\alpha i} = \gamma_{ji\alpha} = -\gamma_{\alpha ij}$.

So, $\gamma_{\alpha ij} = 0$ if $\alpha = j$ or if the indices (α, j, i) are all different from each other. There are then only two remaining cases for which $\gamma_{\alpha ij} \neq 0$:

1) $\alpha \neq j$ and $\alpha = i$
$$h_j h_i \gamma_{iij} = \frac{\partial h_i}{\partial y_j} + \underbrace{h_i h_j \gamma_{iji}}_{=0} ,$$

and thus
$$\gamma_{iij} = \frac{1}{h_i h_j} \frac{\partial h_i}{\partial y_j} ,$$

2) $\alpha \neq j$ and $i = j$

$$\gamma_{\alpha jj} = -\gamma_{jj\alpha} = -\frac{1}{h_j h_\alpha} \frac{\partial h_j}{\partial y_\alpha} ,$$

(using previous relations).

The general result then follows:

$$\gamma_{pij} = \frac{1}{h_i h_j}\frac{\partial h_i}{\partial y_j}\delta_{pi} - \frac{1}{h_p h_i}\frac{\partial h_i}{\partial y_p}\delta_{ji},$$

if we assume: $H_i(j) = \dfrac{1}{h_i h_j}\dfrac{\partial h_i}{\partial y_j} = \dfrac{1}{h_i}\dfrac{\partial h_i}{\partial \xi(j)}$ (Pope's notation),

then comes $\gamma_{pij} = H_i(j)\delta_{pi} - H_i(p)\delta_{ji},$ [3.19]

and after [3.17] the covariant derivative can be written $v_{j,i} = \dfrac{\partial v_j}{\partial \xi(i)} + \sum\limits_p v_p \gamma_{jip}$ and

thus also:

$$v_{j,i} = \frac{\partial v_j}{\partial \xi(i)} + \sum_p \left(H_i(j)\delta_{pi} - H_i(p)\delta_{ji}\right)v_p$$

The expression for the derivative of a vector thus reads:

$$v_{j,i} = \frac{\partial v_j}{\partial \xi(i)} - v_i H_i(j) + \sum_p v_p H_i(p)\delta_{ij}$$ [3.20]

The case of a scalar is simpler:

$$\vartheta_{,i} = \frac{\partial \vartheta}{\partial \xi(i)}$$ [3.21]

The case of higher order tensors can be treated using the same method. In practice, we can note that higher order tensors can be derivated along the same formal rules as the products (non-commutative products) of several vectors, this is providing a mnemonic method for easy calculation of covariant derivatives of any tensor (see for example applications in Chapter 6).

3.4. Conformal transformation

This is the special case of a coordinate transformation defined by a complex function:

$$Z = f(z)$$ [3.22]

where f is an holomorphic function.

The differential reads:

$$\begin{bmatrix} dY_1 \\ dY_2 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} d\xi(1) \\ d\xi(2) \end{bmatrix} \qquad \begin{array}{l} Z = Y_1 + iY_2 \\ z = y_1 + iy_2 \end{array} ,$$

or if we single out the Jacobian matrix:

$$\begin{bmatrix} dY_1 \\ dY_2 \end{bmatrix} = [J] \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix}$$

$$[J] = \begin{bmatrix} \partial Y_1/\partial y_1 & \partial Y_1/\partial y_2 \\ \partial Y_2/\partial y_1 & \partial Y_2/\partial y_2 \end{bmatrix}, \qquad \frac{\partial Y_1}{\partial y_1} = \frac{\partial Y_2}{\partial y_2} \text{ and } \frac{\partial Y_1}{\partial y_2} = -\frac{\partial Y_2}{\partial y_1}$$

Let the definition be

$$\mathscr{T} = det([J]).$$

The transformation of the components of a vector $\vec{V}$ is obtained by:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{[J]}{\sqrt{\mathscr{T}}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad h_i = \sqrt{\mathscr{T}} \ (\forall i)$$

and the ds^2 of the space is then:

$$ds^2 = \mathscr{T}\left(dy_1^2 + dy_2^2\right) = d\xi^2(1) + d\xi^2(2).$$

Coefficient $H_i(j) = \dfrac{1}{2\mathscr{T}} \dfrac{\partial \mathscr{T}}{\partial \xi(j)}$ will be denoted simply $\mathscr{T}(j)$, so that

$$v_{j,i} = \frac{\partial v_j}{\partial \xi(i)} - v_i \mathscr{T}(j) + \sum_p v_p \mathscr{T}(p)\delta_{ij} . \qquad [3.23]$$

It thus appears that in the case of orthogonal curvilinear coordinates (sections 3.3 and 3.4), the use of an orthonormal local basis (non-natural) allows us to develop a much simpler formalism, in particular it is no longer necessary to distinguish the variance modes (covariant or contravariant) of tensors. Thus, the calculation can be

conducted directly from the tensorial equations written in Cartesian notation. The calculation method presented in sections 3.3 and 3.4 is simple and efficient while remaining general.

3.5. Invariants

The study of tensor invariants is necessary for the representation theory of tensorial functions (cf. section 3.6).

3.5.1. *Independent invariants of a tensor*

A tensor invariant can be defined as a tensor of order zero or scalar. It can be obtained for example by tensorial contraction from higher order tensors.

Let us consider some usual cases:

1) First order tensor:

A first order tensor is a vector (for example the velocity vector U_i, or the pressure gradient $p_{,i}$).

The only invariant attached to a vector A_i is its length:

$$I = A_i A^i \qquad\qquad\qquad [3.24]$$

2) Second order tensor:

Let A_j^i be a second order tensor, we can look for the eigenvalues of the associated matrix $\left[A_j^i \right]$ by solving:

$$A_j^i B_i = \lambda B_j \qquad\qquad\qquad [3.25]$$

(λ eigenvalues, B_i eigenvectors)

The eigenvalues are obtained from the characteristic equation:

$$\left| A_j^r - \lambda \delta_j^r \right| = 0 \qquad\qquad\qquad [3.26]$$

this equation can be written:

$$\lambda^n - J_1\lambda^{n-1} + J_2\lambda^{n-2} - \ldots\ldots + (-1)^n J_n = 0 \qquad [3.27]$$

n being the dimension of space, all the coefficients J_r are invariants, called principal invariants (cf. Lumley J.L., [LUM 70]).

Taking the derivative of [3.26] and [3.27] with respect to λ, we obtain the mathematical expression of all the invariants[3]:

$$J_1 = \sum_i A_i^i \, ,$$

trace of matrix A,

$$J_2 = \sum_{i,k} \begin{vmatrix} A_i^i & A_i^k \\ A_k^i & A_k^k \end{vmatrix} , \qquad 1 \le i < k \le n$$

sum of the principal minors of order 2 (the principal minors being the determinants of the truncated matrix extracted from A by suppressing rows and columns; cf. Figure 3.2),

$$J_3 = \sum_{i,k,l} \begin{vmatrix} A_i^i & A_i^k & A_i^l \\ A_k^i & A_k^k & A_k^l \\ A_l^i & A_l^k & A_l^l \end{vmatrix} , \qquad 1 \le i < k < l \le n \qquad [3.28]$$

sum of the principal minors of order 3, etc., up to order n,

$$J_n = det(A) \text{ also denoted } |A| \, ,$$

principal minor of order n (determinant).

---

[3] The derivative of a determinant is the sum of the determinants obtained by replacing successively in the given determinant, the elements of each row (or each column) by their derivatives.

Particular case $n = 3$:

$$J_1 = \sum_p A_p^p \,,$$
$$1 \leq p \leq 3$$

$$J_2 = \frac{1}{2!} \sum_{p,q} \left(A_p^p A_q^q - A_p^q A_q^p \right)$$
$$1 \leq p \leq 3 \,, \ 1 \leq q \leq 3$$

$$J_3 = \frac{1}{3!} \sum_{p,q,r} \left(A_p^p A_q^q A_r^r - 3 A_p^p A_r^q A_q^r + 2 A_q^p A_r^q A_p^r \right) \qquad \text{[3.29]}$$
$$1 \leq p \leq 3 \,, \ 1 \leq q \leq 3 \,, \ 1 \leq r \leq 3$$

$$1 \leq i < j < k \leq n$$
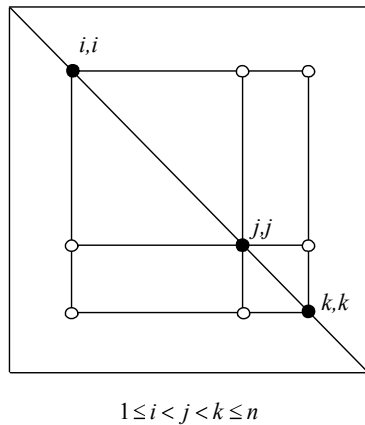
Figure 3.2. *How to get the principal minors of a matrix of order n*

or:

$$J_1 = Tr(A)$$
$$J_2 = \frac{1}{2!} \left[\left(Tr(A) \right)^2 - Tr(A^2) \right]$$
$$J_3 = \frac{1}{3!} \left[\left(Tr(A) \right)^3 - 3.Tr(A).Tr(A^2) + 2.Tr(A^3) \right]$$

Tr denotes the trace.

It is also possible to make use of the generalized Kronecker tensor introduced in section 3.1.4.

$$J_1 = A_m^m$$

$$J_2 = \frac{1}{2!} \delta_{ij}^{lm} A_i^l A_j^m$$

$$J_3 = \frac{1}{3!} \delta_{ijk}^{lmn} A_i^l A_j^m A_k^n$$

Some useful relations:

$$\delta_{ij}^{kl} = \delta_i^k \delta_j^l - \delta_j^k \delta_i^l$$

$$\delta_{ijk}^{lmn} = \delta_i^l \delta_j^m \delta_k^n + \delta_i^m \delta_j^n \delta_k^l + \delta_i^n \delta_j^l \delta_k^m - \delta_i^m \delta_j^l \delta_k^n - \delta_i^l \delta_j^n \delta_k^m - \delta_i^n \delta_j^m \delta_k^l$$

$$\delta_i^l \delta_{ijk}^{lmn} = \delta_{jk}^{mn}, \quad \delta_i^l \delta_{ij}^{lm} = 2\delta_j^m, \quad \delta_{ij}^{ij} = 3!, \quad \delta_{ijk}^{ijk} = 3!$$

Referring once to principal axes:

$$J_1 = \sum_i \lambda^{(i)}$$

$$J_2 = \sum_{i \neq j} \lambda^{(i)} \lambda^{(j)}$$ [3.30]

(they are the sum of principal minors)

$$J_3 = \sum_{i \neq j \neq k} \lambda^{(i)} \lambda^{(j)} \lambda^{(k)}$$

$$J_n = \lambda^{(1)} \lambda^{(2)} \ldots \lambda^{(n)}$$

In the particular case $n = 3$

$$J_1 = \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}$$

$$J_2 = \lambda^{(1)} \lambda^{(2)} + \lambda^{(2)} \lambda^{(3)} + \lambda^{(3)} \lambda^{(1)}$$

$$J_3 = \lambda^{(1)} \lambda^{(2)} \lambda^{(3)}$$ [3.31]

The Cayley-Hamilton theorem states that any second order tensor also satisfies its own characteristic equation.

$$\left(A_i^j\right)^n - J_1 \left(A_i^j\right)^{n-1} + J_2 \left(A_i^j\right)^{n-2} - \ldots + (-1)^n J_n \delta_i^j = 0$$ [3.32]

The most important implication of this theorem is that in a space of dimension n, only the $(n-1)$ first powers of a tensor are independent since [3.32] gives a linear relation involving the n first powers.

Any power of A_i^j can be expressed according to the powers zero, one and two (in a three-dimensional space) and of the principal invariants. The invariants formed from higher powers can also be represented as functions of principal invariants.

We thus have the choice between several possibilities for obtaining the n independent invariants of a second order tensor:

– the n coefficients of the characteristic polynomial: $J_1, J_2, \ldots, J_n$, the so-called principal invariants;

– the traces of the successive powers of A_i^j :

  - $I^A = A_i^i = Tr(A)$,

  - $II^A = A_i^j A_j^i = Tr(A^2)$,

  - $III^A = A_i^j A_j^k A_k^i = Tr(A^3)$, etc.;

– the set of eigenvalues: $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$.

The Reynolds stress tensor R_{ij} is a typical example of second order tensor (symmetrical) involved in turbulence modeling. We also define the anisotropy tensor by $a_{ij} = \dfrac{R_{ij} - 2/3k.\delta_{ij}}{2k}$, which vanishes in isotropic situation, the Reynolds stress tensor being reduced in this case to a spherical tensor $R_{ij} = \dfrac{2}{3} k.\delta_{ij}$.

The invariants of the anisotropy tensor can be defined (using Cartesian notation) by $II = a_{ij} a_{ij}$ and $III = a_{ij} a_{jm} a_{mi}$. The invariant I is zero because of the definition of a_{ij}.

The invariant II is a global measure of the degree of anisotropy of the turbulent field. The invariant III conceals a more subtle meaning.

In principal axes, we get $J_3 = a_{11} a_{22} a_{33}$ and $III = a_{11}^3 + a_{22}^3 + a_{33}^3$, but taking into account relations [3.29], it follows $J_3 = III / 3 = a_{11} a_{22} a_{33}$. The condition $a_{11} + a_{22} + a_{33} = 0$ thus implies $III = -3 a_{11} a_{22} (a_{11} + a_{22})$. It thus appears

(cf. Figure 3.3) that the invariant *III* is negative when the Reynolds tensor in principal axes has two high components and a weak component, and this invariant becomes positive if two components of R_{ij} are weak and the third one is high. The invariant *III* vanishes if one of the components of R_{ij} in principal axes is equal to *2/3 k.*
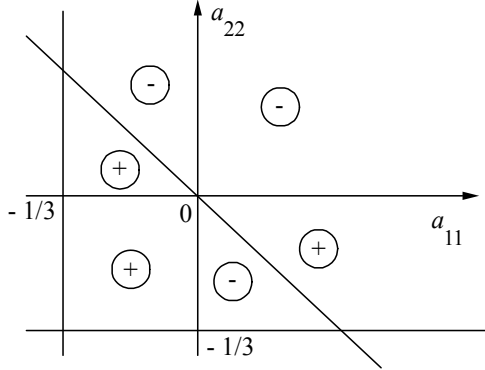
Figure 3.3. *Sign of invariant III of the anisotropy tensor (in principal axes)*

Another usual example would be mean flow strain rate tensor (in Cartesian notation):

$$s_{ij} = \frac{1}{2}\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right).$$

This tensor is symmetric and $s_{jj}=0$ in incompressible flows (because of continuity of the velocity field).

3.5.2. *Invariants of a tensor family*

An invariant can be obtained from a tensor family using summation, contraction of all indices by pairs in such a way that no other free index remains.

Some examples are given in the following.

a) Invariants of an ensemble of vectors $\{A_i, B_i, C_i\}$:

$$A_i A^i, B_i B^i, C_i C^i$$
$$A_i B^i, A_i C^i, B_i C^i, ...$$

b) Invariants of an ensemble of tensors of order one and two $\left\{ A^i_j, B_i \right\}$ which include:

- the proper invariants of A^i_j: $I, II, ...$

- the proper invariants of B_i: $B_j B_j$

- and mixed invariants such as: $A^i_p ... A^p_j B_i B^j$

c) Invariants of an ensemble of general tensors $\left\{ A^i_j, B_i, C_i, E^i_{jp} \right\}$ which include:

- the proper invariants of A^i_j and E^i_{jp}: $I, II, ...$ and $E^i_{jj} E^i_{ss}, ...$

- the proper invariants of B_i and C_i: $B_j B_j$ and $C_j C_j$

- and mixed invariants: $\left(A^i_j \right)^m B_i B^j$, $\left(A^i_j \right)^m B_i C^j$,

$$\left(A^i_j \right)^m C_i C^j, \quad \left(A^i_j \right)^m C_i B^j, \quad E^i_{jp} A^j_i B^p, \text{ etc.}$$

Note: we can define the power of a tensor by a contracted product, which is a tensorial operator, while the usual product of components would not yield a tensor. Thus,

$$\left(A^i_j \right)^2 = A^p_j A^i_p, \quad \left(A^i_j \right)^3 = A^p_j A^h_p A^i_h, \quad \left(A^i_j \right)^m = \underbrace{A^p_j A^h_p ... A^i_s}_{m \text{ times}}.$$

Examples:

1) From the Reynolds stress tensor R_{ij} and the strain rate tensor S_{ij} it is possible to obtain the invariant $P = -R_{ij} S_{ij}$ which is nothing less than the production rate of turbulent kinetic energy (cf. Chapter 2, equation [2.9]).

2) Considering the turbulent flow on a wall, let n_i be the unit normal vector at the wall. The invariant $R_{ij} n_i n_j$ involving the Reynolds stress tensor and the vector n_i, represents the turbulent energy in the direction normal to the wall.

We can also derive new useful identities by introducing in relation [3.32] $A^i_j = S^i_j + T^i_j$ or $A^i_j = S^i_m S^m_j + T^i_j$, or even $A^i_j = S^i_j + T^i_m T^m_j$ and then considering linear combinations. We obtain in this way an extension of the Cayley-Hamilton theorem the so-called Rivlin-Cayley-Hamilton theorem (cf. Kassinos S.C. and

Reynolds W.C., [KAS 94B]), which allows us to express higher order products of two tensors according to lower order terms.

The main three useful relations are:

$$\boldsymbol{bab} = -\boldsymbol{b^2a} - \boldsymbol{ab^2} - \left[tr(\boldsymbol{ba}) - tr(\boldsymbol{a})tr(\boldsymbol{b})\right]\boldsymbol{b} - \tfrac{1}{2}\left[tr(\boldsymbol{b^2}) - tr(\boldsymbol{b})^2\right]\boldsymbol{a}$$

$$-tr(\boldsymbol{b})(\boldsymbol{ba} + \boldsymbol{ab}) - tr(\boldsymbol{a})\boldsymbol{b^2}$$

$$-\left\{tr(\boldsymbol{b^2a}) - tr(\boldsymbol{b})tr(\boldsymbol{ba}) + \tfrac{1}{2}tr(\boldsymbol{a})\left[tr(\boldsymbol{b})^2 - tr(\boldsymbol{a^2})\right]\right\}\boldsymbol{I} \ ,$$

$$\boldsymbol{bab^2} + \boldsymbol{b^2ab} = tr(\boldsymbol{b})\boldsymbol{bab} + tr(\boldsymbol{ba})\boldsymbol{b^2} + \left(tr(\boldsymbol{b^2a}) - tr(\boldsymbol{a})tr(\boldsymbol{ba})\right)\boldsymbol{b}$$

$$-det(\boldsymbol{a})\boldsymbol{b} + det(\boldsymbol{ba})\boldsymbol{I} \ ,$$

$$\boldsymbol{b^2ab^2} = \tfrac{1}{2}\left(tr(\boldsymbol{b})^2 - tr(\boldsymbol{b^2})\right)\boldsymbol{bab} - det(\boldsymbol{b})(\boldsymbol{ba} + \boldsymbol{ab}) + tr(\boldsymbol{b^2a})\boldsymbol{b^2}$$

$$\left\{det(\boldsymbol{b})tr(\boldsymbol{a}) - \tfrac{1}{2}\left[tr(\boldsymbol{b})^2 - tr(\boldsymbol{b^2})\right]tr(\boldsymbol{ba})\right\}\boldsymbol{b} + det(\boldsymbol{b})tr(\boldsymbol{ba})\boldsymbol{I} \ ,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two second order tensors and $\boldsymbol{I}$ the unit tensor.

3.6. Representation of tensorial functions

Let a tensor of arbitrary order be a function of other argument tensors also of arbitrary order:

$$A_{i.....j} = f_{i......j}(x_i, y_j, a_{mp}, ...) \ . \tag{3.33}$$

The problem is to specify which form must have the function $f_{i......j}$ to be compatible with tensorial properties. This is a case which is encountered each time we have to model an unknown correlation according to known tensors (cf. Lumley J.L., [LUM 70]; Siess J., [SIE 75]).

For the tensorial properties to be preserved, it is necessary that the function $f_{i......j}$ remains unchanged under any change of frame, i.e.:

$$\alpha_i^k ... \alpha_j^m A_{k.....m} = f_{i......j}(\alpha_i^k x_k, \alpha_j^m y_m, \alpha_i^k \alpha_j^m a_{km}, ...)$$

In general, for tensors having different variance modes, such as:

$$A_{i \ldots j}^{k \quad m} = f_{i \ldots j}^{k \ldots m}(x_i, y^j, a_i^j, \ldots \ldots)$$

it would be necessary to satisfy the following property:

$$\alpha_i^p \ldots \alpha_j^q \beta_r^k \ldots \beta_r^m A_{p \ldots q}^{r \ldots s} = f_{i \ldots j}^{k \ldots m}(\alpha_i^k x_k, \beta_m^j y^m, \alpha_i^k \beta_m^j a_k^m, \ldots \ldots) \ .$$

In practice, Lumley introduced two basic principles that will be used to establish the specific analytical form of the function.

1) *First basic principle*: an invariant can only be a function of other invariants.

In order to form an invariant from [3.33], let us introduce as many arbitrary vectors as necessary for reducing the left-hand side to a scalar obtained by tensorial contraction. Thus,

$$A_{i \ldots, j} B^i \ldots. C^j = f(B_i B^i, C_i C^i, B_i C^i, x_i x^i, y_i y^i, x_i y^i, B_i x^i, B_i y^i,$$

$$C_i x^i, C_i y^i, I, II, III, \ldots \ldots, a_{ij} B^i C^j, a_{ij} x^i B^j, a_{ij} x^i x^j, a_{ij} x^i y^j, \ldots \ldots) \qquad [3.34]$$

where *I*, *II*, *III* are the independent invariants of a_{mk}. The right-hand side is also a function of the mixed invariants obtained from contracted products of B_i, C_i, x_i, y_i with a_{ij}.

2) *Second basic principle*: if the left-hand side is multilinear in $B^i, \ldots, C^j$, then the right-hand side must also possess the same property.

For this reason, any dependence upon $B_i B^i, C_i C^i$, etc. must be excluded.

Moreover, the terms such as $B_i C^i$ must appear linearly as multipliers.

The application of these two principles leads to represent the tensor $A_{i \ldots, j}$ as the sum of terms which are expressed as products of argument tensors (and also of the Kronecker unit tensor) with a scalar coefficient function of the invariants obtained from the family of argument tensors. The Cayley-Hamilton theorem will be used to reduce this ensemble to a finite number of tensors. The complete set of irreducible independent tensors that make it possible to generate all possible products at any power forms an integrity basis.

3.6.1. *Tensorial function of a scalar: isotropic tensors*

An isotropic tensor can be defined as a tensor which is function of a scalar only. According to Lumley's rules, it is equivalent to define a tensor $A_{i\ldots j}^{k\ldots l}$ of order n as isotropic if the invariant obtained by its tensorial contraction with n arbitrary vectors $B^i \ldots C^j, H_k \ldots G_l$ is only function of the invariants that are proper to the set of the n arbitrary vectors (and satisfying a multilinear relation).

1) Odd order tensors

In the case of a first order tensor or vector, the invariant $A_i B^i = f(\alpha, B_i B^i)$ cannot give rise to a linear relation in B^i. It results that an isotropic vector is identically zero.

Let us consider the case of a third order tensor: $A_{ijk} = f(\alpha)$.

The application of the first basic principle leads to:

$$A_{ijk} B^i C^j D^k = f(\alpha, B_i C^i, B_i D^i, \ldots),$$

where f must be multilinear in B^i, C^j, D^k, but the multiplicative vectors always appear in pairs such as $B_i C^i$; this is thus not possible, and then:

$$A_{ijk} = 0. \qquad\qquad [3.35]$$

The previous reasoning shows that every isotropic odd order tensor is identically zero.

2) Even order tensors

In the case of a second order tensor, the invariant built using the arbitrary vectors, reads:

$$A_i^j B^i C_j = f(\alpha, B^i C_i, \ldots).$$

The bilinearity condition leads to:

$$A_i^j = a \delta_i^j,$$

showing that every isotropic second order tensor is proportional to the Kronecker unit tensor.

For a tensor with two indices having the same mode of variance, we would find $A_{ij} = ag_{ij}$ and $A^{ij} = ag^{ij}$. In the case of Cartesian tensors, $A_{ij} = a\delta_{ij}$.

Let us consider now the case of a fourth order tensor A_{ijkl}. The isotropy condition is $A_{ijkl} = f(\alpha)$ and the invariant built on A thus reads, using multilinearity:

$$A_{ijkl}B^i C^j D^k E^l = f(\alpha, B_i C^i, B_i D^i, B_i E^i ...)$$
$$A_{ijkl}B^i C^j D^k E^l = aB_j C^j D_l E^l + bB_k D^k C_l E^l + cB_l E^l C_k D^k$$

After having raised the indices of the arbitrary vectors to contravariant position, using multiplication by the metric tensor, we find that:

$$A_{ijkl} = a.g_{ij}g_{kl} + b.g_{ik}g_{jl} + c.g_{il}g_{jk} \qquad [3.36]$$

(which are in fact combinations of the metric tensor).

In a space referring to a Cartesian basis, we have:

$$A_{ijkl} = a.\delta_{ij}\delta_{kl} + b.\delta_{ik}\delta_{jl} + c.\delta_{il}\delta_{jk}$$

We shall use for example this relation in section 6.2 for expressing the dissipation tensor $\nu\overline{u_{i,k}u_{j,l}}$.

We can note that the two point correlation tensor or the spectra in homogenous isotropic turbulence (section 3.1.1) are not represented by isotropic tensors in the mathematical sense, since they are functions of the position vectors of the considered points (expression [1.7]). This case is considered in the following.

3.6.2. *Tensorial function of a vector*

Let us consider two examples of application of Lumley's principles, for a vector and a second order tensor.

1ˢᵗ example: $A_i = f_i(x_j)$,

$$A_i B^i = f(x_i x^i, B^i x_i) = a(x_j x^j) B^i x_i ,$$ [3.37]

leads to $A_i B^i = f(x_j x^j)$ and

$$A_i = a(x_j x^j).x_i$$

2ⁿᵈ example: $A_{ij} = f_{ij}(x_k)$,

$$A_{ij} B^i C^j = f(x_k x^k, x_k B^k, x_k C^k, B_k C^k),$$
$$A_{ij} B^i C^j = a(x_k x^k) x_i B^i x_j C^j + b(x_k x^k) B_i C^i ,$$

and restoring the position of indices we find:

$$A_{ij} = a(x_k x^k) x_i x_j + b(x_k x^k) g_{ij}$$ [3.38]

An illustrative example can be given by relation [1.10] in Chapter 1 which expresses the fact that the double velocity correlation second order tensor in HIT is only a function of the vector $\vec{r}$. It is the same for the representation of the spectral tensor in HIT according to the wave vector $\vec{\kappa}$ (relation [1.18]).

The case of a higher order tensor and several argument vectors exactly corresponds to expression [1.7] for higher order correlation tensors in several points.

3.6.3. *Tensorial function of a second order tensor*

The general formula given at the beginning of section 3.6, applied to the particular case of a second order tensor *A* function of another second order tensor *S*, can be written in matrix form:

$$\textbf{\textit{P.A.P}}^{-1} = f\left[\textbf{\textit{P.S.P}}^{-1} \right],$$

where *P* represents the passage matrix $\textbf{\textit{P}} = \left[\alpha_i^j \right]$.

In general, the practical Lumley method, when applied to a relation of type: $A_{i...j} = f_{i...j}(s_{pq})$, leads to:

$$A_{i...j}B^i...C^j = f\left[(s_i^j)^m B^i B_j, (s_i^j)^m B^i C_j, (s_i^j)^m C^i B_j, (s_i^j)^m C^i C_j, ..., II, III, ...\right]$$

for $m = 0$ to 2 (cf. Cayley-Hamilton theorem, section 3.5.1).

If $A_{i...j}$ is of odd order, then $A_{i...j} \equiv 0$ for the same reasons as in section 6.1.

If $A_{i...j}$ is of even order 2α, there are thus 6 groups containing the vector pair $\overline{B}$ and $\overline{C}$, and the same for the other vector pairs. In the expression of $A_{i...j}$, each term will contain the product of α vector pairs, there are thus, all things considered, 6^α possibilities. The coefficients will be functions of the invariants of s_{ij}.

Example: α=1 (this case has been considered previously, it corresponds to a second order tensor function of another second order tensor).

$$A_{ij}B^iC^j = f\left[\begin{matrix}(s_i^k)^0 B^i C_k, (s_i^k)^1 B^i C_k, (s_i^k)^2 B^i C_k, \\ (s_j^k)^0 C^j B_k, (s_j^k)^1 C^j B_k, (s_j^k)^2 C^j B_k, I, II, III\end{matrix}\right],$$

with the definitions: $\left(s_j^i\right)^m = \underbrace{s_j^p s_p^h ... s_q^i}_{m \text{ times}}$ and the understanding $\left(s_{ip}\right)^0 = g_{ip}$,

$$A_{ij}B^iC^j = f\left[\begin{matrix}(s_{ij})^0 B^i C^j, (s_{ij})^1 B^i C^j, (s_{ij})^2 B^i C^j, \\ (s_{ji})^0 C^j B^i, (s_{ji})^1 C^j B^i, (s_{ji})^2 C^j B^i, I, II, III\end{matrix}\right],$$

where we have denoted $\left(s_{ij}\right)^m = \left(s_i^h\right)^m g_{hj}$.

Thus:

$$A_{ij} = \left[\begin{matrix}a(s_{ij})^0 + b(s_{ij})^1 + c(s_{ij})^2 + \\ d(s_{ji})^0 + e(s_{ji})^1 + h(s_{ji})^2\end{matrix}\right],$$

and finally:

$$A_{ij} = (a+d)g_{ij} + bs_{ij} + cs_{im}s_{mj} + es_{ji} + hs_{jm}s_{mi} , \qquad [3.39]$$

where a, b, c, d, e, h are scalar functions of the invariants of s_{ij} .

For illustrative purpose, we could for example consider the expression of the Reynolds stress tensor R_{ij} according to the mean strain rate S_{ij} . We would get in Cartesian notation:

$$R_{ij} = a\delta_{ij} + b.S_{ij} + c.S_{im}S_{mj} ,$$

a relation which also takes into account the fact that these two tensors R_{ij} and S_{ij} are symmetric.

This is in fact a turbulent viscosity model (cf. section 1.5.1.2) with:

$$a = \frac{2}{3}k \quad \text{and} \quad b = -2\nu_t ,$$

the third term being a possible corrective term (cf. also section 12.1.1).

3.6.4. Tensorial function of two second order tensors

The problem is to establish the mathematical form of $A_{i...j} = f_{i...j}(s_{hm}, t_{hm})$ considered according to two second order symmetric tensors. The application of representation theorems (in Cartesian notation) leads in the case of two second order tensors to an expression which is far more complex ([SPE 59 and 60]) than the previous one, involving 19 terms (the total number remains limited by the relations given by the Cayley-Hamilton theorem and the relations of Rivlin-Cayley-Hamilton; cf. section 3.5.2):

$$A_{ij} = \sum_{m=0}^{18} c_m T_{ij}^{(m)}$$

The tensors $T_{ij}^{(m)}$ for $m = 0$ to 18 form an integrity basis and are defined by:

$$T_{ij}^{(0)} = \delta_{ij} \qquad\qquad T_{ij}^{(1)} = s_{ij}$$

$$T_{ij}^{(2)} = t_{ij} \qquad\qquad T_{ij}^{(3)} = \left(s_{ij}\right)^2$$

$$T_{ij}^{(4)} = \left(t_{ij}\right)^2 \qquad\qquad T_{ij}^{(5)} = s_{ip}t_{pj}$$

$$T_{ij}^{(6)} = t_{ip}s_{pj} \qquad\qquad T_{ij}^{(7)} = s_{ip}\left(t_{pj}\right)^2$$

$$T_{ij}^{(8)} = \left(t_{ip}\right)^2 s_{pj} \qquad\qquad T_{ij}^{(9)} = \left(s_{ip}\right)^2 t_{pj}$$

$$T_{ij}^{(10)} = t_{ip}\left(s_{pj}\right)^2 \qquad\qquad T_{ij}^{(11)} = \left(s_{ip}\right)^2 \left(t_{pj}\right)^2$$

$$T_{ij}^{(12)} = \left(t_{ip}\right)^2 \left(s_{pj}\right)^2 \qquad\qquad T_{ij}^{(13)} = t_{ip}s_{ph}\left(t_{hj}\right)^2$$

$$T_{ij}^{(14)} = s_{ip}t_{ph}\left(s_{hj}\right)^2 \qquad\qquad T_{ij}^{(15)} = t_{ip}\left(s_{ph}\right)^2 \left(t_{hj}\right)^2$$

$$T_{ij}^{(16)} = s_{ip}\left(t_{ph}\right)^2 \left(s_{hj}\right)^2 \qquad\qquad T_{ij}^{(17)} = s_{ip}t_{ph}\left(s_{hm}\right)^2 \left(t_{mj}\right)^2$$

$$T_{ij}^{(18)} = t_{ip}s_{ph}\left(t_{hm}\right)^2 \left(s_{mj}\right)^2 .$$

Coefficients c_m are functions of the proper invariants of s_{ij} and t_{ij} and of the mixed invariants such as $s_{ij}t_{ij}$, i.e. more precisely:

– $s_{jj}, t_{jj}, s_{jj}^{(2)}, t_{jj}^{(2)}, s_{jj}^{(3)}, t_{jj}^{(3)}$ (traces of the successive powers);

– and $s_{ij}t_{ij}, s_{ij}^{(2)}t_{ij}, s_{ij}t_{ij}^{(2)}, s_{ij}^{(2)}t_{ij}^{(2)}$ (contracted products of s_{ij} and t_{ij} to the power of 1 or 2).

In the case of a symmetric tensor A_{ij} function of a symmetric tensor S_{ij} and an antisymmetric tensor ω_{ij}, Pope S.B., [POP 75] has shown (cf. also [GAT 93]) that the number of independent tensors forming an integrity basis is reduced to 11 from using relations given by the Cayley-Hamilton theorem and symmetry conditions:

$$A_{ij} = \sum_{\lambda=0}^{10} G^{(\lambda)} T_{ij}^{(\lambda)},$$

where $T_{ij}^{(\lambda)}$ are the tensors of the integrity basis and where the scalar coefficients $G^{(\lambda)}$ are dependent on the invariants built on S_{ij} and ω_{ij} (five in number). More precisely:

$$T_{ij}^{(0)} = \delta_{ij} \qquad\qquad T_{ij}^{(1)} = S_{ij}$$

$$T_{ij}^{(2)} = S_{im}S_{mj} \qquad\qquad T_{ij}^{(3)} = \omega_{im}\omega_{mj}$$

$$T_{ij}^{(4)} = S_{im}\omega_{mj} - \omega_{im}S_{mj} \qquad\qquad T_{ij}^{(5)} = S_{im}S_{mp}\omega_{pj} - \omega_{im}S_{mp}S_{pj}$$

$$T_{ij}^{(6)} = S_{im}\omega_{mp}\omega_{pj} + \omega_{im}\omega_{mp}S_{pj} \qquad T_{ij}^{(7)} = \omega_{im}\omega_{mp}S_{pq}S_{qj} + S_{im}S_{mp}\omega_{pq}\omega_{qj}$$

$$T_{ij}^{(8)} = S_{im}\omega_{mp}S_{pq}S_{qj} \qquad\qquad T_{ij}^{(9)} = \omega_{im}S_{mp}\omega_{pq}\omega_{qj}$$
$$\quad - S_{im}S_{mp}\omega_{pq}S_{qj} \qquad\qquad\qquad - \omega_{im}\omega_{mp}S_{pq}\omega_{qj}$$

$$T_{ij}^{(10)} = \omega_{im}S_{mp}S_{pq}\omega_{qr}\omega_{rj} - \omega_{im}\omega_{mp}S_{pq}S_{qr}\omega_{rj}$$

This representation corresponds, for example, to the expression of the Reynolds stress tensor according to the strain rate tensor and the rotation tensor (cf. section 11.5).

3.6.5. *Linear function of several second order tensors*

We shall find it useful, for the study of pressure-strain correlations, to consider a second order tensor $A_{ij} = f_{ij}\left(a_{hm}, S_{hm}, \omega_{hm}\right)$ of zero trace, according to three second order tensors, the symmetric anisotropy tensor a_{hm}, the symmetric strain rate tensor S_{hm} and the antisymmetric rotation tensor ω_{hm}, limiting the development to linear terms in S_{hm} and ω_{hm}.

The application of the representation theorem and the Rivlin-Cayley-Hamilton relations lead to the following form:

$$A_{ij} = \sum_{m=0}^{7} c_m T_{ij}^{(m)}$$

The tensors $T_{ij}^{(m)}$ for $m = 0$ to 7, forming an integrity basis, are defined by:

$$T_{ij}^{(0)} = a_{ij} \qquad\qquad T_{ij}^{(1)} = a_{im}a_{mj} - \frac{1}{3}a_{mn}a_{mn}\delta_{ij}$$

$$T_{ij}^{(2)} = S_{ij} \qquad\qquad T_{ij}^{(3)} = a_{im}S_{mj} + a_{jm}S_{mi} - \frac{2}{3}a_{mn}S_{mn}\delta_{ij}$$

$$T_{ij}^{(4)} = a_{im}{}^2 S_{mj} + a_{jm}{}^2 S_{mi} - \frac{2}{3}a_{mn}{}^2 S_{mn}\delta_{ij} \qquad T_{ij}^{(5)} = a_{im}\omega_{mj} + a_{jm}\omega_{mi}$$

$$T_{ij}^{(6)} = a_{im}{}^2\omega_{mj} + a_{jm}{}^2\omega_{mi} \qquad T_{ij}^{(7)} = a_{im}{}^2\omega_{mn}a_{nj} + a_{jm}{}^2\omega_{mn}a_{nj}$$

Coefficients c_m are functions of the invariants of the argument tensors. The interested reader will also find useful extensions including argument vectors in [SMI 71] and [PEN 87].

3.6.6. *Extension to pseudo-tensors*

It is possible to use the alternate pseudo-tensor ε_{ijk} in the representation formulae of tensorial functions if we do not assume invariance under axis reflection (Robertson H.P., [ROB 39]). On the example of a second order tensor according to a vector:

$$A_{ij} = f_{ij}(x_m),$$

it is possible to write the generalized development (in Cartesian notation):

$$A_{ij} = a.x_i x_j + b.\delta_{ij} + c.\varepsilon_{ijm} x_m.$$

However, it is necessary to emphasize that in this development, if a and b are true scalars, coefficient c is pseudo-scalar (which changes sign under axis reflection). This is the case in the expression given in section 1.3.1 for the Reynolds stresses in turbulent flows with helicity.

3.7. Fourier transform in the fluctuating field

We consider homogenous turbulence. The Fourier transform operator applied directly in the fluctuating field will be currently used for studies in spectral space.

3.7.1. *Fourier transform of the fluctuating velocity field*

The following definition is usually used:

$$u_j(\vec{x}) = \int_{\mathbb{R}^3} \hat{u}_j(\vec{\kappa}) e^{i\vec{\kappa}.\vec{x}} d\vec{\kappa} \qquad\qquad [3.40]$$

On the physical intuitive point of view, the Fourier transform operator corresponds to a representation of the fluctuating field as a random superposition of sine and cosine wavy motions characterized by a wave vector $\vec{\kappa}$ and a corresponding spectral density. The direction of $\vec{\kappa}$ represents the direction of evolution of the turbulent eddies whereas the modulus κ or wavenumber is related to the size of eddies (inverse of the eddies length scale).

$$\hat{u}_j(\vec{\kappa}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} u_j(\vec{x}) e^{-i\vec{\kappa}.\vec{x}} d\vec{x} \qquad\qquad [3.41]$$

(in the sense of generalized functions or distributions).

3.7.2. *Some useful properties*

These are mainly the properties of derivatives and products in physical space and in Fourier space, explained hereafter:

$$\frac{\partial u_j(\vec{x})}{\partial x_m} = \int_{\mathbb{R}^3} i\kappa_m \hat{u}_j(\vec{\kappa}) e^{i\vec{\kappa}.\vec{x}} d\vec{\kappa}$$

$$\frac{\partial \hat{u}_j(\vec{\kappa})}{\partial \kappa_m} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} -i x_m u_j(\vec{x}) e^{-i\vec{\kappa}.\vec{x}} d\vec{x}$$

$$\widehat{\frac{\partial u_j(\vec{x})}{\partial x_m}} = i\kappa_m \hat{u}_j(\vec{\kappa}) \qquad\qquad \widehat{x_m u_j(\vec{x})} = i\frac{\partial \hat{u}_j(\vec{\kappa})}{\partial \kappa_m}$$

$$\widehat{u_i u_j}(\vec{\kappa}) = \left(\hat{u}_i * \hat{u}_j\right)(\vec{\kappa}) = \int \hat{u}_i(\vec{m}) \hat{u}_j(\vec{h}) \delta(\vec{h} + \vec{m} - \vec{k}) d\vec{m} d\vec{h}$$

δ denoting the Dirac distribution.

This latter equation singles out the wavevector triads $\vec{h} + \vec{m} - \vec{k}$ which play a central role in turbulence non-linear interactions.

3.7.3. *Fluctuating velocity equation in Fourier space*

The equation for the instantaneous velocity u_i in physical space reads:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j - \overline{u_i u_j}\right) + \frac{\partial}{\partial x_j}\left(\overline{U}_i u_j + \overline{U}_j u_i\right) = -\frac{\partial p}{\partial x_i} + v\frac{\partial^2 u_i}{\partial x_j^2}, \qquad\qquad [3.42]$$

and in homogenous turbulent flow:

$$\frac{\partial \overline{u_i u_j}}{\partial x_j} = 0,$$

and $\quad \dfrac{\partial \overline{U}_i}{\partial x_j} = \Lambda_{ij}, \quad \Lambda_{ij}$ constant

$$\overline{U}_i = \overline{U}_i^0 + \frac{\partial \overline{U}_i}{\partial x_m} x_m = \overline{U}_i^0 + \Lambda_{im} x_m,$$

implying: $\dfrac{\partial u_i}{\partial t}+\left(\overline{U}_j+u_j\right)\dfrac{\partial u_i}{\partial x_j}=-u_j\Lambda_{ij}-\dfrac{\partial p'}{\partial x_i}+v\dfrac{\partial^2 u_i}{\partial x_j^2}\,,$ [3.43]

with $\dfrac{\partial u_j}{\partial x_j}=0\,.$

Taking into account the previous properties, the Fourier transform of [3.43] reads:

$$\left(\dfrac{\partial}{\partial t}+v\kappa_j^2\right)\hat{u}_i+i\kappa_j\left(\overline{U}_j^0\hat{u}_i+\hat{u}_i*\hat{u}_j\right)-\kappa_j\Lambda_{jm}\dfrac{\partial\hat{u}_i}{\partial\kappa_m}=-\Lambda_{ij}\hat{u}_j-i\kappa_i\,\hat{p}\,.\quad[3.44]$$

Mass conservation implies:

$$\kappa_i\hat{u}_i=0\,.\tag{3.45}$$

Multiplying each side of equation [3.44] by κ_i and taking into account [3.45] we can show that:

$$\hat{p}=-\dfrac{\kappa_i\kappa_j}{\kappa_m^2}\left(\hat{u}_i*\hat{u}_j\right)+2i\dfrac{\kappa_i}{\kappa_m^2}\Lambda_{ij}\hat{u}_j\tag{3.46}$$

It is thus possible to eliminate pressure from the momentum equations:

$$\left(\dfrac{\partial}{\partial t}+v\kappa_j^2\right)\hat{u}_i+\Lambda_{ij}\hat{u}_j+i\kappa_j\overline{U}_j^0\hat{u}_i-2\dfrac{\kappa_i\kappa_h}{\kappa_m^2}\Lambda_{hj}\hat{u}_j$$

$$-\kappa_j\Lambda_{jm}\dfrac{\partial\hat{u}_i}{\partial\kappa_m}=-i\kappa_j\Delta_{hi}\left(\hat{u}_h*\hat{u}_j\right)$$ [3.47]

where $\Delta_{ij}=\delta_{ij}-\dfrac{\kappa_i\kappa_j}{\kappa_m^2}\,,$

In the particular case $\overline{U}_j\equiv 0$, we find:

$$\left(\dfrac{\partial}{\partial t}+v\kappa_j^2\right)\hat{u}_i=-i\kappa_j\Delta_{hi}\left(\hat{u}_h*\hat{u}_j\right),\tag{3.48}$$

or $\left(\dfrac{\partial}{\partial t}+\nu\kappa_j^2\right)\hat{u}_i=-i\kappa_j\varDelta_{hi}\underbrace{\int\hat{u}_h(\vec{m})\hat{u}_j(\vec{q})\delta(\vec{m}+\vec{q}-\vec{k})\,d\vec{m}d\vec{q}}_{(*)}$.

The product (*) represents the triad interactions:

$$\widehat{u_iu_j}(\vec{\kappa})=\left(\hat{u}_i*\hat{u}_j\right)(\vec{\kappa})=\int\hat{u}_i(\vec{m})\hat{u}_j(\vec{q})\delta(\vec{q}+\vec{m}-\vec{\kappa})\,d\vec{m}d\vec{q}\ .$$

3.7.4. *Correlations*

We can easily show that if $\overline{u_i(\vec{x})u_i(\vec{x'})}=\mathscr{R}_{ij}(\vec{\xi})$ with $\vec{\xi}=\vec{x}-\vec{x'}$, then

$$\overline{\hat{u}_i(\vec{\kappa})\hat{u}_j(\vec{\kappa'})}=\widehat{\mathscr{R}}_{ij}(\vec{\kappa}).\delta(\vec{\kappa}+\vec{\kappa'})\ .\qquad\qquad[3.49]$$

Indeed: $\overline{\hat{u}_i(\vec{\kappa})\hat{u}_j(\vec{\kappa'})}=\dfrac{1}{(2\pi)^6}\iint\overline{u_i(\vec{x})u_i(\vec{x'})}\,e^{+i(\vec{\kappa}.\vec{x}+\vec{\kappa'}.\vec{x'})}d\vec{x}d\vec{x'}$,

$$\overline{\hat{u}_i(\vec{\kappa})\hat{u}_j(\vec{\kappa'})}=\dfrac{1}{(2\pi)^6}\iint\mathscr{R}_{ij}(\vec{\xi})\,e^{+i\left[(\vec{\kappa}+\vec{\kappa'})\,\vec{\varXi}+(\vec{\kappa}-\vec{\kappa'})\,\vec{\xi}/2\right]}d\vec{\varXi}d\vec{\xi}\ ,$$

where we have denoted $\vec{\varXi}=\dfrac{\vec{x}+\vec{x'}}{2}$ and $\vec{\xi}=\vec{x}-\vec{x'}$.

Using the identity: $\dfrac{1}{(2\pi)^3}\int e^{-i\left[(\vec{\kappa}+\vec{\kappa'}).\vec{\varXi}\right]}d\vec{\varXi}=\delta(\vec{\kappa}+\vec{\kappa'})$,

We find for the previous correlation:

$$\dfrac{1}{(2\pi)^3}\int\mathscr{R}_{ij}(\vec{\xi})\,e^{-i\left[(\vec{\kappa}-\vec{\kappa'})\,\vec{\xi}/2\right]}d\vec{\xi}.\delta(\vec{\kappa}+\vec{\kappa'})=$$

$$=\widehat{\mathscr{R}}_{ij}(\dfrac{\vec{\kappa}-\vec{\kappa'}}{2}).\delta(\vec{\kappa}+\vec{\kappa'})=\widehat{\mathscr{R}}_{ij}(\vec{\kappa}).\delta(\vec{\kappa}+\vec{\kappa'})\ .$$

The final result is thus $\overline{\hat{u}_i(\vec{\kappa})\hat{u}_j(\vec{\kappa}')} = \widehat{\mathscr{R}}_{ij}(\vec{\kappa}).\delta(\vec{\kappa}+\vec{\kappa}')$. However, in practice the Dirac factor is generally omitted in this formula considering that this factor always drops out in calculations for homogenous turbulence. Indeed, in a bounded domain, the Dirac distribution is replaced by the Kronecker $\delta_{\vec{\kappa},-\vec{\kappa}'}$ (cf. [*MCC 96]).

So, we simply write:

$$\overline{\hat{u}_i(\vec{\kappa})\hat{u}_j(-\vec{\kappa})} = \widehat{\mathscr{R}}_{ij}(\vec{\kappa}) \text{ , or }\quad \overline{\hat{u}_i(\vec{\kappa})\hat{u}_j^{\;*}(\vec{\kappa})} = \widehat{\mathscr{R}}_{ij}(\vec{\kappa}) \text{ ,}$$

where $\hat{u}_j^{\;*}(\vec{\kappa})$ denotes the complex conjugate.

The following property is also verified:

$$\iint \overline{\hat{u}_i(\vec{\kappa})\hat{u}_j(\vec{\kappa}')}d\vec{\kappa}d\vec{\kappa}' = \int \widehat{\mathscr{R}}_{ij}(\vec{\kappa})d\vec{\kappa}\iint \delta(\vec{\kappa}+\vec{\kappa}')d\vec{\kappa}' = \widehat{\mathscr{R}}_{ij}(0) = R_{ij}$$

3.8. Wavelet transform

More recently, new methods of turbulent signal analysis have appeared that involve wavelet transforms [FAR 92]. The wavelet transform is replacing the sine and cosine basis of the Fourier transform by a family of translations and expansions of a same function $\psi_{u,s}(t)$ with zero mean which is the wavelet. The wavelet transform of f is defined by:

$$Wf(u,s) = \int_{\mathbb{R}} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt \text{ , }\quad \psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

We can consider intuitively this definition as a localized Fourier transform.

The use of orthogonal wavelet bases has allowed the extraction of coherent vortices from two-dimensional and three-dimensional turbulent flows ([FAR 99, FAR 01A]), and has led Farge M. *et al.* to propose a new method which considers a turbulent flow as a superposition of a coherent part and an uncorrelated random part, the first part being simulated and the second part being modeled.

Chapter 4

Methodology for One Point Closures

The methodology for statistical one point closures was systematized by Lumley who introduced a body of principles and hypotheses which make it possible to generate systematically coherent approximations for unknown correlations; this is the method of invariant modeling (Lumley J.L. and Khajeh-Nouri B., [LUM 74]; Siess J., [SIE 75]; Lumley J.L., [LUM 78A]). When high order expansions are used, this method can lead us to introduce numerous numerical constants and it is difficult to determine all of them by referring to experiments, lacking adequate measurements. In practice, we often use simplified versions of these formulations which are similar anyway to those formerly proposed on more intuitive grounds. Launder follows a more phenomenological approach (Launder B.E., [LAU 75C] – "*Simulation of turbulence models and their applications*", vol. 2, Launder B.E. *et al.*, 1984) looking first of all for the efficiency of predetermination methods by relying on turbulence physical bases. Lumley shows that usual one point closures rely upon strong hypotheses such as quasi-homogenity and weak anisotropy. Moreover, in the framework of one point closure methodology the turbulence problem is generally tackled from the point of view of determining the behavior law of a "turbulent material". If this formalism proves to be very useful to mimic turbulent behavior, it does not, however, always convey faithfully the physical essence of the acting phenomena. Indeed, turbulence cannot be only viewed as a non-Newtonian continuous medium. In particular, this medium does not necessarily satisfy the principle of material indifference and turbulent energy can only be maintained by an external energy supply, moreover it is not guaranteed that the local state is a unique functional of the flow variables, (Lumley J.L., [LUM 70]). In spite of these reservations, the approach provides a useful framework for developing turbulence modeling methods directed towards the calculation of practical flows.

4.1. Order of magnitude estimate of terms in the turbulence transport equations

4.1.1. *Velocity and length scales*

Theses characteristic scales are essentially based on $k = \dfrac{1}{2}\overline{u_i u_i}$ and $\varepsilon = \nu\overline{u_{i,j} u_{i,j}}$. They are:

$-u$, characteristic velocity scale, given by $\sqrt{\dfrac{2}{3}k}$;

$-\ell$, characteristic integral length scale of the big energy containing eddies;

$-\lambda$, G. I. Taylor microscale, defined by $\varepsilon = 15\nu u^2/\lambda^2$ (it is in fact, the eddy size down to which the turbulent energy cascades without appreciable viscous dissipation);

$-\eta$, Kolmogorov length scale, defined by $\eta = \left(\nu^3/\varepsilon\right)^{1/4}$ (it is the characteristic scale of dissipating eddies).
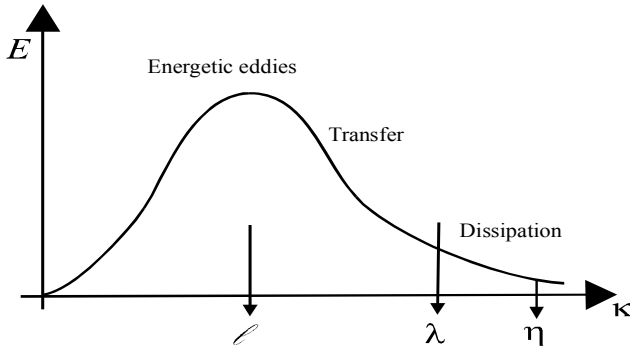
Figure 4.1. *Characteristic length scales in the turbulence energy spectrum*

4.1.2. *Relations between these various scales*

Let us introduce the turbulence Reynolds number $\mathrm{Re}_t = \dfrac{u\ell}{\nu}$.

The dissipation rate $\varepsilon = \dfrac{15\nu}{u\ell}$ is of order u^3/ℓ, from which is deduced the ratio

of the Taylor microscale to the integral scale $\left(\dfrac{\lambda}{\ell}\right)^2 = \dfrac{15\nu}{u\ell}$, and the following

relations, consequences of definitions:

$$\frac{\lambda}{\ell} = \sqrt{15}\,\mathrm{Re}_t^{-1/2}, \qquad \frac{\eta}{\ell} = \mathrm{Re}_t^{-3/4}, \qquad \frac{\eta}{\lambda} = \frac{1}{\sqrt{15}}\,\mathrm{Re}_t^{-1/4}. \qquad\qquad [4.1]$$

We also define the Reynolds number $\mathrm{Re}_\lambda = \dfrac{u\lambda}{\nu}$; this definition implies the relations:

$$\mathrm{Re}_\lambda = \sqrt{15}\,\mathrm{Re}_t^{1/2}, \qquad \frac{\eta}{\lambda} = \frac{1}{(15)^{1/4}}\,\mathrm{Re}_\lambda^{-1/2}, \qquad \frac{\eta}{\lambda} \propto \sqrt{\frac{\lambda}{\ell}} \qquad\qquad [4.2]$$

4.1.3. *Estimate of terms in the dynamic equations*

We will make use of the straightforward estimate rules proposed by Lumley (cf. Siess J., [SIE 75]). Let us consider a quantity $G = \overline{G} + g$.

1) Gradient of fluctuations

$$g_{,i} = \mathscr{O}\left(\frac{g}{\lambda}\right), \qquad\qquad [4.3]$$

where $\mathscr{O}$ represents the order of magnitude of the gradient of a fluctuation according to characteristic scales.

2) Gradient of mean quantities

$$\overline{G}_{,i} = \mathscr{O}\left(\frac{\overline{G}}{\ell}\right). \qquad\qquad [4.4]$$

3) Correlations

The correlation coefficient between two fluctuating quantities is estimated by the ratio of their respective characteristic length scales corresponding to their main spectral domain (cf. Tennekes H. and Lumley J.L., 1972).

Let us consider for example the correlation $\overline{u_i u_{i,j}}$: this term would be overestimated by $\mathcal{O}\left(\dfrac{u^2}{\lambda}\right)$ because u_i and $u_{i,j}$ are not strongly correlated. This problem is solved by putting forward the identity $\overline{u_i u_{i,j}} = \dfrac{1}{2}\left(\overline{u_i u_i}\right)_{,j}$ which is really of order $\mathcal{O}\left(\dfrac{u^2}{\ell}\right)$ and thus:

$$\frac{u^2}{\lambda} \cdot \frac{\lambda}{\ell} = \frac{u^2}{\lambda} . \mathrm{Re}_t^{-1/2} .$$

It appears then that the correlation coefficient between u_i and $u_{i,j}$ can be estimated by $\mathcal{R} = \lambda / \ell$, which is the ratio of their characteristic length scales. Thus,

$$\overline{u_i u_{i,j}} = \mathcal{O}\left(\frac{u^2}{\lambda}\right) . \mathcal{R} = \mathcal{O}\left(\frac{u^2}{\ell}\right) .$$

Generalizing this result leads to the following rule:

$$\overline{g.h_{,j}} = \mathcal{O}\left(\frac{g.h}{\lambda}\right) \times \mathcal{R} = \mathcal{O}\left(\frac{g.h}{\ell}\right), \qquad \mathcal{R}(g,h_{,j}) = \lambda / \ell$$

4) Spatial second derivatives

At high Reynolds numbers, the source and sink terms in the ε equation cancel each other out:

$$\overline{u_{j,m} u_{k,m} u_{j,k}} = \overline{\nu u_{j,km} u_{j,km}} .$$

To support this hypothesis, it can be shown that these two terms are of a higher order than all the other terms in the ε equation (cf. Tennekes H. and Lumley J.L., [*TEN 72]). Thus, in order to satisfy the balance equation at high Reynolds numbers these two terms must cancel each other out, there being no other way to satisfy the equation.

The left-hand side $\overline{u_{j,m}u_{k,m}u_{j,k}}$ is of order $\mathscr{O}\left(\dfrac{u^3}{\lambda^3}\right)$ and the estimate of the right-hand side $\overline{vu_{j,km}u_{j,km}}$ requires introducing an extra length scale characteristic of second derivatives, that we shall denote temporarily ζ, so that this term can be estimated of order $\mathscr{O}\left(v\dfrac{u^2}{\lambda^2\zeta^2}\right)$.

Identifying $\dfrac{u^3}{\lambda^3} = v\dfrac{u^2}{\lambda^2\zeta^2}$, we find $\dfrac{\zeta}{\lambda} = \mathrm{Re}_\lambda^{-1/2}$, it then results from [4.2] that ζ is proportional to the Kolmogorov length scale η.

Generalizing this result leads to the following simple rule:

$$g_{,ij} = \mathscr{O}\left(\dfrac{g}{\lambda\eta}\right) \tag{4.5}$$

4.2. Application to the momentum equations and the k and $\mathcal{E}$ equations

We give in the following the order of magnitude of each term in the transport equations obtained by using the technique presented previously.

$$\underbrace{\frac{\partial \overline{U}_i}{\partial t}}_{\mathscr{O}(u^2/\ell)} + \underbrace{\overline{U}_j\overline{U}_{i,j}}_{\mathscr{O}(u^2/\ell)} = -\underbrace{\left(\frac{\overline{P}}{\rho}\right)_{,i}}_{\mathscr{O}(u^2/\ell)} + \underbrace{v\overline{U}_{i,jj}}_{\substack{\mathscr{O}(vu/\ell^2)\\=\mathscr{O}(\frac{u^2}{\ell}\mathrm{Re}_t^{-1})}} - \underbrace{R_{ij,j}}_{\mathscr{O}(u^2/\ell)} \tag{4.6}$$

$$\underbrace{\frac{\partial k}{\partial t}}_{\mathscr{O}(u^3/\ell)} + \underbrace{\overline{U}_j k_{,j}}_{\mathscr{O}(u^3/\ell)} = \underbrace{-R_{jm}\overline{U}_{j,m}}_{\mathscr{O}(u^3/\ell)} - \underbrace{\left(\overline{u_m\frac{u_j u_j}{2}}\right)_{,m}}_{\mathscr{O}(u^3/\ell)}$$

$$-\underbrace{\left(\overline{\frac{u_j p}{\rho}}\right)_{,j}}_{\mathscr{O}(u^3/\ell)} - \underbrace{\overline{vu_{j,m}u_{j,m}}}_{\substack{\mathscr{O}(u^3/\ell)\\=\mathscr{O}(15vu^2/\lambda^2)}} + \underbrace{vk_{,jj}}_{\substack{\mathscr{O}(vu^2/\ell^2)\\=\mathscr{O}(\frac{u^3}{\ell}\mathrm{Re}_t^{-1})}} \tag{4.7}$$

$$\frac{\partial \varepsilon}{\partial t} \quad + \quad \overline{U}_j \varepsilon_{,j} \quad = - \quad 2\nu \overline{u_{k,j} u_{m,j}} . \overline{U}_{m,k} \quad - \quad 2\nu \overline{u_{j,k} u_{j,m}} . \overline{U}_{m,k}$$

$$\underset{\mathscr{O}(u^4/\ell^2)}{} \quad \underset{\mathscr{O}(u^4/\ell^2)}{} \quad \underset{\mathscr{O}(\nu u^3/\lambda^2 \ell) = \mathscr{O}(u^4/\ell^2)}{} \quad \underset{\mathscr{O}(\nu u^3/\lambda^2 \ell) = \mathscr{O}(u^4/\ell^2)}{}$$

$$- \quad 2\nu \overline{(u_k u_{j,m})} . \overline{U}_{j,km} \quad - \quad 2\nu \overline{u_{j,m} u_{k,m} u_{j,k}}$$

$$\underset{\mathscr{O}(\nu \frac{u^2}{\lambda} \frac{\lambda}{\ell} \frac{u}{\ell^2}) = \mathscr{O}(\nu \frac{u^3}{\ell^3}) = \mathscr{O}(\frac{u^4}{\ell^2} \mathrm{Re}_t^{-1})}{} \qquad \underset{\mathscr{O}(\nu \frac{u^3}{\lambda^3}) = \mathscr{O}(\frac{u^4}{\ell^2} \mathrm{Re}_t^{1/2})}{}$$

$$- \quad \nu \overline{(u_k u_{j,m} u_{j,m})}_{,k} \quad - \quad 2\frac{\nu}{\rho} \overline{u_{j,m} P_{,mj}}$$

$$\underset{\mathscr{O}(\nu u^3/\lambda^2 \ell) = \mathscr{O}(u^4/\ell^2)}{} \qquad \underset{\mathscr{O}(\nu u^3/\lambda^2 \ell) = \mathscr{O}(u^4/\ell^2)}{}$$

$$+ \quad \nu \varepsilon_{,jj} \quad - \quad 2\nu^2 \overline{u_{j,km} u_{j,km}} \qquad \qquad [4.8]$$

$$\underset{\mathscr{O}(\nu \frac{u^3}{\ell^3}) = \mathscr{O}(\frac{u^4}{\ell^2} \mathrm{Re}_t^{-1})}{} \qquad \underset{\mathscr{O}(\nu^2 \frac{u^2}{\lambda^2 \eta^2}) = \mathscr{O}(\frac{u^4}{\ell^2} \mathrm{Re}_t^{1/2})}{}$$

Note 1: it is admitted that the order of magnitude of the mean pressure P is the same as for $\rho \overline{U}^2$ and that the order of magnitude of the fluctuating pressure p is the same as for $\overline{\rho u_j u_j} - \rho \overline{u_j} \overline{u_j}$.

Note 2: the time derivatives are estimated as $\dfrac{\partial \overline{G}}{\partial t} = \mathscr{O}\left(\dfrac{\overline{G}u}{\ell}\right)$ and $\dfrac{\partial g}{\partial t} = \mathscr{O}\left(\dfrac{gu}{\lambda}\right)$.

4.3. Derivation of closure hypotheses

The approach proposed by J.L. Lumley, presupposes in a first stage that known and unknown quantities have been chosen as such. This is clearly dependent on the level of closure which has been chosen. So, we will distinguish the following two types of quantities:

– *Known quantities:* these are the quantities appearing in the left-hand side of the transport equations in the material derivative. For example, in usual second order models: $\{\overline{U}_i, \overline{P}, R_{ij}, \varepsilon\}$ (pressure is also considered as a known quantity, determined by the continuity condition).

– *Unknown quantities:* these are the unknown correlations appearing in the right-hand side of the equations and which need to be modeled.

The closure is achieved by introducing relations between known fields and unknown fields. The modeled terms representing the unknown correlations must have the correct physical dimension and tensorial properties of the unknown term that they replace. They must also fulfill the specific conservative and invariance properties of the original term (for example, the incompressibility condition $u_{j,j} = 0$ or a vanishing tensorial contraction, etc.). Among all the possible approximations it may be necessary to make a choice. For this, the formulation of closure hypotheses is of course guided by the physical intuition of the acting mechanisms.

Lumley developed the systematized approach, introducing a set of principles and hypotheses which make it possible to generate, in a systematic way, coherent approximations of unknown terms: this is known as invariant modeling. This formalist approach largely resorts to the mathematical tools presented in Chapter 3. Unfortunately, the invariant modeling method gives rise to numerous numerical constants and the calibration of all these constants from experimental data is not always completely possible. In practice, simplifications in closure approximations can be generally inferred from physical intuition of acting phenomena, this thus allows us to retain only the main leading terms.

It will be generally assumed, in the possible simplifications, that higher order statistical moments are less influential than the lower order statistical moments. In the case of second order closures, the two groups are:

Known quantities $\qquad\qquad$ Unknown quantities

$\overline{U}_i, \overline{P}, R_{ij}, \varepsilon \qquad\qquad \mathscr{T} = \mathscr{F}\left(\overline{U}_i, \overline{P}, R_{ij}, \varepsilon\right)$

If we take into account the functional relation $R_{ij} = \mathscr{T}_{\mathrm{u}}\left(\overline{U}_i, \overline{P}\right)$ suggested by the equation of mean motion, we are then led to assume along with J.L. Lumley that $\mathscr{T} = \mathscr{F}\left(R_{ij}, \varepsilon\right)$. In these relations, $\mathscr{F}$ represents a general functional dependence involving the whole space and for all instants of time prior to t.

4.4. The formalist approach: Lumley's invariant modeling

Several different steps have to be considered successively but their order of appearance can be modified in order to obtain the result more rapidly.

1) Selecting the known quantities

The point is to draw up the exhaustive list of known fields. As we have seen in the previous section, each unknown correlation will be considered according to these known fields.

2) Quasi-steadiness and weak non-homogenity hypotheses

Let $\mathcal{T}(x_i,t)$ be a tensorial unknown quantity, statistical correlation or group of correlations, thus,

$$\mathcal{T}(x_i,t) = \mathcal{F}(R_{ij},\varepsilon,...), \qquad\qquad [4.9]$$

where $\mathcal{F}(R_{ij},\varepsilon,...)$ is here a general functional involving the whole space and the elapsed time also taking into account the boundary conditions of the problem under consideration.

The hypotheses of quasi-steadiness and weak non-homogenity allow us to assume that every known parameter can be developed in Taylor series in space and time about the point (x_i,t). The series can then be approached by a limited Taylor expansion.

In practice, for every known term *C*, we shall assume:

$$\frac{\partial C}{\partial t} \approx 0, \qquad\qquad C_{,i} = \mathcal{O}(\varepsilon_h \frac{|C|}{\ell}), \qquad\qquad [4.10]$$

i.e.,

$$\ell \frac{C_{,i}}{|C|} \approx \varepsilon_h,$$

most of the time we shall consider that $C_{,ij...n} \approx 0$ for $n \geq 2$.

The symbol ε_h denotes a dimensionless small parameter characterizing the degree of non-homogenity of the field ($\varepsilon_h \ll 1$), thus:

$$C(\vec{x'},t') = C(\vec{x},t) + (x'_j - x_j)C_{,j}(\vec{x},t) +$$

$$\frac{1}{2!}(x'_i - x_i)(x'_j - x_j)C_{,ij}(\vec{x},t) + \underbrace{(t'-t)\partial_t C(\vec{x},t)}_{\approx 0} + ...$$

Thus, we write:

$$C(\vec{x'},t') = C(\vec{x},t) + \underbrace{(x'_j - x_j)C_{,j}(\vec{x},t)}_{\mathscr{O}(\varepsilon_h).|C|} + \underbrace{\frac{1}{2!}(x'_i - x_i)(x'_j - x_j)C_{,ij}(\vec{x},t)}_{\mathscr{O}(\varepsilon_h^2).|C|} + ...$$

in order to make clear the meaning of ε_h in the sense of Taylor expansions.

In the functional $\mathscr{F}$ every argument function C can thus be replaced by $C(\vec{x},t)$ and its successive gradients $C_{,j}(\vec{x},t)$, $C_{,ij}(\vec{x},t)$, etc. considered at point $\vec{x}$; generally we shall limit the expansion to the local couple of functions $C(\vec{x},t), C_{,j}(\vec{x},t)$. The functional $\mathscr{F}$ is thus replaced by an ordinary function F_I, because only the point (x_i,t) is involved and not the entire space. The function F_I is simpler, but at the price of an increased number of tensorial arguments.

$$\mathscr{F}(x_i,t) = F_I\left(R_{ij}(x_m,t), R_{ij,h}(x_m,t), \varepsilon(x_m,t), \varepsilon_{,h}(x_m,t),...\right) \qquad [4.11]$$

3) Weak anisotropy hypothesis

We now assume in addition that the anisotropy of turbulence is weak. It is thus useful to split the second order tensors into a spherical part (isotropic part with non-zero trace) and a deviatoric part (non-isotropic part with zero trace).

For example, the decomposition for the Reynolds stress tensor is:

$$b_{ij} = R_{ij} - \frac{2}{3}k\delta_{ij} \quad \text{and} \quad a_{ij} = \frac{b_{ij}}{2k} . \qquad [4.12]$$

The tensor b_{ij} is the stress deviator of the Reynolds stress tensor and it represents the anisotropic part of the stress tensor and a_{ij} is the anisotropy tensor which is a dimensionless tensor obtained by normalizing b_{ij} by the kinetic energy of turbulence. Thus,

$$\mathscr{F}(x_i,t) = F_I\begin{pmatrix} b_{ij}(x_m,t), b_{ij,h}(x_m,t), k(x_m,t), k_{,h}(x_m,t), \\ \varepsilon(x_m,t), \varepsilon_{,h}(x_m,t),... \end{pmatrix}, \qquad [4.13]$$

all these parameters being understood at point (x_i,t).

We assume:

$$\frac{\left|b_{ij}\right|}{2k} \approx \varepsilon_i \quad \text{is weak.}$$

The symbol ε_i denotes a small anisotropy parameter of the flow (small parameter in the sense of Taylor expansions: $\varepsilon_i \ll 1$), which can be estimated by:

$$\varepsilon_i = \frac{\sqrt{II^b}}{2k} = \frac{\sqrt{b_{ij}b_{ij}}}{2k} = \sqrt{a_{ij}a_{ij}}$$

Examples of orders of magnitude in anisotropy and non-homogenity:

k and ε	are of order	0 in ε_i	and	0 in ε_h
b_{ij}	is of order	1 in ε_i	and	0 in ε_h
$k_{,j}$ and $\varepsilon_{,j}$	are of order	0 in ε_i	and	1 in ε_h
$b_{ij,k}$	is of order	1 in ε_i	and	1 in ε_h
$k_{,ij}$ and $k_{,i}k_{,j}$	are of order	0 in ε_i	and	2 in ε_h

The first two examples ensue directly from the definitions. The quantities k and ε are pure scalars and consequently of order zero in anisotropy and non-homogenity. The stress deviator b_{ij} is of order $\sqrt{b_{ij}b_{ij}} = 2k\,\varepsilon_i$.

For the two next examples, we can write the corresponding expansion:

$$k(\vec{x'},t') = k(\vec{x},t) + \underbrace{(x'_j - x_j)k_{,j}(\vec{x},t)}_{\mathcal{O}(\varepsilon_h).\left|k\right|} + ... \qquad k_{,j} = \mathcal{O}(\varepsilon_h \frac{k}{\ell}),$$

$$b_{ij}(\vec{x'},t') = b_{ij}(\vec{x},t) + \underbrace{(x'_s - x_s)b_{ij,s}(\vec{x},t)}_{\mathcal{O}(\varepsilon_h)\sqrt{b_{ij}b_{ij}}} + ... \qquad b_{ij,s} = \mathcal{O}(\varepsilon_h \frac{\sqrt{b_{ij}b_{ij}}}{\ell}).$$

In the last example, order 2 in non-homogenity is obtained either by a second space derivative or by the square of a first derivative.

4) Exploiting the tensorial nature of terms

Known terms and unknown terms are both tensors, and the mathematical relation between them cannot be of any form. It is thus necessary to refer to the theory of invariants and representation of tensorial functions (cf. Chapter 3).

The application of the tensor representation theorems (section 3.6) will lead $\mathscr{T}(x_i, t)$ to appear as the sum of terms which are products of known tensors $T_{ij...m}$ by scalar functions $A(k, II, III, k_{,j}k_{,j}, ...)$ of the invariants of the list of local arguments of the functional under consideration.

$$\mathscr{T}_{ij...m}(x_p, t) = \sum A(k, II, III, k_{,j}k_{,j}, ...).T_{ij...m}$$

The tensors $T_{ij...m}$ of the representation basis are tensorial products of known argument tensors in [4.13], such as, for example, $T_{ij..m} = b_{ij}...k_{,m}$, etc.

5) Expansions in non-homogenity and anisotropy

Each scalar function introduced in the 4th stage can be expanded in series (truncated) of ε_h and ε_i and thus will be replaced by a polynomial in the invariants of known tensors and whose coefficients are functions of the scalar arguments (k, ε, etc.).

The expansions in anisotropy will involve the invariants of anisotropy II and III. However, this is not limiting the expansion to third order. The expansion can thus be written in the form:

$$A = a + b.II + c.III + d.II^2 + e.II.III + f.III^2 + ...$$

Note: the traces of powers of the anisotropy tensor, at a power higher than three, are seldom used because they are not independent, they are functions of II and III. We can deduce easily from the Cayley-Hamilton theorem that $\left(A_i^j\right)^m - J_1\left(A_i^j\right)^{m-1} + J_2\left(A_i^j\right)^{m-2} - J_3\left(A_i^j\right)^{m-3} = 0$ $(m > 3)$ and consequently: $IV = \frac{1}{2}(II)^2$, $V = \frac{5}{6}II.III$, $VI = \frac{1}{2}II.IV + \frac{1}{3}(III)^2$ and more generally $I^{(m)} = \frac{1}{2}II.I^{(m-2)} + \frac{1}{3}III.I^{(m-3)}$ $(m > 3)$. These invariants, considered as functions of II and III, could be used in the expansions in anisotropy, in place of II^2, $II.III$, III^2, II^3, etc., but even so they must not be included into the argument list of $A(k, II, III, k_{,j}k_{,j}, \text{etc.})$.

For non-homogenity we shall get expansions of the type:

$$A = a + b\ell \frac{\sqrt{g_{,j}g_{,j}}}{g} + b\ell^2 \frac{g_{,j}g_{,j}}{g^2} + d\ell^2 \frac{g_{,jj}}{g} + \dots, \quad \ell = k^{3/2}/\varepsilon .$$

6) Dimensional analysis

The terms generated by the previous steps must have the correct physical dimension. The coefficients which are functions of the scalar arguments (k, ε, etc.) can be determined in this way. We then get at this stage the necessary mathematical form of every function apart from a multiplicative purely numerical coefficient.

7) Exploiting the possible properties of symmetry, incompressibility, etc.

If the original term satisfies some properties such as symmetry, zero contraction, etc., the corresponding modeled term must also verify these properties. Specific to each particular term, these properties allow us to reduce the number of numerical constants.

The order of the previous steps in the process can be sometimes more or less modified in order to get the result more quickly.

4.5. Examples of application

4.5.1. *First example: the pressure-strain correlations*

Let A_{ik} be the non-linear part of the pressure-strain correlations which appear in the evolution equations of the Reynolds stresses (return to isotropy terms). This term is not directly influenced by the mean velocity gradients (see Chapter 6).

Let us resume the different steps in invariant modeling in this particular case:

1) List of known fields, arguments of the functional: $A_{ik} = \mathscr{F}(R_{ij}, \varepsilon)$.

2) Quasi-steadiness, weak non-homogenity.

The term under consideration is now active in homogenous turbulence, so we shall limit the expansions in non-homogenity to order zero. The previous functional is thus replaced by an ordinary function with the same number of arguments: $A_{ik} = f(R_{ij}, \varepsilon)$.

3) Weak anisotropy.

We shall distinguish in the Reynolds stresses a spherical part corresponding to the turbulence kinetic energy k and the anisotropy represented in a non-dimensional form by the anisotropy tensor $a_{ij} = \dfrac{R_{ij} - \dfrac{2}{3}k\delta_{ij}}{2k}$. Consequently, $A_{ik} = \mathscr{F}(a_{ij}, k, \varepsilon)$.

4) Tensorial nature of the mathematical relations and application of representation theorems.

$$A_{ik}B_iC_k = f\left(II^a, III^a, k, \varepsilon, (a_{ik})^m B_iC_k, (a_{ik})^m B_iB_k, (a_{ik})^m C_iC_k\right)$$

$m = 0$ to 2

$$A_{ik}B_iC_k = f_m(a_{ik})^m B_iC_k$$

$m = 0, 1, 2$ (note that a_{ik} is symmetric).

where f_m depends on II, III, k and ε (obviously, f_m is a mere coefficient with no tensorial property).

$$A_{ik} = f_0 g_{ik} + f_1 a_{ik} + f_2 a_i^h a_h^m g_{km}$$

and in Cartesian notation:

$$A_{ik} = f_0 \delta_{ik} + f_1 a_{ik} + f_2 a_{ij} a_{jk}$$

5) Expansion with respect to the anisotropy (at 3rd order).

Using Taylor expansions of the scalar functions with respect to the invariants II, III, etc. every scalar function f_j is replaced by a sum of terms, each having the following form: product of an invariant by a scalar function of the scalar argument k, ε only.

The small quantities are thus expressed using the homogenous anisotropic arguments $II^a = a_{ij}a_{ij}$ (order ε_i^2) and $III^a = a_{ij}a_{jk}a_{ki}$ (order ε_i^3) (let us recall that the first order invariant I^a is not involved because it is identically zero from the definition of the anisotropy tensor, $a_{jj} = 0$).

$$A_{ik} = \delta_{ik} \left[f_0^0 + f_0^1 II^a + f_0^2 III^a \right] + a_{ik} \left[f_1^0 + f_1^1 II^a \right] + a_{ij} a_{jk} f_2^0$$

The quantities f_m^p are mere coefficients (without any tensorial property) which are functions of k and ε.

6) Dimensional analysis.

Each of these latter scalar functions is replaced by its expression as a monomial of the scalar arguments, as required by dimensional analysis. This monomial is then multiplied by an undetermined numerical coefficient.

$$f_i^m = \varepsilon.\alpha_i^m ,$$

(α_i^m is a purely numerical coefficient without any tensorial property).

Then,

$$A_{ik} = \varepsilon.\delta_{ik} \left[\alpha_0^0 + \alpha_0^1 II^a + \alpha_0^2 III^a \right] + \varepsilon.a_{ik} \left[\alpha_1^0 + \alpha_1^1 II^a \right] + \varepsilon.a_{ij} a_{jk} \left[\alpha_2^0 \right].$$

7) Zero trace property.

This is a specific property (redistribution of the energy among the different components of the Reynolds stress tensor) of the pressure-strain correlation which also has to be verified by its approximation model:

$$A_{ii} = 3\varepsilon \left[\alpha_0^0 + \alpha_0^1 II^a + \alpha_0^2 III^a \right] + \alpha_2^0 \varepsilon.II^a = 0 ,$$

and consequently

$$\alpha_0^0 = 3\alpha_0^1 + \alpha_2^0 = \alpha_0^2 = 0 .$$

Thus, there remains:

$$A_{ik} = \varepsilon.\delta_{ik} \left[\alpha_0^1 II^a \right] + \varepsilon.a_{ik} \left[\alpha_1^0 + \alpha_1^1 II^a \right] - 3\alpha_0^1 \varepsilon.a_{ij} a_{jk} ,$$

$$A_{ik} = \left[\alpha_1^0 + \alpha_1^1 II^a \right] \varepsilon.a_{ik} - 3\alpha_0^1 \varepsilon. \left[a_{ij} a_{jk} - \frac{1}{3} II^a \delta_{ik} \right] , \qquad [4.14]$$

i.e., finally:

$$A_{ik} = \left[\alpha_1^0 + \alpha_1^1 II^a \right] \frac{\varepsilon}{k} \cdot \left(R_{ik} - \frac{2}{3} k \delta_{ik} \right)$$

$$-3\alpha_0^1 \frac{\varepsilon}{k^2} \cdot \left[R_{ij} R_{jk} - \frac{4}{3} k R_{ik} + \frac{4}{9} k^2 \delta_{ik} - \frac{1}{3} \left(R_{pm} R_{pm} - \frac{4}{3} k^2 \right) \delta_{ik} \right].$$

In a second order approximation coefficient α_1^1 will not be considered, and in a first order approximation there will remain only α_1^0:

$$A_{ik} = \alpha_1^0 \frac{\varepsilon}{k} \cdot \left(R_{ik} - \frac{2}{3} k \delta_{ik} \right). \tag{4.15}$$

We recover in this way, as a particular case of invariant modeling approach, an approximation introduced formerly by Rotta (cf. Chapter 6) on purely phenomenological grounds.

4.5.2. Second example: behavior law for the Reynolds stress tensor

We consider in this example modeling of the Reynolds stress tensor according to the mean strain rate:

$$R_{ij} = F(k, \varepsilon, S_{ij}) \quad \text{with} \quad S_{ij} = \frac{1}{2} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right),$$

in terms of an approximation of order zero in non-homogenity.

The different steps of the invariant modeling lead to an expansion such as:

$$R_{ij} = f_0 \delta_{ij} + f_1 S_{ij} + f_2 S_{im} S_{mj}$$

where coefficients f_h are functions of the scalars k and ε and of the invariants $II^S = S_{ij} S_{ij}$ and $III^S = S_{im} S_{mp} S_{pi}$.

Taking into account dimensional analysis and the contraction property $R_{jj} = 2k$, we obtain:

$$R_{ij} = \frac{2}{3}k\delta_{ij} - c_1 \frac{k^2}{\varepsilon}S_{ij} + c_2 \frac{k^3}{\varepsilon^2}S_{im}S_{mj}$$

where c_1 and c_2 are functions of $\frac{k^2}{\varepsilon^2}II^S$ and $\frac{k^3}{\varepsilon^3}III^S$.

The first term corresponds to the spherical part in isotropic turbulence, the second term represents the gradient law, whereas the last term is involved in non-linear models.

The models by Townsend A.A., 1956 and Lightill M.J., 1956 (cf. Mathieu, [MAT 71]), formulated as follows:

$$R_{ij} = \frac{2}{3}k\delta_{ij} - 0.4k\frac{S_{ij}}{\sqrt{II^S}} ,$$

appear as a particular case of the previous general expansion.

If more general dependencies are considered, such as:

$$R_{ij} = F(k, \varepsilon, S_{ij}, \omega_{ij})$$

with: $S_{ij} = \frac{1}{2}\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right)$ and $\omega_{ij} = \frac{1}{2}rot_{ij}\overline{U} = \frac{1}{2}\left(\overline{U}_{i,j} - \overline{U}_{j,i}\right)$

the result we obtain is far more complex (cf. section 3.6.4). Considering that the first invariants verify $I^S = I^\omega = 0$, that S_{ij} is symmetric and also that ω_{ij} is symmetric, the number of tensors of the representation basis can be reduced from 19 to 10 (this corresponds to explicit algebraic stress models; cf. section 11.5).

We can quote for illustration, the $k - \omega^2$ model by Wilcox D.C. and Rubesin M.W., [WIL 80] (cf. Chapter 11) which makes use of a non-linear behavior law for expressing the Reynolds stress tensor, and also all the non-linear models which are more recent (cf. section 11.4).

For examples relative to non-homogenous flows, see exercises no. 3 and no. 14.

4.6. Realizability problem

Any sequence of real numbers is not in general a hierarchy of statistical moments of a probability distribution: variances must be always positive, correlation coefficients must be less than unity, etc. The mathematical conditions to satisfy in order that a sequence of the given real numbers can be interpreted as the set of statistical moments of a probability distribution take the form of realizability inequalities (Schumann U., [SCH 77]; Du Vachat R., [DUV 77, AND 78]). The tensorial modeling method presented in sections 4.4 and 4.5 does not guarantee the realizability of the approximations generated in such a way. It is thus advisable to satisfy these realizability properties, or at least make sure that the error introduced remains limited.

4.6.1. *General results on the problem of moments*

The problem of moments is a complex mathematical problem, and so we shall restrict ourselves to recalling some useful results for the applications under consideration. These results have been emphasized by Du Vachat R., [DUV 78], in particular.

Let X be a random variable distributed on $\mathscr{D} \subset \mathbb{R}$, its statistical moments are defined in probability calculus by:

$$a_k = \int_{\mathscr{D}} x^k p(x) dx,$$

where p is the probability density.

If:

$$P(x) = \sum_{k=0}^{m} c_k x^k,$$

is a positive polynomial in $\mathscr{D}$, the quantity:

$$M = \sum_{k=0}^{m} c_k a_k$$

is also positive, indeed:

$$\sum_{k=0}^{m} c_k a_k = \int_{\mathscr{D}} P(x)p(x)dx \geq 0 .$$

It can be shown mathematically that for a sequence of given real numbers a_k, a necessary and sufficient condition for realizability is given by:

$$M = \sum_{k=0}^{m} c_k a_k \geq 0 \qquad\qquad [4.16]$$

for every m and for every positive polynomial $P(x) = \sum_{k=0}^{m} c_k x^k$ in $\mathscr{D}$.

This condition also ensures the existence of $p(x)$.

In the case of several dimensions, a mathematical condition for the existence of $p(x,y)$ in $\mathscr{D}$ such that $a_{pq} = \int_{\mathscr{D}} x^p y^q p(x,y)dxdy$ can be written:

$$M = \sum_{p=0}^{m} \sum_{q=0}^{n} c_{pq} a_{pq} \geq 0 \qquad\qquad [4.17]$$

for every integer m, n and for every positive polynomial $P(x,y) = \sum_{p,q=0}^{m,n} c_{pq} x^p y^q$

in $\mathscr{D}$.

We shall denote symbolically this condition by $M\big[P(x,y)\big] \geq 0$. It is a necessary and sufficient condition.

Particular cases

1) Unbounded variable in one dimension: $\mathscr{D} \equiv \mathbb{R}$.

It can be shown that every positive polynomial $\forall x$ is the sum of two squared real polynomials, and so the realizability conditions are (necessary and sufficient conditions):

$$M\left[Q(x)^2\right] \geq 0 \text{ for every real polynomial } Q(x).$$

$$Q(x) = \sum_{k=0}^{m} c_k x^k \, ,$$

$$M = \sum_{i=0}^{m} \sum_{j=0}^{m} c_i c_j a_{i+j} \geq 0 \qquad\qquad [4.18]$$

m integer, c_i and c_j any real number.

   2) Variable in n dimensions:

$$\mathscr{D} \equiv \mathbb{R}^2 , \qquad M\left[Q(x,y)^2 \right] \geq 0 \, , \qquad\qquad [4.19]$$

for any polynomial in two variables.

and $$\mathscr{D} \equiv \mathbb{R}^3 , \qquad M\left[Q(x,y,z)^2 \right] \geq 0 \, , \qquad\qquad [4.20]$$

for any polynomial in three variables (these two latter conditions are necessary but not sufficient).

   3) Bounded variables (temperature or concentration): $\mathscr{D} \equiv \mathbb{R}^+$.

   A positive polynomial in $\mathbb{R}^+$ can be decomposed into $S(x) + x.T(x)$ where S and T are the sum of squared real polynomials. The realizability conditions thus become:

$$\begin{cases} M\left[Q(x)^2 \right] \geq 0 \\ M\left[x.R(x)^2 \right] \geq 0 \end{cases} \qquad\qquad [4.21]$$

where Q and R denote any real polynomial.

4.6.2. *Conditions for a quadratic form to be positive*

   Let us consider a quadratic form: $F = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j$. There are several equivalent ways for expressing the positivity conditions of a quadratic form. These traditional results will be useful in the following and summed up in Table 4.1.

4.6.3. *Realizability conditions for the Reynolds stress tensor*

The components of the Reynolds stress tensor $R_{ij} = \overline{u_i u_j}$ are the second order moments of the fluctuating velocity u_i. Considering that $\overline{u_i} = 0$, it is possible to use condition [4.20]:

$$Q(u,v,w) = a.u + b.v + c.w,$$

and thus:

$$M = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \geq 0$$

To express that this quadratic form is positive, we can write the three conditions relative to the principal diagonal determinants.

$$\begin{cases} R_{11} \geq 0 \\ R_{11}R_{22} - R_{12}^2 \geq 0 \text{ (Schwarz's condition)} \\ \det(R_{ij}) = R_{11}R_{22}R_{33} + 2R_{12}R_{23}R_{31} - R_{22}R_{13}^2 - R_{33}R_{12}^2 - R_{11}R_{23}^2 \geq 0 \end{cases} \qquad [4.22]$$

This latter condition can also be written:

$$\frac{R_{12}R_{23}R_{31}}{R_{11}R_{22}R_{33}} \geq \frac{1}{2}\left[\frac{R_{13}^2}{R_{11}R_{33}} + \frac{R_{12}^2}{R_{11}R_{22}} + \frac{R_{23}^2}{R_{22}R_{33}} - 1 \right]$$

Note: the inequalities $R_{22} \geq 0$, $R_{33} \geq 0$, $R_{22}R_{33} - R_{23}^2 \geq 0$, $R_{11}R_{33} - R_{13}^2 \geq 0$ are also true but they are redundant.

Equivalent conditions	$F > 0, \quad \forall \vec{x} \neq 0 \in \mathbb{R}_n$ F definite positive	$F \geq 0, \quad \forall \vec{x} \in \mathbb{R}_n$ F positive
n eigenvalues of matrix $[\alpha_{ij}]$	$\lambda(i) > 0$	$\lambda(i) \geq 0$
n principal diagonal determinants: $D_m = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mm} \end{vmatrix}$ $m=1$ to n	$D_m > 0$	$D_m \geq 0$
n principal invariants of α_{ij}	$\mathscr{T}_i > 0$	$\mathscr{T}_i \geq 0$

Table 4.1. *Conditions for a quadratic form to be positive*

Equivalent conditions for the realizability of the Reynolds stress tensor are obtained using the principal invariants of the Reynolds stress tensor:

$$\mathcal{T}_1^R \geq 0, \ \mathcal{T}_2^R \geq 0, \ \mathcal{T}_3^R \geq 0.$$

4.6.4. *Triple velocity correlations*

We can obtain an interesting inequality on the triple velocity correlations in the following way: the Schwarz's inequality $a_{12}^2 \leq a_{11}a_{22}$ is applied to the two variables u_i and $u_j u_k - \overline{u_j u_k}$, which implies:

$$\overline{u_i u_j u_k}^2 \leq \overline{u_i^2} \left(\overline{u_i^2 u_i^2} - \overline{u_j u_k}^2 \right). \tag{4.23}$$

4.6.5. *Realizability conditions for the invariants*

Application of invariant modeling to the example of pressure-strain correlations had led in section 4.5 to the approximation:

$$A_{ik} = f_0(II^a, III^a, k, \varepsilon)\delta_{ij} + f_1(II^a, III^a, k, \varepsilon)a_{ij} + f_2(II^a, III^a, k, \varepsilon)a_{ij}a_{jk} \tag{4.24}$$

It is thus interesting and useful in a general purpose, to study the turbulence state characterized by the two invariants II and III or by the principal invariants $\mathcal{T}_2, \mathcal{T}_3$. We first consider for this, the principal invariants of the R_{ij} and a_{ij} tensors:

$$\mathcal{T}_1^R = R_{ii},$$

$$\mathcal{T}_2^R = \frac{1}{2}\left[(R_{ii})^2 - R_{ij}R_{ij} \right],$$

$$\mathcal{T}_3^R = \frac{1}{6}\left[(R_{ii})^3 - 3R_{mm}R_{ij}R_{ij} + 2R_{ij}R_{jm}R_{mi} \right],$$

$$\mathcal{T}_1^a = 0,$$

$$\mathcal{T}_2^a = -\frac{1}{2}a_{ij}a_{ij} = -\frac{1}{2}II^a,$$

$$\mathcal{T}_3^a = \frac{1}{3}a_{ij}a_{jm}a_{mi} = \frac{1}{3}III^a.$$

Considering after [4.12] that $R_{ij} = 2k(a_{ij} + \delta_{ij}/3)$, it can easily be shown that:

$$\mathcal{T}_1^R = 2k ,$$

$$\mathcal{T}_2^R = 2k^2 \left(\frac{2}{3} + 2\mathcal{T}_2^a \right),$$

$$\mathcal{T}_3^R = \frac{8}{6}k^3 \left(\frac{2}{9} + 2\mathcal{T}_2^a + 6\mathcal{T}_3^a \right).$$

The realizability conditions for the Reynolds stress tensor:

$$\mathcal{T}_1^R \geq 0, \; \mathcal{T}_2^R \geq 0, \; \mathcal{T}_3^R \geq 0$$

can also be written as:

$$k \geq 0, \; \mathcal{T}_2^a \geq -\frac{1}{3}, \; \mathcal{T}_2^a + 3\mathcal{T}_3^a + \frac{1}{9} \geq 0 . \tag{4.25}$$

Following the approach of Lumley J.L. and Newmann G.R., [LUM 77], from now on we shall consider the principal axes in which:

$$[R_{ij}] = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{bmatrix}, \; [a_{ij}] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$R_{11} \geq 0, \; R_{22} \geq 0, \; R_{33} \geq 0, \; \Rightarrow a_{11} \geq -\frac{1}{3}, \; a_{22} \geq -\frac{1}{3}, \; a_{33} \geq -\frac{1}{3} \tag{4.26}$$
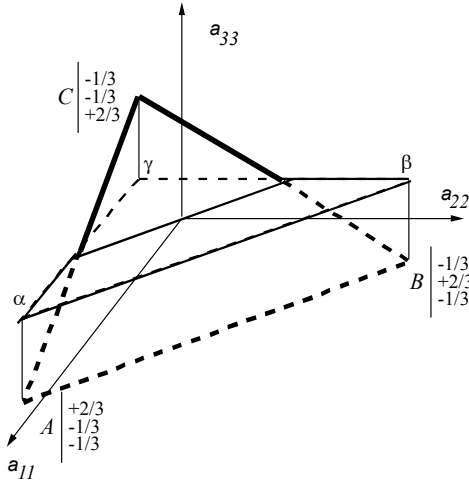
Figure 4.2. *Lumley's anisotropy triangle in perspective view*

The third equation [4.25] is the equation of a plane in the (a_{11}, a_{22}, a_{33}) space. Conditions [4.26] show (cf. Figure 4.2) that the possible values of a_{11}, a_{22}, a_{33} are located inside a triangle $A\,B\,C$ that projects in $\alpha\,\beta\,\gamma$ in the (a_{11}, a_{22}) plane.

The second invariant can be written in two equivalent forms:

$$\mathscr{T}_2^a = -\frac{1}{2}\left(a_{11}^2 + a_{22}^2 + a_{33}^2\right) = a_{11}a_{22} + a_{11}a_{33} + a_{33}a_{22}.$$

The first expression shows that $\mathscr{T}_2^a$ is negative, and the second expression, considering that $\mathscr{T}_1^a = 0$, leads to: $-\mathscr{T}_2^a = a_{11}^2 + a_{22}^2 + a_{11}a_{22}$.

The curve $\mathscr{T}_2^a = C$ is an ellipse (cf. Figure 4.3) in the (a_{11}, a_{22}) plane and its equation is:

$$\left(\frac{a_{11} + a_{22}}{2/3}\right)^2 + \left(\frac{a_{11} - a_{22}}{2/\sqrt{3}}\right)^2 = -3C$$

The largest of these ellipses is obtained for $C = -1/3$, which goes through the points α, β, γ.
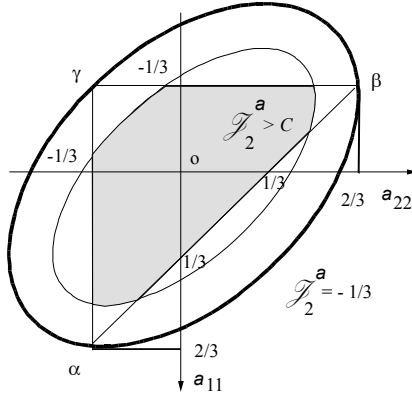
Figure 4.3. *Second principal invariant of anisotropy in principal axes: Lumley's anisotropy triangle*

On the other hand $\mathscr{T}_3^a = a_{11}a_{22}a_{33} = -a_{11}^2 a_{22} - a_{22}^2 a_{11}$. Eliminating a_{11} between the expressions of $\mathscr{T}_2^a$ and $\mathscr{T}_3^a$ we find: $\mathscr{T}_3^a = \mathscr{T}_2^a a_{22} + a_{22}^3$. Thus, we look for the variations of $\mathscr{T}_3^a$ according to a_{22} for $\mathscr{T}_2^a$ fixed. A relative extremum is obtained for:

$$\frac{d\mathscr{T}_3^a}{da_{22}} = \mathscr{T}_2^a + 3a_{22}^2 = 0$$

This extremum corresponds to $a_{22} = \pm\sqrt{-\dfrac{\mathscr{T}_2^a}{3}}$ and $\mathscr{T}_3^a = \pm 2\left(-\dfrac{\mathscr{T}_2^a}{3}\right)^{3/2}$ (cf.

Figure 4.4). On the other hand, on the ellipse $\mathscr{T}_2^a = C$, the value of a_{22} remains contained between $-2\sqrt{-\dfrac{\mathscr{T}_2^a}{3}}$ and $+2\sqrt{-\dfrac{\mathscr{T}_2^a}{3}}$.

At these points, $\mathscr{T}_3^a$ reaches the respective values: $-2\left(-\dfrac{\mathscr{T}_2^a}{3}\right)^{3/2}$ and

$+2\left(-\dfrac{\mathscr{T}_2^a}{3}\right)^{3/2}$. We find in this way the limiting curves $\mathscr{T}_3^a = \pm 2\left(-\dfrac{\mathscr{T}_2^a}{3}\right)^{3/2}$.
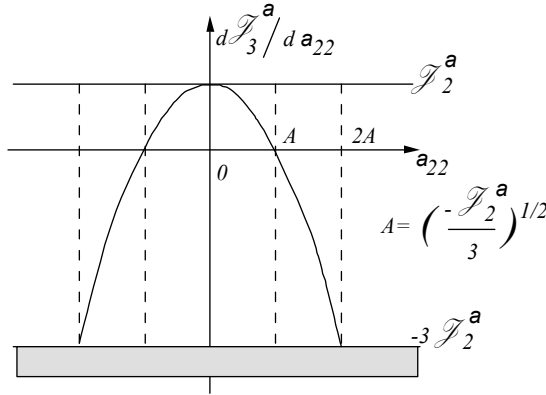
Figure 4.4. *Variations of the third principal invariant of anisotropy*

The previous results are summed up in Figure 4.5 which shows that the values of the principal invariants $\mathscr{T}_2^a$, $\mathscr{T}_3^a$ must lie inside the curvilinear triangle *OMN*. The segment *MN* corresponds to a two component turbulence, for example $R_{11} = 0 \Rightarrow a_{11} = -1/3$.

In such conditions:

$$\mathscr{T}_2^a = -\frac{1}{9} - \left(a_{22}^2 - \frac{a_{22}}{3} \right), \qquad \mathscr{T}_3^a = \frac{\left(a_{22}^2 - \dfrac{a_{22}}{3} \right)}{3},$$

and this implies $\mathscr{T}_2^a + 3.\mathscr{T}_3^a + 1/9 = 0$ or $III^a - \dfrac{1}{2}II^a + \dfrac{1}{9} = 0$.

The arcs *MO* and *NO* correspond to axisymmetric turbulence: for example $a_{22} = a_{33}$,

$a_{11} + a_{22} + a_{33} = 0 \Rightarrow a_{22} = -a_{11}/2,$

and thus:

$$\left[a_{ij} \right] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & -a_{11}/2 & 0 \\ 0 & 0 & -a_{11}/2 \end{bmatrix}, \quad \mathscr{T}_2^a = -\frac{3a_{11}^2}{4}, \quad \mathscr{T}_3^a = \frac{a_{11}^3}{4}$$

which implies:

$$A = \frac{\mathscr{T}_3^{\boldsymbol{a}} / 2}{\left(\mathscr{T}_2^{\boldsymbol{a}} / 3\right)^{3/2}} = \pm 1$$

Depending on the sign of A, i.e. also the sign of $\mathscr{T}_3^{\boldsymbol{a}}$, the representative point will fall on OM or ON. In the case $\mathscr{T}_3^{\boldsymbol{a}} < 0$ (corresponding to $a_{11}<0$ in the previous example) the Reynolds stress tensor has one weak component and two strong components ("saucer" type eddies), while in the case $\mathscr{T}_3^{\boldsymbol{a}} > 0$ (corresponding to $a_{11}>0$ in the previous example) the Reynolds stress tensor has one strong component and two weak components ("cigar" type eddies).
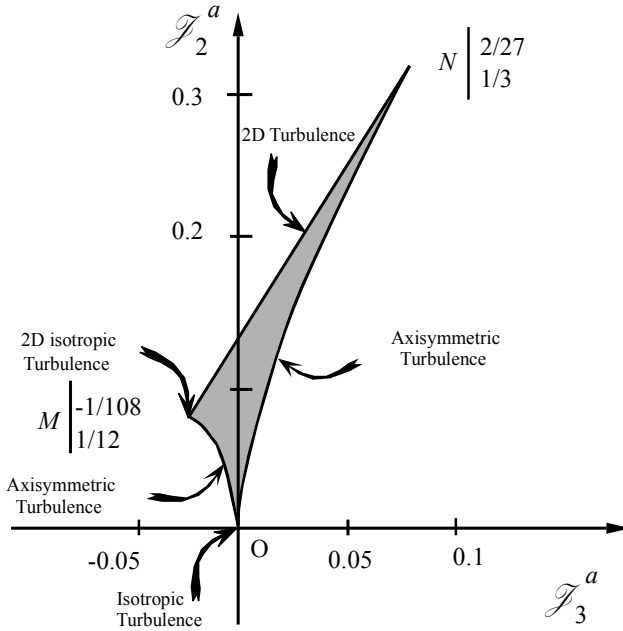
Figure 4.5. *Lumley's invariant diagram of realizability*

MN	2D isotropic turbulence	$\mathscr{T}_2^a + 3\mathscr{T}_3^a + 1/9 = 0$
OM	axisymmetric turbulence	$\mathscr{T}_3^a = -2\left(-\mathscr{T}_2^a/3\right)^{3/2}$
ON	axisymmetric turbulence	$\mathscr{T}_3^a = +2\left(-\mathscr{T}_2^a/3\right)^{3/2}$
O	isotropic 3D state	$\mathscr{T}_2^a = 0 , \mathscr{T}_3^a = 0$
M	isotropic 2D state	$\mathscr{T}_2^a = -1/12 , \mathscr{T}_3^a = -1/108$
N	1D "turbulence"	$\mathscr{T}_2^a = -1/3 , \mathscr{T}_3^a = +2/27$

Table 4.2. *Limiting states of turbulence in the realizability diagram*

Note: some authors choose a different definition for the anisotropy tensor with $a_{ij} = \left(R_{ij} - \dfrac{2}{3}k\delta_{ij} \right)/k$ in place of [4.12]. We shall use here a different typographic font in order to distinguish each of them $a_{ij} = 2\boldsymbol{a}_{ij}$. In this case $II^a = 4II^{\boldsymbol{a}}$, $III^a = 8III^{\boldsymbol{a}}$ and the equation of segment MN then writes $II^a - III^a = 8/9$. We will also recognize the parameter $A = 1 - \dfrac{9}{8}\left(II^a - III^a \right)$ used in many second order models (cf. Chapter 8) and which vanishes near a wall and goes to unity in isotropic turbulence. Also, the equations of the curvilinear sides of the realizability triangle become $III^a = \pm\left(II^a \right)^{3/2}/\sqrt{6}$, and the coordinates of points M and N, $M(II^a = 1/6, III^a = -1/36)$ and $N(II^a = 2/3, III^a = 2/9)$.

The anisotropy invariants and the invariant diagram by Lumley can be applied to any symmetric second order tensor characterizing the turbulent field. Lee M.J. and Reynolds W.C., [LEE 85] apply this concept to the three tensors:

$$a_{ij} = \frac{R_{ij}}{2k} - \frac{1}{3}\delta_{ij}, \quad d_{ij} = \frac{\varepsilon_{ij}}{2\varepsilon} - \frac{1}{3}\delta_{ij}, \quad v_{ij} = \frac{V_{ij}}{2V} - \frac{1}{3}\delta_{ij}$$

with $V_{ij} = \overline{\eta_i\eta_j}$ (double correlation for vorticity $\eta_i = \varepsilon_{ijk}u_{k,j}$).

The invariants $\mathscr{T}_2^a$, $\mathscr{T}_2^d$, $\mathscr{T}_2^v$ and $\mathscr{T}_3^a$, $\mathscr{T}_3^d$, $\mathscr{T}_3^v$ can thus be defined ($\mathscr{T}_1$ is always zero) using equations similar to the ones introduced for the anisotropy

invariants of the Reynolds stress tensor. It is also possible to define invariants for axisymmetry:

$$A = \frac{\mathscr{T}_3 / 2}{\left(-\mathscr{T}_2 / 3\right)^{3/2}} = \pm 1 , \quad |A| \leq 1$$

and for two-dimensionality:

$$G = \mathscr{T}_2 + 3\mathscr{T}_3 + 1/9 \qquad\qquad [4.27]$$

Previous authors have proposed a structural model based on anisotropy invariants (for the large-scales) and the vorticity invariants (for the small scales). According to this description, homogenous and isotropic turbulence is characterized by $\mathscr{T}_2 = \mathscr{T}_3 = 0$, an axisymmetric contraction leads to $A_a = -A_v = -1$ and at the limit an axisymmetric two-dimensional turbulence with $G_a = G_v = 0$, an axisymmetric expansion leads to $A_a = -A_v = +1$ and at the limit an axisymmetric "one-dimensional" turbulence with $G_a = G_v = 0$. A two-dimensional turbulence (or more exactly in this case a two-component turbulence) is characterized by $A_v = 1$ and $G_a = G_v = 0$. A "one-dimensional" turbulence is characterized by $A_a = 1$ and $G_a = G_v = 0$.

4.6.6. Consequences for turbulence transport models

The Reynolds stress transport models are composed of equations of the type:

$$\frac{dR_{ij}}{dt} = \frac{\partial R_{ij}}{\partial t} + \overline{U}_m R_{ij,m} = P_{ij} + \Phi_{ij} + D_{ij} - \varepsilon_{ij} , \qquad\qquad [4.28]$$

the stress tensor R_{ij} being a continuous function and derivable in space and time. We want the model to always give realizable Reynolds stresses (cf. Lumley J.L., [LUM 83]).

If $R_{11} = 0$, it is necessary that $\dfrac{\partial R_{11}}{\partial t} = 0$ and $\dfrac{\partial R_{11}}{\partial x_1} = 0$, i.e. $\dfrac{dR_{11}}{dt} = 0$, because R_{11} is a positive quantity. Moreover, for R_{11} to remain positive we require the second derivative to be positive, $\dfrac{d^2 R_{11}}{dt^2} > 0$.

This is also true, for example, for a quantity such as $F = R_{11}R_{22} - R_{12}^2$ with similar conditions on F. More precisely, it is necessary that $\dfrac{\partial F}{\partial t} = 0$ and also $\dfrac{\partial F}{\partial x_1} = 0$, i.e. $\dfrac{dF}{dt} = 0$ (because F is always positive; cf. equation [4.22]) and $\dfrac{d^2 F}{dt^2} > 0$ when $F \to 0$.

If $F = det(R_{ij}) = 0$, it is also necessary that $\dfrac{\partial F}{\partial t} = 0$, $\dfrac{\partial F}{\partial x_1} = 0$, $\dfrac{dF}{dt} = 0$ ($F \geq 0$; equation [4.22]) and $\dfrac{d^2 F}{dt^2} > 0$.

In practice, it is usual to distinguish a strong formulation and a weak formulation for the realizability conditions ([SPE 94A]). The strong condition given above requires that when a principal component of the Reynolds stress tensor vanishes, its first derivative also vanishes while its second derivative remains positive. Also, for every positive quantity F:

$$F = 0 \quad \Rightarrow \quad \frac{dF}{dt} = 0 \text{ and } \frac{d^2 F}{dt^2} > 0$$

The weak condition only requires that when a principal component of the Reynolds stress tensor vanishes, its first derivative must be positive. Also, for every positive quantity F:

$$F = 0 \quad \Rightarrow \quad \frac{dF}{dt} \geq 0$$

The function F may be a quantity such as a component of the Reynolds stress tensor, the determinant of its representative matrix, an eigenvalue of the Reynolds stress tensor, etc. For example, $R_{11} = 0 \quad \Rightarrow \quad \dfrac{dR_{11}}{dt} \geq 0$.

It can be shown that some of the terms in the right-hand side of the general equation [4.28] satisfy these conditions. For the other terms, it cannot be generally proved and the problem of the realizability of closure hypotheses is posed (some models may appear to be not realizable). Some closure hypotheses (for example, on

the pressure-strain correlations) which assume a weak anisotropy and weak non-homogenity may violate realizability conditions when the anisotropy strongly increases.

We can note that the behavior law $R_{ij} = \frac{2}{3}k\delta_{ij} - 2c_\mu \frac{k^2}{\varepsilon}S_{ij}$ used in the traditional k-ε model is not realizable. Indeed, considering the normal stress $R_{11} = \frac{2}{3}k - 2c_\mu \frac{k^2}{\varepsilon}\overline{U}_{1,1}$, the condition $R_{11} \geq 0 \Leftrightarrow \frac{k\overline{U}_{1,1}}{\varepsilon} \leq 3.7$ is no longer verified if the strain is too high. Also, the Schwarz inequality is no longer satisfied for R_{12} when the shear increases too much.

For the pressure-strain correlations which involve the fourth order tensor,

$$M_{ijks} = \int \frac{\kappa_i \kappa_j}{\kappa^2}\varphi_{ks}(\vec{\kappa})d\vec{\kappa} \text{ (cf. Chapters 6, 7 and 8).}$$

Reynolds W.C., [REY 87], also gives a realizability condition which is useful for model development:

$$R_{11} = 0 \quad \Rightarrow \quad M_{ij1s} = 0, \quad M_{ijk1} = 0 .$$

In the work of Girimaji S. [GIR 04], giving a deeper insight, the realizability conditions are extended in order to ensure that all the statistical moments that contribute in the Reynolds stresses evolution are individually realizable. The aim is to try to reach a better degree of compatibility with the Navier-Stokes equations than using only the usual conditions.

4.6.7. *Methods for ensuring model realizability*

The mathematical conditions introduced previously can be used to develop realizable modeling closure hypotheses. Another point of view consists of devising corrective actions to non-realizable existing models in order to improve their properties and by the way their universality. The simplest method to do this is the clipping method which consists of using the maximal realizable value as a limiting value. For example, when the inequality $R_{12}^2 \leq R_{11}R_{22}$ is not satisfied, we choose to take as a new value for R_{12}, the value corresponding to the equal sign.

A more elegant method consists of considering two classes of terms: one of them includes the terms that satisfy the realizability conditions, the other one includes the

terms that do not guarantee the realizability. These latter terms are then multiplied by a corrective function $F(\xi)$, ξ being a measure of the difference with the extremal realizability condition. F is zero at this limit and near unity otherwise. In this spirit, Schumann U., [SCH 77] proposes, for example, making use of the function A_R:

$$A_R = \frac{\mathscr{I}_3^R}{2} \cdot \left(\frac{\mathscr{I}_2^R}{3} \right)^{-3/2}. \qquad [4.29]$$

Among the methods based on limiting values let us cite the approximation of triple correlations (for velocity or temperature) of André J.C. *et al.*, [AND 77] based on inequality [4.23] known as the clipping approximation:

$$\left| \overline{abc} \right| \leq \min \left\{ \begin{array}{l} \overline{a^2} \left(\overline{b^2}\,\overline{c^2} + \overline{bc}^2 \right)^{1/2} \\[2mm] \overline{b^2} \left(\overline{a^2}\,\overline{c^2} + \overline{ac}^2 \right)^{1/2} \\[2mm] \overline{c^2} \left(\overline{a^2}\,\overline{b^2} + \overline{ab}^2 \right)^{1/2} \end{array} \right\} \qquad [4.30]$$

(the quasi-normal hypothesis $\overline{abcd} = \overline{ab}.\overline{cd} + \overline{ac}.\overline{bd} + \overline{ad}.\overline{bc}$ was introduced in [4.23]).

4.6.8. *Return to isotropy and realizability*

The work of Lumley J.L. and Newmann G.R., [LUM 77] studies the return to isotropy phenomenon referring both to invariant modeling and realizability properties.

These authors write the evolution equation for R_{ij} and ε in the form:

$$\frac{dR_{ij}}{dt} = -\varepsilon \Phi_{ij} - \frac{2}{3} \varepsilon \delta_{ij}$$

$$\frac{d\varepsilon}{dt} = -\Psi \frac{\varepsilon^2}{k}$$

Some restrictions on the mathematical form of the functions $\Phi_{ij}(R_{ij}, \varepsilon, \nu)$ and $\Psi(R_{ij}, \varepsilon, \nu)$ are obtained by considering the limiting behavior at high Reynolds numbers with weak anisotropy and at low Reynolds numbers with any degree of anisotropy.

In particular, this work deduces from the analysis an expression for Φ_{lj}.

Figure 4.6 gives the variations of Φ_{11}/a in the case of axisymmetric turbulence (contraction of duct flow, experiment by Uberoi, [UBE 56]; cf. section 5.7.3).

The parameter "a" defines the anisotropy tensor:

$$\left[a_{ij} \right] = \begin{bmatrix} a & 0 & 0 \\ 0 & -a/2 & 0 \\ 0 & 0 & -a/2 \end{bmatrix}$$

In the general case, the application of the method turns out to be rather complex because it needs to find the eigenvalues and the eigenvectors of $\left[a_{ij} \right]$ in order to refer to principal axes.
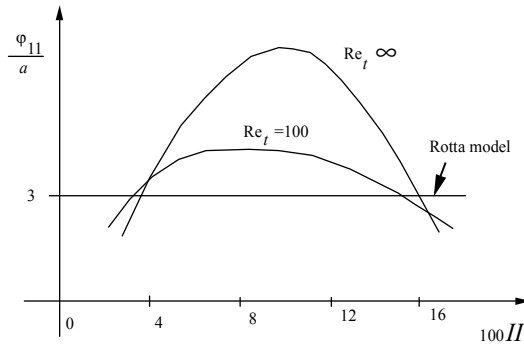
Figure 4.6. *Return to isotropy function by Lumley and Newman*

4.7. Objectivity and material indifference

When we consider a moving frame of reference, the question has to be raised about the indifference property of the equations and tensors with respect to the change in frame of reference (the frame of reference is defined as the particular frame with respect to which the fundamental laws of mechanics are written). The concept of material indifference concerns the physical laws: a behavior law satisfies the principle of material indifference if it is independent of the frame of reference. We can note that the Navier-Stokes equation, as a direct consequence of Newtonian mechanics formulated in a Galilean frame of reference, does not satisfy this

principle. However, the behavior laws connecting, for example, the stresses to the mean velocity field will possibly satisfy this principle. The concept of objectivity concerns physical quantities, generally tensorial quantities. A tensor is said to be objective if it remains identical (unchanged) in a change of frame of reference (its components are simply transformed according to the tensorial rules of purely geometrical change of frame). Thus, for example, the mean velocity vector is not an objective quantity (because a relative velocity appears in a change of frame of reference), whereas the stresses are assumed to be objective. Tensoriality is a purely geometrical concept whereas objectivity is a mechanical concept (because it involves the Galilean frame of reference). We will also notice that the concept of material indifference is more restrictive than the concept of objectivity.

The material derivative of an objective vector is no longer objective. The same occurs for the partial derivatives in time and space. We thus introduce the Oldroyd and Jaumann derivatives which conserve the objectivity:

$$\text{Oldroyd derivative,} \qquad \overset{o}{\frac{D}{Dt}} A_i = \partial_t A_i + U_m A_{i,m} - A_m U_{i,m} . \qquad [4.31]$$

We will note that the Oldroyd derivative can also be written:

$$\overset{o}{\frac{D}{Dt}} A_i = \partial_t A_i + \mathscr{L}_U A_i$$

where $\mathscr{L}_U A_i = U_m A_{i,m} - A_m U_{i,m}$ is the Lie derivative ([*CHO 68]) of the tensor A_i relative to U_i.

$$\text{Jaumann derivative,} \qquad \overset{J}{\frac{D}{Dt}} A_i = \partial_t A_i + U_m A_{i,m} - A_m \omega_{mi} . \qquad [4.32]$$

These derivatives follow the fluid particle not only in position but also in its orientation, contrarily to the material derivative.

Note: in the case of higher order tensors the definitions are:

$$\overset{o}{\frac{D}{Dt}} S_{ij} = \partial_t S_{ij} + U_m S_{ij,m} - S_{mj} U_{i,m} - S_{im} U_{j,m} ,$$

$$\overset{J}{\frac{D}{Dt}} S_{ij} = \partial_t S_{ij} + U_m S_{ij,m} - S_{mj}\omega_{mi} - S_{im}\omega_{mj} \,,$$

$$\overset{o}{\frac{D}{Dt}} S_{ij...m} = \partial_t S_{ij...m} + U_q S_{ij...m,q} - S_{qj...m}U_{i,q} - S_{iq...m}U_{j,q}\cdots - S_{ij...q}U_{m,q} \,,$$

$$\overset{J}{\frac{D}{Dt}} S_{ij...m} = \partial_t S_{ij...m} + U_q S_{ij...m,q} - S_{qj...m}\omega_{iq} - S_{iq...m}\omega_{jq}\cdots - S_{ij...q}\omega_{mq} \,.$$

The equation for the fluctuating velocity in any frame reads:

$$\partial_t u_i + U_j u_{i,j} = -u_j U_{i,j} - \left(u_i u_j - R_{ij}\right)_{,j} - \left(p/\rho\right)_{,i} - 2\varepsilon_{ijk}\Omega_j u_k + \nu u_{i,jj}$$

with $u_{j,j} = 0$, or equivalently:

$$\overset{o}{D_t}\, u_i = -2u_j\left(S_{ij} + \omega_{ij}^{(absol)}\right) - \left(u_i u_j - R_{ij}\right)_{,j} - \left(p/\rho\right)_{,i} + \nu u_{i,jj}$$

where $\omega_{ij}^{(absol)} = \underbrace{\frac{1}{2}\left(U_{i,j} - U_{j,i}\right)}_{\omega_{ij}} + \varepsilon_{ikj}\Omega_k$ represents the intrinsic vorticity and

$$U_{i,j} = S_{ij} + \omega_{ij}, \qquad S_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right),$$

$$\omega_{ij}^{(absol)} = \varepsilon_{ikj}\Omega_k + \omega_{ij}, \qquad \Omega_i^{(absol)} = \frac{1}{2}\varepsilon_{ijk}\omega_{kj}^{(absol)}$$

The evolution of the fluctuating velocity is dependent on the frame of reference only through the intrinsic rotation $\omega_{ij}^{(absol)}$. It then results that any closure model must be of the form:

$$R_{ij}(\vec{x},t) = \mathcal{F}\left[S_{ij}(\vec{x}',\tau), \omega_{ij}^{(absol)}; \vec{x},t\right], \qquad \vec{x}' \in \mathcal{D}, \ \tau \leq t$$

We also note that any dependency according to mean velocity is excluded because of Galilean invariance; only the velocity gradients can be involved.

Although this functional relation connects objective tensors, in general it does not satisfy the material indifference property because of the occurrence of the intrinsic rotation as an argument.

We note on flows submitted to high rotation rates about the Oz axis: vorticity equation [1.33], written in a relative frame of reference and neglecting the viscosity, reads:

$$\frac{\partial \Omega_i}{\partial t} + U_j \frac{\partial \Omega_i}{\partial x_j} = \Omega_j{}^{(absol)} \frac{\partial U_i}{\partial x_j}$$

In a steady flow it reduces to the single term $\Omega_z{}^{(absol)} \frac{\partial U_i}{\partial z} = 0$ and consequently $\frac{\partial U_i}{\partial z} = 0$, constituting the Taylor-Proudman theorem.

4.8. Diffusive correlations

For the diffusive correlations of the general form $\overline{u_j \theta}$ where θ is the turbulent fluctuation of any quantity Θ attached to the particle, the invariant modeling is generally not very efficient. Indeed, because of non-homogenity, the approximations rapidly become very complicated and the number of numerical constants is great: the expansions generally give approximations with several gradients and it is thus difficult to determine the value of each numerical constant by reference to known experiments.

We first present a more phenomenological approach in quasi-homogenous anisotropic turbulence (cf. [SIE 75]).

4.8.1. *Lagrangian approach*

To remain as simple as possible, we consider the case of anisotropic turbulence with $\Theta_{,j}$ constant.

Let $\xi_i(x_j, t/t_0)$ denote the position at the initial instant of time t_0 of the particle which will be located in x_j (Lagrangian coordinates) at a later instant of time t.

The trajectory is described by the equation:

$$x_i = \xi_i + \int_{t_0}^{t} U_i\left[\xi_j\left(x_m, t/t_0\right), t_1\right] dt_1$$
[4.33]

where U_i is the Lagrangian velocity of the particle.

The fluctuation θ is then approximated by:

$$\theta\left(x_j, t\right) = \overline{\Theta}\left[\xi_j\left(x_i, t/t_0\right)\right] - \overline{\Theta}\left[x_j\right] \approx -\left(x_j - \xi_j\right)\overline{\Theta}_{,j}$$

It would be possible to consider higher order expansions such as:

$$\theta\left(x_j, t\right) \approx -\left(x_j - \xi_j\right)\overline{\Theta}_{,j} + \frac{1}{2}\left(x_j - \xi_j\right)\left(x_h - \xi_h\right)\overline{\Theta}_{,jh} + \mathcal{O}\left(x_p - \xi_p\right)^3$$

However, we shall limit ourselves to first order, which implies

$$\theta\left(x_j, t\right) = -\int_{t_0}^{t} \overline{\Theta}_{,j} U_j\left[\xi_j\left(x_m, t/t_1\right), t_1\right] dt_1 \ .$$

and thus:

$$\overline{\theta u_j}\left(x_j, t\right) = -\overline{\Theta}_{,j} \int_{t_0}^{t} \overline{U_j\left[\xi_j\left(x_m, t/t_1\right), t_1\right] U_i\left[x_j, t\right]} dt_1 \ .$$
[4.34]

For sufficiently large time intervals, the Lagrangian time scale τ is defined by the integral of the Lagrangian covariance for $t \to \infty$.

$$\overline{u_i u_j} . \tau = \int_{t_0}^{\infty} \overline{U_j\left[\xi_j, t_1\right] U_i\left[\xi_j, t\right]} dt_1 \ .$$
[4.35]

On dimensional reasoning and considering the similar order of magnitudes it is possible to approximate τ by $c \, k/\varepsilon$ where c is a numerical constant. The usual gradient approximation then follows:

$$\overline{\theta u_i} = -c \frac{k R_{ij}}{\varepsilon} \overline{\Theta}_{,j}$$
[4.36]

This gradient approximation with tensorial coefficient (sometimes called "GGDH", generalized gradient diffusion hypothesis) is the most usual and almost

exclusively used in practical second order closures. If we introduce a hypothesis of isotropic diffusivity (also called "SGDH", simple gradient diffusion hypothesis) the following more simple approximation is obtained, but with a less general field of application:

$$\overline{\theta u_i} = -c_\theta \frac{k^2}{\varepsilon} \Theta_{,i}$$

Corrsin S., [COR 74] examined the necessary conditions for the gradient approximation to be acceptable.

The Lagrangian approach is the basis of the theory of turbulent dispersion in a flow field (cf. Monin and Yaglom, 1971; Corrsin S., [COR 62]).

Other directions of approach have been also explored. Launder B.E. and Spalding D.B., 1972, [*LAU 72] suggest the possibility of using a two flux model in turbulence. Let us also quote other types of approach such as that by Prudnikov A.G. *et al.*, [PRU 72] based on a vortex model. In order to take into account the non-local character of turbulent diffusion by the big eddies, a possible approach would be to express the flux of a transported quantity as a weighted integral applied to the mean field (cf. [CSA 66]). Thus,

$$F_{\theta i} = \overline{u_i \theta} = -\sigma \int f(\vec{y} - \vec{x}) \Theta_{,i}(\vec{y}) d\vec{y} = -\sigma \int f_{,i}(\vec{y} - \vec{x}) \Theta(\vec{y}) d\vec{y},$$

where f is a bell-shaped curve with the normalizing condition $\int f(\vec{y} - \vec{x}) d\vec{y} = 1$, analogous to a filter. The particular case of gradient law local diffusion can be recovered for $f(\vec{y} - \vec{x}) = \delta(\vec{y} - \vec{x})$, Dirac distribution at point $\vec{y} = \vec{x}$.

However, the more commonly used alternative consists of introducing diffusion laws with several gradients, which will be considered in the following section.

4.8.2. *Multiple gradient closures*

Lumley J.L., [LUM 75C] has proposed expressions for the turbulent fluxes involving several gradients that can be obtained using invariant modeling methods. This type of approximation will be used in particular for approaching the triple velocity correlations. For example, Donaldson C. du P., [DON 71] introduces the approximation:

$$\overline{u_i u_j u_k} = -L\sqrt{k} \left(R_{ij,k} + R_{jk,i} + R_{ik,j} \right), \qquad\qquad [4.37]$$

where $L\sqrt{k}$ represents the turbulent diffusivity and where several gradients are involved.

Multiple gradient modeling is also encountered in approximating the turbulent diffusion of a scalar, such as for example the hypothesis of Deardorff, [DEA 73] for the triple correlation velocity-scalar:

$$\overline{u_i u_j \theta} = -\frac{L}{\sqrt{k}}\left[R_{jm}\left(\overline{u_i \theta}\right)_{,m} + R_{im}\left(\overline{u_j \theta}\right)_{,m} + \left(\overline{u_m \theta}\right) R_{ij,m} \right].$$

In addition, Lumley J.L., [LUM 75A] is led to introduce expressions for the turbulent fluxes of a scalar C in non-homogenous turbulence, involving not only the usual gradient term but also a convective type term which is proportional to the gradient of the transport coefficient. The reasoning is based on an integral expression of the flux obtained from the probability density of the particles displacement for which a Taylor expansion is performed. The turbulent flux is then expressed as:

$$\phi_i = -\frac{1}{2}\partial_t \overline{\zeta_i \zeta_j}.C_{,j} - \frac{1}{4}\overline{C}.\left(\partial_t \overline{\zeta_i \zeta_j}\right)_{,j}.$$

In this expression, $\zeta_i(\vec{x},t)$ denotes the displacement at time t of a fluid particle which was initially located at point $\vec{x}$. Townsend, 1956 had already proposed to represent the turbulent diffusion by the sum of a gradient diffusion term and a convective term corresponding to the effect of transport by big eddies. It is remarkable that this approach by Lumley led to an expression of the convection velocity suggested by Townsend, 1956 (cf. [*HIN 75]) but for which no explicit formulation was available.

4.9. Probability densities and stochastic models

The probability density method ([POP 85, HAW 86, POP 94A, POP 94B) initially developed for applications to reactive flows, brings interesting knock-on effects on stochastic models and realizability. The one point distribution function is defined as the probability for each velocity component U_i to be less than a given value V_i:

$$F(\vec{V},\vec{x},t) = \mathscr{P}\{U_i < V_i\} \qquad\qquad [4.38]$$

The Probability Density Function (PDF) is given by its derivative:

$$f(\vec{V}, \vec{x}, t) = \partial^3 F / \partial V_1 \partial V_2 \partial V_3 \, ,$$
[4.39]

which satisfies the normalization condition $\int f . d\vec{V} = 1$.

The realizability of f can be very simply expressed by $f \geq 0 \quad \forall \vec{V}$.

It is thus possible to obtain the moments in the following way:

$$\overline{U}_j = \int V_j f(\vec{V}, \vec{x}, t) d\vec{V} \, ,$$
[4.40]

$$R_{ij} = \overline{u_i u_j} = \int (V_i - \overline{U}_i)(V_j - \overline{U}_j) f d\vec{V} \, .$$
[4.41]

By noting that it is formally possible to write equivalently:

$$f(\vec{V}, \vec{x}, t) = \overline{\delta\left(\vec{U}(\vec{x}, t) - \vec{V} \right)},$$
[4.42]

using the properties of the Dirac distribution, then

$$f(\vec{V}, \vec{x}, t) = \int \delta\left(\vec{V}' - \vec{V} \right) f(\vec{V}') d\vec{V}' \, ,$$

and we can deduce from the Navier-Stokes equations, an equation for the probability density:

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} \frac{\partial f}{\partial V_i} - \frac{\partial}{\partial V_i} \left[f \overline{\left(\nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \middle| \vec{V} \right)} \right]$$
[4.43]

where $\overline{\left(\phi | \vec{V} \right)}$ stands for the conditional mean of ϕ when $\vec{U} = \vec{V}$. More precisely, $\overline{\left(\phi | \vec{V} \right)} = \int \psi f_{\phi|U} (\psi | \vec{V}, \vec{x}, t) d\psi$. The conditional probability density function $f_{\phi|U}$ is given by the ratio of the joint probability density $f_{\phi,U}$ to the probability density of $\vec{U}$: $f_{\phi|U} = \dfrac{f_{\phi,U}}{f}$.

This equation is open and a hypothesis is necessary for the second term in the right-hand side for closure. The more usual closure is the generalized Langevin model:

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} \frac{\partial f}{\partial V_i} - \frac{\partial}{\partial V_j} \left[f.G_{ij}(V_i - \overline{U}_i) \right] + \frac{C_L}{2} \varepsilon \frac{\partial^2 f}{\partial V_j \partial V_j} , \qquad [4.44]$$

where $C_L(\vec{x},t)$ and $G_{ij}(\vec{x},t)$ are coefficients which determine the particular model, $C_L(\vec{x},t)$ being positive.

It is possible to easily deduce from the previous equation, a transport equation for the Reynolds stresses (defined as the second moments [4.41]):

$$\frac{dR_{ij}}{dt} = P_{ij} - (\overline{u_i u_j u_m})_{,m} + G_{im} R_{jm} + G_{jm} R_{im} + C_L \varepsilon \delta_{ij} , \qquad [4.45]$$

and by tensorial contraction, an equation for turbulence kinetic energy:

$$\frac{dk}{dt} = P - \frac{1}{2} (\overline{u_j u_j u_m})_{,m} + G_{jm} R_{jm} + \frac{3}{2} C_L \varepsilon , \qquad [4.46]$$

along with the condition, which is implied $G_{jm} R_{jm} + \left(\frac{3}{2} C_L + 1 \right) \varepsilon = 0$.

The particular choice of $C_L(\vec{x},t)$ and $G_{ij}(\vec{x},t)$ make it possible to recover the usual Reynolds stress transport models.

The idea of stochastic models consists of introducing a random force directly into the Navier-Stokes equation and after deriving the PDF model, instead of making the closure directly on the PDF.

The generalized Langevin model is represented by the stochastic differential process:

$$\begin{cases} dU^{\#}_i(t) = -\frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} dt + G_{ij} \left(U^{\#}_j(t) - U_j \right) dt + \left(C_L \varepsilon \right)^{1/2} dW_i(t) \\ dX^{\#}_i(t) = U^{\#}_i(t) dt \end{cases} \qquad [4.47]$$

where $dU^{\#}{}_i(t)$ denotes the velocity increment from t to $t+\Delta t$:

$$dU^{\#}{}_i(t) = U^{\#}{}_i(t + \Delta t) - U^{\#}{}_i(t) \, ,$$

and $dW_i(t)$ is an isotropic Wiener process (centered Gaussian variable, with standard deviation equal to unity and uncorrelated with earlier times), which satisfies:

$$\overline{dW_i} = 0 \, , \quad \overline{dW_i dW_j} = dt.\delta_{ij} \, , \quad \overline{U^{\#}{}_i dW_j} = 0 \, .$$

In order to reach the PDF, it is necessary to use Lagrangian description. The joint PDF of $X^{\#}{}_i$ and $U^{\#}{}_i$ conditioned by $X^{\#}{}_i(t_0) = Y_i$ is denoted $f^{\#}{}_L(\vec{V}, \vec{x}, t \,|\, \vec{Y})$. Its evolution equation will be:

$$\frac{\partial f^{\#}{}_L}{\partial t} + V_i \frac{\partial f^{\#}{}_L}{\partial x_i} = \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} \frac{\partial f^{\#}{}_L}{\partial V_i} - G_{ij} \frac{\partial}{\partial V_i} \Big[f^{\#}{}_L . (V_j - \overline{U}_j) \Big] + \frac{1}{2} C_L \varepsilon \frac{\partial^2 f^{\#}{}_L}{\partial V_j \partial V_j}$$

We switch to Eulerian description by using integration:

$$f^{\#}(\vec{V}, \vec{x}, t) = \int f^{\#}{}_L(\vec{V}, \vec{x}, t \,|\, \vec{Y}) d\vec{Y} \qquad\qquad [4.48]$$

in order to obtain the PDF equation:

$$\frac{\partial f^{\#}}{\partial t} + V_i \frac{\partial f^{\#}}{\partial x_i} = \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} \frac{\partial f^{\#}}{\partial V_i} - G_{ij} \frac{\partial}{\partial V_i} \Big[f^{\#} . (V_j - \overline{U}_j) \Big] + \frac{1}{2} C_L \varepsilon \frac{\partial^2 f^{\#}}{\partial V_j \partial V_j} \quad [4.49]$$

The stochastic Lagrangian models are realizable if the coefficients are real and bounded. Consequently, the models that can be deduced from them for $f^{\#}$ and for R_{ij} will also be realizable themselves. The Langevin equation (cf. Haworth D.C. and Pope S.B., [HAW 86]; Pope S.B., [POP 94B]) which includes a random force and a deterministic force, has the same statistical properties as the closure model which corresponds to it. This approach thus offers a very efficient method of establishing the realizability of Reynolds stress transport closures. This approach to realizability has been developed by Durbin P.A. and Speziale C.G., [DUR 94A], who show that the realizability of the second order closure models can be established simply from the stochastic Langevin equation for which they are exact.

4.10. Intermittency

Boundary intermittency (this type of intermittency has been studied in detail by Corrsin and Kistler, [COR 54]) is generally not accounted for explicitly in traditional models. The idea of transposing conditional sampling used by experimentalists for the statistical treatment of the moment equations for modeling purpose was however developed by Libby P.A., [LIB 75]. The system of equations he proposes allows us to calculate the conditioned quantities, the non-conditioned quantities being assumed to be known. Lundgren T.S. and Wang F.C., [LUN 73] have also introduced a new modeling approach of turbulence viscosity which allows the generation of discontinuity fronts at the turbulent/non-turbulent interface in free flows. In this direction, we can also quote the work of Duhamel P., [DUH 81], the work of Byggstoyl S. *et al.*, [BYG 81]. The k-ε-γ model by Cho J.R. and Chung M.K., [CHO 92], is based on the solution of an equation for the intermittency factor γ in addition to the equations for k and ε, this description may be useful for improving the numerical prediction of free flows in the external boundary region.

If a quantity q is averaged only during the turbulent periods ($I=1$), we obtain the conditioned mean in the turbulent region $q^T = \overline{Iq}/\overline{I}$ where $\gamma = \overline{I}$. Similarly, we define the conditioned mean in the non-turbulent region $q^L = \overline{(1-I)q}/\overline{(1-I)}$. The conventional mean value is then determined by $\overline{q} = \gamma q^T + (1-\gamma)q^L$. For a statistical correlation, such as the Reynolds stress tensor, we get:

$$\overline{u_i u_j} = \gamma \left(u_i u_j\right)^T + (1-\gamma)\left(u_i u_j\right)^L + \gamma(1-\gamma)\left(U_i^T - U_i^L\right)\left(U_j^T - U_j^L\right).$$

The last term shows that the differences in the mean velocities between the irrotational zones (L) and turbulent zones (T) are contributing to the transport of momentum by convection. According to a hypothesis by Lumley [LUM 80], the velocity jump can be expressed by:

$$U_i^T - U_i^L = -F_2(\gamma)\frac{\overline{k}^2}{\overline{\varepsilon}}\gamma_{,i} - F_3(\gamma)\frac{\overline{k}}{\overline{\varepsilon}}\left(U_j^T - U_j^L\right)U_{i,j} \qquad [4.50]$$

The closure hypotheses used in the model [CHO 92] read:

$$\overline{u_i u_j} = 2c_\mu \frac{\overline{k}^2}{\overline{\varepsilon}}\left[1 + \gamma(1-\gamma)\frac{\overline{k}^3}{\overline{\varepsilon}^2}\frac{F_1^2 F_2}{2c_\mu}\frac{\partial\gamma}{\partial x_m}\frac{\partial\gamma}{\partial x_m}\right]S_{ij} \qquad [4.51]$$

the factors F_1 and F_2, like F_3 are empirical functions of γ and the following approximation can be used: $\dfrac{F_1^2 F_2}{2c_\mu} = c_{\mu\gamma}\gamma^{-4}$.

The generic transport equation of γ has been established by Dopazo [DOP 77] by writing the continuity of the turbulent fluid.

$$\frac{d\gamma}{dt} = D + S \qquad \begin{cases} D = -\left[\gamma(1-\gamma)\left(U_j^T - U_j^L\right)\right]_{,j} \\[3mm] S = \lim_{vol \to 0} \dfrac{1}{vol}\int_S V_e dS \end{cases}$$

The term D represents the spatial transport due to the velocity jump and the term S is the rate of conversion of the irrotational fluid into turbulent fluid, V_e being the interface velocity. The proposed model is the following:

$$D = \left[(1-\gamma)\frac{v_t}{\sigma_\gamma}\gamma_{,j}\right]_{,j}$$

$$S = c_{\gamma 1}\gamma(1-\gamma)\frac{P_N + P_S}{\overline{k}} + c_{\gamma 2}\frac{\overline{k}^2}{\overline{\varepsilon}}\gamma_{,j}\gamma_{,j} - c_{\gamma 3}\gamma(1-\gamma)\frac{\overline{\varepsilon}}{k}\Gamma$$

The term P_N represents the production by normal stresses and P_S the production by shear stresses. The last term which involves $\Gamma = \dfrac{\overline{k}^{-5/2}}{\overline{\varepsilon}^{-2}}\dfrac{\overline{U}_i}{\sqrt{\overline{U_m}\overline{U}_m}}\overline{U}_{i,j}\gamma_{,j}$ represents the entrainment effect.

Thus, the modeled transport equation of γ takes the same form as the other transport equations of turbulence:

$$\frac{d\gamma}{dt} = c_{\gamma 1}\gamma(1-\gamma)\frac{P_N + P_S}{\overline{k}} + \left[(1-\gamma)\frac{v_t}{\sigma_\gamma}\gamma_{,j}\right]_{,j} + c_{\gamma 2}\frac{\overline{k}^2}{\overline{\varepsilon}}\gamma_{,j}\gamma_{,j}$$

$$- c_{\gamma 3}\gamma(1-\gamma)\frac{\overline{\varepsilon}}{k}\Gamma \qquad\qquad\qquad\qquad [4.52]$$

The dissipation rate equation used in the k-ε-γ model is written:

$$\frac{d\varepsilon}{dt} = \left(\frac{\nu_t}{\sigma_\varepsilon}\varepsilon_{,j}\right)_{,j} + \frac{\overline{\varepsilon}^2}{\overline{k}}C_{\varepsilon 1}\frac{P_N + 3P_S}{\overline{\varepsilon}} - C_{\varepsilon 2} + C_{\varepsilon 3}\chi + C_{\varepsilon 4}\Gamma$$

which also includes a modification by Pope (cf. section 11.2.3).

The model by Byggstoyl S. *et al.*, [BYG 81] is more complex, it makes use not only of the transport equation of intermittency rate, but also equations for conditioned mean velocities in the turbulent zone and in the laminar zone, and the equations for k and ε in the turbulent zone.

4.11. Practicing with the development tools

The present chapter has thrown some light on the main tools that can be used for developing new turbulence models. To sum up, the bases on which we can rely, hinge on the following points: dimensional analysis, physical intuition of the acting mechanisms, which leads to phenomenological relations, tensorial properties, invariance, objectivity (see Chapter 16), which leads to a more formalist approach, realizability, compatibility of equations, which lead to restrictive conditions on the moments, taking advantage of contributions of higher order theories or models, in particular two point closures. In all cases, however, it will be necessary to test a new model on refined experimental data, before more extensive applications.

Chapter 5

Homogenous Anisotropic Turbulence

The case of homogenous isotropic turbulence (HIT) considered previously in Chapter 1 is the simplest case but it does not allow us to study turbulence production by shear. In the case of homogenous anisotropic turbulence (HAT), in contrast, it is possible to account for the existence of a mean velocity gradient $\Lambda_{ij} = \overline{U}_{i,j}$. This type of turbulence is interesting from a fundamental point of view because the governing statistical equations are tractable theoretically and it allows us to consider the process of turbulence production originating from interactions with the mean flow.

The first studies by Courseau R. and Loiseau M., [COU 78] considered straightaway the problem of anisotropy by giving a subsidiary role to the non-linear mechanisms which were essential in isotropic turbulence. When the triple correlations are neglected, the solution obtained in the case of sudden strain is analogous to the one given by Batchelor G.K. and Proudman I., [BAT 54] using the rapid distortion theory. More complete modeling developments taking into account the non-linear effects were proposed afterwards by Jeandel D. *et al.*, [JEA 78] and then by Cambon C., [CAM 79].

5.1. The Craya equation

The Craya equation ([CRA 58]) is the dynamic equation for HAT. We can derive the evolution equation of $\varphi_{ij} = \widehat{\overline{u_i u'_j}}$ either from the equation of $\hat{u}_i(\vec{\kappa})$ (cf. section 3.7) or directly by using Fourier transform of the transport equations for the two

point double velocity correlations $\overline{u_i u'_j} = \overline{u_i(\vec{x}) u_j(\vec{x}')}$. It is this latter method that will be used here. In the present chapter we shall denote u_i the component of the fluctuating velocity at point $A(\vec{x})$ and u'_i the component at point $B(\vec{x}')$.

The transport equations for the two point double velocity correlations are obtained from the equation for fluctuating velocity, by using $\dfrac{\partial}{\partial t}\overline{u_i u'_j} = \overline{u_i \dfrac{\partial u'_j}{\partial t}} + \overline{u'_j \dfrac{\partial u_i}{\partial t}}$ where we have denoted $u'_i = u_i(\vec{x}')$.

$$\frac{\partial}{\partial t}\overline{u_i u'_j} + \overline{u_k u'_j}\left(\frac{\partial \overline{U}_i}{\partial x_k}\right) + \overline{u_i u'_k}\left(\frac{\partial \overline{U}_j}{\partial x_k}\right)' + \overline{U}_k \frac{\partial \overline{u_i u'_j}}{\partial x_k} + \overline{U}'_k \frac{\partial \overline{u_i u'_j}}{\partial x'_k}$$

$$= -\frac{\partial}{\partial x_k}\left(\overline{u'_j u_i u_k}\right) - \frac{\partial}{\partial x'_k}\left(\overline{u_i u'_j u'_k}\right) - \frac{1}{\rho}\left[\frac{\partial}{\partial x_i}\left(\overline{p u'_j}\right) + \frac{\partial}{\partial x'_j}\left(\overline{p' u_i}\right)\right]$$

$$+\nu\left[\frac{\partial^2}{\partial x_m \partial x_m} + \frac{\partial^2}{\partial x'_m \partial x'_m}\right]\left(\overline{u_i u'_j}\right), \tag{5.1}$$

and after a change in variables defined by $r_i = x'_i - x_i$, $X_i = \dfrac{1}{2}\left(x_i + x'_i\right)$ and $\dfrac{\partial}{\partial x} = -\dfrac{\partial}{\partial r} + \dfrac{1}{2}\dfrac{\partial}{\partial X}$, $\dfrac{\partial}{\partial x'} = \dfrac{\partial}{\partial r} + \dfrac{1}{2}\dfrac{\partial}{\partial X}$ we find:

$$\frac{\partial}{\partial t}\overline{u_i u'_j} + r_m \Lambda_{lm}\frac{\partial}{\partial r_l}\overline{u_i u'_j} + \Lambda_{il}\overline{u_l u'_j} + \Lambda_{jl}\overline{u_i u'_l} - \frac{\partial}{\partial r_l}\left[\overline{u_i u_l u'_j} - \overline{u_i u'_l u'_j}\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial r_i}\left(\overline{p u'_j}\right) - \frac{\partial}{\partial r_j}\left(\overline{p' u_i}\right)\right] + 2\nu \frac{\partial^2 \overline{u_i u'_j}}{\partial r_m \partial r_m}. \tag{5.2}$$

In the second term, we have used the simple expansion $\overline{U}_k - \overline{U}'_k = -\Lambda_{km} r_m$.

Thus, using Fourier transform:

$$\frac{\partial \varphi_{ij}}{\partial t} - \kappa_l \Lambda_{lm}\frac{\partial \varphi_{ij}}{\partial \kappa_m} + \Lambda_{il}\varphi_{lj} + \Lambda_{jl}\varphi_{li}$$

$$-\left(\kappa_l \theta_{ilj} + \kappa_l \theta^*_{jli}\right) - \left(\kappa_i \Sigma_j + \kappa_j \Sigma^*_i\right) + 2\nu\kappa^2 \varphi_{ij} = 0, \tag{5.3}$$

with the notations:

$$\widehat{u_i u'_j} = \varphi_{ij} , \qquad \widehat{u_i u_l u'_j} = -i\theta_{ilj} , \qquad \widehat{pu'_j} = -i\Sigma_j$$

and where (*) denotes the conjugate complex number.

The contracted multiplication of [5.3] by κ_j thus gives:

$$\Sigma_j = 2\Lambda_{lm} \frac{\kappa_l}{\kappa^2} \varphi_{mj} - \frac{\kappa_l \kappa_m}{\kappa^2} \theta_{mlj} . \tag{5.4}$$

This relation allows us to eliminate pressure from [5.3].

Equation [5.3] taking into account [5.4] can then be written:

$$\frac{\partial \varphi_{ij}}{\partial t} + \Lambda_{il}\varphi_{lj} + \Lambda_{jl}\varphi_{il} - S_{ij}^{(1)} - S_{ij}^{(2)} - P_{ij}^{(1)} - P_{ij}^{(2)} + 2\nu\kappa^2\varphi_{ij} = 0 \tag{5.5}$$

with the definitions:

$$S_{ij}^{(1)} = \Lambda_{lm}\kappa_l \frac{\partial \varphi_{ij}}{\partial \kappa_m} = \Lambda_{lm} \frac{\partial}{\partial \kappa_m}\left(\kappa_l \varphi_{ij}\right),$$

fast transfer term (appearing suddenly due to the action of mean flow gradients);

$$S_{ij}^{(2)} = \kappa_l \theta_{ilj} + \kappa_l \theta_{jli}^* ,$$

non-linear transfer (slow term due to inertial cascade appearing with some delay due to the action of turbulence on itself);

$$P_{ij}^{(1)} = 2\Lambda_{lm} \frac{\kappa_l}{\kappa^2}\left(\kappa_i \varphi_{mj} + \kappa_i \varphi_{mi}^*\right),$$

fast pressure term (linear with respect to mean velocity gradients and appearing suddenly);

$$P_{ij}^{(2)} = -\frac{\kappa_l \kappa_m}{\kappa^2}\left(\kappa_i \theta_{mlj} + \kappa_i \theta_{mli}^*\right),$$

non-linear part of pressure term (slow term appearing with some delay).

The incompressibility condition implies: $\kappa_i \varphi_{il} = 0$, $\kappa_i \theta_{mli} = 0$, $\kappa_i \Sigma_i = 0$, and we get the properties $P_{jj}^{(1)} = 0$, $P_{jj}^{(2)} = 0$.

Note: if ψ is real, then $\widehat{\Psi}(-\vec{\kappa}) = \widehat{\Psi}^*(\vec{\kappa})$ and consequently:

$$\overline{u_i u'_j}(\vec{r}) = \overline{u'_i u_j}(-\vec{r}) \quad \Rightarrow \varphi_{ij}(\vec{\kappa}) = \varphi_{ji}(-\vec{\kappa}) = \varphi_{ji}^*(\vec{\kappa})$$

Similarly:

$$i\theta_{ijk}(-\vec{\kappa}) = \left(i\theta_{ijk}\right)^*(\vec{\kappa}) \text{ or } \theta_{ijk}(-\vec{\kappa}) = -\left(\theta_{ijk}\right)^*(\vec{\kappa})$$

Craya introduced the notations:

$$\Omega_{ij} = P_{ij}^{(2)} + S_{ij}^{(2)} = \Delta_{mi}\kappa_l\theta_{mlj} + \Delta_{mj}\kappa_l\theta_{mli}^*, \text{ with } \Delta_{ij} = \delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2},$$

$$\Psi_{ij} = \Lambda_{il}\varphi_{lj} + \Lambda_{jl}\varphi_{il}^* - P_{ij}^{(1)} - S_{ij}^{(1)}$$

$$= -\Lambda_{lm}\left(\delta_{li}\varphi_{mj} + \delta_{lj}\varphi_{mi}^*\right) + 2\Lambda_{lm}\left(\Delta_{li}\varphi_{mj} + \Delta_{li}\varphi_{mi}^*\right) - \Lambda_{lm}\frac{\partial}{\partial\kappa_m}\left(\kappa_l\varphi_{ij}\right),$$

from which follows the Craya equation:

$$\frac{\partial \varphi_{ij}}{\partial t} + \Psi_{ij} = \Omega_{ij} - 2\nu\kappa^2\varphi_{ij}. \tag{5.6}$$

Note: in the particular case of isotropic turbulence the following relations hold:

$$\varphi_{ij} = \frac{E(\kappa,t)}{4\pi\kappa^2}\Delta_{ij}, \quad E(\kappa,t) = 2\pi\kappa^2\varphi_{jj}, \quad P_{jj}^{(1)} = 0, \quad P_{jj}^{(2)} = 0,$$

$$S_{ij}^{(2)} = \frac{T(\kappa,t)}{4\pi\kappa^2}\Delta_{ij}, \quad S_{jj}^{(2)} = \frac{T(\kappa,t)}{2\pi\kappa^2}, \quad \text{because } \Delta_{jj} = 2,$$

and we recover the Lin equation $\dfrac{\partial E}{\partial t} - T + 2\nu\kappa^2 E = 0$.

Courseau P. and Loiseau M., [COU 78] have obtained linear solutions of the simplified Craya equation, Ω_{ij} being neglected. For spectral space modeling in the general case we can refer to Jeandel D. *et al.*, [JEA 78]; Cambon, C., [CAM 79]; Cambon C. *et al.*, [CAM 78 and 81].

5.2. One-dimensional spectral properties in homogenous turbulent shear flows

Consider a turbulent flow submitted to constant shear Λ_{12}. The transport equation for the turbulence kinetic energy spectrum reads in this case:

$$\frac{\partial \varphi_{jj}}{\partial t} + \Lambda_{12}\varphi_{21} - S_{jj}^{(1)} - S_{jj}^{(2)} + 2\nu\kappa^2 \varphi_{jj} = 0. \qquad [5.7]$$

We shall only consider one-dimensional properties and with this objective in mind introduce the definition of one-dimensional spectrum:

$$E(\kappa,t) = \mathscr{M}\left[\varphi_{jj}(\vec{\kappa},t)\right] = \iint_{S(\kappa)} \varphi_{jj}(\vec{\kappa},t)dA(\vec{\kappa}), \qquad [5.8]$$

(corresponding to averaging on the sphere of radius $\left\|\vec{\kappa}\right\|$).

Equation [5.7], after application of operator $\mathscr{M}$ thus implies:

$$\underbrace{\frac{\partial E(\kappa,t)}{\partial t}}_{(a)} = -\underbrace{\Lambda_{12}E_{21}}_{(b)} + \underbrace{\Lambda_{12}\mathscr{M}\left[\frac{\partial}{\partial \kappa_2}(\kappa_l \varphi_{jj})\right]}_{(c)} + \underbrace{T}_{(d)} - \underbrace{2\nu\kappa^2 E(\kappa,t)}_{(e)}, \qquad [5.9]$$

with $E_{21} = \mathscr{M}(\varphi_{21})$ and $T = \mathscr{M}(S_{jj}^{(2)})$.

The schematic balance of spectral energy $E(\kappa, t)$ in steady homogenous turbulent shear flow is given in Figure 5.1.
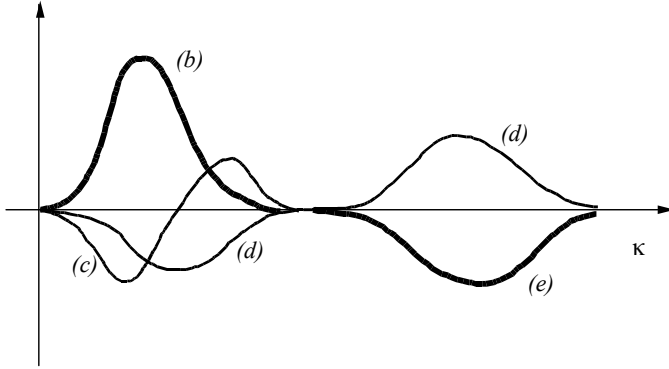
Figure 5.1. *Sketch of spectral balance of turbulent kinetic energy in homogenous turbulent shear flow (after Hinze J.O., [*HIN 75]). Terms in equation [5.9]: (b) production, (c) fast transfer, (d) non-linear slow transfer, (e) viscous dissipation*

Spherical averaging implies the loss of all directional information. This lost information may however be reintroduced by approximate modeling by using tensorial representation methods in Cambon C. *et al.*, [CAM 78]; Bertoglio J-P. *et al.*, [BER 79] in order to achieve closure of the equations.

5.3. Rapid part of pressure correlations in the rapid distortion of isotropic turbulence

The rapid part of the pressure-strain correlations is given by the definition:

$$\Phi_{ij} = \int_{\mathbb{R}^3} P_{ij}^{(1)} d\mathscr{V} = 2\Lambda_{lm} \int_{\mathbb{R}^3} \frac{\kappa_l}{\kappa^2} \left(\kappa_i \varphi_{mj} + \kappa_j \varphi_{mi}^* \right) d\vec{\kappa} \qquad [5.10]$$

This expression takes a remarkable simple form in the case of isotropic turbulence submitted to sudden distortion.

In isotropic turbulence, the spectral tensor of double velocity correlations is:

$$\varphi_{ij} = \frac{E}{4\pi\kappa^2} \left(\delta_{ij} - \frac{\kappa_i \kappa_j}{\kappa^2} \right), \qquad \varphi_{mi}^* = \varphi_{im} = \varphi_{mi},$$

and this definition inserted in [5.10] gives:

$$\Phi_{ij} = 2\Lambda_{lm} \int_{\mathbb{R}^3} \frac{E}{4\pi\kappa^2} \left(\frac{\kappa_l \kappa_i \delta_{mj}}{\kappa^2} - \frac{\kappa_l \kappa_i \kappa_m \kappa_j}{\kappa^4} + \frac{\kappa_l \kappa_j \delta_{mi}}{\kappa^2} - \frac{\kappa_l \kappa_j \kappa_m \kappa_i}{\kappa^4} \right) d\vec{\kappa},$$

$$\Phi_{ij} = 2\Lambda_{lm} \int_0^\infty \frac{Ed\kappa}{4\pi\kappa^2} \iint_{S(\kappa)} \left(\frac{\kappa_l \kappa_i \delta_{mj}}{\kappa^2} - 2\frac{\kappa_l \kappa_i \kappa_m \kappa_j}{\kappa^4} + \frac{\kappa_l \kappa_j \delta_{mi}}{\kappa^2} \right) dA(\kappa) ,$$

where $dA(\kappa)$ denotes the infinitesimal surface element on the sphere $S(\kappa)$ with radius κ. Thus,

$$\Phi_{ij} = 2\Lambda_{lm} \left[\frac{\delta_{mj}\delta_{li}}{3} + \frac{\delta_{mi}\delta_{lj}}{3} - \frac{2}{15}\left(\delta_{li}\delta_{mj} + \delta_{lm}\delta_{ij} + \delta_{lj}\delta_{im} \right) \right] . \int_0^\infty Ed\kappa ,$$

(see note hereafter*)

$$\Phi_{ij} = 2k\Lambda_{lm} \left[\frac{3}{15}\delta_{mj}\delta_{li} + \frac{3}{15}\delta_{mi}\delta_{lj} - \frac{2}{15}\delta_{lm}\delta_{ij} \right] ,$$

$$\Phi_{ij} = 0.4k\left(\Lambda_{ij} + \Lambda_{ji} \right) . \qquad [5.11]$$

(*) Note about spherical integration of tensorial functions.

Consider the analytical derivation of $\mathscr{T}_{i_1 \ldots i_{2n}} = \dfrac{1}{4\pi\kappa^2} \iint_{S(\kappa)} \dfrac{\kappa_{i_1} \kappa_{i_2} \ldots \kappa_{i_{2n}}}{\kappa^{2n}} dA(\kappa)$ for any integer n. It can be shown that $\mathscr{T}_{i_1 \ldots i_{2n}}$ is an isotropic function and thus it can be expressed as a linear combination of products of Kronecker unit tensors (cf. section 3.6).

$$\mathscr{T}_{i_1 \ldots i_{2n}} = A^{(2n)} \sum \delta_{i_h i_k} . \delta_{i_m i_s} \ldots$$

Consider the particular cases $n = 1$ and $n = 2$.

$$\mathscr{T}_{ij} = \frac{1}{4\pi\kappa^2} \iint_{S(\kappa)} \frac{\kappa_i \kappa_j}{\kappa^2} dA(\kappa) = A^{(2)}\delta_{ij}$$

$$\mathscr{T}_{jj} = \frac{1}{4\pi\kappa^2} \iint_{S(\kappa)} dA(\kappa) = 1 = 3A^{(2)} \quad \text{and thus} \quad A^{(2)} = 1/3$$

$$\mathscr{T}_{ijkl} = \frac{1}{4\pi\kappa^2} \iint_{S(\kappa)} \frac{\kappa_i \kappa_j \kappa_k \kappa_l}{\kappa^4} dA(\kappa) = A^{(4)}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right)$$

$$\mathscr{T}_{iimm} = 1 = 15A^{(4)} \quad \text{and thus} \quad A^{(4)} = 1/15$$

5.4. Spectral models

Spectral modeling introduced by Mathieu J. and Jeandel D., [MAT 75] is based on the evolution equation of the mean spectral tensor $E_{ij} = \mathcal{M}\left(\varphi_{ij}\right)$:

$$\left(\frac{\partial}{\partial t} + 2\nu\kappa^2\right)E_{ij} + \mathcal{M}\left(\Psi_{ij}\right) = \mathcal{M}\left(\Omega_{ij}\right)$$

These authors follow an approach similar to the one developed by Lumley in physical space. The unknown terms in this equation are supposed to be functions of E, h_{ij} and κ, with $E = \frac{1}{2}E_{jj}$ and $h_{ij} = \dfrac{E_{ij} - (2/3)E\delta_{ij}}{E}$ (deviatoric spectral tensor), and then approximated using the representation theorems of tensorial functions.

This research work on spectral closures has been developed in particular by Jeandel D. *et al.*, [JEA 78]; Jeandel D. and Mathieu J., [JEA 77]; Cambon C., [CAM 82] and also by Yassour Y. and Wolfshtein M., [YAS 83]. The extension of these methods to non-homogenous turbulence, as is encountered in industrial practice, remains a difficult matter and requires additional hypotheses for approximate description, Jeandel D., [JEA 76]; Laporta A., [LAP 95]. A synthetic review of these methods is given in Mathieu J., [MAT 92].

5.5. Turbulence associated with a passive scalar

It is possible to make a spectral analysis for the turbulence associated with a passive scalar similar to the one developed for the dynamics. With the same notations as in section 5.1 and using equation [2.14] in Chapter 2 we find the transport equations for $\overline{\gamma\gamma'}$ and its Fourier transform:

$$\frac{\partial\overline{\gamma\gamma'}}{\partial t} + \overline{\gamma'u_k}\frac{\partial\overline{\Gamma}}{\partial x_k} + \overline{\gamma u'_k}\frac{\partial\overline{\Gamma}}{\partial x_k} + \overline{U'_k\gamma\frac{\partial\gamma'}{\partial x'_k}} + \overline{U_k\gamma\frac{\partial\gamma'}{\partial x'_k}}$$

$$+\overline{\gamma u'_k\frac{\partial\gamma'}{\partial x'_k}} + \overline{\gamma'u_k\frac{\partial\gamma}{\partial x_k}} - \overline{\sigma\gamma\frac{\partial^2\gamma'}{\partial x'_k\partial x'_k}} - \overline{\sigma\gamma'\frac{\partial^2\gamma}{\partial x_k\partial x_k}} = 0, \quad [5.12]$$

and after the change in variables:

$$\frac{\partial\overline{\gamma\gamma'}}{\partial t} + \left(\overline{\gamma'u_k} + \overline{\gamma u'_k}\right)\frac{\partial\overline{\Gamma}}{\partial X_k} + \eta_l\Lambda_{kl}\frac{\partial\overline{\gamma\gamma'}}{\partial r_k} + \frac{\partial}{\partial r_k}\left(\overline{\gamma\gamma'u'_k} - \overline{\gamma\gamma'u_k}\right) - 2\sigma\frac{\partial^2\overline{\gamma\gamma'}}{\partial r_k\partial r_k} = 0.$$

$$[5.13]$$

Applying Fourier transform we then get:

$$\widehat{\gamma\gamma'} = \varphi_\gamma , \qquad \widehat{u_l\gamma\gamma'} = -i\theta_{\gamma l} , \qquad \widehat{u_l\gamma'} = \psi_{\gamma l}$$

$$\frac{\partial\varphi_\gamma}{\partial t} - \kappa_l \Lambda_{lm} \frac{\partial\varphi_\gamma}{\partial\kappa_m} + \left(\psi_{\gamma l} + \psi_{\gamma l}^*\right)\frac{\partial\overline{\Gamma}}{\partial X_l} - \left(\kappa_l\theta_{\gamma l} + \kappa_l\theta_{\gamma l}^*\right) + 2\sigma\kappa_m^2\varphi_\gamma = 0 , \quad [5.14]$$

an equation that can be written in condensed form:

$$\frac{\partial\varphi_\gamma}{\partial t} + \Psi_\gamma = \Omega_\gamma - 2\sigma\kappa^2\varphi_\gamma .$$

(cf. Mathieu J. *et al.*, [MAT 74]; Rey C. *et al.*, [REY 76A and B]; Rey C. and Mathieu J., [REY 78]).

5.6. One point correlation equations in HAT

We can recover the one point equations by integrating over the entire Fourier space;

$$R_{ij} = \int_{\mathbb{R}^3} \varphi_{ij} d\mathcal{V} ,$$

$$\int_{\mathbb{R}^3} S_{ij}^{(1)} d\mathcal{V} = 0 , \quad \int_{\mathbb{R}^3} S_{ij}^{(2)} d\mathcal{V} = 0 , \quad \int_{\mathbb{R}^3} 2\nu\kappa^2\varphi_{ij} d\mathcal{V} = 2\nu\overline{u_{i,m}u_{j,m}} = \varepsilon_{ij} ,$$

$$\int_{\mathbb{R}^3} \left(P_{ij}^{(1)} + P_{ij}^{(2)}\right) d\mathcal{V} = -\frac{1}{\rho}\left(\overline{u_j p_{,i}} + \overline{u_i p_{,j}}\right) = \frac{1}{\rho}\overline{p\left(u_{i,j} + u_{j,i}\right)} = \Phi_{ij} ,$$

$$\Phi_{ij} = \underbrace{-\int_{\mathbb{R}^3} \frac{\kappa_l\kappa_m}{\kappa^2}\left(\kappa_i\theta_{mlj} + \kappa_j\theta_{mli}^*\right) d\mathcal{V}}_{\Phi_{ij}^{(1)}} + \underbrace{2\Lambda_{lm}\int_{\mathbb{R}^3} \frac{\kappa_l}{\kappa^2}\left(\kappa_i\varphi_{mj} + \kappa_j\varphi_{mi}^*\right) d\mathcal{V}}_{\Phi_{ij}^{(2)}} ,$$

giving the Reynolds stress equations:

$$\frac{\partial R_{ij}}{\partial t} + \Lambda_{il}R_{lj} + \Lambda_{jl}R_{li} = \frac{1}{\rho}\overline{p(u_{i,j} + u_{j,i})} - 2\nu\overline{u_{i,l}u_{j,l}} . \qquad [5.15]$$

5.7. Examples of anisotropic homogenous turbulent flows

5.7.1. *Homogenous turbulent shear flow*

It is an homogenous turbulent flow with imposed constant mean velocity gradient (cf. Figure 5.2).

$$\frac{\partial\overline{U}_1}{\partial x_2} = \Lambda_{12} \qquad \text{constant.}$$

Such a flow can be achieved experimentally only within a limited portion of space and of course only in an approximate way. Indeed, perfect homogenity would imply an infinite extent.

Equation [5.15] for R_{ij} reads:

$$P_{ij} + \Phi_{ij} - \varepsilon_{ij} = 0$$

with $P_{ij} = -\Lambda_{il}R_{lj} - -\Lambda_{jl}R_{il}$.

$$\begin{cases} -2R_{12}\Lambda_{12} + \Phi_{11} - \varepsilon_{11} = 0 \\ \Phi_{22} - \varepsilon_{22} = 0 \\ \Phi_{33} - \varepsilon_{33} = 0 \\ -R_{22}\Lambda_{12} + \Phi_{12} - \varepsilon_{12} = 0 \end{cases} \qquad -R_{12}\Lambda_{12} - \varepsilon = 0$$

Roughly, the turbulent energy is divided up as follows among the normal components of the Reynolds stresses:

$$-2R_{12}\Lambda_{12}$$
$$\downarrow$$

$$2\overline{\frac{p}{\rho}u_{2,2}} \qquad \leftarrow \qquad 2\overline{\frac{p}{\rho}u_{1,1}} \qquad \rightarrow \qquad 2\overline{\frac{p}{\rho}u_{3,3}}$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$-2\nu\overline{u_{2,s}u_{2,s}} \approx \frac{2}{3}\varepsilon \qquad -2\nu\overline{u_{1,s}u_{1,s}} \approx \frac{2}{3}\varepsilon \qquad -2\nu\overline{u_{3,s}u_{3,s}} \approx \frac{2}{3}\varepsilon$$

whereas for the shear stress component, the main sink is not due to viscous dissipation but to the pressure-strain correlation term.
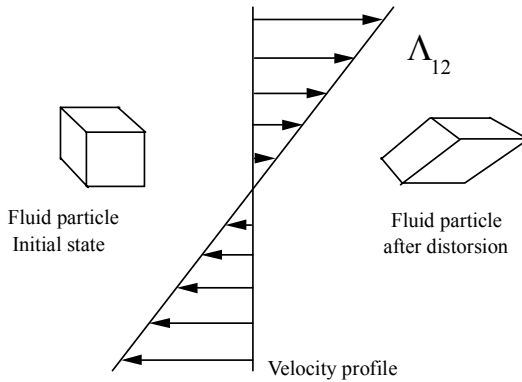
Figure 5.2a. *Distortion of a fluid particle in homogenous turbulent shear flow*
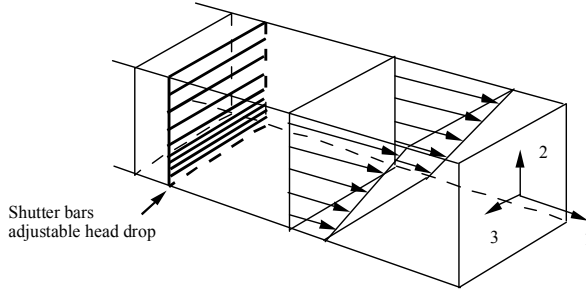
Experiments: ROSE(1965),
CHAMPAGNE, HARRIS & CORRSIN (1970),
HARRIS, GRAHAM & CORRSIN (1976)

Figure 5.2b. *Homogenous turbulent shear flow, sketch of the experimental set-up*

The transfers for the R_{12} equation can be viewed in the following way:

$$-R_{22}\Lambda_{12}$$
$$\downarrow$$
$$\overline{\frac{p}{\rho}\left(u_{1,2}+u_{2,1}\right)}$$
$$\downarrow$$
$$-2\nu\overline{u_{1,s}u_{2,s}} \approx 0$$

this type of flow has been studied experimentally by Champagne F.H., Harris V.G. and Corrsin S., [CHA 70] and the measurements give:

$$\frac{R_{11}-\dfrac{2}{3}k}{k}=0.30\,,\qquad \frac{R_{22}-\dfrac{2}{3}k}{k}=-0.18\,,\qquad \frac{R_{33}-\dfrac{2}{3}k}{k}=-0.12\,,$$

$$-\frac{R_{12}}{k}=0.33$$

5.7.2. *Pure plane distortion*

The mean velocity flow field is given by:

$$\left[\Lambda_{ij}\right]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$\overline{U}_1 = C^{st}\,,\quad \overline{U}_2 = -ax_2\,,\quad \overline{U}_3 = ax_3$$

The fluid particle is distorted in directions x_2 and x_3 (cf. Figures 5.3a and b).

This flow has been studied experimentally by Townsend A.A., [TOW 54]; Tucker M.J. and Reynolds A.J., [TUC 68]; Maréchal J., [MAR 70 and 72]. These latter authors have not shown evidence of the development of an equilibrium structure for this flow.

Fluid particle
Initial state

Distorting channel

Fluid particle
after distorsion
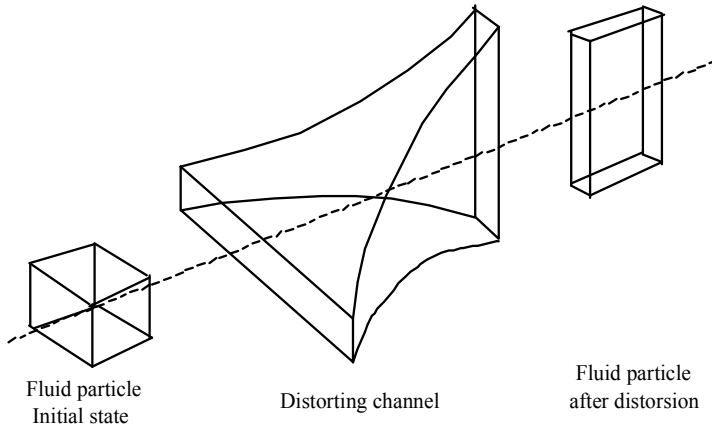
Figure 5.3a. *Stretching of a fluid particle in pure plane distortion of homogenous turbulence*

Maréchal experiment
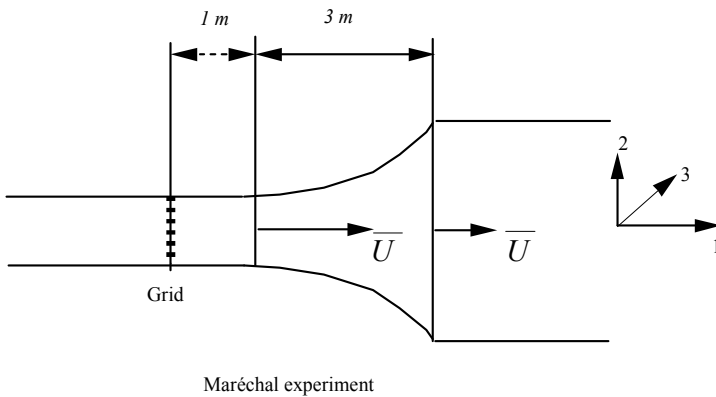
Experiments : TOWNSEND (1956),
TUCKER & REYNOLDS (1968),
MARECHAL (1970,1972)

Figure 5.3b. *Pure plane distortion of homogenous turbulence,
sketch of the experimental set-up*

The governing equations for the Reynolds stress tensor read:

$$\frac{\partial R_{11}}{\partial t} = \Phi_{11} - \varepsilon_{11},$$

$$\frac{\partial R_{22}}{\partial t} = 2aR_{22} + \Phi_{22} - \varepsilon_{22},$$

$$\frac{\partial R_{33}}{\partial t} = -2aR_{33} + \Phi_{33} - \varepsilon_{33},$$

$$\frac{\partial k}{\partial t} = a(R_{22} - R_{33}) - \varepsilon_{23},$$

$$\frac{\partial R_{12}}{\partial t} = aR_{12} + \Phi_{12} - \varepsilon_{12},$$

$$\frac{\partial R_{13}}{\partial t} = -aR_{13} + \Phi_{13} - \varepsilon_{13},$$

$$\frac{\partial R_{23}}{\partial t} = \Phi_{23} - \varepsilon_{23}.$$

An analytical study of this flow has been carried on by Courseau P. and Loiseau M, [COU 72A and B], [COU 78] (cf. section 5.9).

5.7.3. *Axisymmetric contraction of homogenous turbulence*

This flow can be approximately produced in the central region of a duct whose cross-section is suddenly reduced (cf. Figure 5.4)

$$\left[\Lambda_{ij} \right] = \begin{bmatrix} b & 0 & 0 \\ 0 & -b/2 & 0 \\ 0 & 0 & -b/2 \end{bmatrix}$$

Let us quote the experiments by Uberoi M.S., [UBE 56]; Uberoi M.S. and Wallis S., [UBE 66].
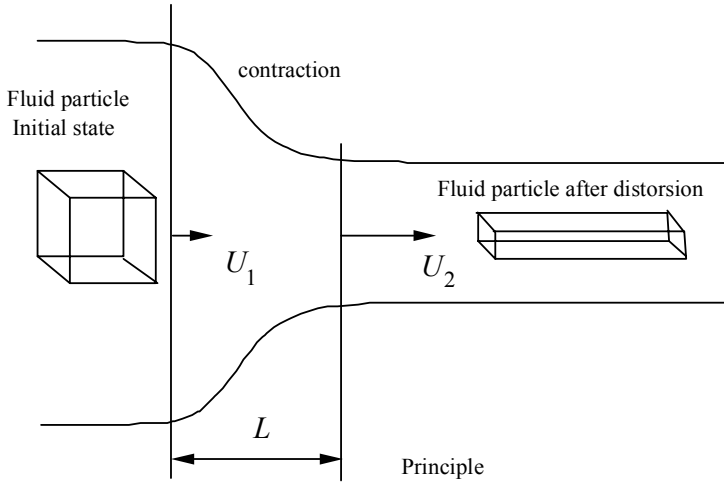
Figure 5.4a. *Lengthening of the fluid particle in the axisymmetric contraction*

Experiment : UBEROI (1956)

U_1=1.1 *m/s* U_1/U_2=16 - 9 and 4   *L*=50 *cm* Re_t=10^3
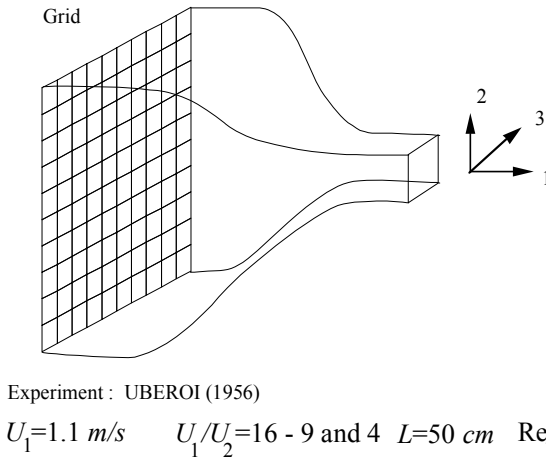
Figure 5.4b. *Axisymmetric contraction of duct flow, sketch of the experimental set-up*

We also quote the experiment by Gence (Gence J.N., [GEN 79A]; Gence J.N. and Mathieu J., [GEN 79B and C]) composed of two successive plane distortions which are coplanar but with a relative angle of their principal axes that can be varied. On the other hand, the experimental work of Reynolds A.J. and Tucker H.J., [REY 75B] extends the study of the distortion of turbulence by considering a general irrotational distortion combining plane distortion, contraction and expansion.

5.8. Rapid distortion theory for an homogenous turbulent flow

The Batchelor G.K. and Proudman I. theory [BAT 54] allows us to calculate the modifications produced in an homogenous turbulent flow submitted to a sudden uniform distortion (rapid distortion theory or RDT).

The usual analysis is based on the Cauchy equation for vorticity:

$$\omega_i(\vec{\xi},t) = \frac{\partial x_i}{\partial \xi_j}\omega_j(\vec{\xi},0)\,,$$
[5.16]

in Lagrangian variables.

Indeed, considering the Eulerian equation $\dfrac{d\omega_i}{dt} = \Lambda_{ij}\omega_j$ (cf. section 1.4), for which the equivalent Lagrangian formulation is:

$$\frac{\partial \omega_i}{\partial t}(\xi_l,t) = \Lambda_{ij}\omega_j(\xi_l,t)\,,$$
[5.17]

the solution of this linear system composed of three equations in three unknowns is the following:

$$\omega_i(\xi_l,t) = e^{\left[\Lambda_{ij}\right]}\omega_j(\xi_l,0)$$
[5.18]

Considering also:

$$v_i(\xi_l,t) = \frac{\partial x_i}{\partial t}(\xi_l,0)\,,$$

We obtain from taking the derivative:

$$\frac{\partial v_i}{\partial \xi_m} = \frac{\partial^2 x_i}{\partial t \partial \xi_m} \qquad \text{or} \qquad \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial \xi_m} = \frac{\partial^2 x_i}{\partial t \partial \xi_m}$$

or even:

$$\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial \xi_m}\right) = \Lambda_{ij}\left(\frac{\partial x_j}{\partial \xi_m}\right)\,,$$

a system whose solution is:

$$\frac{\partial x_i}{\partial \xi_m} = e^{\left[A_{ij} \right]t} \left(\frac{\partial x_j}{\partial \xi_m} \right)(0) = e^{\left[A_{ij} \right]t} \delta_{jm}$$

and thus:

$$e^{\left[A_{ij} \right]t} = \left(\frac{\partial x_i}{\partial \xi_j} \right)$$

we recover relation [5.16] in this way.

We will use the shorthand notation:

$$\omega"_i = \left(\frac{\partial x_i}{\partial \xi_j} \right) \omega'_j \qquad\qquad [5.19]$$

where $(.)' = (.)(\vec{\xi}, 0)$ denotes the particle at point $\vec{a}$ and $(.)" = (.)(\vec{\xi}, t)$ the particle at point $\vec{x}$. Relation [5.19] thus reads, after making use of the definition of the rotational vector or curl:

$$\varepsilon_{ipq} \frac{\partial u"_q}{\partial x_p} = \varepsilon_{jpq} \frac{\partial x_i}{\partial \xi_j} \frac{\partial u'_q}{\partial \xi_p}, \qquad\qquad [5.20]$$

where $\dfrac{\partial x_i}{\partial \xi_j}$ is assumed to be constant.

Taking the curl in both sides of equation [5.20] we find:

$$\varepsilon_{rsi}\varepsilon_{ipq} \frac{\partial^2 u"_q}{\partial x_p \partial x_s} = \varepsilon_{rsi}\varepsilon_{jpq} \frac{\partial x_i}{\partial \xi_j} \cdot \frac{\partial x_i}{\partial x_s} \left(\frac{\partial u'_q}{\partial \xi_p} \right), \qquad\qquad [5.21]$$

or:

$$\Delta_{xx} u"_r = \varepsilon_{rsi}\varepsilon_{jpq} \frac{\partial x_i}{\partial \xi_j} \cdot \frac{\partial \xi_m}{\partial x_s} \left(\frac{\partial^2 u'_q}{\partial \xi_p \partial \xi_m} \right). \qquad\qquad [5.22]$$

(we have used the identity $\varepsilon_{irs}\varepsilon_{ipq} \equiv \delta_{rp}\delta_{sq} - \delta_{rq}\delta_{sp}$).

The linear relation between x_i and ξ_j being $x_i = \dfrac{\partial x_i}{\partial \xi_j}\xi_j$, we define the

corresponding Fourier relations $\chi_i = \kappa_j \dfrac{\partial \xi_j}{\partial x_i}$ satisfying $\dfrac{\partial}{\partial \chi_i} = \dfrac{\partial x_i}{\partial \xi_j}\dfrac{\partial}{\partial \kappa_j}$ and

$\vec{x}.\vec{\chi} = \vec{\xi}.\vec{\kappa}$ and also the Fourier transforms:

$$\hat{u}"_i(\vec{\chi}) = \widehat{u"_i(x)} \qquad \hat{u}'_i(\vec{\kappa}) = \widehat{u'_i(\vec{\xi})}$$

we then deduce from [5.22]:

$$\chi^2 \hat{u}"_r(\vec{\chi}) = -\varepsilon_{rsi}\varepsilon_{jpq}\frac{\partial x_i}{\partial \xi_j}\frac{\partial \xi_m}{\partial x_s}\kappa_p\kappa_m\hat{u}'_q(\vec{\kappa}) , \qquad [5.23]$$

The spectra before distortion $\Psi_{qs}(\vec{\kappa})$ and after distortion $\Phi_{rl}(\vec{\chi})$ being defined by:

$$\overline{\hat{u}"_r(\vec{\chi})\hat{u}"_l(-\vec{\chi}')} = \Phi_{rl}(\vec{\chi})\delta(\vec{\chi}+\vec{\chi}')$$
$$\overline{\hat{u}'_q(\vec{\kappa})\hat{u}'_s(-\vec{\kappa}')} = \Psi_{qs}(\vec{\kappa})\delta(\vec{\kappa}+\vec{\kappa}')$$

we finally get the turbulence spectrum after distortion:

$$\Phi_{rl}(\vec{\chi}) = \frac{\varepsilon_{rsi}\varepsilon_{jpq}\dfrac{\partial x_i}{\partial \xi_j}\dfrac{\partial \xi_h}{\partial x_s}\varepsilon_{lmn}\varepsilon_{vab}\dfrac{\partial x_u}{\partial \xi_v}\dfrac{\partial \xi_n}{\partial x_m}}{\left(\kappa_c\kappa_d\dfrac{\partial \xi_c}{\partial x_e}\dfrac{\partial \xi_d}{\partial x_e}\right)^2}\kappa_p\kappa_h\kappa_a\kappa_n\Psi_{qb}(\vec{\kappa}) . \qquad [5.24]$$

Let us consider for illustration the particular case: $(\kappa_1,\kappa_2,\kappa_3) = (e_1\chi_1, e_2\chi_2, e_3\chi_3)$.

Relation $e_1e_2e_3 = 1$ is a direct consequence of the continuity equation $\dfrac{D(x_1,x_2,x_3)}{D(\xi_1,\xi_2,\xi_3)} = 1$, with

$$e_1 = \frac{\partial x_1}{\partial \xi_1}, \quad e_2 = \frac{\partial x_2}{\partial \xi_2}, \quad e_3 = \frac{\partial x_3}{\partial \xi_3}, \quad e_1e_3^2 = \frac{e_3}{e_2}, \quad e_1e_2^2 = \frac{e_2}{e_3} .$$

We deduce, for example, from equation [5.23]:

$$\chi^2 \hat{u}"_1(\vec{\chi}) = \frac{1}{e_1}\left(\chi^2 - \frac{\kappa_1^2}{e_1^2}\right)\hat{u}'_1 - \frac{\kappa_1\kappa_2}{e_1e_2^2}\hat{u}'_2 - \frac{\kappa_1\kappa_3}{e_1e_3^2}\hat{u}'_3,$$

and analogous relations for $\hat{u}"_2(\vec{\chi})$ and $\hat{u}"_3(\vec{\chi})$ and also $\Phi_{11}, \Phi_{22}, \Phi_{33}$,

$$\chi^2 = \frac{\kappa_1^2}{e_1^2} + \frac{\kappa_2^2}{e_2^2} + \frac{\kappa_3^2}{e_3^2}.$$

In the case of pure plane distortion $e_2 = 1$ and $e_1 = \frac{1}{e_3} = c > 1$.

Batchelor G.K. and Proudman I., [BAT 54] show that when c is large:

$$\mu_1 = \frac{(R_{11})"}{(R_{11})'} \approx \frac{3}{4c}(\log 4c - 1) + \mathcal{O}(\frac{\log c}{c^3}),$$

$$\mu_2 = \frac{(R_{22})"}{(R_{22})'} \approx \frac{3}{4}c - \frac{3}{8c}(\log 4c - \frac{3}{2}) + \mathcal{O}(\frac{\log c}{c^3}),$$

$$\mu_3 = \frac{(R_{33})"}{(R_{33})'} \approx \frac{3}{4}c - \frac{3}{8c}(\log 4c - \frac{1}{2}) + \mathcal{O}(\frac{\log c}{c^3}),$$

$$\mu = \frac{(k)"}{(k)'} = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3) \approx \frac{1}{2}c + \frac{1}{4c}(\log 4c - \frac{1}{2}) + \mathcal{O}(\frac{\log c}{c^3}).$$

The evolution obtained from these equations is schematically represented in Figure 5.5.
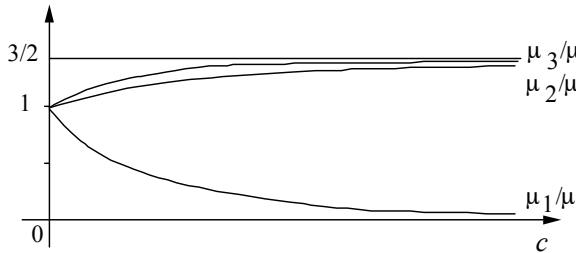
Figure 5.5. *Evolution of the Reynolds stress components in plane homogenous distortion of turbulence by RDT*

5.9. Additional information on linear solutions

The rapid distortion approximation consists of neglecting the non-linear terms and the dissipative terms (the effect of which occurs with some delay). Thus, the Reynolds stress equation can be written in this case:

$$\frac{dR_{ij}}{dt} = -R_{ki}\overline{U}_{j,k} - R_{kj}\overline{U}_{i,k} + \left(a_{lj}^{mi} + a_{li}^{mj}\right)\overline{U}_{l,m},$$

with $a_{lj}^{mi} = -\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^3}\frac{\partial^2 \overline{u'_m u_i}}{\partial\xi_l \partial\xi_j}\cdot\frac{d\mathscr{V}}{r}$, $\Phi_{ij}^{(1)}$ and ε_{ij} being neglected.

The solution of this equation presupposes the knowledge of the two point double velocity correlations $\overline{u'_m u_i}$. The problem is indeed closed at spectral level.

Considering the distortion of homogenous turbulence with $\Lambda_{ij} = \partial\overline{U}_i/\partial x_j$. Let us denote F_{ij} the linear tangent mapping with

$$dx_i = \frac{\partial x_i}{\partial\xi_j}d\xi_j \qquad\qquad F_{ij} = \frac{\partial x_i}{\partial\xi_j} = e^{\left[\Lambda_{ij}\right]t}.$$

The linear solutions of the simplified Craya equation $\dfrac{\partial\varphi_{ij}}{\partial t} + \Psi_{ij} = 0$ can be derived more easily in the moving local frame of Craya A., [CRA 58].

Denoting φ'_{ij} the components of the tensor φ_{ij} in the local frame, we simply come to:

$$\varphi'_{11} = N_1, \quad \varphi'_{22} = N_2, \quad \varphi'_{12} = \left(\varphi'_{21}\right)^* = P + iQ.$$

The incompressibility condition implies $\varphi'_{ij} = 0$ if $i=3$ or $j=3$, taking into account $\kappa'_1 = \kappa'_2 = 0$ and $\kappa'_3 = 1$ (components of the normalized wavevector in the Craya frame; see Figure 5.6), four real scalars N_1, N_2, P, Q are sufficient to define the tensor φ_{ij} .

Note that $\varphi'_{11} = \left(\varphi'_{11}\right)^*$ and $\varphi'_{22} = \left(\varphi'_{22}\right)^*$ from Hermitian symmetry.
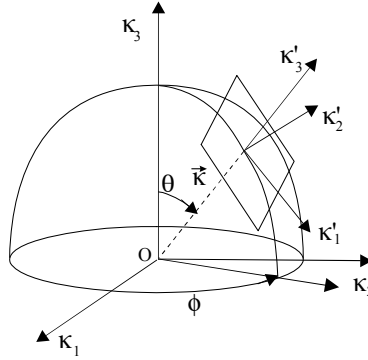
Figure 5.6. *The Craya moving local frame*

In local axes, the simplified Craya equation reads (Loiseau M., [LOI 73]):

$$\mathscr{D} N_1 + 2F_1 N_1 + 2G_1 P = 0 \,,$$
$$\mathscr{D} N_2 + 2F_2 N_2 + 2G_2 P = 0 \,,$$
$$\mathscr{D} P + (F_1 + F_2)P + G_1 N_2 + G_2 N_1 = 0 \,,$$
$$\mathscr{D} Q + (F_1 + F_2)Q = 0 \,,$$

with $\mathscr{D} = \dfrac{\partial}{\partial t} - \Lambda_{pq}\kappa_p \dfrac{\partial}{\partial \kappa_q}$.

The quantities F_1, F_2, G_1, G_2 are functions of Λ_{ij} and the matrix of frame change. The system can be solved using the method of the associated characteristic system. It is thus possible to return to the fixed frame. We thus find: $\varphi_{ij} = \dfrac{E(\kappa)}{4\pi\kappa^2} F_{ij}$.

In homogenous and isotropic turbulence we would recover:

$$\kappa = \chi \,, \qquad \varphi_{ij} = \frac{E(\chi)}{4\pi\chi^2}\Delta_{ij} \,, \qquad \Delta_{ij} = \delta_{ij} - \frac{\chi_i \chi_j}{\chi^2} \,.$$

5.10. Interdependency between differing closure levels: the spectral integral approach

Two point theories, generally considered in spectral space for homogenous turbulence lead to spectral models which aim at determining the two point double

velocity spectral tensor $\Phi_{ij}(\vec{\kappa})$. This description level is very heavy to handle because each component of the spectral tensor, generally complex, is dependent on a vector. An important simplification is allowed by the spherical mean [5.8] which leads to one-dimensional models, in which $\Phi_{ij}(\kappa)$ is now the only function of the wavenumber. It is finally possible, by integrating over spectral space, to recover the one point correlations:

$$R_{ij} = \int_0^\infty \Phi_{ij}(\kappa)d\kappa = \iiint_{\mathbb{R}^3} \Phi_{ij}(\vec{\kappa})d\vec{\kappa} \, .$$

The integration of the three-dimensional spectral equations depending on wavevectors or integration of their spherical means depending on wavenumbers allows us to recover the one point moment equations, by introducing some additional closure hypotheses. We can sum up this unified description using the following scheme:

- two point correlations $\overline{u_i(A)u_j(B)}$

    $\downarrow$ F   (Fourier transform)

- three-dimensional spectral tensor $\Phi_{ij}(\vec{\kappa})$

    $\downarrow$ $\oiint_{S(r)}(.)ds$ (spherical mean)

- one-dimensional spectral tensor $\Phi_{ij}(\kappa)$

    $\downarrow$ $\int_0^\infty(.)d\kappa$ (integration)

- Reynolds stress tensor R_{ij}

Building one point models through integration performed on spectral equations represents the spectral integral method. This type of approach is developed in [AUP 87]. We shall give here a simple illustrative example, the one for turbulence decay.

For convenience, we consider a simplified form of the energy spectrum in homogenous and isotropic turbulence including a power law distribution in the low wavenumbers region followed by a logarithmic zone with $-5/3$ slope to represent the inertial region at higher wavenumbers and then a cut-off at the Kolmogorov scale κ_η.

Thus,

$$
\begin{cases}
E(\kappa) = A\kappa^m & \text{for } \kappa < \kappa_M \\
E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3} & \text{for } \kappa_M < \kappa < \kappa_\eta \\
E(\kappa) = 0 & \text{for } \kappa > \kappa_\eta
\end{cases}
$$

The connection point joining the first two parts corresponds to the maximum of the spectrum. The joining condition at this point κ_M implies $\kappa_M = \left(\dfrac{C\varepsilon^{2/3}}{A} \right)^{\frac{3}{3m+5}}$ and assuming that κ_η is sufficiently large compared to κ_M, the total energy under the spectrum is simply expressed by $k = \int_0^\infty E(\kappa)d\kappa = C\left(\dfrac{1}{m+1} + \dfrac{3}{2} \right)\kappa_M^{-2/3}\varepsilon^{2/3}$,

or:

$$
k = A^{\frac{2}{3m+5}} \left(\frac{1}{m+1} + \frac{3}{2} \right)\left(C^{3/2}\varepsilon \right)^{\frac{2m+2}{3m+5}}.
$$

The hypothesis of permanence of big eddies (cf. Chapter 1) suggests that we assume that $A = $ constant.

Under these conditions, the derivative of the previous equation leads to:

$$
\frac{dk}{k} = \frac{2m+2}{3m+5}\frac{d\varepsilon}{\varepsilon}
$$

and taking into account $\dfrac{dk}{dt} = -\varepsilon$, we then find a modeled equation for the dissipation rate:

$$
\frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k} \quad \text{with} \quad C_{\varepsilon 2} = \frac{3m+5}{2m+2},
$$

which is nothing more than the equation of the k-ε model in the particular case of decay.

The value of this coefficient $C_{\varepsilon 2}$ controls the speed of turbulence decay, which in fact is found to be linked to the value of *m*. Thus, the turbulence evolution is governed by the large-scale part of the spectrum (low wavenumbers). For *m* = 2, we find $C_{\varepsilon 2} = 11/6$, a value which is close to the one traditionally used in the *k-ε* model (cf. Chapters 7 and 11).

The fact that $C_{\varepsilon 2}$ in the dissipation equation (dissipation phenomenon occurs at the small scales level) is dependent only on the large-scale part of the spectrum (through *m*) may be surprising at first sight. This is, however, only the expression of traditional phenomenology of turbulence: the evolution of turbulence is governed by the non universal part of the spectrum whereas the universal part of the spectrum only makes the junction with the dissipative zone. This approach also singles out the underlying meaning of ε as representing the spectral flux of the energy cascade rather than viscous dissipation into heat.

It is also possible to use, at the price of more complex algebra, more realistic spectral distributions, such as those proposed by Von Karman (cf. Hinze, 1975, 1987, [*HIN 75]):

$$E(\kappa,t) = \alpha \frac{\left(\kappa/\kappa_M\right)^4}{\left[1 + \left(\kappa/\kappa_M\right)^2\right]^{17/6}} .$$

This type of approach can be in principle extended to sheared turbulence [AUP 87]; this is supposing, however, a preliminary development of spectral models.

The previous formulae allow us to calculate the integral length scale defined by Rotta (1951) (cf. Chapter 7):

$$L = \frac{1}{k} \int_0^\infty \frac{E(\kappa)d\kappa}{\kappa} = \frac{2m+2}{5m} \frac{1}{\kappa_M} .$$

This type of approach will be in particular also applied in Chapter 18 for the development of multiple scale models. The spectral integral approach gives a unifying view of turbulence modeling by revealing the analytical connection between two point closures and one point closures.

This page intentionally left blank

Chapter 6

Modeling of the Reynolds Stress Transport Equations

One point second order models or Reynolds stress modeling (RSM) is the level of closure most often necessary for a realistic representation of the turbulent phenomena occurring in practical complex flows in the industry or also in the domain of environment. They offer both the potentials to account for the main turbulent interactions with effects of possible additional phenomena (thermal effects, gravity forces, etc.) and also a moderate mathematical complexity that allows efficient numerical treatment on a computer. Indeed, the convection-diffusion type equations with sources/sink terms to which they give rise, can be solved using appropriate numerical methods (cf. Chapter 20). In the present chapter, we shall present mainly the Reynolds stress transport model [LAU 75C], following an essentially phenomenological approach. Indeed, this early model introduces in its founding concepts the main turbulent interaction phenomena modeled in a relatively simple way typical of any second order closure. Thus, we can consider this model as a prototype for reference, all subsequent more advanced models will keep the same basic phenomenological structure, but using higher order expansion approximates that can be established for example by using invariant modeling.

6.1. The Reynolds stress transport equations and their trace

The general transport equations of Reynolds stresses are obtained from the fluctuating velocity equations (cf. Chapter 2).

These equations read:

$$\underbrace{\frac{\partial R_{ij}}{\partial t} + \overline{U}_m R_{ij,m}}_{(a)} = \underbrace{-R_{im}\overline{U}_{j,m} - R_{jm}\overline{U}_{i,m}}_{(b)} + \underbrace{\frac{\overline{p}}{\rho}\left(u_{i,j} + u_{j,i}\right)}_{(c)} - \underbrace{\left(\overline{u_i u_j u_m}\right)_{,m}}_{(d)}$$

$$\underbrace{-\left[\left[\frac{\overline{p}}{\rho}\left(u_i \delta_{jm} + u_j \delta_{im}\right)\right]\right]_{,m}}_{(e)} \underbrace{- 2\nu\overline{u_{i,m} u_{j,m}}}_{(f)} + \underbrace{\nu R_{ij,mm}}_{(g)} \qquad [6.1]$$

This tensorial equation essentially contains two types of terms: those appearing in the form of gradients of tensors and which represent the transport of a quantity from one place to another (their overall integral remains zero for the entire flow), and those that play the role of source and sink terms (and whose integral is generally non-zero).
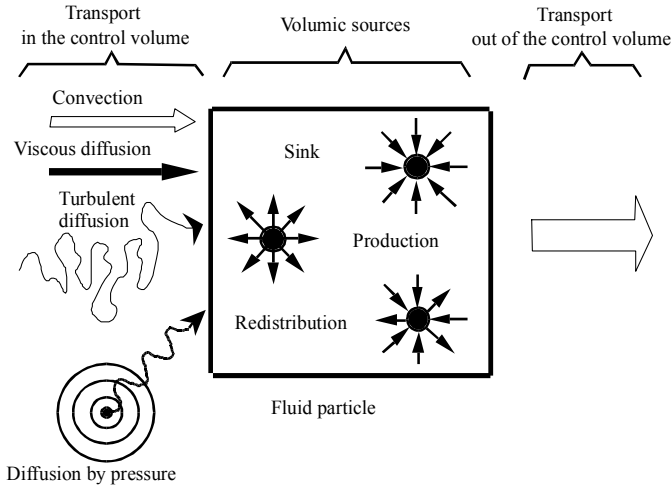
Figure 6.1. *Balance of the Reynolds stress tensor equation*

The physical interpretation of each term is detailed below (cf. Figure 6.1):

1) Material derivative

$$(a) \begin{cases} \dfrac{\partial R_{ij}}{\partial t} & (1) \\[2mm] \overline{U}_m R_{ij,m} & (2) \end{cases}$$

These terms include the time derivative (*a.1*), which vanishes in steady flow regimes, and the spatial convective derivative following the mean flow (*a.2*), which vanishes in homogenous flows.

2) Gradient terms

(d) $\left(\overline{u_i u_j u_m}\right)_{,m}$, turbulent diffusion which is in fact a convective process at the level of turbulent fluctuations (triple correlations).

(e) $\left[\left[\overline{\dfrac{p}{\rho}\left(u_i \delta_{jm} + u_j \delta_{im}\right)}\right]\right]_{,m} = \dfrac{1}{\rho}\left[\overline{(pu_i)}_{,j} + \overline{(pu_j)}_{,i}\right]$,   turbulent   diffusion

due to pressure fluctuations.

(g) $\nu R_{ij,mm}$, molecular diffusion.

3) Source and sink terms

(b) $-R_{im}\overline{U}_{j,m} - R_{jm}\overline{U}_{i,m}$, production due to the work of the turbulent Reynolds stresses submitted to the action of mean velocity gradients.

(c) $\overline{\dfrac{p}{\rho}\left(u_{i,j} + u_{j,i}\right)}$, pressure-strain correlation term with zero tensorial contraction and responsible for redistribution of turbulent energy among the different components of the Reynolds stresses.

(f) $2\nu \overline{u_{i,m} u_{j,m}}$, dissipation rate term due to viscosity of the fluid.

The trace of the R_{ij} equation yields the turbulent kinetic energy equation:

$$\underbrace{\frac{\partial k}{\partial t} + \overline{U}_m k_{,m}}_{(a)} = \underbrace{-R_{ij}\overline{U}_{i,j}}_{(b)} - \underbrace{\left(\overline{u_j \frac{u_i u_i}{2}}\right)_{,j}}_{(c)} - \underbrace{\left[\overline{\left[\frac{p}{\rho}u_j\right]}\right]_{,j}}_{(d)}$$

$$\underbrace{-\nu \overline{u_{i,j} u_{i,j}}}_{(e)} + \underbrace{\nu k_{,jj}}_{(f)} \qquad\qquad [6.2]$$

The interpretation of terms is similar to the one given for the R_{ij} equation. We can distinguish:

1) Material derivative

(a) $\dfrac{\partial k}{\partial t} + \overline{U}_m k_{,m}$, time derivative and convective term.

2) Gradient terms

(c) $\left(\overline{u_j \dfrac{u_i u_i}{2}} \right)_{,j}$, turbulent diffusion due to triple velocity correlations.

(d) $\left[\overline{\left[\dfrac{p}{\rho} u_j \right]} \right]_{,j}$, turbulent diffusion due to pressure fluctuations.

(f) $\nu k_{,jj}$, molecular diffusion.

3) Source and sink terms

(b) $-R_{ij} \overline{U}_{i,j}$, production of turbulence energy by the action of mean flow on turbulence.

(e) $\nu \overline{u_{i,j} u_{i,j}}$, viscous dissipation.

Note: strictly speaking, the viscous dissipation rate is given by:

$$\frac{1}{2} \nu \overline{\left(u_{i,j} + u_{j,i} \right)^2} = \nu \overline{u_{i,j} \left(u_{i,j} + u_{j,i} \right)} = \nu \overline{u_{i,j} u_{i,j}} + \nu R_{ij,ij} ,$$

the second term is negligible at high turbulence Reynolds numbers, and in this case only the first term in fact represents the true dissipation rate.

In the Reynolds stress transport equations, closure hypotheses are necessary for the following terms: viscous dissipation *(f)*, redistribution of energy by pressure-strain correlations *(c)*, turbulent diffusion *(d)* + *(e)*. In the present chapter, we shall consider only fully developed turbulence at high Reynolds numbers Re_t .

Numerous models have been proposed by different authors, but we shall give here a more complete description of the Launder B.E., Reece G.J. and Rodi W., [LAU 75C] model (denoted LRR) which has been among the most useful and served as prototype for subsequent developments considered in Chapter 8 (let us also quote the contributions such as Naot D., [NAO 77 and 78] in this type of closure). Some observations will be given for other types of closures. The reader will find in the bibliography the references to the original articles.

The source and sink terms can be modeled with reference to homogenous turbulence. The other terms appear only in non-homogenous flows and in particular in wall flows. They are mainly turbulent diffusion terms and a part of the pressure-strain correlations and they will be considered separately.

6.2. Modeling viscous dissipation terms

Viscous dissipation occurs at the level of small eddies, in the spectral zone of large wavenumbers in which turbulence is classically assumed to approach isotropy. If the Reynolds number is sufficiently high for the dissipation zone to be clearly separated from the production zone, the viscous dissipation process can be assumed to be isotropic. This is modeled using a second order isotropic tensor through the hypothesis:

$$\varepsilon_{ij} = 2\nu\overline{u_{i,k}u_{j,k}} = \frac{2}{3}\varepsilon\delta_{ij} \quad \text{with} \quad \varepsilon = \nu\overline{u_{i,j}u_{i,j}} , \quad\quad [6.3]$$

ε will be obtained by a separate evolution equation (cf. Chapter 7).

We note that in isotropic turbulence it is possible to express the fourth order tensor $\nu\overline{u_{i,k}u_{j,k}}$. This tensor takes the form: $a\delta_{ik}\delta_{jl} + b\delta_{ij}\delta_{kl} + c\delta_{il}\delta_{jk}$ (equation [3.36], Chapter 3), coefficients a, b, c are determined using the symmetry properties of $\nu\overline{u_{i,k}u_{j,l}}$, the incompressibility condition and the relation of definition of ε. We find,

$$\nu\overline{u_{i,k}u_{j,l}} = \frac{2\varepsilon}{30}(2\delta_{ij}\delta_{kl} - \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{il}\delta_{kj}) . \quad\quad [6.4]$$

Note: some former models, Donaldson C. du P., [DON 69 and 71]; Daly B.J. and Harlow F.H., [DAL 70], did not use relation [6.3] but instead $\varepsilon_{ij} = \dfrac{R_{ij}}{k}\varepsilon$, a relation which does not underlie a tendency to local isotropy. However, it is sufficient to modify the constant c_1 in the Rotta approximation in the pressure-strain correlations (see section 6.4) to recover a global approximation identical to the LRR model. Thus, the terms *(c)* and *(f)* considered individually have different approximates, but considered as a whole their behavior will be the same.

Let us note that hypothesis [6.3] is certainly too strong because the turbulence Reynolds number remains finite and the small eddies have not forgotten entirely the anisotropy of the larger ones. Lumley J.L., [LUM 78A] suggests writing:

$$\frac{\nu \overline{u_{i,k} u_{j,k}}}{\varepsilon} - \frac{\delta_{ij}}{3} \sim \mathcal{O}\left[\frac{\lambda}{\ell}\left(\frac{R_{ij}}{2k} - \frac{\delta_{ij}}{3}\right)\right] \sim \mathrm{Re}_t^{-1/2} a_{ij},$$

i.e.

$$\nu \overline{u_{i,k} u_{j,k}} = \frac{\varepsilon}{3}\delta_{ij} + \mathcal{O}\left[\frac{a_{ij}}{\mathrm{Re}_t^{1/2}}\right], \qquad\qquad [6.5]$$

so, the influence of anisotropy is dependent on the ratio of characteristic length scales. It appears that when the Reynolds number is moderate it is necessary to take into account of departures from hypothesis [6.3] (cf. Chapter 13). This hypothesis, largely used in practice, is thus not rigorous and may even be controversial ([DUR 91B], [SAD 94]).

6.3. Modeling turbulent diffusion terms

Diffusion terms in the R_{ij} equation read:

$$D_{ij} = -\left[\underbrace{T_{ijk} + \frac{\overline{p}}{\rho}(\overline{u_i}\delta_{jk} + \overline{u_j}\delta_{ik})}_{(*)} - \underbrace{\nu R_{ij,k}}_{(**)}\right]_{,k},$$

(*) terms to model, (**) negligible terms at high Re_t.

6.3.1. *Triple velocity correlations*

These can be determined either algebraically or by solving transport equations (cf. Chapter 2, equation [2.10]). Triple correlations can be measured in some cases; experimental results often exhibit large gradients in their distribution.

Chou P.Y., [CHO 45A and B]; Kolovandine B.A. and Vatutine I.A., [KOL 69 and 72]; André J.C. *et al.*, [AND 77] have proposed a modeled transport equation for T_{ijk}. However, in most practical turbulent flows, the turbulent diffusion due to triple velocity correlations, although important, does not play a central role. Furthermore, devising closure hypotheses for the higher order correlations appearing in the T_{ijk} transport equations is an awkward matter. Therefore, using such an elaborate treatment, consisting of solving the full triple correlation transport equations does not seam to be generally adequate for the usual cases.

We shall first quote the algebraic approximation proposed by Hanjalic K. and Launder B.E. [HAN 72].

$$T_{ijk} = -c_s \frac{k}{\varepsilon} \left(R_{il} R_{jk,l} + R_{jl} R_{ik,l} + R_{kl} R_{ij,l} \right).$$ [6.6]

This expression can be established from the transport equations of T_{ijk} (equation [2.10], Chapter 2) considering several simplifying hypotheses which are:

– Flow at high Reynolds number Re_t, negligible viscous diffusion.

– The quadruple correlations can be approximated using a quasi-normal hypothesis (Millionshtchikov hypothesis) which is equivalent to assuming that the fourth order cumulant vanishes:

$$\overline{u_i u_j u_k u_l} = R_{ij} R_{kl} + R_{ik} R_{jl} + R_{il} R_{jk},$$

(this approximation is only valid when the triple correlations are weak).

– Terms having the form $T_{ijl} \overline{U}_{k,l}$ are neglected (a hypothesis suggested [HAN 72] by experimental results in a non-symmetric plane channel).

– The pressure correlation can be approximated by a term proportional to

$$-\frac{\varepsilon}{k} T_{ijk}$$

– The convective transport of T_{ijk} is neglected.

– The viscous dissipation is neglected (isotropic tensor of third order).

Thus, in the equation:

$$\frac{dT_{ijk}}{dt} = \underbrace{-T_{ijl} \overline{U}_{k,l} - T_{kil} \overline{U}_{j,l} - T_{jkl} \overline{U}_{i,l}}_{(b)} + \underbrace{R_{ij} R_{kl,l} + R_{jk} R_{il,l} + R_{ik} R_{jl,l}}_{(c)}$$
$$\underbrace{\phantom{\frac{dT_{ijk}}{dt}}}_{(a)}$$

$$\underbrace{-\left(\overline{u_i u_j u_k u_l} \right)_{,l}}_{(d)} - \underbrace{\frac{1}{\rho} \left(\overline{u_i u_j p_{,k}} + \overline{u_j u_k p_{,i}} + \overline{u_i u_k p_{,j}} \right)}_{(e)},$$

the Millionshtchikov hypothesis leads to the approximation:

$$(c) + (d) = -R_{il} R_{jk,l} - R_{jl} R_{ki,l} - R_{kl} R_{ij,l},$$

the term (*b*) is neglected (by reference to experimental results obtained in non-symmetric turbulent flows) and we assume:

$$(e) = -c\frac{\varepsilon}{k}T_{ijk} \,,$$

(that being obtained by analogy with the approximation of the energy redistribution terms through pressure-strain correlations in the R_{ij} transport equation; see section 6.4).

If the term (*a*) is neglected, then we recover approximation [6.6]. Coefficient c_s in [6.6] is determined by referring to experimental data relative to various turbulent flows, the value $c_s = 0.11$ is obtained in Launder B.E., Reece G.J., Rodi W., [LAU 75C].

Note: differing schemes have been proposed by other authors. Let us cite Donaldson C. du P., [DON 71] who proposes:

$$T_{ijk} = -L\sqrt{k}\left(R_{ij,k} + R_{jk,i} + R_{ik,j}\right), \tag{6.7}$$

in which L is a macro lengthscale. The scheme by Mellor and Herring, 1973 ([MEL 73]) is similar: $T_{ijk} = -c\dfrac{k^2}{\varepsilon}\left(R_{ij,k} + R_{jk,i} + R_{ik,j}\right).$

The scheme by Daly B.J. and Harlow F.H., [DAL 70] keeps only one term but with a tensorial diffusion coefficient:

$$T_{ijk} = -c'_s\frac{k}{\varepsilon}R_{kl}R_{ij,l}\,. \tag{6.8}$$

The hypothesis is not compatible with symmetry properties of T_{ijk}, however, it has been successfully used by Launder B.E., Reece G.J. and Rodi W., [LAU 75C] in the prediction of various free flows. It is, of course, admissible in thin shear flows only, in which the transverse gradients prevail $(.)_{,2} \gg (.)_{,1}$, thus:

$$T_{ij2,2} = -c'_s\left(\frac{k}{\varepsilon}R_{22}R_{ij,2}\right)_{,2}\,. \tag{6.9}$$

To be compatible with the expression of the diffusive term in the k equation (obtained by tensor contraction of T_{ijk}) it is necessary that $c'_s \approx 1.8c_s$ i.e. $c'_s \approx 0.22$.

With the aim of studying turbulent flows with free surface, in the vicinity of which the normal velocity fluctuation vanishes while the fluctuation parallel to the free surface remains non-zero, Magnaudet J., [MAG 93], is led to introduce a more complex formulation for the triple correlation involving multiple gradients:

$$
\begin{aligned}
T_{ijk} = -\frac{k}{\varepsilon}\Big[& c_s \left(R_{il}R_{jk,l} + R_{jl}R_{ik,l} + R_{kl}R_{ij,l} \right) \\
& -c_{s2}\left(R_{ij}R_{kl,l} + R_{ik}R_{jl,l} + R_{jk}R_{il,l} \right) \\
& +\frac{1}{k}\left(R_{ij}R_{kl} + R_{ik}R_{jl} + R_{jk}R_{il} \right)\left(c_{s3}k_{,l} + c_{s4}\frac{k}{\varepsilon}\varepsilon_{,l} \right)\Big].
\end{aligned}
$$

The study of the shearless turbulent mixing layer suggests that the turbulent diffusivity is higher in the case of vanishing shear than in usual shear flows (cf. [SHA 92 and 94], [WEN 03]). This feature is not fully explained but the phenomenon of "shear sheltering" exhibited by Hunt J.C.R. and Durbin P.A. [HUN 99] could help to explain the difference in these behaviors.

6.3.2. Pressure diffusion

Very little is known about diffusion due to pressure fluctuations. These correlations are not directly attainable by measurements using present time experimental means.

Hirt C.W., [HIR 69] has interpreted $\overline{pu_j}$ as the work done by the fluctuating pressure in direction j and is thus led to assume that this correlation is proportional to the turbulent energy gradient on component j.

$$
\frac{1}{\rho}\overline{pu_j} = \frac{k^2}{\varepsilon}R_{jm,m} . \tag{6.10}
$$

Lumley J.L., [LUM 75B] suggested another decomposition of the pressure terms, and put forward the idea that the form of the redistributive term must be:

$$
-\frac{1}{\rho}\left(\overline{p_{,i}u_j} + \overline{p_{,j}u_i} \right) + \frac{2}{3\rho}\left(\overline{pu_k} \right)_{,k} \delta_{ij} ,
$$

and that consequently the diffusive part becomes $-\dfrac{2}{3\rho}\left(\overline{pu_k} \right)_{,k} \delta_{ij} ,$

and thus the turbulent diffusion by pressure would not appear for the shear components of the stress tensor. Lumley thus suggests the hypothesis:

$$\frac{1}{\rho}\overline{pu_j} = -\frac{2}{5}\overline{u_j u_l u_l} \; .$$

Nevertheless most of the authors have neglected the contribution of the turbulent diffusion term due to pressure, mainly because of the lack of experimental data. This term has been neglected in the LRR model.

Kim S.K. and Chung M.K., [KIM 95] introduce the concept of convective velocity and estimate making use of the turbulence intensity $\mathcal{T}$, in order to model the correlation $\overline{p'u_j}$. They thus get the following model:

$$\frac{\overline{pu_i}}{\rho} = c_p \left[c_{c1} e^{-c_{c2}\mathcal{T}} \frac{\overline{u_l u_m u_m}}{\varepsilon} \overline{U}_{j,l} R_{ij} + \frac{1}{2}\overline{u_i u_j u_j} \right],$$

$$c_p = 0.4, \; c_{c1} = 0.28, \; c_{c2} = 0.6, \; \mathcal{T} = \sqrt{\frac{k}{\overline{U}^2}} \; .$$

6.4. Pressure-strain correlations

6.4.1. *Integral expression*

The essential feature of the pressure-strain correlation (c) in equation [6.1] is the energy redistributing effect among the different components of the Reynolds stress tensor (cf. Hinze J.O., 1975).

It is possible to eliminate pressure in this correlation by using the Poisson equation governing the fluctuating pressure. This Poisson equation is obtained by taking the divergence of the fluctuating velocity equation (equation [2.6], Chapter 2); it allows us to relate the pressure fluctuation p to the velocity fluctuation u_i .

$$\frac{1}{\rho} p_{,ii} = -2u_{i,j}\overline{U}_{j,i} - \left(u_i u_j - R_{ij} \right)_{,ij} \tag{6.11}$$

The general solution of this equation is given by:

$$\frac{p}{\rho}(\vec{x}) = \frac{1}{2\pi}\iiint_\Omega \overline{U}_{m,k} u_{k,m} \frac{d\mathcal{V}}{r} - \frac{1}{4\pi}\iiint_\Omega \left(R_{mk} - u_m u_k \right)_{,mk} \frac{d\mathcal{V}}{r}$$

$$+\frac{1}{4\pi\rho}\iint_{\partial\Omega}\left[\frac{1}{r}\frac{\partial p}{\partial n}-p\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right].d\mathscr{S} \qquad\qquad [6.12]$$

where Ω stands for the whole flow domain and $\partial\Omega$ its boundary, $d\mathscr{V}=dy_1dy_2dy_3$ the volume element and $d\mathscr{S}$ the surface element.

The integrands are evaluated at current point $\vec{y}$ and r stands for the distance between points $\vec{x}$ and $\vec{y}$: $r=\left\|\vec{x}-\vec{y}\right\|$, $\dfrac{\partial}{\partial n}$ is the normal derivative at the boundary.

This relation reminds us that the Navier-Stokes equations are indeed integro-differential equations (cf. Favre A. *et al.*, [*FAV 76]).

NOTE: We mention the Green formulae (cf. Courant R. and Hilbert D., [*COU 66]):

Let u and v be two continuous functions, derivable into $\Omega+\partial\Omega$ and with continuous derivatives, which satisfy the two Green formulae:

$$\iiint_{\Omega}u_{,i}v_{,i}d\mathscr{V}+\iiint_{\Omega}v\Delta u\,d\mathscr{V}=\iint_{\partial\Omega}v\frac{\partial u}{\partial n}d\mathscr{S}$$

$$\iiint_{\Omega}\left(u\Delta v-v\Delta u\right)d\mathscr{V}=\iint_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)d\mathscr{S}$$

If v exhibits a singularity of *1/r* type and point $\vec{p(x)}\in\Omega$, we denote $v(\vec{y})=1/r+w(\vec{y})$, $r=\left\|\vec{x}-\vec{y}\right\|$

w being two times differentiable, the previous formulae are modified according to:

$$\iiint_{\Omega}u_{,i}v_{,i}d\mathscr{V}+\iiint_{\Omega}v\Delta w\,d\mathscr{V}=\alpha u+\iint_{\partial\Omega}v\frac{\partial u}{\partial n}d\mathscr{S}$$

$$\iiint_{\Omega}\left(u\Delta w-v\Delta u\right)d\mathscr{V}=\alpha u+\iint_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)d\mathscr{S}$$

$$\alpha=\begin{cases}4\pi\ if\ P\in\Omega\\2\pi\ if\ P\in\partial\Omega\\0\ \ if\ P\,is\,outside\ \Omega+\partial\Omega\end{cases}$$

Considering now the particular case $w = 0$, the latter formula thus reads:

$$\vec{u(x)}_{\vec{x}\in\Omega} = -\frac{1}{4\pi}\iiint_{\Omega}\frac{\Delta u}{r}d\mathcal{V} + \frac{1}{4\pi}\iint_{\partial\Omega}\left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right].d\mathcal{S}$$

This relation expresses that on the physical point of view, the function u can be represented as the superposition on volumic density potential function, a simple layer potential and a double layer potential (dipolar) on the boundary.

By multiplying the two sides of equation [6.12] by $u_{i,j}(\vec{x})$ and then taking the mean value, we get:

$$\overline{\left(\frac{p}{\rho}u_{i,j}\right)}(\vec{x}) = \frac{1}{2\pi}\iiint_{\Omega}\overline{U'_{m,k}u'_{k,m}u_{i,j}}\frac{d\mathcal{V}}{r} + \frac{1}{4\pi}\iiint_{\Omega}\overline{(u_m u_k)'_{,mk}u_{i,j}}\frac{d\mathcal{V}}{r}$$

$$+ \frac{1}{4\pi\rho}\iint_{\partial\Omega}\overline{\left[\frac{1}{r}\frac{\partial p}{\partial n} - p\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]'.u_{i,j}}\ d\mathcal{S} \qquad [6.13]$$

The quantities denoted with primes are taken at current point $\vec{y}$ and the quantities denoted without prime are taken at point $\vec{x}$.

The contribution $\phi_{ij}^{(1)} = \frac{1}{4\pi}\iiint_{\Omega}\overline{(u_m u_k)'_{,mk}u_{i,j}}\frac{d\mathcal{V}}{r}$ is caused by purely turbulent interactions.

The contribution $\phi_{ij}^{(2)} = \frac{1}{2\pi}\iiint_{\Omega}\overline{U'_{m,k}u'_{k,m}u_{i,j}}\frac{d\mathcal{V}}{r}$ is caused by interactions between turbulence and mean velocity gradients.

The term $\phi_{ij}^{(s)} = \frac{1}{4\pi\rho}\iint_{\partial\Omega}\overline{\left[\frac{1}{r}\frac{\partial p}{\partial n} - p\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]'.u_{i,j}}\ d\mathcal{S}$ is a surface integral which is influential only when a wall is present.

So, the use of the formal solution of the Poisson equation deduced from the Green formulae in potential theory, allows us to build an integral expression in which only the velocity components are involved through its two point correlations. The surface integral has to be taken into account only if the domain is limited by an external boundary (this is particularly the case of confined flows with walls). In homogenous flows (HAT), only the volume integrals are involved. Moreover, in this

case the pressure field can be resolved in spectral space (cf. sections 3.7.3 and 5.1). Thus, taking the integral of [6.13] brings us again to a one point correlation.

When r is sufficiently large, the correlations appearing in [6.13] vanish. If we assume that the non-homogenity is weak and can be neglected locally, we can then write:

$$\left(\overline{\frac{p}{\rho} u_{i,j}}\right)(\vec{x}) = -\frac{1}{2\pi} \iiint_\Omega \frac{\partial \overline{U}'_m}{\partial \xi_k} \frac{\partial^2 \overline{u'_k u_i}}{\partial \xi_m \partial \xi_j} \frac{d\mathscr{V}}{r} - \frac{1}{4\pi} \iiint_\Omega \frac{\partial^3 \overline{u'_m u'_k u_i}}{\partial \xi_m \partial \xi_k \partial \xi_j} \frac{d\mathscr{V}}{r}$$

$$+\frac{1}{4\pi\rho} \iint_{\partial\Omega}\left[\frac{1}{r}\frac{\partial}{\partial n}\overline{p'u_{i,j}} - \overline{p'u_{i,j}}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\mathscr{S} \,, \qquad [6.14]$$

where $\vec{\xi} = \vec{y} - \vec{x}$.

Indeed, in homogenous turbulence, $\overline{\dfrac{\partial u'_k}{\partial y_m} \dfrac{\partial u_i}{\partial x_j}} = \dfrac{\partial^2 \overline{u'_k u_i}}{\partial y_m \partial x_j}$, where $u'_k = u_k(\vec{y})$

and $u_i = u_i(\vec{x})$.

This result can be proven using a variable change:

$$\begin{cases} \xi_h = y_h - x_h \\ X_h = x_h \end{cases} \begin{vmatrix} \dfrac{\partial}{\partial y_h} = \dfrac{\partial}{\partial \xi_h} \\ \dfrac{\partial}{\partial x_h} = \dfrac{\partial}{\partial X_h} - \dfrac{\partial}{\partial \xi_h} \end{vmatrix} \Rightarrow \frac{\partial}{\partial \xi_m}\left(\frac{\partial}{\partial X_j} - \frac{\partial}{\partial \xi_j}\right)\overline{u'_k u_i} = -\frac{\partial^2 \overline{u'_k u_i}}{\partial \xi_m \partial \xi_j} \,.$$

We now consider separately and successively each contribution $\phi_{ij}^{(1)}$, $\phi_{ij}^{(2)}$ and $\phi_{ij}^{(s)}$.

6.4.2. *Modeling in homogenous turbulence*

1) Non-linear part $\phi_{ij}^{(1)}$

The term $\Phi_{ij}^{(1)} = \left(\phi_{ij}^{(1)} + \phi_{ji}^{(1)}\right)$ favors the return to isotropy through the effect of non-linear interactions. The Rotta J.C. hypothesis, [ROT 51A and B] accounts for this effect:

$$\Phi_{ij}^{(1)} = -c_1 \frac{\varepsilon}{k} \left(R_{ij} - \frac{2}{3} k \delta_{ij} \right), \qquad\qquad [6.15]$$

coefficient c_1 being a numerical constant which determines the speed at which the return to isotropy occurs and k/ε representing the characteristic time scale of the phenomenon.

We can show that the realizability requires $c_1 > 1.0$. Indeed, if we consider the decay of turbulence in the absence of any mean velocity gradient:

$$\frac{dR_{ij}}{dt} = -c_1 \frac{\varepsilon}{k} R_{ij} + \frac{2}{3}(c_1 - 1)\varepsilon \delta_{ij} \, .$$

For $i=j$, the condition $\dfrac{dR_{ij}}{dt} \geq 0$ when $R_{ij} \to 0$ implies $c_1 \geq 1$.

Approximation [6.15] is the most often used and it has also been retained in the LRR model ([LAU 75C]).

Lumley J.L. and Newmann G.R., [LUM 77] (cf. also section 4.5) have proposed a more elaborate modeling by introducing higher order terms (cf. Chapter 8):

$$\Phi_{ij}^{(1)} = \varepsilon \left[c'_1(II, III)a_{ij} + c''_1(II, III)\left(a_{ik}a_{kj} - \frac{1}{3} II \delta_{ij} \right) \right] \, . \qquad [6.16]$$

2) Linear part $\phi_{ij}^{(2)}$

The term $\Phi_{ij}^{(2)} = \left(\phi_{ij}^{(2)} + \phi_{ji}^{(2)} \right)$ produces a return to isotropy of turbulence through the effect of interactions between turbulent fluctuations and mean velocity gradients.

Townsend A.A., [TOW 54] and Crow J., [CRO 68A] have shown that the effect of $\phi_{ij}^{(2)}$ may be stronger than the effect of $\phi_{ij}^{(1)}$ in the case of a rapid distortion for the pressure-velocity correlation. Naot D., Shavit A. and Wolfshtein M., [NAO 73] have approximated this term by inserting in the integral expression of $\phi_{ij}^{(2)}$ the relations for two point velocity correlations in isotropic turbulence and then integrating in space (quasi-isotropic approximation). Launder B.E. *et al.*, [LAU 75C] have achieved the same result using a different and simpler method. It is this

latter approach which is described now. The integral expression deduced from [6.14] is written as:

$$\Phi_{ij}^{(2)} = -\frac{1}{2\pi} \iiint_\Omega \frac{\partial \overline{U'}_m}{\partial \xi_k} \left[\frac{\partial^2 \overline{u'_k u_i}}{\partial \xi_m \partial \xi_j} + \frac{\partial^2 \overline{u'_k u_j}}{\partial \xi_m \partial \xi_i} \right] \frac{d\mathscr{V}}{r} \qquad [6.17]$$

where $d\mathscr{V} = d\xi_1 d\xi_2 d\xi_3$ is the volume element.

Rotta J.C., [ROT 51A and B] had introduced the following approximation:

$$\phi_{ij}^{(2)} = a_{lj}^{mi} \overline{U}_{l,m} \quad \text{with} \quad a_{lj}^{mi} = -\frac{1}{2\pi} \iiint_\Omega \frac{\partial^2 \overline{u'_m u_i}}{\partial \xi_l \partial \xi_j} \frac{d\mathscr{V}}{r} \qquad [6.18]$$

This relation is exact in homogenous turbulence. In the case of homogenous turbulence equation [5.10] can be compared to [6.18] in order to show that the following relation also holds $a_{lj}^{mi} = \iiint_{\mathbb{R}^3} \frac{\kappa_l \kappa_j}{\kappa^2} \varphi_{mi} d\vec{\kappa}$. In the case of non-homogenous turbulence, considering that the two point double correlations vanish at high ξ, they can be considered as an acceptable approximation when the second derivatives of mean velocity are negligible. Rotta J.C., [ROT 51A and B] has shown that the fourth order tensor a_{lj}^{mi} must satisfy some kinematic constraints which are detailed hereafter:

1) Symmetry: $a_{lj}^{mi} = a_{lj}^{im} = a_{jl}^{im}$, $\qquad\qquad\qquad$ [6.19]

Indeed, in homogenous turbulence $\overline{u'_k u_i}(\vec{\xi}) = \overline{u'_i u_k}(-\vec{\xi})$ and the integrals extended in $\mathbb{R}^3$ for $\overline{u'_k u_i}(\vec{\xi})$ and $\overline{u'_i u_k}(\vec{\xi})$ will be identical.

2) Continuity: $a_{li}^{mi} = 0$, $\qquad\qquad\qquad\qquad\qquad$ [6.20]

This is a consequence of zero divergence of the velocity fluctuations: $\partial u_i / \partial \xi_i = 0$.

3) Consequence of Green theorem: $a_{jj}^{mi} = 2R_{mi}$, $\qquad\qquad$ [6.21]

Indeed, $a_{jj}^{mi} = -\dfrac{1}{2\pi}\iiint_{\Omega} \overline{\Delta u'_m u_i}\,\dfrac{d\mathscr{V}}{r} = 2\left(\overline{u'_m u_i}\right)(0) = 2R_{mi}$.

This latter equation suggests that a_{jj}^{mi} can be approximated by a linear combination of components of the Reynolds stress tensor. The most general tensor satisfying the symmetry constraints [6.19] can be written in the form:

$$a_{lj}^{mi} = \alpha\delta_{lj}R_{mi} + \beta(\delta_{ml}R_{ij} + \delta_{mj}R_{il} + \delta_{il}R_{mj} + \delta_{ij}R_{ml}) + c_2\delta_{mi}R_{lj}$$
$$k\left(\eta\delta_{mi}\delta_{lj} + \vartheta(\delta_{ml}\delta_{ij} + \delta_{mj}\delta_{il})\right) \qquad [6.22]$$

where α, β, c_2, η and ϑ are numerical constants.

We can easily deduce expression [6.22] from the methods presented in section 3.6. Condition [6.20] implies the following relations between coefficients:

$$\begin{cases} \alpha + 5\beta + c_2 = 0 \\ 2\beta + \eta + 4\vartheta = 0 \end{cases} \qquad [6.23]$$

Likewise, condition [6.21] implies the relations:

$$\begin{cases} 3\alpha + 4\beta = 2 \\ 2c_2 + 3\eta + 2\vartheta = 0 \end{cases} \qquad [6.24]$$

It is thus possible to express all the coefficients according to only one of them, c_2 for example. We find:

$$\alpha = \frac{4c_2 + 10}{11}, \quad \beta = -\frac{3c_2 + 2}{11}, \quad \eta = -\frac{50c_2 + 4}{55}, \quad \vartheta = \frac{20c_2 + 6}{55} \qquad [6.25]$$

Thus, there now remains a single modeling constant to determine. Combining [6.22] and [6.25] we find for $\Phi_{ij}^{(2)}$, after rearranging the terms:

$$\Phi_{ij}^{(2)} = \frac{c_2 + 8}{11}\left(R_{mi}\overline{U}_{j,m} + R_{mj}\overline{U}_{i,m}\right) + \frac{8c_2 - 2}{11}\left(R_{il}\overline{U}_{l,j} + R_{jl}\overline{U}_{l,i}\right)$$
$$-2\frac{3c_2 + 2}{11}R_{ml}\overline{U}_{l,m}\delta_{ij} + \frac{2 - 30c_2}{55}k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right) \qquad [6.26]$$

Introducing the notations:

$$P_{ij} = -R_{mi}\overline{U}_{j,m} - R_{mj}\overline{U}_{i,m}$$

$$D_{ij} = -R_{mi}\overline{U}_{m,j} - R_{mj}\overline{U}_{m,i}$$

with

$$P_{ii} = D_{ii} = 2P, \quad P = -R_{ij}\overline{U}_{i,j}$$

it is possible to write [6.26] under the compact form:

$$\Phi_{ij}^{(2)} = -\frac{c_2 + 8}{11}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) - \frac{8c_2 - 2}{11}\left(D_{ij} - \frac{2}{3}P\delta_{ij}\right)$$
$$-\frac{30c_2 - 2}{55}k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right) \tag{6.27}$$

Note 1: if R_{ij} is isotropic,

$$R_{ij} = 2/3k\delta_{ij} \text{ and } P_{ij} = -\frac{2}{3}k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right), \quad D_{ij} = -\frac{2}{3}k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right),$$

whereas *P = 0*.

The non-linear term is zero $\Phi_{ij}^{(1)} = 0$ but considering [6.27] we find for the linear term $\Phi_{ij}^{(2)} = \frac{2}{5}k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right)$.

Thus, in the case of isotropic turbulence submitted to sudden distortion, the value of Φ_{ij} is no longer dependent on c_D and reduces to:

$$\Phi_{ij} = 0.4k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right), \tag{6.28}$$

and this is exactly the result obtained by Crow J., [CRO 68A] from a more elaborate analysis. This result has been also demonstrated in Chapter 5 with another approach.

Note 2: Launder B.E., Reece G.J. and Rodi W., [LAU 75C] mention that in expression [6.27] each of the three groups of terms vanishes through tensorial contraction and has the essential property of redistributing energy. The first group

appeared as a leading term, and considering its clear physical meaning, it may be interesting to discover to what extent this term alone is able to account for the effect of strain on the pressure correlation. These authors then introduce the simplified closure:

$$\Phi_{ij}^{(2)} = -\gamma\left(P_{ij} - \frac{2}{3}P\delta_{ij} \right),$$

[6.29]

in which γ is different from $\dfrac{c_2 + 8}{11}$, in order to compensate for the neglected terms. This scheme is called "isotropization of production" (IP).

In isotropic turbulence we recover [6.28] for $\gamma = 0.6$.

6.4.3. *Near wall modeling*

It is necessary to model the surface integral also called the wall echo term:
$$\Phi_{ij}^{(s)} = \left(\phi_{ij}^{(s)} + \phi_{ji}^{(s)} \right).$$

It is admitted that the influence of the surface integral extends to distances from the wall which are approximately of the order of the integral macroscale, Bradshaw P., [BRA 73]. For example, in a turbulent boundary layer on a flat plate, this term will be significant in the region $0 < y < \delta/4$ approximately where δ is the boundary layer thickness.

This term is redistributive because its trace is zero, it redistributes energy among the normal components of the Reynolds stresses.

Daly B.J. and Harlow F.H., [DAL 70] have proposed the following approximation:

$$\Phi_{ij}^{(s)} = c_w \frac{\varepsilon}{k}\left(W_{il}R_{lj} + W_{jl}R_{li} - \frac{2}{3}W_{lm}R_{lm}\delta_{ij} \right),$$

[6.30]

in which $W_{il}(\vec{r})$ is the wall effect tensor given by:

$$W_{il}(\vec{r}) = \int_S n_i(\vec{z})n_l(\vec{z})F\left(\frac{s(\vec{r})}{\left\| \vec{r} - \vec{z} \right\|} \right)\frac{dS(\vec{z})}{\left\| \vec{r} - \vec{z} \right\|^2},$$

and where $s(\vec{r})$ denotes a microscale of turbulence, $\vec{r}$ the position of the point under consideration and $n_i(\vec{z})$ the unit vector normal to the wall at the position $\vec{z}$.

For calculating the turbulent flow between two parallel plates, Daly and Harlow used a simplified form such as $F = (s/y)^2$, y being the distance along the normal to the wall.

Shir C.C., [SHI 73] has replaced the wall effect tensor of Daly and Harlow by a scalar function of the r from the wall relative to the integral scale ℓ.

$$W_{hm} = n_h n_m \Psi(\ell / r)$$

Thus, in the case of the turbulent flow between two parallel plates or in the boundary layer on a flat plate:

$$\Phi_{ij}^{(s)} = c_w \frac{\varepsilon}{k}\left(n_i n_m R_{mj} + n_j n_m R_{mi} - \frac{2}{3} n_m n_h R_{mh}\delta_{ij} \right) f\left(\frac{\ell}{y}\right), \qquad [6.31]$$

where n_i denotes the unit normal vector and y the normal distance normal to the wall.

For example, if we assume $f \approx 1$ in the boundary layer:

$$\vec{n}\begin{cases} 0 \\ 1 \\ 0 \end{cases} \Rightarrow \Phi_{11}^{(s)} = \Phi_{33}^{(s)} = -\frac{2}{3}c_w \frac{\varepsilon}{k} R_{22}, \qquad \Phi_{22}^{(s)} = \frac{4}{3}c_w \frac{\varepsilon}{k} R_{22}.$$

This latter result is, however, not in agreement with experimental data which rather suggests that R_{33} is practically not modified by the presence of the wall.

The approach of Launder B.E., Reece G.J. and Rodi W., [LAU 75C] is now presented in more details. In the case of a plane wall relation [6.13] can be favorably written into the following equivalent form which no longer involves the surface integral:

$$\phi_{ij} = \frac{1}{4\pi}\iiint_\Omega \left[\frac{\partial^2 (u'_l u'_m)}{\partial y_l \partial y_m} \frac{\partial u_i}{\partial x_j} + 2\left(\frac{\partial \overline{U}_1}{\partial y_2}\right)' \frac{\overline{\partial u'_2 \partial u_i}}{\partial y_1 \partial x_j} \right] \times \left[\frac{1}{\|\vec{x} - \vec{y}\|} + \frac{1}{\|\vec{x} - \overline{y*}\|} \right] d\mathcal{V},$$

$$[6.32]$$

where $\vec{y*}$ is the image of point $\vec{y}$ by symmetry about the wall (cf. Figure 6.2).
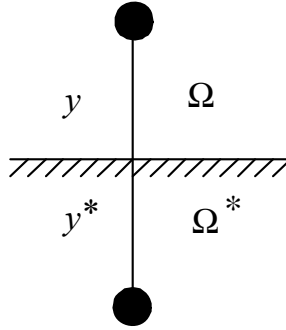
Figure 6.2. *Reflected contribution of the pressure term near a wall*

Equation [6.32] is obtained from a mathematical transformation which allows us to replace the surface integral by a volume integral more easy to grasp and thus more easy to model.

Indeed, if the function φ is symmetric about the plane separating Ω from Ω^*,

$$\varphi = -\frac{1}{4\pi}\iiint_{\Omega+\Omega^*}\frac{\Delta\varphi}{r}d\mathscr{V}$$

$$\varphi = -\frac{1}{4\pi}\iiint_{\Omega}\frac{\Delta\varphi}{r}d\mathscr{V} -\frac{1}{4\pi}\iiint_{\Omega^*}\frac{\Delta\varphi}{r}d\mathscr{V}$$

$$= -\frac{1}{4\pi}\iiint_{\Omega}\frac{\Delta\varphi}{r}d\mathscr{V} -\frac{1}{4\pi}\iiint_{\Omega}\frac{\Delta\varphi}{r*}d\mathscr{V}$$

Form [6.32] immediately suggests that there are two contributions to the wall effect, from the reflected influence of $\phi_{ij}^{(1)}$ and $\phi_{ij}^{(2)}$. The experiments show that the wall acts towards an increase of anisotropy of the normal stresses but also to diminish the shear stress. We are thus led to assume:

$$\Phi_{ij}^{(s)} = \left[c'_1\frac{\varepsilon}{k}\left(R_{ij} -\frac{2}{3}k\delta_{ij}\right)+\left(b_{lj}^{mi} +b_{li}^{mj}\right)\overline{U}_{l,m}\right]f\left(\frac{\ell}{x_2}\right) \qquad [6.33]$$

where ℓ denotes the scale of energy-containing eddies.

The experiment suggests that the wall effect extends to regions where the strain rate is negligible. Thus, a term similar to $\Phi_{ij}^{(1)}$ is necessary to account for this latter effect which remains finite even when the strain rate vanishes. Moreover, a term similar to $\Phi_{ij}^{(2)}$ is necessary to account for the anisotropy of the normal stresses and the reduction of shear stress.

Exactly like a_{lj}^{mi}, the quantity b_{lj}^{mi} is a symmetric tensor expressed as a linear combination of the components of the Reynolds stress tensor.

$$b_{lj}^{mi} = \alpha'\delta_{lj}R_{mi} + \beta'(\delta_{ml}R_{ij} + \delta_{mj}R_{il} + \delta_{il}R_{mj} + \delta_{ij}R_{ml}) + c'_2\,\delta_{mi}R_{lj}$$
$$+k\left(\eta'\delta_{mi}\delta_{lj} + \vartheta'(\delta_{ml}\delta_{ij} + \delta_{mj}\delta_{il})\right). \qquad [6.34]$$

This tensor must also satisfy:

$$b_{li}^{mi} = 0 \quad \text{because } \Phi_{ij}^{(s)} \text{ must also be redistributive.} \qquad [6.35]$$

Condition [6.35] implies relations between the coefficients appearing in [6.34]:

$$\begin{cases} \alpha' + 5\beta' + c'_2 = 0 \\ 2\beta' + \eta' + 4\vartheta' = 0 \end{cases} \qquad [6.36]$$

The experiment in the boundary layer shows that R_{33} remains uninfluenced by the presence of a wall. We then assume $b_{13}^{23} = 0$ (component which appears in the term $b_{13}^{23}\overline{U}_{1,2}$).

Note: the contribution of $c'_1\dfrac{\varepsilon}{k}\left(R_{33} - \dfrac{2}{3}k\delta_{ij}\right)$ is not important in this case, because R_{33} is near $2k/3$.

This condition implies:

$$b_{13}^{23} = \beta'R_{12} = 0 \qquad \Rightarrow \beta' = 0. \qquad [6.37]$$

After [6.36] we thus get: $\begin{cases} \alpha' = -c'_2 \\ \vartheta' = -\eta'/4 \end{cases}$ and the following result:

$$b_{lj}^{mi} = -c'_2\, \delta_{lj} R_{mi} + c'_2\, \delta_{mi} R_{lj} + k\left[\eta'\delta_{mi}\delta_{lj} - \frac{\eta'}{4}\left(\delta_{ml}\delta_{ij} + \delta_{mj}\delta_{il}\right)\right], \qquad [6.38]$$

and thus:

$$\left(b_{lj}^{mi} + b_{li}^{mj}\right)\overline{U}_{l,m} = c'_2\left(P_{ij} - D_{ij}\right) + \zeta'k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right) \qquad [6.39]$$

where $\zeta' = 3\eta'/4$.

Finally, we arrive at the expression:

$$\Phi_{ij}^{(s)} = \left[c'_1\frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right) + c'_2\left(P_{ij} - D_{ij}\right) + \zeta'k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right)\right]f(\frac{\ell}{x_2}) \quad [6.40]$$

In this expression, the function $f(\frac{\ell}{x_2})$ must go to zero when $x_2 \to \infty$.

Launder B.E. *et al.*, [LAU 75C] suppose that $f(\frac{\ell}{x_2})$ is directly proportional to

$\frac{\ell}{x_2}$, i.e.,

$$f(\frac{\ell}{x_2}) = C\frac{\ell}{x_2} \qquad [6.41]$$

The constant C is chosen in such a way that f goes to unity in wall turbulence (in the logarithmic layer). The length scale ℓ is considered as a characteristic length equal to $k^{3/2}/\varepsilon$, and thus:

$$f(\frac{\ell}{x_2}) = C\frac{k^{3/2}}{\varepsilon x_2}.$$

In wall turbulence, the mean velocity profile $\overline{U}_1$ is given by the logarithmic law:

$$\overline{U}_1 = \frac{u_*}{K}\mathrm{Log}(Ex_2^+),$$

(K, Karman constant) and thus,

$$\overline{U}_{1,2} = \frac{u_*}{Kx_2}, \quad P = \varepsilon = -R_{12}\overline{U}_{1,2} = \frac{u_*^3}{Kx_2}.$$

The condition $f=1$ is written as $CK\dfrac{k^{3/2}}{u_*^3} = 1$, i.e. $C = \dfrac{u_*^3}{K.k^{3/2}}$.

The ratio $-\dfrac{R_{12}}{k} = \dfrac{u_*^2}{k}$ keeping a constant value denoted a, we thus obtain

$C = \dfrac{a^{3/2}}{K}$, and thus:

$$f\left(\frac{\ell}{x_2}\right) = \frac{a^{3/2}}{K}\frac{k^{3/2}}{\varepsilon.x_2}, \quad \text{or} \quad f\left(\frac{\ell}{x_2}\right) \simeq 0.4\frac{k^{3/2}}{\varepsilon.x_2} \qquad [6.42]$$

Note 1: the previous relations imply $-R_{12} = a^3\dfrac{k^3}{R_{12}^2}\dfrac{\overline{U}_{1,2}^2}{\overline{U}_{1,2}^2} = a^3\dfrac{k^3\overline{U}_{1,2}^2}{\varepsilon^3}$ and

$-R_{12} = \left(a^{3/2}\ell\right)^2 \overline{U}_{1,2}^2$.

We see that in the logarithmic zone (in dynamic equilibrium $P = \varepsilon$) the scale $a^{3/2}\ell$ behaves like the mixing length (cf. Chapter 12). Let us note however that the mixing length hypothesis is by no means necessary to establish [6.42].

Note 2: in the general case of flows limited by curved walls or irregular and multiple walls, the extension of the previous methods becomes very complex. In fact, the fluctuating pressure, solution of the Poisson equation $p_{,jj} = -2\rho u_{i,j}\overline{U}_{j,i}$ with boundary conditions $\partial p/\partial n|_{\partial\Omega} = 0$, can still be obtained in principle by an integral relation such as:

$$\frac{p}{\rho}(\vec{x}) = -\iiint_\Omega u_{i,j}(\vec{y})\overline{U}_{j,i}(\vec{y}).\mathscr{G}(\vec{x},\vec{y})dV(\vec{y})$$

where $\mathscr{G}(\vec{x},\vec{y})$ is the Green function ([*COU 66], [*DAU 84]) on the domain Ω which satisfies equation $\mathscr{G}(\vec{x},\vec{y})_{,jj} = \delta(\vec{x}-\vec{y})$ in the sense of Schwartz distributions

([*SCH 84]), with the boundary conditions $\partial \mathscr{G}(\vec{x}, \vec{y})/\partial n\big|_{\partial \Omega} = 0$. This approach is far too complex in practice and approximate techniques are generally used by introducing a generalized distance in equations [6.40].

Thus, Launder B.E. *et al.*, [LAU 75C] propose to replace $\dfrac{k^{3/2}}{\varepsilon.x_2}$ by coefficient

$\dfrac{1}{2\pi}\int \dfrac{k^{3/2}}{\varepsilon}\dfrac{d\varphi}{\delta}$ where $d\varphi$ is the solid angle element under which we see the subtended surface element located at a distance δ from the point under consideration.

Other modeling approaches

Gibson M.M. and Launder B.E., [GIB 78B] applied the ideas of Shir (see equation [6.31]) and suggested the hypothesis:

$$\Phi_{ij}^{(s)} = C_{w1}\frac{\varepsilon}{k}\left(R_{km}n_k n_m \delta_{ij} - \frac{3}{2}R_{ki}n_k n_j - \frac{3}{2}R_{kj}n_k n_i\right)f\left(\frac{\ell}{n_i r_i}\right)$$

$$C_{w2}\left(\Pi_{km}n_k n_m \delta_{ij} - \frac{3}{2}\Pi_{ki}n_k n_j - \frac{3}{2}\Pi_{kj}n_k n_i\right)f\left(\frac{\ell}{n_i r_i}\right). \qquad [6.43]$$

In this expression, we have denoted $\Pi_{ij} = -0.6\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right)$ with $C_{w1} = 0.5$, $C_{w2} = 0.3$ and $f\left(\dfrac{\ell}{n_i r_i}\right) = C\dfrac{k^{3/2}}{\varepsilon x_2}$, n_i being the unit vector normal, t the wall and r_i the position of the considered point.

The Craft T.J. and Launder B.E. model, [CRA 92], given hereafter, introduces new additional terms.

Its expression is the following:

$$\Phi_{ij}^{(s)} = C_{w1}\frac{\varepsilon}{k}\left(R_{km}n_k n_m \delta_{ij} - \frac{3}{2}R_{ki}n_k n_j - \frac{3}{2}R_{kj}n_k n_i\right)\frac{k^{3/2}}{2.5\varepsilon x_2}$$

$$-C_{w2}\overline{U}_{l,m}R_{lm}\left(\delta_{ij} - 3n_i n_j\right)\frac{k^{3/2}}{2.5\varepsilon x_2}$$

$$-C_{w3}ka_{lm}\left(\overline{U}_{k,m}n_kn_l\delta_{ij} - \frac{3}{2}\overline{U}_{i,m}n_ln_j - \frac{3}{2}\overline{U}_{j,m}n_ln_i\right)\frac{k^{3/2}}{2.5\varepsilon x_2}$$

$$+C_{w4}\overline{U}_{l,m}n_ln_m\left(n_in_j - \frac{1}{3}\delta_{ij}\right)\frac{k^{3/2}}{2.5\varepsilon x_2},$$

with

$$C_{w1} = 0.5, \quad C_{w2} = 0.08, \quad C_{w3} = 0.1, \quad C_{w4} = 0.4.$$

It allows us to correctly reproduce the behavior of an impinging turbulent jet on a wall whereas the earlier Gibson M.M. and Launder B.E. model, [GIB 78B] fails in this difficult case. This hypothesis is applied in conjunction with the RS2 model using the degenerated form for $\Phi_{ij}^{(2)}$.

In addition to the traditional models by Hanjalic K. and Launder B.E., [HAN 72]; Naot D. *et al.*, [NAO 73]; Launder B.E., Reece G.J. and Rodi W., [LAU 75C]; Lumley J.L., [LUM 75C and 78A] we will quote the work by Gallagher B. *et al.*, [GAL 81] for the linear term, based on a fourth order expansion in the components of the eigenvectors of the one point double correlations tensor which includes 81 constants (determined from symmetry and realizability conditions) and also the approach of Speziale C.G., [SPE 79,80] who proposes a scheme satisfying the required invariance property in any change of frame of reference (Euclidian transformation group). Furthermore, Lumley J.L., [LUM 78A] and also Leslie D.C., [LES 80] show that the traditional LRR scheme cannot be of complete general use.

The simplest approach for the non-linear term of return to isotropy remains the one proposed by Rotta. Lumley J.L. and Newman G.R., [LUM 77] (cf. section 4.6) have proposed a more rigorous modeling taking into account realizability conditions. Also, Gence J.N. and Mathieu J., [GEN 80] show the importance of the invariant *III* for describing the return to isotropy. The study by Weinstock J., [WEI 81 and 82] shows however that modeling the non-linear terms can only find a satisfactory solution in spectral context.

The present directions of research and trends, in the aim to improve the basic models, lead to non-linear closures which allow better account of realizability (cf. Chapter 7).

6.5. Determination of numerical constants

a) Constants c_1 and c_2

Constants c_1 and c_2 are determined by reference to the homogenous turbulent flow with uniform mean velocity gradient. This flow has been studied experimentally in particular by Champagne F.H., Harris V.G. and Corrsin S., [CHA 70] (cf. Figure 5.2).

With hypotheses [6.3], [6.15], [6.27] the Reynolds stress equations can be written in the present case (using the notation $\Lambda_{12} = \overline{U}_{1,2}$ constant):

$$0 = -2R_{12}\Lambda_{12} - \frac{2}{3}\varepsilon - c_1\frac{\varepsilon}{k}\left(R_{11} - \frac{2}{3}k\right)$$
$$-\frac{c_2+8}{11}\left(-2R_{12}\Lambda_{12} + \frac{2}{3}R_{12}\Lambda_{12}\right) - \frac{8c_2-2}{11}\left(\frac{2}{3}R_{12}\Lambda_{12}\right) \quad [6.44]$$

$$0 = -\frac{2}{3}\varepsilon - c_1\frac{\varepsilon}{k}\left(R_{22} - \frac{2}{3}k\right) - \frac{c_2+8}{11}\left(\frac{2}{3}R_{12}\Lambda_{12}\right)$$
$$-\frac{8c_2-2}{11}\left(-2R_{12}\Lambda_{12} + \frac{2}{3}R_{12}\Lambda_{12}\right) \quad [6.45]$$

$$0 = -\frac{2}{3}\varepsilon - c_1\frac{\varepsilon}{k}\left(R_{33} - \frac{2}{3}k\right) - \frac{c_2+8}{11}\left(\frac{2}{3}R_{12}\Lambda_{12}\right)$$
$$-\frac{8c_2-2}{11}\left(\frac{2}{3}R_{12}\Lambda_{12}\right) \quad [6.46]$$

$$0 = -R_{22}\Lambda_{12} - c_1\frac{\varepsilon}{k}R_{12} - \frac{c_2+8}{11}\left(-R_{22}\Lambda_{12}\right)$$
$$-\frac{8c_2-2}{11}\left(-R_{11}\Lambda_{12}\right) - \frac{30c_2-2}{55}\left(k\Lambda_{12}\right) \quad [6.47]$$

Taking into account the kinetic energy equation written as $-R_{12}\Lambda_{12} = \varepsilon$, we easily deduce:

$$\frac{R_{11} - \frac{2}{3}k}{k} = \frac{12c_2+8}{33c_1}, \quad \frac{R_{22} - \frac{2}{3}k}{k} = \frac{2-30c_2}{33c_1}, \quad \frac{R_{33} - \frac{2}{3}k}{k} = \frac{18c_2-10}{33c_1} \quad [6.48]$$

By multiplying both sides of equation [6.47] by R_{12} and also taking into account $-R_{12}\Lambda_{12} = \varepsilon$, we then obtain:

$$\left(\frac{R_{12}}{k}\right)^2 = \frac{3-c_2}{11c_1}\left(\frac{R_{22}-\frac{2}{3}k}{k}\right) - \frac{8c_2-2}{11c_1}\left(\frac{R_{11}-\frac{2}{3}k}{k}\right) + \frac{4}{15c_1} \qquad [6.49]$$

The experimental values of Champagne, Harris and Corrsin are correctly represented by the model if we choose: $c_2 = 0.4$ and $c_1 = 0.5$.

For the simpler closure:

In the framework of hypothesis [6.29] we would obtain in a similar way

$$\frac{R_{11}-\frac{2}{3}k}{k} = \frac{4(1-\gamma)}{3c_1}, \quad \frac{R_{22}-\frac{2}{3}k}{k} = -\frac{2(1-\gamma)}{3c_1}, \quad \frac{R_{33}-\frac{2}{3}k}{k} = -\frac{2(1-\gamma)}{3c_1}. \quad [6.50]$$

	$\dfrac{R_{11}-\frac{2}{3}k}{k}$	$\dfrac{R_{22}-\frac{2}{3}k}{k}$	$\dfrac{R_{33}-\frac{2}{3}k}{k}$	$-\dfrac{R_{12}}{k}$
Champagne, Harris, Corrsin experiment	0.30	−0.18	−0.12	0.33
calculation LRR model	0.26	−0.20	0.6	0.57

Table 6.1. *Anisotropy levels in homogenous turbulent shear flow*

It thus results that $R_{22} = R_{33}$. It is thus not possible to reproduce exactly the experimental values. However, the discrepancies are not sufficiently large for this hypothesis to be disqualified. If we choose $\gamma = 0.6$ (see section 6.4.2) and $c_1 = 1.8$ the experimental values are approached with acceptable approximation.

b) Constants c'$_1$ and c'$_2$

The reference flow will be in this case the turbulent boundary layer on a flat plate (cf. Figure 6.3) in the logarithmic zone (experiment of Klebanoff P.S., [KLE 54]) or also the turbulent flow between two parallel planes (experiment by Comte-Bellot G., [COM 65]). We thus consider the constant shear zone $-R_{12} = u_*^2$.
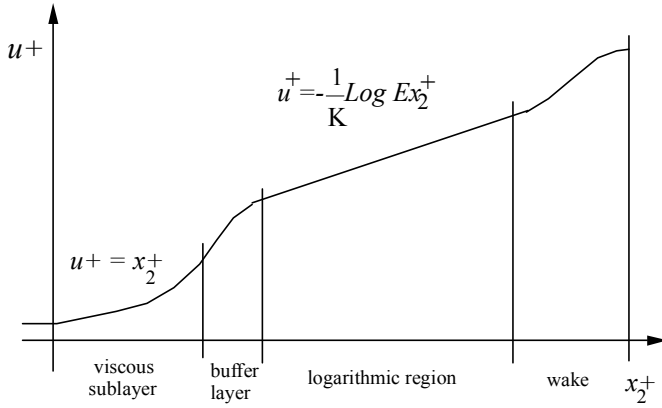
Figure 6.3. *Mean velocity profile in a turbulent boundary layer on a flat plate (with zero pressure gradient)*

with hypotheses [6.3], [6.15], [6.27] and [6.40] the modeled equations for the Reynolds stresses can be written (with $f = 1$):

$$0 = -2R_{12}\overline{U}_{1,2} - \frac{2}{3}\varepsilon - c_1\frac{\varepsilon}{k}\left(R_{11} - \frac{2}{3}k\right) + \frac{c_2+8}{11}\left(\frac{4}{3}R_{12}\overline{U}_{1,2}\right) - \frac{8c_2-2}{11}\left(\frac{2}{3}R_{12}\overline{U}_{1,2}\right)$$

$$+c'_1\frac{\varepsilon}{k}\left(R_{11} - \frac{2}{3}k\right) - 2c'_2\,R_{12}\overline{U}_{1,2}\,, \qquad\qquad [6.51]$$

$$0 = -\frac{2}{3}\varepsilon - c_1\frac{\varepsilon}{k}\left(R_{22} - \frac{2}{3}k\right) - \frac{c_2+8}{11}\left(\frac{2}{3}R_{12}\overline{U}_{1,2}\right) + \frac{8c_2-2}{11}\left(\frac{4}{3}R_{12}\overline{U}_{1,2}\right)$$

$$+c'_1\frac{\varepsilon}{k}\left(R_{22} - \frac{2}{3}k\right) + 2c'_2\,R_{12}\overline{U}_{1,2}\,, \qquad\qquad [6.52]$$

$$0 = -\frac{2}{3}\varepsilon - c_1\frac{\varepsilon}{k}\left(R_{33} - \frac{2}{3}k\right) - \frac{c_2+8}{11}\left(\frac{2}{3}R_{12}\overline{U}_{1,2}\right) - \frac{8c_2-2}{11}\left(\frac{2}{3}R_{12}\overline{U}_{1,2}\right)$$

$$+c'_1\frac{\varepsilon}{k}\left(R_{33} - \frac{2}{3}k\right), \qquad\qquad [6.53]$$

$$0 = -R_{22}\overline{U}_{1,2} - c_1\frac{\varepsilon}{k}R_{12} + \frac{c_2+8}{11}\left(R_{22}\overline{U}_{1,2}\right) + \frac{8c_2-2}{11}\left(R_{11}\overline{U}_{1,2}\right) - \frac{30c_2-2}{55}\left(k\overline{U}_{1,2}\right)$$

$$+c'_1\frac{\varepsilon}{k}R_{12} + c'_2(R_{11} - R_{22})\overline{U}_{1,2} + \zeta'k\overline{U}_{1,2}\,, \qquad\qquad [6.54]$$

R_{11}, R_{22}, R_{33}, R_{12}, being constant, because the turbulent diffusion terms are zero. Taking into account $P = \varepsilon = -R_{12}\overline{U}_{1,2} = \dfrac{u_*^3}{Kx_2}$, we thus show that:

$$\frac{R_{11} - \dfrac{2}{3}k}{k} = \frac{0.39 + 2c'_2}{c_1 - c'_1}, \qquad\qquad \frac{R_{22} - \dfrac{2}{3}k}{k} = -\frac{0.30 + 2c'_2}{c_1 - c'_1},$$

$$\frac{R_{33} - \dfrac{2}{3}k}{k} = -\frac{0.09}{c_1 - c'_1} \qquad\qquad \text{(with } c_2 = 0.4) \qquad\qquad [6.55]$$

By multiplying both sides of equation [6.54] by R_{12} and taking onto account $P = \varepsilon$, we find:

$$(c_1 - c'_1)\left(\frac{R_{12}}{k}\right)^2 = \left(\frac{3 - c_2}{11} + c'_2\right)\left(\frac{R_{22} - \dfrac{2}{3}k}{k}\right)$$

$$-\left(\frac{8c_2 - 2}{11} + c'_2\right)\left(\frac{R_{11} - \dfrac{2}{3}k}{k}\right) + \frac{4}{15} - \zeta'. \qquad\qquad [6.56]$$

The choice $c'_1 = 0.5$ and $c'_2 = 0.06$ with $\zeta' = 0$ allows us to get:

	$\dfrac{R_{11} - \dfrac{2}{3}k}{k}$	$\dfrac{R_{22} - \dfrac{2}{3}k}{k}$	$\dfrac{R_{33} - \dfrac{2}{3}k}{k}$	$-\dfrac{R_{12}}{k}$
Experiment: consensus of measurements	0.51	−0.42	−0.09	0.24
Calculation LRR model	0.51	−0.42	−0.09	0.285

Table 6.2. *Anisotropy levels in the turbulent boundary layer*

Finally, we will write:

$$\Phi_{ij}^{(s)} = \left[0.5\frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right) + 0.06\left(P_{ij} - D_{ij}\right)\right]\frac{a^{3/2}}{K}\frac{k^{3/2}}{\varepsilon x_2},$$

and taking into account : $\dfrac{K}{a^{3/2}} = 4.0$,

$$\Phi_{ij}^{(s)} = \left[0.125 \frac{\varepsilon}{k} \left(R_{ij} - \frac{2}{3} k \delta_{ij} \right) + 0.015 \left(P_{ij} - D_{ij} \right) \right] \frac{k^{3/2}}{\varepsilon x_2} . \qquad [6.57]$$

Let us note finally that the Launder B.E., Reece G.J. and Rodi W. model, [LAU 75C], may lead to non-realizable states, in particular when the anisotropy is high. The modifications presented in Chapter 8, using, for example, coefficients which are functions of anisotropy invariants and non-linear relations, may reinforce the realizability at the price of greater complexity.

6.6. The realizability of the basic models

6.6.1. *Method of principal axes*

The Reynolds stress transport equation written in shorthand notation as $\dfrac{dR_{ij}}{dt} = G_{ij}$ must be rewritten in the principal axes of the Reynolds stress tensor. Indeed, the terms involving the mean velocity gradients and in particular the production terms will appear in a simplified form which will be directly tractable for applying the realizability conditions given in section 4.6.6.

The change of vector basis for the Reynolds stress tensor satisfies the following transformation rules (cf. section 3.1.1):

$$R^*_{ij} = \alpha_{il} R_{lm} \beta_{mj} , \qquad R_{lm} = \beta_{li} R^*_{ij} \alpha_{jm} \qquad \text{with } \alpha_{il} \beta_{lj} = \delta_{ij} .$$

The Reynolds stress tensor being symmetrical, we also get $\left[\alpha_{ij} \right] = {}^{transp} \left[\beta_{ij} \right]$. The Reynolds stress transport equation in principal axes is thus obtained:

$$\frac{dR^*_{ij}}{dt} = \frac{d\alpha_{il}}{dt} R_{lm} \beta_{mj} + \alpha_{il} R_{lm} \frac{d\beta_{mj}}{dt} + \alpha_{il} G_{lm} \beta_{mj} , \qquad [6.58]$$

or $\qquad \dfrac{dR^*_{ij}}{dt} = \dfrac{d\alpha_{il}}{dt} \beta_{lv} R^*_{vj} + R^*_{ih} \alpha_{hm} \dfrac{d\beta_{mj}}{dt} + G^*_{ij} .$

The representative matrix of the tensor R^*_{ij} is diagonal in the new basis.

Considering that $\dfrac{d\alpha_{il}}{dt}\beta_{lj} + \alpha_{il}\dfrac{d\beta_{lj}}{dt} = 0$ and that $\left[\alpha_{ij}\right] = \,^{transp}\left[\beta_{ij}\right]$, it follows that the tensor $\gamma_{ij} = \dfrac{d\alpha_{il}}{dt}\beta_{lj} = -\alpha_{il}\dfrac{d\beta_{lj}}{dt}$ is antisymmetric.

In equation $\dfrac{dR^*_{ij}}{dt} = \gamma_{iv}R^*_{vj} - R^*_{ih}\gamma_{hj} + G^*_{ij}$ we will thus distinguish between the diagonal components and the non-diagonal components:

$$\begin{cases} \dfrac{dR^*_{ij}}{dt} = G^*_{ij} & i = j \\ 0 \quad = \gamma_{iv}R^*_{vj} - R^*_{ih}\gamma_{hj} + G^*_{ij} & i \ne j \end{cases}$$

[6.59]

The second equation allows us to determine γ_{ij}.

If R^*_{11} is the smallest eigenvalue, the realizability condition can be written as:

$$R^*_{11} = 0 \quad \Rightarrow \dfrac{dR^*_{11}}{dt} \ge 0 \quad \text{or} \quad \Rightarrow G^*_{11} \ge 0$$

[6.60]

In the case of the RS1 model:

$$\dfrac{dR_{ij}}{dt} = -\dfrac{c_1}{T}\left(R_{ij} - \dfrac{2}{3}k\delta_{ij} \right) - C_P\left(P_{ij} - \dfrac{2}{3}P\delta_{ij} \right) - C_D\left(D_{ij} - \dfrac{2}{3}P\delta_{ij} \right)$$
$$-C_M kS_{ij} + P_{ij} - \dfrac{2}{3}\varepsilon\delta_{ij},$$

with $T = k/\varepsilon$ and $C_P = \dfrac{c_2 + 8}{11}$, $C_D = \dfrac{8c_2 - 2}{11}$, $C_M = \dfrac{60c_2 - 4}{55}$.

We first note that $P^*_{11} = -2R^*_{11}U^*_{1,1}$ vanishes along with R^*_{11} and the same for $D^*_{11} = -2R^*_{11}U^*_{1,1}$. The right-hand side in transformed axes can thus be written:

$$G^*_{11} = +\dfrac{2}{3}c_1\varepsilon + \dfrac{2}{3}C_P P + \dfrac{2}{3}C_D P - C_M kS^*_{11} - \dfrac{2}{3}\varepsilon,$$

and this gives rise to the condition:

$$\frac{2}{3}(c_1 - 1) + \frac{2}{3}(C_P + C_D)\frac{P}{\varepsilon} - C_M \frac{kS^*_{11}}{\varepsilon} \geq 0 .$$ [6.61]

However, the principal axes of strain S^*_{ij} are not aligned with the principal axes of the Reynolds stress tensor. In order to get a condition which is independent of the orientation of the strain rate tensor, we will replace S^*_{11} by its largest eigenvalue λ_1 (which, of course, is not dependent on the choice of axes). The condition becomes:

$$c_1 \geq 1 - (C_P + C_D)\frac{P}{\varepsilon} + \frac{3}{2}C_M \frac{k\lambda_1}{\varepsilon} .$$ [6.62]

With the values $c_1 = 1.5$ and $c_2 = 0.4$ we arrive at the practical condition:

$$c_1 = \max\left[1.5, 1. - 0.873\frac{P}{\varepsilon} + 0.545\frac{k}{\varepsilon}\lambda_1\right].$$ [6.63]

The method could be also applied, using the same principles, to more advanced complex models.

6.6.2. *Method of stochastic models*

We have seen in section 4.9 that the stochastic Langevin equation ([HAW 86], [POP 94B]) including a random forcing and a deterministic forcing, has the same statistical properties as the corresponding closure model. There results from this fact an elegant and very efficient method for establishing the realizability of turbulence models [DUR 94A]. Thus, the stochastic process whose statistics are described by the RS2 model can be realized by solving:

$$dU^{\#}_i = -\frac{c_1}{2T}U^{\#}_i dt + (c_2 - 1)U^{\#}_m U_{i,m} dt + (C_0\varepsilon)^{1/2} dW_i(t),$$ [6.64]

with $C_0 = \frac{2}{3}\left(\frac{c_1 k}{T\varepsilon} - 1 + c_2 \frac{P}{\varepsilon}\right)$, $T = k/\varepsilon$.

The existence of such a process is sufficient to guarantee realizability. It is however not unique, the condition is not a necessary condition. The condition $C_0 \geq 0$ implies $\frac{P}{\varepsilon} \geq \frac{1 - c_1}{c_2}$ and this is the realizability condition of the model (however, the initial conditions themselves must also be realizable). In the case of the RS1 model, Durbin P.A. and Speziale C.G., [DUR 94A] introduce the process:

$$dU^{\#}_i = -\frac{c_1}{2T}U^{\#}_i dt + (C_P - 1)U^{\#}_m U_{i,m} dt + C_D U^{\#}_m U_{m,i} dt$$
$$+ (C_0 \varepsilon)^{1/2} dW_i(t) + (C_M \varepsilon)^{1/2} M_{im} dW'_m(t)$$

[6.65]

in which $W_i(t)$ and $W'_i(t)$ are two independent Wiener processes.

The tensor M_{ij} is the generalized square root of the strain rate tensor $S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ after normalizing by k/ε and it verifies $M_{im}M_{mj} - \frac{1}{3}M^2\delta_{ij} = -\frac{k}{\varepsilon}S_{ij}$. In practice, it can be obtained from diagonalization of S_{ij}.

The corresponding statistical model is:

$$\frac{dR_{ij}}{dt} = -\frac{c_1}{T}R_{ij} - C_P P_{ij} - C_D D_{ij} + C_M \varepsilon M_{im}M_{mj} + P_{ij} + C_0 \varepsilon \delta_{ij}.$$

[6.66]

Compatibility with the kinetic energy equation for k leads to the relation:

$$C_0 = \frac{2}{3}\left[c_1 - 1 + (C_P + C_D)\frac{P}{\varepsilon}\right] - \frac{C_M}{3}M^2,$$

and using this value in equation [6.66] it becomes:

$$\frac{dR_{ij}}{dt} = -\frac{c_1}{T}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right) - C_P\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) - C_D\left(D_{ij} - \frac{2}{3}P\delta_{ij}\right)$$
$$-C_M kS_{ij} + P_{ij} - \frac{2}{3}\varepsilon\delta_{ij},$$

[6.67]

with $C_P = \frac{c_2 + 8}{11}$, $C_D = \frac{8c_2 - 2}{11}$, $C_M = \frac{60c_2 - 4}{55}$.

The realizability condition $C_0 \geq 0$ leads to:

$$c_1 \geq 1 - (C_P + C_D)\frac{P}{\varepsilon} + \frac{C_M}{2}M^2$$

[6.68]

We can show [DUR94] that $M^2 = 3\dfrac{k}{\varepsilon}\lambda_1$, λ_1 being the largest eigenvalue of S_{ij}.

With $c_1 = 1.5$ and $c_2 = 0.4$ we get condition [6.63] exactly.

RS1 model

$$\frac{dR_{ij}}{dt} = -R_{ik}\overline{U}_{j,k} - R_{jk}\overline{U}_{i,k} - \frac{2}{3}\varepsilon\delta_{ij} - c_1\frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right)$$

$$-\frac{c_2+8}{11}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) - \frac{8c_2-2}{11}\left(D_{ij} - \frac{2}{3}P\delta_{ij}\right) - \frac{30c_2-2}{55}k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right)$$

$$+\left[c'_w\frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right) + c''_w\left(P_{ij} - D_{ij}\right)\right]\frac{k^{3/2}}{\varepsilon x_2}$$

$$+c_s\left[\frac{k}{\varepsilon}\left(R_{il}R_{jk,l} + R_{jl}R_{ki,l} + R_{kl}R_{ij,l}\right)\right]_{,k} \quad \text{or} \quad +c'_s\left[\frac{k}{\varepsilon}R_{kl}R_{ij,l}\right]_{,k}$$

RS2 model

$$\frac{dR_{ij}}{dt} = -R_{ik}\overline{U}_{j,k} - R_{jk}\overline{U}_{i,k} - \frac{2}{3}\varepsilon\delta_{ij} - c_1\frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right)$$

$$-\gamma\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) + \left[c'_w\frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right) + c''_w\left(P_{ij} - D_{ij}\right)\right]\frac{k^{3/2}}{\varepsilon x_2}$$

$$+c_s\left[\frac{k}{\varepsilon}\left(R_{il}R_{jk,l} + R_{jl}R_{ki,l} + R_{kl}R_{ij,l}\right)\right]_{,k} \quad \text{or} \quad +c'_s\left[\frac{k}{\varepsilon}R_{kl}R_{ij,l}\right]_{,k}$$

Value of constants	Basis for determination
$RS1\begin{cases} c_1 = 1.5 \\ c_2 = 0.4 \end{cases}$ ----------------	homogenous shear flow
$RS2\begin{cases} c_1 = 1.8 \\ \gamma = 0.6 \end{cases}$ ---------------	homogenous shear flow and rapid distortion of isotropic turbulence
$\left.\begin{matrix} c_s = 0.11 \\ c'_s = 0.22 \end{matrix}\right\}$ --------------------	numerical optimization
$\left.\begin{matrix} c'_w = 0.125 \\ c''_w = 0.015 \end{matrix}\right\}$ ----------------	turbulent stresses near a wall

Table 6.3. *Recapitulative table of the prototype model*

Chapter 7

Turbulence Scales

The Reynolds stress equations have physical dimension u^3/ℓ, thus it is necessary to introduce at least one turbulence length scale, for achieving closure. The dissipation rate of turbulence kinetic energy is a quantity that can be naturally chosen (its physical meaning is obvious) to provide this length scale, and so $\ell = k^{3/2}/\varepsilon$. Modeled equations for ε have been proposed by several authors (Davidov B.I., [DAV 61]; Harlow F.H., Nakayama P.I., [HAR 68B]; Hirt C.W., [HIR 69]; Daly B.J. and Harlow F.H., [DAL 70]), and this method of approach remains the most widely used.

Other authors have, however, chosen different definitions for the characteristic length scale; let us cite for example:

– Rotta (1951) $\mathscr{L} = \dfrac{1}{k}\displaystyle\int_0^\infty \dfrac{E(\kappa)}{\kappa}d\kappa$, E mean energy spectrum;

– Glushko (1972) $\mathscr{L} = \dfrac{3}{16\pi}\displaystyle\iiint_\Omega Q\dfrac{d\mathscr{V}}{\xi^2}$, with $Q = \dfrac{\overline{u_j(\vec{x})u_j(\vec{x}+\vec{\xi})}}{u_j(\vec{x})u_j(\vec{x})}$;

– Rotta (1972) $\mathscr{L} = \dfrac{3}{16k}\displaystyle\int_{-\infty}^{+\infty} R_{jj}(r)dr$, with $R_{jj}(r) = \overline{u_j(x,y,z)u_j(x,y+r,z)}$;

– Lin and Wolfshtein (1980) $\mathscr{U}_{ij} = 3\displaystyle\iiint_\Omega Q_{ij}d\mathscr{V}$ with $Q_{ij} = \dfrac{\overline{u_i(\vec{x})u_j(\vec{x}+\vec{\xi})}}{u_m(\vec{x})u_m(\vec{x})}$.

The tensor $\mathscr{U}_{ij}$ is also called "tensorial volume". The evolution equation of the dissipation rate of the turbulence kinetic energy ε modeled by Hirt C.W., [HIR 69]

has been widely used from that time, by various authors such as Hanjalic K. and Launder B.E., [HAN 72] and Lumley J.L., [LUM 75C] who have improved the model. However the formulation of this equation has undergone relatively little changes since the first proposals. The unknown correlations appearing in the right-hand side of this equation are complicated and cannot be measured. We have at our disposal very little information for modeling these terms. The traditional modeling for the ε equation does not seem to be of general application and various empirical modifications have been proposed by Hanjalic K. and Launder B.E., [HAN 80] and Pope S.B., [POP 78A] for example.

The equation for ε_γ, the dissipation rate of the variance of a passive scalar, more recently introduced in modeling also poses similar problems (cf. Chapter 10).

7.1. The turbulent kinetic energy dissipation rate equation

The exact general equation for ε is deduced from the fluctuating velocity equation (cf. Chapter 2, equation [2.11]). It is recalled below:

$$\underbrace{\frac{\partial \varepsilon}{\partial t} + \overline{U}_j \varepsilon_{,j}}_{(a)} = \underbrace{-2\nu \left(\overline{u_{k,j} u_{m,j}} + \overline{u_{j,k} u_{j,m}} \right) . \overline{U}_{m,k}}_{(b)} - \underbrace{2\nu \left(\overline{u_k u_{j,m}} \right) . \overline{U}_{j,km}}_{(c)}$$

$$\underbrace{-\nu \left(\overline{u_k u_{j,m} u_{j,m}} \right)_{,k}}_{(d)} - \underbrace{2\frac{\nu}{\rho} \left(\overline{u_{m,j} p_{,j}} \right)_{,m}}_{(e)}$$

$$\underbrace{+\nu \varepsilon_{,jj}}_{(f)} - \underbrace{2\nu \overline{u_{j,m} u_{k,m} u_{j,k}}}_{(g)} - \underbrace{2\nu^2 \overline{u_{j,km} u_{j,km}}}_{(h)} . \tag{7.1}$$

The order of magnitude of individual terms has been studied in section 4.2.

Some additional simplifications are now introduced, they are based on the local isotropy hypothesis:

$$\overline{\nu u_{i,j} u_{k,j}} \xrightarrow[\mathrm{Re}_t \to \infty]{} \approx \frac{1}{3} \varepsilon \delta_{ik} \, ,$$

$$\nu \overline{U}_{i,j} \overline{u_{k,i} u_{k,j}} \xrightarrow[\mathrm{Re}_t \to \infty]{} \approx \frac{1}{3} \varepsilon \overline{U}_{i,j} \delta_{ik} = \frac{1}{3} \varepsilon \overline{U}_{j,j} \approx 0 \, , \text{ (cf. section 6.2)}$$

$$\nu \overline{U}_{k,j} \overline{u_{k,i} u_{j,i}} \xrightarrow[\mathrm{Re}_t \to \infty]{} \approx \frac{1}{3} \varepsilon \overline{U}_{k,j} \delta_{kj} = \frac{1}{3} \varepsilon \overline{U}_{j,j} \approx 0 \, ,$$

$$\overline{u_{k,i}p_{,ik}} \xrightarrow{\ \text{Re}_t \to \infty\ } = \left(\overline{u_{k,i}p_{,i}}\right)_{,k} \approx 0 \,, \tag{7.2}$$

because $\overline{u_{k,i}p_{,i}}$ is a first order isotropic tensor, and thus zero.

Physical interpretation of terms

    1) Material derivative

        (a) $\dfrac{\partial \varepsilon}{\partial t} + \overline{U}_j \varepsilon_{,j}$,   time derivative and spatial convective derivative.

    2) Gradient terms

        (d) $-\nu \overline{\left(u_k u_{j,m} u_{j,m}\right)}_{,k}$,   turbulent diffusion due to fluctuating velocities, term

                    of the general form $\overline{\left(u_k \varepsilon'\right)}_{,k}$.

        (e) $2\dfrac{\nu}{\rho} \overline{\left(u_{m,j}p_{,j}\right)}_{,m}$,   turbulent diffusion due to pressure fluctuations.

        (f) $\nu \varepsilon_{,jj}$,   molecular diffusion.

    3) Source and sink terms

        (g)+(h) $-2\nu \overline{u_{j,m}u_{k,m}u_{j,k}} - 2\nu^2 \overline{u_{j,km}u_{j,km}}$,   terms which are globally representative of the production by eddy interactions (energy cascade process) and the action of viscosity (destruction).

        (b) $-2\nu \left(\overline{u_{k,j}u_{m,j}} + \overline{u_{j,k}u_{j,m}}\right).\overline{U}_{m,k}$, additional production due to the action of mean flow.

        (c) $-2\nu \overline{\left(u_k u_{j,m}\right)}.\overline{U}_{j,km}$, additional production due to the action of mean flow in non-homogenous flows.

    At high Reynolds numbers, the terms *(b)*, *(c)*, *(f)* and *(e)* can be neglected. In addition, if the terms *(g)* and *(h)* considered separately vary according to $\text{Re}_t^{1/2}$ (i.e., they go to infinity when $\text{Re}_t \to \infty$), when considered as a whole, their sum remains of order 1. These terms are likely to be modeled as a whole.

7.2. Modeling of diffusive terms

7.2.1. *Turbulent diffusion due to velocity fluctuations*

The problem is modeling the correlation $Q_j = \overline{\nu u_j u_{i,l} u_{i,l}} = \overline{\nu u_j \varepsilon'}$ in which ε' denotes the fluctuating part of the dissipation rate. Davidov B.I., [DAV 61] has modeled the transport equations for Q_j. Such an elaborate treatment will be seldom used in practice. However, Hanjalic K. and Launder B.E., [HAN 72] have shown that these equations, after some simplifying hypotheses, can lead to a simple algebraic approximation. These authors have introduced a process similar to the one used for approximating the triple correlations $\overline{u_i u_j u_k}$ (section 6.3).

We can easily establish the transport equation for Q_j by using the relation:

$$\frac{\partial}{\partial t}\overline{u_m u_{i,l} u_{i,l}} = 2\overline{u_m u_{i,l}\frac{\partial u_{i,l}}{\partial t}} + \overline{u_{i,l} u_{i,l}\frac{\partial u_m}{\partial t}},$$

together with the fluctuating velocity equation:

$$\underbrace{\frac{\partial Q_k}{\partial t} + \overline{U}_j Q_{k,j}}_{(a)} = \underbrace{-Q_j \overline{U}_{k,j}}_{(b)} \underbrace{-2\overline{\nu U}_{j,l} \overline{u_k u_{i,j} u_{i,l}}}_{(c)} - 2\overline{U}_{i,j}\overline{u_k u_{i,l} u_{j,l}} \underbrace{-2\overline{\nu U}_{i,jl}\overline{u_k u_j u_{i,l}}}_{(d)}$$

$$\underbrace{-2\overline{\nu u_k u_{i,l} u_{j,l} u_{i,j}}}_{(e)} \underbrace{-2\overline{\nu u_k u_{i,l} u_j u_{i,jl}}}_{(f)} \underbrace{-\overline{\nu u_{i,l} u_{i,l} u_j u_{k,j}}}_{(g)} \underbrace{+\overline{\nu u_{i,l} u_{i,l} R_{kj,j}}}_{(h)} \underbrace{+2\overline{\nu u_k u_{i,l} R_{ij,jl}}}_{(i)}$$

$$\underbrace{-2\frac{\nu}{\rho}\overline{u_k u_{i,l} p_{,il}}}_{(j)} \underbrace{-\frac{\nu}{\rho}\overline{u_{i,j} u_{i,j} p_{,k}}}_{(k)} \underbrace{+\nu Q_{k,jj}}_{(l)} \underbrace{-2\nu^2 \overline{u_k u_{i,lj} u_{i,lj}}}_{(m)} \underbrace{-4\nu^2 \overline{u_{i,l} u_{i,lj} u_{k,j}}}_{(n)}. \qquad [7.3]$$

We note that $(f) = -\overline{\nu u_j u_k \left(u_{i,l} u_{i,l}\right)_{,j}} \approx -R_{jk}\varepsilon_{,j}$.

The remaining source terms have been modeled by Hanjalic and Launder using an approximation of the type $-C_{Q1}\varepsilon R_{kl,l} - C_{Q2}\dfrac{\varepsilon}{k}Q_k$. They thus get the equation:

$$\frac{dQ_k}{dt} = Q_l \overline{U}_{k,l} - R_{kl}\varepsilon_{,l} - C_{Q1}\varepsilon R_{kl,l} - C_{Q2}\frac{\varepsilon}{k}Q_k \qquad [7.4]$$

Neglecting the convective transport, this equation yields an implicit algebraic expression for Q_k .

$$Q_k = -\frac{1}{C_{Q2}}\frac{k}{\varepsilon}\left(R_{kl}\varepsilon_{,l} + Q_l\overline{U}_{k,l} + C_{Q1}\varepsilon R_{kl,l}\right)$$ [7.5]

The correlation Q_k is important mainly in the vicinity of a wall, where the gradients of the dissipation rate are high. We can reasonably imagine that the last term in this expression is weak compared to the first one. Moreover, in boundary layer flows the second term is negligible; indeed, index k will correspond to the direction normal to the wall and consequently $\overline{U}_k$ and its gradient will be negligible. In the framework of these approximations Hanjalic and Launder are led to the explicit simpler hypothesis:

$$Q_k = \overline{u_k\varepsilon'} = -C_\varepsilon \frac{k}{\varepsilon}R_{kl}\varepsilon_{,l} .$$ [7.6]

If this hypothesis turns out to be too restrictive in some applications, the more general formulation [7.5] can be used without appreciable increase in the total calculation time.

7.2.2. *Turbulent diffusion due to pressure fluctuations*

The term *(e)* in equation [7.1] involves the flux $2\frac{\nu}{\rho}\overline{p_{,i}u_{k,i}}$. It is possible to derive an exact expression for this term by using the formal solution of the Poisson equation for pressure p. This way we obtain an expression similar to the one developed for Φ_{ij} (Chapter 6) and containing two groups of terms: one of them describes the interaction between turbulence and the mean flow and the other describes the interaction of turbulence with itself. However, the resulting expressions involve derivatives of u_k and $\overline{U}_k$ at higher orders than in Φ_{ij} . For this reason, in the context of second order closures, the HL (Hanjalic K. and Launder B.E.) model neglects this term. It is not taken into account explicitly in the LRR model either.

7.3. Modeling of source and sink terms

These are generation and destruction terms of the dissipation rate ε. Taking into account the study of the orders of magnitude of the different terms (cf. section 4.2)

and the notes made in section 7.1, the equation for ε can be written in the following simplified form:

$$\frac{\partial \varepsilon}{\partial t} + \overline{U}_j \varepsilon_{,j} + Q_{j,j} = -2\nu \overline{u_{j,m} u_{k,m} u_{j,k}} - 2\nu^2 \overline{u_{j,km} u_{j,km}} \qquad [7.7]$$

The two terms *(g)* and *(h)* on the right-hand side are of order $\mathrm{Re}_t^{1/2}$ but their sum considered as a whole is of order 1. The other terms in equation [7.7] are also of order unity. The neglected terms (equation [7.1]) are of higher order. The first term on the right-hand side of [7.7] is related to the production of velocity gradients by "vortex stretching" due to the fluctuating strain field, whereas the second term is related to the decay of these gradients by viscous effects.

Several authors (Daly B.J., Harlow F.H., [DAL 70]; Jones W.P., Launder B.E., [JON 72]; Hanjalic K. and Launder B.E., [HAN 72]; Ng K.H. and Spalding D.B., [NG 72]) have retained the term *(b)* $-2\nu\left(\overline{u_{k,j} u_{m,j}} + \overline{u_{j,k} u_{j,m}}\right).\overline{U}_{m,k}$ in equation [7.1] considering it (rightly) as a production of ε due to the action of mean flow. These authors have thus proposed the approximation:

$$2\nu\left(\overline{u_{k,j} u_{m,j}} + \overline{u_{j,k} u_{j,m}}\right) = (C_{\varepsilon 1} \frac{R_{km}}{k} + \tilde{C}_{\varepsilon 1} \delta_{km})\varepsilon \,,$$

i.e.:

$$(b) = -C_{\varepsilon 1} \frac{\varepsilon R_{km}}{k} \overline{U}_{m,k} \quad \text{(HL)} \qquad [7.8]$$

Hanjalic K. and Launder B.E. then approximate the terms *(g)* and *(h)* globally by:

$$(g) + (h) = -C_{\varepsilon 2} \frac{\varepsilon^2}{k} \qquad [7.9]$$

However, Lumley has shown that these terms are of order $\mathrm{Re}_t^{1/2}$ since

$$\overline{\nu u_{l,i} u_{j,i}} = \frac{1}{3} \varepsilon \delta_{lj} + \mathscr{O}\left(\frac{a_{lj}}{\mathrm{Re}_t^{1/2}}\right) \quad \text{(see section 6.2)},$$

and similarly:

$$\overline{\nu u_{i,l} u_{i,j}} = \frac{1}{3} \varepsilon \delta_{lj} + \mathscr{O}\left(\frac{a_{lj}}{\mathrm{Re}_t^{1/2}}\right).$$

Term *(b)* is thus really negligible at high Re_t numbers.

The real source term for ε comes from the first term on the right-hand side of [7.7]. The analysis by Lumley J.L. and Khajeh-Nouri B., [LUM 74] is based on the following reasoning.

The right-hand side of equation [7.7] represents the equilibrium between the vortex stretching and viscous dissipation effects. It may occur that the difference between these two terms is positive or negative as well. If the spectral energy flux increases then the first term should prevail until the dissipation has increased in order to balance this flux, and conversely. There is, in addition, an unsteady effect: in the turbulent fluctuating velocity field the equilibrium is never exactly reached since the velocity gradients change before being able to reach this equilibrium. So, there is always a fluctuating non-equilibrium state, and we may expect some slight loss due to non-linear effects.

The response to these effects must depend on the ratio of their characteristic scales. The characteristic time scale of the dissipating eddies is $(v/\varepsilon)^{1/2} = \tau_d = \eta/v$. In the decay of isotropic turbulence, the characteristic time scale of the energy cascade controlled by "vortex stretching" would be k/ε. But, if there is some production of turbulent kinetic energy which is also a source of spectral flux, there will be another time scale associated with this energy supply. In homogenous turbulence, the energy supply is characterized by P (production) and the corresponding time scale is k/P. In the non-homogenous case, it is more difficult to find a simple characteristic scale for this energy supply since a part of the energy k is transported (convection and diffusion). However, we will retain parameter P which will be considered as an approximation in the case of weak non-homogenity. The inverse of the effective time scale can thus be written:

$$\frac{1}{\tau_{transf}} = \frac{\varepsilon}{k} F\left(\frac{P}{\varepsilon}\right) \quad F, \text{ a function to be specified.} \qquad [7.10]$$

If the production by shear is weak (and consequently also the anisotropy) we can expand [7.10] to first order only and then take a linear approximation:

$$\frac{1}{\tau_{transf}} = \frac{\varepsilon}{k}\left(1 - a\frac{P}{\varepsilon}\right) \qquad [7.11]$$

It is thus not justified to use [7.11] when P is of same order as ε, but expression [7.11] may nevertheless be useful to predict the qualitative behavior. The right-hand side of [7.7] is thus approached by:

$$(g)+(h) = -\frac{\varepsilon}{\tau_d}\left[0+b\frac{\tau_d}{\tau_{transf}}+\mathcal{O}\left(\frac{1}{\mathrm{Re}_t}\right)\right] \tag{7.12}$$

or

$$(g)+(h) = -\varepsilon\left(\frac{\varepsilon}{\nu}\right)^{1/2}\left[0+b\frac{\varepsilon}{k}\left(1-a\frac{P}{\varepsilon}\right)\left(\frac{\nu}{\varepsilon}\right)^{1/2}+\mathcal{O}\left(\frac{1}{\mathrm{Re}_t}\right)\right].$$

The notation zero stands for the symbolic equality of terms when $\mathrm{Re}_t \rightarrow \infty$.

This approximation thus implies:

$$(g)+(h) = b\left(a\frac{P\varepsilon}{k}-\frac{\varepsilon^2}{k}\right). \tag{7.13}$$

We recover the form used by the authors previously cited (Daly B.J., Harlow F.H., [DAL 70]; Jones W.P., Launder B.E., [JON 72]; Hanjalic K., Launder B.E., [HAN 72]; Ng K.H. and Spalding D.B., [NG 72]). In this way, we can say that these authors had obtained the right approximation for the "wrong" reason (see Lumley J.L. and Khajeh-Nouri B., [LUM 74]).

Coefficient b can be determined by reference to the decay of turbulence behind a grid (see section 7.4), we then find $b \approx 2$.

Launder B.E., Reece G.J. and Rodi W., [LAU 75C] are also using the same type of formulation:

$$(g)+(h) = -C_{\varepsilon 1}\frac{\varepsilon R_{ij}}{k}\overline{U}_{i,j}-C_{\varepsilon 2}\frac{\varepsilon^2}{k}. \tag{7.14}$$

Making use of the invariant modeling method by Lumley (cf. section 4.4) developed from:

$$(g)+(h) = \mathscr{F}(R_{ij},\varepsilon), \tag{7.15}$$

leads to the formulation:

$$(g)+(h)=\frac{\varepsilon^2}{k}\left(-b+\alpha_1 II^a + \alpha_2 III^a\right),$$ [7.16]

where b, α_1 and α_2 are numerical closure coefficients, and with $II^a = a_{ij}a_{ij}$,

$III^a = a_{ij}a_{jk}a_{ki}$ where a_{ij} is   the   anisotropy   tensor   defined

by $a_{ij} = \left(R_{ij} - \frac{2}{3}k\delta_{ij}\right)/2k$.

If we keep only the second order terms in equation [7.16], then we see that $II^a.\frac{\varepsilon^2}{k} = a_{ij}a_{ij}\frac{\varepsilon^2}{k}$ is replacing P in the previous approximation. It is reasonable to associate II with the spectral flux since $II \neq 0$ is an indicator that the turbulence is constrained. The invariant II plays a role analogous to P, but in reality, in many flows II remains non-zero at places where P vanishes. Thus, there is always a source of spectral flux in these zones.

In practice, formulation [7.16] leads to a response time much higher than [7.13], because the production coming from II is no longer directly related to the mean velocity field, and this brings difficulties in non-equilibrium flows. Launder B.E., [LAU 75A]; Zeman O. and Lumley J.L., [ZEM 77] use the more general formulation:

$$(g)+(h)=-b\frac{\varepsilon^2}{k}-\alpha_0\frac{\varepsilon}{k}R_{ij}\overline{U}_{i,j}-\alpha_1 f(II)\frac{\varepsilon^2}{k}$$ [7.17]

where b, α_0 and α_1 are numerical coefficients and $f(II) = \dfrac{II^a}{1+3\sqrt{II^a}}$.

In this way the strong link with the mean velocity gradients is preserved but the coefficients are sensitized to the turbulence anisotropy.

Hypotheses such as [7.13] or [7.14] remain, in practice, the most common, and we shall retain this form in the remainder of the chapter. Other modifications made in the modeling of these source terms will be considered in section 11.2.3 in connection with the k-ε model.

The reasoning on the order of magnitude of eddy interaction terms can also be developed on the enstrophy equation (cf. Tennekes H. and Lumley J.L., 1972 and section 1.7 and also Chapter 2); indeed, the enstrophy $J = \dfrac{1}{2}\overline{\omega_j \omega_j}$ at high Reynolds numbers can be identified with the dissipation rate. It can be shown, actually, that $\varepsilon = 2\nu J + \nu R_{ij,ij}$ and consequently $\varepsilon = 2\nu J$ in homogenous turbulence when $\mathrm{Re}_t \to \infty$.

7.4. Determination of numerical constants

a) Constant $C_{\varepsilon 2}$

The value of $C_{\varepsilon 2}$ is determined by reference to the experiment of decay of isotropic turbulence behind a grid at high Reynolds numbers (initial period of decay). The experiments by Batchelor G.K. and Townsend A.A., [BAT 48A and B] have shown that the decay of turbulence energy behind a grid follows a power law with distance $k = \alpha x^{-n}$, α being a numerical constant depending on the particular experiment under consideration, in addition, the origin of abscissa x must be adequately chosen, the fictitious origin depending on the initial conditions and on the grid itself.

Batchelor and Townsend give $n = 1$, but a more accurate and extensive analysis considering several experiments on turbulence decay shows that most measurements suggest the consensus $n = 1.1$ (Comte-Bellot G. and Corrsin S., [COM 66]).

The coupled equations of turbulence kinetic energy and its dissipation rate written in the case of turbulence decay behind a grid reduce to:

$$U \frac{dk}{dx} = -\varepsilon$$

$$\Rightarrow U^2 \frac{d^2 k}{dx^2} = -U \frac{d\varepsilon}{dt},$$

$$U \frac{d\varepsilon}{dx} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}, \qquad\qquad\qquad [7.18]$$

the kinetic energy k thus satisfying the equation:

$$k \frac{d^2 k}{dx^2} = C_{\varepsilon 2} \left(\frac{dk}{dx} \right)^2. \qquad\qquad\qquad [7.19]$$

The analytical solution of this differential equation is basic, we find the solution:

$$k = k_0 \left[1 + \frac{1}{k_0} \left(\frac{dk}{dx} \right)_0 (1 - C_{\varepsilon 2})(x - x_0) \right]^{\frac{1}{1 - C_{\varepsilon 2}}},$$

where x_0 is the abscissa of the point where $k = k_0$ and $\dfrac{dk}{dx} = \left(\dfrac{dk}{dx} \right)_0 = -\dfrac{\varepsilon_0}{U}$ (initial conditions).

It is thus a power law decay with fictitious origin.

The condition $\dfrac{1}{1 - C_{\varepsilon 2}} = -1.1$ implies:

$$C_{\varepsilon 2} = 1.9 \tag{7.20}$$

b) Constants C_ε *and* $C_{\varepsilon 1}$

The reference flow in this case is the turbulent boundary layer on a flat plate in the zone of constant shear (logarithmic region of the velocity profile).

In this region, we can write:

$$\overline{U}_1 = \frac{u_*}{K} \mathrm{Log}\left(Ex_2^+ \right), \quad \overline{U}_{1,2} = \frac{u_*}{Kx_2} \quad \text{and} \quad -R_{12} = u_*^2 \text{ constant value} \tag{7.21}$$

Moreover, the turbulence is in equilibrium: $P = \varepsilon$, i.e.:

$$\varepsilon = -R_{12}\overline{U}_{1,2} = u_*^2 \overline{U}_{1,2} = \frac{u_*^3}{Kx_2} \tag{7.22}$$

The flow is assumed to be steady and fully developed, the material derivative of each turbulent quantity is zero, and in particular $d\varepsilon / dt$.

Equation [7.7] thus writes under these conditions:

$$0 = \left(C_{\varepsilon 1} - C_{\varepsilon 2} \right) \frac{\varepsilon^2}{k} + C_\varepsilon \left(\frac{k}{\varepsilon} R_{22} \varepsilon_{,2} \right)_{,2} \tag{7.23}$$

From [7.22] we can deduce by taking the derivative,

$$\varepsilon_{,2} = u_*^2 \overline{U}_{1,22} \qquad\qquad [7.24]$$

Expressions [7.22] and [7.24] reported in [7.23] give:

$$C_\varepsilon = \frac{(C_{\varepsilon 1} - C_{\varepsilon 2}) u_*^4 \left(\overline{U}_{1,2}\right)^2}{k \left[k R_{22} \dfrac{\overline{U}_{1,22}}{\overline{U}_{1,2}} \right]_{,2}},$$

and taking into account [7.21] and also the fact that k and R_{22} are constants, we find:

$$C_\varepsilon = \frac{(C_{\varepsilon 1} - C_{\varepsilon 2}) u_*^6}{k^2 K^2 R_{22}} = \frac{(C_{\varepsilon 1} - C_{\varepsilon 2})}{K^2 \left(\dfrac{k}{u_*^2}\right)^2 \left(\dfrac{R_{22}}{u_*^2}\right)}. \qquad [7.25]$$

From experimental data (section 6.5): $\dfrac{k}{u_*^2} \approx 4.2$ and $\dfrac{R_{22}}{u_*^2} \approx 1$ with $K \approx 0.41$ we

get from [7.25] the relation:

$$C_{\varepsilon 1} \approx 1.9 - 3 C_\varepsilon \qquad\qquad [7.26]$$

The choice of C_ε and $C_{\varepsilon 1}$ is thus determined by numerical optimization considering various traditional turbulent flows in order to satisfy [7.26] as closely as possible. Launder B.E., Reece G.J. and Rodi W., [LAU 75C] recommend $C_{\varepsilon 1} \approx 1.44$ and $C_\varepsilon \approx 0.15$.

7.5. Corrective changes introduced on the dissipation equation

When the wall shear stress vanishes or in the case of separated flows, the length scale calculated from the dissipation equation is proved to be excessive, and this fact can lead for example to a large overprediction of thermal transfer coefficients. This is the case for the impinging jet on a wall or for the flow around an airfoil. A correction is thus necessary. With this aim, Yap C., [YAP 87] introduced an additional corrective term in the $\tilde{\varepsilon}$ equation (see Chapter 13 for a precise definition of $\tilde{\varepsilon}$):

$$\mathcal{T}_y = C_w \frac{\tilde{\varepsilon}^2}{k} \left(\frac{\ell}{C_\ell y} - 1 \right) \left(\frac{\ell}{C_\ell y} \right)^2 , \qquad\qquad [7.27]$$

with $\ell = k^{3/2} / \varepsilon$, $C_w = 0.83$, $C_\ell = 2.5$ and y denoting the distance from the wall.

This correction is often used under the variant form:

$$\mathcal{T}_y = C_w \frac{\tilde{\varepsilon}^2}{k} \max\left[\left(\frac{\ell}{C_\ell y} - 1 \right) \left(\frac{\ell}{C_\ell y} \right)^2 , 0 \right] .$$

The k-ε model, which is used by Yap, predicts in the separated zone very high local Nusselt numbers. The proposed Yap correction allows us to temper this tendency and provides good agreement with the experiments. The length scale is drawn back towards its equilibrium value $C_\ell y$ from the effect of $\mathcal{T}_y$.

A somewhat different formulation has been proposed by Launder B.E., [LAU 89A]:

$$\mathcal{T}_y = C'_\varepsilon \frac{\varepsilon}{\sqrt{k}} \frac{\partial k}{\partial n} \frac{\partial}{\partial n} \left(\frac{k^{3/2}}{\varepsilon} \right) .$$

This formulation produces the same qualitative effect but has the advantage of being a local correction and self-adaptive with no direct reference to the distance from the wall. Let us also quote the correction proposed by Hanjalic *et al.*, [HAN 97]:

$$\mathcal{T}_y = \max\left\{ \left[\left(\frac{1}{C_\ell} \frac{\partial \ell}{\partial y} \right)^2 - 1 \right] \left(\frac{1}{C_\ell} \frac{\partial \ell}{\partial y} \right)^2 ; 0 \right\} \frac{\varepsilon \tilde{\varepsilon}}{k} A ,$$

with $C_\ell = 2.5$, $A = 1 - \dfrac{9}{8}(II^a - III^a)$.

Another technique, which replaces Yap's correction, has been proposed by Craft T.J. *et al.*, [CRA 95B]; it consists of introducing the following additional term in the ε equation:

$$\mathcal{T}_c = C_{\varepsilon s} \left(\overline{U}_{p,m} \ell_{,p} \ell_{,m} \right)^2 \frac{k \tilde{\varepsilon}}{\varepsilon} , \qquad C_{\varepsilon s} = 35 .$$

The modifications introduced in the Launder and Shima model (cf. section 13.6.3) on the production terms in the dissipation rate equation may also be beneficial in this context.

We also mention the model by Wilcox D.C. and Rubesin W.M., [WIL 80] which contains a term of a similar type:

$$\mathcal{T}_c = \alpha \ell^2_{,j} \omega^3 \ .$$

7.6. Reconsidering the ε equation: an asymptotic behavior with finite energy?

The usual ε equation based on [7.14] leads, in the case of homogenous turbulent shear flow, to a continuous increase in turbulent energy. Bernard P.S. and Speziale C.G., [BER 92] propose a different point of view from which the homogenous shear flow could go to an asymptotic state. For this, the ε equation is modified by taking into account a "vortex-stretching" effect to bring an asymptotic equilibrium state. The experimental data cannot give the behavior of homogenous turbulent shear flows during sufficiently long durations, thus, it is not possible to come to a definite conclusion for the moment. Thus, the proposal by Bernard P.S. and Speziale C.G., [BER 92] remains only a possible research method.

These authors write ε equation [7.1] in homogenous turbulence in the following form:

$$\underbrace{\frac{d\varepsilon}{dt}}_{(a)} = \underbrace{2\nu \overline{\omega_i \omega_j} \, \overline{U}_{i,j}}_{(b)} + \underbrace{2\nu \overline{\omega_i \omega_j u_{i,j}}}_{(g)} \underbrace{- 2\nu^2 \overline{\omega_{i,j} \omega_{i,j}}}_{(h)} \ , \qquad [7.28]$$

with $\omega_i = \varepsilon_{ijk} u_{j,k}$.

The development is inspired by the equation of $\omega^2 = \overline{\omega_j \omega_j} = \dfrac{10k}{\lambda^2}$ given by Batchelor and Townsend ([BAT 48A and B]) in isotropic turbulence:

$$\frac{d\omega^2}{dt} = \frac{7}{3\sqrt{15}} \omega^3 S_K - 3\sqrt{15}\omega^3 \frac{G}{\mathrm{Re}_\lambda} \ ,$$

with $S_K = \overline{\left(\dfrac{\partial u}{\partial x}\right)^3} \bigg/ \left[\overline{\left(\dfrac{\partial u}{\partial x}\right)^2}\right]^{3/2}$ and $G = \lambda^4 f^{IV}(r=0)$ which involves the fourth

derivative of the longitudinal double velocity correlation f defined in section 1.3.1. S_K is the skewness coefficient for velocity derivatives and G the coefficient of the palinstrophy.

We are then led to (noting that $\varepsilon = v\omega^2$):

$$(h) \approx \frac{7}{15} G \frac{\varepsilon^2}{k} \quad \text{and} \quad (g) \approx \frac{7}{3\sqrt{15}} \frac{S_K}{\sqrt{v}} \varepsilon^{3/2} = \frac{7}{3\sqrt{15}} S_K \sqrt{\mathrm{Re}_t} \frac{\varepsilon^2}{k}.$$

With a hypothesis of Taylor expansions:

$$G = \text{constant and } S_K = S_K(0) + S_K'(0)\mathrm{Re}_\lambda^{-1} + \dots,$$

we then find in the case of unsheared turbulence:

$$\frac{d\varepsilon}{dt} = \frac{7}{3\sqrt{15}} \frac{S_K(0)}{\sqrt{v}} \varepsilon^{3/2} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}.$$

In anisotropic turbulence, it is also necessary to introduce a production term:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1} \frac{\varepsilon P}{k} + \frac{7}{3\sqrt{15}} \frac{S_K(0)}{\sqrt{v}} \varepsilon^{3/2} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \qquad [7.29]$$

The second term (on the right-hand side) which is non-zero in the previous equation [7.29] replaces the term which had been supposed to vanish in equation [7.12] and denoted symbolically by zero.

This equation with $S_K(0) = 0.01$ allows us to obtain in the turbulent homogenous shear flow an exponential increase of energy during the first period (as is shown in the experiments) followed by a progressive saturation of the system finally reaching an equilibrium state with finite energy. This approach has thus been developed in [SPE 92B] with the aim of analyzing the decay laws of isotropic turbulence by identification of fixed points of the transport equation of the turbulence Reynolds number deduced from [7.29].

7.7. Tensorial volumes

Lin A. and Wolfshtein M., [LIN 80] suggest relating the turbulence length scales to a tensorial volume defined by:

$$\mathscr{U}_{ij} = \iiint_{\Omega} \frac{\overline{u_{iA} u_{jB}}}{2k/3} d\vec{\xi} .$$

[7.30]

A length scale can then be related to $\mathscr{U}_{ij}$ by:

$$\mathscr{L}_{ij} = \frac{\mathscr{U}_{ij}}{\left(\mathscr{U}_{ll}\right)^{2/3}}$$

[7.31]

For calculating $\mathscr{U}_{ij}$, the transport equation of the two point (A and B) double velocity correlations, $Q_{ij} = \overline{u_{iA} u_{jB}}$ will be integrated inside a volume surrounding point A.

We recall the equation for Q_{ij} (cf. Hinze J.O., 1975):

$$\frac{dQ_{ij}}{dt} = -\overline{U}_{kA} \frac{\partial Q_{ij}}{\partial x_k} - \left(\overline{U}_{kB} - \overline{U}_{kA}\right)\frac{\partial Q_{ij}}{\partial \xi_k}$$

$$-\frac{\partial S_{ik/j}}{\partial x_k} + \frac{\partial}{\partial \xi_k}\left(S_{ik/j} - S_{ij/k}\right) - \frac{1}{\rho}\frac{\partial K_{p/j}}{\partial x_i}$$

$$+\frac{1}{\rho}\frac{\partial K_{p/j}}{\partial \xi_i} - \frac{1}{\rho}\frac{\partial K_{i/p}}{\partial \xi_j} - Q_{kj}\left(\frac{\partial \overline{U}_i}{\partial x_k}\right)_A$$

$$-Q_{ik}\left(\frac{\partial \overline{U}_j}{\partial x_k}\right)_B + \nu \frac{\partial^2 Q_{ij}}{\partial x_k^2} + \left(\frac{\partial^2 Q_{ij}}{\partial \xi_k^2} - \frac{\partial^2 Q_{ij}}{\partial x_k \partial \xi_k}\right),$$

[7.32]

with $S_{ik/j} = \overline{u_{iA}u_{kA}u_{jB}}$ and $K_{p/j} = \overline{p_A u_{jB}}$

$$\xi_k = x_{kB} - x_{kA} \quad , \quad x_k = x_{kA} .$$

Integrating equation [7.32], using the continuity equation and taking into account the fact that any integral of the form $\iiint_\Omega \frac{\partial}{\partial \xi_j}(X)d\vec{\xi}$ vanishes if X itself is also zero on the boundaries, will give the evolution equation of the tensorial volumes:

$$\frac{d}{dt}\left(k\mathcal{U}_{ij}\right) = -k\left(\mathcal{U}_{mj}\frac{\partial U_i}{\partial x_m} + \mathcal{U}_{im}\frac{\partial U_j}{\partial x_m} + B_{ij}\right) + T_{ij}\,, \qquad [7.33]$$

where B_{ij} is a corrective term coming from the fact that $\left(\partial \overline{U}_j/\partial x_m\right)_B \neq \left(\partial \overline{U}_j/\partial x_m\right)_A$, which will be neglected in the following, and T_{ij} contains all the diffusion terms:

$$T_{ij} = \frac{\partial}{\partial x_k}\left[\nu\frac{\partial\left(k\mathcal{U}_{ij}\right)}{\partial x_k} - \frac{3}{2}\iiint_\Omega\left(S_{ik/j} + \frac{1}{\rho}K_{p/j}\delta_{ik}\right)d\vec{\xi}\right].$$

Equation [7.33] is strictly valid in homogenous turbulence, no numerical constant has been introduced. Only the diffusion terms need to be modeled in the case of non-homogenous flows.

We can show, however, that the tensorial volumes can exist only if the energy spectrum behaves like κ^2 at the origin.

Indeed, in homogenous flow:

$$\iiint_\Omega \hat{Q}_{ij}d\Omega = \iiint_\Omega\iiint Q_{ij}(\vec{\xi})e^{-i\vec{\kappa}\cdot\vec{\xi}}d\vec{\xi}d\Omega \Rightarrow \left[\iiint_\Omega \hat{Q}_{ij}d\Omega\right]_{\vec{\kappa}=0} = \frac{8\pi}{3}k\mathcal{U}_{ij}\,.$$

In addition, in HIT,

$$\iiint_\Omega \hat{Q}_{ij}d\Omega = \iiint_\Omega \Delta_{ij}(\vec{\kappa})\frac{E(\kappa)}{4\pi\kappa^2}d\Omega = \frac{2}{3}\frac{E(\kappa)}{\kappa^2}\delta_{ij}\,,$$

and so: $k\mathcal{U}_{ij} = \lim_{\kappa\to 0}\left[\frac{E(\kappa)}{4\pi\kappa^2}\delta_{ij}\right].$

Using a tensorial equation for the length scales in turbulence allows a more detailed description of the anisotropy of the turbulent field. We can then distinguish

the velocity anisotropy (Reynolds stresses anisotropy) from the anisotropy of length scales (tensorial volumes anisotropy).

7.8. Case of generation of turbulence injected at a fixed wavenumber

When turbulence is produced at a given wavenumber or in a given interval of wavenumbers, then the usual epsilon equation cannot be valid. This is the case for example of a turbulence field fed by vortex shedding with constant size of the emitted vortices. However, it is a case which is mainly encountered in theoretical studies in order to get an equilibrium spectrum in homogenous isotropic turbulence (without shear). If we assume that turbulence energy is injected at a given wavenumber κ_0, in the asymptotic state the energy under the spectrum will be approximately:

$$k = \int_{\kappa_0}^{\infty} C \varepsilon^{2/3} \kappa^{-5/3} d\kappa = \frac{3}{2} C \varepsilon^{2/3} \kappa_0^{-2/3}.$$

The usual epsilon equation does not yield an asymptotic state if $C_{\varepsilon 1} \neq C_{\varepsilon 2}$. If, on the contrary $C_{\varepsilon 1} = C_{\varepsilon 2}$, the initial energy is conserved and thus in general it does not go to the previous value. It is suggested that a different form of the equation could be used in such cases:

$$\frac{d\varepsilon}{dt} = 1.5 \left(\frac{2}{3C} \right)^{3/2} P_{\kappa_0} . \kappa_0 \sqrt{k} - 1.5 \frac{\varepsilon^2}{k},$$

where P_{κ_0} is the production rate injected at the wavenumber κ_0. This simple equation leads to an equilibrium state in homogenous turbulence and predicting the correct energy level. This note leads us to emphasize that the usual epsilon equation, from the particular form of its source term ($C_{\varepsilon 1} \varepsilon P / k$) is only applicable to turbulence created by shear or strain, the characteristic production scale κ_0 being replaced by $\varepsilon / k^{3/2}$. The value of this scale is thus no longer fixed, but it becomes self-adaptive in the interaction between turbulence and the mean field.

7.9. Modeling the dissipation tensor

In the model introduced by Reynolds W.C. (cf. [*COLL 84A]) the closure is made at the level of the Reynolds stresses R_{ij} and of the vorticity tensor

$\Omega_{ij} = \nu \overline{\omega_i \omega_j}$ where ω_i is the fluctuating curl of velocity, the model thus makes it possible to describe the anisotropy of the small scales by using the tensor Ω_{ij} which is quite similar to the dissipation tensor.

In the line of the work by W.C. Reynolds, and with the aim of better estimating the effects of the anisotropy of the dissipation tensor, Speziale C.G. and Gatski T. B., [SPE 92A] have proposed to model the equation for ε_{ij} or equivalently the equation of its deviator:

$$a_{ij}^{\varepsilon} = \frac{\varepsilon_{ij} - \dfrac{2}{3}\varepsilon\delta_{ij}}{2\varepsilon} \qquad\qquad [7.34]$$

The equilibrium hypothesis $da_{ij}^{\varepsilon}/dt = 0$ allows us to derive an algebraic relation giving a_{ij}^{ε} according to $S_{ij} = \dfrac{1}{2}\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right)$ and $\omega_{ij} = \dfrac{1}{2}\left(\overline{U}_{i,j} - \overline{U}_{j,i}\right)$. One of the main consequences on the dissipation rate equation is that coefficient $C_{\varepsilon 1}$ is no longer a constant and that it becomes a function of II^{S} and II^{ω}.

This approach has been further considered in [SPE 97] by establishing first a transport equation for ε_{ij} in homogenous flow:

$$\frac{d\varepsilon_{ij}}{dt} = -\varepsilon_{ik}\overline{U}_{j,k} - \varepsilon_{jk}\overline{U}_{i,k} + 2\left(f_{ikjl} + f_{jkil} - f_{lkij}\right)\overline{U}_{k,l} + \mathcal{N}_{ij}, \qquad [7.35]$$

with: $f_{ijkl} = 2\nu\overline{u_{k,i}u_{l,j}}$ and

$$\mathcal{N}_{ij} = 2\nu\overline{(u_{i,j} + u_{j,i})u_{m,l}u_{l,m}} - 2\nu\overline{u_{i,l}u_{m,l}u_{j,m}} - 2\nu\overline{u_{j,l}u_{m,l}u_{i,m}} - 4\nu^2\overline{u_{i,ml}u_{i,ml}}.$$

The tensor f_{ijkl} is assumed to be a function of the deviator of the dissipation tensor to be determined from tensorial representation techniques (cf. Chapter 3), taking into account the symmetry and contraction constraints:

$$f_{ijkl} = \frac{4\varepsilon}{15}\delta_{ij}\delta_{kl} - \frac{\varepsilon}{15}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \left(\frac{4\alpha}{11} + \frac{10}{11}\right)\varepsilon\delta_{ij}a_{kl}^{\varepsilon}$$

$$+ \alpha\varepsilon\delta_{kl}a_{ij}^{\varepsilon} - \left(\frac{3\alpha}{11} + \frac{2}{11}\right)\varepsilon\left(\delta_{ik}a_{jl}^{\varepsilon} + \delta_{jk}a_{il}^{\varepsilon} + \delta_{il}a_{jk}^{\varepsilon} + \delta_{jl}a_{ik}^{\varepsilon}\right), \qquad [7.36]$$

the resulting expression involves only one numerical constant.

If we introduce $\varepsilon^{*}{}_{ij} = 2\nu\overline{u_{k,i}u_{k,j}}$ and $a_{ij}^{\varepsilon^{*}} = \dfrac{\varepsilon^{*}{}_{ij} - \dfrac{2}{3}\varepsilon\delta_{ij}}{2\varepsilon}$, the previous equation

implies $a_{ij}^{\varepsilon^{*}} = \dfrac{1}{2}\left(\dfrac{21\alpha}{11} - \dfrac{8}{11}\right)a_{ij}^{\varepsilon}$.

The term $\mathcal{N}_{ij}$ is decomposed into $\mathcal{N}_{ij} = \dfrac{2}{3}\mathcal{N}\delta_{ij} + D\mathcal{N}_{ij}$ with $\mathcal{N} = \dfrac{1}{2}\mathcal{N}_{jj}$.

The ε equation can be written $\dfrac{d\varepsilon}{dt} = -\varepsilon_{ij}\overline{U}_{i,j} + 2\varepsilon_{ij}^{*}\overline{U}_{i,j} + \mathcal{N}$ where $\mathcal{N}$ is modeled using the usual approximation:

$\mathcal{N} = C_{\varepsilon 1}\dfrac{\varepsilon P}{k} - C_{\varepsilon 2}\dfrac{\varepsilon^{2}}{k}$, the authors chose $C_{\varepsilon 1} = 1.$ and $C_{\varepsilon 2} = 1.83$.

The isotropic limit of f_{ijkl} is given by [6.4].

The deviator of $\mathcal{N}_{ij}$ brings a return to isotropy effect which is modeled by a relaxation hypothesis inspired by the Rotta hypothesis for the Reynolds stresses:

$$D\mathcal{N}_{ij} = -C_{\varepsilon 5}\dfrac{\varepsilon}{k}\left(\varepsilon_{ij} - \dfrac{2}{3}\varepsilon\delta_{ij}\right).$$

The experiment by Maréchal ([MAR 72]) leads to $C_{\varepsilon 5} \approx 5.0$.

Thus, the modeled transport equation for the dissipation tensor can be written:

$$\dfrac{d\varepsilon_{ij}}{dt} = -\varepsilon_{ik}\overline{U}_{j,k} - \varepsilon_{jk}\overline{U}_{i,k} + \dfrac{16}{15}\varepsilon\overline{S}_{ij}$$
$$+\left(\dfrac{30}{11}\alpha + \dfrac{20}{11}\right)\varepsilon\left(a_{ik}^{\varepsilon}\overline{S}_{jk} + a_{jk}^{\varepsilon}\overline{S}_{ik} - \dfrac{2}{3}a_{kl}^{\varepsilon}\overline{S}_{kl}\delta_{ij}\right)$$
$$-\left(\dfrac{14}{11}\alpha - \dfrac{20}{11}\right)\varepsilon\left(a_{ik}^{\varepsilon}\overline{\omega}_{jk} + a_{jk}^{\varepsilon}\overline{\omega}_{ik}\right)$$
$$-\left(\dfrac{14}{11}\alpha - \dfrac{16}{33}\right)\varepsilon a_{kl}^{\varepsilon}\overline{S}_{kl} + \dfrac{2}{3}\left(C_{\varepsilon 1}\dfrac{\varepsilon P}{k} - C_{\varepsilon 2}\dfrac{\varepsilon^{2}}{k}\right)\delta_{ij}$$

$$-C_{\varepsilon 5}\frac{\varepsilon}{k}\left(\varepsilon_{ij}-\frac{2}{3}\varepsilon\delta_{ij}\right),$$ [7.37]

with $\overline{S}_{ij}=\frac{1}{2}\left(\overline{U}_{i,j}+\overline{U}_{j,i}\right)$, $\overline{\omega}_{ij}=\frac{1}{2}\left(\overline{U}_{i,j}-\overline{U}_{j,i}\right)$ and $\alpha{=}0.6$.

At this stage, it would be necessary to complement the equation by additional turbulent diffusion terms in order to extend its applicability to non-homogenous turbulent flows. This way leads, however, to formulations that are too complicated for practical applications. These authors have thus proposed an algebraic simpler form. To do this, the equilibrium hypothesis is used to obtain a linear system which can be solved analytically using an integrity basis (cf. section 3.6.4) in the same way as can be done for the Reynolds stress tensor (cf. section 11.5). The result is even simpler in two-dimensional flows:

$$a_{ij}^{\varepsilon}=-2C_{\mu\varepsilon}\left[\frac{k}{\varepsilon}\overline{S}_{ij}+\left(\frac{\frac{7\alpha}{11}+\frac{1}{11}}{C_{\varepsilon 5}+\frac{P}{\varepsilon}-1}\right)\frac{k^2}{\varepsilon^2}\left(\overline{S}_{ik}\overline{\omega}_{kj}+\overline{S}_{jk}\overline{\omega}_{ki}\right)\right.$$
$$\left.-\left(\frac{\frac{30\alpha}{11}-\frac{2}{11}}{C_{\varepsilon 5}+\frac{P}{\varepsilon}-1}\right)\frac{k^2}{\varepsilon^2}\left(\overline{S}_{ik}S_{kj}-\frac{1}{3}\overline{S}_{kl}\overline{S}_{kl}\delta_{ij}\right)\right],$$ [7.38]

with $C_{\mu\varepsilon}=\dfrac{1}{C_{\varepsilon 5}+\dfrac{P}{\varepsilon}-1}\left[1+2\left(\dfrac{\frac{7\alpha}{11}+\frac{1}{11}}{C_{\varepsilon 5}+\frac{P}{\varepsilon}-1}\right)^2\xi^2-\dfrac{2}{3}\left(\dfrac{\frac{15\alpha}{11}-\frac{1}{11}}{C_{\varepsilon 5}+\frac{P}{\varepsilon}-1}\right)^2\eta^2\right]^{-1}$, $\eta=\left(\overline{S}_{ij}\overline{S}_{ij}\right)^{1/2}\dfrac{k}{\varepsilon}$

and $\xi=\left(\overline{\omega}_{ij}\overline{\omega}_{ij}\right)^{1/2}\dfrac{k}{\varepsilon}$.

The energy dissipation rate equation is modeled on the basis of the work of Speziale and Bernard ([SPE 92B]). The modeled equation:

$$\frac{d\varepsilon}{dt}=C_{\varepsilon 1}\frac{\varepsilon P}{k}-2\left(1+\alpha_{\varepsilon}\right)\varepsilon a_{ij}^{\varepsilon}\overline{S}_{ij}-C_{\varepsilon 2}\frac{\varepsilon^2}{k},$$ [7.39]

with $\alpha_\varepsilon = \dfrac{3}{4}\left(\dfrac{14}{11}\alpha - \dfrac{16}{33}\right)$, shows direct evidence of the effect of the anisotropy of the dissipation tensor which will be approximated using the previous algebraic model.

The work by Hallbäck, Groth and Johansson, [HAL 90] also proposes an algebraic model for the anisotropy of dissipation, obtained from invariant modeling methods of J.L. Lumley. The general form of the model is the following:

$$a_{ij}^\varepsilon = c_1 a_{ij} + c_2\left(a_{ik}a_{kj} - \frac{1}{3}II^a\delta_{ij}\right)$$

$$+ c_3\left(a_{ik}a_{km}a_{mj} - \frac{1}{3}III^a\delta_{ij}\right)$$

$$+ c_4\left(a_{ik}a_{km}a_{mh}a_{hj} - \frac{1}{3}IV^a\delta_{ij}\right) + ...$$

with $a_{ij}^\varepsilon = \dfrac{\varepsilon_{ij} - \dfrac{2}{3}\varepsilon\delta_{ij}}{\varepsilon}$.

As has been shown in Chapters 3 and 4, the independent invariants are limited to II^a and III^a (IV^a can be expressed in terms of the previous invariants) and the contracted tensors $a_{ij}^{(m)}$ of order 3 and higher are not independent (they can be expressed from the Cayley-Hamilton theorem). After reduction, the proposed model takes the form:

$$a_{ij}^\varepsilon = \left[1 + \alpha\left(\frac{II^a}{2} - \frac{2}{3}\right)\right]a_{ij} - \alpha\left(a_{ik}a_{kj} - \frac{II^a}{3}\delta_{ij}\right), \quad \alpha \approx 3/4$$

This has been tested against numerical simulation data in homogenous turbulence. In this approach, the anisotropy of the dissipation can be separated from the anisotropy of the other terms involved in the transport equation for the Reynolds stresses, and thus it can be directly tested against numerical simulations or experiments.

In addition, the work by Oberlack M. [OBE 97] has developed a model for the dissipation tensor in non-homogenous turbulence, and also for a tensorial turbulence scale obtained from integration of the transport equations of two point correlations.

LRR model

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 1}\frac{\varepsilon R_{ij}}{k}\overline{U}_{i,j} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + C_{\varepsilon}\left(\frac{kR_{ij}}{\varepsilon}\varepsilon_{,i}\right)_{,j}$$

Value of constants   Basis for determination

$C_{\varepsilon 1} = 1.44$ Numerical optimization

$C_{\varepsilon 2} = 1.90$ Decay of grid turbulence

$C_{\varepsilon} = 0.15$ Zone of constant shear near a wall

(logarithmic region)

Table 7.1. *Recapitulative table of the basic prototype model*

This page intentionally left blank

Chapter 8

Advanced Closures: New Directions in Second Order Modeling

Recent progress (cf. Launder B.E., [LAU 89A]; Hanjalic K., [HAN 94]) in modeling of turbulent flows in the framework of Reynolds stress closures puts emphasis on the problems of realizability and ability of the model to yield a realistic prediction even in extreme conditions (two-dimensionality, strong shear, etc.). Compared to the "prototype" model by Launder, Reece and Rodi, 1975 [LAU 75C], these new models, while still retaining the basic closure terms presented in Chapter 6, extend the expansions in approximation at higher orders. These new modeling approaches also benefit from information stemming from direct numerical simulations of turbulence which now provide a valuable addition to experimental data. We will find a critical comparison of models in Shih T.H. and Lumley J.L., [SHI 93] and Speziale C.G. and Gatski T.B., [SPE 94B]. Considering its major importance in the underlying physical mechanisms, the pressure-strain correlation term still plays a central role. The introduction of new wall "detectors" is also a characteristic of these models. In this connection, considering again the physical mechanisms occurring in the near wall region will lead us to underline the complexity of acting phenomena, their identification and their modeling. The wall region is influenced by several phenomena. The first of these is the existence of a strong mean velocity gradient giving rise to high anisotropies and turbulence production. It is also a highly non-homogenous region in which occur important interactions (cf. section 14.1.2) following the scenario of Kline S.J. *et al.* ([KLI 67]). The "vortex stretching" is modified by the presence of walls. In addition, the molecular viscosity becomes dominant in the viscous sublayer region, producing viscous damping. Moreover, the decrease in turbulent Reynolds number no longer allows the tendency to isotropy of the fine scales. The strong decay of the fluctuating velocity component normal to the wall (cf. Chapter 13) is associated with

the "splatting effect" and leads to a two component turbulence. The work by Manceau R. ([MAN 99] and [MAN 01]) using the elliptic relaxation method has pointed out that the "splatting effect" was the main factor responsible for reducing the redistribution near a wall and that the wall echo term (surface integral term), contrary to what was assumed in the early second order closures, did indeed reinforce the redistribution.

8.1. A new generation of second order models

Very roughly, advanced second order closures will generally take the form:

$$\frac{dR_{ij}}{dt} = P_{ij} - c_1 \frac{\varepsilon}{k} a_{ij} + \text{higher order terms}$$

$$-c_2 (P_{ij} - \frac{2}{3} P \delta_{ij}) + \text{higher order terms}$$

$$-\frac{\varepsilon}{k} R_{ij} + \text{anisotropy terms} + \mathscr{D}(R_{ij})$$

8.1.1. *Non-linear closures*

The use of non-linear closure relations is a natural consequence of Lumley's approach (cf. Chapter 4 and Lumley J.L., [LUM 78A]). The new generation of second order closures developed since the 1980s is based on approximations using higher order expansions in anisotropy. This trend takes on two aspects, on the one hand the numerical constants are now instead becoming functions of invariants that are characteristic of the turbulent field, on the other hand, the order in expansions in the sense of J.L. Lumley invariant modeling is pushed up. The central terms in this approach are the redistribution terms and the dissipation terms.

Generally speaking, as regards the slow part of the redistribution term for the pressure-strain correlation, the most general expansion (cf. Chapter 4) to fourth order, satisfying the zero tensorial contraction condition, can be written:

$$\Phi_{ij}^{(1)} = a_0 II^a \delta_{ij} + (a_1^0 + a_1^0 II^a + a_1^0 III^a) a_{ij} - 3 a_0 a_{im} a_{mj}$$
,

which is a quadratic term with respect to the anisotropy tensor.

As regards the rapid part of the redistribution term for the pressure-strain correlation, it is a question of getting an good approximate for the fourth order

tensor a_{lj}^{mi} involved in [6.18]. Its expression in homogenous turbulence (cf. Chapter 5):

$$a_{ml}^{ij} = \int \frac{\kappa_m \kappa_l}{\kappa^2} \varphi_{ij}(\vec{\kappa}) d\vec{\kappa} ,$$

shows that, strictly speaking, it is linear with respect to the spectral tensor $\varphi_{ij}(\vec{\kappa})$. Most models, nevertheless, make use of non-linear approximations that are quadratic or cubic, in order to account for the effect of integration in the previous expression.

8.1.2. *Modeling for the slow part of the pressure-strain correlation*

Several new closures have been proposed by the University of Manchester team in the UK (Launder B.E., [LAU 89A]). We give a general survey in the following.

An improvement of the J.C. Rotta hypothesis can be founded on an approach inspired by the work by Lumley J.L. and Newmann G.R., [LUM 77] who are led to a non-linear expression of the type:

$$\Phi_{ij}^{(1)} = -c_1 \varepsilon \left[a_{ij} + c_1^* \left(a_{ik} a_{kj} - \frac{1}{3} II \delta_{ij} \right) \right] , \qquad [8.1]$$

with, in the present case, the definition $a_{ij} = \dfrac{R_{ij} - \dfrac{2}{3} k \delta_{ij}}{k}$ or $a_{ij} = b_{ij}/k$.

These latter authors are imposing that $\Phi_{\alpha\alpha}^{(1)}$ (without summation) vanishes when $R_{\alpha\alpha}$ also vanishes. This is realized by an adequate choice of coefficient c_1:

$$c_1 = A^n f(II, III, \mathrm{Re}_t)$$
$$II = a_{ij} a_{ij} , \quad III = a_{ij} a_{jk} a_{ki}$$

The flatness factor A which is defined by:

$$A = 1 - \frac{9}{8} (II - III)$$

has the interesting property of vanishing in the limit of two-dimensional turbulence and taking the value unity in the limit of isotropic turbulence. The limit $A = 0$

corresponds, indeed, to the rectilinear upper side of the realizability triangle of Lumley (Figure 4.4) which corresponds to turbulence states with two components. Parameter A is moreover proportional to the expression G introduced in section 4.6.5.

More precisely, Launder proposes:

$$c_1 = 3.1\left(A.II\right)^{1/2}, \quad c_1^* = 1.2 , \tag{8.2}$$

to be used in the expression $\Phi_{ij}^{(1)} = -c_1 \varepsilon \left[a_{ij} + c_1^* \left(a_{ik} a_{kj} - \frac{1}{3} II \delta_{ij} \right) \right] - \varepsilon a_{ij}$.

Another method consists of using an approximation of the following type:

$$\Phi_{ij}^{(1)} = -\tilde{c}_1 \varepsilon \left[(1 + \alpha II) a_{ij} + \beta (a_{im} a_{mj} - \frac{1}{3} II \delta_{ij}) \right], \tag{8.3}$$

provided that $\alpha \to -3/4$ and $\beta \to 3/2$ in the limit of two component turbulence (cf. Reynolds W.C., [*COLL 84A]).

Sarkar S. and Speziale C.G., [SAR 90] also used a non-linear of the following type similar to [8.1]:

$$\Phi_{ij}^{(1)} = -c_1 \varepsilon a_{ij} + c_2 \varepsilon (a_{im} a_{mj} - \frac{1}{3} II^a \delta_{ij}) , \tag{8.4}$$

with $c_1 = 3.4$, $c_2 = 4.2$.

Ristorcelli J.R. *et al.*, [RIS 94] proposed the same kind of expansion given by the following expression:

$$\Phi_{ij}^{(1)} = -\left(2 - 31 \mathscr{T}_2^a F^{1/2}\right) \varepsilon a_{ij} + c_2 \varepsilon \left(a_{im} a_{mj} + \frac{2}{3} \mathscr{T}_2^a \delta_{ij} \right), \tag{8.5}$$

with $a_{ij} = b_{ij} / 2k$ and $F = 9G = 1 + 9 \mathscr{T}_2^a + 27 \mathscr{T}_3^a$ according to the principal invariants (cf. section 4.6.5).

All these various formulations are still under development and more extensive tests for different turbulent flows will be necessary before definitely concluding about their degree of generality.

In particular, Weinstock J., [WEI 81 and 82] has shown that the apparent "Rotta constant":

$$c_1^{[ij]} = \frac{\varPhi_{ij}^{(1)}}{\dfrac{\varepsilon}{k}\left(R_{ij} - \dfrac{2}{3}k\delta_{ij}\right)} , \qquad (i \text{ and } j \text{ fixed})$$

may also vary strongly depending on the considered component of $\varPhi_{ij}^{(1)}$.

8.1.3. *Modeling for the rapid part of the pressure-strain correlation*

8.1.3.1. *Direct modeling of the fourth order tensor* a_{ml}^{ij}

As regards the linear part of the pressure-strain correlations, the simplified model of "isotropization" of production (equation [6.29], Chapter 6) often appears better in practice than the *LRR* model (equation [6.27], Chapter 6). The models however both have some failings: they are not compatible with the 2D turbulence limit and when $P/\varepsilon >> 1$ their behavior is not exact.

The most natural way to proceed is to generalize expansion [6.22] for the fourth order tensor a_{lj}^{mi} including higher order terms. This is the approach proposed by Reynolds W.C., (cf. [*COLL 84A]) who thus introduces quadratic expansions for the rapid pressure-strain term.

The most general expansion according to the anisotropy tensor, deduced from representation theorems can be written:

$$
\begin{aligned}
X_{milj} =\ & c_1\delta_{mi}\delta_{lj} + c_2\left(\delta_{ml}\delta_{ij} + \delta_{mj}\delta_{il}\right) \\
& + c_3\delta_{mi}a_{lj} + c_4\delta_{lj}a_{mi} + c_5\left(\delta_{ml}a_{ij} + \delta_{mj}a_{il} + \delta_{ij}a_{ml} + \delta_{il}a_{mj}\right) \\
& + c_6\delta_{mi}a_{lj}^{\ 2} + c_7\delta_{lj}a_{mi}^{\ 2} + c_8\left(\delta_{ml}a_{ij}^{\ 2} + \delta_{mj}a_{il}^{\ 2} + \delta_{ij}a_{ml}^{\ 2} + \delta_{il}a_{mj}^{\ 2}\right) \\
& + c_9 a_{mi}a_{lj} + c_{10}\left(a_{ml}a_{ij} + a_{mj}a_{il}\right) \\
& + c_{11}a_{mi}a_{lj}^{\ 2} + c_{12}a_{lj}a_{mi}^{\ 2} + c_{13}\left(a_{ml}a_{ij}^{\ 2} + a_{mj}a_{il}^{\ 2} + a_{ij}a_{ml}^{\ 2} + a_{il}a_{mj}^{\ 2}\right) \\
& + c_{14}a_{mi}^{\ 2}a_{lj}^{\ 2} + c_{15}\left(a_{ml}^{\ 2}a_{ij}^{\ 2} + a_{mj}^{\ 2}a_{il}^{\ 2}\right),
\end{aligned}
\qquad [8.6]
$$

using the notation $X_{milj} = a_{lj}^{mi} / 2k$ (cf. Chapter 6), a tensor which can be expressed in homogenous turbulence (cf. Chapter 5 and section 6.4.2) as:

$$a_{lj}^{mi} = \int \frac{\kappa_l \kappa_j}{\kappa^2} \varphi_{mi}(\vec{\kappa}) d\vec{\kappa}.$$

Coefficients c_α, $\alpha=1$ to 15 are functions of the invariants II and III of the anisotropy tensor.

This approach has been used in particular in [LEE 90] and then by Johansson A.V. and Hallbäck M., [JOH 94]. These latter authors, accounting for the various constraints (cf. section 6.4) on X_{milj}, are then led to the following model for the rapid term:

$$
\begin{aligned}
\frac{1}{k}\Phi_{ij}^{(2)} = S_{pq} & \left[A_1 \delta_{ip}\delta_{jq} + A_2\left(a_{ip}\delta_{jq} + a_{jp}\delta_{iq} - \frac{2}{3}a_{pq}\delta_{ij} \right) \right. \\
& + A_3 a_{pq} a_{ij} + A_4\left(a_{iq}a_{jp} - \frac{1}{3}a_{pk}a_{kq}\delta_{ij} \right) \\
& \left. + A_5 a_{pl}a_{lq}a_{ij} + \left(A_5 a_{pq} + A_6 a_{pl}a_{lq} \right)\left(a_{ik}a_{kj} - \frac{1}{3}II^a\delta_{ij} \right) \right] \\
& + \omega_{pq}\left[A_7\left(a_{ip}\delta_{jq} + a_{jp}\delta_{iq} \right) + A_8 a_{pk}\left(a_{jk}\delta_{iq} + a_{ik}\delta_{jq} \right) \right. \\
& \left. + A_9 a_{pk}\left(a_{jk}a_{iq} + a_{ik}a_{jq} \right) \right]. \hspace{2cm} [8.7]
\end{aligned}
$$

The use of strong realizability constraints makes it possible to reduce the number of indeterminate coefficients. The remaining parameters are then determined by comparison with RDT Lee M.J., [LEE 90] also relies upon RDT for establishing a model for the rapid pressure-strain term but does not use realizability constraints. We will also find an approach of this kind in the work of Ristorcelli J.R. *et al.*, [RIS 95] and of Shih T-H. [SHI 97B].

An even more general approach in principle is proposed by Cadiou A., [CAD 95 and 96] who models a transport equation for the tensor a_{ml}^{ij}. This approach allows us to preserve all the information contained in the fourth order tensor, valuable in particular for describing the behavior of rotating turbulence. The transport equation for a_{ml}^{ij} obtained after integrating in spectral space, takes the form:

$$\frac{da_{pq}^{ij}}{dt} = -\varepsilon_{pq}^{ij} + \phi_{pq}^{Sij} + \overline{U}_{m,n}\left[-a_{mp}^{ij}\delta_{nq} - a_{mq}^{ij}\delta_{np} - a_{pq}^{in}\delta_{jm} - a_{pq}^{nj}\delta_{im} \right]$$

$$+2B_{ijpqmn} + 2B_{inpqmj} + 2B_{njpqmi} \Big]$$

$$-2\varepsilon_{mkn}\Omega_k \Big[-a_{pq}^{in}\delta_{jm} - a_{pq}^{nj}\delta_{im} + B_{inpqmj} + B_{njpqmi} \Big],$$ [8.8]

in which we can distinguish on the right-hand side successively, a dissipation term, a slow term, a rapid term and the effect of rotation. The rapid term $B_{ijpqmn} = \int \dfrac{\kappa_p \kappa_q \kappa_m \kappa_n}{\kappa^4} \varphi_{ij}(\vec{\kappa})d\vec{\kappa}$ is modeled through an expansion of the following type:

$$\boldsymbol{B} = \{\boldsymbol{\delta}\otimes\boldsymbol{\delta}\otimes\boldsymbol{\delta}\} + \{\boldsymbol{a}\otimes\boldsymbol{\delta}\otimes\boldsymbol{\delta}\} + \{\boldsymbol{y}\otimes\boldsymbol{\delta}\otimes\boldsymbol{\delta}\} + \{\boldsymbol{X}\otimes\boldsymbol{\delta}\},$$

where $\boldsymbol{a}$ and $\boldsymbol{y}$ are anisotropy tensors defined in section 8.7 and the terms in braces stand for a linear combination of all similar terms (cf. [CAD 96]). The slow term is approximated by a linear and isotropic expression in a_{ml}^{ij} and the dissipation tensor is supposed to be isotropic. The resulting model, although not completely realizable, allows an interesting account of rotation effects (evolution of II^a and III^a in particular; cf. section 8.7).

8.1.3.2. *Quadratic modeling for the rapid pressure-strain correlation term*

Another approach, very similar as regards the final result, consists of performing an expansion directly on the tensor Φ_{ij} itself. Accounting for quadratic terms, Shih T.H. and Lumley J.L., [SHI 85A and B] are led to the more elaborate formulation (with $a_{ij} = b_{ij}/2k$):

$$\Phi_{ij} = -c_1 \varepsilon a_{ij} + \frac{4}{5}k S_{ij} + 12\alpha\left(a_{im}S_{jm} + a_{jm}S_{im} - \frac{2}{3}a_{lm}S_{lm}\delta_{ij} \right)$$

$$+ \frac{4}{5}k\left(a_{il}a_{lm}S_{jm} + a_{jl}a_{lm}S_{im} - 2a_{ik}a_{lj}S_{kl} \right)$$

$$+ \frac{4}{3}(2-7\alpha)k\left(a_{im}\omega_{jm} + a_{jm}\omega_{im} \right) + \frac{4}{5}k\left(a_{il}a_{lm}\omega_{jm} + a_{jl}a_{lm}\omega_{im} \right),$$ [8.9]

with $S_{ij} = \dfrac{1}{2}\left(\overline{U}_{i,j} + \overline{U}_{j,i} \right)$ and $\omega_{ij} = \dfrac{1}{2}\left(\overline{U}_{i,j} - \overline{U}_{j,i} \right),$

$$F = 9G = 1 + 9\mathscr{T}_2^a + 27.\mathscr{T}_3^a,$$

$$\alpha = \frac{1}{10}(1+\frac{4}{5}F^{1/2}),\ \ \mathrm{Re}_t = k^2/\nu\varepsilon,$$

$$c_1 = 2 + \frac{F}{9} \exp\left(-7.77 \middle/ \sqrt{\frac{9}{4} \mathrm{Re}_t}\right) \cdot \left\{ 72 \middle/ \sqrt{\frac{9}{4} \mathrm{Re}_t} + 80.1 \mathrm{Log}\left[1 + 62.4\left(-\mathscr{T}_2^a + 2.3\mathscr{T}_3^a\right)\right] \right\}$$

In order to compare to the Fu S. *et al.* model [FU 87B], the expression of the rapid part $\Phi_{ij}^{(2)}$ can be rearranged into the form (with at present $a_{ij} = b_{ij}/k$):

$$\Phi_{ij}^{(2)} = -0.6\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) + 0.6\varepsilon a_{ij}\frac{P}{\varepsilon}$$
$$-0.2\left[\frac{R_{mj}R_{li}}{k}\left(\overline{U}_{m,l} + \overline{U}_{l,m}\right) - \frac{R_{lm}}{k}\left(R_{im}\overline{U}_{j,l} + R_{jm}\overline{U}_{i,l}\right)\right]$$
$$-0.8A^{1/2}\left[\frac{1}{15}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) + \frac{8}{15}\left(D_{ij} - \frac{2}{3}P\delta_{ij}\right) + \frac{2}{5}k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right)\right]. \quad [8.10]$$

Following the same analytical approach, Fu S., Launder B.E. and Tselepidakis D.P., [FU 87B], proposed a formulation limited to quadratic terms (with $a_{ij} = b_{ij}/k$):

$$\Phi_{ij}^{(2)} = -0.6\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) + 0.6\varepsilon a_{ij}\frac{P}{\varepsilon}$$
$$-0.2\left[\frac{R_{mj}R_{li}}{k}\left(\overline{U}_{m,l} + \overline{U}_{l,m}\right) - \frac{R_{lm}}{k}\left(R_{im}\overline{U}_{j,l} + R_{jm}\overline{U}_{i,l}\right)\right], \quad [8.11]$$

used in conjunction with the following approximation:

$$\Phi_{ij}^{(1)} = -7.5\, II.A^{1/2}\varepsilon\left[a_{ij} + 0.6\left(a_{im}a_{mj} - \frac{1}{3}II\delta_{ij}\right)\right],$$

for the slow term. This formulation however proved to be inadequate for homogenous sheared turbulence.

One of the objectives in the approach of Shih T.H. and Lumley J.L., [SHI 85A and B] was to account for the strong realizability condition of Schumann U., [SCH 77]. However, Speziale C.G., Abid R. and Durbin P.A., [SPE 94A] have shown that this model cannot satisfy the strong realizability conditions, while the Fu S., Launder B.E. and Tselepidakis D.P. model [FU 87B] satisfies the weak form of the realizability conditions.

In the approach by Shih T.H. and Lumley J.L., [SHI 85A] and Shih T.H., Lumley J.L. and Chen J.Y., [SHI 85B] (cf. also Launder B.E., [LAU 89A]) it is the additional corrective term $\Phi_{ij}^{(2+)}$:

$$\Phi_{ij}^{(2+)} = -0.8 A^{1/2} \left[\frac{1}{15} \left(P_{ij} - \frac{2}{3} P \delta_{ij} \right) + \frac{8}{15} \left(D_{ij} - \frac{2}{3} P \delta_{ij} \right) + \frac{2}{5} k \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right) \right],$$

present in the model [8.10] which allows the improvement of the prediction of anisotropy (comparative values of R_{22} and R_{33} components in particular) in homogenous free shear flows. The use of coefficient $A^{1/2}$ seems however difficult to justify, because the resulting expression no longer appears as an expansion in anisotropy with terms in increasing order.

We also have to mention the quadratic model by Speziale C.G., Sarkar S. and Gatski T.B., [SPE 91B] which is based on the following non-linear approximation:

$$\Phi_{ij}^{(1)} = -c_1 \varepsilon a_{ij} + c_2 \varepsilon \left(a_{im} a_{mj} - \frac{1}{3} II \delta_{ij} \right)$$

$$\Phi_{ij}^{(2)} = 2 c_1^* k a_{mp} S_{mp} a_{ij} + \left(c_3 - c_3^* \sqrt{II} \right) k S_{ij}$$

$$+ c_4 k \left(a_{im} S_{jm} + a_{jm} S_{im} - \frac{2}{3} a_{lm} S_{lm} \delta_{ij} \right)$$

$$+ c_5 k \left(a_{im} \omega_{jm} + a_{jm} \omega_{im} \right), \qquad\qquad [8.12]$$

$$c_1 = 3.4, \quad c_1^* = 1.8, \quad c_2 = 4.2, \quad c_3 = 0.8, \quad c_3^* = 1.3, \quad c_4 = 1.25, \quad c_5 = 0.4 .$$

We will note that $P = -2 k a_{mp} S_{mp}$, so that $2 c_1^* k a_{mp} S_{mp} a_{ij} = c_1^* P a_{ij}$.

8.1.3.3. *Cubic modeling for the rapid pressure-strain correlation term*

Afterwards, Fu S., [FU 88] and then Craft T.J., Fu S., Launder B.E. and Tselepidakis D.P., [CRA 89] introduced new cubic terms that led to a more rational formulation for it contains all the generating terms, but it is also more complex. The model reads:

$$\Phi_{ij}^{(2)} = -0.6 \left(P_{ij} - \frac{2}{3} P \delta_{ij} \right) + 0.6 \varepsilon a_{ij} \frac{P}{\varepsilon}$$

$$-0.2\left[\frac{R_{mj}R_{li}}{k}\left(\overline{U}_{m,l}+\overline{U}_{l,m}\right)-\frac{R_{lm}}{k}\left(R_{im}\overline{U}_{j,l}+R_{jm}\overline{U}_{i,l}\right)\right]$$

$$-\lambda\left[II\left(P_{ij}-D_{ij}\right)+3a_{mi}a_{nj}\left(P_{mn}-D_{mn}\right)\right]$$

$$-\lambda'\left\{\left(\frac{7}{15}-\frac{II}{4}\right)\left(P_{ij}-\frac{2}{3}P\delta_{ij}\right)\right.$$

$$+0.2\varepsilon\left[a_{ij}-\frac{1}{2}\left(a_{im}a_{mj}-\frac{1}{3}II\delta_{ij}\right)\right]\frac{P}{\varepsilon}-0.05a_{ij}a_{lm}P_{lm}$$

$$+0.1\left[\left(\frac{R_{im}}{k}P_{mj}+\frac{R_{jm}}{k}P_{mi}\right)-\frac{2}{3}\frac{R_{lm}}{k}P_{lm}\delta_{ij}\right]$$

$$+0.1\left(\frac{R_{li}R_{kj}}{k^2}-\frac{1}{3}\frac{R_{lm}R_{km}}{k^2}\delta_{ij}\right)\cdot\left[6D_{kl}+13k\left(\overline{U}_{l,k}+\overline{U}_{k,l}\right)\right]$$

$$+0.2\frac{R_{li}R_{kj}}{k^2}\left(D_{kl}-P_{kl}\right)\right\},\qquad\qquad\text{[8.13]}$$

with $\lambda\approx0.6$, $\lambda'\approx0$.

This approximation is used in conjunction with the hypothesis:

$$\Phi_{ij}^{(1)}=-c_1\varepsilon\left[a_{ij}+c'_1\left(a_{ik}a_{kj}-\frac{1}{3}II\delta_{ij}\right)\right]-\varepsilon a_{ij},\ c_1=3.1\left(A.II\right)^{1/2},\ c'_1=1.2,$$

for the slow term, given previously. Both hypotheses are included in the so-called TCL (Two-Component Limit) model which is compatible with the tendency towards two component turbulence near a wall ([CRA 01B]).

Launder B.E., [LAU 89A] proposes for practical purposes, an improved form suggested in Fu S. *et al.*, [FU 87B] and relatively simpler, retaining only the first terms:

$$\Phi_{ij}^{(2)}=-0.6\left(P_{ij}-\frac{2}{3}P\delta_{ij}\right)+0.6\varepsilon a_{ij}\frac{P}{\varepsilon}$$

$$-0.2\left[\frac{R_{mj}R_{li}}{k}\left(\overline{U}_{m,l}+\overline{U}_{l,m}\right)-\frac{R_{lm}}{k}\left(R_{im}\overline{U}_{j,l}+R_{jm}\overline{U}_{i,l}\right)\right]$$

$$-\lambda\left[II\left(P_{ij}-D_{ij}\right)+3a_{mi}a_{nj}\left(P_{mn}-D_{mn}\right)\right]\qquad\qquad\text{[8.14]}$$

with $\lambda \approx 0.6$.

This model satisfies realizability conditions (weak formulation) and thus never yields negative values for the principal components of the Reynolds stress tensor. Moreover, the authors emphasize that this property is beneficial for the prediction of turbulent flows submitted to high strain rates.

8.1.4. *Improvements in the dissipation rate equation*

For the ε equation, the natural ways of improvement consists of sensitizing the coefficients in the equation to anisotropy invariants. This practice allows us to account for the effect of big eddies anisotropy on small scales.

The formulation proposed by Launder B.E., [LAU 89A] is relatively simple:

$$\frac{d\varepsilon}{dt} = \frac{P\varepsilon}{k} - \frac{1.92}{1+0.7\, II^{1/2}A} \cdot \frac{\varepsilon^2}{k} + C_\varepsilon \left(\frac{kR_{ij}}{\varepsilon} \varepsilon_{,i} \right)_{,j} . \qquad [8.15]$$

The ε equation in this form, allows us in particular to solve the anomaly in the spreading rates for the round jet and the plane jet (cf. Chapter 11).

In wall flows, let us cite the modification of Yap C., [YAP 87] which will be presented in section 7.5.

With these new modifications now included for Φ_{ij} and for ε, it seems that the term $\Phi_{ij}^{(s)}$ for wall reflection effect is not going to be as strongly influential as in the basic LRR model, and may even be neglected in a first approach.

These new closures proved to give better predictions in particular for calculating a turbulent wall impinging jet (Craft T.J. and Launder B.E., [CRA 91B]) and they have also been extended to the application to the three-dimensional boundary layer by Shima N., [SHI 91].

Parameter A also offers a new method to go progressively from the wall region to the internal core of the flow, in the approximation of the dissipation tensor (cf. Fu S. *et al.*, [FU 87B]):

$$\varepsilon_{ij} = \frac{2}{3} A^{1/2} \varepsilon \delta_{ij} + (1 - A^{1/2}) \varepsilon \frac{R_{ij}}{k} . \qquad [8.16]$$

The approach of Ristorcelli J.R., Lumley J.L. and Abid R., [RIS 94] is based on a non-linear model for the rapid part of the pressure-strain correlation tensor which satisfies material invariance in the limit of two component turbulence and which takes into account the realizability constraints, this then making it possible to determine some of the numerical constants. The remaining coefficients are thus determined by reference to known asymptotic states in homogenous shear turbulence.

8.2. Constraints related to the invariance properties with respect to the frame of reference

The work by Speziale C.G., [SPE 89] considers, from the fundamental point of view, the effect of change in the frame of reference for turbulence models. Several important constraints, which must be satisfied by the turbulence models in non-inertial frames of reference, are obtained as direct consequences of the Navier-Stokes equations.

The analytical study makes use of the Oldroyd derivatives [4.31] and intrinsic vorticity $\overrightarrow{\overline{W}}$ defined as $\overline{W}_{ij} = \omega_{ij}^{(absol)}$ (cf. Chapter 4):

$$\omega_{ij}^{(absol)} = \frac{1}{2}\left(\overline{U}_{i,j} - \overline{U}_{j,i}\right) + \varepsilon_{ikj}\Omega_k$$

or in vector form as $\overline{W}_j = 2\Omega_j^{(absol)}$:

$$\overline{W}_j = 2\Omega_j^{(absol)} = \varepsilon_{jml}\omega_{lm}^{(absol)} = \varepsilon_{jml}\overline{U}_{l,m} + 2\Omega_j$$

with $\overline{W}_j = \varepsilon_{jml}\overline{W}_{lm}$ and conversely $\overline{W}_{ij} = \frac{1}{2}\varepsilon_{imj}\overline{W}_m$.

The following main constraints are then established:

1) The Reynolds stress tensor models must be invariant under accelerations in translation of the frame of reference. These models are sensitive to the rotation of the frame of reference only through the effect of the intrinsic vorticity.

2) Any dependency on the frame of reference (and thus on W) must vanish in the limit of 2D turbulence (material indifference property). The relation in section 4.7 becomes:

$$R_{ij}(\vec{x},t) = \mathscr{T}\left[S_{ij}(\vec{x}',\tau);\vec{x},t\right], \quad \vec{x}' \in \mathscr{D}, \ \tau \leq t$$

3) Turbulence submitted to strong rotation and far from any wall must go to the *2D* limit.

4) Isotropic turbulence submitted to sudden rotation must undergo a linear viscous isotropic decay implying a very reduced dissipation rate.

Speziale C.G., [SPE 89] then proposes an approach for improving turbulence models in non-inertial frames of reference taking into account the previous properties.

In this way, a general form compatible with constraints 1) to 4) is proposed:

$$\overset{o}{D_t}\, R_{ij} = \alpha\left[R_{im}\overline{W}_{jm} + R_{jm}\overline{W}_{im} - \frac{1}{2k}\left(R_{im}R_{mn}\overline{W}_{nj} + R_{jm}R_{mn}\overline{W}_{ni} \right) \right]$$

$$-\left(\beta\frac{\varepsilon}{k} + \gamma I \right)\left(R_{ij} - \frac{2}{3}k\delta_{ij} \right) + T_{ijm,m}^{(R,\nabla R,\varepsilon)} + \Pi_{ij}^{(R,\overline{S},\varepsilon)} + \nu R_{ij,ll} , \qquad\qquad [8.17]$$

in which I is an invariant equal to Ω in rotating isotropic turbulence and vanishing in *2D* turbulence:

$$I = \left[\frac{3}{2}\left(\overline{W}_{il}\overline{W}_{jl}\,\frac{R_{ij}}{k} + \overline{W}_{il}\overline{W}_{il} \right) \right]^{1/2} .$$

The term Π_{ij} contains the parts of production, pressure-strain correlation and dissipation which are independent of rotation. The term $T_{ijm}^{(R,\nabla R,\varepsilon)}$ corresponds to turbulent diffusion.

Finally, the ε equation is written in the form:

$$D_t\varepsilon = C\left(\frac{k}{\varepsilon}R_{ij}\varepsilon_{,i} \right)_{,j} - C_1 f_1\left(\frac{I.k}{\varepsilon} \right)\frac{\varepsilon R_{ij}}{k}\overline{U}_{i,j} - C_2 f_2\left(\frac{I.k}{\varepsilon} \right)\frac{\varepsilon^2}{k}$$

$$f_1 = 1 - \gamma_1\frac{I.k}{\varepsilon} , \quad f_2 = 1 - \gamma_2\frac{I.k}{\varepsilon} , \quad \gamma_1 \text{ and } \gamma_2 \text{ numerical constants.}$$

This latter closure accounts for the reduction in dissipation rate caused by strong rotation.

The model by Ristorcelli J.R., Lumley J.L. and Abid R., [RIS 95] in particular, satisfies property 2). We can also quote [FU 97B].

8.3. Other methods of approach for the pressure-strain correlations

8.3.1. *Introduction of an effective velocity gradient*

In the approach of section 6.4 the hypothesis of weak non-homogenity is used in order to suppose that the mean velocity gradient in the integrand can be calculated at the local point in which the pressure correlation is considered. An improvement to this hypothesis is proposed by Launder B.E. and Tselepidakis D.P., [LAU 91A and B] who introduce an effective gradient:

$$\overline{U}_{k,h}^{effective} = \overline{U}_{k,h} + c_I \ell.\tilde{\ell}_{,m} \overline{U}_{k,hm} \,, \qquad\qquad [8.18]$$

with $\ell = k^{3/2}/\varepsilon$, $\tilde{\ell} = \ell.A^{1/2}$, $c_I = 0.15$.

A similar approach is used in Launder B.E. and Li S.P., [LAU 94]:

$$\overline{U}_{k,h}^{effective} = \overline{U}_{k,h} + c_I \ell^2 .f(A)_{,m} \overline{U}_{k,hm} \,,$$

with $\ell = k^{3/2}/\varepsilon$, $f(A) = A^{0.3}(1 + 2.5A^3)$, $c_I = 0.07$.

8.3.2. *Elliptic relaxation method*

Some authors have introduced a partial differential equation for the pressure-strain correlation Φ_{ij} itself. Already, Kolovandine B.A and Vatutine I.A., [KOL 69 and 72] had proposed a very elaborate system of equations including in addition to transport equations for double and triple velocity correlations, new equations for the pressure correlations.

More recently, Durbin P.A., [DUR 91A and 93] has introduced the so-called elliptic relaxation model for calculating the pressure-strain term and more generally the redistributive terms in the Reynolds stress equation:

$$\mathscr{P}_{ij} = -\frac{1}{\rho}\left[\overline{u_i\,p_{,j}} + \overline{u_j\,p_{,i}} - \frac{2}{3}\overline{u_m\,p_{,m}}\delta_{ij}\right] - \varepsilon_{ij} + \frac{R_{ij}}{k}\varepsilon \,,$$

denoted $\mathscr{P}_{ij} = \Phi_{ij} - \varepsilon_{ij} + \dfrac{R_{ij}}{k}\varepsilon$.

The Reynolds stress transport equations are solved in the form:

$$\frac{dR_{ij}}{dt} = \mathscr{P}_{ij} + P_{ij} - \frac{R_{ij}}{k}\varepsilon - \left(\overline{u_i u_j u_k}\right)_{,k} - \frac{2}{3\rho}\delta_{ij}\left(\overline{pu_k}\right)_{,k} + \nu R_{ij,mm} ,$$

with $\left(\overline{u_l u_i u_j}\right)_{,l} = -\left(\dfrac{\nu_{Tlm}}{\sigma_k}R_{ij,m}\right)_{,l}$ and $\nu_{Tij} = C_\mu R_{ij}T$ ($C_\mu = 0.23$), and T denotes the characteristic time scale (cf. section 13.5.3) relative to the previous length scale L. More precisely, $T = \max\left[\dfrac{k}{\varepsilon}, C_T\sqrt{\dfrac{\nu}{\varepsilon}}\right]$ with $C_T = 6$.

The equation for dissipation rate is considered as:

$$\frac{d\varepsilon}{dt} = \frac{C'_{\varepsilon 1}P - C_{\varepsilon 2}\varepsilon}{T} + \left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\varepsilon_{,j}\right]_{,j} .$$

The tensor f_{ij} defined from $\mathscr{P}_{ij} = k.f_{ij}$ is obtained by solving an elliptic equation which express the non-local character of wall effects:

$$L^2 f_{ij,hh} - f_{ij} = -\Pi_{ij} \qquad\qquad [8.19]$$

The elliptic relaxation equation [8.19] accounts for non-local effects of pressure. In this equation, the length scale is obtained by analogy with the hypothesis already introduced by Durbin P.A., [DUR 91A] (cf. Chapter 13):

$$L = c_L \max\left[\frac{k^{3/2}}{\varepsilon}, c_\eta\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}\right],$$

where c_L and c_η are numerical constants ($c_L = 0.2$, $c_\eta = 80$). For the source term Π_{ij}, it is possible to use a quasi-homogenous model such as the Launder B.E. *et al.* model, [LAU 75C] in its simplified form:

$$\Pi_{ij} = \frac{1-c_1}{kT}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right) - c_2\frac{1}{k}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right).$$

The elliptic relaxation model has been applied with success by its author to the case of boundary layers on a curved wall.

The coefficients are given by:

$$C_\mu = 0.23,\ C_{\varepsilon 1} = 1.44,\ C^*_{\varepsilon 1} = 0.1,\ C_{\varepsilon 2} = 1.9,\ c_1 = 1.22,\ c_2 = 0.6,$$

$$\sigma_k = 1.2,\ \sigma_\varepsilon = 1.65,\ C'_{\varepsilon 1} = C_{\varepsilon 1}\left(1 + C^*_{\varepsilon 1}\frac{P}{\varepsilon}\right).$$

We will notice however that equation [8.19] is not strict, rather it is a model which can be justified heuristically ([DUR 93]). To do this, we consider the formal solution of the Poisson equation for the fluctuating pressure,

$$p_{,jj} = -2\rho u_{i,j}\overline{U}_{j,i},$$

which reads $\dfrac{p}{\rho}(\vec{x}) = \dfrac{1}{2\pi}\iiint_\Omega \dfrac{u_{i,j}(\vec{y})\overline{U}_{j,i}(\vec{y})}{\|\vec{x}-\vec{y}\|}dV(\vec{y})$.

The quasi-homogenous approximation for the correlation:

$$\frac{\overline{pu_{l,m}}}{\rho}(\vec{x}) = \frac{1}{2\pi}\iiint_\Omega \frac{\overline{u_{i,j}(\vec{y})u_{l,m}(\vec{x})}.\overline{U}_{j,i}(\vec{y})}{\|\vec{x}-\vec{y}\|}dV(\vec{y}),$$

presented in Chapter 6 is no longer used here. The proposed alternative consists of assuming that the two point correlation can be approximated by an exponential:

$$\overline{u_{i,j}(\vec{y})u_{l,m}(\vec{x})} \approx \overline{u_{i,j}u_{l,m}}(\vec{y}).e^{-\dfrac{\|\vec{x}-\vec{y}\|}{L}}$$

and consequently:

$$\phi_{lm}(\vec{x}) = \overline{\frac{pu_{l,m}}{\rho}}(\vec{x}) = \frac{1}{2\pi} \iiint_{\Omega} \overline{U}_{j,i}(\vec{y}).\overline{u_{i,j}u_{l,m}}(\vec{y}) \frac{e^{-\frac{\|\vec{x}-\vec{y}\|}{L}}}{\|\vec{x}-\vec{y}\|} dV(\vec{y}).$$

If we note that $e^{-\frac{\|\vec{x}-\vec{y}\|}{L}}\Big/\|\vec{x}-\vec{y}\|$ is the Green function of the operator $-\nabla^2 + \dfrac{1}{L^2}$
(assuming that L varies only slowly), the previous expression is the solution of the following equation:

$$\phi_{lm,jj} - \frac{\phi_{lm}}{L^2} = \rho\overline{u_{i,j}u_{l,m}}.\overline{U}_{j,i} \quad \text{or} \quad L^2\phi_{lm,jj} - \phi_{lm} = \rho L^2\overline{u_{i,j}u_{l,m}}.\overline{U}_{j,i},$$

which is the elliptic relaxation equation.

In the particular case of homogenous turbulence, this equation becomes $\phi_{lm} = -\rho L^2\overline{u_{i,j}u_{l,m}}.\overline{U}_{j,i}$, and this leads to a simple interpretation of the right-hand side, which thus can be modeled using any pointwise traditional model for $\phi^{(2)}{}_{lm}$ (cf. Chapter 6).

The relaxation model by Durbin [DUR 93] is in fact also applied to the dissipation tensor through the global quantity $\mathscr{P}_{ij} = k.f_{ij}$. The boundary conditions on f_{ij} are determined in order to get a correct asymptotic behavior for the distribution of the Reynolds stresses components near the wall.

Several variants of the elliptic relaxation method and improvements are discussed in [WIZ 96], [MAN 99] and [MAN 01].

8.4. Elimination of topographical parameters

Most of the usual models use the distance from the wall to account for the damping of normal velocity fluctuations from the wall. This practice becomes difficult when we deal with confined flows involving several walls or complex shaped walls. The contribution of Launder B.E. and Li S.P., [LAU 94] offers an alternative to this approach. These authors show on various applications for confined flows that the realizable cubic model (equation [8.13]) correctly accounts for the reflected contribution to the pressure-strain correlations in the presence of a wall, if

a different choice of numerical constant is made (λ=0.55, λ'=0.6). In this approach, the wall "detection" is then made through the flatness parameter A rather than through the wall distance (cf. section 13.6.5). So, the proposed model no longer contains a wall-reflection term in the pressure-strain correlation and consequently no longer uses the distance to the wall. The model is thus based on hypotheses [8.1]-[8.2] for the slow part and on hypothesis [8.13] with non-zero coefficients for the rapid part.

The model by Shima N., [SHI 97 and 98] also does not include any term for the wall reflection term in the pressure-strain correlation. This has become useless because of an appropriate choice of the model coefficients as functions in the anisotropy invariants. The application of this model to complex geometries is thus made easier.

In the Shima N. model [SHI 98], the slow terms for pressure redistribution are grouped together with the dissipation anisotropy:

$$-\varepsilon_{ij} + \Phi_{ij} = -\frac{2}{3}\varepsilon\delta_{ij} + \Phi_{ij}^{*(1)} + \Phi_{ij}^{*(2)}$$

$$\Phi_{ij}^{*(1)} = -c_1\frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right)$$

$$\Phi_{ij}^{*(2)} = -c_2\frac{\varepsilon}{k}\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) - c_3\frac{\varepsilon}{k}\left(D_{ij} - \frac{2}{3}P\delta_{ij}\right) - c_4 k\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right), \text{ [8.20]}$$

with a coefficient c_1 function of the anisotropy invariants and of the turbulence Reynolds number $c_1 = c_1(A, II^a, \text{Re}_t)$. According to Lumley, [LUM 78A] this coefficient must satisfy the following constraints:

$$c_1(0, II^a, \text{Re}_t) = c_1(1, 0, \infty) = c_1(A, II^a, 0) = 1,$$

and at the limit of two component turbulence, we must get $A = 1 - \frac{9}{8}\left(II^a - III^a\right) \to 0$, $c_2 + c_3 = 0$ and $c_4 = 0$.

The numerical values of coefficients are first calibrated in the logarithmic boundary layer after the DNS data of Kim:

$$c_1 = 2.50, \quad c_2 = 0.45, \quad c_3 = 0.23, \quad c_4 = 0.22.$$

Thus, taking into account the previous constraints, the final choice is given below:

$$c_1 = 1 + 2.45 II^{a^{1/4}} A^{3/4} \left[1 - \exp\left\{ -(7A)^2 \right\} \right] \cdot \left[1 - \exp\left\{ -\left(\frac{Re_t}{60} \right)^2 \right\} \right]$$

$$c_2 = 0.7A, \quad c_3 = 0.3A^{1/2}, \quad c_4 = 0.65A(0.23c_1 + c_2 - 1) + 1.3 II^{a^{1/4}} .c_3 \quad [8.21]$$

The dissipation rate equation is identical to the one in the Launder B.E. and Shima N. model, [LAU 89B] but with different coefficients:

$$\tilde{\varepsilon} = \varepsilon - 2\nu \left(\sqrt{k} \right)_{,j} \left(\sqrt{k} \right)_{,j}, \quad C_{\varepsilon 2} = 1.92, \quad C_{\varepsilon 1} = 1.44 + \beta_1 + \beta_2,$$

$$C_\varepsilon = 0.15, \quad \beta_1 = 0.25A.\min\left[\lambda/2.5 - 1; 0 \right] - 1.4A.\min\left[\frac{P}{\varepsilon} - 1; 0 \right],$$

$$\beta_2 = 1.0A\lambda^2 .\max\left[\lambda/2.5 - 1; 0 \right],$$

$$\lambda = \min\left[\lambda^*; 4 \right], \quad \lambda^* = \left[\left(\frac{k^{3/2}}{\varepsilon} \right)_{,j} \left(\frac{k^{3/2}}{\varepsilon} \right)_{,j} \right]^{1/2}, \quad [8.22]$$

A quantity related to the length scale $\ell = \dfrac{k^{3/2}}{\varepsilon}$.

Coefficient β_1 allows us to recover the correct rate of spread of turbulent jets and β_2 to suppress the too large increase in the length scale in boundary layers in adverse pressure gradients. The model has been tested on various turbulent flows including jets and boundary layers.

Another type of approach consists of retaining the wall effect terms but introducing non-homogenity indicators which are replacing the distance from the wall and the normal vector to the wall (cf. section 13.6.5).

8.5. Models based on the renormalization group (RNG models)

Second order models have been deduced from the renormalization group theory (Yakhot V. and Orszag S.A., [YAK 86]) by Rubinstein R. and Barton J.M., [RUB 92]. The method allows us to determine the pressure-strain correlation together with the value of numerical constants. The resulting first order

approximation is similar to the Launder B.E. *et al.*, [LAU 75C] model for the linear part and to the Rotta J. model for the non-linear part. The most usual RNG models are however two equation k-ε type models (cf. section 11.2.5).

8.6. Memory effects

When the turbulent field is submitted during a sufficiently long time to the action of strain (mean velocity gradients), the characteristic time scale of turbulence will go to $1/\sqrt{S_{ij}S_{ij}}$ in which S_{ij} is the mean strain tensor. If the strain varies in time (unsteadiness) or in space (non-homogenity) the equilibrium will be broken. The characteristic scales of the turbulent field will then be influenced by the past history of the fluid elements which, statistically, are going by this point.

This concept is the basis of the model proposed by J.L. Lumley (cf. [*GAT 96]) who introduces the quantity S which is the inverse of a characteristic time, obtained from a relaxation equation:

$$\frac{dS}{dt} = \frac{1}{T}\left(\sqrt{S_{ij}S_{ij}} - S\right) + \nu_t S_{,jj} \, , \qquad [8.23]$$

with $T = c.k/\varepsilon$ as relaxation time and ν_t as diffusivity coefficient. In steady (in the mean) homogenous turbulence, this equation gives again $S = \sqrt{S_{ij}S_{ij}}$. The quantity S is then used as a source term in the dissipation rate equation:

$$\frac{d\varepsilon}{dt} = \frac{C_1 k.S - \varepsilon}{T} + Diff(\varepsilon) \, .$$

From the physical point of view, the quantity S also acts like an "effective" strain rate.

In this respect, this modeling approach can be compared to the model by Maxey, [MAX 82] who introduces a transport equation for "effective strain":

$$\frac{\partial \alpha_{eff}}{\partial t} = \frac{\partial}{\partial y}\left(\nu_t \frac{\partial \alpha_{eff}}{\partial y}\right) - \frac{\alpha_{eff}}{T} + \frac{\partial \overline{U}}{\partial y} \, , \qquad [8.24]$$

in boundary layer type unsteady flows. This model relies on the rapid distortion theory (cf. section 5.8) of turbulence submitted to a uniform shear $A_{12} = \beta(t)$. The

strain parameter $\alpha(t)$ characterizes the cumulative effect of the turbulent mean shear on the structure of turbulence and it can be defined by:

$$\alpha(t) = \int_0^t \beta(t')dt'.$$

If we take into account the fact that the characteristic time scale T of the energy cascade implies a limiting value to the time interval during which the eddy will be strained by mean velocity gradients, then an effective strain α_{eff} naturally emerges. At short time intervals, in the limit of rapid distortion theory, the equation for α_{eff} proposed by Maxey reduces to the usual previous definition of $\alpha(t)$. This type of model has been applied to various homogenous flows and shear flows by Cotton M.A. and Ismael J.O. [COT 98].

8.7. Pressure-velocity correlations

The pressure-velocity correlations have been relatively scarcely studied until now because the experimental data do not allow us to deduce precise information. The development of DNS will probably be useful in this respect. Most often, these correlations were merged into the diffusion term by triple velocity correlations or even just neglected. In order to improve the prediction of the flow behind an obstacle, Suga K. [SUG 04] introduces the possibility of modeling the pressure-velocity correlations using the Green function approach. From the expression of fluctuating pressure, the pressure-velocity diffusive correlation is approximated in the same way as in the approach for the pressure-strain correlations.

Thus, two contributions appear, a slow term and a rapid term:

$$-\frac{\overline{u_i p}}{\rho} = \varphi^{(p1)}{}_i + \varphi^{(p2)}{}_i ,$$

with the approximates:

– slow term:

$$\varphi^{(p1)}{}_i = 0.2\overline{u_k u_k u_i} ,$$

– rapid term:

$$\varphi^{(p2)}{}_i = \overline{U}_{k,l}\gamma_k^{lj}(R_{ij},\ell_k), \; \ell_j = \ell n_j ,$$

$$\gamma_k^{ij} = k\Big[\beta_1 \ell_k \delta_{ij} + \beta_2\big(\ell_i \delta_{jk} + \ell_j \delta_{ki}\big) + \beta_3 \ell_k a_{ij} + \beta_4\big(\ell_i a_{jk} + \ell_j a_{ki}\big)$$

$$+ \beta_5 \ell_m a_{km} \delta_{ij} + \beta_6 \ell_m \left(a_{im} \delta_{jk} + a_{jm} \delta_{ki} \right) \Big].$$

The continuity condition and the two component limiting condition allow us to determine the set of constants as functions of one of them:

$$\beta_2 = -(1/4)\beta_1, \quad \beta_3 = (3/2)\beta_1, \quad \beta_4 = \beta_5 = 0, \quad \beta_6 = -(3/8)\beta_1.$$

This model is used in conjunction with the TCL model by the University of Manchester. The model can be applied in particular to the turbulent flow around an obstacle.

8.8. Internal variable models, structural models

The application of the usual second order closures to the case of homogenous turbulence submitted to solid rotation leads to the following specific form for the rapid term in the pressure-strain correlations:

$$\begin{aligned}
\Phi_{ij}^{(2)} &= C_1 \left(a_{im}\omega_{mj} - \omega_{im}a_{mj} \right) + C_2 \left(a_{im}a_{mp}\omega_{pj} - \omega_{im}a_{mp}a_{pj} \right) \\
&\quad + C_3 \left(a_{im}\omega_{mp}a_{pq}a_{qj} - a_{im}a_{mp}\omega_{pq}a_{qj} \right)
\end{aligned}$$

[8.25]

with $\omega_{ij} = \dfrac{1}{2}\left(\overline{U}_{i,j} - \overline{U}_{j,i} \right)$. This is the most general expression which is a function of a symmetric tensor a_{ij} and an antisymmetric tensor ω_{ij} and also linear with respect to this latter tensor (cf. sections 3.6.4 and 16.5). Coefficients C_1, C_2, C_3 are functions of II and III. When this latter expression [8.25] is used ([KAS 94B]), the resulting equations for the invariants become:

$$\frac{dII^a}{dt} = 0 \quad \text{and} \quad \frac{dIII^a}{dt} = 0,$$

[8.26]

because the tensorial contraction of the antisymmetric tensor ω_{ij} with any symmetric tensor such as $a_{ij}^{(n)}$ vanishes (the evolution equations for II^a and III^a being obtained through contracting the equation for a_{ij} by a_{ij} and a_{ij}^2, respectively). However, the RDT analysis by Mansour et al. ([MAN 91]) shows that a_{ij} exhibits damped oscillations in the presence of rotation and that consequently the invariants II^a and III^a are in reality evolving. This result shows that second

order models which assume that $\Phi_{ij}^{(2)}$ is a function of the single argument a_{ij} are basically incorrect and that it is necessary to involve other arguments (hidden variables).

From these findings, an entirely original approach has been developed by Reynolds W.C. and Kassinos S.C., [REY 94] (cf. also [KAS 94B], [KAS 01]) based on a structural description of the turbulent field which allows compatibility with the rapid distortion theory (cf. section 5.8). The essential key idea which guides this work is the distinction between the concepts of dimension and component in the turbulent field ("dimensionality" and "componentality" in the terminology of the authors). The Reynolds stress tensor only embodies the componental property of turbulent velocities, related to the sharing out of energy among each component, whereas the dimensionality represents the spatial properties of turbulence structures such as elongated or flattened eddies. This latter property is conveyed by the structure tensor defined in homogenous turbulence by:

$$D_{ij} = 2k \, d_{ij} = \int \frac{\kappa_i \kappa_j}{\kappa^2} \varphi_{mm}(\vec{\kappa}) d\vec{\kappa} \,. \tag{8.27}$$

These authors also introduce the "circularity" defined by:

$$F_{ij} = 2k \, f_{ij} = \varepsilon_{inm} \varepsilon_{jhs} \int \frac{\kappa_n \kappa_h}{\kappa^2} \varphi_{ms}(\vec{\kappa}) d\vec{\kappa} = \int \frac{\widehat{\omega_i}\,\widehat{\omega_j}^*}{\kappa^2} d\vec{\kappa} \,, \tag{8.28}$$

and show the characteristic identity:

$$R_{ij} + D_{ij} + F_{ij} = 2k\delta_{ij} \,,$$

also considering that $F_{mm} = D_{mm} = R_{mm} = 2k$.

The turbulent field is then described using several tensors, in addition to the Reynolds stress tensor. These new tensors are not directly attainable by measurements, which is why they are called "hidden variable" or "internal variables".

Higher order tensors can be defined, such as the fourth order tensor $a_{ml}^{ij} = \int \frac{\kappa_m \kappa_l}{\kappa^2} \varphi_{ij}(\vec{\kappa}) d\vec{\kappa}$ already considered in section 8.1.2. Using tensorial contraction, we can recover in particular $a_{mm}^{ij} = R_{ij}$ and $a_{ml}^{jj} = D_{ml}$.

It is also possible to define several types of anisotropies:

$$
a_{ij} = \frac{R_{ij} - \frac{2}{3}k\delta_{ij}}{2k}, \quad y_{ij} = \frac{D_{ij} - \frac{2}{3}k\delta_{ij}}{2k}, \quad x_{ij} = \frac{F_{ij} - \frac{2}{3}k\delta_{ij}}{2k},
$$

with $a_{ij} + y_{ij} + x_{ij} = 0$ in homogenous turbulence.

In non-homogenous turbulence, there is an additional non-homogenity tensor $C_{ij} = \overline{\psi'_{i,m}\,\psi'_{j,m}}$ where ψ'_i is the vector stream function satisfying $u_i = \varepsilon_{ihs}\psi'_{s,h}$.

Moreover, we have the extended definitions:

$$
D_{ij} = \overline{\psi'_{n,i}\,\psi'_{n,j}}, \qquad F_{ij} = \overline{\psi'_{ni,n}\,\psi'_{j,n}}
$$

and also the identity $R_{ij} + D_{ij} + F_{ij} + C_{ij} + C_{ji} = 2k\delta_{ij}$.

These authors have studied the evolution of these tensors in rapid distortion (cf. section 5.8). The general RDT solution for a turbulence spectrum submitted to mean rotation can be obtained analytically (Reynolds W.C. and Kassinos S.C., [REY 94]).

The solution can be expressed as:

$$
\varphi_{ij}\left(\vec{\kappa},t\right) = \frac{1}{2}\left[\varphi_{ij}\left(\vec{\kappa},0\right) + \mathscr{F}_{ij}\left(\vec{\kappa},0\right)\right]
$$

$$
+\frac{1}{2}\left[\varphi_{ij}\left(\vec{\kappa},0\right) - \mathscr{F}_{ij}\left(\vec{\kappa},0\right)\right]\cos\left(2\vec{\omega}\frac{\vec{\kappa}}{\kappa}t\right)
$$

$$
+\frac{1}{2}\left[\varepsilon_{ihs}\frac{\kappa_h}{\kappa}\varphi_{sj}\left(\vec{\kappa},0\right) + \varepsilon_{jhs}\frac{\kappa_h}{\kappa}\varphi_{is}\left(\vec{\kappa},0\right)\right]\sin\left(2\vec{\omega}\frac{\vec{\kappa}}{\kappa}t\right),
$$

where $\vec{\omega}$ is the rotation vector and $\mathscr{F}_{ij}\left(\vec{\kappa},t\right) = \varepsilon_{inm}\varepsilon_{jpq}\frac{\kappa_n\kappa_p}{\kappa^2}\varphi_{mq}\left(\vec{\kappa},t\right)$.

We can deduce by integration the relation:

$$
r_{ij} = f_{ij} = (\delta_{ij} - d_{ij})/2 \quad \text{with} \quad r_{ij} = R_{ij}/2k. \tag{8.29}
$$

Devising a closure model for the general tensor a^{ij}_{ml} would of course give a means to represent the RDT. The usual closures in terms of a_{ij} do not provide the correct response to imposed rotation. Extensions in terms of a_{ij} and y_{ij} appear natural, but the linear model $a^{ij}_{ml}(a_{sh}, y_{sh})$, while being correct for low anisotropies, fails for high anisotropies by giving non-realizable results. The use of group theory allows any tensor to be split into symmetric and antisymmetric sub-tensors. Thus, like the velocity gradient tensor can be split into:

$$\overline{U}_{i,j} = S_{ij} - \varepsilon_{ijm}\omega_m,$$

it is also possible to split the tensor a^{ij}_{ml} into the sum of 10 sub-tensors distributed among three groups:

  – a completely symmetric part that is not directly involved in Φ_{ij};

  – a part that can be expressed according to δ_{ij}, R_{ij} and D_{ij};

  – a part involving a third order symmetric pseudo-tensor denoted Q_{ijk} and defined by $Q_{ijk} = \varepsilon_{ipq}a^{jq}_{pk}$ and that the authors have called "stropholysis" (meaning "symmetry break up"). This tensor cannot be expressed according to preceding tensors and contains the missing information in rotation turbulence.

Indeed, in the case of pure rotation, the rapid redistribution term reads:

$$\Phi^{(2)}_{ij} = a^{im}_{lj}\omega_{ml},$$

and the fact that a^{im}_{lj} is not symmetric in m and l, implies that $\Phi^{(2)}_{ij}$ is non-zero, the antisymmetric part $\left(a^{im}_{lj} - a^{il}_{mj}\right)\omega_{ml} \neq 0$ giving rise to stropholysis.

The method of calculating Q_{ijk} from its transport equation proved to be hard and heavy to carry on and the authors proposed another method of approach.

A particle representation model is thus proposed as the conceptual basis of a structural model. Hypothetical "particles" are characterized by the group of properties (V_i velocity vector, W_i vorticity vector, N_i gradient vector, A_i eddy-axis vector and P the pressure). These properties are determined referring to RDT and thus statistical ensemble averaging can be performed, leading to one point equations.

The structure-based model first refers to the eddy-axis vector:

$$A_i = \frac{\ell_i}{\sqrt{\ell_j \ell_j}} \, ,$$

where ℓ_i denotes a line element along the eddy axis. On this basis, the tensors A_{ij} and α_{ij} are defined by:

$$A_{ij} = 2k.\alpha_{ij} = \overline{V^2 A_i A_j} \, .$$

The particle velocity vector V_i is decomposed respectively to a component aligned with A_i and a component normal to A_i, thus introducing the scalar parameters ϕ (jettal mode) and γ (vortical mode). Tensor r_{ij} is related to α_{ij}, ϕ and γ through an algebraic, very simple, constitutive law, whereas the structure parameters (α_{ij}, ϕ and γ) are obtained from transport equations. It is worth noting that the transport equation of R_{ij} is thus no longer used. In the presence of high rotation, the model has a fixed point ($\phi = 1/2$, $\gamma = 0$) corresponding to the random-Fourier spectrum due to spectral scrambling effect:

$$\varphi_{ij}(\vec{\kappa}) = \frac{1}{2}\varphi_{hh}(\vec{\kappa})\left(\delta_{ij} - \frac{\kappa_i \kappa_j}{\kappa^2} \right), \qquad\qquad [8.30]$$

for which $a_{ij} = -\frac{1}{2}y_{ij}$ (cf. equation [8.29]).

However, the system gives rise to undamped oscillations. Damping must thus be modeled explicitly by introducing an additional correction. This approach presents many similarities with the approach by C. Cambon *et al.*, [CAM 92A] on rotating turbulence.

The definition of structure tensors in non-homogenous turbulence involves the turbulent vector stream function such that $u_i = \varepsilon_{ijk}\Psi_{k,j}$. Relying on this basis, transport equations can be established. Kassinos S.C. *et al.* [KAS 00] propose in this framework a transport model for stropholysis Q_{ijk} in non-homogenous turbulence:

$$\frac{dQ_{ijk}}{dt} = -G_{jm}Q_{imk} - 0.5G_{tm}\varepsilon_{tik}R_{mj} + 0.5G_{jm}(Q_{imk} + Q_{kmi})$$

$$-\frac{\varepsilon}{k}Q_{ijk} + kf_{ijk} + \left[\left(\nu\delta_{rs} + \frac{C_\nu}{\sigma_Q}R_{rs}\tau\right)Q_{ijk,s}\right]_{,r} ,$$

where G_{ij} is the mean velocity gradient, and τ is a characteristic time scale of turbulence.

The term deriving from pressure is obtained by elliptic relaxation in a form similar to the Durbin model (cf. section 8.3.2):

$$L^2 f_{ijk,mm} - f_{ijk} = -\mathscr{P}_{ijk}/k ,$$

$$\mathscr{P}_{ijk} = Q_{ijk} + G_{jm}Q_{imk} + 0.5G_{tm}\varepsilon_{tik}R_{mj} - 0.5G_{mj}(Q_{imk} + Q_{kmi}) + \frac{1}{\tau}Q_{ijk} .$$

This page intentionally left blank

Chapter 9

Modeling the Turbulent Flux Evolution Equations for a Passive Scalar

The turbulent fluxes of a passive scalar are represented by the double correlations for the fluctuating velocities and scalar $F_{\gamma i} = \overline{u_i \gamma}$. They are the closure term in the evolution equations for the mean passive scalar $\overline{\Gamma}$ such as temperature or concentration of a contaminant having no mechanical action on the fluid. Modeling these fluxes through a transport equation allows a more refined approach of the problem of turbulent diffusion than is possible from the usual gradient theories.

9.1. Evolution equations of the turbulent fluxes of a passive scalar

These equations are derived from the evolution equations of the fluctuating velocity and of the fluctuation of the passive scalar (cf. Chapter 2). We shall only consider in the present chapter the transfer of a passive scalar, i.e. the hydrodynamic field is supposed to be unaffected by the scalar field (this is the case for example for a temperature or concentration field with small differences behaving as a passive contaminant).

$$\underbrace{\frac{\partial F_{\gamma i}}{\partial t} + \overline{U}_j F_{\gamma i,j}}_{(a)} = \underbrace{-R_{ij}\,\overline{\Gamma}_{,j}}_{(b)} \underbrace{-F_{\gamma j}\overline{U}_{i,j}}_{(c)} \underbrace{-\left(\overline{u_i u_j \gamma}\right)_{,j}}_{(d)} \underbrace{-\frac{1}{\rho}\left(\overline{\gamma p}\right)_{,i}}_{(e)} + \underbrace{\frac{1}{\rho}\,\overline{p\gamma_{,i}}}_{(f)}$$

$$\underbrace{-(\nu + \sigma)\overline{u_{i,j}\gamma_{,j}}}_{(g)} + \underbrace{\left(\nu\overline{\gamma u_{i,j}} + \sigma\overline{u_i\gamma_{,j}}\right)_{,j}}_{(h)} . \tag{9.1}$$

This equation (cf. [2.15]) is obtained from the combination of the fluctuating velocity equation [2.6] and for the scalar fluctuation [2.14].

Interpretation of terms:

1) Material derivative

(a) $\left\{ \begin{array}{l} \dfrac{\partial F_{\gamma i}}{\partial t} \\[2mm] +\overline{U}_j F_{\gamma i,j} \end{array} \right.$ time derivative and spatial convective derivative.

2) Gradient terms

(d) $-\left(\overline{u_i u_j \gamma}\right)_{,j}$ turbulent diffusion due to velocity fluctuations;

(e) $-\dfrac{1}{\rho}\left(\overline{\gamma p}\right)_{,i}$ turbulent diffusion due to pressure fluctuations;

(h) $+\left(\nu\overline{\gamma u_{i,j}} + \sigma\overline{u_i\gamma_{,j}}\right)_{,j}$ molecular diffusion (if $Pr = 1$, then $(h) = \nu F_{\gamma i,jj}$).

3) Source and sink terms

(b) $-R_{ij}\overline{\Gamma}_{,j}$ production due to mean gradients of $\overline{\Gamma}$ which enhance the scalar fluctuations;

(c) $-F_{\gamma j}\overline{U}_{i,j}$ production due to mean velocity gradients which enhance velocity fluctuations;

(f) $+\dfrac{1}{\rho}\overline{p\gamma_{,i}}$ term similar to the pressure-strain correlation in the Reynolds stress equation, this term corresponds to a mechanism which limits the increase of the turbulent flux;

(g) $-(\nu + \sigma)\overline{u_{i,j}\gamma_{,j}}$ dissipation term.

9.2. Order of magnitude of terms

In the spirit of section 4.1 we introduce the characteristic scales:

– $\gamma_T = \sqrt{\overline{\gamma^2}}$, characteristic scale of the passive scalar fluctuations, square root of variance;

– $\ell_\gamma = \dfrac{u\gamma_T^2}{\varepsilon_\gamma}$, macroscale relative to the estimate of gradients of mean scalar quantities and correlations. Considering that turbulence is a single phenomenon with dynamic origin (there is no "thermal turbulence" without a sustaining velocity turbulence!), we can easily admit $\dfrac{\ell_\gamma}{\ell} \sim 1$;

– λ_γ , microscale defined by $\varepsilon_\gamma = 12\sigma \dfrac{\gamma_T^2}{\lambda_\gamma^2}$;

– η_γ , Kolmogorov scale for the scalar $\eta_\gamma = \left(\dfrac{\nu\sigma^2}{\varepsilon} \right)^{1/4}$.

The scales allow us, along the same principles as those presented in section 4.1, to estimate the orders of magnitude of correlations and their gradient involving scalar γ.

Thus we assume $\gamma_{,i} = \mathcal{O}\left(\gamma / \lambda_\gamma \right)$ and $\overline{\Gamma}_{,i} = \mathcal{O}(\Gamma / \ell_\gamma) = \mathcal{O}(\Gamma / \ell)$.

It can be easily shown that:

$\dfrac{\lambda_\gamma}{\ell} \sim Pr^{-1/2} Re_t^{-1/2}$ (indeed $12\sigma \dfrac{\gamma_T^2}{\lambda_\gamma^2} = \dfrac{\gamma_T^2 u}{\ell}$), $Pr = \nu / \sigma$ and $\dfrac{\lambda_\gamma}{\lambda} = Pr^{-1/2}$.

Similarly, $\dfrac{\eta_\gamma}{\eta} = Pr^{-1/2}$.

9.3. Modeling dissipative terms

The correlation *(g)* vanishes in isotropic turbulence. As in section 6.2 for the correlation $\overline{vu_{i,j}u_{k,l}}$ in dynamics, we will get the following order of magnitude for the dissipation term:

$$\overline{vu_{i,j}\gamma_{,j}} = \mathscr{O}(\mathrm{Re}_t^{-1/2}) , \qquad\qquad [9.2]$$

indeed $\overline{vu_{i,j}\gamma_{,j}} = \mathscr{O}\left(\dfrac{\lambda}{\ell}(\overline{u_i\gamma}-0)\right)$ in which the zero stands for the vanishing correlation in isotropic turbulence. We see that the term *(g)* remains negligible in anisotropic turbulence if the Reynolds number is high and if the Prandtl number remains near unity.

9.4. Modeling the turbulent diffusion terms

The diffusion terms in the transport equation of $F_{\gamma i}$ reads:

$$D_{\gamma ij} = -\left[\underbrace{\overline{u_i u_j \gamma} + \frac{1}{\rho}\overline{\gamma p}\delta_{ij}}_{(*)} \underbrace{-\left(\overline{v\gamma u_{i,j}} + \overline{\sigma u_i \gamma_{,j}}\right)}_{(**)} \right]_{,j} , \qquad\qquad [9.3]$$

(*) terms to be modeled
(**) terms that are negligible at high Reynolds numbers Re_t .

a) Triple correlations $\overline{u_i u_j \gamma}$

Samaraweera D.S.A., [SAM 78] has developed a technique similar to the one presented in section 6.3a for the triple velocity correlations. The approximation is thus derived from the evolution equations of the triple correlations $T_{ik\gamma} = \overline{u_i u_k \gamma}$ using simplifying hypotheses. This equation can be written at high turbulence Reynolds numbers:

$$\underbrace{\frac{dT_{ik\gamma}}{dt}}_{(a)} = \underbrace{-T_{il\gamma}\overline{U}_{k,l} - T_{lk\gamma}\overline{U}_{i,l}}_{(b)} \underbrace{-T_{ikl}\overline{\Gamma}_{,l}}_{(c)} \underbrace{-\left(\overline{u_i u_k u_l \gamma}\right)_{,l}}_{(d)}$$

$$-\frac{1}{\rho}\left(\overline{u_i\gamma p_{,k}}+\overline{u_k\gamma p_{,i}}\right)+\underbrace{F_{\gamma i}R_{kl,l}+R_{ik}F_{\gamma l,l}+F_{\gamma k}R_{il,l}}_{(f)}\;. \qquad [9.4]$$

The following hypotheses are introduced:

– approximation of the quadruple correlations using the Millionshtchikov hypothesis

$$\overline{u_i u_k u_l \gamma}=F_{\gamma i}R_{kl}+R_{ik}F_{\gamma l}+F_{\gamma k}R_{il}\,, \qquad [9.5]$$

and also $(d)+(f)=-\left(R_{kl}F_{\gamma i,l}+F_{\gamma l}R_{ik,l}+R_{il}F_{\gamma k,l}\right)$;

– pressure correlation *(e)* is approximated from a term proportional to $-\dfrac{\varepsilon}{k}T_{\gamma ik}$

$$(e)=-c\frac{\varepsilon}{k}T_{\gamma ik}\;;$$

– term *(b)* is neglected, as in section 6.3 for the equation of T_{ijk}.

If term *(a)* is now also neglected, then we find the approximation:

$$T_{\gamma ik}=-c_\gamma\frac{k}{\varepsilon}\left(R_{kl}F_{\gamma i,l}+R_{il}F_{\gamma k,l}+F_{\gamma l}R_{ik,l}+T_{ikl}\,\overline{\Gamma}_{,l}\right). \qquad [9.6]$$

Launder B.E., [LAU 76] has only retained the first two terms in this approximation and assumed:

$$T_{\gamma ik}=-c_\gamma\frac{k}{\varepsilon}\left(R_{kl}F_{\gamma i,l}+R_{il}F_{\gamma k,l}\right), \qquad [9.7]$$

where c_γ is a constant determined from "numerical optimization", found to be of order 0.11.

The various expressions given by different authors (Donaldson C. du P. *et al.*, [DON 72]; Lumley J.L. and Khajeh-Nouri B., [LUM 74]; Wyngaard J.C. and Cote O.R., [WYN 74]) are particular cases of [9.6].

Donaldson C. du P. *et al.*, [DON 72] have introduced the approximation:

$$T_{\gamma ik} = -c'_{\gamma}\frac{k^2}{\varepsilon}\left(F_{\gamma i,k} + F_{\gamma k,i}\right).$$ [9.8]

Lumley J.L. and Khajeh-Nouri B. [LUM 74] are led to a similar expression using purely dimensional reasoning.

b) Turbulent diffusion by pressure

No precise information is available for the value of the term *(e)* in equation [9.1]. Thus, as we did in section 6.3 this term will be neglected.

9.5. Modeling the pressure-passive scalar gradient correlations

The term *(f)* in equation [9.1] is the analog of the pressure-strain correlation in the Reynolds stress equations.

After multiplying expression [6.12] in section 6.4 based on Green theorem for fluctuating pressure by γ_j and after averaging, we get:

$$\overline{\left(\frac{p}{\rho}\gamma_{,j}\right)}(\vec{x}) = \frac{1}{2\pi}\iiint_{\Omega}\overline{U'_{m,k}\,u'_{k,m}\,\gamma_{,j}}\,\frac{d\mathscr{V}}{r} + \frac{1}{4\pi}\iiint_{\Omega}\overline{(u_m u_k)'_{,mk}\,\gamma_{,j}}\,\frac{d\mathscr{V}}{r}$$

$$+ \frac{1}{4\pi\rho}\iint_{\partial\Omega}\overline{\left[\frac{1}{r}\frac{\partial p}{\partial n} - p\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]'\cdot\gamma_{,j}}\,d\mathscr{S}.$$ [9.9]

The modeling technique is in every respect, similar to the one used in Chapter 6 for approximating the pressure-strain correlations.

When *r* is sufficiently large the two point statistical correlations appearing in [9.9] vanish. If we suppose that the non-homogeneities are not very influential in the integrals, we will get the expression:

$$\overline{\left(\frac{p}{\rho}\gamma_{,j}\right)}(\vec{x}) = -\frac{1}{2\pi}\iiint_{\Omega}\frac{\partial\overline{U}'_m}{\partial\xi_k}\frac{\partial^2\overline{u'_k\,\gamma}}{\partial\xi_m\partial\xi_j}\frac{d\mathscr{V}}{r} - \frac{1}{4\pi}\iiint_{\Omega}\frac{\partial^3\overline{u'_m\,u'_k\,\gamma}}{\partial\xi_m\partial\xi_k\partial\xi_j}\frac{d\mathscr{V}}{r}$$

$$+ \frac{1}{4\pi\rho}\iint_{\partial\Omega}\left[\frac{1}{r}\frac{\partial}{\partial n}\overline{p'\gamma_{,j}} - \overline{p'\gamma_{,j}}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\mathscr{S},$$ [9.10]

where we have denoted $\vec{\xi} = \vec{y} - \vec{x}$.

Far from any wall, the last term $\Phi_{\gamma j}^{(s)}$ will be negligible.

1) Non-linear part $\Phi_{\gamma j}^{(1)}$

The most common approximation was proposed by Monin A.S., [MON 65]. It is inspired by the Rotta hypothesis for returning to isotropy due to pressure-strain correlations in the Reynolds stress equations:

$$\Phi_{\gamma j}^{(1)} = -c_{\gamma 1} \frac{\varepsilon}{k} F_{\gamma j}.$$ [9.11]

This approximation has been retained by various authors, Launder B.E., [LAU 75A]; Lumley J.L., [LUM 75C]; Donaldson C. du P. *et al.*, [DON 72]. In fact, this term would have to involve not only the scale $\dfrac{k}{\varepsilon} = \dfrac{\overline{u_i u_i}}{2\varepsilon}$ but also $\dfrac{q}{\varepsilon_\gamma} = \dfrac{\overline{\gamma\gamma}}{2\varepsilon_\gamma}$.

However, with a hypothesis of spectral equilibrium these two time scales can be assumed to be proportional, thus the use of k/ε only has the advantage of not introducing new unknowns.

Lumley J.L. and Khajeh-Nouri B., [LUM 74] have proposed the approximation:

$$\frac{1}{\rho} \overline{\gamma p_{,j}} = \mathscr{F}\left(F_{\gamma i}, R_{ij}, \varepsilon\right),$$

in the absence of gravity forces and have obtained an expression similar to:

$$\frac{1}{\rho} \overline{\gamma p_{,j}} = \left[\left(a_0 + a_1 \frac{b_{kl} b_{kl}}{k^2}\right)\delta_{ij} + a_2 \frac{b_{ij}}{k} + a_3 \frac{b_{jm} b_{mi}}{k^2}\right] \frac{F_{\gamma i}\varepsilon}{k},$$ [9.12]

where b_{ij} is the deviator of the Reynolds stress tensor.

The fact to consider $\dfrac{1}{\rho}\overline{\gamma p_{,j}}$ rather than $\dfrac{1}{\rho}\overline{p\,\gamma_{,j}}$ is not really essential. We shall use for $\Phi_{\gamma j}^{(1)}$ a truncated form of [9.12] (Launder B.E., [LAU 76]; Samaraweera D.S.A., [SAM 78]):

$$\Phi_{\gamma j}^{(1)} = -c_{\gamma 1} \frac{\varepsilon}{k} F_{\gamma j} - c'_{\gamma 1} \frac{\varepsilon}{k^2} b_{jm} F_{\gamma m}$$ [9.13]

2) Linear part $\Phi_{\gamma j}^{(2)}$

Neglecting non-homogenity effects in [9.10], we can express $\Phi_{\gamma j}^{(2)}$ in the following way (Samaraweera D.S.A., [SAM 78]):

$$\Phi_{\gamma j}^{(2)} = c_{nj}^m \overline{U}_{n,m} ,$$
[9.14]

with $c_{nj}^m = -\dfrac{1}{2\pi} \iiint_{\Omega} \dfrac{\partial^2 \overline{u'_m \gamma}}{\partial \xi_n \partial \xi_j} \dfrac{d\mathcal{V}}{r} .$
[9.15]

The third order tensor c_{nj}^m must satisfy several constraints:

– symmetry: $c_{nj}^m = c_{jn}^m$
[9.16]

– continuity: $c_{mj}^m = 0$
[9.17]

(coming from the zero divergence of the fluctuating velocities)

– Green theorem: $c_{jj}^m = 2F_{\gamma m}$
[9.18]

This latter relation suggests approximating c_{nj}^m as a linear combination of the turbulent fluxes:

$$c_{nj}^m = \alpha_\gamma \delta_{nj} F_{\gamma m} + \beta_\gamma \delta_{mj} F_{\gamma n} + \eta_\gamma \delta_{nm} F_{\gamma j} .$$
[9.19]

Conditions [9.16], [9.17], [9.18] thus imply:

$$\begin{cases} \beta_\gamma = \eta_\gamma \\ 0 = \alpha_\gamma + \beta_\gamma + 3\eta_\gamma \\ 2 = 3\alpha_\gamma + \beta_\gamma + \eta_\gamma \end{cases}$$
[9.20]

We thus find:

$$\Phi_{\gamma j}^{(2)} = 0.8 F_{\gamma m} \overline{U}_{j,m} - 0.2 F_{\gamma m} \overline{U}_{m,j} .$$
[9.21]

This expression (quasi-isotropic approximation) was also given by Lumley J.L., [LUM 75C] on the basis of arguments, similar to the previous ones. In addition, Launder B.E., [LAU 75A] has also given an approximate expression which is the analog of expression [6.29] introduced in Chapter 6:

$$\Phi_{\gamma j}^{(2)} = -c_{\gamma 2} P_{\gamma j} \,, \tag{9.22}$$

where $P_{\gamma j} = -F_{\gamma m} \overline{U}_{j,m}$ is the production rate of $F_{\gamma j}$ coming from the part of the equation of $\overline{u_j \gamma}$ responsible for pressure fluctuations, i.e. $\overline{\gamma \dfrac{du_j}{dt}}$.

Thus,

$$\Phi_{\gamma j}^{(2)} = c_{\gamma 2} F_{\gamma m} \overline{U}_{j,m} \,. \tag{9.23}$$

3) Near wall modeling

In the case of a plane wall, relation [9.9] can be favorably written in the form (Samaraweera D.S.A., [SAM 78]) of a volume integral using similar reasoning as in section 6.4.3:

$$\phi_{\gamma j} = \frac{1}{4\pi} \iiint_{\Omega} \left[\overline{\frac{\partial^2 \left(u'_l u'_m\right)}{\partial y_l \partial y_m} \frac{\partial \gamma}{\partial x_j}} + 2 \left(\frac{\partial \overline{U}_m}{\partial y_k} \right)' \overline{\frac{\partial u'_k}{\partial y_m} \frac{\partial \gamma}{\partial x_j}} \right] \times$$

$$\left[\frac{1}{\left\| \vec{x} - \vec{y} \right\|} + \frac{1}{\left\| \vec{x} - \vec{y}^* \right\|} \right] d\mathcal{V} \,, \tag{9.24}$$

(same notations as for relation [6.32]).

This expression suggests that there are two contributions to $\Phi_{\gamma j}^{(s)}$ coming from the reflected influence of $\Phi_{\gamma j}^{(1)}$ and $\Phi_{\gamma j}^{(2)}$. We are then led to set the approximation:

$$\Phi_{\gamma j}^{(s)} = \left[-c_{\gamma 1}^w \frac{\varepsilon}{k} F_{\gamma j} + d_{lj}^m \overline{U}_{l,m} \right] f\left(\frac{\ell}{x_2} \right) \,. \tag{9.25}$$

The third order tensor is assumed to be represented by a linear combination of the turbulent fluxes:

$$d_{lj}^{m} = \alpha'_{\gamma}\, \delta_{lj} F_{\gamma m} + \beta'_{\gamma}\, \delta_{mj} F_{\gamma l} + \eta'_{\gamma}\, \delta_{ml} F_{\gamma j}\,. \qquad [9.26]$$

This tensor is symmetric $d_{lj}^{m} = d_{jl}^{m}$, moreover it must satisfy $d_{mj}^{m} = 0$ (mass conservation for equation [9.24]). It then follows that:

$$\begin{cases} \beta'_{\gamma} = \eta'_{\gamma} \\ 0 = \alpha'_{\gamma} + 4\beta'_{\gamma} \end{cases}$$

and thus:

$$\Phi_{\gamma j}^{(s)} = \left[-c_{\gamma 1}^{w}\, \frac{\varepsilon}{k} F_{\gamma j} - 4\beta'_{\gamma}\, \delta_{lj} F_{\gamma m} \overline{U}_{l,m} + \beta'_{\gamma}\, \delta_{mj} F_{\gamma l} \overline{U}_{l,m} \right] f\!\left(\frac{\ell}{x_2} \right). \qquad [9.27]$$

This kind of second order closure presented in the present chapter is necessary to handle complex flow cases with heat or mass transfer, in particular flows in which the thermal boundary conditions are different from the dynamic ones, excluding any analogy between the thermal and dynamic fields. Applications of this type of closure to heated dissymmetric free flows have been considered by Dekeyser I., [DEK 82].

4) Other models

Another model was introduced by Gibson M.M. and Launder B.E., [GIB 78B] which is similar to the model by the same authors for the wall effect in the pressure-strain correlations (cf. section 6.4, equation [6.43]). It is formulated as follows:

$$\Phi_{\gamma j}^{(s)} = c_{\gamma 1}^{w}\, \frac{\varepsilon}{k} F_{\gamma k} n_k n_j f\!\left(\frac{\ell}{n_i r_i} \right) + c_{\gamma 2}^{w} \Phi_{\gamma k}^{(2)} n_k n_j f\!\left(\frac{\ell}{n_i r_i} \right). \qquad [9.28]$$

9.6. Determination of numerical constants

a) Constants $c_{\gamma 1}$ and $c_{\gamma 2}$ in formulations [9.11] and [9.23]

Let us consider the homogenous turbulent shear flow with constant mean velocity gradient $\Lambda_{12} = \overline{U}_{1,2}$ and constant mean scalar gradient $\overline{\Gamma}_{,2}$. Equations [9.1] can be written in this case:

$$-R_{12}\overline{\Gamma}_{,2} - F_{\gamma 2}\Lambda_{12} - c_{\gamma 1}\frac{\varepsilon}{k} F_{\gamma 1} + c_{\gamma 2} F_{\gamma 2}\Lambda_{12} = 0\,, \qquad [9.29]$$

$$-R_{22}\overline{\Gamma}_{,2} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma 2} = 0 .$$ [9.30]

Equation [9.30] implies immediately:

$$F_{\gamma 2} = -\frac{1}{c_{\gamma 1}}\frac{kR_{22}}{\varepsilon}\overline{\Gamma}_{,2} .$$ [9.31]

Thus we deduce from [9.29]:

$$\frac{F_{\gamma 1}}{F_{\gamma 2}} = \frac{R_{12}}{R_{22}} + \frac{1}{c_{\gamma 1}}\frac{k}{\varepsilon}(c_{\gamma 1}-1)\Lambda_{12} .$$ [9.32]

Supposing spectral equilibrium $\varepsilon = -R_{12}\Lambda_{12}$, we find:

$$\frac{F_{\gamma 1}}{F_{\gamma 2}} = \frac{R_{12}}{R_{22}} + \frac{k(1-c_{\gamma 2})}{c_{\gamma 1}R_{12}}$$ [9.33]

It is possible to define a turbulence Prandtl number by $\mathrm{Pr}_t = \dfrac{R_{12}/\overline{U}_{1,2}}{F_{\gamma 2}/\overline{\Gamma}_{,2}}$, which,

of course, is not a constant in the general case. We can then deduce from [9.31] taking into account the hypothesis of spectral equilibrium:

$$c_{\gamma 1} = \frac{k.R_{22}}{R_{12}{}^2}\mathrm{Pr}_t \quad \text{or} \quad c_{\gamma 1} = \left(\frac{k}{R_{12}}\right)^2\frac{R_{22}}{k}\mathrm{Pr}_t ,$$ [9.34]

Equation [9.33] thus implies:

$$\frac{F_{\gamma 1}}{F_{\gamma 2}} = \frac{k}{c_{\gamma 1}R_{12}}(1-c_{\gamma 2}+\mathrm{Pr}_t)$$ [9.35]

We recall here the Reynolds stress values obtained by Champagne, Harris and Corrsin (cf. section 6.5):

$$-R_{12}/k = 0.33 \qquad (R_{22}-\tfrac{2}{3}k)/k = -0.18$$

The measurements by Webster C.A.G., [WEB 64] suggest $F_{\gamma 1}/F_{\gamma 2} \simeq -1.1$.

Moreover, the turbulence Prandtl number obtained in various free flows is of order 0.66 in the mean. These experimental values lead, using [9.34] and [9.35], to the following estimates of the constants (Launder B.E., [LAU 75A]; Samaraweera D.S.A., [SAM 78]):

$$c_{\gamma 1} = 2.9, \quad c_{\gamma 2} = 0.5 \tag{9.36}$$

Launder B.E., [LAU 76] rather suggests $c_{\gamma 2} = 0.33$ because the values of R_{11}/R_{22} given by Webster are 25% lower than those usually obtained in homogenous turbulent flows, we may also suppose that the corresponding values of $F_{\gamma 1}/F_{\gamma 2}$ are also too low. The choice $c_{\gamma 1} = 3.0$ and $c_{\gamma 2} = 0.33$ gives the values $\mathrm{Pr}_t = 0.67$ and $F_{\gamma 1}/F_{\gamma 2} = -1.3$.

b) Quasi-isotropic model, formulations [9.13] and [9.21]

Using approximations [9.11] and [9.21] leads to two incompatible values for $c_{\gamma 1}$. If we choose to use [9.21] it is thus necessary to introduce the non-linear formulation [9.13].

The flux equations in the case of the homogenous turbulent shear flow reads:

$$-R_{12}\overline{\Gamma}_{,2} - F_{\gamma 2}\varLambda_{12} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma 1} - c'_{\gamma 1}\frac{\varepsilon}{k^2}b_{11}F_{\gamma 1} - c'_{\gamma 1}\frac{\varepsilon}{k^2}b_{12}F_{\gamma 2} + 0.8F_{\gamma 2}\varLambda_{12} = 0,$$
$$\tag{9.37}$$

$$-R_{22}\overline{\Gamma}_{,2} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma 2} - c'_{\gamma 1}\frac{\varepsilon}{k^2}b_{12}F_{\gamma 1} - c'_{\gamma 1}\frac{\varepsilon}{k^2}b_{22}F_{\gamma 2} - 0.2F_{\gamma 1}\varLambda_{12} = 0. \tag{9.38}$$

Taking into account $\varepsilon = -R_{12}\varLambda_{12}$ and $R_{12}\overline{\Gamma}_{,2} = \mathrm{Pr}_t\,F_{\gamma 2}\varLambda_{12}$ we will get:

$$\mathrm{Pr}_t\frac{k}{R_{12}} + 0.2\frac{k}{R_{12}} - c_{\gamma 1}\frac{F_{\gamma 1}}{F_{\gamma 2}} - c'_{\gamma 1}\frac{b_{11}}{k}\frac{F_{\gamma 1}}{F_{\gamma 2}} - c'_{\gamma 1}\frac{b_{12}}{k} = 0, \tag{9.39}$$

$$\mathrm{Pr}_t\frac{R_{22}}{k}\left(\frac{k}{R_{12}}\right)^2 - c_{\gamma 1} - c'_{\gamma 1}\frac{b_{12}}{k}\frac{F_{\gamma 1}}{F_{\gamma 2}} - c'_{\gamma 1}\frac{b_{22}}{k} + 0.2\frac{k}{R_{12}}\frac{F_{\gamma 1}}{F_{\gamma 2}} = 0. \tag{9.40}$$

After substituting the experimental values for Pr_t and $F_{\gamma 1}/F_{\gamma 2}$ and also the values for the Reynolds stresses, we deduce from [9.39] and [9.40] the estimates of $c_{\gamma 1}$ and $c'_{\gamma 1}$. (cf. Table 9.1).

	Pr_t	$-F_{\gamma 1}/F_{\gamma 2}$
Mean values of experimental data Webster, 1964 in particular	$0.6 \sim 0.7$	$1.1 \sim 1.3$
Model $c_{\gamma 1} = 5.7$ and $c'_{\gamma 1} = -3.2$ Samaraweera, 1978	0.66	1.07
Model $c_{\gamma 1} = 5.7$ and $c'_{\gamma 1} = -6.1$ Gibson and Launder, 1978	0.73	1.23

Table 9.1. *Values of the turbulence Prandtl number in free flows*

Launder B.E., [LAU 76] also suggests the couple of values $c_{\gamma 1} = 3.8$, $c'_{\gamma 1} = -2.2$. Neither of the two formulations [9.11]+[9.23] or [9.13]+[9.21] has been sufficiently tested on various flows to decide definitely of the best choice between the two.

c) Wall effect

In a turbulent wall flow the values of $F_{\gamma 1}/F_{\gamma 2}$ and Pr_t reach approximatively – 2.0 and 0.9 respectively while in free flows these values where respectively –1.2 and 0.66 (a great dispersion however exists in the measurements).

1) Model [9.11], [9.23] + [9.27]

The flux equations in wall flow (logarithmic region) reads:

$$-R_{12}\overline{\Gamma}_{,2} - F_{\gamma 2}\overline{U}_{1,2} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma 1} + c_{\gamma 2}F_{\gamma 2}\overline{U}_{1,2} - c_{\gamma 1}^w\frac{\varepsilon}{k}F_{\gamma 1} - 4\beta'_\gamma F_{\gamma 2}\overline{U}_{1,2} = 0, \quad [9.41]$$

$$-R_{22}\overline{\Gamma}_{,2} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma 2} - c_{\gamma 1}^w\frac{\varepsilon}{k}F_{\gamma 2} - \beta'_\gamma F_{\gamma 1}\overline{U}_{1,2} = 0. \quad [9.42]$$

Taking into account $\varepsilon = -R_{12}\overline{U}_{1,2}$ and $R_{12}\overline{\Gamma}_{,2} = \Pr_t F_{\gamma 2}\overline{U}_{1,2}$, we deduce:

$$
\begin{cases}
\dfrac{F_{\gamma 2}}{F_{\gamma 1}}\dfrac{k}{R_{12}}\left(1+\Pr_t - c_{\gamma 2}+4\beta'_\gamma\right)-\left(c_{\gamma 1}+c_{\gamma 1}^w\right)=0 \\[4mm]
\Pr_t\dfrac{R_{22}}{k}\left(\dfrac{k}{R_{12}}\right)^2-\left(c_{\gamma 1}+c_{\gamma 1}^w\right)-\beta'_\gamma\dfrac{F_{\gamma 1}}{F_{\gamma 2}}\dfrac{k}{R_{12}}=0
\end{cases}
$$

Having retained the values $c_{\gamma 1} = 2.9$ and $c_{\gamma 2} = 0.5$ Samaraweera is then led to $c_{\gamma 1}^w = 0.25$ and $\beta'_\gamma = 0.06$.

On the other hand, for $c_{\gamma 1} = 3.0$ and $c_{\gamma 2} = 0.33$ (Launder B.E., [LAU 76]) the model [9.11]+[9.23] gives from [9.34] and [9.35]: $\Pr_t = 0.81$ and $F_{\gamma 1}/F_{\gamma 2} = -1.9$.

Thus, it appears that the wall effect essentially modifies the Reynolds stresses which then indirectly affect the turbulent fluxes of the passive scalar $F_{\gamma j}$. The direct effect of the wall on $F_{\gamma j}$ remains weak in these conditions.

2) Model [9.13], [9.21]

For $c_{\gamma 1} = 5.7$ and $c'_{\gamma 1} = -6.1$ the modifications in the level of the Reynolds stresses and fluxes due to the presence of a wall also lead us to consider the correction in $\Phi_{\gamma j}^{(s)}$ as useless (cf. Table 9.2).

	$\Pr_t$	$-F_{\gamma 1}/F_{\gamma 2}$
Free flows	0.73	1.23
Wall flows	0.93	1.98

Table 9.2. *Values of the turbulence Prandtl number in wall flows*

It is thus possible that this model does not require any wall correction for the turbulent fluxes of a passive scalar, but more extensive investigations are necessary to conclude more surely.

3) Gibson and Launder model [9.28]

Used in relations [9.11] and [9.23], the effect of [9.28] is to increase the value of $c_{\gamma 1}$ from 3 to 3.5 when $c_{\gamma 1}^w = 0.5$.

In these conditions, $\Pr_t = 0.90$ and $F_{\gamma 1}/F_{\gamma 2} = -2.0$. Samaraweera chooses $c_{\gamma 1}^w = 0.75$.

The term $\Phi_{\gamma j}^{(2)}$ is not active in two-dimensional wall flows, and so the constant $c_{\gamma 2}^w$ has not been evaluated.

Model RC1

$$\frac{dF_{\gamma i}}{dt} = -F_{\gamma j}\overline{U}_{i,j} - R_{ij}\overline{\Gamma}_{,j} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma i} + c_{\gamma 2}F_{\gamma j}\overline{U}_{i,j}$$

$$-\left[c_{\gamma 1}^w\frac{\varepsilon}{k}F_{\gamma i} + 4\beta'_\gamma\,\delta_{li}F_{\gamma m}\overline{U}_{l,m} + \beta'_\gamma\,\delta_{mi}F_{\gamma l}\overline{U}_{l,m}\right]f\left(\frac{\ell}{x_2}\right)$$

$$+c_\gamma\left[\frac{k}{\varepsilon}\left(R_{kl}F_{\gamma i,l} + R_{il}F_{\gamma k,l} + F_{\gamma l}R_{ik,l}\right)\right]_{,k}$$

$$\text{or} \mapsto +c_\gamma\left[\frac{k}{\varepsilon}\left(R_{kl}F_{\gamma i,l} + R_{il}F_{\gamma k,l}\right)\right]_{,k} \quad \text{or} \mapsto +c_\gamma\left[\frac{k}{\varepsilon}R_{kl}F_{\gamma i,l}\right]_{,k}$$

Model RC2

$$\frac{dF_{\gamma i}}{dt} = -F_{\gamma j}\overline{U}_{i,j} - R_{ij}\overline{\Gamma}_{,j} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma i} - c'_{\gamma 1}\frac{\varepsilon}{k^2}b_{im}F_{\gamma m}$$

$$+0.8F_{\gamma m}\overline{U}_{i,m} - 0.2F_{\gamma m}\overline{U}_{m,i}$$

$$+c_\gamma\left[\frac{k}{\varepsilon}\left(R_{kl}F_{\gamma i,l} + R_{il}F_{\gamma k,l} + F_{\gamma l}R_{ik,l}\right)\right]_{,k}$$

$$\text{or} \mapsto +c_\gamma\left[\frac{k}{\varepsilon}\left(R_{kl}F_{\gamma i,l} + R_{il}F_{\gamma k,l}\right)\right]_{,k} \quad \text{or} \mapsto +c_\gamma\left[\frac{k}{\varepsilon}R_{kl}F_{\gamma i,l}\right]_{,k}$$

Value of constants   Basis for determination

RC1 $\begin{cases} c_{\gamma 1} = & 2.9 \\ c_{\gamma 2} = & \underset{\text{Samaraweera}}{\underline{0.5}} \end{cases} \begin{cases} = & 3.0 \\ = & \underset{\text{Launder}}{\underline{0.33}} \end{cases}$

RC2 $\begin{cases} c_{\gamma 1} = & 4.3 \\ c'_{\gamma 1} = & \underset{\text{Samaraweera}}{\underline{-3.2}} \end{cases} \begin{cases} = & 5.7 \\ = & \underset{\text{Gibson}}{\underline{-6.1}} \end{cases} \begin{cases} = & 3.8 \\ = & \underset{\text{Launder}}{\underline{-2.2}} \end{cases}$

Turbulence Prandtl number and ratio of turbulent fluxes in free flows.

$c_\gamma = 0.11$ Numerical optimization

$c'_\gamma = 0.20$ similar value as for diffusion of R_{ij}

RC1 $\begin{cases} c_{\gamma 1}^w = & 0.25 \\ \beta'_\gamma = & \underset{\text{Samaraweera}}{\underline{0.06}} \end{cases}$

Turbulence Prandtl number and ratio of turbulent fluxes in wall flows.

Table 9.3. *Recapitulative table of the prototype models*

9.7. New generation of modeling

The new second order modeling approaches presented in Chapter 8 for the dynamics have also found an extension in the case of the associated turbulent transport of a scalar. The same basic principles have been applied with an account of realizability conditions. Among others, several authors have contributed from the University of Manchester team. Let us quote in particular Craft T., Fu S., Launder B.E. and Tselepidakis D.P., [CRA 89], Shih T.H. and Lumley J.L., [SHI 85A and B], Craft T.J., [CRA 91A].

Exactly like the pressure-strain correlation model given earlier (section 8.1.3 (1)), the model for the linear part of the pressure-scalar gradient correlation $\Phi_{\gamma j}^{(2)}$ can be obtained from invariant modeling and making use of the realizability constraints. The result given by Craft T.J., [CRA 91A] is the following:

$$\Phi_{\gamma i}^{(2)} = 0.8 F_{\gamma m} \overline{U}_{i,m} - 0.2 F_{\gamma m} \overline{U}_{m,i} + \frac{1}{3} \frac{\varepsilon}{k} F_{\gamma i} \frac{P}{\varepsilon}$$

$$-0.4 F_{\gamma m} a_{il} \left(\overline{U}_{m,l} + \overline{U}_{l,m} \right) + 0.1 F_{\gamma p} a_{ip} a_{ml} \left(\overline{U}_{m,l} + \overline{U}_{l,m} \right)$$

$$-0.1 F_{\gamma j} \left(a_{im} P_{mj} + 2 a_{mj} P_{im} \right) / k + 0.15 a_{ml} \left(\overline{U}_{p,l} + \overline{U}_{l,p} \right) \left(a_{mp} F_{\gamma i} - a_{mi} F_{\gamma p} \right)$$

$$-0.05 a_{ml} \left[7 a_{mp} \left(F_{\gamma i} \overline{U}_{p,l} + F_{\gamma p} \overline{U}_{i,l} \right) - F_{\gamma p} \left(a_{ml} \overline{U}_{i,p} + a_{mp} \overline{U}_{i,l} \right) \right]. \qquad [9.43]$$

The recommended formulation for the non-linear part $\Phi_{\gamma j}^{(1)}$ being:

$$\Phi_{\gamma i}^{(1)} = -c_{\gamma 1} \frac{\varepsilon}{k} \left[\left(1 + 0.6 II \right) F_{\gamma i} + c'_{\gamma 1} a_{im} F_{\gamma m} + c''_{\gamma 1} a_{im} a_{mj} F_{\gamma j} \right] - c'''_{\gamma 1} S k a_{ij} \overline{\Gamma}_{,j} ,$$

$$[9.44]$$

with:

$$c_{\gamma 1} = 1.7 \left(1 + 1.2 \left(II.A \right)^{1/2} \right) S^{1/2} , \qquad a_{ij} = \frac{R_{ij} - \frac{2}{3} k \delta_{ij}}{k} ,$$

$$c'_{\gamma 1} = -0.8 , \qquad c''_{\gamma 1} = 1.1 , \qquad c'''_{\gamma 1} = 0.2 A^{1/2} ,$$

$$S = R^{-1} , \quad R = \frac{\varepsilon.q}{\varepsilon_{\gamma}.k} , \quad A = 1 - \frac{9}{8} (II - III) .$$

It allows a good degree of universality, giving satisfactory predictions both in homogenous turbulence and in turbulent free flows.

Chapter 10

The Passive Scalar Variance
and its Dissipation Rate

10.1. Transport equation for the variance of a passive scalar

The evolution equation for $q = \dfrac{1}{2}\overline{\gamma^2}$ can be obtained directly from the transport equation of the passive scalar fluctuation (Chapter 2).

$$\underbrace{\frac{\partial q}{\partial t} + \overline{U}_j q_{,j}}_{(a)} = \underbrace{-F_{\gamma j}\,\Gamma_{,j}}_{(b)} \underbrace{-\left(\overline{u_j \frac{\gamma^2}{2}}\right)_{,j}}_{(c)} \underbrace{+\sigma q_{,jj}}_{(d)} \underbrace{-\sigma\overline{\gamma_{,j}\gamma_{,j}}}_{(e)} . \qquad [10.1]$$

Interpretation of terms

  1) Material derivative

  (a) $\left\{ \begin{array}{l} \dfrac{\partial q}{\partial t} \\[4pt] +\overline{U}_j q_{,j} \end{array} \right.$ time derivative and spatial convective derivative.

2) Gradient terms

(c) $-\left(\overline{u_j \dfrac{\gamma^2}{2}}\right)_{,j}$ turbulent diffusion due to velocity fluctuations;

(d) $+\sigma q_{,jj}$ molecular diffusion.

3) Source and sink terms

(b) $-F_{\gamma j}\Gamma_{,j}$ production through the action mean scalar gradients on the turbulence;

(e) $-\sigma\overline{\gamma_{,j}\gamma_{,j}}$ dissipation rate of the passive scalar.

We notice the similarity with the turbulence kinetic energy equation. The difference comes, however, from the absence of pressure correlation.

10.2. Modeling the turbulent diffusion terms

Using the same method as for approximating the triple correlations T_{ijk} and $T_{\gamma ij}$ (Chapters 6 and 9) we can here rely upon the general evolution equation for $T_{i\gamma\gamma} = \overline{u_i\gamma\gamma}$ which can be written at high turbulence Reynolds numbers:

$$\underbrace{\frac{dT_{i\gamma\gamma}}{dt}}_{(a)} = \underbrace{-T_{l\gamma\gamma}\overline{U}_{i,l}}_{(b)} \underbrace{-2T_{il\gamma}\overline{\Gamma}_{,l}}_{} \underbrace{-\left(\overline{u_i u_l \gamma^2}\right)_{,l}}_{(c)} \underbrace{-\frac{1}{\rho}\overline{\gamma^2 p_{,i}}}_{(d)} \underbrace{+\overline{\gamma^2}R_{il,l}}_{(e)} \underbrace{+2F_{\gamma i}F_{\gamma l,l}}_{(f)}\ . \quad [10.2]$$

Putting forward similar arguments and hypotheses as those used for the approximation of T_{ijk} and $T_{\gamma ij}$ we get the following approximation:

$$T_{i\gamma\gamma} = -c_q \frac{k}{\varepsilon}\left(R_{il}\overline{\gamma^2}_{,l} + 2F_{\gamma l}F_{\gamma i,l} + 2T_{\gamma il}\overline{\Gamma}_{,l}\right). \quad [10.3]$$

We emphasize that the proportionality hypothesis between the dynamic scale k/ε and the scale relative to the passive contaminant q/ε_γ is inherent to this type of formulation. In circumstances for which the dynamic and the scalar fields would be very different (very different $\overline{U}_{i,j}$ and $\overline{\Gamma}_{,j}$ gradients) there would be no reason

for this hypothesis to be correct, then the scales k/ε and q/ε_γ will both be involved.

Most of the approximations used for $T_{i\gamma\gamma}$ are variants or particular cases of expression [10.3]; Wyngaard J.C. and Cote O.R., [WYN 74] use a simplified expression only retaining the first two terms of [10.3]. Samaraweera D.S.A., [SAM 78] uses an approximation retaining only the first term in [10.3]:

$$T_{i\gamma\gamma} = -c_q \frac{k}{\varepsilon} R_{il} \overline{\gamma^2}_{,l} \text{ with } c_q = 0.11 \quad . \tag{10.4}$$

10.3. Modeling the dissipation rate

The dissipation rate is defined by $\varepsilon_\gamma = \sigma \overline{\gamma_{,j}\gamma_{,j}}$. The hypothesis stating that there is an almost constant ratio $R = \dfrac{q}{\varepsilon_\gamma} \bigg/ \dfrac{k}{\varepsilon}$ between the time scales of the dynamic field and the passive contaminant field leads to the expression:

$$\varepsilon_\gamma = \frac{\varepsilon.q}{k.R}, \tag{10.5}$$

in which R is the previous constant (ratio of time scales).

The value of this simple hypothesis can be tested against experimental data. We can show easily that the decay of velocity and temperature fluctuations behind a heated grid follow a power law, the ratio of the exponents in the decay law of k and q must be precisely R. Most of the experiments of turbulence decay behind a grid suggest $k \approx x^{-m}$ with $1.15 \le m \le 1.4$. In addition, the study by Lin S.C. and Lin S.C., [LIN 73] shows that exponent n in the decay law $q \approx x^{-n}$ for half scalar variance q, is spreading over an even larger domain, and this does not allow us to draw precise conclusions for the value of R. However, Launder B.E., [LAU 76] points out that a deeper insight into the experimental data leads us to believe that the value $R = 1$ is adequate.

Spalding D.B., [SPA 71] for the numerical prediction of turbulent jets used $R = 0.5$ whereas Launder B.E., [LAU 75B] uses the value $R = 0.8$ for calculating free shear flows with gravity forces.

Considering a number of free flows and wall flows, Béguier C., Dekeyser I. and Launder B.E., [BEG 78] show that the value of R seems to vary only slightly in these cases and remains close to 0.5.

However, considering that R is not a universal constant at all, the most appropriate method consists of solving a modeled equation for ε_γ. The usual approaches are based on ideas similar to the ones advanced for modeling the ε equation (Chapter 9). But the occurrence of two time scales k/ε and q/ε_γ makes the modeling task more complicated.

10.4. Equation for the dissipation rate of the passive scalar variance

The general equation reads:

$$
\frac{d\varepsilon_\gamma}{dt} = \underbrace{-2\overline{\gamma_{,j}u_{i,j}}\ \overline{\Gamma}_{,i}}_{(a)} \underbrace{-2\overline{\gamma u_i \gamma_{,j}}\ \overline{\Gamma}_{,ij}}_{(b)} \underbrace{-2\sigma\overline{\gamma_{,i}\gamma_{,j}}\ \overline{U}_{i,j}}_{(c)} \underbrace{-2\sigma\overline{\gamma_{,j}u_{i,j}\gamma_{,i}}}_{(d)}
$$
$$
\underbrace{-2\sigma^2\overline{\gamma_{,ij}\gamma_{,ij}}}_{(f)} \underbrace{-\sigma\left(\overline{u_i \gamma_{,j}\gamma_{,j}}\right)_{,i}}_{(g)} \underbrace{+\sigma\varepsilon_{\gamma,jj}}_{(h)} . \qquad [10.6]
$$

An analysis of the order of magnitude of the different terms shows that the source terms (e) and (f) are dominant, and so at high Reynolds numbers we can write:

$$
\frac{d\varepsilon_\gamma}{dt} = -2\sigma\overline{\gamma_{,i}\gamma_{,j}u_{i,j}} - 2\sigma^2\overline{\gamma_{,ij}\gamma_{,ij}} - \left(\overline{u_j \varepsilon'_\gamma}\right)_{,j} . \qquad [10.7]
$$

Lumley J.L. and Khajeh-Nouri B., [LUM 74] following a similar reasoning as the one developed in Chapter 9 for modeling the ε equation, have proposed the closure of sources terms:

$$
(e)+(f) = -a\frac{\varepsilon.\varepsilon_\gamma}{k} - b\frac{\varepsilon_\gamma P}{k} . \qquad [10.8]
$$

Launder B.E., [LAU 76] points out that this closure cannot be generally applicable because the production term for q (term (b) in equation [10.1]) does not appear. Indeed, in grid turbulence ($P = 0$) with a transverse temperature gradient the relation [10.8] would imply that ε_γ decays indefinitely while q would become very large, which does not seem plausible.

Siess J., [SIE 75] used an approximation in the form:

$$
(e)+(f) = -\alpha\frac{\varepsilon_\gamma^2}{q} + \beta\frac{\varepsilon F_{\gamma i}F_{\gamma i}}{k^3} + \eta\frac{\varepsilon\ \varepsilon_\gamma II}{k} . \qquad [10.9]
$$

Devising a modeled transport equation for ε_γ is still nowadays not definitely achieved. Additional new measurements would be necessary in various simple turbulent flows submitted to temperature gradients in order to help choose the closure hypotheses.

Let us also cite the work of Newman G.R., Launder B.E. and Lumley J.L., [NEW 81] who, using invariant modeling and taking into account realizability conditions, were led to the following approaches in homogenous turbulence without mean velocity gradients:

$$\frac{\partial R_{ij}}{\partial t} = -\varepsilon \varphi_{ij} - \frac{2}{3}\varepsilon \delta_{ij}\,, \qquad \frac{\partial F_{\gamma i}}{\partial t} = -R_{ij}\,\overline{\Gamma}_{,j} - G\frac{\varepsilon}{k}F_{\gamma i}\,,$$

$\varphi_{ij} = \varphi_{ij}(b_{ij}, \mathrm{Re}_t)$ accounts for the pressure-strain correlations and for the anisotropic part of viscous dissipation,

$$\frac{\partial \varepsilon}{\partial t} = -\psi\frac{\varepsilon^2}{2k}\,, \qquad \frac{\partial \varepsilon_\gamma}{\partial t} = -\psi_\gamma\frac{\varepsilon.\varepsilon_\gamma}{2q}\,,$$

$$\psi = \psi(b_{ij}, \mathrm{Re}_t)\,, \qquad \psi_\gamma = B\frac{q}{k}\frac{\varepsilon}{\varepsilon_\gamma} + D - E\frac{F_{\gamma i}\,\overline{\Gamma}_{,i}}{\varepsilon_\gamma}\,,$$

with $B = B(II, \mathrm{Re}_t)$ and the terms D, E, G are constant.

The generalized form usually retained at the present time involves both the dynamic time scale k/ε and the time scale of the passive scalar q/ε_γ. The model equation is:

$$\frac{d\varepsilon_\gamma}{dt} = c_{\gamma 1}\frac{\varepsilon_\gamma P_\gamma}{2q} + c_{\gamma 1}^*\frac{\varepsilon_\gamma P}{k} - c_{\gamma 2}\frac{\varepsilon_\gamma^2}{2q} - c_{\gamma 2}^*\frac{\varepsilon}{k}\varepsilon_\gamma + c_{\varepsilon\gamma}\left(\frac{R_{ij}k}{\varepsilon}\varepsilon_{\gamma,i}\right)_{,j}\,, \qquad [10.10]$$

in which P_γ and ε_γ denote respectively the production and the dissipation terms of $q = \overline{\gamma^2}/2$.

There is, however, no consensus for the numerical values of the model constants. We can cite, for example, the values used by Farshchi M., Kollmann W., Dibble R.W. and Schefer R.W., [FAR 85] who have studied the problem of mixing turbulent jets:

$$c_{\gamma 1} = 1.0\,, \quad c_{\gamma 1}^* = 2.45\,, \quad c_{\varepsilon\gamma} = 0.18\,, \quad c_{\gamma 2} = 2.0\,, \quad c_{\gamma 2}^* = 0.9$$

The model equation proposed by Gibson M.M. *et al.*, [GIB 87] is almost similar in its formulation:

$$\frac{d\varepsilon_\gamma}{dt} = -c_{\gamma 2}\frac{\varepsilon_\gamma^2}{2q} - c_{\gamma 2}^*\frac{\varepsilon\,\varepsilon_\gamma}{k} + c_{\gamma 1}^+\frac{\varepsilon P_\gamma}{k} + c_{\gamma 1}^*\frac{\varepsilon_\gamma P}{k} + Diff(\varepsilon_\gamma),$$ [10.11]

$$c_{\gamma 2} = 2.0\,, \quad c_{\gamma 2}^* = 1.8\,, \quad c_{\gamma 1}^+ = 1.7\,, \quad c_{\gamma 1}^* = 1.4$$

Mantel T. and Borghi R., [MAN 94] have modeled each source term with an order of magnitude $Re_t^{1/2}$, as is suggested by dimensional analysis of each correlation considered individually. When $Re_t \to \infty$, the model then yields again the hypothesis R=constant.

We can also cite the work by Jones W.P. and Musonge P. [JON 88] on the transport equation for the dissipation rate of temperature variance.

10.5. New directions of research

When we use the simplified approach in which the dissipation rate is calculated from the factor R, new directions can be explored by considering R according to invariants. For example such an improved formulation has been proposed by Craft T. J. and B.E. Launder (cf. [CRA 91A]):

$$\varepsilon_\gamma = \frac{\varepsilon.q}{k.R},$$

with $R = 1.5(1 + A_F)$, $A_F = \dfrac{F_{\gamma j}F_{\gamma j}}{2q.k}$.

The most general approach remains the development of transport equations for the scalar dissipation. In this direction, the equation for ε_γ developed by Craft T.J., [CRA 91A] and used in conjunction with the models presented in section 9.7, is the following:

$$\frac{d\varepsilon_\gamma}{dt} = \left(c_{\varepsilon\gamma 1}\frac{P_\gamma}{\varepsilon_\gamma} + c'_{\varepsilon\gamma 1}\frac{\nu_t}{\varepsilon}\overline{U}_{i,j}\overline{U}_{i,j}\right)\frac{\varepsilon_\gamma}{S}\frac{\varepsilon}{k} - \left(c_{\varepsilon\gamma 2}\frac{\varepsilon\,\varepsilon_\gamma}{k} + c'_{\varepsilon\gamma 2}\frac{\varepsilon_\gamma^2}{2q}\right)$$

$$+ c_{\varepsilon\gamma}\left(\frac{R_{ij}k}{\varepsilon}\varepsilon_{\gamma,i}\right)_{,j},$$ [10.12]

where

$$c_{\varepsilon\gamma 1} = 1.6, \quad c'_{\varepsilon\gamma 1} = 2.6, \quad c_{\varepsilon\gamma} = 0.18,$$

$$c_{\varepsilon\gamma 2} = \frac{0.92}{1 + 0.5 A II^{1/2}}, \quad c'_{\varepsilon\gamma 2} = \frac{2.0}{1 + 0.5 A II^{1/2}}$$

$$S = R^{-1} = \frac{k.\varepsilon_{\gamma}}{q.\varepsilon}, \quad A = 1 - \frac{9}{8}(II - III), \quad a_{ij} = b_{ij}/k.$$

This more advanced formulation, as for the ε equation, makes use of the anisotropy invariants and the flatness factor in the expressions for the model coefficients.

This page intentionally left blank

Chapter 11

Simplified Closures:
Two and Three Transport Equation Models

The Reynolds stress transport models probably correspond to the optimal level of closure for practical applications because they seem to embody potentials providing a sufficient level of universality to be applied to a wide range of turbulent shear flows. However, they have not always been sufficiently tested, their development is still in progress and they are sometimes cumbersome for numerical treatment. For all these reasons, simplified closures may be interesting for industrial applications in which the details of the turbulence field are not required. Simplified closures are so numerous that it will not be possible to cite all of them; we shall mention here the main directions for their study. The usual classification is made according to the number of transport equations. We shall consider in the present chapter:

– three equation models for application in two-dimensional turbulent thin shear flows;

– two equation models using an isotropic turbulence viscosity coefficient and those using algebraic modeling of the Reynolds stresses.

11.1. The k-R_{12}-$\mathcal{E}$ model for turbulent thin shear flows

For turbulent thin shear flows, in which $\overline{U}_{1,2}$ is the only important mean velocity, R_{12} is the only shear stress component appearing in the momentum

equation. In such circumstances, we can simplify the equations of the RS1 model (Chapter 6) by neglecting lower order terms. In this way we obtain:

$$\frac{dR_{12}}{dt} = P_{12} - c_1 \frac{\varepsilon}{k} R_{12} + c_s \left[\frac{k}{\varepsilon} \left(2R_{22} R_{12,2} + R_{12} R_{22,2} \right) \right]_{,2}$$

$$- \frac{c_2 + 8}{11} P_{12} - \frac{8c_2 - 2}{11} D_{12} - \frac{30c_2 - 2}{55} k \overline{U}_{1,2}$$

$$\frac{dk}{dt} = -R_{12} \overline{U}_{1,2} + c_s \left[\frac{k}{\varepsilon} \left(R_{22} k_{,2} + R_{22} R_{22,2} + R_{12} R_{12,2} \right) \right]_{,2} - \varepsilon$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 1} \frac{\varepsilon}{k} R_{12} \overline{U}_{1,2} + C_\varepsilon \left[\frac{k}{\varepsilon} R_{22} \varepsilon_{,2} \right]_{,2} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \qquad [11.1]$$

with $P_{12} = -R_{22} \overline{U}_{1,2}$ and $D_{12} = -R_{11} \overline{U}_{1,2}$.

In order to close the system, it is then necessary to specify R_{11} and R_{22}. Hanjalic K. and Launder B.E., [HAN 72] suppose that the normal stresses can be approached by a constant fraction of the kinetic energy k. For this, we will use the values observed in a turbulent homogenous shear flow (cf. section 6.5).

$$\frac{R_{11}}{k} \approx 1.0, \qquad \frac{R_{22}}{k} \approx 0.48 .$$

The model equations then read:

$$\frac{dk}{dt} = -R_{12} \overline{U}_{1,2} + c'_s \left[\frac{k^2}{\varepsilon} k_{,2} \right]_{,2} - \varepsilon ,$$

$$\frac{dR_{12}}{dt} = -c'_2 k \overline{U}_{1,2} + c''_s \left[\frac{k^2}{\varepsilon} R_{12,2} \right]_{,2} - c_1 \frac{\varepsilon}{k} R_{12} ,$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 1} \frac{\varepsilon}{k} R_{12} \overline{U}_{1,2} + C_\varepsilon \left[\frac{k^2}{\varepsilon} \varepsilon_{,2} \right]_{,2} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} . \qquad [11.2]$$

Hanjalic K. and Launder B.E., [HAN 72] use the following set of constants:

$$c_1 = 2.8 , \; c''_s = 0.08 , \; c'_2 = 0.2 , \; c'_s = 0.064 ,$$
$$C_{\varepsilon 1} = 1.45 , \; C_{\varepsilon 2} = 2.0 , \; C_\varepsilon = 0.065$$

Launder B.E. *et al.*, [LAU 79A] propose the numerical values:

$$c_1 = 1.5, \quad c''_s = 0.12, \quad c'_2 = 0.19, \quad c'_s = 0.10,$$
$$C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.50, \quad C_\varepsilon = 0.07$$

This model has been used by the latter authors for calculating the turbulent flow in non-symmetric or annular ducts in which the transport of R_{12} is significant. Thus, the model makes it possible to account for the non-coincidence of the vanishing points for R_{12} and for $\overline{U}_{1,2}$ in non-symmetric flows (region of negative turbulent energy production).

11.2. Two equation models

11.2.1. *General points*

The simplest two equation models generally make use of the concept of isotropic turbulence viscosity. The Reynolds stresses are then obtained from a behavior law:

$$R_{ij} = \frac{2}{3} k \delta_{ij} - \nu_t \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right),$$
[11.3]

in which the turbulence viscosity ν_t is given by the Prandtl-Kolmogorov relation:

$$\nu_t = c_\mu \ell \sqrt{k},$$
[11.4]

a formulation in which ℓ denotes the characteristic scale of energy containing eddies.

Usually, two equation models involve: a transport equation for the velocity scale (k) and a transport equation for a turbulence length scale (ℓ) or for a variable Z from which the length scale can be deduced $Z = k^m \ell^n$.

The existing equations for Z are generally under the same form:

$$\frac{dZ}{dt} = C_{Z1} \frac{Z}{k} P + \left(\frac{\ell \sqrt{k}}{h_Z} Z_{,i} \right)_{,i} - C_{Z2} Z \frac{\sqrt{k}}{\ell} + S_Z,$$
[11.5]

S_Z being a secondary source term dependent on the particular choice of Z.

Let us cite for example:

Model	Corresponding Z function
Kolmogorov (1942)	$k^{1/2}\big/\ell$
Rotta (1951)	ℓ
Davidov (1961)	$k^{3/2}\big/\ell \approx \varepsilon$
Harlow, Nakayama (1967)	$k^{3/2}\big/\ell \approx \varepsilon$
Spalding (1969)	ℓ
Saffman (1970)	k/ℓ^2
Rodi and Spalding (1970)	$k\ell$
Rotta (1972)	$k\ell$
Spalding (1972)	k/ℓ^2
Jones, Launder (1972)	$k^{3/2}\big/\ell \approx \varepsilon$
Wilcox (1988)	$k^{1/2}\big/\ell$

Equations such as [11.5] corresponding to the various expressions for Z are all equivalent except for the influence of the secondary source term S_Z and the turbulent diffusion term, but this difference may be essential. The ε equation usually does not need secondary diffusion sources and this is one of the reasons for which the k-ε model may be preferred (see also Chapter 7).

In the same way as previously for R_{ij} (equation [11.3]) a turbulent diffusivity is introduced which relates the thermal fluxes to the mean temperature gradient:

$$F_{\gamma i} = -\sigma_t \overline{\varGamma}_{,i}. \qquad\qquad [11.6]$$

The turbulence Prandtl number is defined by $\mathrm{Pr}_t = \nu_t / \sigma_t$.

The variations in Pr_t must thus be specified in the studied region of the flow.

11.2.2. Derivation of a generic two equation model

In order to synthesize the various formulations cited previously, it is possible to derive a formal approach for a generic model equation for the quantity $\phi = u^{\alpha}\ell^{\beta}$. In comparison with the formula given in the beginning of section 11.2.1, it is suitable to take $m=\alpha/2$ and $n=\beta$, i.e. $Z = k^{\alpha/2}\ell^{\beta}$. The following derivation ([SCH 05B]) is inspired by the invariant modeling method, it is thus purely formal.

We first assume that the function ϕ satisfies a transport equation of the type:

$$\frac{d\phi}{dt} = \Psi \ .$$

Function ϕ will be used for calculating the characteristic turbulence length scale from $\ell = \phi^{1/\beta} k^{-\alpha/2\beta}$ or the dissipation rate from $\varepsilon = \phi^{-1/\beta} k^{(\alpha+3\beta)/2\beta}$. Physically, the right-hand side of the equation for ϕ accounts for the effect of the energy cascade modified by the effect of turbulent convection and diffusion. The characteristic length scale of the energy cascade will be $T = \phi^{1/\beta} k^{-(\alpha+\beta)/2\beta}$.

We then assume that the right-hand side Ψ of this equation is an unknown quantity considered as a general functional in space and time of the known quantities. In the framework of two equation closures, the known quantities are k and ϕ. We can add to these the quantity P, the turbulence kinetic energy production rate, which may modify the value of the characteristic scale T through the dimensionless parameter $G = \dfrac{P}{\varepsilon} = \dfrac{P.\phi^{1/\beta}}{k^{(\alpha+3\beta)/2\beta}}$. Thus,

$$\Psi = \mathscr{F}\{k,\phi,G\} \ .$$

In this expression Ψ is considered at point $\vec{x}$ and at time t, whereas the arguments in the functional all involve the points $\vec{y}$ in the spatial domain occupied by the fluid and all the instants of time before the considered time t.

Following the approach of J.L. Lumley, we shall assume weak non-homogenity and weak anisotropy, in order to allow Taylor expansions about the point $\vec{x}$ and consequently replacing the functional by a true function. Here, G has the role of a small anisotropy parameter (in fact G can rather be interpreted as a small non-equilibrium parameter):

$$\Psi = f\left\{k, k_{,m}, k_{,mp}, \phi, \phi_{,m}, \phi_{,mp}, G\right\}$$

The application of tensorial representation theorems, imposes that the quantity Ψ is a scalar, and it can be a function only of other scalars or invariants.

The invariants built on the tensorial arguments of function f are:

$$k_{,j}k_{,j}, \ k_{,jj}, \ \phi_{,j}\phi_{,j}, \ \phi_{,jj}, \ k_{,j}\phi_{,j} \ ,$$

if the expansions are limited to order two in non-homogenity. To these invariants, we have to add the pure scalars k, ϕ and P.

We can then write:

$$\Psi = f\left\{k, \phi, G, k_{,j}k_{,j}, k_{,jj}, \phi_{,j}\phi_{,j}, \phi_{,jj}, k_{,j}\phi_{,j}\right\}.$$

In the next step, Taylor expansion is performed in the small non-homogenity parameters.

$$\Psi = a_0(k,\phi,G) + a_1(k,\phi,G)k_{,j}k_{,j} + a_2(k,\phi,G)k_{,jj}$$
$$+ a_3(k,\phi,G)\phi_{,j}\phi_{,j} + a_4(k,\phi,G)\phi_{,jj} + a_5(k,\phi,G)k_{,j}\phi_{,j}$$

Then, expansion is carried out with respect to the small anisotropy parameter G:

$$a_m(k,\phi,G) = a_m^0(k,\phi) + a_m^1(k,\phi).G + a_m^2(k,\phi).G^2 + \dots,$$

for $m = 0$ to 5.

However, the expansion will be practically limited to order one for the homogenous term and to order zero for the other terms. Thus:

$$\Psi = a_0^0(k,\phi) + a_0^1(k,\phi).G + a_1^0(k,\phi)k_{,j}k_{,j} + a_2^0(k,\phi)k_{,jj}$$
$$+ a_3^0(k,\phi)\phi_{,j}\phi_{,j} + a_4^0(k,\phi)\phi_{,jj} + a_5^0(k,\phi)k_{,j}\phi_{,j}.$$

Coefficients $a_m^p(k,\phi)$ which are functions of the sole scalars k and ϕ can be obtained from dimensional analysis:

$$a_0^0(k,\phi) = b_0^0\phi^{(\beta-1)/\beta}k^{(\alpha+\beta)/2\beta}$$

$$a_0^1(k,\phi) = b_{00}^1\phi^{(\beta-1)/\beta}k^{(\alpha+\beta)/2\beta}$$

$$a_1^0(k,\phi) = b_1^0\phi^{(\beta-1)/\beta}k^{(\alpha+\beta)/2\beta}\frac{\ell^2}{k^2} = b_1^0\phi^{(\beta+1)/\beta}k^{-(\alpha+3\beta)/2\beta}$$

$$a_2^0(k,\phi) = b_2^0\phi^{(\beta-1)/\beta}k^{(\alpha+\beta)/2\beta}\frac{\ell^2}{k^2} = b_2^0\phi^{(\beta+1)/\beta}k^{-(\alpha+\beta)/2\beta}$$

$$a_3^0(k,\phi) = b_3^0\phi^{(\beta-1)/\beta}k^{(\alpha+\beta)/2\beta}\frac{\ell^2}{\phi^2} = b_3^0\phi^{(1-\beta)/\beta}k^{(\beta-\alpha)/2\beta}$$

$$a_4^0(k,\phi) = b_4^0 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} \frac{\ell^2}{\phi^2} = b_4^0 \phi^{1/\beta} k^{(\beta-\alpha)/2\beta}$$

$$a_5^0(k,\phi) = b_5^0 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} \frac{\ell^2}{k\phi} = b_5^0 \phi^{1/\beta} k^{-(\alpha+\beta)/2\beta}$$

Coefficients b_m^p are pure numerical constants. Of course, they cannot be determined from the method of J.L. Lumley, and they will be calibrated against basic experimental data. Thus we come to the generic formulation of the transport equation:

$$\frac{d\phi}{dt} = b_0^0 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} + b_0^1 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta}.G$$

$$+ b_1^0 \phi^{(\beta+1)/\beta} k^{-(\alpha+3\beta)/2\beta} k_{,j} k_{,j} + b_2^0 \phi^{(\beta+1)/\beta} k^{-(\alpha+\beta)/2\beta} k_{,jj}$$

$$+ b_3^0 \phi^{(1-\beta)/\beta} k^{(\beta-\alpha)/2\beta} \phi_{,j}\phi_{,j} + b_4^0 \phi^{1/\beta} k^{(\beta-\alpha)/2\beta} \phi_{,jj}$$

$$+ b_5^0 \phi^{1/\beta} k^{-(\alpha+\beta)/2\beta} k_{,j}\phi_{,j} .$$

It is useful to introduce the turbulence viscosity and the turbulence diffusivity in the form:

$$\sigma_t = C.\phi^{1/\beta} k^{(\beta-\alpha)/2\beta} ,$$

leading to the equation:

$$\frac{d\phi}{dt} = b_0^0 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} + b_0^1 \frac{\phi P}{k} + b_1^0 \frac{\phi}{Ck^2} \sigma_t k_{,j} k_{,j}$$

$$+ b_2^0 \frac{\phi}{Ck} \sigma_t k_{,jj} + b_3^0 \frac{1}{C\phi} \sigma_t \phi_{,j}\phi_{,j} + b_4^0 \frac{1}{C} \sigma_t \phi_{,jj} + b_5^0 \frac{1}{Ck} \sigma_t k_{,j}\phi_{,j} .$$

Taking into account, mathematical relations such as:

$$\left(\sigma_t \theta_{,j}\right)_{,j} = \sigma_t.\theta_{,jj} + \frac{\beta-\alpha}{2\beta} \frac{\sigma_t}{k} k_{,j}\theta_{,j} + \frac{1}{\beta}\frac{\sigma_t}{\phi}\phi_{,j}\theta_{,j} ,$$

where θ stands for k or ϕ, we can also write:

$$\frac{d\phi}{dt} = C_0 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} + C_1 \frac{\phi P}{k} + C_2 \left(\sigma_t \phi_{,j}\right)_{,j}$$

$$+ C_3 \frac{1}{\phi} \sigma_t \phi_{,j}\phi_{,j} + C_4 \frac{\phi}{k}\left(\sigma_t k_{,j}\right)_{,j} + C_5 \frac{\phi}{k^2} \sigma_t k_{,j} k_{,j} + C_6 \frac{1}{k} \sigma_t k_{,j}\phi_{,j} , \qquad [11.7]$$

a form similar to [11.5] in which the terms in (C_3, C_4, C_5, C_6) correspond to additional diffusion terms. We notice that in homogenous turbulence all the formulations are equivalent; they may however differ in non-homogenous turbulence depending on the choice in the additional diffusion terms. It can be verified easily that the resulting model is indeed generic and that, consequently, any two equation models can be put into the previous form. Indeed, a change of unknown function does not introduce newer terms ([CAT 99]). It can also be easily verified that for $\alpha = 3$ and $\beta = -1$ we recover the k-ε model in which the constants C_3, C_4, C_5, C_6 are usually zero (cf. section 11.2.3).

To extend the model to low Reynolds numbers (cf. Chapter 13) it is necessary to add the molecular viscosity as an additional parameter in the approximation functions, or equivalently, the turbulence Reynolds number defined by $\mathrm{Re}_t = \dfrac{\sigma_t}{\nu}$. We are thus led to expansions of the type:

$$a_0^0(k,\phi,\nu) = b_0^0 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} (1 + h_0^0 \, \mathrm{Re}_t^{-1} + ...)$$

$$a_0^1(k,\phi,\nu) = b_0^1 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} (1 + h_0^1 \, \mathrm{Re}_t^{-1} + ...)$$

$$a_1^0(k,\phi,\nu) = b_1^0 \phi^{(\beta+1)/\beta} k^{-(\alpha+3\beta)/2\beta} (1 + h_1^0 \, \mathrm{Re}_t^{-1} + ...)$$

$$a_2^0(k,\phi,\nu) = b_2^0 \phi^{(\beta+1)/\beta} k^{-(\alpha+\beta)/2\beta} (1 + h_2^0 \, \mathrm{Re}_t^{-1} + ...)$$

$$a_3^0(k,\phi,\nu) = b_3^0 \phi^{(1-\beta)/\beta} k^{(\beta-\alpha)/2\beta} (1 + h_3^0 \, \mathrm{Re}_t^{-1} + ...)$$

$$a_4^0(k,\phi,\nu) = b_4^0 \phi^{1/\beta} k^{(\beta-\alpha)/2\beta} (1 + h_4^0 \, \mathrm{Re}_t^{-1} + ...)$$

$$a_5^0(k,\phi,\nu) = b_5^0 \phi^{1/\beta} k^{-(\alpha+\beta)/2\beta} (1 + h_5^0 \, \mathrm{Re}_t^{-1} + ...)$$

in which we shall limit the expansion to Re_t^{-1} terms.

$$\frac{d\phi}{dt} = b_0^0 \phi^{(\beta-1)/\beta} k^{(\alpha+\beta)/2\beta} + b_0^1 \frac{\phi P}{k} + b_1^0 \frac{\phi}{Ck^2}(\sigma_t + h_1\nu) k_{,j} k_{,j}$$

$$+ b_2^0 \frac{\phi}{Ck}(\sigma_t + h_2\nu) k_{,jj} + b_3^0 \frac{1}{C\phi}(\sigma_t + h_3\nu)\phi_{,j}\phi_{,j}$$

$$+ b_4^0 \frac{1}{C}(\sigma_t + h_4\nu)\phi_{,jj} + b_5^0 \frac{1}{Ck}(\sigma_t + h_5\nu) k_{,j}\phi_{,j}.$$

This formulation allows us to recover as a particular case, the various two-equation models k-ε, k-ω, $k - \ell$, etc. Other variable choices have been proposed, as in the model by J. Cousteix which makes use of the function $\varphi = \varepsilon / \sqrt{k}$, and

developed with the aim of getting a better description of the k distribution near a wall (cf. [CAT 99]).

11.2.3. *The k-ε model*

This is the most widely tested and used two equation transport model (two transport equations for k and ε).

1) Formulation

The transport equations for k and ε can be deduced from the modeled equations presented in Chapters 6 and 7 by introducing a gradient diffusion hypothesis (with isotropic viscosity) in the turbulent diffusion terms for k and ε, and replacing R_{ij} in the production terms by its behavior law [11.3]:

$$\frac{dk}{dt} = P + \left(\frac{v_t}{h_k} k_{,i} \right)_{,i} - \varepsilon ,$$ [11.8]

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1} \frac{P\varepsilon}{k} + \left(\frac{v_t}{h_\varepsilon} \varepsilon_{,i} \right)_{,i} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} ,$$ [11.9]

with $P = v_t \overline{U}_{i,j} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right)$ and $v_t = c_\mu \dfrac{k^2}{\varepsilon}$, h_k and h_ε standing for turbulent Prandtl-Schmidt numbers assumed to be constant. The components of the Reynolds stress tensor are obtained from the behavior law $R_{ij} = \dfrac{2}{3} k \delta_{ij} - v_t \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right)$. If this law gives a good approximation of the shear stresses, the normal stresses are, however, poorly estimated in general.

2) Determination of numerical constants

The decay of grid turbulence allows us to determine the value of constant $C_{\varepsilon 2}$ (see Chapter 7); its value is found to be $C_{\varepsilon 2} = 1.9$.

Wall turbulence (logarithmic region of the turbulent boundary layer on a flat plate) yields the relation:

$$C_{\varepsilon 1} = C_{\varepsilon 2} - \frac{K^2}{h_\varepsilon \sqrt{c_\mu}} , \quad (K = 0.41 \text{ Karman constant})$$

and also $c_\mu = \dfrac{u_*^4}{k^2}$ with $u_* = \sqrt{\tau_p / \rho}$ (wall friction velocity).

Referring to experimental data, it can be deduced $c_\mu \approx 0.09$.

The numerical values of constants recommended by Launder B.E., [LAU 75A] are the following:

$$c_\mu = 0.09, \quad h_k = 1.0, \quad h_\varepsilon = 1.3, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92$$

3) Problems and comments

The k-ε model generally gives good results in simple flows as far as the mean velocities and energies are concerned, but it cannot predict satisfactorily the specific characteristics of complex flows (recirculation regions, secondary flows, etc.). In two-dimensional thin shear flows, the shear stress is the only dominating term in the mean momentum equations and consequently the gradient hypothesis with eddy viscosity may be acceptable even if the normal stresses are poorly predicted. But difficulties may remain also for relatively apparently simple flows. So, using the numerical values of the model constants given previously, the spreading rate of a turbulent plane jet is correctly predicted but the spreading rate of a circular jet is overestimated by about 30%. The same problem appears, in the case of Reynolds stress transport models, even more acute. The reason for this seems to come from the ε equation, and Rodi W., [ROD 72A] has suggested an empirical modification for c_μ and $C_{\varepsilon 2}$ which improves the numerical prediction of the circular jet:

$$c_\mu = 0.09 - 0.04 f \,,$$

$$C_{\varepsilon 2} = 1.92 - 0.0667 f \quad \text{with} \quad f = \left[\frac{\delta}{\Delta \overline{U}_m} \left(\frac{\partial \overline{U}_{CL}}{\partial x} - \left| \frac{\partial \overline{U}_{CL}}{\partial x} \right| \right) \right]^{0.2} ,$$

in which $\overline{U}_{CL}$ denotes the jet centerline mean velocity, $\Delta \overline{U}_m$ the velocity difference between the axis and the external region, δ the jet width (distance from the axis of the point at which the velocity difference is $\Delta \overline{U}_m / 2$).

Pope S.B., [POP 78A] suggested a modification of the source terms in the ε equation:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1} \frac{\varepsilon P}{k} + \left(\frac{v_t}{h_\varepsilon} \varepsilon_{,i} \right)_{,i} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} + C_{\varepsilon 3} \chi \frac{\varepsilon^2}{k} \,,$$

$$\chi = \omega_{ij}\omega_{jk}S_{ki}, \quad S_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right), \quad \omega_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\overline{U}_{i,j} - \overline{U}_{j,i}\right),$$

where $\chi = \omega_{ij}\omega_{jk}S_{ki}$ accounts for an effect of "vortex stretching" by the mean flow. This additional term vanishes in two-dimensional flows (in the mean) but remains non-zero in three-dimensional axisymmetric flows, thus allowing us to give a possible explanation to the round jet/plane jet anomaly. However, none of these modifications are of general application.

Similar to the Pope modification, that proposed by Hanjalic K. and Launder B.E., [HAN 80] makes it possible to enhance the role of irrotational strain by increasing the spectral flux. In practice, this corresponds to an increased influence of normal stresses. The modified equation reads:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1}\frac{\varepsilon P}{k} + \left(\frac{\nu_t}{h_\varepsilon}\varepsilon_{,i}\right)_{,i} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} - C_{\varepsilon 3}k\overline{U}_{i,j}\left(\overline{U}_{i,j} - \overline{U}_{j,i}\right)$$

with $C_{\varepsilon 3} = 4.44$.

The additional term can be equivalently written:

$$C_{\varepsilon 3}k\overline{U}_{i,j}\left(\overline{U}_{i,j} - \overline{U}_{j,i}\right) = C_{\varepsilon 3}k\left(\varepsilon_{ijk}\overline{U}_{i,j}\right)^2,$$

thus showing the curl of mean velocity. This modification improves the round jet anomaly together with the prediction of turbulent boundary layers in adverse pressure gradient, but here also its application is not of general use.

The problem of the round jet anomaly can be solved using other alternatives (cf. Craft T.J., [CRA 91A]) involving modifications such as those presented in Chapter 8. Thus, a satisfactory prediction of the spreading rate of turbulent jets is obtained if we use the equation:

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon^2}{k}\left[0.35\left(\frac{P}{\varepsilon} + \frac{\nu_t}{\varepsilon}\overline{U}_{i,j}^2\right) - \frac{1.92}{1+1.65A\,II^{1/2}}\right] + Diff(\varepsilon).$$

Let us also cite the multiscale models (cf. Chapter 18) which allow an improvement in the numerical prediction of the spreading rates of free turbulent flows.

The constant c_μ has been determined by reference to equilibrium flows in which $P = \mathcal{E}$. When this is not the case (for example in flows with weak production of turbulence kinetic energy such as wakes), this constant must be modified in order to get a good agreement between prediction and experiments. Rodi W., [ROD 72A] proposes an empirical relation $c_\mu(P/\varepsilon)$ deduced from experimental data, in which P/ε is the mean value of the ratio production/dissipation in a cross-section of the flow (cf. Figure 11.1).

This modification improves the numerical prediction of turbulent wakes.

In spite of its limitations, the k-$\mathcal{E}$ model often remains a good compromise between on the one hand simplicity and flexibility of use and on the other hand degree of universality and performance.
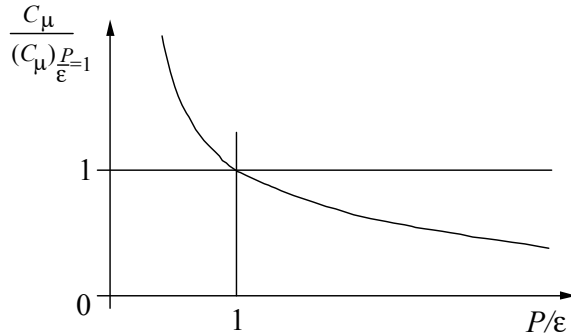

Figure 11.1. *Variations in coefficient c_μ as a function of non-equilibrium state*

11.2.4. *Realizable k-$\mathcal{E}$ models*

The standard k-$\mathcal{E}$ model, presented previously, even if it guarantees the positivity of the kinetic energy k, still remains non-realizable in the case of very high shear which may entail negativity of some normal components. Empirical corrections may cure this failing, but another better practice is to use non-linear k-$\mathcal{E}$ models presented in the following (section 11.4).

Among the empirical techniques developed in order to satisfy the realizability condition:

$$\frac{R_{12}}{k} = c_\mu \frac{k}{\varepsilon} \overline{U}_{1,2} < a \qquad (a=1),$$

the simplest consists of limiting the shear stress in the case of high velocity gradients by introducing a corrective factor to coefficient c_μ:

$$R_{ij} = \frac{2}{3} k \delta_{ij} - c_\mu \frac{a}{a + c_\mu \frac{k}{\varepsilon} \sqrt{\overline{U}_{l,m} \overline{U}_{l,m}}} \frac{k^2}{\varepsilon} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right).$$

It is along this line of thought that the model by Shih et al. [SHI 95] has been developed by introducing a turbulence viscosity satisfying the realizability constraints based on positivity of normal stresses and Schwarz inequality for the shear stresses:

$$\overline{u_\alpha^2} \geq 0 \quad \text{and} \quad \frac{\overline{u_\alpha u_\beta}}{\sqrt{\overline{u_\alpha^2} \cdot \overline{u_\beta^2}}} \leq 1, \qquad \alpha, \beta = 1, 2, 3$$

The resulting behavior law is given below:

$$R_{ij} = \frac{2}{3} k \delta_{ij} - c_\mu^* \frac{k^2}{\varepsilon} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right), \qquad c_\mu^* = \frac{1}{A_0 + A_s \frac{U^* k}{\varepsilon}},$$

$$A_0 = 4, \quad A_s = \sqrt{6} . \cos \phi, \quad \phi = \frac{1}{3} \arccos \left(\sqrt{6} W \right), \quad W = \frac{S_{ij} S_{jk} S_{ki}}{\tilde{S}},$$

$$\tilde{S} = \sqrt{S_{ij} S_{ij}}, \quad U^* = \sqrt{S_{ij} S_{ij} + \tilde{\omega}_{ij} \tilde{\omega}_{ij}}, \quad \tilde{\omega}_{ij} = \omega_{ij} - 2\varepsilon_{ijk} \omega'_k,$$

$$\omega_{ij} = \frac{1}{2} (\overline{U}_{i,j} - \overline{U}_{j,i}), \quad S_{ij} = \frac{1}{2} (\overline{U}_{i,j} + \overline{U}_{j,i}), \quad \omega'_i = \varepsilon_{ijk} u_{k,j}.$$

This formulation also includes the effect of solid rotation (cf. Chapter 16).

The dissipation rate equation is modeled in the following way:

$$\frac{d\varepsilon}{dt} = C_1 S \varepsilon - C_2 \frac{\varepsilon^2}{k + \sqrt{v\varepsilon}} + Diff(\varepsilon),$$

$$C_1 = \max \left[0.43, \frac{\eta}{\eta + 5} \right], \quad \eta = \frac{Sk}{\varepsilon}, \quad S = \sqrt{2 S_{ij} S_{ij}}, C_2 = 1.9$$

In this equation, the sink term has been chosen in order to recover $C_2 \dfrac{\varepsilon^2}{k}$ at high Reynolds numbers, in a similar way to the Durbin correction (cf. section 13.5.3).

11.2.5. *The k-ε RNG models*

The renormalization group methods of Yakhot V. and Orszag S.A., [YAK 86] (cf. Chapter 1) have led to the development of k-ε type models which in many aspects are similar to traditional k-ε models, providing them with an additional justification. But this method also allows us to complement them on the one hand by explicitly calculating the numerical constants and on the other hand by introducing new terms.

In the original version of the k-ε RNG model proposed by Yakhot V. and Orszag S.A., [YAK 86] the coefficients (explicitly derived from the theory) are numerical constants. The numerical values are given below:

$c_\mu = 0.085$, $\sigma_k = \sigma_\varepsilon = 0.7179$ (turbulent Prandtl-Schmidt numbers)

$C_{\varepsilon 1} = 1.063$, $C_{\varepsilon 2} = 1.7215$ in the ε equation.

This version proved to be limited in its field of applications. A new version proposed by Yakhot V. *et al.*, [YAK 92] based on the double expansion technique, allowed us to improve the model for situations in which the strain rate is high. The formulation of this new version is summarized here:

$$\frac{dk}{dt} = v_t S^2 - \varepsilon + \left(\frac{v_e}{\sigma_k} k_{,j} \right)_{,j},$$

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1} \frac{\varepsilon v_t S^2}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} - R + \left(\frac{v_e}{\sigma_\varepsilon} \varepsilon_{,j} \right)_{,j}.$$

The term R stands for the correlation $R = 2\nu \overline{u_{l,i} u_{l,j}} S_{ij}$ approximated by:

$$R = \frac{c_\mu \eta^3 (1 - \eta/\eta_0)}{1 + \beta \eta^3} \frac{\varepsilon^2}{k},$$

with $\eta = Sk/\varepsilon$, $S^2 = 2 S_{ij} S_{ij}$, $v_e = v + v_t$, $v_t = c_\mu k^2 / \varepsilon$,

$c_\mu = 0.085$, $C_{\varepsilon 1} = 1.42$, $C_{\varepsilon 2} = 1.68$,
$\sigma_k = 0.72$, $\sigma_\varepsilon = 0.72$, $\eta_0 = 4.38$, $\beta = 0.012$.

The ε equation can be written in an equivalent way:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1}^* \frac{\varepsilon v_t S^2}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} + \left(\frac{v_e}{\sigma_\varepsilon} \varepsilon_{,j} \right)_{,j} \text{ with } C_{\varepsilon 1}^* = C_{\varepsilon 1} - \frac{\eta (1 - \eta/\eta_0)}{1 + \beta \eta^3}.$$

The value of η_0 appears as the fixed point corresponding to the homogenous turbulent shear flow and the constant β is related to the Karman constant ($K = 0.4$). This new model gives distinctly superior predictions compared to the original model, in particular for the turbulent flow over a backward facing step.

11.2.6. *Stagnation point anomaly*

It turns out that the traditional k-ε model gives rise to energy levels that are far too high near a stagnation point. Even in cases for which this specific region of the flow is not the precise aim of the study, the consequences on the numerical prediction of the whole flow may pose a problem. This is the case for flows over cylindrical obstacles, for impinging jets on a wall and the flow around airfoil profiles. In order to overcome this drawback Kato *et al.* [KAT 93] proposed to replace the turbulence kinetic energy production term $P = 2\nu_t S_{ij} S_{ij} = 2\nu_t S^2$ by $P = 2\nu_t |\omega||S|$. The production term being strict and not issuing from modeling, the modification can be understood as a modification in the turbulence viscosity:

$$P = 2\nu^*_{\,t} S_{ij} S_{ij} \quad \text{with} \quad \nu^*_{\,t} = c_\mu \left|\frac{\omega}{S}\right| \frac{k^2}{\varepsilon}, \quad S = \sqrt{\frac{1}{2} S_{ij} S_{ij}}, \quad \omega = \sqrt{\frac{1}{2} \omega_{ij} \omega_{ij}} \; .$$

Other variants have also been proposed, in particular by Chen *et al.* [CHE 94]. The application of the model by Kato and Launder [KAT 93] to the case of a wake behind a cylinder of square cross-section placed near a wall, which was further developed by Bosch and Rodi [BOS 96], rightly predicted the vortex shedding phenomenon present in the experiment. This vortex shedding is not obtained, or only with considerable damping, when the standard k-ε model by Jones and Launder is solved.

Durbin [DUR 96] shows that the stagnation point anomaly can be improved by enforcing the realizability conditions on the Reynolds stresses components given by the usual behavior law $R_{ij} = \frac{2}{3} k \delta_{ij} - 2\nu_t S_{ij}$. In principal axes (cf. sections 4.6 and 6.6.1), this relation becomes $\overline{u_\alpha^2} = -2\nu_t \lambda^S_{(\alpha)} + \frac{2}{3} k$ in which $\lambda^S_{(\alpha)}$ are the eigenvalues of the strain rate tensor. The realizability condition $0 \le \overline{u_\alpha^2} \le 2k$ thus leads to $2\nu_t \lambda^S_{(M)} \le \frac{2}{3} k$ where $\lambda^S_{(M)}$ is the largest eigenvalue (it can be shown [DUR 96], taking into account the relation $\lambda^S_1 + \lambda^S_2 + \lambda^S_3 = 0$ coming from incompressibility, that the first inequality implies the second one). If we write the turbulence viscosity

under the $\nu_t = c_\mu T.k$ in which $T = k/\varepsilon$ is the characteristic turbulence time scale, the preceding condition leads to a time scale limitation:

$$T \leq \frac{1}{3c_\mu \lambda_{(M)}^S} .$$

Considering that it is possible to show in three-dimensional space ([DUR 96]) that $\left|\lambda_\alpha^S\right| \leq \sqrt{\dfrac{2\left|S^2\right|}{3}}$, the condition can be written: $T \leq \dfrac{1}{c_\mu \sqrt{6}\left|S\right|}$. In practice, everywhere the time scale is involved, we will apply the limiting $T = \min\left[\dfrac{k}{\varepsilon}; \dfrac{1}{c_\mu \sqrt{6}\left|S\right|}\right]$. The condition can also be expressed on the viscosity coefficient as well: $\nu_t = \min\left[c_\mu \dfrac{k^2}{\varepsilon}; \dfrac{k}{\sqrt{6}\left|S\right|}\right]$.

In the case of second order models, other types of modifications can be introduced in order to modify the length scale (cf. section 7.5).

11.2.7. *Two equation models based on a variable different from ε for calculating the length scale*

We encounter in practice numerous two equation models using a variable different from ε for calculating the length scale. This variable is generally of the form $Z = k^m \ell^n$ (cf. section 11.2.1). These models are often more empirical in their derivation and the equation for Z is thus generally built on a purely intuitive basis. The reasons for this choice may come from the need for better numerical stability or an easier way to implement the boundary conditions. Often dedicated to a particular class of turbulent flows for which each model has been carefully tested, these models are very useful for engineering applications. They are mainly encountered for boundary layer applications in external or internal aerodynamics. We shall give only an overview in the following.

11.2.7.1. *The Wilcox model*

The use of $\omega = k/\ell$ for calculating the length scale was proposed for the first time by Kolmogorov in 1942, but this model, in its original formulation, did not give rise to effective applications. Other authors, Saffman P.G., [SAF 70], Wilcox D.C.

and Rubesin W.M., [WIL 80], have extended the approach by using a formulation in $k - \omega^2$.

After this, Wilcox D.C., [WIL 88A] proposed a $k - \omega$ model which has been quite extensively tested and for which we give the formulation:

$$\frac{dk}{dt} = -R_{ij}\overline{U}_{i,j} - \beta^* k\omega + \left[\left(\nu + \sigma^* \nu_t\right)k_{,j}\right]_{,j},$$

$$\frac{d\omega}{dt} = -\alpha \frac{\omega}{k} R_{ij}\overline{U}_{i,j} - \beta\omega^2 + \left[\left(\nu + \sigma\nu_t\right)\omega_{,j}\right]_{,j},$$

$$\nu_t = k/\omega, \quad \alpha = 5/9, \quad \beta = 3/40, \quad \beta^* = 9/100, \quad \sigma = 1/2, \quad \sigma^* = 1/2.$$

The boundary conditions on a wall are specific to this type of model. Supposing that on the wall, the molecular diffusion term balances the sink term:

$$\beta\omega^2 = \nu\omega_{,jj},$$

it is thus possible to find a power law solution: $\omega \to \dfrac{6\nu}{\beta} y^{-2}$ which is used as a boundary condition at the wall. The energy equation $\beta^* k\omega = \nu k_{,jj}$ leads to $k \approx y^m$ with $m(m-1) = \dfrac{6\beta^*}{\beta}$.

11.2.7.2. The Menter models

These models [MEN 94] were developed in order to combine the advantages of the k-ε model and the Wilcox model. The first model so-called "baseline" uses the k-ω model by Wilcox in the internal region of the boundary layer and then is joined to the k-ε model (rewritten in terms of k-ω variables) in the external region of the boundary layer and in free flows. The performances are similar to those of the k-ω model with its numerical robustness while avoiding the too high sensibility of this model to free stream turbulence [MEN 92] by using the k-ε model, which has the advantage of remaining unaffected by the free stream turbulence. The k-ω and k-ε models are weighted by a function of the distance from the wall.

$$\frac{d\rho k}{dt} = -R_{ij}\overline{U}_{i,j} - \beta^* \rho k\omega + \left[\left(\mu + \sigma_k \mu_t\right)k_{,j}\right]_{,j},$$

$$\frac{d\rho\omega}{dt} = -\frac{\gamma}{\nu_t} R_{ij}\overline{U}_{i,j} - \beta\rho\omega^2 + \left[\left(\mu + \sigma_\omega \mu_t\right)\omega_{,j}\right]_{,j} + 2(1 - F_1)\rho\sigma_{\omega 2}\frac{1}{\omega}k_{,j}\omega_{,j}$$

The different constants in model ϕ are obtained from the constants of each of the initial models ϕ_1 and ϕ_2 by interpolation $\phi = F_1\phi_1 + (1 - F_1)\phi_2$. The constants ϕ_1 are relative to the Wilcox model:

$$\beta_1 = 3/40, \quad \beta^* = 9/100, \quad \sigma_{k1} = 1/2, \quad \sigma_{\omega 1} = 1/2,$$

$$\gamma_1 = \frac{\beta_1}{\beta^*} - \sigma_{\omega 1}\frac{K^2}{\sqrt{\beta^*}}, \quad K = 0.41,$$

while the constants ϕ_2 are relative to the standard k-ε model (after transformation):

$$\beta_2 = 0.0828, \quad \beta^* = 9/100, \quad \sigma_{k2} = 1., \quad \sigma_{\omega 2} = 0.856, \quad \gamma_2 = \frac{\beta_2}{\beta^*} - \sigma_{\omega 2}\frac{K^2}{\sqrt{\beta^*}}$$

with the definitions:

$$\nu_t = k/\omega, \quad R_{ij} = \frac{2}{3}\rho k\delta_{ij} - \mu_t\left(\overline{U}_{i,j} + \overline{U}_{j,i} - \frac{2}{3}\overline{U}_{m,m}\delta_{ij}\right),$$

$$F_1 = \tanh\left(Arg_1^4\right), \quad Arg_1 = \min\left[\max\left(\frac{\sqrt{k}}{0.09\omega y}; \frac{500\nu}{y^2\omega}\right); \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2}\right],$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}k_{,j}\omega_{,j}; 10^{-20}\right),$$

y being the distance to the nearest wall.

The recommended boundary condition on a wall is $\omega = 10\dfrac{6\nu}{\beta_1(\Delta y_1)^2}$.

The second model so-called "SST" (shear stress transport model) introduces a modification in the turbulence viscosity which takes into account the influence of the transport of turbulent shear stress. It gives an improvement in the prediction of the turbulent boundary layer in an adverse pressure gradient.

The Bradshaw hypothesis (cf. section 12.1.3) would lead to $\tau = -R_{12} = \rho a_1 k$

whereas the usual turbulence eddy viscosity models lead to $\tau = \mu_t\overline{U}_{1,2} = \dfrac{(\rho a_1 k)^2}{\varepsilon}\overline{U}_{1,2}$

which can also be written in the form $\tau = \rho\sqrt{\dfrac{P}{\varepsilon}}a_1 k$. The ratio P/ε occurs to be markedly greater than unity in the boundary layer in an adverse pressure gradient, which is leading to an overestimate of viscosity, the Bradshaw formula is then preferable. A weighted formulation is thus adopted in the SST model. Its formulation is identical to the previous one except for the constants ϕ_1 which are modified:

$$\beta_1 = 0.0750\,,\quad \beta^* = 9/100\,,\quad \sigma_{k1} = 0.85\,,\quad \sigma_{\omega 1} = 1/2\,,\quad a_1 = 0.31$$

$$\gamma_1 = \frac{\beta_1}{\beta^*} - \sigma_{\omega 1}\frac{K^2}{\sqrt{\beta^*}}\,,\quad K = 0.41\,,$$

the viscosity being defined by:

$$\nu_t = \frac{a_1 k}{\max\left(a_1\omega,\Omega F_2\right)}\,,\qquad \Omega \text{ is the absolute value of vorticity,}$$

$$\Omega = \left|\overline{U}_{1,2}\right| \text{ in a boundary layer,}$$

$$F_2 = \tanh\left(Arg_2^{\,2}\right)\,,\quad Arg_2 = \max\left(2\frac{\sqrt{k}}{0.09\omega y}; \frac{500\nu}{y^2\omega}\right).$$

The cross term like $k_{,j}\omega_{,j}/\omega$ coming from the mathematical transformation of the formulation k-ε to the formulation k-ω, was also introduced by Wilcox [WIL 93] in a modified version of the original model. This variant of the model reads:

$$\frac{dk}{dt} = -R_{ij}\overline{U}_{i,j} - \beta^* k\omega + \left[\left(\nu + \sigma^* \nu_t\right)k_{,j}\right]_{,j}\,,$$

$$\frac{d\omega}{dt} = -\alpha\frac{\omega}{k}R_{ij}\overline{U}_{i,j} - \beta\omega^2 + \left[\left(\nu + \sigma\nu_t\right)\omega_{,j}\right]_{,j} + \sigma_d\frac{\omega_{,j}k_{,j}}{\omega}\,,$$

$$\nu_t = k/\omega\,,\quad \alpha = 1/2\,,\quad \beta = 3/40\,,\quad \beta^* = 9/100\,,\quad \sigma = 3/5\,,$$

$$\sigma^* = 1 \text{ and } \sigma_d = \begin{cases} 0.3 & \text{if } \omega_{,j}k_{,j} > 0 \\ 0 & \text{if } \omega_{,j}k_{,j} < 0 \end{cases}$$

Another variant has been proposed by Kok J.C. [KOK 00] in order to solve the problem of dependency of the k-ω formulation relative to the level of ω outside the boundary layer. The formulation based on the introduction of the tensor $\omega_{,j}k_{,j}$ is similar to the previous one but with different numerical coefficients.

Bredberg J. *et al.* [BRE 02] propose a form of the *k-ω* model which makes clear the importance of cross diffusion terms to get a correct behavior of the model at the boundaries with the external current and in recirculating flows.

11.2.7.3. *The k-kl model and the k-l model by Smith*

The models of type *k-l* or *k-kl*, after Rotta J.C. [ROT 51A and B], [ROT 72] have been developed by Spalding D.B., [SPA 69], Rodi W. and Spalding D.B., [ROD 70] and then by Ng K.H. and Spalding D.B., [NG 72] and are using an equation of the type:

$$\frac{d(k\ell)}{dt} = B_P \nu_t \ell (\overline{U}_{i,j} + \overline{U}_{j,i})\overline{U}_{i,j} - B_D k^{3/2} + B_Q \left(\nu_t (k\ell)_{,i} + B_\ell \nu_t \ell k_{,i} \right)_{,i}$$

The transport equation for $(k\ell)$ was obtained previously by Rotta [ROT 51A and B], [ROT 72] by analytical means on the basis of two point statistics. We find again this type of formulation in [MEL 73] then reconsidered by Smith [SMI 90]. The difficulties encountered for developing the model $k - k\ell$ in the near wall region and in particular the viscous sublayer have led Smith [SMI 94] to switch to a $k - \ell$ formulation which is more adequate on the numerical point of view for describing the wall region. Indeed, ℓ varies like y near the wall whereas $k\ell$ varies far more rapidly (like y^3). The formulation is also simpler than the one of the model based on the $k\ell$ equation.

The model equations read:

$$\nu_t = \mu\chi\Phi(\chi, f), \qquad \chi = \frac{\rho\ell\sqrt{2k}}{\mu B_1^{1/3}},$$

$$\frac{dk}{dt} = \nu_t \left(\overline{U}_{i,j} \right)^2 - \frac{(2k)^{3/2}}{B_1 \ell} - 2\nu \left(\sqrt{k} \right)_{,j}^2 + \left[(\nu + S_k \nu_t)k_{,j} \right]_{,j},$$

$$\frac{d\ell}{dt} = -\frac{2 - E_2}{B_1}\sqrt{2k}\left[\left(\frac{\ell}{Ky} \right)^2 - 1 \right] + \left[(\nu + S_k.\nu_t)\ell_{,j} \right]_{,j} - S_k \nu_t \left(\frac{\ell}{Ky} \right)^2 \frac{\ell_{,j}\ell_{,j}}{\ell}$$

$$+2S_k \nu_t \frac{\ell_{,j}k_{,j}}{k},$$

with $\Phi = \left(\dfrac{C_1^4 f_1 + C_2^2 \chi^2 + \chi^4}{C_1^4 + C_2^2 \chi^2 + \chi^4} \right)^{1/4}$, $f_1 = \exp\left[-50\left(\dfrac{\ell}{Ky} \right)^2 \right]$, $B_1 = 18.0$, $E_2 = 1.2$,

$S_k = 0.7$, $C_1 = 25.5$, $C_2 = 2.0$.

11.2.7.4. *Turbulence scale based on the enstrophy of energetic eddies*

We also mention the work by Reynolds W.C. *et al.* [REY 02] which uses the enstrophy of energy containing eddies as a basis of determination of the length scale of the turbulent field. The two equation model using k and the enstrophy of large eddies also refers to structure tensors (cf. section 8.7). The simplicity of its formulation makes the model intended for practical applications in engineering.

11.2.8. *Transport of a passive scalar*

In most two equation models, we make use of a constant turbulence Prandtl number Pr_t = constant. The experiments, however, show that Pr_t is not a constant valid for all flows, even for the simplest ones; thus for example:

– mixing layer and plane jet $\qquad\qquad Pr_t \approx 0.5$

– circular jet $\qquad\qquad\qquad\qquad Pr_t \approx 0.7$

– wall flows $\qquad\qquad\qquad\qquad\quad Pr_t \approx 0.9$

Reynolds A.J., [REY 76C] has also shown that Pr_t varies inside a cross-section of a free flow. In wall flows, we also measure some variations in Pr_t according to the distance from the wall (Kestin J. and Richardson P.D., [KES 63]).

11.3. Algebraic modeling of the Reynolds stresses and the turbulent fluxes of a passive scalar

These are economical methods for the numerical prediction of the Reynolds stresses and the turbulent fluxes of a passive scalar. If the transport terms (convection and diffusion) in the modeled transport equations of the Reynolds stresses are expressed without involving the Reynolds stress gradients, we then get algebraic relations which can be used to determine the stress components R_{ij}. The same is true for the $F_{\gamma j}$ equations.

Rodi W., [ROD 72A, ROD 76] introduces the hypothesis:

$$\frac{dR_{ij}}{dt} - Diff(R_{ij}) = \frac{R_{ij}}{k}\left(\frac{dk}{dt} - Diff(k)\right). \qquad [11.10]$$

This hypothesis does not assume the energy balance $P = \varepsilon$; it is approximately verified if R_{ij}/k is only slowly varying, indeed, in the expression:

$$\frac{dR_{ij}}{dt} = \frac{R_{ij}}{k}\frac{dk}{dt} + k\frac{d}{dt}\left(\frac{R_{ij}}{k}\right)$$

The second term then becomes weak. This term is exactly zero in homogenous turbulent flow when the anisotropy tensor is in equilibrium $da_{ij}/dt = 0$.

[11.10] can also be written $P_{ij} + \Phi_{ij} - \varepsilon_{ij} = \frac{R_{ij}}{k}(P - \varepsilon)$.

In the framework of the RS1 model (Chapter 6) we get:

$$P_{ij} - \frac{2}{3}\varepsilon\delta_{ij} - c_1\frac{\varepsilon}{k}(R_{ij} - \frac{2}{3}k\delta_{ij}) - (a_{lj}^{mi} + a_{li}^{mj})\overline{U}_{l,m} = \frac{R_{ij}}{k}(P - \varepsilon). \qquad [11.11]$$

The wall effect term in the pressure-strain correlations can also be introduced in this formulation.

In the framework of the simplified RS2 model (Chapter 6) we are led to:

$$\frac{R_{ij} - \frac{2}{3}k\delta_{ij}}{k} = \Lambda\frac{P_{ij} - \frac{2}{3}P\delta_{ij}}{\varepsilon} \text{ or } \Lambda = \frac{1 - \gamma}{c_1 - 1 + P/\varepsilon}. \qquad [11.12]$$

The algebraic modeling of the turbulent fluxes of a passive scalar is based on a relation similar to [11.10] (Gibson M.M. and Launder B.E., [GIB 76]):

$$\frac{dF_{\gamma i}}{dt} - Diff(F_{\gamma i}) = \frac{F_{\gamma i}}{\sqrt{kq}}\left(\frac{d\sqrt{kq}}{dt} - Diff(\sqrt{kq})\right), \qquad [11.13]$$

which is also written:

$$\frac{dF_{\gamma i}}{dt} - Diff(F_{\gamma i}) = \frac{F_{\gamma i}}{2k}(P - \varepsilon) + \frac{F_{\gamma i}}{2q}\left(P_{\gamma} - \varepsilon_{\gamma}\right).$$

In the framework of the RS1 model (Chapter 9) we get:

$$(1 - c_{\gamma 2})P_{\gamma i} - R_{ij}\overline{\Gamma}_{,j} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma i} = \frac{F_{\gamma i}}{2k}(P - \varepsilon) + \frac{F_{\gamma i}}{2q}\left(P_{\gamma} - \varepsilon_{\gamma}\right), \qquad [11.14]$$

with $P_{\gamma i} = -F_{\gamma j}\overline{U}_{i,j}$, $P_{\gamma} = -F_{\gamma j}\overline{\Gamma}_{,j}$, which leads to an expression for $F_{\gamma i}$.

Thus, $-F_{\gamma i} = \Lambda_\gamma \dfrac{k}{\varepsilon} R_{ij} \overline{T},_j - \Lambda'_\gamma \dfrac{k}{\varepsilon} P_{\gamma i}$,   [11.15]

with $\Lambda_\gamma = \left[c_{\gamma 1} + \dfrac{1}{2}\left(\dfrac{P}{\varepsilon}-1\right) + \dfrac{k\varepsilon_\gamma}{2q\varepsilon}\left(\dfrac{P_\gamma}{\varepsilon_\gamma}-1\right)\right]^{-1}$, $\Lambda'_\gamma = \left(1-c_{\gamma 2}\right)\Lambda_\gamma$.

Coefficients Λ_γ and Λ'_γ , like Λ, are numerical or variable coefficients not to be confused with Λ_{ij} used in Chapter 5 with another meaning.

The additional hypothesis $P_\gamma = \varepsilon_\gamma$ allows a simplified treatment, and in this case:

$$\Lambda_\gamma = \left[c_{\gamma 1} + \dfrac{1}{2}\left(\dfrac{P}{\varepsilon}-1\right)\right]^{-1}$$ [11.16]

(flow with equilibrium conditions for q).

In general, the Reynolds stresses and the turbulent heat fluxes are thus obtained by solving a linear system numerically. This implicit system is coupled with the transport equations for k and ε. These models are also called implicit algebraic models.

In the particular case of a turbulent thin shear flow far from any wall, the equations become much simpler. Hypothesis [11.12] thus implies:

$$\dfrac{R_{12}}{k} = -\Lambda \dfrac{R_{22}\overline{U}_{1,2}}{\varepsilon} ,$$ [11.17]

$$\dfrac{R_{22}}{k} = \dfrac{2}{3}\left(1-\Lambda\dfrac{P}{\varepsilon}\right),$$ [11.18]

(taking into account $P_{12} = -R_{22}\overline{U}_{1,2}$, $P = -R_{12}\overline{U}_{1,2}$ and $P_{22} = 0$).

On the other hand [11.15] implies:

$$-F_{\gamma 1} = \Lambda_\gamma \dfrac{kR_{12}}{\varepsilon}\overline{T},_2 - \Lambda'_\gamma \dfrac{kF_{\gamma 2}}{\varepsilon}\overline{U}_{1,2} ,$$ [11.19]

$$-F_{\gamma 2} = \Lambda_\gamma \dfrac{kR_{22}}{\varepsilon}\overline{T},_2 ,$$ [11.20]

(taking into account $P_{\gamma 1} = -F_{\gamma 2}\overline{U}_{1,2}$ and $P_{\gamma 2} = 0$).

After [11.17]:

$$v_t = \Lambda \frac{k R_{22}}{\varepsilon} = \frac{2}{3}\Lambda\left(1-\Lambda\frac{P}{\varepsilon}\right)\frac{k^2}{\varepsilon},$$

and taking into account the expression for Λ, we get:

$$v_t = \frac{2}{3}\frac{(1-\gamma)(c_1-1+\gamma P/\varepsilon)}{(c_1-1+P/\varepsilon)^2}\frac{k^2}{\varepsilon}, \qquad\qquad [11.21]$$

with $c_1 = 1.5, \gamma = 0.6$.

Relation [11.21] expresses the turbulence viscosity according to the parameter P/ε. The corresponding coefficient c_μ defined by $c_\mu = \dfrac{v_t}{k^2/\varepsilon} = \dfrac{-R_{12}}{\left(k^2/\varepsilon\right)\overline{U}_{1,2}}$ has a similar qualitative behavior (cf. Figure 11.1) as the one previously given by Rodi following an empirical approach (cf. section 2.2) for improving the $k\text{-}\varepsilon$ model.

From [11.17] and [11.20] it can be shown that:

$$\Pr_t = \frac{\Lambda}{\Lambda_\gamma} = \frac{(1-\gamma)\left[c_{\gamma 1} + \dfrac{1}{2}(P/\varepsilon - 1)\right]}{c_1 - 1 + P/\varepsilon}, \qquad\qquad [11.22]$$

a relation which gives the variations in turbulence Prandtl number in turbulent free flows.

In the algebraic modeling equations, k and ε are temporarily assumed to be known. They will have to be determined from their modeled transport equations (the global process being iterative).

A generalized formulation of [11.10] has been proposed by Launder B.E., [LAU 82], in which the factor R_{ij}/k is replaced by a weighted mean (of R_{ij}/k and the unit tensor) hypothesis of the type $\left[(1+\alpha)\dfrac{R_{ij}}{k} - \dfrac{2}{3}\alpha\delta_{ij}\right]$.

11.4. Non-linear models

11.4.1. *Non-linear behavior laws*

The J.L. Lumley approach has shown that it is possible to derive formal expansions (cf. section 4.5) at high order for the Reynolds stress tensor in which the gradient model is nothing more than the first term in the expansion (Lumley J.L., [LUM 70]). More precisely, this expansion can be written (cf. also Saffman P.G., [SAF 76]):

$$R_{ij} = \frac{2}{3} k \delta_{ij} - 2\nu_t S_{ij} + B \frac{k^3}{\varepsilon^2} \left(S_{im} S_{mj} - \frac{1}{3} II^S \delta_{ij} \right)$$
$$+ C \frac{k^3}{\varepsilon^2} \left(\omega_{im} \omega_{mj} - \frac{1}{3} II^\omega \delta_{ij} \right) + D \frac{k^3}{\varepsilon^2} \left(S_{im} \omega_{mj} + S_{jm} \omega_{mi} \right), \qquad [11.23]$$

with $S_{ij} = \frac{1}{2} (\overline{U}_{i,j} + \overline{U}_{j,i})$ and $\omega_{ij} = \frac{1}{2} (\overline{U}_{i,j} - \overline{U}_{j,i})$.

The $k - \omega^2$ model by Wilcox D.C. and Rubesin M.W., [WIL 80] belongs to this category.

The particular expression for the Reynolds stresses in this model,

$$R_{ij} = \frac{2}{3} k \delta_{ij} - 2\nu_t \left(S_{ij} - \frac{1}{3} \overline{U}_{h,h} \delta_{ij} \right) - \frac{8}{9} \frac{k \left(S_{im} \omega_{mj} + S_{jm} \omega_{mi} \right)}{\beta^* \omega^2 + 2 S_{ml} S_{ml}},$$

can also apply in compressible flows (the velocity divergence being not zero). The turbulence kinetic energy dissipation rate is given by $\varepsilon = \beta^* \omega k$.

We note that the material indifference hypothesis (also related to the concept of objectivity of tensors, satisfying invariance in a change of frame of reference in relation to which are written the fundamental laws of mechanics) often used in practice would remove any dependency on the antisymmetric rotation tensor ω_{ij}. However, this hypothesis is not justified in the general case (cf. Lumley J.L., [LUM 70]) in turbulent flows, and this makes it possible to retain the expansions of the type [11.23] as functions of both S_{ij} and ω_{ij}.

Several authors have thus contributed to the development of this type of model, in particular Yoshizawa A., [YOS 84], Nisizima S. and Yoshizawa A., [NIS 86], Taulbee D.B. *et al.*, [TAU 93], Craft T.J., Launder B.E. and Suga K., [CRA 93].

This latter reference gives a synthesis of the work made at the University of Manchester showing that a cubic expansion was necessary to reach a sufficient level of generality to overcome the main difficulties and weaknesses of traditional models.

The model by Craft T.J., Launder B.E. and Suga K., [CRA 93] has been applied in particular to the complex case of a curved channel and to the axisymmetric wall jet and it allowed a decisive improvement over the traditional models. A revised version of this model is proposed in [CRA 96C] still directed towards greater generality including in particular the effect of streamline curvature. The formulation is the following:

$$
a_{ij} = -\frac{\nu_t}{k} S_{ij} + c_1 \frac{\nu_t}{\tilde{\varepsilon}} \left(S_{im} S_{mj} - \frac{1}{3} II^S \delta_{ij} \right) + c_2 \frac{\nu_t}{\tilde{\varepsilon}} \left(S_{im} \omega_{mj} + S_{jm} \omega_{mi} \right)
$$

$$
+ c_3 \frac{\nu_t}{\tilde{\varepsilon}} \left(\omega_{im} \omega_{mj} - \frac{1}{3} II^\omega \delta_{ij} \right) + c_4 \nu_t \frac{k}{\tilde{\varepsilon}^2} \left(S_{im} \omega_{lj} + S_{jm} \omega_{li} \right) S_{ml}
$$

$$
+ c_5 \nu_t \frac{k}{\tilde{\varepsilon}^2} \left(\omega_{il} \omega_{lm} S_{mj} + S_{il} \omega_{lm} \omega_{mj} - \frac{2}{3} S_{ml} \omega_{mn} \omega_{nl} \delta_{ij} \right)
$$

$$
+ c_6 \nu_t \frac{k}{\tilde{\varepsilon}^2} II^S S_{ij} + c_7 \nu_t \frac{k}{\tilde{\varepsilon}^2} II^\omega S_{ij} \,, \qquad\qquad [11.24]
$$

$$
c_1 = -0.1, \qquad c_2 = 0.1, \qquad c_3 = 0.26, \qquad c_4 = -10 c_\mu^2,
$$

$$
c_5 = 0, \qquad c_6 = -5 c_\mu^2, \qquad c_7 = 5 c_\mu^2,
$$

$$
c_\mu = \frac{0.3}{1 + 0.35 \left[\max(S, \omega) \right]^{3/2}} \left[1 - \exp \left\{ -0.36 / \exp \left(-0.75 \max(S, \omega) \right) \right\} \right],
$$

$$
f_\mu = 1 - \exp \left[-\left(\frac{\mathrm{Re}_t}{90} \right)^{1/2} - \left(\frac{\mathrm{Re}_t}{400} \right)^2 \right],
$$

$$
\nu_t = c_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}}, \qquad f_\mu = \min \left[1, 0.2 + \frac{0.8 \mathrm{Re}_t}{50} \right],
$$

$$
S = \frac{k}{\varepsilon} \sqrt{\frac{1}{2} S_{ij} S_{ij}} \,, \quad \omega = \frac{k}{\varepsilon} \sqrt{\frac{1}{2} \omega_{ij} \omega_{ij}} \,, \quad S_{ij} = \overline{U}_{i,j} + \overline{U}_{j,i} \,, \quad \omega_{ij} = \overline{U}_{i,j} - \overline{U}_{j,i} \,,
$$

$$
\mathrm{Re}_t = \frac{k^2}{\nu \tilde{\varepsilon}} \text{ (cf. Chapter 13 for } \mathrm{Re}_t \text{ effects and the definition of } \tilde{\varepsilon} \text{).}
$$

The dissipation rate equation used in conjunction with model includes the Yap correction term [YAP 87] (cf. section 14.4).

The model used by Cotton M.A. and Ismael J.O., [COT 93] lays emphasis upon the role of the strain parameter S and uses the gradient law diffusion hypothesis together with a turbulence viscosity sensitized to the parameter S under the general form:

$$\nu_t = c_\mu f_\mu(\mathrm{Re}_t) f_s(S) \frac{k^2}{\varepsilon} \;.$$

The non-linear model by Speziale C.G., [SPE 87B] is written in the following form:

$$R_{ij} = \frac{2}{3} k \delta_{ij} - 2\nu_t S_{ij} - 4c_D c_\mu^2 \frac{k^3}{\varepsilon^2} \left(S_{im} S_{mj} - \frac{1}{3} II^S \delta_{ij} \right)$$

$$-4c_E c_\mu^2 \frac{k^3}{\varepsilon^2} \left(\overset{o}{D} S_{ij} - \frac{1}{3} \overset{o}{D} S_{mm} \delta_{ij} \right), \tag{11.25}$$

where $\overset{o}{D} S_{ij}$ denotes the Oldroyd derivative (cf. section 7.6) and $c_D = c_E = 1.68$. If we make use of the expansion technique by Lumley, it will be necessary to replace usual time derivatives by Oldroyd derivatives, in order to get objective formulations in the general case. The occurrence of Oldroyd derivatives in this formulation confers to this model the properties of a viscoelastic scheme ([CRO 68]).

The formulation of this model guarantees positivity of the kinetic energy of turbulence, and also the material indifference at the limit of two-dimensional turbulence (Speziale C.G., [SPE 89] and [SPE 91A]). The Reynolds stress tensor being represented by a model of type $R_{ij} = \mathscr{T} \left[S_{ij}, \omega_{ij}^{(absol)} \right]$ (cf. section 4.7) in which $\omega_{ij}^{(absol)}$ is the intrinsic rotation, any reference to $\omega_{ij}^{(absol)}$ must disappear in the 2D limit. This latter property is acquired through the introduction of Oldroyd derivatives. This model allows us, in particular, to improve the prediction of recirculating zones by increasing the reattachment length in the flow over a two-dimensional step (cf. Thangam S. and Speziale C.G., [THA 92]).

11.4.2. *Models using an auxiliary equation for anisotropy invariants*

The non-linear model (k-ε -II) of Craft T.J. *et al.*, [CRA 95B] includes, in addition to the equations for k and ε, an evolution equation for the anisotropy invariant II and moreover it is free from any topographical parameter. The non-linear behavior law for the Reynolds stresses is directly inspired by the previous

model by the same authors (Craft T.J. *et al.*, [CRA 93]) which has been complemented by additional terms. The main new contribution relies in the modeling of the transport equation for *II*, developed from the definition of *II*:

$$\frac{dII}{dt} = -\frac{2II}{k}\left(Diff(k) + P - \varepsilon\right) + \frac{2a_{ij}}{k}\left(Diff(R_{ij}) + P_{ij} + \Phi_{ij} - \varepsilon_{ij}\right).$$

The closure of the second factor in the right-hand side is deduced from the second order transport models of the new generation (cf. Chapters 8 and 13) developed at the University of Manchester associated with the non-linear behavior law of Craft *et al.* [CRA 93] giving the Reynolds stresses. The level of generality of the numerical predictions obtained using this model can be comparable to one of the many usual second order models, at a noticeably reduced cost.

Various applications of a non-linear (k-ε-II) model of this type derived from [CRA 97B] have been carried out by Suga K. *et al.*, 2001 for complex three-dimensional flows ([SUG 01]) and have shown their relevance for industrial flows.

Afterwards, Suga K., [SUG 97] proposed a (k-ε-A) model including, this time, a transport equation for the flatness factor A in conjunction with the non-linear behavior law for the Reynolds stresses similar to the one in Craft T.J. *et al.*, [CRA 95B]. The model does not use any topographical reference, the dependency on the flatness factor A giving a means to distinguish free flows from wall flows.

The equation for A which reads formally:

$$\frac{dA}{dt} = -\frac{9}{8k}\left(\frac{3}{2}III.H_{mm} + 2a_{ij}H_{ij} - 3a_{jm}a_{mi}H_{ij}\right)$$
$$\text{with } H_{ij} = Diff(R_{ij}) + P_{ij} + \Phi_{ij} - \varepsilon_{ij},$$

is closed using the hypotheses of the Craft T.J. and Launder B.E. model [CRA 96B].

11.4.3. The $k - \upsilon^2 - f$ model

This model has been developed by Durbin [DUR 91A] and can be considered as a simplified version of the elliptic relaxation model (cf. section 8.3.2). It has been tested against various turbulent flows such as separated flows [DUR 95], the impinging jet flow on a heated wall [BEH 96] and also three-dimensional boundary layers ([PAR 98]). It allows a significant improvement over the traditional k-ε model as regards the velocity profiles or the heat transfers.

This model gives a better distribution of turbulence viscosity near a wall. Indeed, near a wall, the turbulence viscosity is rather due to velocity fluctuations normal to the wall, i.e. to the $\overline{u_n u_n}$ correlation (cf. section 6.3a)) which is supposed to be taken into account through the v^2 variable in the $k - v^2 - f$ model. Thus, the variable v^2 is considered as a scalar (and not as a tensor component) which in the near wall region, behaves like the variance of normal velocity fluctuations $\overline{u_n u_n} = R_{ij} n_i n_j$.

In the $k - v^2 - f$ model, the turbulence viscosity is given by:

$$v_t = C_\mu v^2 T,$$

in order to account for the damping effect of turbulence near a wall. The v^2 variable thus acts like an indicator of near wall anisotropy and in this respect recalls the approaches in section 11.4.2 using an auxiliary equation for the anisotropy invariants. The characteristic time scale T is defined by:

$$T = \max\left[\frac{k}{\varepsilon}; C_T \left(\frac{v}{\varepsilon}\right)^{1/2}\right].$$

The three equation system is composed of the equations:

$$\frac{dk}{dt} = P - \varepsilon + \left[\left(v + \frac{v_t}{\sigma_k}\right)k_{,j}\right]_{,j},$$

$$\frac{d\varepsilon}{dt} = \frac{C'_{\varepsilon 1} P - C_{\varepsilon 2}\varepsilon}{T} + \left[\left(v + \frac{v_t}{\sigma_\varepsilon}\right)\varepsilon_{,j}\right]_{,j},$$

$$\frac{dv^2}{dt} = kf - \frac{v^2}{k}\varepsilon + \left[\left(v + \frac{v_t}{\sigma_k}\right)v^2_{,j}\right]_{,j},$$

in which P is the mean turbulent kinetic energy production term. The variable f, which represents the redistribution effect is obtained from the following elliptic equation (cf. section 8.3.2):

$$f - L^2 f_{,jj} = \left(C_{f1} - 1\right)\frac{2/3 - v^2/k}{T} + C_{f2}\frac{P}{k},$$

in which L is the characteristic length scale given by:

$$L = c_L \max\left[\frac{k^{3/2}}{\varepsilon}, c_\eta \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}\right].$$

The boundary conditions on a wall read:

$$k = 0, \upsilon^2 = 0, \varepsilon = \lim_{y \to 0} \frac{2\nu k}{y^2}, f = \lim_{y \to 0} \frac{-20\nu^2 \upsilon^2}{\varepsilon y^4} \text{ in order to get } \upsilon \propto y^4 \text{ on the wall.}$$

The constants are chosen as:

$$C_\mu = 0.19, \ C_{\varepsilon 1} = 1.55, \ C^*_{\varepsilon 1} = 0.1, \ C_{\varepsilon 2} = 1.9, \ C_{f1} = 1.4, \ C_{f2} = 0.3,$$

$$C_T = 6, \ C_L = 0.3, \ c_\eta = 70, \ \sigma_k = 1, \ \sigma_\varepsilon = 1.3, \ C'_{\varepsilon 1} = C_{\varepsilon 1}\left(1 + C^*_{\varepsilon 1}\sqrt{\frac{k}{\upsilon^2}}\right).$$

The model by Park T.S. and Sung H.J. [PAR 97] introduces a similar concept intended to eliminate the distance from the wall by solving, in addition to the k and ε equations, an elliptic equation for coefficient f_μ appearing in the turbulence viscosity $\nu_t = C_\mu f_\mu k^2 / \varepsilon$.

The Van Driest formula $f_w = 1 - \exp(-y^+ / A)$ is derived twice and some simplifying physical hypotheses are made to get the equation:

$$L^2 f_{w,jj} = \frac{\text{Re}_t^{3/2}}{A^2}(f_w - 1), \quad L^2 = C_p^2\left[\frac{k^3}{\varepsilon^2} + C_\eta^2 \left(\frac{\nu^3}{\varepsilon}\right)^{1/2}\right],$$

with $C_p = 0.2$ and $C_\eta = 70$. This model applies, in particular, to separated flows.

A variant of the $k - \upsilon^2 - f$ model by Durbin is proposed by Hanjalic K. *et al.* [HAN 04B] in which an equation for $\zeta = \upsilon^2 / k$ is solved in place of the equation for υ^2, in order to reinforce the numerical stability of the method and the robustness of the model. The transformed equation reads:

$$\frac{d\zeta}{dt} = f - \frac{\zeta}{k}P + \left[\left(\nu + \frac{\nu_t}{\sigma_\zeta}\right)\zeta_{,j}\right]_{,j} + X.$$

The term $X = \dfrac{2}{\tau\varepsilon}\left(\nu + \dfrac{\nu_t}{\sigma_\zeta}\right)\zeta_{,j}k_{,j}$ coming from the variable change is omitted in practice. The elliptic equation for f is built from the Speziale *et al.* model [SPE 91]:

$$L^2\nabla^2 f - f = \frac{1}{\tau}\left(c_1 - C'_2\frac{P}{\varepsilon}\right)\left(\zeta - \frac{2}{3}\right) - \left(\frac{1}{4}C_4 - C_5\right)\frac{P}{k},$$

with $\quad C_\mu = 0.22$, $\quad c_1 = 0.4$, $\quad C'_2 = 0.65$, $\quad C_4 = 0.625$, $\quad C_5 = 0.2$, $\quad$ and
$C_{\varepsilon 1} = 1.4(1 + 0.012/\zeta)$, $C_{\varepsilon 2} = 1.92$, $\sigma_\zeta = 1.2$.

The turbulence viscosity is obtained from $\nu_t = C_\mu \zeta \dfrac{k^2}{\varepsilon}$. An additional suggestion by the authors consists of solving an equation for $\widetilde{f} = f - 2\nu\left(\sqrt{\zeta}\right)^2_{,j}$ instead of an equation for f in order to use the boundary condition $\widetilde{f} = 0$ on the wall, which is more easy to handle.

11.5. Explicit algebraic models

11.5.1. *Model by Gatski and Speziale*

The Rodi hypothesis developed in section 11.3 associated with a second order transport model leads to an implicit system giving the components of the Reynolds stresses. The formal solution of this system after linearization yields effective viscosity models which generalize the Boussinesq hypothesis. These are the so-called explicit algebraic stress models (EASM).

Gatski T.B. and Speziale C.G., [GAT 93] write the resulting implicit relation to be solved in the form:

$$a^*_{ij} = -S^*_{ij} - a^*_{ip}S^*_{jp} - a^*_{jp}S^*_{ip} + \frac{2}{3}a^*_{ml}S^*_{ml}\delta_{ij} + a^*_{ip}\omega^*_{pj} + a^*_{jp}\omega^*_{pi} \qquad [11.26]$$

The following definitions:

$$S^*_{ij} = \frac{1}{2}g\frac{k}{\varepsilon}(2 - c_3)S_{ij}, \qquad \omega^*_{ij} = \frac{1}{2}g\frac{k}{\varepsilon}(2 - c_4)\omega_{ij}$$

$$a_{ij}^* = \left(\frac{c_3 - 2}{c_2 - 4/3} \right) a_{ij} , \qquad g = \left(\frac{c_1}{2} + \frac{P}{\varepsilon} - 1 \right)^{-1} ,$$

allow us to make the connection with the model by Speziale C.G., Sarkar S. and Gatski T.B., [SPE 91B] (cf. Chapter 8), written in the form:

$$\Phi_{ij} = -c_1 \varepsilon a_{ij} + c_2 k S_{ij} + c_3 k \left(a_{im} S_{jm} + a_{jm} S_{im} - \frac{2}{3} a_{lm} S_{lm} \delta_{ij} \right)$$
$$+ c_4 k \left(a_{im} \omega_{jm} + a_{jm} \omega_{im} \right),$$

in which coefficients c_1 to c_4 are constants or functions of the invariants related to the non-linearities of the model:

$$c_1 = 3.4 + 1.8 P / \varepsilon , \quad c_2 = \frac{4}{5} - 1.30 \left(II^a \right)^{1/2} , \quad c_3 = 1.25 , \quad c_4 = 0.40 .$$

This equation corresponds to a condensed expression of equation [8.12] in which the quadratic term has been neglected. Adaptation to other second order models can, of course, be made in the same way.

Pope S.B., [POP 75] has shown that the general analytic solution of the implicit system (linearized) for the Reynolds stresses such as [11.26] can be written in the form (cf. section 3.6.4):

$$a_{ij}^* = \sum_{\lambda=1}^{10} G^{(\lambda)} T_{ij}^{(\lambda)} , \qquad\qquad [11.27]$$

where $T_{ij}^{(\lambda)}$ are the tensors of the integrity basis (here 10 in number, because the independent irreducible tensorial combinations are limited by the Cayley-Hamilton theorem; cf. Chapter 3) and where the scalar coefficients $G^{(\lambda)}$ are dependent on the invariants built on S_{ij} and ω_{ij} (here five in number). In the general 3D case, the large number of basis tensors makes the formulation very complex ([JON 98]). However, in the two-dimensional case, the formalism is greatly simplified since the number of basis tensors is reduced to three.

They are:

$$S_{ij} , \left(S_{jm}\omega_{im} + S_{im}\omega_{jm} \right) = \left(S_{jm}\omega_{im} - S_{im}\omega_{mj} \right) \text{ and} \left(S_{im}S_{jm} - \frac{1}{3}S_{lm}S_{lm}\delta_{ij} \right).$$

Coefficients $G^{(\lambda)}$ are obtained from identification.

The relation $\alpha a^* - \beta S^* - \left(a^* S^* + a^* S^* - \frac{2}{3}\left| a^* . S^* \right| \delta \right) - \left(a^* W^* + W^* a^* \right) = 0$, or equivalently written in components:

$$\alpha a_{ij}^* - \beta S_{ij}^* - \left(a_{ip}^* S_{jp}^* + a_{jp}^* S_{ip}^* - \frac{2}{3} a_{ml}^* S_{ml}^* \delta_{ij} \right) - \left(a_{ip}^* W_{pj}^* + a_{jp}^* W_{pi}^* \right) = 0 ,$$

in which a^* is an anisotropy tensor, S^* a symmetric tensor and W^* an antisymmetric tensor, can be solved explicitly in the case of two-dimensional flows, into the form:

$$a^* = C_1 S^* + b(W^* . S^* - S^* . W^*) + c(S^{*2} - \frac{1}{3}\left| S^{*2} \right| \delta)$$

or $a_{ij}^* = C_1 S_{ij}^* + b(W_{im}^* S_{mj}^* - S_{im}^* W_{mj}^*) + c(S_{im}^* S_{mj}^* - \frac{1}{3} S_{ml}^* S_{ml}^* \delta_{ij})$

with: $C_1 = \dfrac{-\beta\gamma}{1 - \frac{2}{3}\gamma^2 \left| S^{*2} \right| + 2\gamma^2 \left| W^{*2} \right|}$, $b = \gamma C_1$, $c = -2\gamma C_1$, $\gamma = -\dfrac{1}{\alpha}$.

We will notice that the parameters α and β could have been chosen equal to unity without loss of generality. It would suffice in fact to make the change of functions $a^* = \beta a'^*$ and $S^* = \alpha S'^*$.

Coefficients C_1, b and c are obtained from identification by reporting the explicit relation into the initial equation and taking into account the relations coming from the Cayley-Hamilton theorem. Any linear model can be put into the form of the initial equation, provided an adequate change in unknown function is made.

Thus, $G^{(1)} = C_1$, $G^{(2)} = b$ and $G^{(3)} = c$. The preceding authors [GAT 93], on the basis of the linear second order model and the equilibrium relation $\dfrac{dR_{ij}}{dt} = (P - \varepsilon) \dfrac{R_{ij}}{k}$, obtain in this way, after rearrangement of terms:

$$R_{ij} = \frac{2}{3} k \delta_{ij} - \frac{6\alpha_1}{3 - 2\eta^2 + 6\xi^2} \left[\frac{k^2}{\varepsilon} S_{ij} + \alpha_2 \frac{k^3}{\varepsilon^2} \left(S_{im} \omega_{mj} + S_{jm} \omega_{mi} \right) \right.$$
$$\left. - \alpha_3 \frac{k^3}{\varepsilon^2} \left(S_{im} S_{mj} - \frac{1}{3} II^S \delta_{ij} \right) \right]. \tag{11.28}$$

Coefficients α_1, α_2 and α_3 are dependent on the values of c_m in the initial turbulence model:

$$\alpha_1 = \frac{1}{2} \left(\frac{4}{3} - c_2 \right) g, \quad \alpha_2 = \frac{1}{2} \left(\frac{4}{3} - c_4 \right) g^2, \quad \alpha_3 = (2 - c_3) g^2,$$
$$g = \frac{1}{P/\varepsilon - 1 + c_1 / 2},$$

and also $\eta = \dfrac{1}{2} (2 - c_3) g^2 \dfrac{k}{\varepsilon} \sqrt{II^S}$ and $\xi = \dfrac{1}{2} (2 - c_4) g^2 \dfrac{k}{\varepsilon} \sqrt{II^\omega}$.

In the case of closure hypotheses including non-linearities, it will be necessary to determine these values of coefficients depending on the type of flow under consideration.

The singularity appearing in the denominator can be regularized by the approximation:

$$\frac{1}{3 - 2\eta^2 + 6\xi^2} \approx \frac{1 + \eta^2}{3 + \eta^2 + 6\xi^2 \eta^2 + 6\xi^2}, \tag{11.29}$$

deduced from the approximation $\eta^2 \approx 1 - \dfrac{1}{1 + \eta^2}$ in which η is supposed to be small.

This regularization allows us to use the model in situations which are far from equilibrium ([SPE 96]).

Note: equation [7.36] for the energy dissipation rate developed by Speziale and Gatski [SPE 97] can be used in conjunction with model [11.28]. This model gives the value of P/ε to be used in [7.35]:

$$\frac{P}{\varepsilon} = 2c_\mu^* \eta^2 , \quad c_\mu^* = \frac{3\alpha_1}{3 - 2\eta^2 + 6\xi^2} .$$

Equation [7.36] can thus be simplified in this case:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon1}^* \frac{\varepsilon P}{k} - C_{\varepsilon2} \frac{\varepsilon^2}{k} + Diff(\varepsilon) ,$$

with $C_{\varepsilon1}^* = C_{\varepsilon1} + \frac{2(1+\alpha_\varepsilon)}{15 c_\mu^*} \left[\frac{C_{\varepsilon5} + 2c_\mu^* \eta^2 - 1}{\left(C_{\varepsilon5} + 2c_\mu^* \eta^2 - 1\right)^2 - \frac{2}{3}\beta_2^2 \eta^2 + 2\beta_1^2 \xi^2} \right]$, $\beta_1 = \frac{7}{11}\alpha + \frac{1}{11}$,

$\beta_2 = \frac{15}{11}\alpha - \frac{1}{11}$, $\alpha_\varepsilon = \frac{3}{4}\left(\frac{14}{11}\alpha - \frac{16}{33}\right)$, $\alpha = 0.6$, $C_{\varepsilon5} \approx 5.0$.

We also mention the work [GRU 05] on the development of EASM. The study of realizability of the explicit algebraic models such as the ones by Gatski T. and Speziale C.G., [GAT 93] was carried out by Fu S. *et al.*, [FU 97]. This study has led these authors to propose a new set of coefficients ensuring realizability.

11.5.2. *Model by Abe*, *Kondoh and Nagano*

Abe K., Kondoh T., Nagano Y. [ABE 97] developed a non-linear k-ε model of EASM type taking into account the effects of a low Reynolds number near a wall and applicable to separated flows. The formulation is:

$$R_{ij} = \frac{2}{3}k\delta_{ij} + \frac{1}{1+\left(\dfrac{C_D \nu_t}{k}\right)^2 \left[\dfrac{11}{3}\Omega^2 + \dfrac{1}{3}\left(\Omega^2 - S^2\right)f_B\right]}$$

$$\times \left[-2\nu_t S_{ij} - 4C_D \frac{\nu_t^2}{k}\left(S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}\right) + 4C_D \frac{\nu_t^2}{k}\left(S_{ik}S_{kj} - \frac{1}{3}S_{mn}S_{mn}\delta_{ij}\right)\right],$$

$$\overline{\left(u_i u_i + \frac{p}{\rho}\right)u_j} = -C_s f_{t1}\tau R_{jl}k_{,l} , \qquad \overline{\varepsilon' u_j} + 2\frac{\nu}{\rho}\overline{p_{,i}u_{j,i}} = -C_\varepsilon f_{t2}\tau R_{jl}\varepsilon_{,l} ,$$

$$v_t = C_\mu f_\mu \frac{k^2}{\varepsilon}, \qquad \tau = v_t / k, \quad f_B = 1 + C_\eta C_D^2 \tau^2 \left(\Omega^2 - S^2 \right),$$

$$f_\mu = \left\{ 1 + \frac{35}{\mathrm{Re}_t^{3/4}} \exp\left[-\left(\frac{\mathrm{Re}_t}{30} \right)^{3/4} \right] \right\} \left\{ 1 - \exp\left[-\left(\frac{y^*}{26} \right)^2 \right] \right\},$$

$$f_\varepsilon = \left\{ 1 + 0.3 \exp\left[-\left(\frac{\mathrm{Re}_t}{6.5} \right)^2 \right] \right\} \left\{ 1 - \exp\left[-\left(\frac{y^*}{3.7} \right)^2 \right] \right\},$$

$$C_D = 0.8, \, C_\mu = 0.12, \, y^* = \frac{u_\varepsilon y}{v}, \, u_\varepsilon = (v\varepsilon)^{1/4}, \, \mathrm{Re}_t = k^2 / v\varepsilon,$$

$$C_\eta = 5.0, \quad C_{\varepsilon 1} = 1.45, \quad C_{\varepsilon 2} = 1.9, \quad C_s = C_\varepsilon = 1.4 \ .$$

The authors point out that the main ingredient of the model is the form of the coefficient appearing in the expression of R_{ij} in front of the term in square brackets, while the factor into square brackets simply reuses the basis tensors which were present in the model by Speziale [SPE 87B].

Another characteristic, like in the model by Abe K. *et al.* [ABE 94 and 95], is the use of the Kolmogorov characteristic velocity scale u_ε in place of the friction velocity.

11.5.3. *Explicit algebraic models for the scalar fluxes*

Explicit algebraic modeling can also be developed for the turbulent fluxes of heat or any passive scalar, along the same lines and techniques as previously for the Reynolds stresses.

The tensorial representation theory leads to expressions of the type:

$$\frac{F_{\gamma j}}{\sqrt{kq}} = \sum_{\lambda=1}^{M} H^{(\lambda)} Q_j^{(\lambda)}, \qquad Q_j^{(\lambda)} \text{ being the basis vectors.}$$

This is the equivalent of formulation [11.27] for the scalar. However, the complete integrity basis (So R.M.C. *et al.* [SO 04]) contains 74 terms in the three-dimensional case (cf. also [*SPE 71]). In the 2D case, the number of tensors of the basis is reduced to 7, they are (So R.M.C. *et al.* [SO 04]):

$$Q^{(1)}_j = \Theta^*_j, \qquad\qquad\qquad Q^{(2)}_j = \omega^*_{jh} \Theta^*_h,$$

$$Q^{(3)}{}_j = S^*{}_{jh}\Theta^*{}_h \, , \qquad\qquad Q^{(4)}{}_j = \left(S^*{}_{jp}\omega^*{}_{ph} - \omega^*{}_{jp}S^*{}_{ph} \right)\Theta^*{}_h \, ,$$

$$Q^{(5)}{}_j = a^*{}_{jh}\Theta^*{}_h \, , \qquad\qquad Q^{(6)}{}_j = \left(a^*{}_{jp}\omega^*{}_{ph} - \omega^*{}_{jp}a^*{}_{ph} \right)\Theta^*{}_h \, ,$$

$$Q^{(7)}{}_j = \left(a^*{}_{jp}S^*{}_{ph} - S^*{}_{jp}a^*{}_{ph} \right)\Theta^*{}_h \, , \qquad \text{with } \Theta_i^* = \frac{k}{\varepsilon}\sqrt{\frac{k}{q}}\,\overline{\Gamma}_{,i} \, .$$

The relation $a^* = f(S^*, \omega^*, \delta)$ for the anisotropy tensor given previously allows us to further reduce the number of basis vectors to 4 ($Q^{(1)}{}_j$ to $Q^{(4)}{}_j$). The authors make additional simplifications in the representation relying upon physical arguments and consider the application to the turbulent homogenous shear flow with temperature gradient studying in particular the values of the turbulence Prandtl number. The reader may also refer to [SHI 97B] on the non-linear models of the turbulent heat fluxes.

This page intentionally left blank

Chapter 12

Simplified Closures:
Zero and One Transport Equation Models

Instead of solving a transport equation for ε (or any other quantity related to the length scale), ε can be determined through an explicit relation of the type $\varepsilon = k^{3/2}/\ell$ in which ℓ is a length scale given at each point of the flow field. These models thus have only one single transport equation for the velocity scale of turbulence (k for example).

We will distinguish essentially two types of one equation models:

– on the one hand, the models using the concept of turbulence eddy viscosity: Prandtl L., 1945 (cf. Launder B.E. and Spalding D.B., 1972); Nee W., Kovasznay L.S.G., [NEE 69A and B];

– on the other hand, the model by Bradshaw P. *et al.*, [BRA 67].

In the zero equation models, no transport equation is solved for calculating the turbulence quantities and the mean flow, The Reynolds stresses being directly related to the mean flow.

We will distinguish essentially two types of zero equation models (cf. Launder B.E. and Spalding D.B., 1972 and also Abramovich G.N., 1963):

– the effective viscosity models (Boussinesq J., 1877; Prandtl L., 1942);

– the mixing length models (Prandtl L., 1925).

12.1. One equation models

12.1.1. *Prandtl's model*

We consider the case of turbulent thin shear flows. The shear stress is assumed to be given by an expression based on the turbulence eddy viscosity hypothesis:

$$R_{12} = v_t \overline{U}_{1,2} \,, \tag{12.1}$$

the viscosity v_t is expressed using the Kolmogorov-Prandtl relation:

$$v_t = c_\mu \ell \sqrt{k} \,, \tag{12.2}$$

the turbulence kinetic energy k is obtained from its modeled transport equation:

$$\frac{dk}{dt} = v_t \left(\overline{U}_{1,2} \right)^2 + \left(\frac{v_t}{h_k} k_{,2} \right)_{,2} - \frac{k^{3/2}}{\ell} \,, \tag{12.3}$$

with $c_\mu = 0.09$, $h_k = 1.5$, the scale ℓ must be specified in each particular flow case: in free flows, jets, wakes, mixing layers, ℓ is generally assumed to be proportional to the width of the flow, but the proportionality constant may vary from one flow type to another.

In wall flows, Prandtl assumes $\ell \approx x_2$.

Advantages of the model: v_t is sensitive to the diffusion and convection processes of turbulence kinetic energy (history of the flow).

Drawbacks of the model: in more complex flows it is difficult to make an adequate choice for ℓ.

In this category, are placed various models such as the model by Nevzgljadov, [NEV 45] or Gawain and Prittchett, [GAW 70B]. A somehow different approach is used by Townsend and Lighthill (cf. Townsend A.A., 1956) resorting to the behavior law such as:

$$R_{ij} = \frac{2}{3} k \delta_{ij} - 0.4k \frac{S_{ij}}{\left(S_{lm} S_{lm} \right)^{1/2}} \,.$$

The calculation method of turbulent wakes proposed by Lee S.C. and Auiler J.E., [LEE 70] complemented by Harsha P.T. and Lee S.C., [HAR 70] also makes use of a relation to be compared to the one of Townsend.

12.1.2. *Model by Nee and Kovasznay*

The theory by Nee W. and Kovasznay L.S.G., [NEE 69A] is based on a transport equation for the effective viscosity $v_e = v + v_t$. This equation is derived empirically:

$$\frac{dv_e}{dt} = \left(v_e v_{e,2}\right)_{,2} + G - D , \qquad\qquad [12.4]$$

with $G = A(v_e - v)\left|\overline{U}_{1,2}\right|$ and $D = B(v_e - v)v_e / \ell^2$.

Coefficients A and B are constants and ℓ a characteristic length scale function of the current position and corresponding to the shear layer thickness. In a boundary layer, ℓ will be assumed to be equal to δ in the external region of the flow and proportional to x_2 (direction normal to the wall) near the wall.

Initially applied to the zero pressure gradient boundary layer, the method has been extended by the same authors to the case of flows in a non-zero pressure gradient. For this, an additional term has been introduced to equation [11.4]:

$$P = -C \frac{v_e (v_e - v)}{U_\infty U_{\infty,1}} \overline{U}_{1,2} , \qquad\qquad [12.5]$$

where C is a numerical constant and U_∞ the mean velocity of the fluid in the far field.

12.1.3. *Model by Spalart and Allmaras*

The model by Spalart P.R. and Allmaras S.R. [SPA 94] can be considered as an offshoot of the model by Nee W. and Kovasznay L.S.G., [NEE 69A] (cf. section 12.1.2) and also the model by Baldwin B.S. and Barth T.J. (BAL 90]). This model was developed in the applied context of aerodynamic flows and gives good predictions of boundary layers with pressure gradient. Compared to the model by Nee and Kovasznay, the Spalart and Allmaras model no longer involves the length scale related to the shear layer thickness, but still uses the distance from the wall as a parameter.

The model equations are the following:

$$\frac{d\tilde{v}}{dt} = C_{b1}\tilde{v}\tilde{S} + \left[\frac{\left(v + \tilde{v}\right)}{\sigma}\tilde{v}_{,j}\right]_{,j} + C_{b2}\frac{\tilde{v}_{,j}\tilde{v}_{,j}}{\sigma} - C_{w1}f_w \frac{\tilde{v}^2}{y^2} ,$$

$$\nu_t = \tilde{\nu} f_{v1}, \quad \tilde{\nu} = \nu + \nu_t, \quad \chi = \frac{\tilde{\nu}}{\nu}, \quad \bar{\omega}_{ij} = \frac{1}{2}\left(\overline{U}_{i,j} - \overline{U}_{j,i}\right),$$

$$S = \sqrt{2.\bar{\omega}_{ij}\,\bar{\omega}_{ij}}, \quad \tilde{S} = f_{v3}(\chi)S + \frac{\tilde{\nu}}{K^2 y^2} f_{v2}(\chi),$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + C_{v1}{}^3}, \quad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}, \quad f_{v3} = 1,$$

$$f_w = g\left(\frac{1 + C_{w3}{}^6}{g^6 + C_{w3}{}^6}\right)^{1/6}, \quad g = r + C_{w2}(r^6 - r), \quad r = \frac{\tilde{\nu}}{\tilde{S}K^2 y^2},$$

$$C_{b1} = 0.1355, \quad \sigma = 2/3, \quad C_{b2} = 0.622, \quad C_{w2} = 0.3, \quad C_{w3} = 2,$$

$$C_{w1} = \frac{C_{b1}}{K^2} + \frac{1 + C_{b2}}{\sigma}, \quad C_{v1} = 7.1.$$

The turbulent stresses are then obtained using the Boussinesq hypothesis:

$$\tau_{ij} = -\nu_t\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right).$$

The alternative choice ([CAT 99]) of coefficients $f_{v2} = \left(1 + \dfrac{\chi}{C_{v2}}\right)^{-3}$ and

$f_{v3} = \dfrac{(1 + \chi f_{v1})(1 - f_{v2})}{\chi}$ with $C_{v2} = 5$, allows us to ensure the positivity of $\tilde{S}$.

12.1.4. *One equation model with elliptic relaxation*

Associating a transport equation for the turbulence eddy viscosity derived from the Nee and Kovasznay model with the concept of elliptic relaxation (cf. section 8.3.2) has led Durbin P.A. *et al.* [DUR 94B] to a model of practical interest useful for treating flow cases which are noticeably far from equilibrium.

The transport equation for the turbulence eddy viscosity,

$$\frac{d\nu_t}{dt} = P_\nu - c_2 \nu_t |S| - c_4 \frac{\nu_t^2}{L_\nu^2} + \left[(\nu + \nu_t)\nu_{t,j}\right]_{,j},$$

contains a source term $P_\nu = \nu_t |S|$ whose physical meaning also includes the effects of pressure redistribution. According to the concept introduced in [DUR 91A], this term is obtained from the solution of a Poisson equation (elliptic relaxation):

$$L_p{}^2 P_{\nu,jj} - P_\nu = c_3 \nu_{t,j}{}^2 - |S| \nu_t,$$

with the corresponding definitions and the following values of numerical constants:

$$S_{ij} = \frac{1}{2}\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right), \quad |S| = \sqrt{2 S_{ij} S_{ij}},$$

$$L_\nu{}^2 = \frac{|S|^2}{S_{,j} S_{,j}} + c_m \frac{\nu_{t,j} \nu_{t,j}}{|S|^2}, \quad L_p{}^2 = c_p{}^2 \min\left[L_\nu{}^2; \max\left(\nu_t, c_\ell{}^2 \nu\right)/|S|\right],$$

$$c_2 = 0.85, \; c_4 = 0.2, \; c_\ell = 3.3 \; c_p = 1.2, \; c_m = 2 \text{ and } c_3 = \frac{1-c_2}{K^2} + 1 - \frac{c_4}{1 + K^4 c_m}.$$

In this model, the characteristic length scales are determined locally, without reference to the distance from the wall. The boundary conditions imposed at a wall are $\nu_t = n_j \nu_{t,j} = 0$; it thus follows, after the transport equation of ν_t, that the value of P_ν on the boundary is given by $-\nu \nu_{t,jj}(0)$.

Taking into account that $P_\nu - c_2 \nu_t |S| = (1 - c_2) \nu_t |S|$, we can note that the transport equation of ν_t, in homogenous turbulence can be viewed as a relaxation towards the mixing length (cf. section 12.2.1) with $\nu_t \to \frac{1-c_2}{c_4} L_\nu{}^2 \cdot |S|$.

The authors of this model have shown its good behavior in boundary layer flows with or without pressure gradient, the laminar/turbulent transition and also the flow around an airfoil at large incidence.

12.1.5. Model by Bradshaw

The exact equation for k in a turbulent thin shear flow reads:

$$\frac{dk}{dt} = -R_{12} \overline{U}_{1,2} - \left(\overline{pu_2}/\rho + \overline{k'u_2}\right)_{,2} - \varepsilon. \qquad [12.6]$$

The experiments in turbulent boundary layers show that $-R_{12}/k = a_1$ remains almost constant and equal to 0.3.

Bradshaw P. *et al.*, [BRA 67] thus define:

$$G = \frac{\overline{pu_2}/\rho + \overline{k'u_2}}{-R_{12}\cdot(-R_{12})_{max}^{1/2}} \quad \text{and} \quad L = \frac{(-R_{12})^{3/2}}{\varepsilon}. \qquad [12.7]$$

Bradshaw derives in this way an equation for the turbulent shear stress in a thin shear flow:

$$\frac{d}{dt}\left(-\frac{R_{12}}{a_1}\right) = -R_{12}\overline{U}_{1,2} - \left[G(-R_{12})(-R_{12})_{max}^{1/2}\right]_{,2} - \frac{(-R_{12})^{3/2}}{\varepsilon}. \qquad [12.8]$$
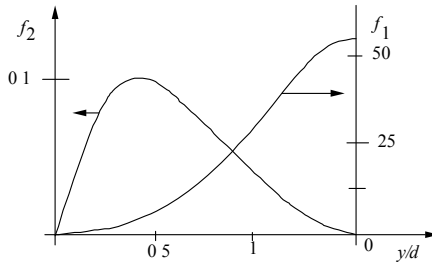

Figure 12.1. *Distribution of the empirical functions in the Bradshaw model for the turbulent boundary layer on a flat plate*

The quantities a_1, L and G are given functions (cf. Figure 12.1) in the boundary layer, such that:

$$a_1 = 0.3, \quad G = \frac{(-R_{12})_{max}^{1/2}}{U_\infty}f_1\left(\frac{y}{\delta}\right), \quad \frac{L}{\delta} = f_2\left(\frac{y}{\delta}\right).$$

The model by Bradshaw gives very good predictions in the calculation of boundary layers with or without pressure gradient. In this model, no gradient diffusion hypothesis is made, the diffusive flux is supposed to be proportional to a convection velocity of the big eddies: $(-R_{12})_{max}^{1/2}$. The use of the convection velocity of big eddies changes the mathematical nature of the equations, the equations become hyperbolic (they can be solved numerically using the method of characteristics). The difficulties in the choice of L in more complex flows still remain however. Let us note also that when R_{12} is changing sign the relation $-R_{12}/k = a_1$ is no longer valid, this is the case for example in wall jets. The method has been extended to the calculation of three-dimensional boundary layers by Nash J.F., [NAS 69] and Wesseling P. and Lindhout J.P.F., [WES 71] in particular.

An interesting extension to flows with maximum in the velocity profile (free jets) is given by the interaction model by Morel T. and Torda T.P., [MOR 74] who consider the plane jet as the interaction of two adjoining mixing layers in which two different equations are solved for the turbulent stresses τ^+ and τ^- in each of the two zones. The true turbulent stress is obtained by $\tau = \tau^+ + \tau^-$.

The comparative study of the three types of models (Patankar-Spalding with Van Driest hypothesis [13.17], Nee-Kovasznay and Kolmogorov-Prandtl) applied to the boundary layer by Wassel A.T. and Catton I., [WAS 73] leads us to conclude that the scheme by Patankar and Spalding derived from Van Driest hypothesis, in spite of the very approximate picture it gives for turbulent mixing, gives good predictions of friction and heat transfer, the two other models being more difficult to use.

12.2. Zero equation models

12.2.1. *Mixing length model*

1) The mixing length concept

In a turbulent shear flow (thin shear layer) a first hypothesis is introduced relatively to the expression of shear stress τ:

$$\frac{\tau}{\rho} = \nu_t \frac{\partial \overline{U}}{\partial y} = \ell_m V_t \frac{\partial \overline{U}}{\partial y}.$$ [12.9]

This expression is inspired by the kinetic theory of gases. However, if in the kinetic theory of gases, the mean free path of molecules can be assumed to be small compared to the size of the flow domain, i.e. the Knudsen number $\mathscr{K} = \xi / L$ is small (ξ being the mean free path of molecules), in contrast, in turbulence the dimensionless number ℓ_m / L remains of order unity. It follows that the gradient transport model is not justified in turbulent flows. The mixing length ℓ_m like V_t and ν_t are not properties of the fluid but they are properties of the flow itself. Relation [12.9] is thus merely a consequence of dimensional analysis in a flow dominated by a single length scale and a single velocity scale.

The second hypothesis is relative to the velocity scale:

$$V_t = \ell_m \left| \frac{\partial \overline{U}}{\partial y} \right|.$$ [12.10]

This relation simply states that the turbulence characteristic time ℓ_m / V_t has the same order of magnitude as the characteristic time scale of the mean flow $\left| \dfrac{\partial \overline{U}}{\partial y} \right|^{-1}$.

We deduce from [12.9] and [12.10] the usual form of the mixing length hypothesis:

$$\frac{\tau}{\rho} = \ell_m^2 \left| \frac{\partial \overline{U}}{\partial y} \right| \frac{\partial \overline{U}}{\partial y} .$$

[12.11]

2) Determination of the mixing length in usual turbulent flows

Free flows

The mixing length ℓ_m is generally assumed to be proportional to the width δ of the flow.

Type of flow	Two-dimensional mixing layer	Plane jet	Circular jet	Radial jet	Plane wake
ℓ_m / δ	0.07	0.09	0.075	0.125	0.16

Table 12.1. *Mixing length in free flows*

Tollmien (cf. Abramovich, [*ABR 63]) applied this hypothesis to the case of plane and circular jets in order to get analytical solutions for the mean velocity profile. The Schlichting theory on turbulent wakes is also based on this hypothesis.

In addition, the Taylor theory (1932) (cf. Abramovich, [*ABR 63]) supposes that the turbulent stress is due to the transfer of the swirl vector, it leads to an equation similar to the one obtained in the framework of Prandtl theory.

Boundary layers

Several empirical expressions have been proposed (cf. Launder B.E. and Spalding D.B., [*LAU 72]).

The Von Karman Th. hypothesis (1930) $\ell_m = C \dfrac{\partial \overline{U} / \partial y}{\partial^2 \overline{U} / \partial y^2}$ is not applicable to profiles with an inflexion point because $\ell_m \to \infty$ (!).

The hypothesis by Escudier M.P., [ESC 66] is written as:

$$\begin{cases} \dfrac{\ell_m}{\delta} = \mathrm{K}\dfrac{y}{\delta} & \text{if} \quad \dfrac{y}{\delta} \le \dfrac{\lambda}{\mathrm{K}} \\[2mm] \dfrac{\ell_m}{\delta} = \lambda & \text{if} \quad \dfrac{y}{\delta} > \dfrac{\lambda}{\mathrm{K}} \end{cases} \qquad \text{(cf. Figure 12.2)} \qquad [12.12]$$

with $\mathrm{K} = 0.43$, $\lambda = 0.09$.
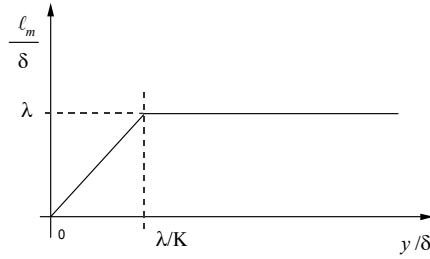
Figure 12.2. *Escudier mixing length hypothesis for the turbulent boundary layer on a flat plate*

Pipes

The hypothesis of Nikuradse J. (1932) can be quoted:

$$\frac{\ell_m}{R} = 0.14 - 0.08\left(1 - \frac{y}{R}\right)^2 - 0.06\left(1 - \frac{y}{R}\right)^4 , \qquad [12.13]$$

where R is the pipe radius.

The mixing length remains difficult to specify in more general flows with complex geometry, empirical practices are used.

Reichardt (1941) assumes that the momentum transport is proportional to the gradient of the longitudinal component of momentum:

$$\overline{u_1 u_2} = -\Lambda(x)\frac{\partial \overline{U}_1^{\,2}}{\partial y} . \qquad [12.14]$$

The Reichardt hypothesis leads to an equation formally identical to the heat equation. However, as it is noticed in Hinze J.O., 1975 this hypothesis does not satisfy the Galilean invariance principle and consequently cannot be theoretically correct even if the predicted velocity profiles are in good agreement with experiments.

3) Connection with one equation models

When the shear is large and the turbulence is in equilibrium state (or near equilibrium state), the kinetic energy k equation is dominated by the production and dissipation terms $P \approx \varepsilon$.

Thus, using the relations in section 11.2:

$$\frac{P}{\varepsilon} = 1 = c_\mu \frac{k^2}{\varepsilon^2} \overline{U}_{i,j} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right).$$

Taking into account $\varepsilon = k^{3/2}/\ell$, we then find:

$$k = c_\mu \ell^2 \overline{U}_{i,j} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right),$$

$$\nu_t = c_\mu \ell \sqrt{k} = c_\mu^{3/2} \ell^2 \left[\overline{U}_{i,j} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right) \right]^{1/2}. \tag{12.15}$$

We thus recover the mixing length hypothesis $\nu_t = \ell_m^2 \left| \overline{U}_{1,2} \right|$ where $\ell_m = c_\mu^{3/4} \ell$ is the mixing length. According to this point of view, the mixing length theory implies local energy equilibrium.

4) Extension to three-dimensional flows

Relation [12.15] suggests the most direct extension which can be written in the form:

$$\nu_t = \ell_m^2 \left(2 S_{ij} S_{ij} \right)^{1/2}, \tag{12.16}$$

with $S_{ij} = \frac{1}{2} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right)$.

There are other variants, such as the one suggested in the model by Baldwin B.S. and Lomax H., [BAL 78] (cf. section 12.2.2):

$$\nu_t = \ell_m^2 \left(2 \omega_{ij} \omega_{ij} \right)^{1/2}, \tag{12.17}$$

where $\omega_{ij} = \frac{1}{2} \left(\overline{U}_{i,j} - \overline{U}_{j,i} \right)$.

We can also refer to Cousteix J., [*COU 88].

12.2.2. *Effective viscosity models*

In these models, directly inspired by the J. Boussinesq hypothesis, the turbulence eddy viscosity ν_t is explicitly specified at every point of the flow field. Generally:

$$\nu_t = \underbrace{\ell_e V_e}_{(*)} \cdot \underbrace{f\left(\frac{y}{\delta}\right)}_{(**)},$$

(*) product of characteristic scales
(**) empirical function of the position in the flow.

These models are numerous (cf. *Proc. Comput. of Turb. Boundary Layers*, Stanford Conf., 1969).

The Prandtl L. (1942) model, intended to be used in free shear flows calculation, is an example (cf. "Prandtl's new theory" Abramovich, 1963). This model has been applied by Görtler H. (1942) to the case of a turbulent plane jet and to the mixing layer. In this scheme, the eddy viscosity ν_t is assumed constant in each cross-section of the flow:

$$\nu_t = C\delta \times \left|\overline{U}_{\max} - \overline{U}_{\min}\right|.$$

In this expression, C is an empirical constant and δ denotes the width of the flow. C varies from one flow to another (cf. Table 12.2).

Type of flow	Two-dimensional mixing layer	Plane jet	Circular jet
Constant C	0.0044	0.0148	0.010

Table 12.2. *Prandtl's constant in turbulent free flows*

Let us also cite the hypothesis by Clauser F.H., 1956, [CLA 56] applied to boundary layers:

$$\nu_t = C^{te} U_\infty \delta_*, \qquad \delta_* \text{ displacement thickness.}$$

This type of hypothesis is applied by Cebeci T., Smith A.M.O. and Mosinski G., [CEB 70] in the external region of a boundary layer. It is also this type of hypothesis which is used by Inouye, Marvin and Sheaffer, [INO 72] for the calculation of wakes.

The Baldwin B.S. and Lomax H., [BAL 78] model also widely used in practice, distinguishes two regions, as in the Cebeci T. and Smith A.M.O., [*CEB 74] model, in the treatment of the boundary layer:

– in the external region:

$$v_{t,ext} = \alpha C_{cp} F_{wake} F_{Kleb.}$$

$$F_{wake} = \min\left(y_M F_M, C_{wk} y_M U_M^2 / F_M\right),$$

with $\alpha = 0.0168$, $C_{cp} = 1.6$, $C_{wk} = 1$, and $F_M = \max\left(\dfrac{\ell |\omega|}{K}\right)$ for $y = y_M$.

$F_{Kleb.}$ is the Klebanoff correction function for intermittency at free boundaries and is equal to:

$$F_{Kleb.} = \left[1 + \left(C_{Kleb.} \frac{y}{y_M}\right)^6\right]^{-1}, \qquad C_{Kleb.} = 0.3,$$

and ω denotes the modulus of vorticity and U_M is the maximum velocity in the boundary layer (or the maximum mean velocity difference in a shear layer in free flows).

– in the internal region:

$$v_{t,int} = \ell_m^2 |\omega|, \quad \text{with} \quad \ell_m = Ky.$$

In the viscous sublayer, we also introduce the Van Driest correction factor (expression [13.17]).

The hypothesis by Gosse J., [GOS 61] is applicable to boundary layers and pipes:

$$v_t = C^{te}\left[1 - \left(1 - \frac{y}{R}\right)^2\right].$$

The main advantage of these models is simplicity. Their drawback comes from the fact that the extent of application of these rudimentary models to different flows is very limited.

Turbulence eddy viscosity models have been used in many turbulent flows in order to find predictions of mean velocity profiles (for example Chue and McDonald, [CHU 70] for Couette flow). Turbulence eddy viscosity models applicable to duct flows are also proposed by Kleinstein G., [KLE 71] who introduces a hypothesis which is valid on the whole cross-section of the pipe, and also by Mei J. and Squire W., [MEI 72]. More complex formulations have been introduced by Pope S.B., [POP 75]; Schonauer W., [SCH 72B].

These simple models can be easily and successfully applied to turbulent flows near equilibrium, such as usual boundary layers, but in more complex cases, like separated flows, it is necessary to account for some memory of turbulence using more advanced closures involving relaxation terms, generally in the form of transport equations.

12.2.3. *Example of application*

The use of very simple models of closure s (zero equation models) in conjunction with similarity hypotheses often leads to analytical solutions, which were very useful before the advent of computers. We consider, as an example, the turbulent wake behind an obstacle (cf. section 2.2.1, Figure 2.2).

The velocity of the external current is constant and equal to U_∞ and the defect velocity $\tilde{u} = U_\infty - u$ is assumed to be small compared to U_∞. The mean momentum equation can thus be linearized:

$$U_\infty \frac{\partial \tilde{u}}{\partial x} = -\frac{1}{\rho} \frac{\partial \tau}{\partial y} ,$$

The momentum conservation is expressed through:

$$\int_{-\infty}^{+\infty} \tilde{u}.dm = C^{te} \text{ with } dm = \rho u dy = \rho \left(U_\infty - \tilde{u} \right) dy ,$$

while the drag is equal to:

$$\rho \int_{-\infty}^{+\infty} \tilde{u}.\left(U_\infty - \tilde{u} \right) dy = D \quad \text{or} \quad \rho U_\infty \int_{-\infty}^{+\infty} \tilde{u}.dy \approx D ,$$

Looking for a similarity solution in the form:

$$\tilde{u}(x,y) = u_e(x)f(\eta) \text{ with } \eta = y/\delta(x),$$

using the eddy viscosity model (Prandtl's theory [*ABR 63]):

$$\tau = -\rho v_t \frac{\partial u}{\partial y}, \quad v_t = \chi\delta(x)u_e(x),$$

leads to the change in variables: $(x,y) \rightarrow (X,\eta)$

$$\begin{cases} \eta = y/\delta(x) \\ X = x \end{cases}, \quad \begin{vmatrix} \dfrac{\partial}{\partial y} = \dfrac{1}{\delta}\dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial X} - \dfrac{\eta}{\delta}\dfrac{d\delta}{dx}\cdot\dfrac{\partial}{\partial \eta} \end{vmatrix},$$

and to the transformed equation:

$$U_\infty\left(\frac{du_e}{dx}f - \frac{\eta u_e}{\delta}\frac{d\delta}{dx}f'\right) = \chi\frac{u_e^2}{\delta}f''.$$

The associated boundary conditions are:

$$f(\eta) \rightarrow 0 \quad \text{for } \eta \rightarrow \infty, \quad f'(\eta) \rightarrow 0 \quad \text{for } \eta = 0,$$

with the integral property:

$$\rho U_\infty \int_{-\infty}^{+\infty}\tilde{u}.dy = \rho U_\infty \delta(x)u_e(x)\int_{-\infty}^{+\infty} f d\eta = D$$

The separation of variables expresses the similarity conditions:

$$\left(U_\infty\frac{du_e}{dx}\frac{\delta}{u_e^2}\right)f - \left(\frac{U_\infty}{u_e}\frac{d\delta}{dx}\right)\eta f' = \chi f'',$$

$$U_\infty\frac{du_e(x)}{dx}\frac{\delta(x)}{u_e(x)^2} = C_1, \quad \frac{U_\infty}{u_e(x)}\frac{d\delta(x)}{dx} = C_2, \quad \frac{D}{\rho U_\infty \delta(x)u_e(x)} = 2$$

Eliminating u_e between the two previous relations gives:

$$\frac{d\delta^2}{dx} = \frac{C_2 D}{\rho U_\infty^2} \quad \text{or} \quad \delta = \sqrt{\frac{C_2 D}{\rho U_\infty^2}}.x,$$

an expression which is then reported in the latter equation to give $u_e = \dfrac{1}{2}\sqrt{\dfrac{D}{\rho C_2 x}}$.

The first relation then implies $C_1 = -C_2$.

The equation for f can then be written $C_2\left(f + \eta f'\right) + \chi f'' = 0$.

Taking into account the boundary conditions, the solution of this equation can be easily derived analytically:

$$f(\eta) = \exp\left[-\eta^2 \Big/ 2\frac{\chi}{C_2}\right],$$

The width δ can be defined arbitrarily by the location of the point such that $f = \dfrac{\tilde{u}}{u_e} = 0.9$, and so the relation $0.9 = \exp\left[-1\Big/2\dfrac{\chi}{C_2}\right]$ allows us to calculate χ from the knowledge of C_2. The constant C_2 can be obtained from calibration against experimental data (the value $C_2 = 0.648$ is deduced from the experimental relation $\delta = 0.805\sqrt{\dfrac{D}{\rho U_\infty^2}}.x$, [*ABR 63]).

12.2.4. *Heat and mass transfer*

The turbulent flux of a passive scalar quantity (temperature or concentration) being expressed by:

$$\overline{u_2\gamma} = -v_\gamma \overline{\Gamma}_{,2},$$

it is possible to introduce the turbulence Prandtl number $\mathrm{Pr}_t = v_t/v_\gamma$ and thus to write: $\overline{u_2\gamma} = -\dfrac{v_t}{\mathrm{Pr}_t}\overline{\Gamma}_{,2}$, or $\overline{u_i\gamma} = -\dfrac{v_t}{\mathrm{Pr}_t}\overline{\Gamma}_{,i}$ in tensorial form, which is the "SGDH" hypothesis introduced in Chapter 4.

The value of the turbulence Prandtl number is often assumed to be constant in the flow field, but its value in reality strongly depends on the type of flow geometry. The turbulent Prandtl number Pr_t is of order unity in duct flows (Reynolds'

analogy) but this is no longer the case in free flows in which Pr_t is quite close to 0.5.

The original Prandtl mixing length hypothesis applied to scalar transfer leads to mean scalar profiles that are identical to the velocity profiles (this result is thus not in accordance with experiments, in particular for free flows). In contrast, the theory of Taylor (1932) leads to $\ell_\gamma = \sqrt{2}.\ell$ and consequently $\dfrac{\overline{\Gamma}}{\overline{\Gamma}_M} = \sqrt{\dfrac{\overline{\overline{U}}}{\overline{\overline{U}}_M}}$.

The work by Fulachier L., [FUL 72] has shown that the turbulence Prandtl number is not exactly equal to unity in a heated boundary layer (this invalidates the Reynolds analogy) and its mean value is of order 0.9 in the fully turbulent region. But on the other hand the quantity $B = -\sqrt{\dfrac{2k}{\gamma^2}}.\dfrac{\overline{\Gamma}_{,2}}{\overline{U}_{,2}}$ remains remarkably constant

and near 1.5 inside the region $0.04 < y/\delta < 0.8$.

The approximation "GGDH" (cf. Chapter 4):

$$\overline{u_i\gamma} = -c_\gamma \frac{kR_{ij}}{\varepsilon}\,\overline{\Gamma}_{,j}\,, \qquad\qquad c_\gamma = 0.18\,,$$

is far more general. However, it assumes that the Reynolds stress components have already been calculated from a second order transport model or an algebraic model. In practice, the passive scalar Γ may stand for temperature or concentration of a constituent. It is a usual practice to couple a Reynolds stress transport model to a "GGDH" hypothesis for temperature.

A totally different approach to turbulent transfer is proposed by Launder B.E. and Spalding D.B., [*LAU 72] who have introduced no longer a gradient hypothesis but a two flux model inspired formally from radiation phenomena.

Chapter 13

Treatment of Low Reynolds Number Turbulence

The aim is to include, in the closure hypotheses, the effect of turbulence Reynolds number in the regions of the flow which low intensity turbulence. The effects of molecular viscosity appear essentially in two circumstances. On the one hand, in the very near wall region (smooth wall or slightly rough wall) the viscous sublayer is governed by molecular viscosity which thus influences the turbulent flow. On the other hand, there may be flows in which fully turbulent regions and laminarized regions coexist. This is the case in transitional boundary layer flows. Coexistence of adjoining turbulent regions and laminarized regions is also encountered in some complex flows such as rotating confined flows. The turbulence models presented previously neglected any direct viscous effect on the turbulence field and thus in particular they were not applicable in the viscous sublayer adjacent to the wall. In the viscous sublayer, the molecular transport reaches the same order of magnitude or higher than turbulent transport. The spectral regions of the energy-containing eddies and the dissipating eddies overlap. The dissipative eddies are then directly influenced by the mean flow and thus they are not isotropic and the large-scale structures are influenced by the molecular viscosity. To account for these phenomena, some of the empirical closure coefficients in the models will no longer be considered as numerical constants but rather as functions of Re_t, the turbulence Reynolds number. This influence is thus usually accounted for in an empirical way. Usual formulations are often limited in practice to the study of the viscous sublayer adjacent to the wall. The first work by Van Driest E.R., [VDR 56] dealt with the modification of the Prandtl mixing length near a wall, as for the work of Cousteix J., Quemard C. and Michel R., [COU 72C]. Glushko G.S., [GLU 71] taking into account the effect of the turbulence Reynolds number in a two equation transport

model has obtained a simulation of a transition-like phenomenon on a flat plate. In the case of transport equation models for the turbulence kinetic energy or for the Reynolds stresses, we can cite, among others, the contributions by Jones W.P. and Launder B.E., [JON 72 and 73]; Harlow F.H., [HAR 68A]; Hoffman G., [HOF 75]; Hanjalic K. and Launder B.E., [HAN 76]; Gibson M.M., Spalding D.B. and Zinser W., [GIB 78C].

13.1. Reynolds stress equations

For a general basic presentation, we shall refer essentially to the work of Hanjalic K. and Launder B.E., [HAN 76] whose model can be considered as a prototype.

1) Pressure-strain correlations

Considering the pressure-strain correlation $\Phi_{ij} = \overline{p\left(u_{i,j} + u_{j,i}\right)}/\rho$ in which the fluctuating pressure p is expressed in terms of the mathematical formal solution of the Poisson equation for the fluctuating pressure (Chapter 6), we see that the viscosity no longer explicitly appears. It is thus natural to think that the effect of viscosity is of second order and that no modification is necessary for this term as a first approach.

2) Molecular diffusion

The terms $\nu R_{ij,mm}$ which had been neglected in the approximation for high Reynolds number Re_t must now be retained in the present case.

3) Turbulent diffusion

The approximation for turbulent diffusion would seem to need modification, because the viscosity is directly involved in the triple velocity correlations equations for T_{ijk}. However, experiments near a wall suggest that:

– for $y+$ > 15 the production and dissipation terms are dominant and the turbulent diffusion remains weak;

– for $y+$ < 8 the molecular diffusion progressively exceeds the turbulent diffusion. Thus, there would remain only a very thin region in which the effect of Re_t could be effective. For this reason, the same approximation for the turbulent diffusion terms will generally be retained without modification for all turbulence Reynolds numbers Re_t. A more refined study is however possible (cf. section 13.4).

4) Viscous dissipation

On the other hand, it is essential to include the viscous effects in the dissipation terms. Considering the two limiting forms:

$- \varepsilon_{ij} = \dfrac{2}{3}\varepsilon\delta_{ij}$ which assumes local isotropy of microturbulence and $\varepsilon_{ij} = \dfrac{\varepsilon R_{ij}}{k}$ which assumes that the dissipative eddies have the same structure as the energy-containing eddies, Hanjalic and Launder proposed the intermediate approximation:

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\left[(1-f_s)\delta_{ij} + \frac{3}{2}\frac{R_{ij}}{k}f_s\right],\qquad\qquad [13.1]$$

$- $ or $\varepsilon_{ij} = \dfrac{2}{3}\varepsilon\delta_{ij} + f_s\varepsilon a_{ij}$ which is in fact a weighted mean between the two previous limiting hypotheses and where f_s is a function of $\mathrm{Re}_t = k^2/\nu\varepsilon$.

Using numerical optimization, these authors propose the empirical relation:

$$f_s = 1\Big/\left(1 + \frac{\mathrm{Re}_t}{10}\right).\qquad\qquad [13.2]$$

13.2. Equation for the dissipation rate

1) Considering the decay of turbulence behind a grid, the ε equation reduces to:

$$\overline{U}\frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k}.$$

In the initial region of decay $k \sim x^{-n}$, $n = 1.1$ and this implies $C_{\varepsilon 2} = 1.9$ (cf. Chapter 7).

In the final phase of decay, the viscous effects appear and n increases up to 2.5 approximately. Hanjalic and Launder propose to introduce a corrective coefficient to account for this effect in the dissipation rate equation $\overline{U}\dfrac{d\varepsilon}{dt} = -C_{\varepsilon 2}f_\varepsilon\dfrac{\varepsilon^2}{k}$ where f_ε is a function of $\mathrm{Re}_t = k^2/\nu\varepsilon$, ($f_\varepsilon \to 1$ when $\mathrm{Re}_t \to \infty$).

The experiments of Batchelor G.K. and Townsend A.A., [BAT 48A and B] suggested the empirical formula:

$$f_\varepsilon = 1.0 - 0.22 \exp\left(-\frac{\text{Re}_t^2}{36}\right).$$ [13.3]

2) Other modifications are necessary in the ε equation. Indeed, ε goes to a finite value on the wall whereas k goes to zero, ε^2 / k would then go to infinity, and this is not acceptable. The difficulty is simply solved by replacing ε^2 by $\varepsilon\tilde\varepsilon$ where $\tilde\varepsilon$ is the isotropic part of the dissipation defined by:

$$\tilde\varepsilon = \varepsilon - 2\nu\left(\sqrt{k}\right)_{,j}\left(\sqrt{k}\right)_{,j}.$$ [13.4]

In the region very close to the wall, the fluctuating velocity components can be represented as Taylor expansions in the wall distance:

$$u_1 = a_1 x_2 + a_2 x_2^2 + \ldots$$
$$u_2 = + b_2 x_2^2 + \ldots$$
$$u_3 = c_1 x_2 + c_2 x_2^2 + \ldots$$

which implies:

$$k = \frac{1}{2}\underbrace{\left(\overline{a_1^2} + \overline{c_1^2}\right)}_{a} x_2^2 + \underbrace{\left(\overline{a_1 a_2} + \overline{c_1 c_2}\right)}_{b} x_2^3 + \mathcal{O}(x_2^4)$$

Coefficients $a_1, a_2, b_2, c_1, c_2 \ldots$ are functions of x_1, x_3 and t. We can thus get:

$$\left(\frac{\partial \sqrt{k}}{\partial x_2}\right)^2 = a + 2bx_2 + \mathcal{O}(x_2^2),$$

and on the other hand $\dfrac{\varepsilon}{\nu} = \overline{\dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_i}{\partial x_j}} = 2a + 4bx_2 + \mathcal{O}(x_2^2).$

Comparing the two results we see that $\tilde\varepsilon = \varepsilon - 2\nu\left(\dfrac{\partial \sqrt{k}}{\partial x_2}\right)^2$ varied like x_2^2 at the wall. k and ε both behave like x_2^2, $\varepsilon\tilde\varepsilon / k$ goes to a constant value (cf. Figure 13.1). Direct numerical simulations seem to show that the maximum of ε is indeed

generally located on the wall (unlike Figure 13.1) rather than apart from the wall as experimental data has suggested.
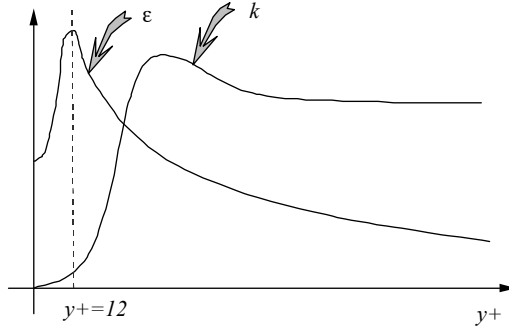
Figure 13.1. *Distribution of k and ε near a wall*

3) A gradient diffusion hypothesis allows us to approximate the term $2\nu \overline{u_k u_{i,l}} \overline{U}_{i,kl}$, thus from:

$$\overline{u_k u_{i,l}} = -\frac{C_{\varepsilon 3}}{2} \frac{kR_{jk}}{\varepsilon} \overline{U}_{i,lj},$$

the following approximation is obtained:

$$-2\nu \overline{u_k u_{i,l}} \, \overline{U}_{i,kl} = C_{\varepsilon 3} \nu \frac{kR_{jk}}{\varepsilon} \overline{U}_{i,lj} \overline{U}_{i,kl} .$$ [13.5]

4) Coefficients $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are kept constant.

We thus arrive at the complete form for the ε equation at low turbulence Reynolds numbers:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1} \frac{\varepsilon P}{k} - C_{\varepsilon 2} f_\varepsilon \frac{\varepsilon \tilde{\varepsilon}}{k} + C_\varepsilon \left(\frac{kR_{ij}}{\varepsilon} \varepsilon_{,i} \right)_{,j} + \nu \varepsilon_{,jj} + C_{\varepsilon 3} \nu \frac{kR_{jk}}{\varepsilon} \overline{U}_{i,lj} \overline{U}_{i,kl} .$$ [13.6]

13.3. The *k-R₁₂-ε* model for wall flows

In two-dimensional boundary layers, the gradient $\overline{U}_{1,2}$ is the only important velocity gradient and R_{12} the only important shear stress component. The RS1

model (Chapter 6), after accounting for the previous modifications yields for the transport equation of the shear stress:

$$\frac{dR_{12}}{dt} = -(c_1 + f_s)\frac{\varepsilon}{k}R_{12} + c'_s\left(\frac{kR_{22}}{\varepsilon}R_{12,2}\right)_{,2} + \nu R_{12,22}$$

$$-\left(0.24\frac{R_{22}}{k} + 0.18 - 0.11\frac{R_{11}}{k}\right)k\overline{U}_{1,2} \qquad\qquad [13.7]$$

(Hanjalic and Launder have neglected the wall effect term in the pressure-strain correlation, which is weak in the case of the shear component R_{12}).

The three equations for the normal stresses will be replaced by an equation for the kinetic energy of turbulence k. However, at low Reynolds numbers Re_t, it is no longer possible (as in Chapter 11) to approximate R_{11} and R_{22} by a constant fraction of the kinetic energy k, since R_{22}/R_{11} goes to zero on the wall.

However, R_{33} remains correctly approximated by $(R_{11} + R_{22})/2$ and it will be possible to use:

$$R_{11} + R_{22} = 4/3k . \qquad\qquad [13.8]$$

The measurements by Eckelmann (cf. Hanjalic K. and Launder B.E., [HAN 76]) show that the correlation coefficient $\dfrac{-R_{12}}{\sqrt{R_{11}R_{22}}}$ remains constant and equal to 0.45 down to a distance of $y+ = 10$ from the wall. Thus:

$$R_{11}R_{22} = \left(\frac{R_{12}}{0.45}\right)^2 . \qquad\qquad [13.9]$$

From relations [13.8] and [13.9] we get R_{11} and R_{22}:

$$R_{11} = \frac{2}{3}k\left[1 + \sqrt{1 - 12\left(\frac{R_{12}}{k}\right)^2}\right] \qquad R_{22} = \frac{2}{3}k\left[1 - \sqrt{1 - 12\left(\frac{R_{12}}{k}\right)^2}\right].$$

or approximately:

$$R_{11} \approx \frac{4}{3}k - 4\frac{R_{12}^2}{k} \quad \text{and} \quad R_{22} \approx 4\frac{R_{12}^2}{k} .$$

Accounting for these relations, we will get the system:

$$\frac{dk}{dt} = -R_{12}\overline{U}_{1,2} + 4.0c'_s\left(\frac{R_{12}^2}{\varepsilon}k_{,2}\right)_{,2} + vk_{,22} - \varepsilon,$$

$$\frac{dR_{12}}{dt} = -(c_1 + f_s)\frac{\varepsilon}{k}R_{12} + 4.0c'_s\left(\frac{R_{12}^2}{\varepsilon}R_{12,2}\right)_{,2} + vR_{12,22}$$

$$-f_{s2}\left[1.4\left(\frac{R_{12}}{k}\right)^2 + 0.03\right]k\overline{U}_{1,2},$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon1}\frac{\varepsilon R_{12}}{k}\overline{U}_{1,2} + 4.0c_\varepsilon\left(\frac{R_{12}^2}{\varepsilon}\varepsilon_{,2}\right)_{,2} + v\varepsilon_{,22} - C_{\varepsilon2}f_\varepsilon\frac{\varepsilon\tilde{\varepsilon}}{k} + 8v\frac{R_{12}^2}{\varepsilon}\left(\overline{U}_{1,22}\right)^2,$$

$$c_1 = 1.5, \quad c'_s = 0.22, \quad c_\varepsilon = 0.15 \qquad\qquad\qquad [13.10]$$

Coefficient f_{s2} should be equal to unity, but numerical results led, however, Hanjalic and Launder to use $f_{s2} = \exp\left[-\dfrac{2}{1 + \mathrm{Re}_t/30}\right]$ to improve the results.

13.4. Modification of the turbulent fluxes in low intensity turbulence

Harlow F.H., [HAR 68A] has studied the modifications to bring to the gradient law for approximating the turbulent diffusion fluxes in the cases of low Reynolds numbers Re_t. Let us follow his reasoning.

Let Q be any transported quantity (function of the point) $Q = \overline{Q} + q$.

If an "eddy" is moving on a distance s and we assume:

$$q(x_j) = \overline{Q}(x_j - sn_j) - \overline{Q}(x_j),$$

with n_j unit vector in the direction of motion, at first order we shall get:

$$q(x_j) = -sn_j\overline{Q}_{,j}$$

and for the turbulent flux $\overline{u_i q} = -\overline{u_i sn_j}\,\overline{Q}_{,j} = -\sigma_{t0}\overline{Q}_{,i},$

σ_{t0} being the isotropic diffusivity coefficient. We thus recover the gradient law.

However, when the molecular diffusivity σ is large, we expect that the value of Q in the "eddy" conforms and follows the value in the surrounding medium.

Harlow considers a spherical model of "eddy", the value of Q attached to it follows the equation:

$$\frac{4}{3}\pi R^3 \frac{dQ}{dt} = 4\pi R^2 F \quad \text{where} \quad F = -\sigma \frac{dQ}{dr} \text{ is the exchanged flux}$$

$$F \approx \sigma \frac{Q_0 - Q}{R} \quad \Rightarrow \quad \frac{dQ}{dt} = \frac{3\sigma}{R^2}(Q_0 - Q),$$

R radius, Q_0 value of Q in the ambient medium.

At first order $Q_0 = u_m \overline{Q}_{,m} t$, (for simplifying we assumed $Q_0 = 0$ at $t = 0$).

The solution of the equation for Q (or q) thus reads:

$$q = Q - Q_0 = \frac{R^2}{3\sigma} u_m \overline{Q}_{,m} \left(e^{-3\sigma t / R^2} - 1 \right),$$

and consequently:

$$\overline{u_j q} = \overline{u_j u_m} \frac{R^2}{3\sigma} \overline{Q}_{,m} \left(e^{-3\sigma t / R^2} - 1 \right),$$

with the hypotheses: $\sigma_{t0} = C\ell\sqrt{k}$, $\overline{u_j u_m} = \frac{2}{3}k\delta_{jm} + ...$

Considering R proportional to ℓ and t proportional to ℓ^2 / σ_{t0}, Harlow arrives at an expression of the type:

$$\overline{u_j q} = A \underbrace{\frac{\sigma_{t0}^2}{\sigma} \left(e^{-\sigma / A\sigma_{t0}} - 1 \right)}_{\sigma_t} \overline{Q}_{,j}, \qquad\qquad [13.11]$$

and so when $\sigma_{t0} / \sigma \to \infty$ then $\overline{u_j q} \to -\sigma_{t0} \overline{Q}_{,j}$ and $\sigma_t \to \sigma_{t0}$

and when $\sigma_{t0}/\sigma \to 0$ we get $\overline{u_j q} \to -A\dfrac{\sigma_{t0}^{\,2}}{\sigma}\overline{Q}_{,j}$ and $\sigma_t \to A\dfrac{\sigma_{t0}^{\,2}}{\sigma}$.

In the case of a vector formulation [13.11] is modified in order for the tensorial contraction to be correct. Thus:

$$\overline{u_j u_m} = \frac{2}{3}k\delta_{jm} - B\frac{\nu_{t0}^{\,2}}{\nu}\left(e^{-\nu/B\nu_{t0}} - 1\right)\left(\overline{U}_{j,m} + \overline{U}_{m,j}\right) \tag{13.12}$$

There then results, if Q is the temperature in [13.11], an expression of the turbulence Prandtl number:

$$\mathrm{Pr}_t = \frac{B}{A}\frac{1}{\mathrm{Pr}}\frac{1-\exp(-1/B.\mathrm{Re}_t)}{1-\exp(-1/A.\mathrm{Pr}.\mathrm{Re}_t)}. \tag{13.13}$$

On this subject, let us also cite the expression for Pr_t obtained by Cebeci T., [CEB 73] from an analysis of Stokes flow schematizing the viscous sublayer near a wall.

$$\mathrm{Pr}_t = \frac{C_m}{C_\gamma}\frac{1-\exp(-y/A)}{1-\exp(-y/B)}, \qquad B = B(\mathrm{Pr}). \tag{13.14}$$

13.5. Lower order models

The Reynolds stress transport models at low Reynolds numbers Re_t have not always been tested very extensively. For industrial applications the simpler models are often preferred.

13.5.1. *The k-ε model*

The behavior laws:

$$R_{ij} = \frac{2}{3}k\delta_{ij} - \nu_t\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right) \quad \text{and} \quad F_{\gamma i} = -\frac{\nu_t}{\mathrm{Pr}_t}\overline{\Gamma}_{,i},$$

involve $c_\mu = c_\mu(\mathrm{Re}_t)$ and $\mathrm{Pr}_t = \mathrm{Pr}_t(\mathrm{Re}_t, \mathrm{Pr})$.

Let us recall that on the wall $k \sim x_2^2$, $\tilde{\varepsilon} \sim x_2^2$ and $\varepsilon \rightarrow$ constant. Jones W.P. and Launder B.E., [JON 72] propose the formulation:

$$\frac{dk}{dt} = -R_{ij}\overline{U}_{i,j} + \left(\nu_t k_{,j}\right)_{,j} + \nu k_{,jj} - \varepsilon, \qquad \nu_t = c_\mu \frac{k^2}{\varepsilon},$$

$$\frac{d\tilde{\varepsilon}}{dt} = -C_{\varepsilon 1}\frac{\tilde{\varepsilon}R_{ij}}{k}\overline{U}_{i,j} + 0.77\left(\nu_t\tilde{\varepsilon}_{,j}\right)_{,j} + \nu\tilde{\varepsilon}_{,jj} - C_{\varepsilon 2}f_\varepsilon\frac{\tilde{\varepsilon}^2}{k} + 2.0\nu\nu_t\overline{U}_{i,jm}\overline{U}_{i,jm} \quad [13.15]$$

The model makes use of $\tilde{\varepsilon}$ rather than ε as the unknown function, the boundary conditions for the equation of $\tilde{\varepsilon}$ are indeed simpler ($\tilde{\varepsilon} = 0$ on the wall). The experimental data of Klebanoff P.S., [KLE 54] for the boundary layer on a flat plate have been used for determining the empirical coefficients of the model, and afterwards recalibrated by Launder B.E. and Sharma B.I., [LAU 74A]:

$$c_\mu = 0.09\exp\left[-3.4/(1+\mathrm{Re}_t/50)^2\right], \qquad \mathrm{Re}_t = \frac{k^2}{\nu\tilde{\varepsilon}},$$

$$f_\varepsilon = 1.0 - 0.3\exp\left(-\mathrm{Re}_t^2\right), \qquad \varepsilon = \tilde{\varepsilon} + 2\nu\left(\sqrt{k}\right)_{,j}\left(\sqrt{k}\right)_{,j}$$

$$C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92.$$

Algebraic Reynolds stress models ASM (coupled with a k-ε model) can also be extended empirically to low Reynolds numbers in order to calculate turbulent flows down to the wall (cf., for example, Iacovides H. and Li H., [IAC 93]).

13.5.2. *The Hassid and Poreh model*

This is the Prandtl-Kolmogorov model extended by Hassid S. and Poreh M., [HAS 78] to the case of low Reynolds numbers.

The model equations are the following:

$$\frac{dk}{dt} = \nu_t\left(\overline{U}_{1,2}\right)^2 + \left(\nu_t k_{,2}\right)_{,2} + \nu k_{,22} - \varepsilon, \qquad\qquad [13.16]$$

$$\nu_t = C.L\sqrt{k}\left[1-\exp(-A_\mu\,\mathrm{Re}_t)\right], \qquad \mathrm{Re}_t = \frac{L\sqrt{k}}{\nu},$$

$$\varepsilon = c_D\frac{k^{3/2}}{L}\left[\frac{2}{c_D\,\mathrm{Re}_t} + 1 - \exp(-A_\mu\,\mathrm{Re}_t)\right],$$

in which it is assumed that $L \sim x_2$ in the internal region of the boundary layer.

Thus, at the wall (when Re_t goes to zero) $\varepsilon \to c_D \dfrac{k^{3/2}}{x_2} \dfrac{2\nu}{c_D x_2 \sqrt{k}} = \dfrac{2\nu k}{x_2^2}$ which

is a constant since $k \approx x_2^2$ and in the fully turbulent region (when Re_t goes to

infinity) $\mathrm{Re}_t \to \infty$, $\varepsilon \to c_D \dfrac{k^{3/2}}{L}$.

The numerical values of constants are:

$$c_D = 0.416, \qquad C = 0.22, \qquad A_\mu = 0.012.$$

For the study of associated heat transfer, Gibson M.M., Spalding D.B. and Zinser N., [GIB 78C] also use $\mathrm{Pr}_t = 0.9$.

13.5.3. *Other one or two equation models*

Numerous variants have been proposed by different authors for the one or two equation models at low turbulence Reynolds numbers. The most usual of these models are presented and compared in the paper by Patel V.C., Rodi W. and Scheuerer G., [PAT 85]. They are in particular: *(a)* Jones W.P. and Launder B.E., [JON 72 and 73]; *(b)* Launder B.E. and Sharma B.I., [LAU 74A]; *(c)* Hassid S. and Poreh M., [HAS 78]; *(d)* Hoffmann G., [HOF 75]; *(e)* Chien K.Y., [CHI 82]; *(f)* Reynolds W.C., [REY 76D]; *(g)* Lam C.K.G. and Bremhorst K., [LAM 81]; *(h)* Dutoya D. and Michard P., [DUT 81].

In all cases, the transport equations for the turbulent field keep the same general form:

$$\frac{dk}{dt} = P - \varepsilon + Diff(k),$$

$$\frac{d\tilde{\varepsilon}}{dt} = \Sigma_1 - \Sigma_2 + E + Diff(\varepsilon),$$

$$\nu_\phi = \nu + \frac{\nu_t}{\mathrm{Pr}_\phi}, \; \nu_t = c_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}}, \qquad \varepsilon = \tilde{\varepsilon} - D,$$

$$f_2(\Sigma_1 = C_{\varepsilon 1} f_1 \frac{\tilde{\varepsilon} P}{k}), \qquad \Sigma_2 = C_{\varepsilon 2} f_2 \frac{\tilde{\varepsilon}^2}{k}.$$

The use of a term such as D allows us to set $\varepsilon = 0$ on the wall.

$$D = 2\nu \left(\partial \sqrt{k} / \partial y\right)^2 \qquad\qquad \text{(ref. } a, b, c\text{)}$$

$$D = 2\nu k / y^2 \qquad\qquad \text{(ref. } e\text{)}$$

The models (f) and (g) use $D = 0$ and assume the condition $\varepsilon = \nu k_{,22}$ on the wall.

f_1 is generally kept constant,   except in (h).

$f_2 = f_2(\text{Re}_t)$ is dependent on Re_t

$f_\mu = f_\mu(\text{Re}_t)$ (ref. a, b, c, d, h)

$f_\mu = f_\mu(y^+)$ (ref. e, f)

$f_\mu = f_\mu(y^+, \text{Re}_t)$ (ref. g)

The term E is an additional source:

$$E = 2\nu\nu_t \left(\overline{U}_{i,jj}\right)^2 \qquad\qquad \text{(ref. } a, b\text{)}$$

$$E = f(\varepsilon) \qquad\qquad \text{(ref. } c, e\text{)}$$

These various models may give different distributions for k and ε in the viscous sublayer, however the predicted values of ν_t remain close.

Other models have also been developed using a transport equation for quantity other than ε for calculating the turbulent length scale. We can cite in particular the model by Wilcox D.C. and Rubesin W.M., [WIL 80] based on a system (k, ω^2) with $\omega = \sqrt{k} / \ell$ and also the model by Tanaka H. *et al.*, [TAN 82] of type (k, W) with $W = k\ell$. The model by Wilcox and Rubesin, presented in Chapter 11, also applies to low Reynolds numbers with the advantage on the numerical point of view to simplify the boundary conditions at a wall, since ω vanishes on the wall while ε remains non-zero.

We can also cite the model by Morse A., [MOR 88] which is a modification of the Launder and Sharma model and which is applied by the same author to the flow in rotating cavities. The formulation is given below:

$$\frac{dk}{dt} = P - \varepsilon - D + \textit{Diff}(k),$$

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon}{k}\left(C_{\varepsilon 1}P - C_{\varepsilon 2}\varepsilon\right) + E - F + Diff\left(\varepsilon\right),$$

$$D = 2v\left(\sqrt{k}\right)_{,j}\left(\sqrt{k}\right)_{,j}, \quad \tilde{\varepsilon} = \varepsilon - D,$$

$$E = 2vv_t \overline{U}_{i,jl}\overline{U}_{i,jl}, \quad F = 2v\left(\sqrt{\tilde{\varepsilon}}\right)_{,j}\left(\sqrt{\tilde{\varepsilon}}\right)_{,j}.$$

The model by Nagano Y. and Tagawa M., [NAG 90] allows us to reconcile the behavior of turbulence in the region of the viscous sublayer near the wall with the behavior in the final period of turbulence decay behind a grid.

In the framework of one equation models, we also have to mention the approach of Wolfshtein M., [WOL 69] who makes use of two different length scales for the modeling of the turbulence kinetic energy equation, a length scale ℓ_μ related to diffusion and a length scale ℓ_D characteristic of viscous dissipation which are defined as follows:

$$\ell_\mu = y\left[1 - \exp\left(-A_\mu \frac{y\sqrt{k}}{v}\right)\right], \quad A_\mu = 0.016,$$

$$\ell_D = y\left[1 - \exp\left(-A_D \frac{y\sqrt{k}}{v}\right)\right], \quad A_D = 0.263.$$

This model has been applied by its author to the turbulent Couette flow and to the turbulent duct flow. This type of approach has been again undertaken by Chen H.C. and Patel V.C., [CHE 88] and also by Goldberg U.C., [GOL 91]. There are also one equation transport models for a quantity different from k, like the Spalart P.R. and Allmaras S.R. model, [SPA 92] based, such as the Nee W. and Kovasznay L.S.G. model, [NEE 69A and B] on a transport equation for the turbulence viscosity, extended to low Reynolds numbers.

The basic equation of this model takes the following form:

$$\frac{dv_t}{dt} = C_1 S v_t + \left[\left(v + C_2 v_t\right)v_{t,j}\right]_{,j} + C_3 v_{t,j}v_{t,j} - C_4 f_w\left(\frac{v_t}{d}\right)^2,$$

in which S is the norm of the strain rate tensor, d the distance from the nearest wall and f_w a wall damping function.

An alternative approach to the use of corrective functions near the wall and proposed by Durbin P.A., [DUR 93] consists of using clipping approximations. Thus, in the sink term in the dissipation rate equation written as $-C_{\varepsilon 2}\varepsilon/T_{\varepsilon}$, Durbin assumes:

$$T_{\varepsilon} = \max\left[\frac{k}{\varepsilon}, C_{Db}\sqrt{\frac{v}{\varepsilon}}\right]$$

This choice reduces, of course, to the usual expression k/ε when the turbulence Reynolds number is large. The same author ([DUR 95]) proposes a $k - v^2 - f$ model (cf. section 11.4.3) which can be considered as an extension of the k-ε model by adjoining an additional equation for the variable v^2, scalar quantity that can be identified with the normal stress R_{22} in the near wall region and which is used for the calculation of the turbulence viscosity in $v_t = c_{\mu D}v^2 T$.

Let us consider at present time the even simpler one equation models.

13.5.4. *Mixing length model and eddy viscosity model*

This type of formulation is more specifically dedicated to the calculation of the viscous sublayer. A simple improvement of the mixing length hypothesis was introduced by Rotta J.C. (1950); it reads:

$$\ell_m = C^{te}(y-\delta_1),$$

where δ_1 is the thickness of the viscous sublayer.

An eddy viscosity hypothesis is then also proposed by Reichardt H. (1951):

$$v_t = C^{te}u_*\left(y-\delta_1.\text{th}(\frac{y}{\delta_1})\right),$$

and by Deissler R.G. (1954):

$$v_t = C^{te}\overline{U}_1 y\left(1-\exp(-A\frac{\overline{U}_1 y}{v})\right),$$

where A is a numerical constant.

Among the more usual, are the following formulations:

1) Correction of Van Driest E.R., [VDR 56]

$$\ell_m = Ky\left[1 - \exp\left(-\frac{yu_*}{\nu A}\right)\right], \qquad u_* = \sqrt{\frac{\tau_p}{\rho}}, \qquad A=26, \qquad\qquad [13.17]$$

in which coefficient A determines the thickness of the viscous sublayer. In practice A is chosen in such a way as a correct value for E is obtained in the logarithmic law $\overline{U}_1 = \frac{u_*}{K} \text{Log} E y^+$ valid in the fully turbulent region. In the presence of strong pressure gradient, coefficient A becomes a function of dp/dx. Derived from the Van Driest hypothesis, the scheme used by Pletcher R.H., [PLE 72] is applicable in compressible flows.

Numerous variants have been introduced, they usually fall into the general form:

$$\nu_t = yu_* f(y^+), \qquad y^+ = \frac{yu_*}{\nu},$$

f being a "universal" function for a given type of flow.

2) Correction of Baker R.J. and Launder B.E., [BAK 74] and Launder B.E. and Jones W.P., [LAU 69A and B]. This correction applies to the mixing length:

$$\ell_m = Ky\left[1 - \exp\left(-\frac{y\tau}{\mu A u_*}\right)\right], \qquad\qquad\qquad [13.18]$$

where τ is the local shear stress $\tau = -\rho R_{12} + \mu \overline{U}_{1,2}$.

This model has been used by Baker R.J. and Launder B.E., [BAK 74] in the turbulent boundary layer with pressure gradient.

This type of correction consists in fact of keeping A as a constant, but modifying the definition of $^+$. In the case of the Baker and Launder model $y^+ = \frac{y\tau/\rho}{\nu u_*}$.

Other formulations of this type exist in which $y^+ = \dfrac{y}{v}\sqrt{\dfrac{\tau}{\rho}}$, ([PAT 67], Patankar S.V. and Spalding D.B., 1970).

If we consider the local stress $\tau(y)$, Cebeci T. *et al.*, [CEB 70] use $\tau = \tau_p + y\,dp/dx$ according to pressure gradient. Let us also cite:

$$- \ y^+ = \frac{y(\tau/\rho)^{3/2}}{vu_*} \, , \qquad \text{Launder B.E. and Priddin C.H., [LAU 73A]}$$

$$- \ y^+ = \frac{y}{v}\sqrt{\frac{\tau_A}{\rho}} \, , \qquad \text{Andersen, 1972, (cf. Launder B.E., [LAU 86])}$$

with $\tau_A = \tau(y^+ = 3A)$.

Note: the model by Kendall and Anderson, 1968 (cf. *Proc. Comput. of Turb. Boundary Layers*, AFOVR-IFP Stanford Conf., 1968) based on the equation:

$$\frac{\partial \ell_m}{\partial y} = \frac{Ky - \ell_m}{y_a}\frac{\sqrt{\tau/\rho}}{v} \, ,$$

leads after integration to $\ell_m = K\left[y - y_a\left(1 - \exp\left(-\dfrac{y}{y_a}\right)\right)\right]$, a relation similar to the previous ones.

3) Improved scheme by Michel (cf. Cousteix J., Quemard C. and Michel R., [COU 72C]). In this model, the mixing length is given by:

$$\frac{\ell_m}{\delta} = 0.085 \,\text{th}\left(\frac{K}{0.085}\frac{y}{\delta}\right), \qquad\qquad\qquad [13.19]$$

$$\tau = \mu\overline{U}_{1,2} + \rho F^2 \ell_m^{\,2}\left|\overline{U}_{1,2}\right|\overline{U}_{1,2} \, , \qquad F = 1 - \exp\left(\frac{-\sqrt{\tau_p}}{26K\mu}\right).$$

Let us cite finally Kleinstein G., [KLE 67] who makes use of the reduced velocity (normalized) as an independent variable and shows that the effect of the variations in the shear stress in the universal velocity law of the wall can be taken into account through a deviation from the logarithmic law.

13.6. Advanced modeling

The recent research work on second order closures at low turbulence Reynolds numbers have been stimulated by the new statistical information derived from the direct numerical simulations of turbulence and by a deeper use of realizability constraints. In this direction, we can cite the work of Launder B.E. and Shima N., [LAU 89B].

The main closure hypotheses to reconsider are those for Φ_{ij} and ε_{ij}.

If we assume that $\Phi_{ij}^{(1)}$ vanishes on the wall, the hypothesis $\Phi_{ij}^{(1)} = -c_1 \varepsilon a_{ij}$ cannot be suitable because the diagonal elements of a_{ij} are non-zero on the wall. A simple solution consists of using the hypothesis:

$$\Phi_{ij}^{(1)} = -c_1 \tilde{\varepsilon} a_{ij} \; .$$

This adaptation does not require important modifications.

13.6.1. *Near wall dissipation schemes*

The Taylor expansion of the fluctuating velocity components at the wall (cf. section 2) shows that the dissipation tensor ε_{ij} is highly anisotropic. Indeed, at the wall $\nu k_{,22} = 2\nu a = \varepsilon_w = 2k / x_2{}^2$ and:

$$\frac{\varepsilon_{11}}{R_{11}} = \frac{\varepsilon_{33}}{R_{33}} = \frac{\varepsilon_{12}}{2R_{12}} = \frac{\varepsilon_{22}}{4R_{22}} = \frac{\varepsilon}{k} \; .$$

Hypothesis [13.1] does not allow us to satisfy these relations (Launder B.E. and Reynolds W.C., [LAU 83]). This behavior at the wall can be reproduced using the approximation (Launder B.E., [LAU 86]) below:

$$\varepsilon_{ij}^{(1)} = \frac{\varepsilon}{k} \frac{R_{ij} + R_{il} n_l n_j + R_{jl} n_l n_i + R_{pl} n_p n_l \delta_{ij}}{1 + \frac{5}{2} \frac{R_{pl} n_p n_l}{k}} , \tag{13.20}$$

whereas at high Reynolds numbers, we must have:

$$\varepsilon_{ij}^{(2)} = \frac{2}{3} \varepsilon \delta_{ij} \; .$$

In general, a smooth change between these two expressions can be obtained by linear interpolation:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(1)} f_s + \varepsilon_{ij}^{(2)} (1 - f_s), \qquad f_s = \exp(-\mathrm{Re}_t / 40).$$

Another possibility consists of making use of the isotropic dissipation $\tilde{\varepsilon}$:

$$\varepsilon_{ij} = \frac{2}{3} \tilde{\varepsilon} \delta_{ij} + \frac{\varepsilon - \tilde{\varepsilon}}{k + \frac{5}{2} R_{22}} \left(R_{ij} + R_{il} n_j n_l + R_{jl} n_i n_l + R_{pl} n_p n_l \delta_{ij} \right) \qquad [13.21]$$

This problem has also been reconsidered in the work of Tagawa M. *et al.*, [TAG 91].

13.6.2. *Turbulent diffusion schemes near a wall*

For turbulent diffusion, which becomes negligible compared to viscous diffusion, it is possible to retain the gradient hypothesis:

$$T_{ijk} = -c_s \frac{k}{\varepsilon} R_{kl} R_{ij,l},$$

in which $\dfrac{k}{\varepsilon} \to C^{st}$ goes to a constant value at the wall.

13.6.3. *Model by Launder and Shima*

The Launder and Shima model [LAU 89B] includes the following hypotheses:

$$\Phi_{ij}^{(1)} = -c_1 \varepsilon a_{ij},$$

$$\Phi_{ij}^{(2)} = -c_2 \left(P_{ij} - \frac{2}{3} P \delta_{ij} \right),$$

$$\varepsilon_{ij} = \frac{2}{3} \varepsilon \delta_{ij}.$$

Using here a practice suggested by Lumley (cf. Lumley J.L. and Newmann G.R., [LUM 77]), the expression $\varepsilon_{ij} = \dfrac{2}{3} \varepsilon \delta_{ij}$ is formally retained whereas Φ_{ij} will

embody not only the approximation for the true pressure-strain correlation but also the anisotropy of ε_{ij}. This approach is thus different from the one considered in section 13.6.1. The analytical expressions of coefficients recommended by the authors are:

$$c_1 = 1 + 2.58 A.II^{1/4} \left\{ 1 - \exp\left[-(0.0067\,\mathrm{Re}_t)^2 \right] \right\},$$

$$c_2 = 0.75 A^{1/2}.$$

The compatibility with limiting relation [5.11] in isotropic turbulence imposes, however, a maximal value of 0.6 for the constant c_2. In this case we rather have to use:

$$c_2 = \min\left[0.75 A^{1/2}, 0.6 \right],$$

in the wall reflection term $\Phi_{ij}^{(s)}$ (equation [6.43]):

$$\Phi_{ij}^{(s)} = -c_{w1}\frac{\varepsilon}{k}\left(R_{km}n_k n_m \delta_{ij} - \frac{3}{2}R_{ki}n_k n_j - \frac{3}{2}R_{kj}n_k n_i \right) f\left(\frac{\ell}{n_i r_i} \right)$$

$$-c_{w2}\left(\Phi_{km}^{(2)}n_k n_m \delta_{ij} - \frac{3}{2}\Phi_{ki}^{(2)}n_k n_j - \frac{3}{2}\Phi_{kj}^{(2)}n_k n_i \right) f\left(\frac{\ell}{n_i r_i} \right),$$

$$c_{w1} = -\frac{2}{3}c_1 + 1.67, \quad c_{w2} = -\max\left[\left(\frac{2}{3}c_2 - \frac{1}{6} \right), 0 \right], \quad f\left(\frac{\ell}{n_i r_i} \right) = C\frac{k^{3/2}}{\varepsilon x_2},$$

n_i being the normal unit vector at the wall and r_i the position of the considered point and $C \simeq 0.4$. Moreover, the equation for ε is solved in the form:

$$\frac{d\varepsilon}{dt} = \left(C_{\varepsilon 1} + \Psi_1 + \Psi_2 \right)\frac{P\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon\tilde{\varepsilon}}{k} + \left[\left(C_\varepsilon \frac{kR_{pl}}{\varepsilon} + \nu\delta_{pl} \right)\varepsilon_{,p} \right]_{,l},$$

$$\Psi_1 = 2.5A\left(\frac{P}{\varepsilon} - 1 \right),$$

$$\Psi_2 = 0.3(1 - 0.3II)\exp\left[(-0.002\,\mathrm{Re}_t)^2 \right].$$

To some extent, this model still retains simplicity in its formulation while allowing some account of more advanced parameters in the coefficients. This type of model thus has to be used for prediction of complex flows ([CHA 02, CHA 06]).

13.6.4. *Model by Launder and Tselepidakis*

We also mention the model by Launder B.E. and Tselepidakis D.P., [LAU 91A and B] for which the constitutive hypotheses are summed up below:

$$\Phi_{ij}^{(1)} = -\tilde{c}_1 \varepsilon \left[a_{ij} + c'_1 \left(a_{im} a_{mj} - \frac{1}{3} II \delta_{ij} \right) \right],$$

$$\Phi_{ij}^{(2)} = -0.6 \left(P_{ij}^* - \frac{2}{3} P^* \delta_{ij} \right) + 0.6 \varepsilon a_{ij} P^* / \varepsilon$$

$$-0.2 \left[\frac{R_{mj} R_{li}}{k} (\Lambda_{ml} + \Lambda_{lm}) - \frac{R_{ml}}{k} (R_{im} \Lambda_{jl} + R_{jm} \Lambda_{il}) \right]$$

$$-\lambda \left[II(P_{ij}^* - D_{ij}^*) + 3 a_{mi} a_{nj} (P_{mn}^* - D_{mn}^*) \right],$$

[13.22]

$$\tilde{c}_1 = \begin{cases} 6.3 \, A.II^{1/2}(1 - f_1) & \text{if} \quad II \leq 0.6 \\ 6.3 \, (0.6)^{1/2}(1 - f_1) & \text{if} \quad II > 0.6 \end{cases},$$

$$c'_1 = 0.7, \quad f_1 = \max \left[(1 - \text{Re}_t / 140.), \, 0. \right],$$

$$\lambda = \min(0.6, A), \quad \Lambda_{lm} = \overline{U}_{l,m} + \alpha \ell \ell_{,p} \overline{U}_{l,mp},$$

$$\ell = \frac{k}{\varepsilon} (R_{pq} n_p n_q)^{1/2}, \quad \alpha = 0.30,$$

For the terms P_{ij}^*, D_{ij}^* and P, the usual velocity gradient $\overline{U}_{i,j}$ will be replaced by Λ_{ij} in the usual formulae.

$$\Phi_{ij}^{(1s)} = 0.$$

$$\Phi_{ij}^{(2s)} = c_{w2} \left(\Phi_{lm}^{(2)} n_l n_m \delta_{ij} - \frac{3}{2} \Phi_{ip}^{(2)} n_p n_j - \frac{3}{2} \Phi_{jp}^{(2)} n_p n_i \right) \frac{\ell}{x_n}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{(1)} f_s + \varepsilon_{ij}^{(2)} (1 - f_s)$$

$$\varepsilon_{ij}^{(1)} = \frac{\varepsilon}{k} \left(R_{ij} + R_{il} n_l n_j + R_{jl} n_l n_i + R_{pl} n_p n_l \delta_{ij} \right) \bigg/ \left(1 + \frac{5}{2} \frac{R_{pl} n_p n_l}{k} \right)$$

$$\varepsilon_{ij}^{(2)} = \left[(1 - f'_s) \frac{2}{3} \delta_{ij} + f'_s \frac{R_{ij}}{k} \right] \varepsilon$$

$$f_s = \exp(-20 A^2), \quad f'_s \approx 0$$

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1}\frac{P\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon\tilde{\varepsilon}}{k} + \left[\left(C_{\varepsilon}\frac{kR_{pl}}{\varepsilon} + \nu\delta_{pl}\right)\varepsilon_{,p}\right]_{,l} + 2C_{\varepsilon 3}\nu\frac{k}{\varepsilon}R_{pl}\overline{U}_{i,pj}\overline{U}_{i,lj}$$

$$C_{\varepsilon} = 0.18, \quad C_{\varepsilon 1} = 1.0, \quad C_{\varepsilon 2} = \frac{1.92}{1 + 0.63(A.II)^{1/2}}, \quad C_{\varepsilon 3} = 0.43$$

These modeling approaches rely on the concept given in section 8.4 which tends to eliminate the topographical parameters and which are extended here in the case of low Reynolds numbers. This makes it possible to reduce the explicit influence of the turbulence Reynolds number in their formulation, in particular in the expression for f_s.

The authors also introduce an additional diffusion term due to pressure and which enhance isotropy in the diffusion process inside the viscous sublayer:

$$\mathscr{D}_{ij}^{(p)} = \frac{1}{3}\nu\left(R_{lh,mm}n_l n_h \delta_{ij} - R_{il,mm}n_l n_j - R_{jl,mm}n_l n_j\right). \qquad [13.23]$$

Another formulation that does not involve the wall normal vector is given by Launder B.E. in Gatski T.B., Hussaini M.Y. and Lumley J.L., 1996 (Chapter 6):

$$\mathscr{D}_{ij}^{(p)} = -\frac{1}{6}\nu\left(R_{jk,ik} + R_{ik,jk}\right).$$

A pressure diffusion term can also be included in the ε equation in the form:

$$\mathscr{D}_{\varepsilon}^{(p)} = C_{\varepsilon 4}\nu\left(\frac{\varepsilon}{k}k_{,m}\right)_{,m}, \qquad C_{\varepsilon 4} = 0.92.$$

This term tends to increase ε in the viscous sublayer.

Let us also mention other authors such as Lai Y.G. and So R.M.C., [LAI 90] whose research work is in this same line of thought. Also, a useful comparative study of various formulations is given in So R.M.C., Lai Y.G., Zhang H.S. and Hwang B.C., [SO 91].

13.6.5. *Non-homogenity indicators used as wall detectors and elimination of topographical parameters*

The elimination of topographical parameters has been thoroughly achieved in the model by Craft T.J. and Launder B.E., [CRA 96B] and then by Craft T.J., [CRA

97A]. This model is dedicated to the calculation of turbulent flows in complex geometries using a low Reynolds number formulation.

The wall normal vector is no longer used, it is replaced by non-homogenity indicators, considered wall "detectors":

$$ d_i = \frac{N_i}{\alpha + \left(N_m N_m \right)^{1/2}} \ , \qquad N_i = \left(\frac{k^{3/2}}{\varepsilon} \right)_{,i} \ , \qquad [13.24] $$

where $\alpha = 0.5$ is a numerical constant which allows us to moderate the value of d_i in regions of weak non-homogeneity. In the ideal case of the boundary layer on a flat plate near the wall, if we assume $k \approx k_0 y^2$ with $k_0 \approx$ constant and $\varepsilon \approx \varepsilon_0 \approx$ constant we get:

$$ \frac{N_i}{\|N_i\|} = \begin{cases} N_x / \|N_x\| = 0 \\ N_y / \|N_y\| = 1 \\ N_z / \|N_z\| = 0 \end{cases} . $$

This shows that $N_i / \|N_i\|$ plays the role of an extended concept of wall normal vector.

Definition [13.24] however yields values of d_i that are too small in the buffer layer of the turbulent boundary layer. This failing is corrected in another definition ([CRA 96B]) which allows a better description of the wall boundary layer. It also involves the flatness factor A:

$$ d_i^A = \frac{N_i^A}{\alpha + \left(N_m^A N_m^A \right)^{1/2}} \ , \qquad N_i^A = \left(\frac{A k^{3/2}}{\varepsilon} \right)_{,i} \qquad [13.25] $$

The Reynolds stresses dissipation rate tensor is obtained from:

$$ \varepsilon_{ij} = (1 - f_h \frac{\varepsilon_{mm}^*}{2\varepsilon}) \left[(1 - f_\varepsilon)(\varepsilon_{ij}' + \varepsilon_{ij}'') / D + \frac{2}{3} f_\varepsilon \varepsilon \delta_{ij} \right] + f_h \varepsilon_{ij}^* $$

with $\varepsilon_{ij}' = \varepsilon \dfrac{R_{ij}}{k} + 2\nu \dfrac{R_{ln}}{k} \dfrac{\partial \sqrt{k}}{\partial x_l} \dfrac{\partial \sqrt{k}}{\partial x_n} \delta_{ij} + 2\nu \dfrac{R_{li}}{k} \dfrac{\partial \sqrt{k}}{\partial x_j} \dfrac{\partial \sqrt{k}}{\partial x_l} + 2\nu \dfrac{R_{lj}}{k} \dfrac{\partial \sqrt{k}}{\partial x_i} \dfrac{\partial \sqrt{k}}{\partial x_l}$

$$\varepsilon''_{ij} = \varepsilon \left[2\frac{R_{kl}}{k} d_l^A d_k^A \delta_{ij} - \frac{R_{il}}{k} d_l^A d_j^A - \frac{R_{jl}}{k} d_l^A d_i^A \right] f_R$$

$$\varepsilon^*_{ij} = 0.2\nu \left[\frac{\partial \sqrt{kA}}{\partial x_k} \frac{\partial \sqrt{kA}}{\partial x_k} \delta_{ij} + 2\frac{\partial \sqrt{kA}}{\partial x_i} \frac{\partial \sqrt{kA}}{\partial x_j} \right]$$

$$D = (\varepsilon'_{kk} + \varepsilon''_{kk})/2\varepsilon, \qquad f_R = (1-A)\min\left[(Re_t/80)^2, 1.0 \right],$$

$$f_\varepsilon = A^{1/2}, \qquad f_h = 1 - \exp(-Re_t/50)$$

Instead of modeling the pressure-strain correlations Φ_{ij} directly, the authors propose to model the quantity:

$$\Phi^*_{ij} = \Phi_{ij} - \mathcal{D}^{(p)}_{mm} \frac{R_{ij}}{2k} \qquad \text{where} \qquad \mathcal{D}^{(p)}_{ij} = -\frac{1}{\rho}\left[\left(\overline{pu_i}\right)_{,j} + \left(\overline{pu_j}\right)_{,i} \right]$$

the behavior of which varies less from one flow to another and thus it seems easier to model. The closure model is composed of the following equations:

$$\Phi^*_{ij} = \Phi^{*(1)}_{ij} + \Phi^{*(2)}_{ij} + \Phi^{inh(1)}_{ij} + \Phi^{inh(2)}_{ij}$$

$$\Phi^{*(1)}_{ij} = -C_1 \tilde{\varepsilon} \left[a_{ij} + C'_1\left(a_{ik}a_{kj} - \frac{1}{3} II\delta_{ij} \right) \right] - \tilde{\varepsilon} A a_{ij}$$

$$\Phi^{*(2)}_{ij} = -0.6\left(P_{ij} - \frac{2}{3}P\delta_{ij} \right) + 0.6\varepsilon a_{ij}\left(P/\varepsilon \right)$$

$$-0.2\left[\frac{R_{mj}R_{li}}{k}(\overline{U}_{m,l} + \overline{U}_{l,m}) - \frac{R_{ml}}{k}(R_{im}\overline{U}_{j,l} + R_{jm}\overline{U}_{i,l}) \right]$$

$$-c_2\left[II(P_{ij} - D_{ij}) + 3a_{mi}a_{nj}(P_{mn} - D_{mn}) \right]$$

$$-c'_2\left\{ \left(\frac{7}{15} - \frac{II}{4} \right)\left(P_{ij} - \frac{2}{3}P\delta_{ij} \right) + 0.2\left[a_{ij} - \frac{1}{2}\left(a_{im}a_{mj} - \frac{1}{3}II\delta_{ij} \right) \right]P \right.$$

$$-0.05a_{ij}a_{lm}P_{lm} + 0.1\left[\left(\frac{R_{im}}{k}P_{mj} + \frac{R_{jm}}{k}P_{mi} \right) - \frac{2}{3}\frac{R_{lm}}{k}P_{lm}\delta_{ij} \right]$$

$$\left. +0.1\left(\frac{R_{li}R_{kj}}{k^2} - \frac{1}{3}\frac{R_{lm}R_{km}}{k^2}\delta_{ij} \right)\left[6D_{kl} + 13k\left(\overline{U}_{l,k} + \overline{U}_{k,l}\right) \right] + 0.2\frac{R_{li}R_{kj}}{k^2}(D_{kl} - P_{kl}) \right\}$$

$$\Phi^{inh(1)}_{ij} = f_{w1} \frac{\varepsilon}{k}\left(R_{lk}d_l d_k \delta_{ij} - \frac{3}{2}R_{ik}d_j d_k - \frac{3}{2}R_{jk}d_i d_k \right)$$

$$+f'_{w1}\frac{k^2}{\varepsilon}\left(R_{kl}\frac{\partial\sqrt{A}}{\partial x_k}\frac{\partial\sqrt{A}}{\partial x_l}\delta_{ij}-\frac{3}{2}R_{ik}\frac{\partial\sqrt{A}}{\partial x_k}\frac{\partial\sqrt{A}}{\partial x_j}-\frac{3}{2}R_{jk}\frac{\partial\sqrt{A}}{\partial x_k}\frac{\partial\sqrt{A}}{\partial x_i}\right),$$

$$\Phi_{ij}^{inh(2)}=f_{w2}k\overline{U}_{l,n}d_ld_n\left(d_id_j-\frac{1}{3}d_kd_k\delta_{ij}\right),$$

$$\tilde{\varepsilon}=\varepsilon-2v\left(\frac{\partial\sqrt{k}}{\partial x_j}\right)^2-\frac{1}{2}f_h\varepsilon_{mm}^*,$$

$$c_1=3.1f'_A\,f_{Rt}\,\min\left[II^{1/2},0.5\right],\qquad\qquad f_{Rt}=\min\left[Re_t/160.1\right],$$

$$c_2=\min\left[A^{1/2},0.55\right],\qquad\qquad\qquad c'_2=\min\left[A^{3/2},0.6\right],$$

$$c'_1=1.1,\qquad f_{w1}=1.9f'_{Rt},\qquad f'_{w1}=0.22,\qquad f_{w2}=5f_A,$$

$$f'_{Rt}=\min\left\{1,\max\left[0,1-(Re_t-55)/20\right]\right\}$$

$$f_A=\begin{cases}(A/14)^{1/2} & A<0.05\\A/0.7^{1/2} & 0.05\le A\le0.7\\A^{1/2} & A>0.7\end{cases}$$

The pressure diffusion term is obtained from:

$$\overline{pu_k}/\rho=-\left(0.5d_k+1.1d_k^A\right)(v\varepsilon kA.II)^{1/2}\left[c_{pd1}.II+c_{pd2}\,Re_t^{-1/4}\exp(-Re_t/40)\right],$$

$$c_{pd1}=1+2.\exp(-Re_t/40),\qquad\qquad c_{pd2}=0.4$$

The dissipation rate equation is modeled in the form:

$$\frac{d\tilde{\varepsilon}}{dt}=C_{\varepsilon1}\frac{\tilde{\varepsilon}P}{k}-C_{\varepsilon2}\frac{\tilde{\varepsilon}^2}{k}-C'_{\varepsilon2}\frac{(\varepsilon-\tilde{\varepsilon})\tilde{\varepsilon}}{k}+\left[\left(v\delta_{ij}+C_\varepsilon\frac{kR_{ij}}{\varepsilon}\right)\tilde{\varepsilon}_{,i}\right]_{,j}$$

$$+C_{\varepsilon3}v\frac{kR_{ij}}{\varepsilon}\overline{U}_{k,il}\overline{U}_{k,jl}+C_{\varepsilon4}v\frac{R_{ij}}{\varepsilon}k_{,i}\overline{U}_{l,k}\overline{U}_{l,kj}$$

$$+C_{\varepsilon5}A^{1/2}(1-A)\frac{\varepsilon}{\sqrt{k}}R_{ij}A_{,i}\left(\frac{k^{3/2}A^{1/2}}{\varepsilon}\right)_{,j}$$

with:

$$C_{\varepsilon 1} = 1.0, \qquad C_{\varepsilon 2} = \frac{1.92}{1+0.7 A_d \sqrt{II}}, \qquad A_d = \max[0.2, A], \quad C'_{\varepsilon 2} = 1.0,$$

$$C_{\varepsilon 3} = 1.55, \qquad C_{\varepsilon 4} = 1.0, \qquad\qquad C_{\varepsilon 5} = 1.0, \qquad\qquad C_\varepsilon = 0.09.$$

Several variants of this model have been proposed (cf. [CRA 95A, CRA 97A, CRA 98]). In these versions, the indicator vector for non-homogenity involves $A^{1/2}$ instead of A in the definition of N_i^A :

$$d_i = \frac{N_i}{\alpha + \left(N_m N_m\right)^{1/2}}, \qquad N_i = \left(\frac{k^{3/2}}{\varepsilon}\right)_{,i}$$

and $\quad d_i^A = \dfrac{N_i^A}{\alpha + \left(N_m^A N_m^A\right)^{1/2}} \quad , \quad N_i^A = \left(\dfrac{A^{1/2} k^{3/2}}{\varepsilon}\right)_{,i} \quad , \quad \alpha = 0.5$

In the version given by Craft T.J., [CRA 98], the Reynolds stresses dissipation rate tensor is obtained from:

$$\varepsilon_{ij} = (1 - f_\varepsilon)(\varepsilon'_{ij} + \varepsilon''_{ij} + \varepsilon'''_{ij}) / D + \frac{2}{3} f_\varepsilon \varepsilon \delta_{ij},$$

with $\quad \varepsilon'_{ij} = \varepsilon \dfrac{R_{ij}}{k} + 2\nu \dfrac{R_{ln}}{k} \dfrac{\partial \sqrt{k}}{\partial x_l} \dfrac{\partial \sqrt{k}}{\partial x_n} \delta_{ij} + 2\nu \dfrac{R_{li}}{k} \dfrac{\partial \sqrt{k}}{\partial x_j} \dfrac{\partial \sqrt{k}}{\partial x_l} + 2\nu \dfrac{R_{lj}}{k} \dfrac{\partial \sqrt{k}}{\partial x_i} \dfrac{\partial \sqrt{k}}{\partial x_l}$,

$$\varepsilon''_{ij} = \varepsilon \left[2 \frac{R_{kl}}{k} d_l^A d_k^A \delta_{ij} - \frac{R_{il}}{k} d_l^A d_j^A - \frac{R_{jl}}{k} d_l^A d_i^A \right],$$

$$\varepsilon'''_{ij} = c_{\varepsilon s} \nu k \left[\frac{\partial \sqrt{kA}}{\partial x_k} \frac{\partial \sqrt{kA}}{\partial x_k} \delta_{ij} + 2 \frac{\partial \sqrt{kA}}{\partial x_i} \frac{\partial \sqrt{kA}}{\partial x_j} \right],$$

$$D = (\varepsilon'_{kk} + \varepsilon''_{kk} + \varepsilon'''_{kk}) / 2\varepsilon, \quad c_{\varepsilon s} = 0.2, \quad f_\varepsilon = A^{1/2}.$$

This formulation allows us to get $\dfrac{\varepsilon_{11}}{R_{11}} = \dfrac{\varepsilon_{33}}{R_{33}} = \dfrac{1}{4} \dfrac{\varepsilon_{22}}{R_{22}} = \dfrac{1}{2} \dfrac{\varepsilon_{12}}{R_{12}} = \dfrac{\varepsilon}{k}$ at the wall.

Modeling also deals with the quantity $\Phi_{ij}^* = \Phi_{ij} - \mathscr{D}_{mm}^{(p)} \dfrac{R_{ij}}{2k}$ and the pressure diffusion term is obtained as previously.

The slow homogenous term is:

$$\Phi_{ij}^{*(1)} = -c_1 \tilde{\varepsilon}\left[a_{ij} + c'_1\left(a_{ik}a_{kj} - \frac{1}{3}II\delta_{ij} \right) \right] - \tilde{\varepsilon}f'_A\, a_{ij} ,$$

whereas the expression for $\Phi_{ij}^{*(2)}$ is not modified.

The non-homogenous terms are written:

$$\Phi_{ij}^{inh(1)} = f_{w1}\frac{\varepsilon}{k}\left(R_{lk}d_l{}^A d_k{}^A \delta_{ij} - \frac{3}{2}R_{ik}d_j{}^A d_k{}^A - \frac{3}{2}R_{jk}d_i{}^A d_k{}^A \right)$$

$$+ f_{w2}\frac{\varepsilon}{k^2}R_{ln}\left(R_{nk}d_k{}^A \delta_{ij} - \frac{3}{2}R_{in}d_j{}^A - \frac{3}{2}R_{jn}d_i{}^A \right)d_l{}^A$$

$$+ f_{w3}\nu a_{il}\left(\frac{\partial\sqrt{k}}{\partial x_j}\frac{\partial\sqrt{k}}{\partial x_l} + a_{jl}\frac{\partial\sqrt{k}}{\partial x_i}\frac{\partial\sqrt{k}}{\partial x_l} - \frac{2}{3}a_{nl}\frac{\partial\sqrt{k}}{\partial x_l}\frac{\partial\sqrt{k}}{\partial x_n}\delta_{ij} \right.$$

$$\left. -\frac{4}{3}a_{ij}\frac{\partial\sqrt{k}}{\partial x_l}\frac{\partial\sqrt{k}}{\partial x_l} \right)$$

$$+ f'_{w1}\frac{k^2}{\varepsilon}\left(R_{kl}\frac{\partial\sqrt{A}}{\partial x_k}\frac{\partial\sqrt{A}}{\partial x_l}\delta_{ij} - \frac{3}{2}R_{ik}\frac{\partial\sqrt{A}}{\partial x_k}\frac{\partial\sqrt{A}}{\partial x_j} - \frac{3}{2}R_{jk}\frac{\partial\sqrt{A}}{\partial x_k}\frac{\partial\sqrt{A}}{\partial x_i} \right),$$

$$\Phi_{ij}^{inh(2)} = f_l k\,\overline{U}_{l,n}d_l d_n\left(d_i d_j - \frac{1}{3}d_k d_k \delta_{ij} \right),\quad \tilde{\varepsilon} = \varepsilon - 2\nu\left(\frac{\partial\sqrt{k}}{\partial x_j} \right)^2 ,$$

with the coefficients:

$$c_1 = 3.1 f_A f_{Rt} II^{1/2},\quad c'_1 = 1.1,\quad f_{Rt} = \min\left[\text{Re}_t/160, 1 \right],$$

$$c_2 = \min\left[0.55\left[1 - \exp\left\{ -A^{3/2}\,\text{Re}_t/100 \right\} \right], 3.2A/(1+S) \right],$$

$$c'_2 = \min\left[A, 0.6 \right] + 3.5(S - \Omega)/(3 + S + \Omega) - 2S_l ,$$

$$S = \frac{k}{\varepsilon}\left(\frac{S_{ij}S_{ij}}{2}\right)^{1/2}, \quad \Omega = \frac{k}{\varepsilon}\left(\frac{\omega_{ij}\omega_{ij}}{2}\right)^{1/2}, \quad S_l = \frac{S_{ij}S_{jk}S_{ki}}{(0.5S_{ij}S_{ij})^{3/2}},$$

$$f_{w1} = 0.4 + 1.6f'_{Rt}, \quad f'_{Rt} = \min\{1, \max[0, 1-(\mathrm{Re}_t - 55)/20]\}$$

$$f'_{w1} = 0.22, \quad f_{w2} = 0.1 + 0.8IIf''_{Rt}, \quad f_{w3} = 2.5A^{1/2}, \quad f_l = 2.5f_A,$$

$$f''_{Rt} = \min\{1, \max[0, 1-(\mathrm{Re}_t - 50)/85]\},$$

f_A is given by the same expression as previously whereas $f'_A = A^{1/2}f_{Rt} + A(1 - f_{Rt})$.

The effective velocity gradient is also used (cf. section 8.3.1):

$$\overline{U}_{k,h}^{\text{effective}} = \overline{U}_{k,h} + c_I \ell.d_m^A \overline{U}_{k,hm},$$

$$\ell = \min\left[k^{3/2}/\varepsilon, 40(\nu^3/\varepsilon)^{1/4}\right], \quad c_I = 0.7[A(1-A)]^{1/2}$$

The dissipation rate equation is modeled in the form:

$$\frac{d\tilde{\varepsilon}}{dt} = C_{\varepsilon 1}\frac{\tilde{\varepsilon}P}{k} - C_{\varepsilon 2}\frac{\tilde{\varepsilon}^2}{k} - C'_{\varepsilon 2}\frac{(\varepsilon - \tilde{\varepsilon})\tilde{\varepsilon}}{k} + \left[\left(\nu\delta_{ij} + C_\varepsilon\frac{kR_{ij}}{\varepsilon}\right)\tilde{\varepsilon}_{,i}\right]_{,j}$$

$$+ C_{\varepsilon 3}\nu\frac{kR_{ij}}{\varepsilon}\overline{U}_{k,il}\overline{U}_{k,jl} + Y_E,$$

which singles out the correction introduced by Iacovides H. and Raise M. [IAC 97] and which recalls the Yap correction term (cf. section 7.5):

$$Y_E = C_{\varepsilon l}\frac{\tilde{\varepsilon}^2}{k}\max\left[F(F+1)^2, 0\right],$$

$$F = \ell_{,j}\ell_{,j} - c_l\left[1 - \exp(-B_\varepsilon \mathrm{Re}_t)\right] + B_\varepsilon c_l \mathrm{Re}_t \exp(-B_\varepsilon \mathrm{Re}_t),$$

$$c_l = 2.55, \quad B_\varepsilon = 0.1069,$$

with: $C_{\varepsilon 1} = 1.0$, $\quad C_{\varepsilon 2} = \dfrac{1.92}{1 + 0.7A_d\sqrt{II}}$, $\quad A_d = \max[0.25, A]$, $\quad C'_{\varepsilon 2} = 1.0$,

$C_{\varepsilon 3} = 0.875$, $\quad C_{\varepsilon l} = 0.5$, $\quad C_\varepsilon = 0.15$.

Also, in the non-linear model by Craft T.J. *et al.* [CRA 97B], the wall effect terms involve a modified length scale that relates it to the same type of approach:

$$\ell_f = \frac{1-\exp\left(-\text{Re}_t/30\right)}{1+3.5\varPi^2}\ell \,, \qquad \ell = k^{3/2}/\varepsilon \,.$$

The wall redistribution term being modeled through:

$$\begin{aligned}
\varPhi_{ij}^w = &-c_{2w}\overline{U}_{l,m}R_{lm}\left(\ell_{f,q}\ell_{f,q}\delta_{ij} - 3\ell_{f,i}\ell_{f,j}\right) \\
&-c'_{2w}\left(a_{lm}\overline{U}_{k,m}\ell_{f,l}\ell_{f,k}\delta_{ij} - \frac{3}{2}a_{lm}\overline{U}_{i,m}\ell_{f,l}\ell_{f,j}\right. \\
&\left. -\frac{3}{2}a_{lm}\overline{U}_{j,m}\ell_{f,l}\ell_{f,i}\right)k \\
&+c''_{2w}\,k\overline{U}_{l,m}\ell_{f,l}\ell_{f,m}\left(\ell_{f,i}\ell_{f,j} - \ell_{f,q}\ell_{f,q}\delta_{ij}\right).
\end{aligned}$$

The non-homogenity indicator approach has been generalized by Gerolymos G. A. *et al.* [GER 04] by distinguishing the two different concepts that are the generalized normal vector at the wall and the generalized distance from the wall which were intimately connected in the models proposed until then. For this, a unit vector in the non-homogenity direction is first defined by:

$$e_j = e_j(\mathscr{T}, \mathscr{T}_{,j}),$$

where $\mathscr{T}(\varPi, A, \text{Re}_t, \ell, f)$ with $\ell = k^{3/2}/\varepsilon$, $f = \varepsilon/k$, $\text{Re}_t = k^2/\nu\varepsilon$ is a function which describes the local state of turbulence. The empirical model proposed by Gerolymos G.A. and Vallet I. ([GER 01, GER 04]) makes use of the following expression for the unit vector:

$$e_j = \frac{\tilde{\ell}_{,j}}{\left\|\tilde{\ell}_{,j}\right\|}, \qquad \tilde{\ell} = \frac{\ell\left(1-e^{-\text{Re}_t/30}\right)}{1+2\sqrt{\varPi}+2A^{16}}.$$

In addition, the effects of distance are included in the coefficients of the slow and the fast wall effect terms in the pressure-strain correlation. In the expression of Launder B.E. and Shima N. (cf. section 13.6.3), coefficients $c_{w1}f$ and $c_{w2}f$ are replaced by expressions of the type:

$$c_{1/2}^{I} = F_{1/2}^{a}(A, II, \mathrm{Re}_t) \left\| \left\{ \frac{\ell F_{1/2}^{b}(\mathrm{Re}_t)}{1 + F_{1/2}^{c}(A, II, \mathrm{Re}_t) II^{F_{1/2}^{d}(A, II, \mathrm{Re}_t)}} \right\}_{,j} \right\|,$$

in which the subscript 1/2 either stands for the indices 1 or 2 corresponding to each coefficient. More precisely:

$$F_1^a = 0.78 \left[1 - \frac{2}{3}(c_1 - 1) \right], \quad F_1^b = \ell \left[1 - e^{-\mathrm{Re}_t/30} \right], \quad F_1^c = 2, F_1^d = 0.8,$$

$$F_2^a = \max \left[\frac{2}{3} - \frac{1}{6c_2}, 0 \right], F_2^b = \ell \left[1 - e^{-\mathrm{Re}_t/30} \right], F_2^c = 1.8, F_2^d = \max[0.6, A],$$

c_1 and c_2 being the constants present in the homogenous part of the pressure-strain correlations,

$$\Phi_{ij}^{(s)} = -c_1^I \frac{\varepsilon}{k} \left(R_{km} e_k e_m \delta_{ij} - \frac{3}{2} R_{ki} e_k e_j - \frac{3}{2} R_{kj} e_k e_i \right)$$

$$- c_2^I \left(\Phi_{km}^{(2)} e_k e_m \delta_{ij} - \frac{3}{2} \Phi_{ki}^{(2)} e_k e_j - \frac{3}{2} \Phi_{kj}^{(2)} e_k e_i \right).$$

The coefficients have been calibrated in order to predict correctly the structure of the turbulent field in a plane channel flow.

This approach would thus make it possible in principle to adapt any usual model with topographical parameters and then convert it without ambiguity into a model which is independent of topographical parameters.

The approach aims at applications including mainly turbulent flows in complex geometries and the model has been applied in particular to three-dimensional flows and confined flows submitted to rotation ([GER 01, GER 02]), flows in ducts with square cross-section and supersonic flows ([GER 04]). The same authors have also outlined a beginning of theory for non-homogenity which could be based on a tensorial representation including the non-homogenity indicator vector. The tensor basis (cf. section 3.6.5) would be built on the anisotropy a_{ij}, the strain rate tensor S_{ij}, the intrinsic rotation tensor $Z_{ij} = \omega_{ij} + \varepsilon_{ikj}\Omega_k$ and the non-homogenity indicator vector e_j. Considering the normalized expressions $S^*_{ij} = \frac{k}{\varepsilon}\left(S_{ij} - \frac{1}{3} S_{mm}\delta_{ij} \right)$

and $Z^*_{ij} = \dfrac{k}{\varepsilon} Z_{ij}$ together with the hypothesis of linearity with respect to S^*_{ij} and

Z^*_{ij} we arrive at ([GER 04], [SMI 71], [PEN 87]):

$$T^{(0)}_{ij} = a_{ij}, \qquad\qquad T^{(1)}_{ij} = a_{im}a_{mj} - \frac{1}{3}a_{mn}a_{mn}\delta_{ij},$$

$$T^{(2)}_{ij} = S_{ij}, \qquad\qquad T^{(3)}_{ij} = a_{im}S_{mj} + a_{jm}S_{mi} - \frac{2}{3}a_{mn}S_{mn}\delta_{ij},$$

$$T^{(4)}_{ij} = a_{im}{}^2 S_{mj} + a_{jm}{}^2 S_{mi} - \frac{2}{3}a_{mn}{}^2 S_{mn}\delta_{ij}, \qquad T^{(5)}_{ij} = a_{im}\omega_{mj} + a_{jm}\omega_{mi},$$

$$T^{(6)}_{ij} = a_{im}{}^2 \omega_{mj} + a_{jm}{}^2 \omega_{mi}, \qquad\qquad T^{(7)}_{ij} = e_i e_j - \frac{1}{3}\delta_{ij},$$

$$T^{(8)}_{ij} = a_{im}e_m e_j + e_i a_{jm}e_m - \frac{2}{3}e_i e_j a_{ij},$$

$$T^{(9)}_{ij} = a_{im}a_{mp}e_p e_j + e_i a_{jm}a_{mp}e_p - \frac{2}{3}e_i e_j a_{im}a_{mj},$$

$$T^{(10)}_{ij} = S^*_{im}e_m e_j + e_i S^*_{jm}e_m - \frac{2}{3}e_i e_j S^*_{ij}, \quad T^{(11)}_{ij} = Z^*_{im}e_m e_j + e_i Z^*_{jm}e_m.$$

This tensorial basis gives rise to generalized expressions for the approximation of the pressure-strain correlations in non-homogenous flows ([GER 04]).

Using non-homogenity indicators in place of topological parameters leads us to adopt a somehow different point of view in modeling, the point of view to account for the effects of non-homogenity rather than the presence of the walls. The resulting approach is similar to the expansions that would be obtained from the Lumley J.L. method (cf. section 4.4) in which the non-homogenity parameters would be retained up to a given order (in the pressure-strain correlations in particular).

13.6.6. *Model by Jakirlic and Hanjalic*

The model by Jakirlic S. and Hanjalic K., [JAK 95] applies to complex flows with walls and subjected to rapid spatial and temporal variations. The asymptotic form of the model at high Reynolds numbers remains relatively traditional, the new contributions deal with the modeling of the Reynolds number effects and the wall effects allowing the flow calculation down to the wall, and likely deal with the laminar/turbulent transition regions. These effects are taken into account through dimensionless parameters (non-topographical) such as the invariants of the anisotropy tensor II^a and III^a and also the invariants of anisotropy of the

dissipation tensor II^ε and III^ε. Indeed, the DNS data show that the anisotropy of the stresses and the anisotropy of the dissipation tensor are very different and a separate account of each of them is retained. An additional term, recalling the Yap C. modification (cf. section 7.5) is introduced to prevent the excessive increase of the turbulence length scale in separated and reattached flows. The equations ([JAK 95], [HAN 97]) are summed up below:

Reynolds stress transport equations:

$$\frac{dR_{ij}}{dt} = P_{ij} + \Phi_{ij} - \varepsilon_{ij} + D_{ijk,k}, \qquad D_{ijk} = T_{ijk} + \nu R_{ij,k},$$

Triple velocity correlation terms: $T_{ijk} = (\nu\delta_{kl} + C_s \frac{kR_{kl}}{\varepsilon})R_{ij,l}$ with $C_s = 0.22$.

Diffusion of ε: $D_k(\varepsilon) = (\nu\delta_{kl} + C_\varepsilon \frac{kR_{kl}}{\varepsilon})\varepsilon_{,l}$ with $C_\varepsilon = 0.18$.

Pressure-strain correlations:

$$\Phi_{ij} = \Phi^{(1)}{}_{ij} + \Phi^{(2)}{}_{ij} + \Phi^{(w)}{}_{ij}$$

$$\Phi^{(1)}{}_{ij} = -C_1\varepsilon a_{ij}, \qquad \Phi^{(2)}{}_{ij} = -C_2\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right),$$

$$\Phi^{(w)}{}_{ij} = \left(\widetilde{\Phi}_{kl}n_k n_l \delta_{ij} - \frac{3}{2}\widetilde{\Phi}_{ik}n_j n_k - \frac{3}{2}\widetilde{\Phi}_{jk}n_k n_i\right)f,$$

$$\widetilde{\Phi}_{ij} = C^{(w)}{}_1 \frac{\varepsilon}{k}R_{ij} + C^{(w)}{}_2\Phi^{(2)}{}_{ij}, \qquad f = \min\left[\frac{k^{3/2}/\varepsilon}{C_\ell y}, 1.4\right],$$

$$C_1 = C + A^{1/2}E^2, \qquad C_2 = 0.8A^{1/2}, \qquad C^{(w)}{}_1 = \max[1 - 0.7C, 0.3],$$

$$C^{(w)}{}_2 = \min[A, 0.3], \quad C_\ell = 2.5, \quad C = 2.5Af_{II}^{1/4}f_{Rt},$$

$$f_{II} = \min[II, 0.6], f_{Rt} = \min\left[\left(\frac{\text{Re}_t}{150}\right)^{3/2}, 1\right],$$

$$A = 1 - \frac{9}{8}(II^a - III^a), \qquad E = 1 - \frac{9}{8}(II^e - III^e), \qquad e_{ij} = \frac{\varepsilon_{ij} - \frac{2}{3}\varepsilon\delta_{ij}}{\varepsilon},$$

$$\varepsilon_{ij} = \frac{2}{3} f_\varepsilon \varepsilon \, \delta_{ij} + (1 - f_\varepsilon) \varepsilon^*_{\ ij} \, ,$$

$$\varepsilon^*_{\ ij} = \frac{\varepsilon}{k} \frac{R_{ij} + \left(R_{ik} n_j n_k + R_{jk} n_i n_k + R_{kl} n_k n_l n_i n_j \right) f_d}{1 + \frac{3}{2} \frac{R_{kl} n_k n_l}{k} f_d} \, ,$$

$$f_\varepsilon = A^{1/2} E^2 \, , \quad f_d = \frac{1}{1 + 0.1 \, \mathrm{Re}_t} \, .$$

Equation for the dissipation rate:

$$\frac{d\varepsilon}{dt} = \left(C_{\varepsilon 1} f_1 P - C_{\varepsilon 2} f_2 \varepsilon \right) \frac{\varepsilon}{k} + S_l + S_\Omega + S_\varepsilon + D_{j,j}(\varepsilon) \, ,$$

$$C_{\varepsilon 1} = 1.44 \, , \ C_{\varepsilon 2} = 1.92 \, , \ f_1 = 1 \, , \ f_2 = \left[1 - \left(\frac{C_{\varepsilon 2} - 1.4}{C_{\varepsilon 2}} \right) e^{-\left(\mathrm{Re}_t / 6 \right)^2} \right] \frac{\tilde{\varepsilon}}{\varepsilon} \, ,$$

$$S_\varepsilon = 0.25 \nu \frac{k}{\varepsilon} R_{jk} \overline{U}_{i,jl} \overline{U}_{i,kl} \, , \quad \tilde{\varepsilon} = \max \left[\varepsilon - 2\nu \left(\sqrt{k} \right)^2_{,j}, 0 \right] , \ S_l = 0 \, ,$$

$$S_l = \max \left\{ \left[\left(\frac{1}{C_\ell} \ell_{,n} \right)^2 - 1 \right] \left(\frac{1}{C_\ell} \ell_{,n} \right)^2 , 0 \right\} \frac{\varepsilon \tilde{\varepsilon}}{k} A \, , \quad S_\Omega = C_{\varepsilon \Omega} f_\Omega k \omega_{ij} \omega_{ij} \, .$$

The factor f_Ω has the same expression as f_μ in the low Reynolds number k-ε model. For $C_{\varepsilon \Omega}$, the authors suggest two possibilities:

– either $C_{\varepsilon 1} = 2.6$ and $C_{\varepsilon \Omega} = 0.1$;

– or a simplified form for near equilibrium flows close to a wall, $S_\Omega = C^*_{\varepsilon \Omega} (R_{22} - R_{11}) \overline{U}_{1,1} \frac{\varepsilon}{k}$, with in this case $C^*_{\varepsilon \Omega} = 1.16$ whereas $C_{\varepsilon 1}$ remains unchanged ($C_{\varepsilon 1} = 1.44$).

This model proved to be efficient for a large variety of turbulent flows ([HAN 97]) including boundary layers with pressure gradient, transition, pulsating flows, the three-dimensional boundary layer, the flow over a step or an obstacle.

A new version of the model was proposed afterwards by Jakirlic S. and Hanjalic K. [JAK 02], which makes use of an equation for the homogenous dissipation $\varepsilon^h = \varepsilon - \frac{1}{2} \nu k_{,mm}$.

Relying on the two point correlation equations, the authors are led to: $\varepsilon_{ij} = \varepsilon^h{}_{ij} + \frac{1}{2}\nu R_{ij,mm}$. The equations of the revised model are thus:

Reynolds stress transport equations:

$$\frac{dR_{ij}}{dt} = P_{ij} + \Phi_{ij} - \varepsilon^h{}_{ij} + \left[\left(\frac{1}{2}\nu\delta_{kl} + C_s\frac{k}{\varepsilon^h}R_{kl}\right)R_{ij,l}\right]_{,k},$$

with $C_s = 0.22$, $C_\varepsilon = 0.18$.

Pressure-strain correlations:

$$\Phi_{ij} = \Phi^{(1)}{}_{ij} + \Phi^{(w1)}{}_{ij} + \Phi^{(2)}{}_{ij} + \Phi^{(w2)}{}_{ij}$$

$$\Phi^{(1)}{}_{ij} = -C_1\varepsilon^h a_{ij}, \quad \Phi^{(2)}{}_{ij} = -C_2\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right),$$

$$\Phi^{(w1)}{}_{ij} = C_{w1}\frac{\varepsilon^h}{k}\left(R_{kl}n_k n_l\delta_{ij} - \frac{3}{2}R_{ik}n_j n_k - \frac{3}{2}R_{jk}n_k n_i\right)f_w,$$

$$\Phi^{(w2)}{}_{ij} = C_{w2}\left(\Phi^{(2)}{}_{kl}n_k n_l\delta_{ij} - \frac{3}{2}\Phi^{(2)}{}_{ik}n_j n_k - \frac{3}{2}\Phi^{(2)}{}_{jk}n_k n_i\right)f_w,$$

$$f = \min\left[\frac{k^{3/2}/\varepsilon^h}{C_\ell y}, 1.4\right],$$

$$C_1 = C + A^{1/2}E^2, \quad C_2 = 0.8A^{1/2}, \quad C^{(w)}{}_1 = \max[1 - 0.7C, 0.3],$$

$$C^{(w)}{}_2 = \min[A, 0.3], C_\ell = 2.5, \quad C = 2.5Af_{II}^{1/4}f_{Rt},$$

$$f_{II} = \min[II, 0.6], f_{Rt} = \min\left[\left(\frac{Re_t}{150}\right)^{3/2}, 1\right],$$

$$A = 1 - \frac{9}{8}(II^a - III^a), \quad E = 1 - \frac{9}{8}(II^e - III^e), \quad e_{ij} = \frac{\varepsilon_{ij} - \frac{2}{3}\varepsilon\delta_{ij}}{\varepsilon},$$

$$\varepsilon^h{}_{ij} = \frac{2}{3}(1 - f_{\varepsilon h})\varepsilon^h.\delta_{ij} + f_{\varepsilon h}\frac{R_{ij}}{k}\varepsilon^h, \quad f_{\varepsilon h} = 1 - A^{1/2}E^2 .$$

Dissipation rate equation:

$$\frac{d\varepsilon^h}{dt} = -C_{\varepsilon 1} R_{ij} \overline{U}_{i,j} \frac{\varepsilon^h}{k} - 2\nu \left(R_{ik,l} \overline{U}_{i,kl} + C_{\varepsilon 3} \frac{k}{\varepsilon^h} R_{kl,j} \overline{U}_{i,k} \overline{U}_{i,jl} \right)$$

$$- C_{\varepsilon 2} f_\varepsilon \frac{\varepsilon^h \tilde{\varepsilon}^h}{k} + \left[\left(\frac{1}{2} \nu \delta_{kl} + C_\varepsilon \frac{\varepsilon^h}{k} R_{kl} \right) \varepsilon^h{}_{,l} \right]_{,k} ,$$

$$C_{\varepsilon 1} = 1.44 , \quad C_{\varepsilon 2} = 1.8 , \quad C_{\varepsilon 3} = 0.32 , f_\varepsilon = \left[1 - \left(\frac{C_{\varepsilon 2} - 1.4}{C_{\varepsilon 2}} \right) e^{-\left(Re_t/6 \right)^2} \right],$$

$$\tilde{\varepsilon}^h = \varepsilon^h - 2\nu \left(\sqrt{k} \right)^2_{,j} .$$

13.6.7. *First order non-linear models*

Along the same lines, the non-linear k-ε models can be extended to the case of low Reynolds numbers (cf. Craft T.J. *et al.*, [CRA 97B]). For example, the model by Craft T.J., Launder B.E. and Suga K., [CRA 96C] presented in section 11.4.1 (equation [11.24]) based on a cubic expansion is in fact a low Reynolds numbers model. The model by Craft *et al.* [CRA 97B] uses an approach similar to the one developed in [CRA 95B] and based on a non-linear scheme (k-ε -*II*) with an auxiliary equation for the anisotropy invariant *II* (section 11.4.2). This version of the model was specially developed for studying turbulent flows far from equilibrium and in particular transitional flows.

The non-linear k-ε model for low Reynolds numbers by Apsley D.D., Leschziner M.A. [APS 98] is obtained from successive approximations of an algebraic model, and involves cubic terms. The formulation is extended to non-equilibrium states by introducing the parameter P/ε in the model coefficients. The model can thus be applied to complex strains.

13.6.8. *Models for scalar transport*

The heat and mass transfer can be treated using similar methods of approach. The extension of model for the scalar variance and its dissipation rate to the case of low intensity turbulence can be developed by analogy with the models for the dynamics (cf. Kenjeres S. and Hanjalic K., [KEN 95]):

$$\frac{dq}{dt} = P_\gamma + D_\gamma - \varepsilon_\gamma ,$$

$$\frac{dk}{dt} = P + G + D_k - \varepsilon \,,$$

$$\frac{d\varepsilon}{dt} = D_\varepsilon + C_{\varepsilon 1}(P+G)\frac{\tilde{\varepsilon}}{k} - C_{\varepsilon 2}f_\varepsilon \frac{\tilde{\varepsilon}^2}{k} + E + S \,,$$

$$\frac{d\varepsilon_\gamma}{dt} = D_{\varepsilon\gamma} + C_{\varepsilon\gamma 1}\frac{P\tilde{\varepsilon}_\gamma}{k} - C_{\varepsilon\gamma 2}\frac{\tilde{\varepsilon}_\gamma^2}{q} + C_{\varepsilon\gamma 3}\frac{P_\gamma\tilde{\varepsilon}_\gamma}{q} - C_{\varepsilon\gamma 4}f_{\varepsilon\gamma}\frac{\tilde{\varepsilon}\tilde{\varepsilon}_\gamma}{k} + E_\gamma \,,$$

$$P = -R_{ij}\overline{U}_{i,j} \,, \quad P_\gamma = -F_{\gamma j}\overline{\Gamma}_{,j} \,, \qquad G = -\beta g_i F_{\gamma i} \text{ (see Chapter 15),}$$

$$E = 2\nu\nu_t\left(\overline{U}_{i,jk}\right)^2 \,, \qquad\qquad E_\gamma = 2\sigma\sigma_t\left(\overline{\Gamma}_{,jk}\right)^2 \,,$$

$$\tilde{\varepsilon} = \varepsilon - 2\nu\left(\sqrt{k}\right)_{,j}^2 \,, \qquad\qquad \tilde{\varepsilon}_\gamma = \varepsilon_\gamma - 2\sigma\left(\sqrt{q}\right)_{,j}^2 \,,$$

$$D_\phi = (c_\phi f_\mu \frac{k^2}{\varepsilon}\phi_{,j})_{,j} + \frac{\nu}{\text{Pr}_\phi}\phi_{,jj} \,.$$

$$S_\ell = 0.85\left(\frac{\ell_e}{\ell}-1\right)\left(\frac{\ell_e}{\ell}\right)^2\frac{\tilde{\varepsilon}^2}{k} \,, \qquad \ell_e = k^{3/2}/\varepsilon \,, \quad \ell = 2.5y \,,$$

$$f_\mu = \exp\left[-3.4/\left(1+\text{Re}_t/50\right)^2\right] \,, \qquad f_\varepsilon = 1 - 0.3\exp(-\text{Re}_t^2)$$

$$F_{\gamma i} = -0.2\frac{k}{\varepsilon}\left(R_{ij}\overline{\Gamma}_{,j} + 0.6F_{\gamma j}\overline{U}_{i,j} + 1.2\beta qg_i\right)$$

$$C_{\varepsilon\gamma} = 0.07 \,, \qquad C_{\varepsilon\gamma 1} = 0.72 \,, \qquad C_{\varepsilon\gamma 2} = 2.2 \,, \qquad C_{\varepsilon\gamma 3} = 1.3 \,,$$

$$C_{\varepsilon\gamma 4} = 0.80 \,, \qquad C_{\varepsilon 1} = 1.44 \,, \qquad C_{\varepsilon 2} = 1.92 \,, \qquad C_\varepsilon = 0.07 \,.$$

Kawamura H. and Wakao Y., [KAW 95] established a modeled equation for ε_γ which satisfies the linearity principle (Pope S.B., [POP 83]), property deriving directly from the fact that the instantaneous scalar transport equation is itself also linear:

$$\frac{d\varepsilon_\gamma}{dt} = \left[\left(\sigma + \frac{\nu_t}{\text{Pr}_t\,\text{Pr}_{\varepsilon\gamma}}\right)\varepsilon_{\gamma,j}\right]_{,j} - C_{\varepsilon\gamma 1}\sqrt{\frac{\text{Pr}_t}{\text{Pr}}}\cdot\frac{\varepsilon}{k}F_{\gamma j}\overline{\Gamma}_{,j} + C_{\varepsilon\gamma 2}\frac{P\tilde{\varepsilon}_\gamma}{q}$$

$$-C_{\varepsilon\gamma 3}\,\text{Pr}_t\frac{\tilde{\varepsilon}\tilde{\varepsilon}_\gamma}{q} + 2C_{\varepsilon\gamma 4}\sigma\frac{\nu_t}{\text{Pr}_t}\overline{\Gamma}_{,jj}^2 + C_{\varepsilon\gamma 5}\sigma\frac{k}{\varepsilon}k_{,j}\overline{\Gamma}_{,m}\overline{\Gamma}_{,jm} \,,$$

$$C_{\varepsilon\gamma1} = 1.65 , \qquad C_{\varepsilon\gamma2} = 0.50 , \qquad C_{\varepsilon\gamma3} = 2.71/\mathrm{Pr}^{0.3} ,$$
$$C_{\varepsilon\gamma4} = 0.30 , \qquad C_{\varepsilon\gamma5} = 0.0035 .$$

The data coming from direct numerical simulations of turbulence also contributes more and more to the refined modeling of turbulent transport (Kasagi N. and Shikazono N., [KAS 94A]).

More recently, the low Reynolds number model by Hwang C.B. and Lin C.A., [HWA 97] has also been calibrated against numerical simulations for wall flows. This model includes pressure diffusion terms in the k and ε equations. The formulation is given below:

$$\frac{dk}{dt} = \nu k_{,jj} + \left(\frac{\nu_t}{\sigma_k} k_{,j} \right)_{,j} - \frac{1}{2}\nu \left(\frac{k}{\varepsilon} \hat{\varepsilon}_{,j} \right)_{,j} + P - \left(\tilde{\varepsilon} + \hat{\varepsilon} \right) ,$$

$$\frac{d\tilde{\varepsilon}}{dt} = \nu \tilde{\varepsilon}_{,jj} + \left(\frac{\nu_t}{\sigma_\varepsilon} \tilde{\varepsilon}_{,j} \right)_{,j} - \nu \left(\frac{\tilde{\varepsilon}}{k} k_{,j} \right)_{,j} + 1.44 \frac{\tilde{\varepsilon}P}{k} - 1.92 \frac{\tilde{\varepsilon}^2}{k} ,$$

$$\frac{dq}{dt} = \sigma q_{,jj} + \left(\frac{\nu_t}{\sigma_q} q_{,j} \right)_{,j} - \frac{1}{2}\sigma \left(\frac{q}{\varepsilon_\gamma} \hat{\varepsilon}_{\gamma,j} \right)_{,j} + P_\gamma - \left(\tilde{\varepsilon}_\gamma + \hat{\varepsilon}_\gamma \right) ,$$

$$\frac{d\tilde{\varepsilon}_\gamma}{dt} = \sigma \tilde{\varepsilon}_{\gamma,jj} + \left(\frac{\nu_t}{\sigma_{\varepsilon\gamma}} \tilde{\varepsilon}_{\gamma,j} \right)_{,j} - \sigma \left(\frac{\tilde{\varepsilon}_\gamma}{q} q_{,j} \right)_{,j} + C_{p1}f_{p1} \frac{\tilde{\varepsilon}_\gamma P_\gamma}{2q} + C_{p2}f_{p2} \frac{\tilde{\varepsilon}_\gamma P}{k}$$

$$- C_{d1}f_{d1} \frac{\tilde{\varepsilon}_\gamma^2}{2q} - C_{d2}f_{d2} \frac{\tilde{\varepsilon}_\gamma \tilde{\varepsilon}}{k} ,$$

$$R_{ij} = \frac{2}{3}k\delta_{ij} - 2\nu_t S_{ij} , \qquad \nu_t = 0.09 f_\mu(y_\lambda) \frac{k^2}{\tilde{\varepsilon}} ,$$

$$\sigma_k = 1.4 - 1.1\exp(-y_\lambda/10) , \qquad \sigma_\varepsilon = 1.3 - 1.0\exp(-y_\lambda/10)$$

$$y_\lambda = \frac{y}{\sqrt{\nu k/\tilde{\varepsilon}}} , \qquad f_\mu = 1 - \exp\left(-0.01y_\lambda - 0.008 y_\lambda^3 \right) ,$$

$$\tilde{\varepsilon} = 2\nu \left(\sqrt{k} \right)_{,j}^2 , \qquad \varepsilon = \tilde{\varepsilon} + \hat{\varepsilon} ,$$

$$\sigma_q = \mathrm{Pr}_t \, \sigma_k , \quad \sigma_{\varepsilon\gamma} = \mathrm{Pr}_t \, \sigma_\varepsilon , \qquad R = \frac{q}{\varepsilon_\gamma} \bigg/ \frac{k}{\tilde{\varepsilon}} , \quad \mathrm{Pr}_t = \frac{0.09}{c_\lambda} \sqrt{\frac{2R}{\mathrm{Pr}}} ,$$

$$c_\lambda = 0.116 , \quad C_{p1} = 1.8 , \qquad\qquad f_{p1} = 1 - \exp(-y_\lambda / 20) ,$$

$$C_{p2} = 1.4 + 17 \exp(-4.5 \, \mathrm{Pr}) , \qquad f_{p2} = 1 ,$$

$$C_{d1} = 2.7 - 0.5 \exp(-\mathrm{Re}_t / 60) , \qquad f_{d1} = 1 ,$$

$$C_{d2} = 0.8 , \qquad\qquad\qquad f_{d2} = 1 , \qquad \mathrm{Re}_t = k^2 / \nu \tilde{\varepsilon} .$$

We can also mention the work by Abe K. *et al.* [ABE 95 and 96], Yoshizawa A. [YOS 88] and also So R.M.C. and Sommer T.P. [SO 95].

The model by Abe K. *et al.* [ABE 94 and 95] applies to the dynamic and thermal fields. It is composed of transport equations for k, $\overline{\theta^2}$, ε, ε_θ, which are written:

$$\nu_t = C_\mu f_\mu \frac{k^2}{\varepsilon} , \quad \mathrm{Pr}_t = \frac{\nu_t}{\sigma_t} , \quad R = \frac{\overline{\theta^2} \, \varepsilon}{2k \, \varepsilon_\theta} ,$$

$$f_\mu = \left\{ 1 - \exp\left(-\frac{y^*}{14} \right) \right\}^2 \left[1 - \frac{5}{\mathrm{Re}_t^{3/4}} \exp\left\{ -\left(\frac{\mathrm{Re}_t}{200} \right)^2 \right\} \right] ,$$

$$f_\varepsilon = \left\{ 1 - \exp\left(-\frac{y^*}{3.1} \right) \right\}^2 \left[1 - 0.3 \exp\left\{ -\left(\frac{\mathrm{Re}_t}{6.5} \right)^2 \right\} \right] ,$$

f_ε being the coefficient of $C_{\varepsilon 2}$ in the ε equation.

The model uses u_ε (Kolmogorov velocity scale) instead of u_*, and thus $y^* = \dfrac{u_\varepsilon y}{\nu}$ with $u_\varepsilon = (\nu \varepsilon)^{1/4}$. The thermal diffusivity is given by:

$$\sigma_t = C_\lambda \left\{ \frac{k^2}{\varepsilon} \left(\frac{2R}{C_m + R} \right) + 3k^{1/2} \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} \frac{(2R)^{1/2}}{\mathrm{Pr}} f_d \right\}$$

$$\times \left\{ 1 - \exp\left(-\frac{y^*}{14} \right) \right\} \left\{ 1 - \exp\left(-\frac{\mathrm{Pr}^{1/2} \, y^*}{14} \right) \right\} ,$$

with $C_\lambda = 0.1 , C_m = 0.5$.

13.7. Transition and laminarization

It has been observed for a long time that most of the low Reynolds number models can, qualitatively at least, mimic transition ([JON 72], [SCH 72A]), in particular transition in a boundary layer or in a pipe or channel according to a Reynolds number. It is rather a "mimetism" produced by the mathematical system of equations than a real account of some transition mechanisms. It thus appear that the location of the transition point in the boundary layer is most of the time very poorly determined because it is highly sensitive to the particular formulation of the low Reynolds number empirical terms in the models and the choice of numerical constants. In practice, for applications in aerodynamics for example, we will artificially impose the transition point location to a correct value experimentally known.

Some flows may also present laminarized regions coexisting with fully turbulent regions. In this case, also, the outline of these zones is highly sensitive to the specific formulation of each particular model and to the type of flow under consideration. A laminarized region is characterized by turbulent energies approaching zero in the calculation, the turbulence cannot be sustained. The turbulent energies cannot however be exactly zero in a numerical solution because the quantities such as k/ε or ε/k involved in most models would become undetermined or infinite.

Chapter 14

Wall Treatment: Methods and Problems

Often considered the weak point of statistical closures, the wall treatment has given rise to numerous works. We present a summary of the main approaches, the techniques to implement wall boundary conditions, their potentials and difficulties.

14.1. The turbulent flow near a wall

14.1.1. *Influence of a wall on the turbulent flow*

The wall effects on the turbulence field are numerous and complex; however, we can bring out the main effects:

– the reduction in the length scales;

– the wall reflection contribution in the pressure-strain correlations;

– the development of a thin layer adjoining the wall (viscous sublayer) in which the molecular viscosity is dominant $Re_t = k^2 / v\varepsilon \leq 100$;

– the strongly anisotropic character of the turbulence near a wall ε_{ij} in particular;

– the strongly non-homogenous character of the turbulence field.

All these effects have important consequences on the transfer properties of the flow, in particular on the wall friction and thermal diffusion.

14.1.2. *Knowledge of the physics in the wall layer*

Numerous experimental studies have been carried out on wall turbulence in order to reveal and understand the interactions that come into play. In particular, the turbulent eddies and structures and the dynamic mechanisms have been extensively studied in the past (Kline S.J. *et al.*, [KLI 67], Willmarth W.W., [WIL 75], etc.). The concept of "streaks", "ejections", "sweeps", "bursts" and "coherent structures" have been introduced, the space-time properties of turbulence have also been studied. In the boundary layer in particular, the coherent structures are formed by vortex stretching to give rise to the so-called "hairpin" eddies or "horseshoe" eddies associated with ejections and which then fray into microturbulence. In the viscous sublayer, "streaks" develop, which have the appearance of longitudinal parallel ridges separated from each other by approximately $\Delta z = 100\nu / u_*$. The "sweeps" correspond to incursions of big eddies into the viscous sublayer and impinging the wall forcefully ("splatting effect"). It is this whole cyclic scenario (cf. Figure 14.1) that is called "bursts".
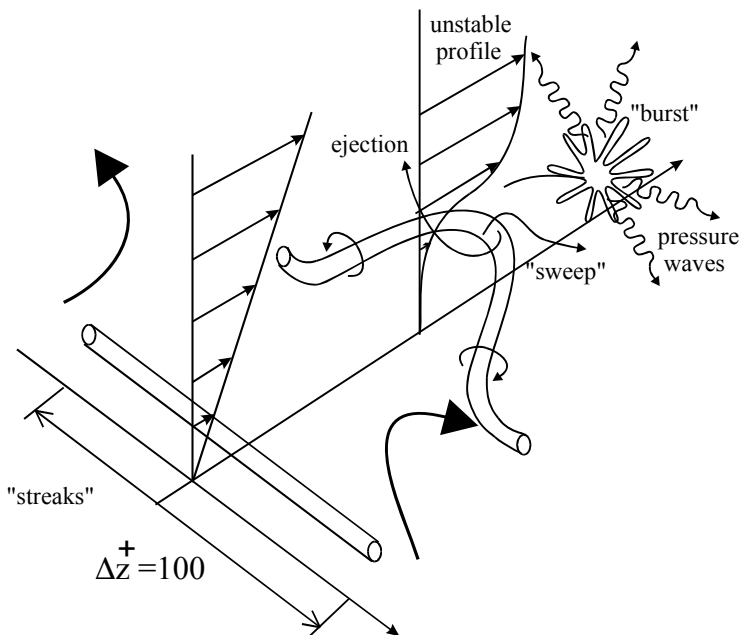
Figure 14.1. *Conceptual sketch of the cyclic "bursting" process in the turbulent boundary layer (after [KLI 67], [*HIN 75])*

In the viscous sublayer, a very thin region in which the molecular viscosity becomes dominant, the nearby wall brings many difficulties in the measurements. We thus mention all the interest in visualizations showing the eddying evolutions. All these studies, just as qualitative as quantitative, are of central interest in modeling, if we remember that in reality any turbulence model can be considered as a synthetic scheme which condenses the knowledge acquired at some stage of progress in time.

Moreover, the direct numerical simulations of turbulence (direct Navier-Stokes solutions) can nowadays provide numerical databases for wall turbulence. This data, after statistical treatment, can be very useful to complement experiments (for example, some correlations are hard or impossible to measure).

14.1.3. *Velocity profiles*

A hypothesis is often necessary for expressing the mean velocity profile according to wall distance.

In many cases, the total shear stress can be supposed constant across the wall layer $\tau = \tau_w$. Under these conditions, there is a universal velocity profile $U^+ = f(y^+)$ in which several zones can be distinguished:

- viscous zone $\qquad\qquad U^+ = y^+ \qquad\qquad\qquad$ for $y^+ < 5$,
- buffer zone $\qquad\qquad\quad$ junction $\qquad\qquad\qquad\quad$ for $5 \le y^+ \le 40$,
- log law zone $\qquad\qquad U^+ = (1/K)\, \text{Log } Ey^+ \quad$ for $y^+ > 40$.

Departures from the hypothesis $\tau = \tau_w$ can be due to different reasons. For example, it is the case for duct flows (impermeable wall) with mean pressure gradient. The momentum equation thus implies $\tau = \tau_w + y\, dp/dx$.

This effect is, however, hardly noticeable on the mean velocity profiles if the Reynolds number is sufficiently high.

Accelerated flows can be encountered in various circumstances: ducts with varying cross-section, boundary layer with imposed pressure gradient, density variations produced by heating. In the case of accelerated flows, the decrease in τ produces a thickening of the viscous sublayer, and the constants in the logarithmic law are modified. The values of τ_w (wall shear stress) and Nu (Nusselt number) are lower than those that would be obtained according to the usual law.

The situation is similar for flows with wall suction ($d\tau/dy < 0$).

The reverse effect ($d\tau / dy > 0$) produces a decrease of the viscous sublayer thickness, but it is more difficult to create and to observe. It can be encountered in the case of boundary layers in adverse pressure gradient (positive) or in the case of wall injection (blowing).

We can also cite the case of mixed convection flows (influence of gravity forces).

If the decrease in τ is very large, relaminarization of the turbulent layer may occur.

We can distinguish several types of numerical approaches for the problem of calculating the wall region (cf. Table 14.1).

Wall functions	Multizonal approach elliptic/parabolic	Low Reynolds number models (low Re_t)
Model for high Re_t coupled to a global (integrated) formulation of the wall layer	Model for high Re_t joining a simpler model for low Re_t in the wall layer (elliptic treatment or parabolic treatment)	Model for low Re_t used in the whole flow

Table 14.1. *Classification of the different approaches of the wall region*

14.2. Wall functions

The detailed calculation of the flow near a wall requires the use of a large number of discretization points in the sublayer region and the intermediate region in which high values of gradients prevail. The practical method to prevent this drawback is to make the junction to a universal law. In usual cases for the boundary layer on a flat plate, we use the logarithmic law for the velocity profile (Patankar S.V. and Spalding D.B., 1970). In more general cases, with recirculating zones or in three-dimensional flows, the problem cannot be solved in an entirely satisfactory way since a universal law no longer exists.

The wall function approach is thus an economical approach based on a global fully integrated formulation of the wall region. It allows the use of computational meshes that are very coarse. It has mainly been implemented in the framework of finite volume methods.

14.2.1. *The Patankar and Spalding formulation*

The wall functions can be obtained by integrating the momentum and the heat transfer equations, in the one-dimensional approximation of plane Couette flow assumed to be verified in the near wall region.

With this hypothesis, the equation of motion is thus written:

$$m\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 U}{\partial y^2} - \rho\frac{\partial}{\partial y}\left(\overline{uv}\right) \quad \text{and} \quad \frac{\partial P}{\partial y} = 0 \,,$$

where m is the mass flux injected at the wall.

If we introduce normalized quantities with respect to u_*, θ_* and μ denoted with superscripts (+), this equation becomes:

$$1 + m^+ U^+ + y^+ p^+ = \frac{dU^+}{dy^+} - \overline{u^+ v^+} \,,$$

where m^+ is the mass flux, p^+ the longitudinal pressure gradient and μ^+ an effective viscosity.

$$\frac{dU^+}{dy^+} = \frac{1 + m^+ U^+ + p^+ y^+}{\mu^+} \,, \qquad\qquad\qquad [14.1]$$

$$\frac{d\theta^+}{dy^+} = \frac{1 + m^+ \theta^+}{\mu^+ / \mathrm{Pr}_t} \,. \qquad\qquad\qquad [14.2]$$

Patankar and Spalding ([PAT 67], Patankar S.V. and Spalding D.B., 1970) use the mixing length hypothesis with Van Driest correction and after integration obtain the practical formulae written in the form:

$$s = s(R, F, M) \,, \qquad\qquad s = \frac{\tau_w}{\rho U_P^2} \,,$$

$$S = S(R, F, M, \mathrm{Pr}) \,, \qquad\qquad S = \frac{J_w}{\rho U_P \theta_P} \,,$$

$$R = (U^+ y^+)_P \,, \qquad M = (m^+ / U^+)_P \,, \qquad F = (p^+ y^+ / U^{+2})_P \,,$$

where P is the boundary point of the Couette flow, τ_w the wall shear stress and J_w the wall heat flux. For example, in the usual boundary layer with $F = M = 0$, Patankar and Spalding propose:

$$s = R^{-1} - 0.1561R^{-0.45} - 0.08723R^{-0.3} + 0.03713R^{-0.18} \quad (0 < R < 10^5).$$

The method remains limited by the restrictive hypotheses of Couette flow and the validity of the velocity laws obtained from integration.

In the framework of the mixing length hypothesis, some useful relations can be obtained. The shear stress in the Couette flow region can be written:

$$\overline{u^+ v^+} = -\left(Ky^+ \frac{dU^+}{dy^+}\right)^2.$$

If the viscous contribution is neglected, we deduce from the previous equation of motion:

$$\frac{dU^+}{dy^+} = \frac{1}{Ky^+}\sqrt{1 + m^+ U^+ + y^+ p^+}.$$

Analytical solutions are possible in some particular cases.

When $p^+ = 0$, we get the law $\dfrac{2}{m^+}\sqrt{1 + m^+ U^+} = \dfrac{1}{K}Log\left(E^* y^+\right)$ where E^* is a function of m. In order to obtain a formulation which reduces transparently to the usual logarithmic law when $m \to 0$, we write in an equivalent way, by changing the integration constant:

$$\frac{2}{m^+}\left(\sqrt{1 + m^+ U^+} - 1\right) = \frac{1}{K}Log\left(Ey^+\right),$$

leading to the bilogarithmic formulation:

$$U^+ = \frac{1}{K}LogEy^+ + \frac{m^+}{4K^2}\left(LogEy^+\right)^2.$$

In the case $m^+ = 0$, we can get the law:

$$U^+ = \frac{1}{K}\left[2\left(\sqrt{1+p^+y^+}-1\right)+Log\left(\frac{4Ey^+}{2+p^+y^++2\sqrt{1+p^+y^+}}\right)\right].$$

14.2.2. Formulation based on y+ = y u*/ν

The formulation based on the universal velocity and temperature profiles is easier to handle and to interpret. The relations to use are:

$$U^+ = y^+ \qquad\qquad \text{for}\quad y^+ \le y_v^+,$$

$$U^+ = \frac{1}{K}Log\,Ey^+ \qquad \text{for}\quad y^+ > y_v^+,$$

$$u_* = \sqrt{\tau_w/\rho}\,, \quad U^+ = U/u_*\,, \quad y_v^+ = \frac{1}{K}Log\,Ey_v^+\,, \quad K = 0.41,\ y_v^+ = 11$$

$$\theta^+ = \mathrm{Pr}.y^+ \qquad\qquad \text{for}\quad y^+ \le y_\theta^+,$$

$$\theta^+ = \mathrm{Pr}_t(U^+ + P) \qquad \text{for}\quad y^+ < y_\theta^+,$$

$$\theta_* = \frac{J_w}{\rho C_p u_*}\,, \qquad \theta^+ = (\theta-\theta_w)/\theta_*\,, \quad y_\theta^+ = \frac{\mathrm{Pr}_t}{K.\mathrm{Pr}}\left(Log\,Ey_\theta^+ + P.K\right).\qquad [14.3]$$

The empirical function P is given in the correlated data from known experimental results:

$$P = 3.68\left(\frac{\mathrm{Pr}}{\mathrm{Pr}_t}\right)^{-1/4}\left(\frac{\mathrm{Pr}}{\mathrm{Pr}_t}-1\right), \quad \text{(Patankar and Spalding)}$$

$$P = 9.24\left[\left(\frac{\mathrm{Pr}}{\mathrm{Pr}_t}\right)^{3/4}-1\right]\left[1+0.28\exp\left(-0.007\frac{\mathrm{Pr}}{\mathrm{Pr}_t}\right)\right] \text{ (Jayatilleke C.L.V., [JAY 69]).}$$

The skin friction coefficient $s = u_*^2/U_P^2$ and the flux ratio $S = \theta_*u_*/\theta_P U_P$ which corresponds to the Stanton number or Margoulis number *Marg* are obtained from:

$$s = \left(\frac{K}{Log\,E.R\sqrt{s}}\right)^2, \qquad S = \frac{s}{\mathrm{Pr}_t\left(1+P\sqrt{s}\right)}, \qquad R = \frac{y_P U_P}{\nu}.$$

These relations are deduced directly from the logarithmic profiles of velocity and temperature applied at point P.

The Margoulis number (or Stanton number) is related to the Nusselt number by $Marg = Nu / \mathrm{Re}.\mathrm{Pr}$. The grouping $\mathrm{Pe} = \mathrm{Re}.\mathrm{Pr}$ is also called the Péclet number, used in liquid metal heat transfer. The Nusselt number being defined by $Nu = J_w.L / \lambda.\Delta\theta = \dfrac{(J_w / \rho C_p).L}{\sigma.\Delta\theta}$ (λ, thermal conductivity coefficient of the fluid; C_p, specific heat of the fluid; σ, thermal diffusivity of the fluid) or by $Nu = \alpha.L / \lambda$ (cf. [*MCA 58]) if we introduce the heat transfer coefficient α, it follows that $Marg = \dfrac{J_w / \rho C_p}{\Delta\theta.U} = \dfrac{\alpha}{\rho C_p.U}$. In the particular case of the wall layer considered here, $Marg = \dfrac{J_w / \rho C_p}{\theta_P.U_P}$.

There are other possibilities: we can deduce S from traditional heat transfer correlations such as $Marg = \mathrm{Re}.\mathrm{Pr}^{1/3}$ or also $Marg = (C_f / 2).\mathrm{Pr}^{-2/3}$ (Chilton-Colburn analogy where C_f is the skin friction coefficient). This formalism can also be used for rough walls by using an appropriate modified logarithmic law.

However, normalizing on $u*$ cannot be suitable for separated flows in which the skin friction comes to vanish.

14.2.3. *Formulation based on a modified y^+*

In the previous formulation, y^+ vanishes at a separation point. We achieve greater generality by using $\sqrt{k_P}$ as the velocity scale, P being the first discretization point near the wall. It is of course necessary in this case to solve a transport equation for k.

The relation $U^+ = f(y^+)$ remains here formally identical but the normalizing scales are different.

Thus,

$$U^+ = \frac{U}{u*}, \qquad u* = \frac{\tau_w}{\rho C_\mu^{1/4} k_P^{1/2}}, \qquad y^+ = \frac{y C_\mu^{1/4} k_P^{1/2}}{\nu}, \qquad \text{[14.4]}$$

this implying $\dfrac{1}{\sqrt{s}} = \dfrac{1}{K}\,\mathrm{Log}\dfrac{C_\mu^{1/4}Ek_P^{1/2}y_P}{\nu}$ or $\tau_w = \nu_a\dfrac{U_P}{y_P}$,

with $\nu_a = \dfrac{\rho C_\mu^{1/4}Ky_Pk_P^{1/2}}{\mathrm{Log}\left(EC_\mu^{1/4}y_Pk_P^{1/2}/\nu\right)}$ ("apparent viscosity").

The same for the heat transfer problem:

$$\theta^+ = \dfrac{\Delta\theta}{\theta_*}\,,\qquad \theta_* = \dfrac{J_w}{\rho C_p C_\mu^{1/4}k_P^{1/2}}$$

$$\Rightarrow\quad S = \dfrac{s}{\mathrm{Pr}_t\left(1+P\sqrt{s}\right)}\quad\text{or}\quad J_w = \dfrac{\rho^2 C_p U_P\theta_P C_\mu^{1/2}k_P s}{\mathrm{Pr}_t\,\tau_w\left(1+P\sqrt{s}\right)}$$

In an equilibrium layer the previous formulation is recovered, making this:

$$k_P = C_\mu^{-1/2}\left(\tau_w/\rho\right).$$

From the practical point of view, point P must always be located in the beginning of the logarithmic zone (cf. Figure 14.2). If, *as an exception*, a point P is located below ($y+ < 11$) we can use the law $U^+ = y^+$ according to a similar formalism. The method has been widely used in the finite volume approaches (cf. Pun W.M. and Spalding D.B., [PUN 77]).
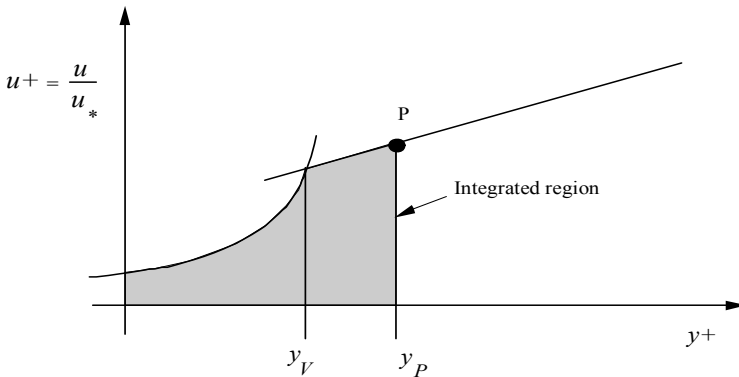
Figure 14.2. *Application of the wall function method, location of the first mesh point near the wall*

This formulation based on $y^+ = y\sqrt{k_P}/v$, and applied to separated flows, has given very fickle results with noticeable, depending on the particular cases, overestimates or underestimates of the wall Nusselt number.

14.2.4. *Formulation based on the values at the boundary of the viscous sublayer*

The previous formulation using k_P at the first discretization point near the wall has the drawback of being dependent on the mesh considered (wall distance y_P). Therefore Chieng C.C. and Launder B.E., [CHI 80] preferred to use k_V at the boundary of the viscous sublayer to define $y^+ = y\sqrt{k_V}/v$.

$$U^+ = \frac{U.k_V^{1/2}}{\tau_w/\rho}, \qquad \theta^+ = \frac{(\theta - \theta_w)\rho C_p k_V^{1/2}}{J_w}, y^+ = \frac{y\sqrt{k_V}}{v} \qquad\qquad [14.5]$$

$$U^+ = y^+ \qquad\qquad \text{for } y^+ \le y_V^+,$$
$$U^+ = \frac{1}{K^*}\text{Log } E^* y^+ \qquad \text{for } y^+ > y_V^+,$$

with $K^* = C_\mu^{1/4}K$ and $E^* = C_\mu^{1/4}E$, and the position $y_V^+ = 20$ defines the thickness of the sublayer with $C_\mu^{1/4} y_V^+ = 11.22$.

Similarly, the temperature law reads:

$$\theta^+ = \text{Pr}_t\left(U^+ + P\right), \qquad P^* = C_\mu^{-1/4}P$$

and k_V is obtained through extrapolation to the boundary of the sublayer of the values for k in two consecutive interior points (cubic equation).

In the sublayer, we assume:

$$k = k_V\left(y/y_V\right)^2, \qquad \varepsilon = 2v k_V/y_V^2 \qquad\qquad y \le y_V,$$
$$k = ay + b, \qquad\qquad \varepsilon = k^{3/2}/cy \qquad\qquad y > y_V.$$

These expressions are used to calculate the source terms in the wall cells (finite volume method):

$$\overline{Prod} = \frac{1}{y_n} \int_{y_v}^{y_n} \frac{\tau}{\rho} \frac{\partial U}{\partial y} dy , \qquad \overline{\varepsilon} = \frac{1}{y_n} \int_0^{y_n} \varepsilon dy ,$$

the position y_n corresponding to the external boundary of the first discretization cell at the wall. The wall shear stress τ_w is obtained by extrapolation to the wall of the values obtained in the first discretization points.

The results obtained by application of this formulation still show weaknesses that are also mainly obvious on the predicted Nusselt numbers. The advantage over the previous method is hardly noticeable.

14.2.5. *Account of the variations in the viscous sublayer thickness*

In the case where τ varies ($\tau \neq \tau_w$) it follows a variation in the viscous sublayer thickness.

Johnson R.W. and Launder B.E., [JOH 82] proposed the following modification:

$$y_V^+ = \frac{y_{V0}^+}{1 + cz} , \qquad z = \frac{y_V}{k_V} \left(\frac{\partial k}{\partial y} \right)_{y_V^+} , \qquad [14.6]$$

in which y_{V0}^+ stands for the sublayer thickness in equilibrium situation $y_{V0}^+ = 20$ (i.e. $y_{V0} k_{V0}^{1/2} / \nu = 20$).

This modification allows an improvement in the prediction of local Nusselt numbers, in particular in the case of a sudden expansion in a pipe (reattachment).

Ciofalo M. *et al.*, [CIO 89] related y_V^+ to the local turbulence intensity:

$$y_V^+ = f(\mathscr{T}_P) , \qquad \mathscr{T}_P = \frac{k_P^{1/2}}{U_P} .$$

Let $\mathcal{I}_E$ be the turbulence intensity in an equilibrium layer $\mathcal{I}_E = k^{+1/2}/U^+$ which can be calculated by using the velocity law $U^+ = f(y^+)$; the hypothesis of Ciofalo *et al.* reads:

$$\frac{y_V^+}{y_{V0}^+} = \left(\frac{\mathcal{I}_P}{\mathcal{I}_E}\right)^{-c}, \qquad c \approx \frac{1}{3} \text{ to } \frac{2}{3},$$

and for the thermal sublayer:

$$y_\theta^+ = y_V^+ \left(\frac{\mathrm{Pr}}{\mathrm{Pr}_t}\right)^{-0.25}.$$

14.2.6. *Formulation based on a three zone scheme*

Amano R.S., Jensen M.K. and Goel P., [AMA 83] and Amano R.S., [AMA 84] introduced a wall scheme using three regions (the sublayer region, the buffer region, logarithmic region) based on $y^+ = y\sqrt{k_B}/\nu$.

The velocity scale $\sqrt{k_B}$ is considered at the boundary between the buffer region and the logarithmic region. The velocity profile is defined from:

$$U^+ = \frac{U}{u_*}, \qquad u_* = \frac{\tau_w}{\rho k_B^{1/2}}, \qquad y^+ = y\frac{k_B^{1/2}}{\nu}, \tag{14.7}$$

$$U^+ = \frac{1}{\mathrm{K}^*}\mathrm{Log}\, E^* y^+, \quad \text{for the logarithmic region.}$$

As shown previously (section 14.2.4) we have $\mathrm{K}^* = C_\mu^{1/4}\mathrm{K}$ and $E^* = C_\mu^{1/4}E$. If we make use of the normalized distance yu_*/ν based on the distance from the wall y and the friction velocity u_*, the three zones in the wall region include the viscous sublayer ($0 < yu_*/\nu < 5$), the buffer zone ($5 < yu_*/\nu < 30$) and the fully turbulent zone ($30 < yu_*/\nu < 400$). The variations of the quantities k, ε and τ are specified within each zone according to y. The layout of the integration zones is illustrated in Figure 14.3.
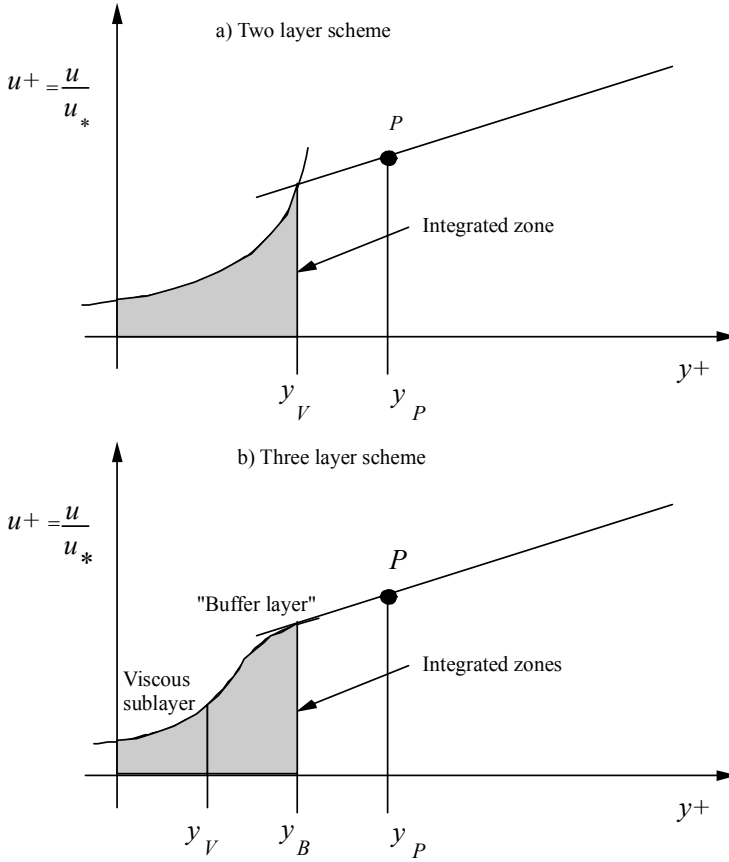
Figure 14.3. *Scheme with several near wall layers*

14.2.7. *Other types of velocity distributions*

Chen H.C. and Patel V.C., [CHE 88] used a generalized velocity law of the wall:

$$
\frac{U}{u_*} = \frac{1}{\mathrm{K}} \left\{ \mathrm{Log}\, \frac{4\left(1 + \Delta_\tau y^+\right)^{1/2} - 1}{\Delta_\tau \left(1 + \Delta_\tau y^+\right)^{1/2} + 1} + 2\left[\left(1 + \Delta_\tau y^+\right)^{1/2} - 1\right] + B + 3.7 \Delta p \right\},
$$

with $\Delta p = \dfrac{dp/dx}{\mathrm{Re}\,.u_*^3}$, Δ_τ being a parameter related to the gradient of the turbulent stress, Re the Reynolds number.

Ciofalo *et al.* refer to the universal profiles for counterflow in recirculating zones, introduced by Simpson (1983):

$$\frac{U}{U_N} = A\left(\frac{y}{N} - \text{Log}\frac{y}{N} - 1\right) - 1, \quad 0.02 < \frac{y}{N} < 1, \quad A \approx 0.3,$$

where U_N is the maximal counterflow velocity and N the distance to the wall for which the velocity U_N is reached.

Let us also mention the velocity law given by Spalding D.B., [SPA 61].

The wall function approach also allows us to tackle easily the problem of the treatment rough walls (Jayatilleke C.L.V., [JAY 69]) whereas the boundary conditions in a low Reynolds number model would remain more tricky to specify (Lawn C.J., [LAW 74], Nikitin J.K., [NIK 72]). Extensions of the laws of the wall to the case of compressible boundary layers have been developed by Saffman P.G. and Wilcox D.C., [SAF 74] and Wilcox D.C., 1993.

14.3. Simple models for the viscous sublayer

These models can be used for example in a multilayer approach, adjoining higher order models for the bulk flow but at high Reynolds number. The reader may refer to Chapter 13. We give here some specific additional information for the wall models.

14.3.1. *Turbulence eddy viscosity models*

They can generally be written in the form $\nu_t = cu_* y C(y)$ where $C(y)$ is a corrective factor for the viscous sublayer. These are quite old methods (cf. *Proc. Comput. of Turb. Boundary Layers*, AFOVR-IFP Stanford Conf., 1968).

We mention:

$$\nu_t = cu_* \left[y - \delta_\ell \, \text{Th}\left(y/\delta_\ell\right)\right], \qquad \text{Reichardt (1951)}$$
$$\nu_t = cUy\left[1 - \exp(-AUy/\nu)\right], \qquad \text{Deissler (1954)}$$

Mellor and Herring (cf. *Proc. Comp. of Turb. Boundary Layers*, AFOVR-IFP Stanford Conf., 1968) introduce three hypotheses for the different regions in the

boundary layer (viscous sublayer, external region and intermediate layer) grouped together into a single analytical formulation:

$$\frac{v+v_t}{U_\infty \delta_*} = \frac{v}{U_\infty \delta_*} f\left(\frac{y}{v}\sqrt{\frac{\tau}{\rho}}\right) + g\left(\frac{y}{U_\infty \delta_*}\sqrt{\frac{\tau}{\rho}}\right) - \frac{y}{U_\infty \delta_*}\sqrt{\frac{\tau}{\rho}}, \qquad [14.8]$$

in which f and g are dimensionless functions deduced from experimental profiles.

14.3.2. *Mixing length models*

They are written in the form $\ell_m = Ky\mathscr{D}$, where $\mathscr{D}$ is a corrective factor.

Let us recall the Van Driest correction [VDR 56] (cf. Chapter 13),

$$\mathscr{D} = 1 - \exp(-y^+ / A), \qquad y^+ = u_* y / v, \qquad A = 26. \qquad [14.9]$$

The Van Driest formula gives good results in a constant shear layer, but is no longer valid in other cases.

a) First extension of the Van Driest formula: variable A

The value of A being related to the sublayer thickness, we can try to account for the variations in the thickness of the sublayer by using a variable value for A. We mention for example:

$$A = \max\left[26, 11 + 2790 P^+\right], \qquad P^+ = \frac{v}{\rho u_*^3}\frac{dp_\infty}{dx},$$

(Launder B.E. and Jones W.P., [LAU 69B])

$$A = 24 \Bigg/ \left[a\left(V_w^+ - \frac{bP^+}{1+cV_w^+}\right) + 1\right], \qquad V_w^+ = \frac{V_w}{u_*}, \qquad a(V_w^+), \ b(P^+) \text{ and } c(P^+).$$

(Kays W.M. and Moffat R.J., [KAY 75]).

This type of formulation has been used also by Cebeci T. *et al.*, [CEB 70].

Parameter A can also be obtained by an evolution equation:

$$\frac{dA}{dx^+} = \alpha(A_E - A) \quad \text{where} \quad A_E \text{ is the equilibrium value.}$$

This relaxation equation introduced by Launder and Jones (cf. Launder B.E., [LAU 86]) accounts for the effect of non-equilibrium.

b) Second extension of the Van Driest formula: y+ redefined

This solution consists of keeping A constant but modifying the definition of y^+ by using the local stress τ (cf. Chapter 13).

We also recall the scheme of Michel (cf. Chapter 13). Various variants in the formulations exist, some of which are compared in Spalding D.B. and Priddin C.H., [SPA 73].

14.3.3. *Adaptation of the Prandtl model*

In the Prandtl model, the velocity scale is related to $\sqrt{k}$. Thus, a transport equation for k must be solved.

Wolfshtein M., [WOL 69] also introduces two characteristic length scales, one for the turbulent diffusion and the other for the viscous dissipation.

$$\nu_t = C_\mu \ell_\mu \sqrt{k} , \qquad \varepsilon = k^{3/2} / \ell_\varepsilon , \qquad\qquad [14.10]$$

$$\ell_\mu = Cy \left[1 - \exp\left(-y^+ / A_\mu \right) \right], \qquad A_\mu = 62$$

$$\ell_\varepsilon = Cy \left[1 - \exp\left(-y^+ / A_\varepsilon \right) \right], \qquad A_\varepsilon = 3.8 , \quad C_\mu = 0.09 , \quad C = 2.44$$

When $y^+ \to 0$, $\varepsilon \to 1.56 \nu k / y^2$, value close to the "exact" asymptotic value which is $\varepsilon = 2\nu k / y^2$. When $y^+ \to \infty$, (high Reynolds number turbulence) the two scales ℓ_μ and ℓ_ε become equal.

We also mention the model by Glushko (1965) applied by Beckwith and Bushnell (cf. *Proc. Comput. of Turb. Boundary Layers*, AFOVR-IFP Stanford Conf., 1968) and then reconsidered by Rubesin W.M., [RUB 76] and extended to compressible flows.

$$\mu_t = \alpha\mu R^+ \times R^+ \qquad\qquad R^+ < 0.75\,,$$

$$= \alpha\mu R^+ \times \left[R^+ - \left(R^+ - 0.75\right)^2 \right] \qquad 0.75 \le R^+ \le 1.25\,,$$

$$= \alpha\mu R^+ \times 1 \qquad\qquad R^+ > 1.25\,,$$

$$R^+ = \frac{L\sqrt{k}}{120\nu}\,,$$

$$L = \delta \times (y/\delta) \qquad\qquad y/\delta < 0.23\,,$$

$$= \delta \times (y/\delta + 0.37)/2.61 \qquad 0.23 \le y/\delta \le 0.57\,,$$

$$= \delta \times (1.48 - y/\delta)/2.52 \qquad 0.57 < y/\delta < 1.48\,.$$

However, Gibson M.M., Spalding D.B. and Zinser W., [GIB 78C] have found that these types of schemes do not correctly account for the relaminarization phenomenon.

14.3.4. *Heat and mass transfer in the wall layer*

The basic Reynolds analogy in a boundary layer, based on the similarity in the velocity and temperature profiles, leads to [*KRE 67]:

$$Marg = \frac{1}{2}C_f\,.$$

This analogy is verified by the experiment only if the Prandtl number is equal to unity and the boundary conditions for velocity and temperature are similar (heated wall). The Prandtl analogy [*KRE 67] accounts for the existence of a viscous sublayer near the wall and leads to the expression:

$$Marg = \frac{C_f/2}{1 + 5\sqrt{\dfrac{C_f}{2}}\,(\mathrm{Pr}-1)}\,, \qquad \mathrm{Pr} \ge 1$$

For the problem of predicting temperature profiles and heat transfer, the leading parameter is the turbulence Prandtl number. Patankar S.V. and Spalding D.B., [PAT 67] use the constant value $\mathrm{Pr}_t = 0.9$ in any case for all values of the turbulence Reynolds number Re_t. These authors have thus obtained using the

mixing length hypothesis, good predictions of Nusselt numbers in a pipe for a wide range of Prandtl numbers.

In addition, various correlation formulae have been proposed by different authors, which give $\Pr_t$ according to parameters such as $\Pr, y, \Re_t$, etc.

$$\Pr_t = f\left(\Pr, \frac{\nu_t}{\nu}, y^+\right) \qquad\qquad \text{Reynolds A.J., [REY 75A],}$$

$$\Pr_t = \max\left[0.86, \left(1.43 - 0.17 y^{+1/4}\right)\right] \text{Kays W.M. and Moffat R.J., [KAY 75],}$$

$$\Pr_t = \frac{K}{K_\theta} \frac{1 - \exp(-y^+ / A)}{1 - \exp(-y^+ / B)} \qquad\qquad \text{Cebeci T., [CEB 73],}$$

where K_θ is the slope in the mixing length distribution near the wall for the temperature.

$$B = B(\Pr)$$

Thermal diffusivity:

$$D_t = K.K_\theta y^2 \left[1 - \exp(-y^+ / A)\right]\left[1 - \exp(-y^+ / B)\right]\left|dU/dy\right|$$

$$\Re_t \to \infty \qquad\qquad \Pr_t \to K/K_\theta \ ,$$

$$\Re_t \to 0 \qquad\qquad \Pr_t \to K.B/K_\theta.A$$

$$\Pr_t = \frac{B}{A} \frac{1}{\Pr} \frac{1 - \exp(-1/B.\Re_t)}{1 - \exp(-1/A.\Pr.\Re_t)} \ , \qquad\qquad \text{(Harlow F.H., [HAR 68A])}$$

$$D_t = A \frac{D_{t\infty}^2}{\sigma}\left[\exp(-\frac{\sigma}{A.D_{t\infty}}) - 1\right],$$

where σ is the molecular thermal diffusivity, and $D_{t\infty}$ is the turbulent thermal diffusivity for $\Re_t \to \infty$.

This analysis is based on the simplified model of a spherical "eddy" exchanging heat with its environment by molecular diffusion (cf. Chapter 13, section 13.4).

Let us also cite the work by Simpson R.L., Whitten D.G. and Moffat P.J., [SIM 70].

14.4. Models using several transport equations for the viscous sublayer

These models were presented in Chapter 13.

In the case where τ is decreasing from the wall (accelerated boundary layer) the ε equation yields satisfactory results. But in the case of separated flows, the calculated length scale is excessive. The Yap C. correction, [YAP 87] allows us to overcome this drawback (cf. section 7.5).

The approach of the wall region using models with several transport equations valid for low Reynolds numbers allows a more general and more universal description but it requires, from the numerical point of view, very fine meshes refined near the wall in order to account for the large gradients developing near a wall.

Direct numerical simulations of turbulence (direct solution of the Navier-Stokes equations) have also provided a detailed description of the wall region, useful for complementing the experimental knowledge. This data has been used in particular by Mansour N.N., Kim J. and Moin P., [MAN 89] for determining the empirical coefficients in the low Reynolds number models near a wall:

$$
f_\mu = \underbrace{C_\mu^{-1} \left(\frac{R_{12}}{k} \right)^2}_{f_1(y^+,\mathrm{Re}_t)} \; \underbrace{\frac{\tilde{\varepsilon}}{P}}_{f_2(y^+)} \; .
$$

This type of work has been extended by Miner E.W. *et al.*, [MIN 91].

14.5. New directions in the wall function formulation

Two approaches were developed during the years 1999 to 2002, the first one ([CRA 01]) consisting of performing a fine numerical solution of the wall layer using a low Reynolds number model (cf. section 14.4) with one-dimensional approximation in the direction perpendicular to the wall (Couette flow). This approach is similar to a zonal approach with a simplified numerical solution in the wall layer. The other strategy developed by Craft T.J. *et al.* [CRA 02] retains an analytical approach which allows an appreciable economy in computer time, but introducing some new characteristics which allow us to extend the applicability domain compared to already existing methods.

The method supposes an implementation in the framework of finite volumes approach. We thus assume that the first discretization cell adjoining the wall and

centered at point P has a height of y_n and that the viscous sublayer thickness is y_v. The turbulence eddy viscosity is assumed zero in the range $[0, y_v]$ and then linear beyond.

$$\frac{\mu_t}{\mu_v} = c_\mu c_\ell (y^* - y^*_v), \quad \text{with} \quad c_\mu = 0.09, \quad c_\ell = 2.55, \quad y^* = \rho_v y k_P^{1/2} / \mu_v.$$

The heat transfer equation written in a simplified form:

$$\frac{\partial}{\partial y^*}\left[\left(\frac{\mu}{\text{Pr}} + \frac{\mu_t}{\text{Pr}_t}\right)\frac{\partial T}{\partial y^*}\right] = \underbrace{\frac{\mu_v^2}{\rho_v^2 k_P}\left(\rho U \frac{\partial T}{\partial x}\right)}_{C_{th}},$$

can be easily integrated in y to give:

$$T = T_w + \frac{\text{Pr}}{\mu_v}\left[\frac{C_{th} y^{*2}}{2} + A_{th} y^*\right] \qquad \text{for} \quad y^* < y^*_v$$

$$T = T_w + \frac{\text{Pr}}{\mu_v \alpha_t} C_{th}(y^* - y^*_v) + \frac{\text{Pr}}{\mu_v \alpha_t}\left[A_{th} + C_{th}\left(y^*_v - \frac{1}{\alpha_t}\right)\right] Log Y_T$$

$$+ \frac{\text{Pr}\, y_v}{\mu_v}\left[\frac{C_{th}}{2} y^*_v + A_{th}\right] \qquad \text{for} \quad y^* > y^*_v$$

with $\alpha_t = \text{Pr}\,\alpha / \text{Pr}_t$, $\alpha = c_\mu c_\ell$, $Y_T = 1 + \alpha_t\left(y^* - y^*_v\right)$, $A_{th} = -J_w \mu_v / C_p \rho_v k_P^{1/2}$.

Similarly, the momentum equation is written in the form:

$$\frac{\partial}{\partial y^*}\left[(\mu + \mu_t)\frac{\partial U}{\partial y^*}\right] = C + b(T - T_{ref}),$$

where $C = \frac{\mu_v^2}{\rho_v^2 k_P}\left[\rho U \frac{\partial U}{\partial x} + \frac{\partial P}{\partial x}\right]$ and $b = -\frac{\mu_v^2}{\rho_v^2 k_P}\rho_{ref} g\beta$ (effect of gravity forces).

The equation is then integrated in y, the result in the absence of gravity forces is the following:

$$\mu_v U = \frac{C y^{*2}}{2} + A_1 y^* \qquad\qquad \text{for} \quad y^* < y^*_v,$$

$$\mu_v U = \frac{C}{\alpha}\left[y^* - \left(\frac{1}{\alpha} - y^*_v\right) Log Y \right] + \frac{A_2}{\alpha} Log Y + B_2 \qquad \text{for } y^* > y^*_v \,,$$

with $Y = 1 + \alpha\left(y^* - y^*_v\right), \qquad B_2 = y^*_v C\left(\frac{y^*_v}{2} - \frac{1}{\alpha}\right) + A_1 y^*_v \,,$

$$A_2 = A_1 = \frac{\mu_v U_n - N}{Log \,{}^{Y_n}\!/\!_\alpha + y^*_v} \,,$$

$$N = \frac{C}{\alpha}\left[y^*_n - y^*_v - \left(\frac{1}{\alpha} - y^*_v\right) Log Y_n + \frac{\alpha y^{*\,2}_v}{2} \right].$$

The solution of the turbulence kinetic energy requires the knowledge of the mean values of production and dissipation. In the turbulent region, the dissipation rate is assumed to be given by solution in the logarithmic zone:

$$\varepsilon = \frac{k_P^{3/2}}{c_\ell y} \,,$$

while in the viscous sublayer:

$$\varepsilon = \frac{2\nu k_P}{y_d^2} \,.$$

The ordinate y_d of the boundary given by the condition at the adjoining point $y^*_d = 2c_\ell$ is different from y^*_v and allows us to place the maximum of ε on the very wall as it is shown in the present numerical simulations. The mean value is obtained by integration:

$$\varepsilon_{moy} = \frac{1}{y_n}\left[\frac{2k_P^{3/2}}{y^*_d} + \frac{k_P^{3/2}}{2.55} Log\left(\frac{y_n}{y_d}\right) \right].$$

The authors also introduce the effect of variations in the viscosity according to temperature and the effect of gravity forces. The implementation of generalized wall functions is made in the framework of finite volume methods. The formulations obtained in the general case are far more complex and are detailed in reference [CRA 02]. When the gradient of the shear stress is significant, the thickness of the

viscous sublayer is changed and empirical modifications are often used (cf. section 14.2.5). The practice proposed here is different, it consists of modifying the mean dissipation by a factor depending on the ratio of the shear stresses at the boundary of the viscous sublayer and at the wall.

This modification is introduced in the form:

$$\varepsilon_{\text{modif}} = F_\varepsilon(\lambda).\varepsilon_{\text{initial}}, \qquad \lambda = \mu_w \sqrt{\left(\overline{U}_{i,j}\right)_w^2} \Big/ \mu_v \sqrt{\left(\overline{U}_{i,j}\right)_v^2}.$$

This technique has been applied to various situations including the turbulent pipe flow, the mixed heat convection in a vertical channel and the turbulent wall jet. In cases for which the flow streamlines are highly curved near the wall like in impinging jets, a zonal approach with a fine one-dimensional numerical solution remains however preferable. The extension to the case of rough walls is considered in [SUG 05].

Chapter 15

Influence of Archimedean Forces

The methods of approach to account for the gravity forces (and the Archimedean forces coming from varying density due to temperature changes) are similar to those used in dynamics but the acting parameters are more numerous. Turbulence being a phenomenon-having dynamic origin, every model of the thermal field must rely upon a satisfactory dynamic model already achieved. Let us cite, among others, the following studies on heat transfer and the Prandtl number of turbulence: Lumley J.L., [LUM 75C]; Launder B.E., [LAU 76]; Béguier C., Dekeyser I. and Launder B.E., [BEG 78]; Jischa M. and Rieke H.B., [JIS 79]; Launder B.E. and Samaraweera D.S.A., [LAU 79A]; Kolovandine B.A., [KOL 70] and on the influence of Archimedean forces: Gibson M.M. and Launder B.E., [GIB 76]; Zeman O. and Lumley J.L., [ZEM 76 and 77]; Lumley J.L., Zeman O. and Siess J., [LUM 78B]; Gibson M.M. and Launder B.E., [GIB 78B]; Rodi W., [ROD 80]; Launder B.E., [LAU 75B] and also Gence J.N., [GEN 77A and B]; Gence J.N. *et al.*, [GEN 78]. [HAN 02] also contains an interesting review including various illustrations of applications to turbulent flows influenced by gravity or buoyancy forces.

15.1. Transport equations of turbulence in the Boussinesq approximation

For an introduction to the Boussinesq approximation the reader may refer to the basic books on the subject (Joseph D.D., 1976; Favre A. *et al.*, 1976).

The variations in the density ρ according to temperature Γ can be written:

$$\rho = \rho_0 \left[1 + \beta(\Gamma - \Gamma_0)\right] \quad \text{with} \quad \beta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial \Gamma}\right)_p < 0,$$

The scalar Γ is no longer a passive contaminant because the Archimedean forces (or buoyancy forces) are acting on the mean flow:

$$g_i(\rho - \rho_0) = \rho_0 g_i \beta(\Gamma - \Gamma_0),$$

where g_i stands for the acceleration of gravity. The gravity is directed vertically but we have retained any possible direction to the vector, in order to make possible transpositions of the modeling relations to volumic forces of a different nature.

In the Navier-Stokes equations:

$$\rho\frac{dU_i}{dt} = -P_{,i} + \rho g_i + \mu U_{i,jj},$$

we will assume $U_{j,j} = 0$ and use the approximation:

$$\rho_0\frac{dU_i}{dt} = -P_{,i} + \rho_0\left[1 + \beta(\Gamma - \Gamma_0)\right]g_i + \nu\rho_0 U_{i,jj},$$

assuming in addition that $P = P_0 + \rho_0 g_j x_j$ gives:

$$\frac{dU_i}{dt} = -\left(\frac{P_0}{\rho_0}\right)_{,i} + \beta(\Gamma - \Gamma_0)g_i + \nu U_{i,jj}. \qquad [15.1]$$

In the following, we shall denote P_0 and ρ_0 by using simply P and ρ. After the relation $\frac{\Delta\rho}{\rho} = \beta(\Gamma - \Gamma_0)$, we see that the fluctuations γ of Γ and the fluctuations ρ' of ρ will be related through $\rho'/\rho = \beta\gamma$.

Taking into account the previous hypotheses, we can easily show that the transport equations for R_{ij}, $F_{\gamma i}$, k, ε (Chapter 2) are modified as follows:

$$\frac{dR_{ij}}{dt} = \underbrace{\diamond - \boxed{+\Phi_{ij}^{(3)}} - \diamond}_{(*)} + \underbrace{\beta\left(F_{\gamma i}g_j + F_{\gamma j}g_i\right)}_{G_{ij}}, \qquad [15.2]$$

(*) additional redistribution term.

$$\frac{dk}{dt} = \underline{\qquad} \diamond \underline{\qquad} + \underbrace{\beta F_{\gamma i} g_i}_{G} \;, \qquad\qquad [15.3]$$

$$\frac{d\varepsilon}{dt} = \underline{\qquad} \diamond \underline{\qquad} + 2\nu\beta g_i \overline{u_{i,l}\gamma_{,l}} \;, \qquad\qquad [15.4]$$

$$\frac{dF_{\gamma i}}{dt} = \underline{\qquad} \diamond \underline{\quad} \boxed{+\Phi_{\gamma i}^{(3)}} \underline{\quad} \diamond \underline{\qquad} + \underbrace{\beta\overline{\gamma^2}g_i}_{G_{\gamma i}} \;, \qquad\qquad [15.5]$$

$\underline{\qquad} \diamond \underline{\qquad}$ stands for the terms already obtained in the absence of gravity forces.

A new parameter appears; the Archimedes number defined as the ratio between buoyancy forces to inertial forces:

$$A_r = \frac{\beta\Delta\Gamma g}{U.dU/dx} \;, \quad \text{(local definition)}$$

We can also define the Archimedes number in a global way by introducing the characteristic length scale of the flow L and the characteristic mean velocity U:

$$\mathscr{A}_r = \beta g \frac{L\Delta\Gamma}{U^2} \;, \quad \text{(global definition)}$$

In heat transfer problems (natural convection and mixed convection), it is usual to introduce the Grashof number:

$$\mathscr{G}r = \mathscr{A}_r.\mathrm{Re}^2 = \rho^2 g\beta\frac{\Delta\Gamma.L^3}{\mu^2} \;, \qquad \mathrm{Re} = \rho\frac{U.L}{\mu} \;,$$

The Grashof number is a similarity parameter in natural convection driven flows. It can be viewed as the ratio of the buoyancy forces to the viscosity forces. The convective instabilities (Rayleigh-Bénard convective rolls) are controlled by the Rayleigh number:

$$\mathscr{R}a = \mathscr{G}r.\mathrm{Pr} = g\beta\frac{\Delta\Gamma.L^3}{\nu\sigma} \;, \qquad \mathrm{Pr} = \frac{\nu}{\sigma} \;,$$

In addition, in the turbulence kinetic energy equation, there are two production terms, the usual production term coming from turbulent shear P and a second term due to Archimedean forces. Their ratio defines the flux Richardson number:

$$\mathscr{R}f = \frac{G}{P} = \beta\frac{g_i F_{\gamma i}}{-R_{ij}\overline{U}_{i,j}}.$$

and thus equation [15.3] can also be written:

$$\frac{dk}{dt} = -R_{ij}\overline{U}_{i,j}(1-\mathscr{R}f) + Diff(k) - \varepsilon.$$

If the turbulent diffusion terms are neglected, we see that the turbulence can be sustained only if the total production term balances the dissipation rate, and for this it is necessary that $\mathscr{R}f < 1$. In reality, the existence of diffusion terms somehow modifies the result and in practice the condition is $\mathscr{R}f < \mathscr{R}f_{critical}$.

If we use a turbulence eddy viscosity hypothesis, then in the thin shear layer comes the equation:

$$\frac{dk}{dt} = \nu_t\left(\frac{\partial\overline{U}}{\partial y}\right)^2 - \beta g\frac{\nu_t}{\mathrm{Pr}_t}\frac{\partial\overline{\Gamma}}{\partial y} - \varepsilon,$$

or $$\frac{dk}{dt} = \nu_t\left(\frac{\partial\overline{U}}{\partial y}\right)^2(1-\mathscr{R}i.\mathrm{Pr}_t^{-1}) - \varepsilon,$$

where $\mathscr{R}i$ is the gradient Richardson number $\mathscr{R}i = \beta g\dfrac{\partial\overline{\Gamma}/\partial y}{\left(\partial\overline{U}/\partial y\right)^2}$.

The condition $\mathscr{R}i < \mathrm{Pr}_t$ is equivalent to $\mathscr{R}f < 1$.

In its global form, the Richardson number can be related to the Archimedes number and to the Grashof number through:

$$\mathscr{R}i = \beta g\frac{\Delta\overline{\Gamma}.L}{U^2} = \frac{\mathscr{G}_r}{\mathrm{Re}^2} = \frac{\mathscr{A}_r}{\mathrm{Re}}.$$

15.2. Influence of buoyancy terms in the pressure-strain correlations

When Archimedean forces are present, the Poisson equation for the pressure fluctuations reads:

$$\left(\frac{p}{\rho}\right)_{,jj} = -2u_{k,i}\overline{U}_{i,k} + \left(R_{ik} - \overline{u_i u_k}\right)_{,ik} + \beta g_i \gamma_{,i} \; , \qquad [15.6]$$

and its solution includes an additional term:

$$\frac{p}{\rho} = \underline{\quad\quad} \Diamond \underline{\quad\quad} - \frac{1}{4\pi}\iiint_\Omega \beta g_i \gamma_{,i} \frac{d\mathcal{V}}{r} \; . \qquad [15.7]$$

Proceeding in the same way as in Chapter 6, we get:

$$\left(\overline{\frac{p}{\rho}u_{i,j}}\right)(\vec{x}) = \underbrace{-\frac{1}{4\pi}\iiint_\Omega \frac{\partial^3 \overline{u'_m u'_k u_i}}{\partial\xi_m \partial\xi_k \partial\xi_j}\frac{d\mathcal{V}}{r}}_{\phi_{ij}^{(1)}} \underbrace{-\frac{1}{2\pi}\iiint_\Omega \overline{U}_{m,k}\frac{\partial^2 \overline{u'_k u_i}}{\partial\xi_m \partial\xi_j}\frac{d\mathcal{V}}{r}}_{\phi_{ij}^{(2)}}$$

$$\underbrace{-\frac{1}{4\pi}\iiint_\Omega \frac{\partial^2 \overline{\gamma' u_i}}{\partial\xi_k \partial\xi_j}\beta' g_k \frac{d\mathcal{V}}{r}}_{\phi_{ij}^{(3)}} \; . \qquad [15.8]$$

As in Chapter 6, the quantities denoted with prime are considered at the current point $\vec{y}$ and the quantities denoted without prime are considered at point $\vec{x}$. In expression [15.8] a surface integral may possibly be added in order to also account for wall effects $\phi_{ij}^{(s)}$. The contributions $\phi_{ij}^{(1)}$, $\phi_{ij}^{(2)}$ and $\phi_{ij}^{(s)}$ have been completely processed in Chapter 6, so we shall consider here only the additional term $\phi_{ij}^{(3)}$.

If βg_k is assumed to be constant in the integrand, we can then write:

$$\phi_{ij}^{(3)} = c_{lj}^i \beta g_l \quad \text{with} \quad c_{lj}^i = \frac{1}{4\pi}\iiint_\Omega \frac{\partial^2 \overline{\gamma' u_i}}{\partial\xi_l \partial\xi_j}\frac{d\mathcal{V}}{r} \; . \qquad [15.9]$$

The third order tensor c_{lj}^i satisfies some mathematical constraints:

$- \; c_{lj}^i = c_{jl}^i \quad$ symmetry property;

– $c_{lj}^j = 0$ continuity;

– $c_{ll}^j = F_{\gamma j}$ consequence of Green theorem.

This latter relation suggests looking for the expression of c_{lj}^i in the form of a linear function with respect to the turbulent fluxes. We then get:

$$c_{lj}^i = 0.4\left[\delta_{lj}F_{\gamma i} - \frac{1}{4}\left(\delta_{il}F_{\gamma j} + \delta_{ij}F_{\gamma l}\right)\right].$$ [15.10]

There is no numerical constant to adjust in the previous result and thus:

$$\Phi_{ij}^{(3)} = \phi_{ij}^{(3)} + \phi_{ji}^{(3)} = 0.3\left(\beta g_i F_{\gamma j} + \beta g_j F_{\gamma i}\right) - 0.2\beta g_l F_{\gamma l}\delta_{ij}.$$ [15.11]

The model generally retained is the following:

$$\Phi_{ij}^{(1)} = -c_1 \frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right),$$

$$\Phi_{ij}^{(2)} = -\gamma_1\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right),$$

$$\Phi_{ij}^{(3)} = -c_3\left(G_{ij} - \frac{2}{3}G\delta_{ij}\right),$$ [15.12]

with $P_{ij} = -R_{ik}\overline{U}_{j,k} - R_{jk}\overline{U}_{i,k}$, $P = -R_{ij}\overline{U}_{i,j}$, $G_{ij} = \beta\left(g_i F_{\gamma j} + g_j F_{\gamma i}\right)$,   and $c_3 \approx 0.3$ but the numerical predictions on the computer suggest $c_3 \approx 0.5$.

In the presence of a wall, it is necessary to add the "reflected contribution" according to the same technique as that given in Chapter 6.

15.3. Influence of buoyancy forces on the pressure-temperature gradient correlations

These terms read:

$$\frac{\overline{p\gamma_{,i}}}{\rho} = -\underbrace{\frac{1}{4\pi}\iiint_\Omega \frac{\partial^3 \overline{u'_l u'_m \gamma}}{\partial\xi_l\partial\xi_m\partial\xi_i}\frac{d\mathcal{V}}{r}}_{\phi_{\gamma i}^{(1)}} - \underbrace{\frac{1}{2\pi}\iiint_\Omega \overline{U}_{l,m}\frac{\partial^2 \overline{u'_m \gamma}}{\partial\xi_l\partial\xi_i}\frac{d\mathcal{V}}{r}}_{\phi_{\gamma i}^{(2)}}$$

$$-\frac{1}{4\pi}\underbrace{\iiint_{\Omega}\beta'g_l\frac{\partial^2\overline{\gamma\gamma'}}{\partial\xi_l\partial\xi_i}\frac{d\mathscr{V}}{r}}_{\phi_{\gamma i}^{(3)}},$$

[15.13]

to which may be added the surface integral $\phi_{ij}^{(s)}$.

Launder B.E., [LAU 76] proposes the approximation:

$$\phi_{\gamma i}^{(3)} = c_{\gamma 3}\beta\overline{\gamma^2}g_i,$$

[15.14]

i.e. $\phi_{\gamma i}^{(3)} = c_{\gamma 3}\beta\overline{\gamma^2}g_m\delta_{mi}$.

Assuming βg_l constant in the integral we get:

$$\frac{1}{4\pi}\iiint_{\Omega}\frac{\partial^2\overline{\gamma\gamma'}}{\partial\xi_l\partial\xi_i}\frac{d\mathscr{V}}{r} = c_{\gamma 3}\overline{\gamma^2}\delta_{il},$$

this relation implies through tensorial contraction (on i and l) $c_{\gamma 3} = 1/3$, taking into account the relation valid in homogenous turbulence:

$$\frac{1}{4\pi}\iiint_{\Omega}\frac{\partial^2\overline{\gamma\gamma'}}{\partial\xi_i\partial\xi_i}\frac{d\mathscr{V}}{r} = \overline{\gamma^2} \quad \text{(from Green theorem)}.$$

Relation [15.14] can also be written $\phi_{\gamma i}^{(3)} = c_{\gamma 3}G_{\gamma i}$ where $G_{\gamma i} = \beta\overline{\gamma^2}g_i$.

If we choose $c_{\gamma 2} = c_{\gamma 3}$ the simple model is obtained:

$$\phi_{\gamma i}^{(2)} + \phi_{\gamma i}^{(3)} = c_{\gamma 2}\left(P_{\gamma i} + G_{\gamma i}\right),$$

[15.15]

with $P_{\gamma i} = -F_{\gamma j}\overline{U}_{i,j}$ and $G_{\gamma i} = \beta g_i\overline{\gamma^2}$.

Gibson M.M. and Launder B.E., [GIB 78B] have also discussed the wall effect on the three contributions in $\phi_{\gamma i}$.

15.4. Influence of buoyancy forces on the turbulence length scales or the dissipation rate

There would be no reason to think that buoyancy forces are only acting on the turbulence energy (Rodi W., [ROD 72B]). Figure 15.1 suggests how the gravity forces may act to increase the eddy size in an unstable density stratification, the effect being reversed in a stable stratification. It is, however, not possible to obtain an exact form for this term.
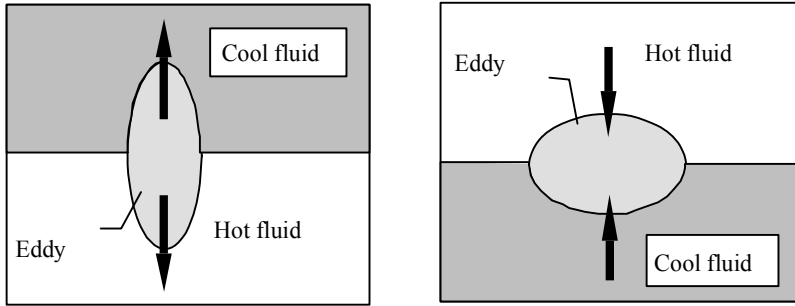
Figure 15.1. *Sketch of the influence of Archimedean forces on the turbulence length scales*

Rodi W., [ROD 80] uses the equation:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1}\frac{\varepsilon}{k}(P+G)(1+C_{\varepsilon 3}.\mathscr{R}f)+\left(\frac{\nu_t}{h_\varepsilon}\varepsilon_{,i}\right)_{,i} - C_{\varepsilon 2}\frac{\varepsilon^2}{k},$$ [15.16]

where $\mathscr{R}f = -G/P$.

The numerical applications carried on using this equation have suggested using $C_{\varepsilon 3} \approx 0$ in vertical flows and $C_{\varepsilon 3} \approx 1$ in horizontal flows.

Zeman O. and Lumley J.L., [ZEM 77] use a more complex model of the type:

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 1}\frac{\varepsilon}{k}R_{ij}\overline{U}_{i,j} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + C_\varepsilon\left(\frac{k.R_{ij}}{\varepsilon}\varepsilon_{,i}\right)_{,j}$$

$$-C_{\varepsilon 3}\frac{\varepsilon^2}{k}a_{ij}a_{ij} - C_{\varepsilon 4}\frac{\varepsilon}{k}\beta g_i F_{\gamma i}$$ [15.17]

The application of more recent models based on non-linear closures (cf. Chapter 7) has led to complicated formulations (cf. for example: Ristorcelli J.R. *et al.*, [RIS 93]).

15.5. Two-dimensional horizontal flows in the presence of buoyancy forces

Let $g_i = \begin{vmatrix} 0 \\ 0 \\ -g \end{vmatrix}$ denote the acceleration of gravity.

We shall use the abridged notation $\mathscr{T} \equiv \dfrac{d}{dt} - Diff$.

In such conditions, the Reynolds stress and the turbulent heat flux transport equations are written:

$$\begin{cases} \mathscr{T}(R_{11}) = -2R_{13}\overline{U}_{1,3} + \Phi_{11} - \dfrac{2}{3}\varepsilon \\[2mm] \mathscr{T}(R_{22}) = \qquad\quad + \Phi_{22} - \dfrac{2}{3}\varepsilon \\[2mm] \mathscr{T}(R_{33}) = -2\beta g F_{\gamma 3} + \Phi_{33} - \dfrac{2}{3}\varepsilon \\[2mm] \mathscr{T}(R_{13}) = -R_{33}\overline{U}_{1,3} - \beta g F_{\gamma 1} + \Phi_{13} \end{cases} \quad \begin{cases} \mathscr{T}(F_{\gamma 1}) = -R_{13}\overline{\Gamma}_{,3} - F_{\gamma 3}\overline{U}_{1,3} + \Phi_{\gamma 1} \\[2mm] \mathscr{T}(F_{\gamma 3}) = -R_{33}\overline{\Gamma}_{,3} - \beta g \overline{\gamma^2} + \Phi_{\gamma 3} \\[2mm] \mathscr{T}(\overline{\gamma^2}) = -2F_{\gamma 3}\overline{\Gamma}_{,3} - \varepsilon_\gamma \end{cases}$$

The introduction of the flux Richardson number $\mathscr{R}f = -\dfrac{G}{P} = -\dfrac{\beta g F_{\gamma 3}}{R_{13}\overline{U}_{1,3}}$ allows us to write:

$$P + G = -R_{13}\overline{U}_{1,3}\left(1 - \mathscr{R}f\right),$$

and if we consider an equilibrium homogenous flow $\mathscr{T} \equiv 0$ and $P + G = \varepsilon$, it then results:

$$P = \frac{\varepsilon}{1 - \mathscr{R}f}, \qquad G = \frac{-\mathscr{R}f.\varepsilon}{1 - \mathscr{R}f} .$$

Taking into account approximations [15.12] we then get the relations:

$$\frac{R_{11}-\frac{2}{3}k}{k} = \frac{4}{3}\frac{1-\gamma_1}{c_1} + \frac{1}{c_1}\left[\frac{4}{3}(1-\gamma_1)+\frac{2}{3}(1-c_3)\right]\frac{\mathscr{R}f}{1-\mathscr{R}f},$$

$$\frac{R_{22}-\frac{2}{3}k}{k} = -\frac{2}{3}\frac{1-\gamma_1}{c_1} + \frac{1}{c_1}\left[-\frac{2}{3}(1-\gamma_1)+\frac{2}{3}(1-c_3)\right]\frac{\mathscr{R}f}{1-\mathscr{R}f},$$

$$\frac{R_{33}-\frac{2}{3}k}{k} = -\frac{2}{3}\frac{1-\gamma_1}{c_1} - \frac{1}{c_1}\left[\frac{2}{3}(1-\gamma_1)+\frac{4}{3}(1-c_3)\right]\frac{\mathscr{R}f}{1-\mathscr{R}f}. \qquad [15.18]$$

Gibson and Launder obtained a good agreement with the experimental results of Webster C.A.G., [WEB 64] by choosing $c_1 = 2.2$ and $\gamma_1 = c_3 = 0.55$. Thus:

$$\frac{R_{11}-\frac{2}{3}k}{k} = 0.94 + 0.41\frac{\mathscr{R}f}{1-\mathscr{R}f},$$

$$\frac{R_{22}-\frac{2}{3}k}{k} = 0.53,$$

$$\frac{R_{33}-\frac{2}{3}k}{k} = 0.53 - 0.41\frac{\mathscr{R}f}{1-\mathscr{R}f}, \qquad [15.19]$$

and also:

$$-R_{13} = \frac{1-\gamma_1}{c_1}\left(1-\frac{1-c_3}{1-\gamma_1}\mathscr{R}f\right)\frac{R_{33}}{k}\cdot\frac{k^2}{\varepsilon}\overline{U}_{1,3}.$$

15.6. Algebraic modeling

The Rodi hypotheses on the Reynolds stresses and the turbulent heat fluxes can be written:

$$\frac{dR_{ij}}{dt} - Diff(R_{ij}) = \frac{R_{ij}}{k}\left(\frac{dk}{dt} - Diff(k)\right) = \frac{R_{ij}}{k}(P+G-\varepsilon),$$

$$\frac{dF_{\gamma i}}{dt} - Diff\left(F_{\gamma i}\right) = \frac{F_{\gamma i}}{2k}\left(P + G - \varepsilon\right) + \frac{F_{\gamma i}}{2q}\left(P_\gamma - \varepsilon_\gamma\right).$$

Gibson and Launder assumed that the coupling between the equations for $\overline{\gamma^2}$ and for $F_{\gamma i}$ is weak and in this context made the simplifying hypothesis $P_\gamma = \varepsilon_\gamma$.

Taking into account hypotheses [15.12] and [15.15] the resulting algebraic relations can be deduced:

$$R_{ij} = \frac{2}{3}k\delta_{ij} + k\frac{(1-\gamma_1)\dfrac{P_{ij} - \dfrac{2}{3}P\delta_{ij}}{\varepsilon} + (1-c_3)\dfrac{G_{ij} - \dfrac{2}{3}G\delta_{ij}}{\varepsilon}}{c_1 + \dfrac{P+G}{\varepsilon} - 1}, \qquad [15.20]$$

$$F_{\gamma i} = \frac{k}{\varepsilon}\frac{R_{il}\,\overline{\Gamma}_{,l} + (1-c_{\gamma 2})\left(F_{\gamma l}\overline{U}_{i,l} + \beta g_i\overline{\gamma^2}\right)}{c_{\gamma 1} + \dfrac{1}{2}\left(\dfrac{P+G}{\varepsilon} - 1\right)}. \qquad [15.21]$$

If we assume $c_3 = \gamma_1$ it is possible to write equivalently:

$$\frac{R_{ij} - \dfrac{2}{3}k\delta_{ij}}{k} = \Lambda\frac{P_{ij} + G_{ij} - \dfrac{2}{3}\left(P+G\right)\delta_{ij}}{\varepsilon}, \quad \Lambda = \frac{1-\gamma_1}{c_1 - 1 + (P+G)/\varepsilon},$$

$$-F_{\gamma i} = \Lambda_\gamma\frac{k}{\varepsilon}R_{ij}\,\overline{\Gamma}_{,j} - \Lambda'_\gamma\frac{k}{\varepsilon}(P_{\gamma i} + G_{\gamma i}), \quad \Lambda_\gamma = \left[c_{\gamma 1} + \frac{1}{2}\left(\frac{P+G}{\varepsilon} - 1\right)\right]^{-1},$$

$$\Lambda'_\gamma = \left(1-c_{\gamma 2}\right)\Lambda_\gamma.$$

In the particular case of horizontal flows:

$$-R_{13} = \frac{k}{\varepsilon}\Lambda\left(R_{33}\overline{U}_{1,3} + \beta g F_{\gamma 1}\right),$$

$$R_{33} = \frac{2}{3}k\left(1 - \frac{P+G}{\varepsilon}\Lambda\right) - 2k\Lambda\frac{\mathscr{R}f}{1-\mathscr{R}f},$$

$$-F_{\gamma 1} = \Lambda_\gamma\frac{k}{\varepsilon}R_{13}\overline{\Gamma}_{,3} + \Lambda'_\gamma\frac{k}{\varepsilon}F_{\gamma 3}\overline{U}_{1,3},$$

$$-F_{\gamma 3} = \Lambda_\gamma \frac{k}{\varepsilon} R_{33} \overline{\Gamma}_{,3} - \Lambda'_\gamma c'_\gamma \frac{k^2}{\varepsilon^2} \beta g F_{\gamma 3} \overline{\Gamma}_{,3} ,$$

$$\mathscr{R}f = -\frac{G}{P} = \frac{-\beta g F_{\gamma 3}}{R_{13} \overline{U}_{1,3}} , \quad \overline{\gamma^2} = -c'_\gamma \frac{k}{\varepsilon} F_{\gamma j} \overline{\Gamma}_{,j} , \quad c'_\gamma \approx 1.6 ,$$

$$-F_{\gamma 3} = a \frac{k R_{33}}{\varepsilon} \overline{\Gamma}_{,3} , \quad a = \frac{\Lambda_\gamma}{1 + \Lambda'_\gamma c'_\gamma B} , \quad B = -\beta g \frac{k^2}{\varepsilon^2} \overline{\Gamma}_{,3} , \qquad [15.22]$$

$$-R_{13} = b \frac{k R_{33}}{\varepsilon} \overline{U}_{1,3} , \quad b = \frac{\Lambda}{1 + \Lambda \left(\Lambda_\gamma + \dfrac{\Lambda'_\gamma}{\mathrm{Pr}_t} \right) B} , \quad \mathrm{Pr}_t = \frac{b}{a} . \qquad [15.23]$$

In the absence of buoyancy forces the turbulence Prandtl number is thus equal to the limiting value $\mathrm{Pr}_{t0} = \dfrac{\Lambda}{\Lambda_\gamma}$.

In the presence of buoyancy forces, it will vary according to:

$$\mathrm{Pr}_t = \mathrm{Pr}_{t0} \frac{1 + \Lambda'_\gamma (c'_\gamma - \Lambda_\gamma) B}{1 + \Lambda \Lambda_\gamma B} . \qquad [15.24]$$

In addition, we also get the expression for the turbulence eddy viscosity:

$$\nu_t = b \frac{k R_{33}}{\varepsilon} = \frac{\dfrac{2}{3} \Lambda \left(1 - \dfrac{P+G}{\varepsilon} \Lambda \right) - \dfrac{2}{3} \dfrac{P+G}{\varepsilon} \Lambda^2 \dfrac{\mathscr{R}f}{1 - \mathscr{R}f}}{1 + \Lambda \left(\Lambda_\gamma + \dfrac{\Lambda'_\gamma}{\mathrm{Pr}_t} \right) B} \cdot \frac{k^2}{\varepsilon} . \qquad [15.25]$$

The explicit algebraic models can be extended to flows influenced by buoyancy forces. The model by So R. *et al.* [SO 02] is established on the basis of representation theorems forming an integrity basis and involving the tensor

$$\gamma_{ij} = \frac{G_{ij} - \dfrac{2}{3} G \delta_{ij}}{G} \quad \text{in addition to } a_{ij} \text{ with } G_{ij} = \beta \left(F_{\gamma i} g_j + F_{\gamma j} g_i \right).$$

15.7. Simplified models

15.7.1. *Mixing length models*

The most well known formulations express the mixing length according to the Richardson number.

For $\mathscr{R}i > 0$ (stable stratification):

$$\frac{\ell_m}{\ell_{m0}} = 1 - \beta_1 \mathscr{R}i \text{ (Monin-Obukhov relation), } \beta_1 = 7 .$$ [15.26]

For $\mathscr{R}i < 0$ (unstable stratification):

$$\frac{\ell_m}{\ell_{m0}} = \left(1 - \beta_2 \mathscr{R}i\right)^{-1/4} \text{ (Keyps formula), } \beta_2 = 14$$ [15.27]

$$\mathscr{R}i = -\frac{g}{\rho} \frac{\partial\rho/\partial z}{\left(\partial U/\partial z\right)^2} .$$

15.7.2. *Turbulence eddy viscosity models*

We can estimate the influence of $\mathscr{R}f$ on ν_t by considering the turbulence kinetic energy equation:

$$\frac{dk}{dt} = -R_{ij}\overline{U}_{i,j}(1 - \mathscr{R}f) - \varepsilon + Diff(k) .$$

At equilibrium, in a thin shear flow:

$$-R_{12}\overline{U}_{1,2}(1 - \mathscr{R}f) = \varepsilon = c.k^{3/2}/\ell .$$

Taking into account $R_{12} = -\nu_t\overline{U}_{1,2}$ and $\nu_t = \ell\sqrt{k}$ we find that:

$$\frac{R_{12}^2}{\nu_t}(1 - \mathscr{R}f) = c\frac{k^{3/2}}{\ell} = c\frac{\nu_t^3}{\ell_4} ,$$

this relation implies:

$$\nu_t = c^{-1/4}\ell u_* (1 - \mathscr{R}f)^{1/4}$$

i.e. $\nu_t = \nu_{t0}(1 - \mathscr{R}f)^{1/4}$. [15.28]

If $\mathscr{R}f > 1$, the formula is no longer applicable, since the relation $\varepsilon = ck^{3/2}/\ell$ in itself is no longer valid. In practice, we would rather use empirical formulae such as:

$$\nu_t = \nu_{t0}(1 - \beta_1 \mathscr{R}i)^\alpha,$$ [15.29]

$$\sigma_t = \sigma_{t0}(1 - \beta_{1\gamma} \mathscr{R}i)^{\alpha_\gamma},$$

$$\alpha = -0.5, \quad \beta_1 = 10, \quad \alpha_\gamma = -1.5, \quad \beta_{1\gamma} = 3.33,$$

$$\frac{\Pr_t}{\Pr_{t0}} = \frac{(1 + \beta_1 \mathscr{R}i)^\alpha}{(1 + \beta_{1\gamma} \mathscr{R}i)^{\alpha_\gamma}} \approx \frac{(1 + 3.33 \mathscr{R}i)^{1.5}}{(1 + 10 \mathscr{R}i)^{0.5}} \quad \text{(formula by Munk-Anderson)}$$

$\dfrac{dR_{ij}}{dt} = \text{———}\diamond\text{———} - c_3\left(G_{ij} - \dfrac{2}{3}G\delta_{ij}\right) + \beta\left(F_{\gamma i}g_j + F_{\gamma j}g_i\right)$							
$\dfrac{dk}{dt} = \text{———}\diamond\text{———} + \beta F_{\gamma i}g_i$							
$\dfrac{dF_{\gamma i}}{dt} = \text{———}\diamond\text{———} + c_{\gamma 3}G_{\gamma i} + \beta\overline{\gamma^2}g_i$							
Reference	c_1	c_2	c_3	γ_1	$c_{\gamma 1}$	$c_{\gamma 2}$	$c_{\gamma 3}$	R
Launder, Reece, Rodi, 1975	1.50	0.40		0.60				
Launder, 1975	2.20		0.55	0.55	3.20	0.50	0.50	0.80
Gibson, Launder, 1978	1.80		0.50	0.60	3.00	0.33	0.33	0.80

Table 15.1. *Recapitulative table of the basic models*

15.8. Advanced models of the new generation

We give the example of the model by Craft T.J., [CRA 91A] already mentioned in Chapter 9. We consider here its extension to the case of buoyancy forces. The development of the model is made according to the same principles. Thus, the additional term in the pressure-strain correlation is written:

$$
\begin{aligned}
\Phi_{ij}^{(3)} = &-\left(\frac{4}{10} + \frac{3II}{80}\right)\left(G_{ij} - \frac{2}{3}G\delta_{ij}\right) + \frac{1}{2}a_{ij}G \\
&+\frac{3}{20}\left(\beta g_i \frac{R_{mj}}{k} + \beta g_j \frac{R_{mi}}{k}\right)F_{\gamma m} - \frac{1}{10}\delta_{ij}\beta g_k \frac{R_{mk}}{k}F_{\gamma m} \\
&-\frac{1}{4}\beta g_k\left(\frac{R_{ki}}{k}F_{\gamma j} + \frac{R_{kj}}{k}F_{\gamma i}\right) + \frac{1}{20}\delta_{ij}\beta g_k \frac{R_{mn}R_{mk}}{k^2}F_{\gamma n} \\
&-\frac{1}{8}\beta g_k\left(\frac{R_{mj}}{k}F_{\gamma i} + \frac{R_{mi}}{k}F_{\gamma j}\right)\frac{R_{mk}}{k} + \frac{1}{8}\beta g_k\left(\frac{R_{ki}R_{mj}}{k^2} + \frac{R_{kj}R_{mi}}{k^2}\right)F_{\gamma m} \\
&-\frac{3}{40}\left(\beta g_i \frac{R_{mj}}{k} + \beta g_j \frac{R_{mi}}{k}\right)\frac{R_{mn}}{k}F_{\gamma n} + \frac{1}{4}\beta g_k \frac{R_{mk}R_{ij}}{k^2}F_{\gamma m},
\end{aligned}
\qquad [15.30]
$$

whereas the additional term in the pressure-scalar gradient is given more simply by:

$$
\Phi_{ij}^{(3)} = \frac{1}{3}\overline{\gamma^2}\beta g_i - \overline{\gamma^2}a_{im}\beta g_m,
\qquad [15.31]
$$

(cf. also Craft T.J. *et al.*, [CRA 96A] and [CRA 04]).

In the case of low turbulence levels, near a wall, the transfer models can be adapted for taking into account the effect of low turbulence Reynolds number (cf. Peeters T.W.J. and Henkes R.A.W.M., [PEE 92]; Murakami S. *et al.*, [MUR 96]; Rhee G.H. and Sung H.J., [RHE 96]).

This page intentionally left blank

Chapter 16

Notes on the Problems Posed
by the Study of Complex Flows

We consider here essentially the numerical prediction of complex turbulent flows in the sense of Bradshaw P. [BRA 75]; turbulent shear flows whose structures are modified by an additional effect which is further added to the mean strain rate $\overline{U}_{1,2}$ (for example, volumic forces, additional strain rate, interaction with another shear flow, etc.).

Some examples of the type of situation are as follows:

(1) Turbulent thin shear flow with longitudinal curvature.

(2) Asymmetric turbulent flow in a duct.

(3) Interaction of two boundary layers.

(4) Jets issuing and developing into a turbulent flow.

(5) "Internal" boundary layer.

(6) Convergent and divergent effects.

(7) Ducts with non-circular cross-section.

(8) Effects of rotation, swirling flows.

(9) Effect of buoyancy forces.

In the Reynolds stress equations, the additional production due to an additional strain rate (e) is of order $e.P/\overline{U}_{1,2}$, the value of (e) being assumed to be weak. While this modification also remains weak, the effect on the other terms does not appear explicitly but can be significant. In reality, the effect of (e) is often an order

of magnitude higher than it would be assumed from the sole consideration of the explicit additional terms in the R_{ij} equations.

The behavior of these complex flows is generally not well predicted by simple turbulence models and raises problems for the modeler. Only the second order closures can generally explain the phenomena that are experimentally revealed and provide a framework for development of appropriate modeling.

The case of flows with non-symmetric velocity profiles (wall jets, channel flow in which one wall is smooth and the other is corrugated, flows in annular passages, etc.) had exhibited a shift between the location of the maximum of the mean velocity profile and the zero of the turbulent shear stress, clearly causing failure in any model based on the eddy viscosity concept (Mathieu J., [MAT 61]). Only the turbulence stress transport models may account for the peculiarity (Hanjalic K., [HAN 74]; Gosse J. and Schiestel R., [GOS 75]; Schiestel R. and Gosse J., [SCH 74B]).

On the general point of view, the appearance of coherent structures in turbulent flows, leads to a dichotomy in the turbulent field concept in which structured elements in the form of organized large eddies on a random fluctuating background of finer scales coexist. The presence of organized structures, specific to each type of flow, can restrict the degree of universality of the traditional statistical descriptions.

16.1. Curvature effect

The curvature effect has been studied in particular by Gibson M.M., [GIB 78A, GIB 79A]; Launder B.E. *et al.*, [LAU 77]; Gibson M.M. and Rodi W., [GIB 81B]; Gibson M.M., Jones W.P. and Younis B.A., [GIB 81A]; Gibson M.M. and Younis B.A., [GIB 82]; Patankar S.V., Pratap V.S. and Spalding D.B., [PAT 75]. The curvature effect presents several similarities with the effect of buoyancy forces and can be described through a curvature "Richardson number". Townsend A.A., [TOW 80] also analyzes the curvature effect and the effect of diverging flow on the Reynolds stresses using the rapid distortion theory.

16.1.1. *Second order models*

We consider for example the boundary layer on a wall with longitudinal curvature.

The governing equations will be written in polar coordinates (r, θ, z). To get these equations, the methods presented in Chapter 3 give a very easy mean to calculate the covariant derivatives.

$$h_r = 1, \quad h_\theta = r, \quad h_z = 1, \quad H_i(j) = 0 \;\; \forall i, j \text{ except } H_\theta(r) = 1/r,$$

and thus we get for any vector V_i and for a second order tensor R_{ij} (such as the Reynolds stress tensor):

$$V_{i,j} = \frac{\partial V_i}{\partial y^j}, \quad \text{except}$$

$$V_{\theta,\theta} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}, \quad V_{r,\theta} = \frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} \quad \text{and} \quad V_{z,\theta} = \frac{1}{r}\frac{\partial V_z}{\partial \theta},$$

$$R_{ij,m} = \frac{\partial R_{ij}}{\partial y^m}, \quad \text{except}$$

$$R_{rr,\theta} = \frac{1}{r}\frac{\partial R_{rr}}{\partial \theta} - 2\frac{R_{r\theta}}{r}, \quad R_{\theta\theta,\theta} = \frac{1}{r}\frac{\partial R_{\theta\theta}}{\partial \theta} + 2\frac{R_{r\theta}}{r}, \quad R_{zz,\theta} = \frac{1}{r}\frac{\partial R_{zz}}{\partial \theta},$$

$$R_{r\theta,\theta} = \frac{1}{r}\frac{\partial R_{r\theta}}{\partial \theta} + \frac{R_{rr} - R_{\theta\theta}}{r}, \quad R_{z\theta,\theta} = \frac{1}{r}\frac{\partial R_{z\theta}}{\partial \theta} + \frac{R_{zr}}{r}, \quad R_{zr,\theta} = \frac{1}{r}\frac{\partial R_{zr}}{\partial \theta} - \frac{R_{z\theta}}{r}.$$

In the boundary layer approximation (cf. Schlichting H., [*SCH 68]), the equations for the Reynolds stresses are thus written as:

$$\frac{U_\theta}{r}\frac{\partial R_{\theta\theta}}{\partial \theta} + U_r \frac{\partial R_{\theta\theta}}{\partial r} = -2R_{\theta\theta}\frac{\partial U_\theta}{\partial r} - 2\frac{U_\theta}{r}R_{r\theta} + 2\frac{\overline{p\,\partial u_\theta}}{r\,\partial \theta} - \frac{2}{3}\varepsilon,$$

$$\frac{U_\theta}{r}\frac{\partial R_{rr}}{\partial \theta} + U_r \frac{\partial R_{rr}}{\partial r} = 4\frac{U_\theta}{r}R_{r\theta} + 2\overline{p\,\frac{\partial u_r}{\partial r}} - \frac{2}{3}\varepsilon,$$

$$\frac{U_\theta}{r}\frac{\partial R_{zz}}{\partial \theta} + U_r \frac{\partial R_{zz}}{\partial r} = 2\overline{p\,\frac{\partial u_z}{\partial z}} - \frac{2}{3}\varepsilon,$$

$$\frac{U_\theta}{r}\frac{\partial R_{r\theta}}{\partial \theta} + U_r \frac{\partial R_{r\theta}}{\partial r} = -R_{rr}\frac{\partial U_\theta}{\partial r} + 2\frac{U_\theta}{r}R_{\theta\theta} - \frac{U_\theta}{r}R_{rr} + \overline{p\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r}\right)}.$$

$$[16.1]$$

The terms involving the longitudinal acceleration $\partial U_\theta / \partial \theta$ have been neglected, the structure of turbulence is supposed to be mainly sensitive to the effect of curvature U_θ / r (Gibson M.M., [GIB 78A]).

Assuming that the radius of curvature $\mathscr{R}$ is large compared to the boundary layer thickness, we will choose variables x, y, z such that:

$$\begin{cases} y = r - \mathscr{R} \\ x = \mathscr{R}\theta \end{cases}, \qquad dX = \left(1 + \frac{y}{\mathscr{R}}\right)dx ,$$

and we shall denote: $(1,2,3) = (x,y,z)$.

Equations [16.1] can be written more simply:

$$U_1 \frac{\partial R_{11}}{\partial x} + U_2 \frac{\partial R_{11}}{\partial y} = -2R_{12}\frac{\partial U_1}{\partial y} - 2\frac{U_1}{\mathscr{R}}R_{12} + 2\overline{p\frac{\partial u_x}{\partial x}} - \frac{2}{3}\varepsilon ,$$

$$U_1 \frac{\partial R_{22}}{\partial x} + U_2 \frac{\partial R_{22}}{\partial y} = 4\frac{U_1}{\mathscr{R}}R_{12} + 2\overline{p\frac{\partial u_y}{\partial y}} - \frac{2}{3}\varepsilon ,$$

$$U_1 \frac{\partial R_{33}}{\partial x} + U_2 \frac{\partial R_{33}}{\partial y} = 2\overline{p\frac{\partial u_z}{\partial z}} - \frac{2}{3}\varepsilon ,$$

$$U_1 \frac{\partial R_{12}}{\partial x} + U_2 \frac{\partial R_{12}}{\partial y} = -R_{22}\frac{\partial U_1}{\partial y} + 2\frac{U_1}{\mathscr{R}}R_{11} - \frac{U_1}{\mathscr{R}}R_{22} + \overline{p\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)} . \quad [16.2]$$

and for kinetic energy:

$$U_1 \frac{\partial k}{\partial x} + U_2 \frac{\partial k}{\partial y} = \underbrace{-R_{12}\frac{\partial U_1}{\partial y} + \frac{U_1}{\mathscr{R}}R_{12}}_{P} - \varepsilon ,$$

$$P = -R_{12}\frac{\partial U_1}{\partial y}(1-S) \quad \text{with} \quad S = \frac{U_1/\mathscr{R}}{\partial U_1/\partial y} , \qquad [16.3]$$

S plays the role of a curvature parameter.

We also usually introduce the flux Richardson numbers and the gradient Richardson numbers defined as follows:

$$\mathscr{R}f = \frac{2S}{1+S} = \frac{2U_1/\mathscr{R}}{\partial U_1/\partial y + U_1/\mathscr{R}} , \qquad [16.4]$$

$$\mathscr{R}i = 2S(1+S) = \frac{2U_1}{\mathscr{R}^2}\frac{\partial U_1 \mathscr{R}}{\partial y} \bigg/ \left(\frac{\partial U_1}{\partial y}\right)^2 . \qquad [16.5]$$

These two Richardson numbers are equivalent when S is small.

The boundary layer of a convex wall corresponds to a stable situation, i.e. that the turbulence energy tends to decrease. On a concave wall, the curvature effect is destabilizing, the turbulence energy tends to increase (cf. Figure 16.1).

a) Stable case   b) Unstable case
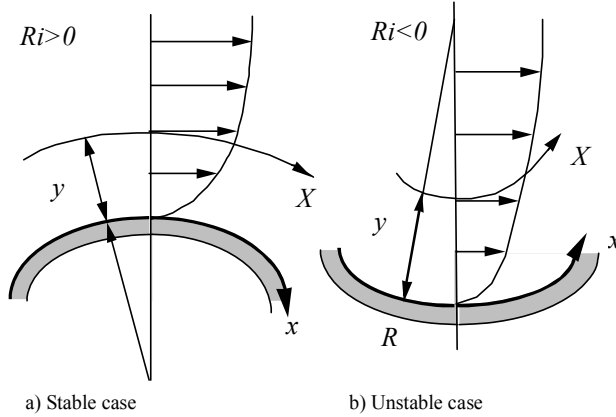
Figure 16.1. *Effect of curvature on stability*

Bradshaw P., [BRA 73] emphasizes that the longitudinal curvature effect is not of order S but at least 10 times S.

On the basis of modeled Reynolds stress equations, Gibson M.M., [GIB 78A] using the Rodi hypothesis (cf. Chapter 11), derives an algebraic modeling of the Reynolds stresses taking into account the curvature effect. He thus introduces a turbulence eddy viscosity which is a function of the curvature parameter S.

$$R_{12} = -v_t \overline{U}_{1,2}, \qquad v_t = \frac{k^2}{\varepsilon} f\left(S, \frac{P}{\varepsilon}\right). \qquad [16.6]$$

(The method is similar to the one developed in section 11.3, but starting from equations [16.2].)

16.1.2. *Simpler models*

Bradshaw in his one equation model (cf. Chapter 12) has proposed, in order to account for the curvature effect, to modify the dissipation scale involved in:

$$\varepsilon = \left(-R_{12}\right)^{3/2} / L \qquad \text{using:}$$

$$L = L_0(1-\alpha S), \qquad (S \text{ small}) \qquad\qquad [16.7]$$

$\alpha = 9$ on a concave wall, $\alpha = 14$ on a convex wall.

Launder B.E., [LAU 75A] modifies the constant $C_{\varepsilon 2}$ of the k-ε model by multiplying it by a factor:

$$1 - 0.25 \frac{k^2}{\varepsilon^2} \frac{U}{r^2} \frac{\partial(Ur)}{\partial r} = 1 - \frac{1}{8} \frac{k^2}{\varepsilon^2} \mathscr{R}i\left(\frac{\partial U}{\partial r}\right)^2, \qquad\qquad [16.8]$$

this expression can be obtained by replacing the time scale $\left(\dfrac{\partial U}{\partial r}\right)^{-1}$ by k/ε in definition [16.5].

In mixing length models, we use in general:

$$\ell = \ell_0(1-\beta S), \qquad\qquad [16.9]$$

S and $\mathscr{R}f$ are small, or $\ell = \ell_0(1+\alpha\mathscr{R}f)^{-1}$ by analogy with the buoyancy forces.

Coefficient α is of course dependent on the type of flow under consideration.

The study of the curvature effect on turbulent heat transfer can be developed using algebraic modeling of the turbulent heat fluxes (Gibson M.M., [GIB 78A]). It is thus possible to predict the variations of the turbulence Prandtl number according to curvature:

$$\mathrm{Pr}_t = \mathrm{Pr}_t(S, P/\varepsilon). \qquad\qquad [16.10]$$

16.2. Secondary motions

In ducts with non-circular cross-section the turbulence itself produces secondary motions in the cross-section of the duct (cf. Figure 16.2). Although these secondary velocities generally do not exceed 1 to 2% of the axial mean velocity, they may modify the axial velocity up to 30% in some locations, and thus influence the skin friction and the heat transfer. These secondary motions are generated by the Reynolds stresses and do not exist at all in laminar flow. Let us cite the work by Naot P., [NAO 74]; Pratap V.S. and Spalding D.B., [PRA 76] for ducts with non-

circular cross-section, that by Gessner F.B. and Emery A.F., [GES 77] for the study of the flow in a rectangular channel and Gosse J. *et al.*, [GOS 79] for the flow around tube bundles with longitudinal attack. On the contrary, the secondary recirculating motions in curved ducts do exist both in laminar regime and in turbulent regime as well (cf. Patankar S.V., Pratap V.S. and Spalding D.B., [PAT 75] and Gosse J. and Schiestel R., [GOS 78B]).
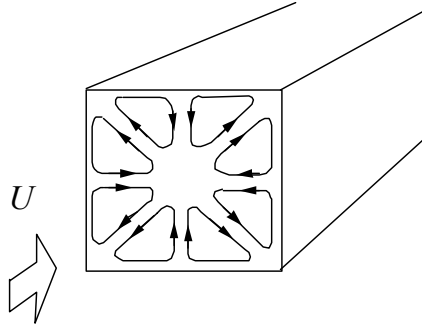
Figure 16.2. *Development of secondary motions in a square cross-section duct*

Launder B.E. and Ying W.M., [LAU 73B]; Gosman A.D. and Rapley C., [GOS 78A] relying on the Reynolds stress transport model by Launder, Reece and Rodi, together with the energy equilibrium hypothesis, proposed an algebraic modeling of the Reynolds stresses.

The components of the Reynolds stress tensor are then given by:

$$R_{13} = -c_4 \frac{k^2}{\varepsilon} \overline{U}_{3,1}, \quad R_{23} = -c_4 \frac{k^2}{\varepsilon} \overline{U}_{3,2},$$

$$R_{11} = c_3 k - c_2 c_4 \frac{k^3}{\varepsilon^2} \overline{U}_{3,1}^2, \quad R_{22} = c_3 k - c_2 c_4 \frac{k^3}{\varepsilon^2} \overline{U}_{3,2}^2,$$

$$R_{12} = -2 c_2 c_4 \frac{k^3}{\varepsilon^2} \overline{U}_{3,1} \overline{U}_{3,2}, \quad c_2 = 0.0185, \; c_3 = 0.57, \; c_4 = 0.09 \qquad [16.11]$$

We see in these expressions that the shear stress R_{12} in the plane of the cross-section is produced by longitudinal velocity gradients in perpendicular planes. The secondary flow is not obtained if we use the usual isotropic eddy viscosity hypothesis. Formulae [16.11] show that the principal axes of the strain rate tensor $\left(\overline{U}_{i,j} + \overline{U}_{j,i} \right)$ are not aligned with those of the Reynolds stress tensor R_{ij}. The model used for Φ_{ij} proved to have a great influence on the results.

The non-linear k-ε model by Speziale C.G., [SPE 87B] also allows us to recover these secondary motions in a duct with square cross-section.

16.3. Rotation effects

We can sum up the global effect of a superimposed rotation on the flow in a boundary layer (cf. Figure 16.3) in the following way:

– negative rotation produces stabilizing effect (k decreasing);

– positive rotation produces destabilizing effect (k increasing).
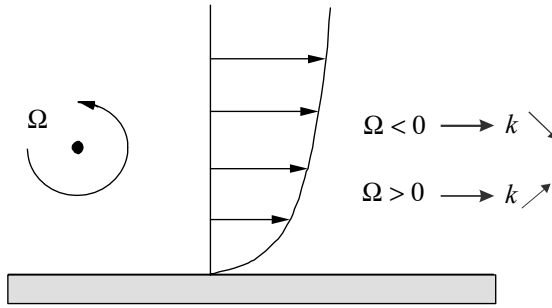
Figure 16.3. *Effect of rotation on the boundary layer on a flat plate*

It is possible to carry a stability analysis by considering a fluid particle in the boundary layer (cf. [*SWI 81], Chapter 8, Instability in geophysical fluid dynamics, D.J. Tritton and P.A. Davies).

A fluid particle located at ordinate y_2 is submitted to the action of the Coriolis force $2\rho\Omega U(y_2)$ balanced by the pressure gradient $\dfrac{\partial P}{\partial y}(y_2)$. If a fluid particle located at ordinate y_1 is moved towards ordinate y_2, its velocity will be modified according to the equation of motion:

$$\rho\frac{dU}{dt} = 2\rho\Omega V \text{ , } V \text{ being the velocity in the } Oy \text{ direction,}$$

implying $U(y_1 + dy) - U(y_1) = 2\Omega V dt = 2\Omega dy$, relation that can be compared to the equation at equilibrium $U(y_2) - U(y_1) = \dfrac{\partial U}{\partial y}dy$. It thus follows that:

$$U(y_1 + dy) - U(y_2) = \left(2\Omega - \frac{\partial U}{\partial y} \right) dy .$$

If $U(y_1 + dy) < U(y_2)$, the Coriolis force $2\rho\Omega U(y_1 + dy)$ is not sufficient to balance the pressure at point y_2, so there is instability for $2\Omega < \dfrac{\partial U}{\partial y}$ and stability for $2\Omega > \dfrac{\partial U}{\partial y}$. In the case where Ω and $\partial U / \partial y$ have opposite signs (i.e. the vorticities Ω and $-\partial U / \partial y$ have the same sign), there is always stability. To sum up, there is instability when $0 < 2\Omega < \dfrac{\partial U}{\partial y}$ or $\mathscr{R}i < 0$, or also $0 < \dfrac{\Omega}{S} < 1$.

The correction due to rotation can be analyzed on the Reynolds stress equations using a method similar to the one used for considering curvature effects. The Coriolis terms are introduced in the momentum equations. The rotation parameter is thus in this case:

$$S_\Omega = -\frac{\Omega}{\partial U_1 / \partial y} .$$ [16.12]

The corresponding Richardson number is also introduced as:

$$\mathscr{R}i = -\frac{2\Omega}{\partial U / \partial y} \left(1 - \frac{2\Omega}{\partial U / \partial y} \right).$$

As for the curvature effect, it is possible to derive an algebraic modeling from the Reynolds stress equations, and thus to bring out the influence of rotation on the eddy viscosity:

$$\nu_t = \frac{k^2}{\varepsilon} f(S_\Omega) .$$

The study of swirling flows requires the use of non-isotropic viscosity coefficients or the calculation of individual Reynolds stress components through their transport equations (Koosinlin M.L. and Lockwood F.C., [KOO 74]; Sharma R.N. and Patankar S.V., [SHA 82]).

The influence of the rotation of the frame of reference on a turbulent flow is apparent at two levels. First, there are the effects of the Coriolis terms which appear in the right-hand side of the Reynolds stress transport equations; these are explicit effects. Next, there are the direct more subtle effects of the rotation on the structure of the turbulent field itself; these are implicit effects. The Reynolds stress transport equations and the extension of the LRR model to the case of rotating frame of reference have been studied by Cousteix J. and Aupoix B., [COU 81B] (cf. section 16.5). They show the explicit effect of the Coriolis terms in the production rate and the redistribution terms. Moreover, the experiments in rotating homogenous turbulence and the numerical simulations show that rotation inhibits the spectral transfers, it is among one of the first implicit effects to be recognized. In order to take into account the effect in modeling, the numerical coefficients in the ε equation are usually sensitized to the turbulence Rossby number $Ro = \varepsilon / \Omega k$. For example, the modification proposed by Aupoix B., [AUP 87] bears on coefficient $C_{\varepsilon 2}$ (cf. section 16.7).

16.4. Examples of complex turbulent flows for which the traditional one point closures fail

We shall mention several known problems, some of which still remain open:
– Grid turbulence submitted to strong acceleration through a duct contraction.
– Circular jet issuing in a still atmosphere or a moving atmosphere.
– Swirling jets.
– Turbulent boundary layer in adverse pressure gradient.
– Non-symmetric turbulent flows.
– Separated flows, recirculating zones.
– Flows in non-circular ducts.
– Flows in a channel with free surface.
– Boundary layer submitted to a sudden transverse shear.
– Turbulence Prandtl number in various free flows.

Let us also mention the work by HaMinh H. and Chassaing P., [HAM 78] on the study of separated regions with recirculation. Daly B.J., [DAL 74] has studied the convective cells in free convection. Three-dimensional boundary layers also bring much complexity (cf. Bradshaw P., [BRA 71]; Nash J.F., [NAS 69], Cousteix J., Quemard C. and Michel R., [COU 72C]; Rotta J.C., [ROT 77, 79]; Cousteix, J., [COU 81A]). The turbulent flow in ducts with undulated wavy wall (Chauve M.P. and Schiestel R., [CHA 81]; Schiestel R. and Chauve M-P., [SCH 82]) relates both

to the curvature effects and the effect of change in the cross-section (convergent-divergent effects).

Still nowadays, the second order one point closures only reach a modest degree of universality. The main weaknesses, that appear in the numerical prediction of complex flows, seem to be due to the approximation of the pressure-strain correlations and the method of obtaining the dissipation rate, in particular.

It is also necessary to carry out further adequate experiments in order to facilitate the development or the test of closure hypotheses.

We will find in Rodi W., [ROD 80, 82] numerous examples of applications of one point models for the prediction of turbulent flows in real situations.

16.5. More on the Navier-Stokes equations in a relative frame of reference

The Navier-Stokes equations in a rotating frame of reference can be written:

$$\rho \frac{dU_i}{dt} = -P_{,i} + \rho \left(\frac{\Omega^2 r^2}{2} \right)_{,i} - 2\rho \varepsilon_{ipq} \omega_p U_q + \mu U_{i,jj} + \rho g_i \,,$$

with $\Omega = |\vec{\omega}|$ constant, and where ε_{ipq} stands for the alternate pseudo-tensor.

A modified pressure can be defined as $P^* = P - \rho \dfrac{\Omega^2 r^2}{2}$.

It can be shown easily that if r is the distance of point $\vec{x}$ to the rotation axis, we get:

$$r^2 = x_j x_j - \frac{\left(x_j \omega_j \right)^2}{\Omega^2} \,,$$

and $\left(\dfrac{\Omega^2 r^2}{2} \right)_{,j} = \Omega^2 x_j - \omega_j \left(x_i \omega_i \right)$.

The fluctuating velocity equations in the relative frame of reference will allow the statistical treatment.

$$\partial_t u_i + \overline{U}_j u_{i,j} = -u_j \overline{U}_{i,j} - \left(u_i u_j - R_{ij}\right)_{,j} - (p/\rho)_{,i} - 2\varepsilon_{ijk}\Omega_j u_k + \nu u_{i,jj} ,$$

$$u_{j,j} = 0 .$$

Compared to the equations in a fixed frame of reference, the transport equations for R_{ij}, $F_{\gamma i}$, k and ε are modified according to:

$$\frac{dR_{ij}}{dt} = \underbrace{\text{———}\diamond\text{—}\boxed{+\Phi_{ij}^{(R)}}\text{—}\diamond\text{———}}_{(*)} + \Omega_{ij} , \qquad [16.13]$$

(*) additional redistribution term.

$$\Omega_{ij} = -2\varepsilon_{i\alpha\beta}\omega_\alpha R_{\beta j} - 2\varepsilon_{j\alpha\beta}\omega_\alpha R_{\beta i} ,$$

$$\frac{dk}{dt} = \text{———}\diamond\text{———} \quad \text{(no explicit modification as } \Omega_{jj} = 0 \text{)},$$

$$\frac{dF_{\gamma i}}{dt} = \text{———}\diamond\text{—}\boxed{+\Phi_{\gamma i}^{(R)}}\text{—}\diamond\text{———} + \Omega_{\gamma i} , \qquad [16.14]$$

$$\Omega_{\gamma i} = -2\varepsilon_{i\alpha\beta}\omega_\alpha F_{\gamma\beta} ,$$

$$\frac{d\varepsilon}{dt} = \text{———}\diamond\text{———} - 4\nu\varepsilon_{mpq}\omega_p\varepsilon_{mq} , \qquad [16.15]$$

where ε_{mq} is the dissipation rate tensor of the Reynolds stresses, there is no explicit modification at high Reynolds numbers Re_t.

Cousteix J. and Aupoix B., [COU 81B] have shown that the Launder, Reece and Rodi (LRR) model is invariant through non-Galilean transformation. It then results in a natural modeling for terms such as $\Phi_{ij}^{(R)}$.

The fluctuating pressure equation reads:

$$\left(\frac{p}{\rho}\right)_{,jj} = \text{———}\diamond\text{———} - 2\varepsilon_{kij}\omega_i u_{j,k} ,$$

and its formal solution:

$$
\frac{p}{\rho}(\vec{x}) = \underline{\qquad} \Diamond \underline{\qquad} + \frac{1}{4\pi} \iiint_{\Omega} \varepsilon_{kij} \omega_i u'_{j,k} \frac{d\mathscr{V}}{r} .
$$

We see that it suffices to replace in $\Phi_{ij}^{(2)}$ and in $\Phi_{\gamma i}^{(2)}$ (rapid terms), $\overline{U}_{i,j}$ by $\varepsilon_{ikj}\omega_k$ to get the expression of $\Phi_{ij}^{(R)}$ and that of $\Phi_{\gamma i}^{(R)}$. To obtain the complete term, we shall thus replace $\overline{U}_{i,j}$ by $\overline{U}^{*}_{i,j} = \overline{U}_{i,j} + \varepsilon_{ikj}\omega_k$. We shall note however that the Poisson equation of pressure deduced from the Navier-Stokes equations in a rotating frame of reference transforms differently:

$$
\left(\frac{P^{*}}{\rho} \right)_{,jj} = -\left(U_{i,j} + 2\varepsilon_{ikj}\omega_k \right) U_{j,i} .
$$

The previous analysis thus leads to the approximation:

$$
\Phi_{ij}^{(2)} + \Phi_{ij}^{(R)} = -c_2 \left(P_{ij} + \frac{1}{2}\Omega_{ij} - \frac{2}{3}P\delta_{ij} \right) .
$$

More precisely, the equations for the Reynolds stresses and the turbulent fluxes of a scalar can be written:

$$
\frac{dR_{ij}}{dt} = P_{ij} + \Omega_{ij} - c_1 \frac{\varepsilon}{k}\left(R_{ij} - \frac{2}{3}k\delta_{ij} \right) - \frac{2}{3}\varepsilon\delta_{ij} + Diff(R_{ij}) - c_2 \left(\tilde{P}_{ij} - \frac{2}{3}P\delta_{ij} \right), \quad [16.16]
$$

$\tilde{P}_{ij}$ denotes the expression for P_{ij} in which $\overline{U}_{i,j}$ is replaced by $\overline{U}^{*}_{i,j} = \overline{U}_{i,j} + \varepsilon_{ikj}\omega_k$, i.e. $\tilde{P}_{ij} = P_{ij} + \frac{1}{2}\Omega_{ij}$. Let us note that $\tilde{P} = P$.

$$
\frac{dF_{\gamma i}}{dt} = P_{\gamma i}^{U} + P_{\gamma i}^{\Gamma} + \Omega_{\gamma i} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma i} + Diff(F_{\gamma i}) - c_2 \tilde{P}_{\gamma i}^{U}, \qquad [16.17]
$$

where $\tilde{P}_{\gamma i}^{U}$ denotes the expression for $P_{\gamma i}^{U}$ in which $\overline{U}_{i,j}$ is replaced by $\overline{U}_{i,j} + \varepsilon_{ikj}\omega_k$, i.e. $\tilde{P}_{\gamma i}^{U} = P_{\gamma i}^{U} + \frac{1}{2}\Omega_{\gamma i}$.

Other types of modeling are possible, but they must involve only objective tensors (cf. section 4.7 for the concept of objectivity). Thus, $\widetilde{P}_{ij} = P_{ij} + \frac{1}{2}\Omega_{ij}$ is an objective tensor. It can be shown (Fu S., Launder B.E. and Leschziner M.A., [FU 87A]) that any combination of the type $(1+\alpha)P_{ij} + (1+\beta)\Omega_{ij} - (1-\alpha+2\beta)D_{ij}$ where $D_{ij} = -R_{im}\overline{U}_{m,j} - R_{jm}\overline{U}_{m,i}$ (definition given in Chapter 6) is objective. In particular, the following tensors are objective:

$$P_{ij} + D_{ij}, \quad P_{ij} + \frac{1}{2}\Omega_{ij}, \quad D_{ij} - \frac{1}{2}\Omega_{ij}, \quad P_{ij} - D_{ij} + \Omega_{ij}.$$

Moreover, the quantity $d_t R_{ij} - \frac{1}{2}\Omega_{ij}$ is objective whereas the material derivative $d_t R_{ij}$ is not. In the particular case of a steady flow (in the relative frame of reference) $d_t R_{ij}$ can be replaced by the convection tensor $C_{ij} = \overline{U}_m R_{ij,m}$. Other combinations again yield objective tensors. Thus, $d_t R_{ij} + P_{ij}$ is nothing more than the Oldroyd derivative $\overset{o}{\frac{D}{Dt}} R_{ij}$ (cf. Chapter 4), and $d_t R_{ij} + \frac{1}{2}(P_{ij} - D_{ij}) = \overset{J}{\frac{D}{Dt}} R_{ij}$ is the Jaumann derivative. In steady flow, the quantities $C_{ij} + P_{ij}$, $2C_{ij} + P_{ij} - D_{ij}$ are objectives.

Fu *et al.* [FU 87A] propose, however, another approach by taking:

$$\Phi_{ij}^{(R)} = -c_2(\Omega_{ij} - C_{ij} - \frac{2}{3}C\delta_{ij}), \qquad [16.18]$$

where C_{ij} is the convection term in the Reynolds stress equations.

Thus,

$$\Phi_{ij}^{(2)} + \Phi_{ij}^{(R)} = -c_2\left[P_{ij} + \Omega_{ij} - C_{ij} - \frac{2}{3}(C + P)\delta_{ij} \right].$$

The tensor $P_{ij} + \Omega_{ij} - C_{ij}$ is indeed objective, exactly like $d_t R_{ij} - P_{ij} - \Omega_{ij}$.

This latter approach is better suited to the prediction of swirling turbulent jets, a calculation considered by the authors. In an equilibrium situation, $C_{ij} = 0$ and we recover the usual approximation of isotropization of production (IP). In usual free

flows (without rotation) the term $\left(C_{ij} - \dfrac{2}{3} C \delta_{ij} \right)$ is generally weak and the numerical predictions are only slightly modified by this term.

16.6. Algebraic modeling of turbulence submitted to rotation

16.6.1. *Objective algebraic models*

Equation [16.6] can be written:

$$\partial_t R_{ij} + C_{ij} = P_{ij} + \Omega_{ij} + \Phi_{ij} - \varepsilon_{ij} + D_{ij} \,,$$

with $C_{ij} = \overline{U}_m R_{ij,m}$, $\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(R)} + \Phi_{ij}^{(s)}$, $D_{ij} = c_s \left(\dfrac{k}{\varepsilon} R_{kl} R_{ij,l} \right)_{,k}$,

$$\Phi_{ij}^{(1)} = -c_1 \frac{\varepsilon}{k} \left(R_{ij} - \frac{2}{3} k \delta_{ij} \right), \qquad \Phi_{ij}^{(2)} + \Phi_{ij}^{(R)} = -c_2 \left(\tilde{P}_{ij} - \frac{2}{3} \tilde{P} \delta_{ij} \right),$$

and $\Phi_{ij}^{(s)}$ is given by equation [6.43] in which P_{ij} is replaced by $\tilde{P}_{ij}$.

The hypothesis [LAU 82] (cf. section 11.3) is then revised in order to involve only objective tensors:

$$C_{ij} - \frac{1}{2} \Omega_{ij} = C \left[(1+\alpha) \frac{R_{ij}}{k} - \frac{2}{3} \alpha \delta_{ij} \right], \quad \text{with} \quad \Omega_{jj} = 0 \,, \ C = \frac{1}{2} C_{jj} \,,$$

$$D_{ij} = D \left[(1+\beta) \frac{R_{ij}}{k} - \frac{2}{3} \beta \delta_{ij} \right], \ D = \frac{1}{2} D_{jj} \,.$$

This approach allows us to satisfy the conditions of section 4.7 which shows that a behavior law for the Reynolds stresses can be a function of rotation only through the intrinsic rotation tensor.

These hypotheses lead to the algebraic model:

$$\frac{R_{ij} - \dfrac{2}{3} k \delta_{ij}}{k} = \frac{1-c_2}{r_1} \frac{\tilde{P}_{ij} - \dfrac{2}{3} P \delta_{ij}}{\varepsilon} + \frac{\Phi_{ij}^{(s)}}{r_1 \varepsilon} \,,$$

$$r_1 = c_1 + (1+\alpha)\left(\frac{P}{\varepsilon} - 1\right) + (\alpha + \beta)\frac{D}{\varepsilon} \quad (c_1 = 1.8, \; c_2 = 0.6, \; c_s = 0.22).$$

This is thus an implicit model for the Reynolds stresses in a non-inertial frame of reference.

Similarly, the Rodi hypothesis can be extended to the case of a rotating frame of reference by using:

$$\frac{dR_{ij}}{dt} - \frac{1}{2}\Omega_{ij} - Diff(R_{ij}) = \frac{R_{ij}}{k}\left(\frac{dk}{dt} - Diff(k)\right).$$

All the models obtained using this approach are dependent on rotation only through the intrinsic rotation tensor.

The same approach can be developed for explicit models and in particular for non-linear models. The realizable k-ε model presented in section 11.2.4, which was formulated in a relative frame of reference, is also an example. The non-linear models in section 11.4.1 can be reformulated to the case of a rotating frame of reference by replacing every occurrence of the rotation tensor $\omega_{ij} = \frac{1}{2}(\overline{U}_{i,j} - \overline{U}_{j,i})$ by the intrinsic rotation tensor $\omega_{ij}^{(absol)} = \varepsilon_{ikj}\Omega_k + \omega_{ij}$.

16.6.2. *Equilibrium algebraic models*

Equilibrium algebraic models consider another point of view ([DUR 99]). The modeled equations for the Reynolds stresses are written in a relative frame. The model by Speziale C.G., Sarkar S. and Gatski T.B., [SPE 91B] (cf. Chapter 7), is used in its linear form (the quadratic term for the slow return to isotropy is neglected):

$$\frac{dR_{ij}}{dt} = -\omega^{(ref)}{}_{im}R_{mj} - \omega^{(ref)}{}_{jm}R_{mi} - \omega^{(abs)}{}_{im}R_{mj} - \omega^{(abs)}{}_{jm}R_{mi}$$

$$-R_{im}S_{jm} - R_{jm}S_{im} - c_1\varepsilon a_{ij} + c_2 k S_{ij} + c_3 k\left(a_{im}S_{jm} + a_{jm}S_{im} - \frac{2}{3}a_{lm}S_{lm}\delta_{ij}\right)$$

$$+c_4 k\left(a_{im}\omega^{(abs)}{}_{jm} + a_{jm}\omega^{(abs)}{}_{im}\right) - \frac{2}{3}\varepsilon\delta_{ij}.$$

This form of equation shows, on the one hand the rotation of the frame of reference $\omega^{(ref)}{}_{ij}$ appearing in the Coriolis terms and on the other hand the intrinsic rotation $\omega^{(abs)}{}_{ij}$ appearing in the modeled terms, the behavior laws involving only objective tensors.

$$-\omega^{(ref)}{}_{im}R_{mj} - \omega^{(ref)}{}_{jm}R_{mi} = \frac{1}{2}\Omega_{ij},$$

$$-\omega^{(abs)}{}_{im}R_{mj} - \omega^{(abs)}{}_{jm}R_{mi} - R_{im}S_{jm} - R_{jm}S_{im} = \tilde{P}_{ij} = P_{ij} + \frac{1}{2}\Omega_{ij}.$$

The equilibrium hypothesis for the anisotropy tensor equation $\dfrac{da_{ij}}{dt} = 0$ in homogenous turbulence, equivalent to $\dfrac{dR_{ij}}{dt} = (P - \varepsilon)\dfrac{R_{ij}}{k}$, leads to:

$$\left(1 - \frac{c_1}{2} - \frac{P}{\varepsilon}\right)a_{ij} + \frac{1}{2}\left(c_2 - \frac{4}{3}\right)\frac{k}{\varepsilon}S_{ij} + \frac{1}{2}(c_4 - 2)\frac{k}{\varepsilon}\left(\omega^{(abs)}{}_{im}a_{mj} + \omega^{(abs)}{}_{jm}a_{mi}\right)$$

$$-\frac{k}{\varepsilon}\left(\omega^{(ref)}{}_{im}a_{mj} + \omega^{(ref)}{}_{jm}a_{mi}\right)$$

$$+\frac{1}{2}(c_3 - 2)\frac{k}{\varepsilon}\left(S_{im}a_{mj} + S_{jm}a_{mi} - \frac{2}{3}S_{ml}a_{ml}\delta_{ij}\right) = 0.$$

In order to recover the form given in section 11.5 similar to [11.26], we make the following changes to unknown functions:

$$a^*{}_{ij} = \frac{1}{2}(2 - c_3)a_{ij}, \quad S^*{}_{ij} = \frac{1}{2}\frac{k}{\varepsilon}(2 - c_3)S_{ij},$$

$$W^*{}_{ij} = \frac{k}{\varepsilon}\left(\omega^{(ref)}{}_{ij} - \frac{1}{2}(c_4 - 2)\omega^{(abs)}{}_{ij}\right),$$

which yields:

$$\left(1 - \frac{c_1}{2} - \frac{P}{\varepsilon}\right)a^*{}_{ij} + \frac{1}{2}\left(c_2 - \frac{4}{3}\right)S^*{}_{ij} - \left(W^*{}_{im}a^*{}_{mj} + W^*{}_{jm}a^*{}_{mi}\right)$$

$$-\left(S^*{}_{im}a^*{}_{mj} + S^*{}_{jm}a^*{}_{mi} - \frac{2}{3}S^*{}_{ml}a^*{}_{ml}\delta_{ij}\right) = 0,$$

with $-2a^*{}_{ij}S^*{}_{ij} = \frac{1}{4}(2 - c_3)^2\frac{P}{\varepsilon}, \qquad P = -2ka_{ij}S_{ij}.$

The explicit algebraic model in two-dimensional flows (cf. section 11.5) is thus given by:

$$a_{ij}^* = C_1 S_{ij}^* + b(W_{im}^* S_{mj}^* - S_{im}^* W_{mj}^*) + c(S_{im}^* S_{mj}^* - \frac{1}{3} S_{ml}^* S_{ml}^* \delta_{ij}) \, ,$$

with:

$$C_1 = \frac{\frac{1}{2}(c_2 - \frac{4}{3})\gamma}{1 - \frac{2}{3}\gamma^2 \left| S^{*2} \right| + 2\gamma^2 \left| W^{*2} \right|} \, , \quad \gamma = -\frac{1}{1 - \frac{c_1}{2} - \frac{P}{\varepsilon}} \, , \quad b = \gamma C_1, \quad c = -2\gamma C_1,$$

or $\quad C_1 = -\dfrac{3\alpha_1}{3 - 2\eta^2 + 6\xi^2} \quad$ with $\quad \alpha_1 = \dfrac{1}{2}(\dfrac{4}{3} - c_2)\gamma, \quad \eta = \gamma \sqrt{\left| S^{*2} \right|} = \gamma \sqrt{II^{S^*}} \quad$ and

$$\xi = \gamma \sqrt{\left| W^{*2} \right|} = \gamma \sqrt{II^{W^*}} \, .$$

We will note that this model involves not only the intrinsic rotation but also the rotation of the frame of reference in the expression for W_{ij}^*. Consequently, the model obtained in this way does not satisfy the conditions given in section 4.7 and thus cannot be considered as a general behavior law. It has to be considered rather as an equilibrium model that faithfully mirrors the properties of the second order closure from which it derives, but that is valid only in a particular relative frame of reference. These models proved to be particularly useful in the stability studies of turbulent flows.

The two-dimensionality brings some additional simplifying relations, deduced from the two-dimensional Cayley-Hamilton theorem:

$$\left(S_{ij} \right)^2 = -J^S{}_2 \delta_{ij} = \frac{1}{2} \left(S^2 \right)_{mm} \delta_{ij} \quad \text{which implies} \quad S_{im}^* S_{mj}^* S_{ij}^* = 0 \, ,$$

$$\left(W_{ij} \right)^2 = -J^W{}_2 \delta_{ij} = \frac{1}{2} \left(W^2 \right)_{mm} \delta_{ij} = -\frac{1}{2} \left| W^2 \right| \delta_{ij} \, .$$

We also have $W_{im}^* S_{mj}^* - W_{jm}^* S_{mi}^* = 0$.

The contracted multiplication of the explicit model by S_{ij}^*, taking into account the vanishing of the power 3 of the tensor S_{ij}^* and of the relation $W_{im}^* S_{mj}^* S_{ij}^* = 0$ coming from the antisymmetry of W_{ij}^*, we deduce $a_{ij}^* S_{ij}^* = C_1 S_{ij}^* S_{ij}^*$.

This relation can also be written in the form:

$$(2-c_3)^2 \frac{P}{\varepsilon} = \frac{4(\frac{4}{3}-c_2)\gamma S_{ij}^* S_{ij}^*}{1-\frac{2}{3}\gamma^2 |S^{*2}| + 2\gamma^2 |W^{*2}|},$$

or $\quad \dfrac{1}{\gamma^2 S^{*2}} = \dfrac{2}{3} - 2\dfrac{|W^{*2}|}{S^{*2}} + \dfrac{4(\frac{4}{3}-c_2)}{(2-c_3)^2\,\gamma\,\dfrac{P}{\varepsilon}},$

which   express $\quad \dfrac{1}{\gamma^2 S^{*2}} = \dfrac{2}{3} - 2\dfrac{|W^{*2}|}{S^{*2}} + \dfrac{4(\frac{4}{3}-c_2)}{(2-c_3)^2\,\gamma\,\dfrac{P}{\varepsilon}} \quad$ according   to

$$\left|\frac{W^{*2}}{S^{*2}}\right| = \left(\frac{2}{2-c_3}\right)^2 \left(\frac{II^{\omega ref}}{II^S} - \frac{1}{2}(c_4-2)\frac{II^{\omega abs}}{II^S}\right).$$

In the case of a plane flow submitted to a rotation with perpendicular axis, this relation can then be considered as an expression for $\left(\dfrac{\varepsilon}{kS}\right)^2$ according to

$$\left[\frac{\Omega^{(ref)}}{S} - \frac{1}{2}(c_4-2)\frac{\Omega^{(abs)}}{S}\right]:$$

$$\frac{\left(\dfrac{\varepsilon}{kS}\right)^2}{\gamma^2\left[\dfrac{1}{2}(2-c_3)\right]^2} = \frac{2}{3} + \frac{4(\frac{4}{3}-c_2)}{(2-c_3)^2\,\gamma\,\dfrac{P}{\varepsilon}} - 2\frac{\left[\dfrac{\Omega^{(ref)}}{S} - \dfrac{1}{2}(c_4-2)\dfrac{\Omega^{(abs)}}{S}\right]^2}{\left[\dfrac{1}{2}(2-c_3)\right]^2},$$

or, as well
$$\frac{\left(\dfrac{\varepsilon}{kS}\right)^2}{\gamma^2\left[\dfrac{1}{2}(2-c_3)\right]^2}=\frac{2}{3}+\frac{4(\dfrac{4}{3}-c_2)}{(2-c_3)^2\,\gamma\,\dfrac{P}{\varepsilon}}-\frac{\dfrac{1}{2}(c_4-2)^2}{\left[\dfrac{1}{2}(2-c_3)\right]^2}\,\mathcal{R}^2,$$

with $\mathcal{R}=\dfrac{2}{(c_4-2)}\,\dfrac{\Omega^{(ref)}}{S}-\dfrac{\Omega^{(abs)}}{S}$.

The equation for $\left(\dfrac{\varepsilon}{k}\right)$ in homogenous flow:

$$\frac{d}{dt}\left(\frac{\varepsilon}{k}\right)=\left(\frac{\varepsilon}{k}\right)^2\left[(C_{\varepsilon1}-1)\frac{P}{\varepsilon}-(C_{\varepsilon2}-1)\right],$$

coupled with $\dfrac{dk}{dt}=\varepsilon\left(\dfrac{P}{\varepsilon}-1\right)$ shows that two equilibrium solutions exist,

$\dfrac{P}{\varepsilon}=\dfrac{C_{\varepsilon2}-1}{C_{\varepsilon1}-1}$ and $\left(\dfrac{\varepsilon}{kS}\right)_\infty=0$. In the first case, the kinetic energy and the

dissipation rate are increasing exponentially according to:

$$k\propto e^{\lambda t},\quad \varepsilon\propto e^{\lambda t}\quad\text{with}\quad \lambda=\frac{C_{\varepsilon2}-C_{\varepsilon1}}{C_{\varepsilon1}-1}\left(\frac{\varepsilon}{k}\right)_\infty.$$

The relation $\left(\dfrac{\varepsilon}{kS}\right)^2=f(\mathcal{R})$ corresponds to half an ellipse (Figure 16.4) on the interval $\in(A,B)$.

In the second case, the solution for k and ε follows a power law and the condition $\left(\dfrac{\varepsilon}{kS}\right)_\infty=0$ reported in the relation $\left(\dfrac{\varepsilon}{kS}\right)^2=f(\mathcal{R})$, leads to the relation:

$$\frac{\left(1-\dfrac{c_1}{2}-\dfrac{P}{\varepsilon}\right)}{P/\varepsilon}=\frac{(2-c_3)^2}{17(\dfrac{4}{3}-c_2)}-\frac{2}{(\dfrac{4}{3}-c_2)}\left[\frac{\Omega^{(ref)}}{S}-\frac{1}{2}(c_4-2)\frac{\Omega^{(abs)}}{S}\right]^2,$$

which can be considered as an expression for P/ε according to $\mathcal{R}$:

$$\frac{\left(1-\frac{c_1}{2}-\frac{P}{\varepsilon}\right)}{P/\varepsilon} = \frac{(2-c_3)^2}{17(\frac{4}{3}-c_2)} - \frac{(c_4-2)^2}{2(\frac{4}{3}-c_2)}\mathcal{R}^2 \,,$$

and which is plotted on Figure 16.4. The value of $\mathcal{R}$ for $P/\varepsilon = 1$ corresponds to the stability points As and Bs, appearing nearby the bifurcation points A and B and located on both sides.

In the elliptic zone $\in (A,B)$ in which k and ε increase exponentially, then $\left(\dfrac{\varepsilon}{kS}\right)_\infty \neq 0$, whereas in the zone $\notin (A,B)$ where $\left(\dfrac{\varepsilon}{kS}\right)_\infty = 0$, the time decay of k and ε prevails.
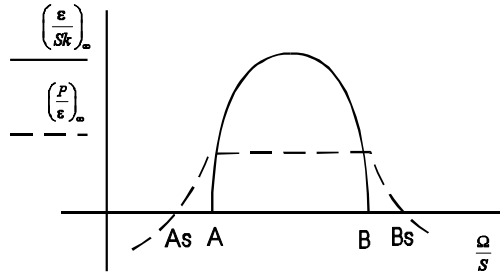
Figure 16.4. *Equilibrium diagram, bifurcation points A and B, stability points As and Bs*

In the case of homogenous turbulent shear flow $\Omega/S = -1$ and thus $\mathcal{R} = 1 - \dfrac{2}{(2-c_4)}\dfrac{\Omega^{(ref)}}{S}$ makes it possible to determine the abscissa of the bifurcation points and the stability points which should not be very different from the values obtained using linear analysis. This study shows that the key element in the model, for accounting for the effects of rotation combined with shear is the occurrence of a variable viscosity coefficient C_1 ([DUR 99]).

The work by Pettersson Reif B.A. *et al.* [PET 99] relies on the idea of modifying coefficient C_μ to become a function of the invariants of the mean velocity gradients in order to reproduce the bifurcations of an EASM model, in conjunction with the elliptic relaxation $k - v^2 - f$ model by Durbin.

16.6.3. *Algebraic models for the thermal field*

The hypothesis $\dfrac{dF_{\gamma i}}{dt} - \dfrac{1}{2}\Omega_{\gamma i} - Diff\left(F_{\gamma i}\right) = \dfrac{F_{\gamma i}}{2k}\left(\dfrac{dk}{dt} - Diff(k)\right)$ adapted from [11.13] allows us to derive an objective model. It leads to the relation:

$$\left(1-c_2\right)\widetilde{P}_{\gamma i}^{U} + P_{\gamma i}^{\Gamma} - c_{\gamma 1}\frac{\varepsilon}{k}F_{\gamma i} = \frac{F_{\gamma i}}{2k}\left(P-\varepsilon\right) \text{ which yields } F_{\gamma i}.$$

In the case of the boundary layer, the relation $F_{\gamma i} = \dfrac{k}{\varepsilon}\dfrac{\left(1-c_2\right)\widetilde{P}_{\gamma i}^{U} + P_{\gamma i}^{\Gamma}}{\dfrac{1}{2}\left(\dfrac{P}{\varepsilon}-1\right)+c_{\gamma 1}}$ gives:

$$F_{\gamma 1} = -\frac{k}{\varepsilon}\frac{\dfrac{R_{12}}{\alpha} - \left(\dfrac{k}{\varepsilon}\right)\dfrac{\left(1-c_2\right)}{\alpha^2}\left(U_{1,2}-\Omega_3\right)R_{22}}{\chi}\,\overline{\Gamma}_{,2},$$

$$F_{\gamma 2} = -\frac{k}{\varepsilon}\frac{\dfrac{R_{22}}{\alpha} - \left(\dfrac{k}{\varepsilon}\right)\dfrac{\left(1-c_2\right)}{\alpha^2}\Omega_3 R_{12}}{\chi}\,\overline{\Gamma}_{,2},$$

with $\quad \alpha = \dfrac{1}{2}\left(\dfrac{P}{\varepsilon}-1\right)+c_{\gamma 1} \quad$ and $\quad \chi = 1 - \left(\dfrac{k}{\varepsilon}\right)^2\dfrac{\left(1-c_2\right)^2}{\alpha^2}\Omega_3\left(\overline{U}_{1,2}-\Omega_3\right).$

16.7. Implicit effects of rotation on the turbulent field

The Coriolis acceleration forces act not only in an explicit way on the mean equations but also in an indirect and more subtle way on the turbulent field itself, thus modifying its structure and properties. This is why the phenomenology of rotating turbulence is so complex from the fundamental point of view.

Among these effects, we shall mention the blocking of the inertial cascade and also the appearance of an anisotropy at the length scale structural level (two-dimensionalization). The numerical simulations by Bardina J. *et al.* [BAR 85] show, in particular, the decrease in the dissipation rate and an increase of the integral length scale in the direction of the rotation axis. This same result is shown experimentally in the experiment of Jacquin *et al.* [JAC 90]. It is possible to account empirically for the first effect in the equation for ε by taking $C_{\varepsilon 2}$ according to the rotation rate Ω (Reynolds W.C., 1984, cf. *Simulation of turbulence models and their applications*, 1984): the rotation slows down the decay of turbulence.

Aupoix [AUP 87] proposes similarly, the modification $C_{\varepsilon 2} = C_{\varepsilon 20} + C_{\varepsilon 2}^{*}$ in which $C_{\varepsilon 20}$ is the usual value for this coefficient and $C_{\varepsilon 2}^{*}$ is a correction deduced from the EDQNM theory which reads:

$$C_{\varepsilon 2}^{*} = \frac{0.2236\, \omega^{*2} + 0.0303\, \omega^{*}}{0.2540\, \omega^{*2} + 0.1567\, \omega^{*} + 1},$$

$$\omega^{*} = \frac{\omega.k}{\varepsilon}, \text{ dimensionless rotation rate.}$$

The study of rotating turbulence finds its natural framework at the spectral level (Bertoglio J.-P. *et al.*, [BER 78A and B]). Analytical theories such as EDQNM, extended by Cambon C., [CAM 82] to the case of rotating sheared turbulence, exhibit a highly anisotropic structuring of the tensor $\hat{R}_{ij}(\vec{\kappa})$. Cambon C. *et al.*, [CAM 92A] introduce, for describing the effects of rotation on turbulence, two anisotropy tensors: the polarization anisotropy and directional anisotropy. Indeed, the rotation has a subtle influence on the directional properties of turbulence in spectral space (Jacquin L. *et al.*, [JAC 90]; Cambon C. *et al.*, [CAM 89, CAM 97, CAM 04]) that cannot be represented in the traditional models. In particular, these studies make clear the role of the D_{ij} tensor or its deviator Y_{ij} (cf. Kida S. and Hunt J.C.R., [KID 89]) defined by:

$$D_{ij} = \int_{\mathbb{R}^3} \frac{\kappa_i \kappa_j}{\kappa^2} \varphi_{mm}(\vec{\kappa}) d\vec{\kappa} \quad \text{and} \quad Y_{ij} = D_{ij} - \frac{1}{3} D_{mm} \delta_{ij}, \quad\quad\quad [16.19]$$

and which are directly related to the directional anisotropy in the three-dimensional turbulence energy spectrum (cf. section 8.7).

Cambon C. *et al.*, [CAM 92A] consider the restriction of the tensor $\varphi_{ij}(\vec{\kappa})$ to the plane normal to the wavevector (the Craya frame):

$$\varphi'_{\alpha\beta} = \varphi_{ij}(\vec{\kappa})\, e_i^{(\alpha)}(\vec{\kappa})\, e_j^{(\beta)}(\vec{\kappa}),$$

where $\quad \vec{e}^{(1)} = \dfrac{\vec{\kappa} \wedge \vec{\omega}}{\left|\vec{\kappa} \wedge \vec{\omega}\right|} \quad$ and $\quad \vec{e}^{(2)} = \dfrac{\vec{\kappa} \wedge \vec{e}^{(1)}}{\left|\vec{\kappa} \wedge \vec{e}^{(1)}\right|}.$

It can be shown that the spectral tensor can then be put into the form:

$$\varphi_{ij}(\vec{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right) + \left(e(\vec{\kappa}) - \frac{E(\kappa)}{4\pi\kappa^2}\right)\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right)$$

$$+ \mathcal{R}\left[Z(\vec{\kappa})N_i(\vec{\kappa})N_j(\vec{\kappa})\right], \quad \mathcal{R} \text{ real part}$$

$$\varphi_{ij}(\vec{\kappa}) = \varphi_{ij}^{iso}(\vec{\kappa}) + \varphi_{ij}^{e}(\vec{\kappa}) + \varphi_{ij}^{Z}(\vec{\kappa}), \qquad\qquad [16.20]$$

with $\overrightarrow{N}(\vec{\kappa}) = \vec{e}^{(2)} - i\vec{e}^{(1)}$, $\overrightarrow{N^*}(\vec{\kappa}) = \vec{e}^{(2)} + i\vec{e}^{(1)} = \overrightarrow{N}(-\vec{\kappa})$,

   $Z(\vec{\kappa}) = \frac{1}{2}\varphi_{ij}(\vec{\kappa})N_i^* N_j^*$ $e(\vec{\kappa}) = \frac{1}{2}\varphi_{jj}(\vec{\kappa})$.

Expression [16.20] allows us to distinguish three terms with a clear physical meaning:

– $\varphi_{ij}^{iso}(\vec{\kappa})$, the isotropic part;

– $\varphi_{ij}^{e}(\vec{\kappa})$, the departure of $e(\vec{\kappa})$ from the spherical distribution;

– $\varphi_{ij}^{Z}(\vec{\kappa})$, describes the tensorial structure of $\varphi_{ij}(\vec{\kappa})$ in the plane normal to the wavevector.

The term $e(\vec{\kappa})$ describes the directional properties ("dimensionality" according to W.C. Reynolds terminology) whereas $Z(\vec{\kappa})$ describes the polarization (which in the terminology of C. Cambon corresponds to the "componentality" of W.C. Reynolds).

The integrated form of [16.20] leads to:

$$R_{ij} = \frac{2}{3}k\delta_{ij} + R_{ij}^{(e)} + R_{ij}^{(Z)}, \qquad\qquad [16.21]$$

where $R_{ij}^{(e)}$ is nothing more than Y_{ij} defined by W.C. Reynolds. In this form, the problem is how to determine $e(\vec{\kappa})$ and $Z(\vec{\kappa})$. The in viscid RDT solution in the case of rapid block rotation is given by:

$$e(\vec{\kappa},t) = e(\vec{\kappa},0),$$
$$Z(\vec{\kappa},t) = Z(\vec{\kappa},0).\exp(4i\omega t.\cos\theta),$$

in which θ is given by $\cos\theta = \overline{\omega.\kappa}/\omega.\kappa$. It shows that only Z is modified by rotation. This explains the spectral jamming (scrambling effect) that appears in the damping of oscillations in the RDT solution.

In order to show this, let us consider the equation for the velocity fluctuations, in a relative rotating frame, in the absence of mean velocities $\overline{U}_i = 0$, in which the non-linear terms have been neglected, retaining only the linear terms (RDT approximation):

$$\frac{\partial u_j}{\partial t} = -\pi_{,j} - 2\varepsilon_{jkm}\Omega_k u_m .$$

We have denoted π the ratio of pressure to density supposed to be constant.

After Fourier transform, we get in spectral space:

$$\frac{\partial \hat{u}_j}{\partial t} = -i\kappa_j \hat{\pi} - 2\varepsilon_{jkm}\Omega_k \hat{u}_m .$$

The incompressibility condition reads $\kappa_j \hat{u}_j = 0$ and consequently, the previous equation implies:

$$i\hat{\pi} = -\frac{2}{\kappa^2}\varepsilon_{jkm}\kappa_j\Omega_k\hat{u}_m ,$$

which allows us to eliminate the pressure in the momentum equation:

$$\frac{\partial \hat{u}_j}{\partial t} = 2\left(\frac{\kappa_j\kappa_l}{\kappa^2} - \delta_{jl}\right)\varepsilon_{lkm}\Omega_k\hat{u}_m .$$

The identity $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ (cf. section 3.1.4) makes it possible to transform the term in the right-hand side into: $\dfrac{\kappa_j\kappa_l}{\kappa^2} - \delta_{jl} = -\varepsilon_{jpm}\varepsilon_{lhm}\dfrac{\kappa_p\kappa_h}{\kappa^2}$ and thus:

$$\frac{\partial \hat{u}_j}{\partial t} = -2\varepsilon_{jpz}\varepsilon_{lhz}\varepsilon_{lkm}\frac{\kappa_p\kappa_h}{\kappa^2}\Omega_k\hat{u}_m .$$

Using once again the previous identity for the product $\varepsilon_{lhz}\varepsilon_{lkm}$, we easily find the equivalent form:

$$\frac{\partial \hat{u}_j}{\partial t} = -2\left(\frac{\kappa_h \Omega_h}{\kappa}\right) \varepsilon_{jpm} \frac{\kappa_p}{\kappa} \hat{u}_m \ .$$

This equation shows [MAN 91] that in the framework of the rapid distortion theory, a block rotation causes the rotation of the velocity vector $\hat{u}_j(\vec{\kappa})$ in Fourier space about its own wavevector $\vec{\kappa}$ at a rate depending on the projection $\kappa_h \Omega_h$ of $\vec{\kappa}$ on the rotation axis (cf. Figure 16.5). It then results in a decorrelation of the Fourier modes which rotates at different speeds. When the vectors $\hat{u}_j(\vec{\kappa})$ are statistically distributed uniformly around their wavevector, the spectrum takes the form:

$$\varphi_{ij}(\vec{\kappa}) = \frac{1}{2}\varphi_{mm}(\vec{\kappa})\left(\delta_{ij} - \frac{\kappa_i \kappa_j}{\kappa^2}\right),$$

and for this particular spectrum, the so-called "random Fourier spectrum" [KAS 94B] we get $R_{ij} = k\delta_{ij} - \frac{1}{2}C_{ij}$.

In the modeling point of view for the tensor a_{kl}^{ij}, the "stropholysis" $Q_{ijk} = \varepsilon_{ipq}a_{pk}^{jq}$ (cf. Chapter 8) can be related to Z. In the model proposed by C. Cambon *et al.*, [CAM 92A] composed of transport equations for R_{ij} and for $R_{ij}^{(e)}$, the linear terms are obtained from invariant modeling (linear) and the spectral jamming (linear dispersive effect of inertial waves) is explicitly supplemented in the form of an empirical relaxation term.
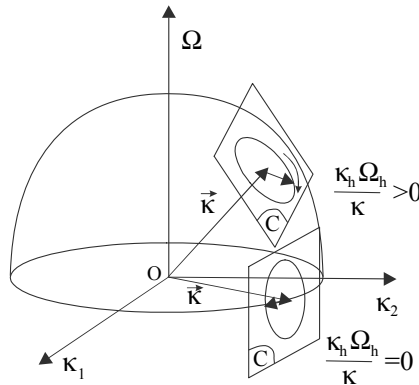
Figure 16.5. *Spectral scrambling effect due to rotation Ω,*
(C is the Craya plane at the point under consideration)

In the equation for the spectral tensor, this term can be assumed to be modeled like:

$$-\frac{1}{T}\Big[E_{ij}(\kappa) - E_{ij}^{\,e}(\kappa)\Big]\,,$$ [16.22]

with $E_{ij}(\kappa) = \oiint_{S(\kappa)} \varphi_{ij}(\vec{\kappa})dS$, $E_{ij}^{\,e}(\kappa) = \oiint_{S(\kappa)} \frac{1}{2}\varphi_{mm}(\vec{\kappa})\left(\delta_{ij} - \frac{\kappa_i \kappa_j}{\kappa^2}\right)dS$,   (cf.

[8.30]) and T, a characteristic time scale.

After integration in Fourier space, term [16.22] becomes in physical space in the Reynolds stress equations:

$$-\frac{1}{T}\left(R_{ij} - k\delta_{ij} + \frac{1}{2}C_{ij}\right)\quad\text{(notations of section 8.7).}$$ [16.23]

An homogenous anisotropic turbulence submitted to rotation evolves under the effect of spectral scrambling, the components of Reynolds stress tensor then undergo damped oscillations (Mansour N.N. *et al.*, [MAN 91]) and the asymptotic anisotropy of the Reynolds stress tensor is proportional to the initial anisotropy of the structure tensor (cf. relation [8.19], section 8.7):

$$a_{ij}^{\infty} = -\frac{1}{2}y_{ij}\,,$$ [16.24]

with $a_{ij} = \left(R_{ij} - 2/3k\delta_{ij}\right)/2k$, $y_{ij} = \left(D_{ij} - 2/3k\delta_{ij}\right)/2k$.   [16.25]

In addition, the non-linear effects are responsible for the blocking of spectral transfer and also concentrating energy in the spectral plane normal to the rotation axis with a polarization of this energy. This latter effect (cf. C. Cambon, [CAM 92A]) only occurs at intermediate Rossby numbers such as $0.01 < Ro < 1$, $Ro = \varepsilon/2k\omega$. Schiestel R. and Elena L., [SCH 97] proposed for practical applications, a simplified model using a behavior law for the structure tensor associated with a spectral tensor representation which generalizes the one proposed by Cambon C. *et al.*, [CAM 81] and then by Shih T.H. *et al.*, [SHI 90]. The use of a term such as [16.23] added to the linear terms, allows us to account roughly the effect of spectral scrambling and thus to predict the decrease in the invariants II^{a} and III^{a} in pure rotation (cf. section 8.7). With the aim of calculating confined flows submitted to strong rotation, the structure tensor is approximated in this model by an empirical formula according to intrinsic rotation:

$$C_{ij} = \frac{2}{3}\left(1 + \frac{f_C}{2}\right)k\delta_{ij} - f_C k \frac{\Omega^*_i \Omega^*_j}{\Omega^{*2}}, \quad f_C = \frac{Ro_T^{-1}}{5 + Ro_T^{-1}}, \quad Ro = \frac{\varepsilon}{k\Omega^*}.$$

This formulation, although very simplified, includes a non-linear effect and allows us to recover $\dfrac{dII^a}{dt} \neq 0$ and $\dfrac{dIII^a}{dt} \neq 0$ in homogenous turbulence. This model has given good predictions of the turbulent flow in rotating disk systems and in particular corotating disk systems ([RAN 01], [RAN 04]) and rotor-stator systems ([PON 05]). It emerges that second order closures generally seem to be the adequate level of closure to study in practice turbulent flows submitted to strong rotation. The practical applications deal in particular with problem of flow and cooling in rotating disk systems in turbomachinery ([IAC 95], [ELE 96]).

If the traditional k-ε model is "blind" to any rotation effect, on the other hand the non-linear models, which inherit some characteristics of the second order models, may account for some of the effects of rotation that are useful in practice. Nagano Y. and Hattori H. [NAG 02] used a modified version of the non-linear model [ABE 97] in order to study the case of the plane channel submitted to rotation (rotation axis in the transverse direction) and in particular the dissymmetry of the turbulence field near the walls.

Let us mention at last the work by Speziale C.G., [SPE 89, 91A] who, studying theoretically the effect of an arbitrary change in frame of reference on the formulation of turbulence models, shows that the fluctuating velocity is modified in the frame change only through the Coriolis acceleration. It is then possible to specify the constraints that must be satisfied by the closure hypotheses in the turbulence models and their consequences in rotating flows (Speziale C.G., [SPE 87A]). It is shown in particular that the usual closures violate the principle of material indifference at the limit of two-dimensional turbulence (cf. section 4.7) imposed by the Taylor-Proudman theorem. The author proposes a modification of the stochastic model by Haworth D.C. and Pope S.B. [HAW 86] which allows us to make it compatible.

16.8. Rotating turbulence in the presence of active thermal effects

If we assume that the temperature differences are sufficiently small, the interaction terms between the Coriolis forces, the centrifugal forces and the temperature gradients can be approached using the Boussinesq approximation.

Thus, we shall get:

$$\frac{d\overline{U}_i}{dt} = \underline{\quad\Diamond\quad} + \beta\Delta\overline{\Gamma}\zeta_i \,,$$

with

$$\begin{bmatrix} \zeta_z \\ \zeta_r \\ \zeta_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\overline{U}_\theta + \Omega r\right)^2 / r \\ -2\left(\overline{U}_\theta + \Omega r\right)\overline{U}_r / r \end{bmatrix} \quad \text{(in a fixed frame)}$$

$$\frac{dk}{dt} = \underline{\quad\Diamond\quad} + \beta F_{\gamma i}\zeta_i \,,$$

$$\frac{d\varepsilon}{dt} = \underline{\quad\Diamond\quad} + C_{\varepsilon 1}\beta F_{\gamma j}\zeta_j \tilde{\varepsilon}/k \,,$$

$$\frac{dR_{ij}}{dt} = \underline{\quad\Diamond\quad} \underbrace{\boxed{+\Phi_{ij}^{(T)}}}_{(*)} \underline{\quad\Diamond\quad} + G^T{}_{ij} \,,$$

$$\Phi_{ij}^{(T)} = -c_3\left(G^T{}_{ij} - \frac{2}{3}G^T\delta_{ij}\right), \qquad\qquad\qquad \text{[16.26]}$$

$$G^T{}_{ij} = \beta\left(F_{\gamma i}\zeta_j + F_{\gamma j}\zeta_i\right), \qquad\qquad G^T = \frac{1}{2}G^T{}_{jj} \,.$$

In addition to the rotation Richardson number defined previously, it is possible to introduce a Grashof number by replacing in its definition already given in natural convection, the gravity by the centrifugal acceleration:

$$Gr_\Omega = \frac{\rho^2 r \Omega^2 \beta.\Delta\Gamma.L^3}{\mu^2} \,,$$

the results obtained in the case of natural convection can thus be transposed in the case of rotation.

16.9. Coherent structures and modeling

Coherent structures appear like identifiable deterministic elements. Numerous studies have been pursued in this direction, in order to detect and describe these structures in various types of flows. These structures are indeed characteristic of each type of flow (Brown G.L. and Roshko A., [BRO 74] for free flows, Kline S.J. *et al.*, [KLI 67] for boundary layers). At the free boundary of a turbulent flow, these structures explain the boundary intermittency at the edge formed by the turbulent/non-turbulent interface. Coherent structures that are orderly elements evolving into an apparently disorderly background, appear to be a specific property of turbulence. It seems to explain, at least partly, the lack of universality of one point statistical models which cannot easily account for this aspect of the phenomenology. Some research works however exist in this direction, such as that by Perry A.E., [PER 87] who uses a superposition of horse shoe vortices to describe the turbulent field, the behavior of which can be related to simpler methods based on turbulence eddy viscosity.

The dichotomic description of the turbulent field composed of structured elements, the coherent structures, against a random fluctuating background of finer scales, can be related to the semi-deterministic point of view proposed by Ha Minh [HAM 99]. This concept must be also connected to the work of Farge M. *et al.* ([FAR 99, FAR 01A and B]), which allows us to reveal the coherent contribution by using wavelet bases. According to this point of view, the turbulent field would be represented as the superposition of coherent structures with non-Gaussian PDF on a random background with Gaussian PDF, without any reference to the size of the structures. The proper orthogonal decomposition (POD) introduced by Lumley ([*GAT 96], Chapter 1) also proceeds from this type of concept.

16.10. Laminar/turbulent interface, free boundaries

The work by Townsend A.A. [*TOW 56] has revealed some fundamental characteristics of the free turbulent shear flows (jets, wakes, mixing layers, etc.). It appears that the fully turbulent region of the flow which is highly rotational is bounded by an undulated and contorted surface which is fluttering through the convective action of the big eddies whose size reaches the thickness of the whole flow domain. In contrast, the non-turbulent external region is irrotational, only fluctuating by the potential flow due to the motion of the external boundary. The interface itself, which was studied by Corrsin S. and Kistler A.L. [COR 54], appears like a propagating turbulent front involving the smallest scales. The interface is thus reduced to a very thin layer, the so-called "superlayer" whose thickness should be near the Kolmogorov scale. This interface is subjected to large-scale fluctuations which are related to the boundary intermittency (cf. sections 2.2.1 and 4.10 and also

[BIS 02]). The study of the behavior of statistical models through the laminar/turbulent interface at a free boundary is thus of major interest. To do this, we can consider the diffusion phenomenon in a simplified one-dimensional geometry in the absence of strain.

The first works by Lundgren T.S. and Wang F.C. ([LUN 69, LUN 71, LUN 73]) considered the propagation of a turbulent front and the prediction of the laminar/turbulent interface using closure models. The analysis of the rate of return to isotropy has led the authors to suggest that the characteristic time for the return to isotropy would be better estimated by $\dfrac{1}{\tau} \approx \dfrac{\varepsilon + dk/dt}{k}$ than by the usual expression $\dfrac{1}{\tau} \approx \dfrac{\varepsilon}{k}$, which overestimates the return to isotropy in homogenous turbulence (grid turbulence for example). The one-dimensional study of the turbulence eddy viscosity model based on the previous scale leads to:

$$v_t = C \frac{k^2}{\varepsilon + \dfrac{\partial k}{\partial t}},$$
    [16.27]

is carried on the kinetic energy equation:

$$\varepsilon + \frac{\partial k}{\partial t} = -\frac{\partial Q}{\partial y} + v \frac{\partial^2 k}{\partial y^2}, \quad \text{with} \quad Q = -\alpha v_t \frac{\partial k}{\partial y}.$$
    [16.28]

In this equation (cf. [2.9] and section 11.2.3), the production term has been neglected since the shear is supposed to be weak and Q stands for the turbulence diffusion flux.

After eliminating v_t and reporting the value of Q in equation [16.27] in which the molecular viscosity is neglected, we find:

$$Q \frac{\partial Q}{\partial y} = \alpha C k^2 \frac{\partial k}{\partial y},$$

this equation can be integrated as $Q^2 = \dfrac{2}{3} \alpha C k^3 + F(t)$. If k and Q go to zero when y goes to infinity, then we can assume $F(t) = 0$. Reporting this value in the initial

equation, Lundgren T.S. and Wang F.C. are led to the following propagation equation:

$$\frac{\partial \sqrt{k}}{\partial t} + \left(\frac{3\alpha C}{2}\right)^{1/2} \sqrt{k}\, \frac{\partial \sqrt{k}}{\partial y} = -\frac{\varepsilon}{2\sqrt{k}}, \qquad [16.29]$$

which singles out the propagation velocity $\left(\dfrac{3\alpha Ck}{2}\right)^{1/2}$, the initial profile becoming steeper and steeper until a discontinuity appears like in shock waves. The introduction of a viscous term will prevent the appearance of discontinuities by giving a finite thickness to the transition zone. However, the use of a model of the type $\nu_t = C\dfrac{k^2}{\varepsilon}$, does not give any discontinuous jump in the kinetic energy profile.

The study by Cazalbou J.B. *et al.* [CAZ 94] is based on purely diffusive one-dimensional equations:

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial y}\right) \quad \text{and} \quad \frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) \quad \text{with } \nu_t = C_\mu \frac{k^2}{\varepsilon}, \qquad [16.30]$$

and looks for the existence of mathematical solutions such that k and ε vanish on one side of the interface (with propagation). The study could be made in an equivalent way in a moving frame attached to the interface:

$$U\frac{\partial k}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial y}\right) \quad \text{and} \quad U\frac{\partial \varepsilon}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right).$$

The following relations express the solutions corresponding to ramp flows moving at a constant speed and satisfying equations [16.30] on each side of the interface:

$$k = k_0 H(ct - y)\left|\frac{ct - y}{\delta_0}\right|^{\frac{\sigma_k}{2\sigma_k - \sigma_\varepsilon}}, \quad \varepsilon = \varepsilon_0 H(ct - y)\left|\frac{ct - y}{\delta_0}\right|^{\frac{\sigma_\varepsilon}{2\sigma_k - \sigma_\varepsilon}}, \qquad [16.31]$$

with a propagation velocity $c = \dfrac{C_\mu}{2\sigma_k - \sigma_\varepsilon} \dfrac{k_0{}^2}{\varepsilon_0 \delta_0}$. The quantities k_0, ε_0 and δ_0 are characteristic scales of the problem and $H(x)$ the Heaviside function defined by:

$$\begin{cases} H(x) = 0 & \text{if } x > 0 \\ H(x) = 1 & \text{if } x \geq 0 \end{cases}$$

The derivatives are not defined on the very interface, and so the resulting solutions can only be weak solutions on the mathematical point of view. The authors also show that the production and dissipation terms can indeed be neglected in equations [16.30] if the condition

$$\frac{\sigma_\varepsilon}{2} < \sigma_k < 2,$$

is satisfied. Another condition $2\sigma_k - 1 \leq \sigma_\varepsilon$ makes it possible to prevent an infinite slope appearing in the velocity profile at the interface. It is shown that, with a few exceptions, most of the usual models are compatible with the existence of ramp solutions propagating at finite speed. The existence of fronts at the free boundaries is a desirable property in a turbulence model, making it possible in particular to prevent a too large and unphysical influence of the external flow. It is the k-ω model which seems to have the least appropriate form for the behavior at a free boundary, and this is to be related to the high sensitivity of the k-ω model to the external flow (cf. section 11.2.7). Cazalbou J.B. *et al.* [CAZ 02] have extended the previous analysis to the case of second order closures. The authors show that solutions similar to [16.31] can be obtained if there is a value of $(R_{22}/k)_{front}$ at the interface for

which: $f\left(\dfrac{R_{22}}{k}\right) = f_{22}\left(\dfrac{R_{22}}{k}\right)$ and $2h\left(\dfrac{R_{22}}{k}\right) - f\left(\dfrac{R_{22}}{k}\right) > 0$.

The functions:

$$f\left(\frac{R_{22}}{k}\right) = \frac{D_{(k)}}{k^2 / \varepsilon}, \quad h\left(\frac{R_{22}}{k}\right) = \frac{D_{(\varepsilon)}}{k^2 / \varepsilon}, \quad f_{22}\left(\frac{R_{22}}{k}\right) = \frac{D_{(R_{22})}}{k^2 / \varepsilon},$$

are linearized expressions of the turbulent diffusion coefficients for k, ε and R_{22}, respectively.

This page intentionally left blank

Chapter 17

Variable Density Turbulent Flows

The pioneering works on turbulence modeling were first devoted essentially to incompressible flows. However, many problems from industry or aeronautics involve effects of density variations. These variations in density may have various origins: thermal dilatation, compression, supersonic regime, mixing of fluids, chemical reactions, etc.

Thus, the present chapter will be dealing as well with dilatable turbulent flows, in which the density would be a function of temperature only, as truly compressible flows in which the density also varies according to pressure. Some variable density flows can also be produced by the mixing of miscible fluids but with differing densities. Chemical reaction, like combustion, induces complex phenomena coupled with turbulence.

In weakly compressible flows, it will often be possible to assume in the equations that only the mean density is variable and neglect its fluctuations. When this hypothesis is no longer valid, the problem becomes far more complex. We will find in Favre A. *et al.*, 1976 an introduction to the theory of Kovasnay turbulence modes which applies when the fluctuation levels are low. These modes interact when the fluctuation levels are no longer weak.

Most of the modeling methods used in practice still rely on a direct extension of existing methods for incompressible flows. The effects of compressibility are however complex and may require us to reconsider most of closure hypotheses. The first fundamental work by Feiereisen *et al.* [FEI 81] has shed light on many problems of statistical modeling that are specific to compressible flow, by

considering the data obtained by direct numerical simulation. But the account of compressibility in closure methods still remains a current research problem.

17.1. Averaging

The formalism of mass weighted averaging (or Favre averaging) introduced by Favre (cf. Favre A. *et al.*, 1976, Favre A., [FAV 89]) is often used to describe these flows.

We will split (Favre decomposition) any turbulent quantity Φ (other than ρ and p):

$$\Phi = \widetilde{\Phi} + \varphi', \qquad\qquad\qquad [17.1]$$

where $\widetilde{\Phi}$ is the mass weighted average $\widetilde{\Phi} = \dfrac{\overline{\rho\Phi}}{\overline{\rho}}$.

In the particular case of a constant density the previous splitting becomes identical to the usual Reynolds decomposition:

$$\Phi = \overline{\Phi} + \varphi'', \qquad \overline{\varphi''} = 0$$

(in the present chapter we shall denote φ'' the non-weighted usual turbulent fluctuation).

We mention below some useful properties of Favre averaging:

$$\overline{\widetilde{\Phi}} = \widetilde{\Phi}, \qquad\qquad \varphi' = \frac{(\rho\varphi)''}{\rho},$$

$$\overline{\varphi'} \neq 0, \qquad\qquad \overline{\varphi'} = -\frac{\overline{\rho'\varphi''}}{\overline{\rho}}, \qquad\qquad [17.2]$$

and moreover $\overline{\rho'\varphi''} = \overline{\rho'\varphi'}$.

This scheme does not apply to p nor ρ, for which we write in all cases:

$$p = \overline{p} + p', \qquad \rho = \overline{\rho} + \rho'.$$

We denote D/Dt the material derivative defined by:

$$\frac{D\Phi}{Dt} \equiv \frac{\partial \Phi}{\partial t} + \widetilde{U}_j \Phi_{,j} \qquad\qquad [17.3]$$

and also:

$$\frac{D}{Dt}\left(\overline{\rho}\widetilde{\Phi}\right) \equiv \overline{\rho}\frac{D\widetilde{\Phi}}{Dt} - \overline{\rho}\widetilde{\Phi}\widetilde{U}_{j,j} = \frac{\partial}{\partial t}\left(\overline{\rho}\widetilde{\Phi}\right) + \left(\overline{\rho}\widetilde{U}_j\widetilde{\Phi}\right)_{,j} - \overline{\rho}\widetilde{\Phi}\widetilde{U}_{j,j}.$$

17.2. Transport equations

The transport equations for variable density turbulence in mass weighted averages in their general form have been developed by Favre (cf. Favre A. *et al.*, 1976). We give here the main results useful for modeling.

17.2.1. *The mean field equations*

The starting point is the instantaneous Navier-Stokes equation:

$$\rho\frac{dU_i}{dt} = -p_{,i} + \rho g_i + f_{ij,j}, \qquad\qquad \partial_t\rho + \left(\rho U_m\right)_{,m} = 0,$$

$$f_{ij} = \mu S_{ij}, \qquad\qquad S_{ij} = U_{i,j} + U_{j,i} - \frac{2}{3}U_{m,m}\delta_{ij}.$$

The momentum and energy equations are the following:

– mass conservation equation:

$$\frac{\partial\overline{\rho}}{\partial t} + \left(\overline{\rho}\widetilde{U}_k\right)_{,k} = 0 \quad \text{or} \quad \frac{D\overline{\rho}}{Dt} + \overline{\rho}\widetilde{U}_{k,k} = 0; \qquad\qquad [17.4]$$

– mean momentum equation:

$$\overline{\rho}\frac{D\widetilde{U}_i}{Dt} = \frac{\partial}{\partial t}\left(\overline{\rho}\widetilde{U}_i\right) + \left(\overline{\rho}\widetilde{U}_i\widetilde{U}_j\right)_{,j} = -\overline{p}_{,i} - R_{ij,j} + \overline{f}_{ij,j} + \overline{\rho}g_i,$$

$$R_{ij} = \overline{\rho u'_i u'_j} = \overline{\rho}\,r_{ij},$$

$$r_{ij} = \widetilde{u'_i u'_j}, \qquad \overline{f}_{ij} \approx \overline{\mu}\widetilde{S}_{ij}; \qquad\qquad [17.5]$$

– total energy equation:

$$E = e + \frac{1}{2} U_j U_j, \qquad \tilde{E} = \tilde{e} + \frac{1}{2} \tilde{U}_j \tilde{U}_j + \frac{1}{2} \widetilde{u'_j u'_j},$$

with $de = C_v d\theta$,

$$\bar{\rho} \frac{D\tilde{E}}{Dt} = -\left(\overline{\rho u'_j E'}\right)_{,j} + \left(\overline{f_{ij} u'_j}\right)_{,j} + \left(\overline{p u'_j}\right)_{,j} - \bar{F}_{j,j}, \qquad [17.6]$$

$$F_j = -\lambda \theta_{,j}$$

and an equation of state such as $p = (\gamma - 1)\rho e$.

In the case of a strongly heated dilatable fluid flow, we can define an equation for temperature. The density ρ is in this case, only a function of θ:

$$\bar{\rho} \frac{D\tilde{\theta}}{Dt} = \frac{\partial}{\partial t}(\bar{\rho}\tilde{\theta}) + \left(\overline{\rho \tilde{U}_j \tilde{\theta}}\right)_{,j} = -F_{\theta j,j} + \bar{h}_{j,j},$$

$$F_{\theta j} = \overline{\rho u'_j \theta'} = \bar{\rho} \, \widetilde{u'_j \theta'}, \qquad h_j = \frac{\mu}{\Pr} \theta_{,j}, \qquad \bar{h}_j = \frac{\bar{\mu}}{\Pr} \tilde{\theta}_{,j}. \qquad [17.7]$$

This equation can be obtained from a simplification of the enthalpy equation $h = e + p/\rho = C_p \theta$, $dh = C_p d\theta$.

17.2.2. The turbulent field equations

17.2.2.1. *Reynolds stress equations:* $R_{ij} = \bar{\rho} r_{ij}$

$$\frac{D}{Dt}\left(\bar{\rho} r_{ij}\right) = \boxed{-\overline{\rho u'_i u'_j} \tilde{U}_{l,l}} - \overline{\rho u'_j u'_l} \tilde{U}_{i,l} - \overline{\rho u'_i u'_l} \tilde{U}_{j,l}$$

$$-\left(\overline{\rho u'_i u'_j u'_l}\right)_{,l} - \left(\overline{u'_i p'}\right)_{,j} - \left(\overline{u'_j p'}\right)_{,i} + \overline{p'(u'_{i,j} + u'_{j,i})}$$

$$+\left(\overline{\mu S_{il} u'_j} + \overline{\mu S_{jl} u'_i}\right)_{,l} - \overline{\mu S_{il} u'_{j,l}} - \overline{\mu S_{jl} u'_{i,l}} - \boxed{\left(\overline{u'_i \cdot \bar{p}_{,j}} + \overline{u'_j \cdot \bar{p}_{,i}}\right)}. \qquad [17.8]$$

The terms within frames are new terms compared to the usual equations for incompressible flow (cf. Chapter 2).

17.2.2.2. *Kinetic energy equation*

$$\frac{D}{Dt}\left(\overline{\rho}\,\widetilde{\frac{u'_i u'_i}{2}}\right) = \boxed{-\frac{1}{2}\overline{\rho u'_j u'_j}.\widetilde{U}_{l,l}} - \overline{\rho u'_j u'_l}.\widetilde{U}_{j,l}$$

$$-\frac{1}{2}\left(\overline{\rho u'_j u'_j u'_l}\right)_{,l} - \left(\overline{u'_j p'}\right)_{,j} + \overline{p' u'_{j,j}}$$

$$+\left(\overline{\mu S_{jl} u'_j}\right)_{,l} - \overline{\mu S_{jl} u'_{j,l}} - \boxed{\overline{u'_j.\overline{P}_{,j}}}. \tag{17.9}$$

This equation presents a new term for dilatation production and also an additional pressure term due to the fact that the Favre fluctuation is not zero in the mean. We shall also note the occurrence of a pressure-dilatation correlation term $\overline{p' u'_{j,j}}$ (cf. section 17.7).

17.2.2.3. *Turbulent heat fluxes* $\varphi_{\theta i} = \widetilde{u'_i \theta'}$

$$\frac{D}{Dt}\left(\overline{\rho}\varphi_{\theta i}\right) = \boxed{-\overline{\rho u'_i \theta'}.\widetilde{U}_{l,l}} - \overline{\rho \theta' u'_l}.\widetilde{U}_{i,l} - \overline{\rho u'_i u'_l}.\tilde{\theta}_{,l}$$

$$-\left(\overline{\rho u'_i u'_l \theta'}\right)_{,l} - \left(\overline{\theta' p'}\right)_{,i} + \overline{p' \theta'_{,i}} - \boxed{\overline{\theta'.\overline{p}_{,i}}}$$

$$+\overline{\mu S_{il} \theta'_{,l}} + \left(\overline{\mu S_{il} \theta'}\right)_{,l} + \frac{1}{\overline{C_v}}\left[\underbrace{\overline{u'_i \mathbf{\Phi}} - \overline{u'_i\, p' u'_{l,l}}}_{(*)} + \overline{u'_i(\lambda\theta'_{,l})_{,l}}\right], \tag{17.10}$$

where $\mathbf{\Phi} = f_{ij} U_{i,j}$ denotes the dissipation function.

The terms (*) will be neglected if equation [17.7] is used, in this case C_p and C_v are equivalent anyway.

17.2.2.4. *Variance of temperature fluctuations*

$$\frac{D}{Dt}\left(\overline{\rho.\widetilde{\theta'^2}}\right) = \boxed{-\overline{\rho\theta'^2}.\widetilde{U}_{l,l}} - \left(\overline{\rho\theta'^2 u'_l}\right)_{,l} - 2\overline{\rho u'_l \theta'}\tilde{\theta}_{,l}$$

$$+\frac{2}{C_v}\left[\underbrace{\overline{\theta'\mathbf{\Phi}} - \overline{\theta'\, pu_{l,l}}}_{(*)} + \left(\overline{\theta'\lambda\theta_{,l}}\right)_{,l} - \overline{\lambda\theta'_{,l}\tilde{\theta}_{,l}} - \overline{\lambda\theta'_{,l}\,\theta'_{,l}}\right] \tag{17.11}$$

(*) see previous example.

17.2.2.5. *Dynamic and thermal dissipations*

The dissipation rate of turbulent energy reads:

$$\in = \overline{\mu S_{il} u'_{i,l}} , \qquad\qquad S_{il} = \tilde{S}_{il} + s_{il} ,$$

$$\in = \frac{1}{2}\overline{\mu s_{ij} s_{ij}} + \frac{1}{2}\overline{\mu s_{ij} \tilde{S}_{ij}} ,$$

and at high Reynolds numbers $\in \approx \dfrac{1}{2}\bar{\nu}.\overline{\rho s_{ij} s_{ij}}$.

Some authors distinguish in this expression a usual rotational contribution (solenoidal dissipation) and a dilatation dissipation which is specific to the effects of the variations in density:

$$\in \approx \bar{\nu}.\bar{\rho}\,\overline{u'_{i,j}\, u'_{i,j}} - \frac{4}{3}\bar{\nu}.\bar{\rho}\,\overline{u'_{l,l}\, u'_{l,l}} ,$$

or $\in = \in_s + \in_d = \rho(\varepsilon_s + \varepsilon_d)$.

In order to convey this distinction, we can use the decomposition of the fluctuating velocity in a solenoidal part and a dilatable part that is then reported in the expression for the statistical correlation for dissipation $u'_i = u'^{(s)}_i + u'^{(d)}_i$.

The solenoidal dissipation is associated with the energy cascade process whereas the dilatation dissipation is related to the volume fluctuations. Zeman O., [ZEM 90], [ZEM 91B] has associated this latter with "shocklets", shock waves appearing at the level of eddies and proposes modeling it in the following way:

$$\in_d = f(M_t, K) ,$$

where M_t is the turbulent Mach number defined by $M_t = \sqrt{2k}/a$ (a, sound velocity) and K is the flatness factor $K = \overline{u'^4}/\left(\overline{u'^2}\right)^2$ which characterizes the departure from Gaussianity. Thus, the eddies for which the turbulent Mach number is larger than unity develop local shocks which take part in energy transfers. The Sarkar S. and Balakrishnan L. model, [SAR 91] is similar. It is based on the relation:

$$\in_d = \alpha_1 M_t^2 \in_s , \quad \alpha_1 = 0.5 ,$$

whereas $\in_s$ is given by a traditionally modeled transport equation.

An improved form for f has been proposed by Zeman O., [ZEM 91]:

$$f(M_t) = 1 - \exp\left\{ -\left[(M_t - 0.25)/80 \right]^2 \right\}$$
$$f(M_t) = 0 \quad \text{if} \quad M_t < 0.25$$
$$\text{and} \quad K \approx 4.$$

Compressed turbulence is also an important particular case of application. It can be an isotropic compression in which $U_i = -c\, x_i$ or a one-dimensional compression such as $U_1 = -c\, x_1$ and $U_i = 0$ if $i \neq 1$. Rapid compression can be studied from the rapid distortion theory (Durbin P.A. and Zeman O., [DUR 92], Cambon C. *et al.*, [CAM 93]). For modeling purposes, it is necessary to involve the dilatation compressible dissipation $\in_d$ introduced previously. Moreover, Cambon C. *et al.*, [CAM 93] have proposed, in order to have a better account of the results of the RDT analysis, modifying the modeling of the redistribution term Φ_{ij} by introducing a multiplicative coefficient equal to $\exp(-M_g /40)$, M_g being a compressibility parameter defined by $M_g = \left(\overline{U}_{i,j}\overline{U}_{i,j} \right) \ell / a$ where ℓ is the turbulence macroscale and a the local sound velocity.

The same approach is used for the dissipation rate of a transported scalar

$$\in_\gamma = \overline{\rho}\varepsilon_\gamma = \frac{\overline{\lambda}}{C_p}\overline{\theta'_{,l}\,\theta'_{,l}} = \frac{\overline{\nu}}{\text{Pr}}\overline{\rho\theta'_{,l}\,\theta'_{,l}}\,.$$

To derive the evolution equations for $\in$ and $\in_\gamma$ we will use the formal equations established by Favre for the transport of arbitrary correlations:

$$\frac{D}{Dt}\left(\overline{\rho\xi'\eta'} \right) = \dots \text{ and } \frac{D}{Dt}\left(\overline{\rho\xi'^2} \right) = \dots \text{ (cf. Favre A. \textit{et al.}, 1976).}$$

We will find:

$$\frac{D\in}{Dt} = -\boxed{\in \tilde{U}_{l,l}} - \left(\overline{u'_l \in'} \right)_{,l} - \overline{\nu\rho u'_l\, s_{ij}}\,\tilde{S}_{ij,l} - \boxed{\overline{\nu\rho s_{ij}}\,\frac{D\tilde{S}_{ij}}{Dt}}$$

$$+\boxed{2\nu\frac{\overline{-s_{ij}\rho_{,j}}}{\rho}(p_{,i}-f_{ij,j})}-2\overline{\nu\rho s_{ij}u'_{j,m}u'_{m,i}}-2\overline{\nu\rho s_{ij}u'_{j,m}\tilde{U}_{m,i}}-2\overline{\nu\rho s_{ij}u'_{m,i}\tilde{U}_{j,m}}$$

$$-2\overline{\nu s_{ij}p_{,ij}}-\boxed{2\overline{\nu\rho s_{ij}\tilde{U}_{j,m}\tilde{U}_{m,i}}}+2\overline{\nu s_{ij}f_{im,mj}}\ ,\qquad\qquad[17.12]$$

$$\frac{D\in_{\gamma}}{Dt}=-\boxed{\in_{\gamma}\tilde{U}_{l,l}}-\left(\overline{u'_{l}\in_{\gamma}'}\right)_{,l}-2\overline{\sigma\rho u'_{l}\,\theta'_{,m}\tilde{\theta}_{,ml}}-\boxed{2\overline{\sigma\rho q'}_{,m}\frac{D\tilde{\theta}_{,m}}{Dt}}$$

$$-\boxed{2\overline{\sigma\frac{\rho_{,m}}{\rho}\theta'_{,m}h_{j,j}}}-2\overline{\sigma\rho\theta'_{,m}u'_{j,m}\theta'_{,j}}-2\overline{\sigma\rho\theta'_{,m}u'_{j,m}\tilde{\theta}_{,j}}-2\overline{\sigma\rho\theta'_{,m}\theta'_{,j}\tilde{U}_{j,m}}$$

$$-2\overline{\sigma\theta'_{,m}h_{j,jm}}-\boxed{2\overline{\sigma\rho\theta'_{,m}\tilde{U}_{j,m}\tilde{\theta}_{,j}}}\ .\qquad\qquad[17.13]$$

The pressure equation is more complex than in the incompressible case and can be written in different forms. Propagation phenomena can be put in light:

$$\Delta p'-\frac{\partial^{2}\rho'}{\partial t^{2}}=sources\ .$$

17.3. Reynolds stress transport modeling in the framework of mass weighted averaging

The main advantage of the Favre formalism is to lead to transport equations that can be written in a form very similar to those equations in incompressible fluid. This formal similarity has led many authors to model these equations being inspired by closures already established in incompressible turbulence.

For example, Vandromme D. *et al.*, [VAN 83] propose this type of closure:

$$\frac{\partial}{\partial t}\left(\overline{\rho}r_{ij}\right)+\left(\overline{\rho\tilde{U}}_{m}r_{ij}\right)_{,m}=P_{ij}+c_{s}\left(\overline{\rho}\frac{k}{\tilde{\varepsilon}}r_{km}r_{ij,m}\right)_{,k}-c_{1}\overline{\rho}\frac{\varepsilon}{k}\left(r_{ij}-\frac{2}{3}k\delta_{ij}\right)$$

$$-\frac{c_{2}+8}{11}\left(P_{ij}-\frac{2}{3}P\delta_{ij}\right)-\frac{8c_{2}-2}{11}\left(D_{ij}-\frac{2}{3}P\delta_{ij}\right)-\frac{30c_{2}-2}{55}\overline{\rho}k\tilde{S}_{ij}$$

$$+\left[c_{1w}\overline{\rho}\frac{\varepsilon}{k}\left(r_{ij}-\frac{2}{3}k\delta_{ij}\right)+c_{2w}\left(P_{ij}-D_{ij}\right)+c_{3w}\overline{\rho}k\left(\tilde{U}_{i,j}+\tilde{U}_{j,i}-\frac{2}{3}\tilde{U}_{m,m}\delta_{ij}\right)\right]\frac{k^{3/2}}{\varepsilon y_{n}}$$

$$-\overline{u'_{i}}.\overline{P}_{,j}-\overline{u'_{j}}.\overline{P}_{,i}-\frac{2}{3}\overline{\rho\varepsilon}\delta_{ij}\ ,$$

with $P_{ij} = -\overline{\rho}(r_{il}\widetilde{U}_{j,l} + r_{jl}\widetilde{U}_{i,l})$, $D_{ij} = -\overline{\rho}(r_{il}\widetilde{U}_{l,j} + r_{jl}\widetilde{U}_{l,i})$,   [17.14]

and $P = \dfrac{1}{2}P_{jj} = \dfrac{1}{2}D_{jj}$, $\in = \overline{\rho}\varepsilon$.

We could also use a simpler "isotropization of production" model:

$$\frac{\partial}{\partial t}\left(\overline{\rho}r_{ij}\right) + \left(\overline{\rho}\widetilde{U}_m r_{ij}\right)_{,m} = P_{ij} + c_s\left(\overline{\rho}\frac{k}{\varepsilon}r_{km}r_{ij,m}\right)_{,k} - c_1\overline{\rho}\frac{\varepsilon}{k}\left(r_{ij} - \frac{2}{3}k\delta_{ij}\right)$$

$$-\gamma_1\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right) - \overline{u'_i}.\overline{P}_{,j} - \overline{u'_j}.\overline{P}_{,i} - \frac{2}{3}\overline{\rho}\varepsilon\delta_{ij}.$$

(in the absence of wall).

More recent models such as those by Speziale C.G., Sarkar S. and Gatski T.B., [SPE 91B] (cf. section 17.7) can be formally extended to the compressible case. The weak point of these approaches, however, is that they remain based on the Poisson equation for the pressure fluctuation obtained in incompressible flow and which thus takes no account of the wave operator.

It is now necessary to model the mean value of the mass weighted velocity fluctuation involved in the last two terms. Several types of hypothesis have been proposed and we present some possibilities.

First, we can note the identity:

$$\overline{u'_j} = -\frac{\overline{\rho'u'_j}}{\overline{\rho}}.$$ [17.15]

We can, considering this form, immediately suggest the approximation:

$$\overline{u'_j} = -\frac{1}{\overline{\rho}}\frac{\nu_t}{\sigma_1}\overline{\rho}_{,j},$$

where σ_1 is a numerical constant, or also makes use of the tensorial diffusion coefficient:

$$\overline{u'_j} = -c_\rho\frac{kr_{ij}}{\in}\overline{\rho}_{,i}.$$

If we assume polytropic behavior,

$$\frac{P}{\rho^n} = const \qquad \Rightarrow \frac{\rho'}{\overline{\rho}} = \frac{1}{n-1}\frac{\theta'}{\widetilde{\theta}}, \qquad \text{and then} \quad \overline{u'_j} \approx -\frac{1}{n-1}\frac{\varphi_{\theta j}}{\widetilde{\theta}}.$$

In the simpler case for which ρ is dependent only on θ (dilatable fluid) we can make the estimate directly:

$$\overline{u'_j} \approx -\frac{1}{\overline{\rho}}\left(\frac{\partial\overline{\rho}}{\partial\theta}\right)\overline{\theta'u'_j} \approx -\frac{1}{\overline{\rho}}\left(\frac{\partial\overline{\rho}}{\partial\theta}\right)\varphi_{\theta j}.$$

Taulbee D. and Van Osdol J., [TAU 91] propose a more refined approach based on the modeling of the transport equation for $\overline{u'_j}$.

The new advanced second order closures presented in Chapter 8 have also been extended and applied to the compressible regime in particular by Elbaz A.M., [ELB 91].

In compressible turbulence it is no longer possible to distinguish clearly the slow part from the rapid part in the pressure-strain correlations, and Fujiwara H. *et al.* propose for their modeling ([FUJ 00]) a global expression such as:

$$\Phi_{ij} = \overline{\rho}\varepsilon f(M_t)\left(C_1 + C_2\frac{P}{\overline{\rho}\varepsilon}\right)F(a_{ij}, \chi_{ij}, ...),$$

C_1, C_2, being numerical constants of order unity, and $\chi_{ij} = \dfrac{P_{ij} - \dfrac{2}{3}P\delta_{ij}}{P}$,

$P_{ij} = -\rho\overline{u'_j u'_k}\widetilde{U}_{i,k} - \rho\overline{u'_i u'_k}\widetilde{U}_{j,k}$, $\quad P = P_{jj}/2$, using Favre averaging, M_t turbulent Mach number.

17.4. Dissipation rate equation

We can refer to equations [17.12] but in this case also, the modeling methods are largely inspired by the formulations in incompressible regime. The problem of the approximation of the additional terms in equation [17.12] remains a complex problem, still open nowadays.

We will generally retain the following type of formulation:

$$\frac{\partial}{\partial t}\left(\overline{\rho}\varepsilon\right)+\left(\overline{\rho\tilde{U}}_j\varepsilon\right)_{,j}=-C_{\varepsilon 1}\overline{\rho}\frac{\varepsilon}{k}r_{ij}\tilde{U}_{i,j}+C_\varepsilon\left(\overline{\rho}\frac{k}{\varepsilon}r_{ij}\varepsilon_{,i}\right)_{,j}$$

$$-C_{\varepsilon 2}\overline{\rho}\frac{\varepsilon^2}{k}-C_{\varepsilon 3}\frac{\varepsilon}{k}\overline{u'_i}.\overline{P}_{,i}\,.\tag{17.16}$$

Coleman G.N. and Mansour N.N., [COL 91] have shown that for low Mach numbers, the effects of variation of the fluid viscosity according to temperature may be important. They thus propose to take them into account in the case of isotropic compression by introducing an additional term in equation [17.16]:

$$S_{CM}=\overline{\rho}\omega^2\frac{d\tilde{v}}{dt}-c_{2v}\frac{\overline{\rho}}{k}\frac{d\tilde{v}}{dt}\varepsilon_s\omega,\quad\omega=\sqrt{\overline{\omega'_i\omega'_i}}\approx\sqrt{\frac{\varepsilon_s}{v}}\text{ being the enstrophy}$$

$$S_{CM}=\frac{\overline{\rho}\varepsilon_s}{\tilde{v}}\frac{d\tilde{v}}{dt}-c_{2v}\frac{\overline{\rho}}{k}\frac{d\tilde{v}}{dt}\varepsilon_s\sqrt{\frac{\varepsilon_s}{\tilde{v}}}\,.$$

A similar method is followed by Le Penven L. and Serre G., [LEP 93] who distinguish the spectral flux ε_f from the true energy dissipation ε, in the case of compressed flows with varying viscosity. The compression strongly reduces ε without changing ε_f anymore since it is related to energetic eddies. This effect is then represented by a relaxation term in the equations for ε_f and ε.

17.5. Turbulent heat flux equations

By reference to RC models, we can suggest the following formulation:

$$\frac{\partial}{\partial t}\left(\overline{\rho}\varphi_{\theta i}\right)+\left(\overline{\rho}\tilde{U}_m\varphi_{\theta i}\right)_{,m}=-\overline{\rho u'_j\theta'}\tilde{U}_{i,j}-\overline{\rho u'_i u'_j}\tilde{\theta}_{,j}+c_\gamma\left(\overline{\rho}\frac{k}{\varepsilon}r_{km}\varphi_{\theta i,m}\right)_{,k}$$

$$-c_{\gamma 1}\overline{\rho}\frac{\varepsilon}{k}\varphi_{\theta i}-c_{\gamma 2}\overline{\rho}\varphi_{\theta j}\tilde{U}_{i,j}-\overline{\theta'}.\overline{P}_{,i}\,.\tag{17.17}$$

It is then necessary to approximate $\overline{\theta'}$.

For a polytropic fluid, we would get:

$$\overline{\theta'}=-\frac{\overline{\rho'\theta'}}{\overline{\rho}}\approx-\frac{1}{n-1}\frac{\widetilde{\theta'^2}}{\tilde{\theta}}\,.$$

If ρ is a function of θ only (dilatable fluid):

$$\overline{\theta'} = -\frac{1}{\overline{\rho}}\left(\frac{\partial\overline{\rho}}{\partial\overline{\theta}}\right)\overline{\theta'^2} \approx -\frac{1}{\overline{\rho}}\left(\frac{\partial\overline{\rho}}{\partial\widetilde{\theta}}\right)\widetilde{\theta'^2} \ .$$

17.6. Equation for the variance of temperature fluctuations

This equation reads:

$$\frac{\partial}{\partial t}\left(\overline{\rho}\widetilde{\theta'^2}\right) + \left(\overline{\rho}\widetilde{U}_m\widetilde{\theta'^2}\right)_{,m} = -2\overline{\rho u'_j\,\theta'}\,\widetilde{\theta}_{,j}$$

$$+c_{\widetilde{\theta^2}}\left(\overline{\rho}\frac{k}{\widetilde{\varepsilon}}r_{km}\widetilde{\theta'^2}_{,m}\right)_{,k} - 2\varepsilon_\gamma\,. \quad [17.18]$$

The dissipation rate ε_γ can be obtained from a hypothesis such as $\varepsilon_\gamma = \dfrac{\varepsilon}{R.k}\dfrac{\widetilde{\theta'^2}}{2}$ or through the use of a transport equation.

In the case of low Reynolds number turbulence or in the case of the very near wall region, modifications similar to those used in incompressible turbulence models must be introduced in the equations. They are directly inspired by them.

In general, it is also necessary to model energy equation [17.6].

Some authors (cf. Taulbee D. and Van Osdol J., [TAU 91]) have also modeled an equation for $\overline{\rho'^2}$. Similarly, with the aim of studying the influence of a shock on the turbulent flow going through, Zeman O., [ZEM 91] also uses an equation for the mass flux:

$$\frac{d\overline{\rho'u'_i}}{dt} = -\frac{\overline{\rho'u'_i}}{T_a} - r_{ij}\overline{\rho}_{,j} - \overline{\rho'u'_j}\,\overline{U}_{i,j}\,.$$

(T_a is a characteristic time).

According to the same formalism as Le Penven L. and Serre G., [LEP 93] (cf. section1 7.4), the two scale model presented by Pinheiro J. *et al.*, [PIN 97] applies to isotropic compression in the case of a scalar.

17.7. Two equation models and simplified models

The k-ε model relies on a behavior law giving the Reynolds stresses:

$$\overline{\rho \widetilde{u_i u_j}} = \frac{2}{3}\left(\rho k + \mu_t \widetilde{U}_{l,l}\right)\delta_{ij} - \mu_t\left(\widetilde{U}_{i,j} + \widetilde{U}_{j,i}\right),\tag{17.19}$$

in which $\widetilde{U}_{l,l}$ is generally non-zero while the trace of [17.19] remains equal to $2\rho k$.

So $P = \overline{\rho \widetilde{u_i u_j}}\,\widetilde{U}_{i,j}$ with $\mu_t = \overline{\rho}\dfrac{k^2}{\varepsilon}$ is written:

$$P = \frac{2}{3}\overline{\rho}k\widetilde{U}_{j,j} + \frac{2}{3}\mu_t\widetilde{U}_{j,j}{}^2 - \mu_t\left(\widetilde{U}_{i,j} + \widetilde{U}_{j,i}\right)\widetilde{U}_{i,j},$$

i.e. $P = P_{dil.} + P_{shear}$, in order to make clear the dilatation contribution and the shear contribution in the production term.

The first two terms are often relatively weak and are thus considered as negligible.

In highly compressible flows, the pressure-dilatation correlation term must be modeled separately. The approach of Sarkar S., [SAR 92] leads to:

$$\overline{p'u'_{j,j}} = \alpha_1\overline{\rho}r_{ij}\widetilde{U}_{i,j}M_t + \alpha_2\overline{\rho}\varepsilon_s M_t{}^2, \qquad \alpha_1 = 0.15, \ \alpha_2 = 0.20 \quad.$$

The application of the k-ε model to supersonic flows has also been studied by Guézengar D., *et al.* [GUE 99], considering the modifications to bring to the usual incompressible form of the model in the case of the supersonic turbulent boundary layer. We also mention the k-ω^2 model introduced by Saffman P.G., [SAF 70] and then by Wilcox D.C., [WIL 88A] applicable to compressible turbulent flows. In this model, ω is a characteristic frequency of the turbulent field which allows us to express the dissipation rate through $\varepsilon = \beta^* k\omega$ and the turbulence eddy viscosity by $\mu_t = \gamma k/\omega$.

Simpler models have also been used in particular in boundary layers, such as the Baldwin B.S. and Lomax H. model, [BAL 78] (cf. Chapter 12). Aupoix B., [AUP 92] gives useful comparisons between various models in compressible turbulence applied to the boundary layer.

As regards the boundary conditions, the universal velocity law no longer exists, but it is possible to use relations deduced from the Van Driest transformation ([VDR 51]) verifying:

$$u^+ = \frac{1}{u_*} \int_0^u \sqrt{\frac{\rho}{\rho_p}} . du \ , \quad u_* = \sqrt{\frac{\tau_p}{\rho_p}} \ ,$$

(the index p denotes quantities at the wall),

or also more complex laws (Viegas J.R. *et al.*, [VIE 83, VIE 85]). In this form the experiment also shows the existence of a logarithmic law in the case of the compressible boundary layer. A correct description of compressible boundary layers can only be obtained by supplementing additional diffusion terms in two equation models (cf. [CAT 99] and section 11.2.7).

With the aim of studying the effects of compression and dilatation, Chomiak J., [CHO 89A] relying on the work of Reynolds W.C. is led to modify the formulation of the dissipation rate equation:

$$\overline{\rho} \frac{d\varepsilon}{dt} = C_{\varepsilon 1} \overline{\rho} \frac{\overline{P}_{shear} \varepsilon}{k} - C_{\varepsilon 2} \overline{\rho} \frac{\varepsilon^2}{k} - \frac{1}{3} \overline{\rho} \frac{\overline{P}_{dil.} \varepsilon}{k} + Diff(\varepsilon) , \qquad [17.20]$$

in order to dissociate the shear production from the dilatation production.

The study of the effect of compression and dilatation (important for example in the applications to alternative piston machines) has been further pursued by Cambon C. *et al.*, [CAM 92B] who introduce a time dependent renormalization to make the Navier-Stokes equations invariant with respect to compression. The phenomenon, however, generally requires second order modeling to be correctly represented. Let us also mention the work by Coleman G.N. and Mansour N.N., [COL 91] based on numerical simulations of turbulence.

Explicit algebraic models and non-linear models presented in Chapter 11 can be extended in principle to compressible flows, considering that the terms containing the velocity divergence (which is non-zero in the present case) have to be kept in the expansions of invariant modeling.

17.8. Approach in non-weighted variables

Modeling based on mass weighted averaging, if it retains the advantage of relative simplicity, remains nevertheless more or less formal in the derivation of

closure hypotheses. Some authors (Ha Minh H. *et al.*, [HAM 81]; MacInnes J., [MCI 85]) have preferred to use traditional averaging and keep more explicitly the appearance of variable density terms in the equations.

We now present the hybrid formalism introduced by MacInnes.

A new variable "mass velocity" is introduced in short-hand notation $G_m = \rho U_m$; it thus follows that:

$$\overline{G}_m = \overline{\rho}\,\overline{U}_m + \overline{\rho' u_m}\;,$$

$$g_m = \overline{\rho}\, u_m + \rho'\overline{U}_m + \rho' u_m + \overline{\rho' u_m}\;. \tag{17.21}$$

This new variable allows us to reduce the cubic non-linearity of the convection terms to a quadratic non-linearity. Thus $G_m U_i = \rho U_m U_i$.

The advantage of the hybrid notation becomes clear if we take the mean value:

$$\overline{G}_m \overline{U}_i + \overline{g_m u_i} = \overline{\rho}\,\overline{U}_m \overline{U}_i + \overline{\rho' u_m}\,\overline{U}_i + \overline{\rho' u_i}\,\overline{U}_m + \overline{\rho}\,\overline{u_i u_m} + \overline{\rho' u_i u_m}\;. \tag{17.22}$$

17.9. Continuity

The instantaneous equation $G_{m,m} = -\partial_t \rho$ implies for the mean part and the fluctuating part:

$$\overline{G}_{m,m} = -\partial_t \overline{\rho}\;,$$

$$g_{m,m} = -\partial_t \rho'\;.$$

If the equation of state which links θ to ρ is transferred into the transport equation for θ we get:

$\theta = \dfrac{A}{\rho} + B$ where A and B are numerical constants,

$$\partial_t(\rho\theta) + (\rho U_m \theta)_{,m} = -J_{m,m}\;, \qquad J_m = -\lambda\theta_{,m}$$

$$\Rightarrow\quad \partial_t\left[\rho(A/\rho + B)\right] + \left[U_m \rho(A/\rho + B)\right]_{,m} = -J_{m,m}\;,$$

and then $\overline{U}_{m,m} = \dfrac{\lambda}{A}\overline{\theta}_{,mm}$.

We see that the divergence of the velocity field is generally non-zero, but it becomes negligible at high Péclet numbers.

At the level of fluctuations,

$$u_{m,m} = \frac{\lambda}{A}\theta_{,mm} \, ,$$

but even if λ is very small, the derivatives of the fluctuation θ may become very large for the small eddies and $u_{m,m}$ is not zero. It will however be possible to neglect $u_{m,m}$ if only the energy-containing eddies are considered.

17.10. Statistical equations and modeling

– Mean field equations:

$$\partial_t\left(\overline{\rho U_i}\right) + \left(\overline{G_k \overline{U_i}}\right)_{,k} = -\partial_t\left(\overline{\rho' u_i}\right) - \left(\overline{g_k u_i}\right)_{,k} - P_{,i} \, , \qquad [17.23]$$

$$\partial_t\left(\overline{\rho\theta}\right) + \left(\overline{G_k \overline{\theta}}\right)_{,k} = -\partial_t\left(\overline{\rho'\theta}\right) - \left(\overline{g_k\theta}\right)_{,k} \, , \qquad [17.24]$$

$$\partial_t\left(\overline{\rho u_i}\right) + \left(\overline{G_k u_i}\right)_{,k} = -\partial_t\left(\overline{\rho'\overline{U}_i + \rho' u_i - \overline{\rho' u_i}}\right) - p_{,i}$$
$$-\left(g_k\overline{U}_i + g_k u_i - \overline{g_k u_i}\right)_{,k} + \tau'_{ik,k} \, , \qquad [17.25]$$

τ_{ik} is the molecular stress tensor,

$$\partial_t\left(\overline{\rho\theta}\right) + \left(\overline{G_k\theta}\right)_{,k} = -\partial_t\left(\rho'\overline{\theta} + \rho'\theta - \overline{\rho'\theta}\right) - \left(g_k\overline{\theta} + g_k\theta - \overline{g_k\theta}\right)_{,k} - j_{k,k} \, , \qquad [17.26]$$

j_k is the molecular heat flux (or scalar flux).

– Poisson equation for the pressure fluctuations:

$$-p_{,nn} = \left(g_m u_n - \overline{g_m u_n}\right)_{,nm} + \partial_t\left(\rho' u_n - \overline{\rho' u_n}\right)_{,n} \qquad]A$$

$$+\overline{G}_{m,n}u_{n,m} + \overline{U}_{m,n}g_{n,m} + \overline{\rho}_{,n}\partial_t u_n \qquad]B$$

$$+\overline{G}_m u_{n,mn} + g_n\overline{U}_{n,mn} + \overline{\rho}\partial_t u_{n,n} + \rho'\partial_t\overline{U}_{n,n} + \partial_t\overline{U}_m\rho'_{,m} - \tau'_{mn,mn} \qquad]C$$

Group C can be neglected if it is assumed that the velocity field has zero divergence.

Group A essentially corresponds to turbulent interactions.

Group B involves the mean gradients of velocity and density.

The equations for the second order moments read:

$$\partial_t \overline{\rho u_i u_j} + \left(\overline{G_k u_i u_j}\right)_{,k} = -\overline{g_k u_i}\overline{U}_{j,k} - \overline{g_k u_j}\overline{U}_{i,k} - \overline{\tau'_{ik} u_{j,k}} - \overline{\tau'_{jk} u_{i,k}}$$

$$+\overline{p(u_{i,j} + u_{j,i})} - \left(\overline{g_k u_i u_j} + \overline{p u_i}\delta_{jk} + \overline{p u_j}\delta_{ik} - \overline{u_i \tau'_{jk}} - \overline{u_j \tau'_{ik}}\right)_{,k}$$

$$-\partial_t \left(\overline{\rho' u_i u_j}\right) - \overline{\rho' u_j}\partial_t \overline{U}_i - \overline{\rho' u_i}\partial_t \overline{U}_j ,$$

$$\partial_t \overline{\rho u_i \theta} + \left(\overline{G_k u_i \theta}\right)_{,k} = -\overline{g_k u_i}\overline{U}_{j,k} - \overline{g_k u_j \theta}_{,k} - \overline{\tau'_{ik} \theta}_{,k} - \overline{\tau'_{jk}\theta}_{,k} + \overline{p\theta}_{,i}$$

$$-\left(\overline{g_k u_i \theta} + \overline{p\theta}\delta_{ik} - \overline{\theta\tau'_{ik}} - \overline{u_i j'_k}\right)_{,k} - \partial_t \left(\overline{\rho' u_i \theta}\right) - \overline{\rho'\theta}\partial_t \overline{U}_i - \overline{\rho' u_i}\partial_t \overline{\theta} .$$

Each of these equations contains on its right-hand side five groups of terms: production, dissipation, pressure effect, diffusion and unsteadiness.

MacInnes reconsiders the form of the fluctuating pressure equation in the case of a zero divergence velocity field, then expanding each term, he is led to:

$$-p_{,nn} = \overline{\rho}\left(u_m u_n - \overline{u_m u_n}\right)_{,mn} + \left(\rho' u_m u_n - \overline{\rho' u_m u_n}\right)_{,mn}$$

$$+2\overline{\rho U}_{m,n}u_{m,n} + \left(\partial_t \overline{U}_m + \overline{U}_n \overline{U}_{m,n}\right)\rho'_{,m}$$

$$+2\overline{\rho}_{,m}\left(u_m u_n - \overline{u_m u_n}\right)_{,n} + 2\overline{U}_{m,n}\left(\rho' u_n - \overline{\rho' u_n}\right)_{,m} + S_p .$$

Source S_p contains terms which, while not being negligible, yield a negligible contribution when they are correlated with a velocity or scalar gradient.

The pressure correlations are then expressed using the green theorem. To each term in the Poisson equation for p corresponds a contribution in the pressure correlations:

$$\overline{p\left(u_{i,j} + u_{j,i}\right)} = \phi^{(0)}_{ij(1)} + \phi^{(\rho)}_{ij(1)} + \phi^{(0)}_{ij(2)} + \phi^{(\rho a)}_{ij(2)} + \phi^{(\rho b)}_{ij(2)} + \phi^{(\rho c)}_{ij(2)} ,$$

$$\overline{p\theta}_{,i} = \phi^{(0)}_{\theta i(1)} + \phi^{(\rho)}_{\theta i(1)} + \phi^{(0)}_{\theta i(2)} + \phi^{(\rho a)}_{\theta i(2)} + \phi^{(\rho b)}_{\theta i(2)} + \phi^{(\rho c)}_{\theta i(2)} .$$

Approximation of the non-linear terms:

$$\phi_{ij(1)}^{(0)} = -c_1 \bar{\rho} \frac{\varepsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} k \delta_{ij} \right),$$

$$\phi_{ij(1)}^{(\rho)} = -c_1^{\rho} \frac{\varepsilon}{k} \left(\overline{\rho' u_i u_j} - \frac{1}{3} \overline{\rho' u_m u_m} \delta_{ij} \right) - c_1^{\prime \rho} \frac{\varepsilon}{\rho k} \left(\overline{\rho' u_i}\, \overline{\rho' u_j} - \frac{1}{3} \overline{\rho' u_m}^2 \delta_{ij} \right),$$

$$\phi_{\theta i(1)}^{(0)} = -c_{\theta 1} \bar{\rho} \frac{1}{T} \overline{u_i \theta}, \quad T = \sqrt{\frac{k}{\varepsilon} \frac{\overline{\theta^2}}{\varepsilon_\gamma}},$$

$$\phi_{\theta i(1)}^{(\rho)} = -c_{\theta 1}^{\rho} \frac{1}{T} \overline{\rho' u_i \theta} - c_{\theta 1}^{\prime \rho} \frac{1}{T} \overline{\rho' u_i}\, \overline{\rho' \theta}.$$

Approximation of the linear terms:

$$\phi_{ij(2)}^{(0)} = c_2 \bar{\rho} \left(\overline{u_i u_n} \bar{U}_{j,n} + \overline{u_j u_n} \bar{U}_{i,n} - \frac{2}{3} \overline{u_m u_n} \bar{U}_{m,n} \delta_{ij} \right),$$

$$\phi_{ij(2)}^{(\rho a)} = \frac{3}{10} \left[\overline{\rho' u_i} \left(\partial_t \bar{U}_j + \bar{U}_n \bar{U}_{j,n} \right) + \overline{\rho' u_j} \left(\partial_t \bar{U}_i + \bar{U}_n \bar{U}_{i,n} \right) \right.$$
$$\left. - \frac{2}{3} \overline{\rho' u_m} \left(\partial_t \bar{U}_m + \bar{U}_n \bar{U}_{m,n} \right) \delta_{ij} \right],$$

$$\phi_{ij(2)}^{(\rho b)} = c_2^{\rho} \left(\overline{\rho' u_i u_n} \bar{U}_{j,n} + \overline{\rho' u_j u_n} \bar{U}_{i,n} - \frac{2}{3} \overline{\rho' u_m u_n} \bar{U}_{m,n} \delta_{ij} \right),$$

$$\phi_{ij(2)}^{(\rho c)} = D \left(\overline{u_i u_j u_n \rho}_{,n} - \frac{1}{3} \overline{u_k^2 u_n \rho}_{,n} \delta_{ij} \right)$$
$$+ E \left(\overline{u_k^2 u_i \rho}_{,j} + \overline{u_k^2 u_j \rho}_{,i} - \frac{2}{3} \overline{u_k^2 u_n \rho}_{,n} \delta_{ij} \right),$$

$$\phi_{\theta i(2)}^{(0)} = c_{\theta 2} \overline{\rho u_n \theta} \bar{U}_{i,n},$$

$$\phi_{\theta i(2)}^{(\rho a)} = \frac{1}{3} \overline{\rho' \theta} \left(\partial_t \bar{U}_i + \bar{U}_n \bar{U}_{i,n} \right),$$

$$\phi_{\theta i(2)}^{(\rho b)} = c_{\theta 2}^{\rho} \overline{\rho' u_n \theta} \bar{U}_{i,n},$$

$$\phi_{\theta i(2)}^{(\rho c)} = 0.$$

These approximations are all obtained using the same reasoning as presented in Chapters 6 and 9.

Let us note that the expansions such as [17.22] also involve triple correlation terms which can be either approximated or neglected depending on the considered case.

17.11. Dissipation rate equation

Formally, the modeled equation for ε can be written:

$$\partial_t(\rho\varepsilon) + \left(\overline{G_j}\varepsilon\right)_{,j} = \overline{\rho}\frac{\varepsilon^2}{k}\varPsi_\varepsilon + C_\varepsilon\left(\overline{g_ju_i}\frac{k}{\varepsilon}\varepsilon_{,j}\right)_{,i},$$

$\varPsi_\varepsilon$ being a dimensionless function of the invariants which govern the spectral transfer. This function can be divided into two parts $\varPsi_\varepsilon = \varPsi_\varepsilon^{(0)} + \varPsi_\varepsilon^{(\rho)}$.

$\varPsi_\varepsilon^{(0)}$ contains the usual terms already present in the incompressible case.

$$\varPsi_\varepsilon^{(0)} = C_{\varepsilon 1}\frac{P}{\overline{\rho}\varepsilon} - C_{\varepsilon 2},$$

P being the total production. $P = -\overline{g_m u_n}\overline{U}_{n,m} - \overline{\rho'u_n}\partial_t\overline{U}_n$.

MacInnes, relying on stability considerations in a non-uniform density field, is led to propose for the second part, which accounts for density effects:

$$\varPsi_\varepsilon^{(\rho)} = C_{\varepsilon 1}^\rho\frac{k}{\varepsilon}\frac{\overline{\rho'u_l}}{\overline{\rho}^2}\overline{\rho}_{,l} + C_{\varepsilon 2}^\rho\frac{\overline{u_m^2 u_l}}{\overline{\rho}\varepsilon}\overline{\rho}_{,l}. \qquad [17.27]$$

17.12. Other approaches

17.12.1. *Other decomposition schemes*

We mention a proposition by Chassaing P., [CHA 85] which is based on a three term decomposition.

$$\mathcal{M}_{ij} = \rho U_i U_j,$$

$$\overline{\mathscr{M}_{ij}} = \overline{M_{ij}} + \overline{m_{ij}} + \left(\overline{K_i^j} + \overline{K_j^i} \right),$$

$$M_{ij} = \rho \overline{U}_i \overline{U}_j, \qquad m_{ij} = \rho u_i u_j, \qquad K_i^j = \rho u_i \overline{U}_j .$$

In this splitting, the first term M_{ij} is characteristic of the mean flow, the term m_{ij} represents the "main flux" due to turbulent fluctuations whereas the last additional terms $K_i^j + K_j^i$, which come from the source term in the continuity equation, are a type of "external flux" interpreted through the account of a thermodynamic state law. Chassaing P. *et al.*, [CHA 94] on the basis of this ternary splitting, make the closure of the equations of moments $\overline{\rho u_i u_j}$, $\overline{\rho \theta u_j}$ and $\overline{\rho \theta^2}$ where θ is representing here the concentration.

It is also possible to devise transport equations for statistical turbulence moments that involve only Reynolds centered averaging, derived from the Navier-Stokes equations after having divided each side by the density. It is this type of formulation which is used by Shih T.H. *et al.*, [SHI 87] and Panchapakesan W.R. and Lumley J.L., [PAN 93A and B]. This approach is based on a Taylor expansion of $1/(\overline{\rho} + \rho')$ by supposing $\rho'/\overline{\rho}$ to be small.

17.12.2. *Using approximate expansions*

This previous formulation can also be related to that proposed by Rey C., [REY 85], Rey C. and Rosant J.M., [REY 90] and Rey C. *et al.*, [REY 91] which is based on the transport equation of the instantaneous velocity written in the form:

$$\frac{dU_i}{dt} = g_i - \frac{RT}{\overline{P}}(1 - \frac{p'}{\overline{P}})(P_{,i} - \tau_{ij,j}) , \qquad [17.28]$$

which is deduced from a first order expansion of the fluctuating pressure given by $\dfrac{1}{P} = \dfrac{1}{\overline{P}(1+\frac{p'}{\overline{P}})} \approx \dfrac{1}{\overline{P}}(1 - \frac{p'}{\overline{P}})$ and leading to $\dfrac{1}{\rho} \approx \dfrac{RT}{\overline{P}}(1 - \frac{p'}{\overline{P}})$. This formulation can

be viewed as a balance equation with constant mass, for any quantity (momentum, concentration, energy). The statistical treatment in compressible turbulent regime thus allows a formal structuring of the equations containing deviatoric terms in addition to the incompressible usual equations. The deviatoric terms which embody the effects of volume fluctuations in the moment equations are separated and

organized into a hierarchy in the order to be approximated. The resulting models apply to strongly heated turbulent flows, with large density fluctuations for which the usual Boussinesq approximation is not justified. Another important interest in this formulation, is that it lends itself to a spectral analysis in the sense of Craya A., [CRA 58] (cf. Chapter 5). This approach has been systematized in Rey C. [REY 00] who proposes to use balance equations established for a fluid unit of mass rather than a unit of volume as made usually. The continuity condition then introduces a new parameter $\Theta = Log \dfrac{\mathscr{U}}{\mathscr{U}_r}$, where $\mathscr{U} = 1/\rho$ is the volume per unit mass, and $\mathscr{U}_r$ its reference value. If $|\Theta| < Log\,2$, it can be shown that two contributions can be distinguished in the equations and modeled separately: a first contribution remains formally identical to the incompressible case whereas the second is directly related to the volume variations. This formalism applies in particular to free convection ([PAV 02]).

17.12.3. *Not truly compressible variable density flows*

When the density varies according to temperature or the composition of mixing but remains independent of pressure, the effects of compressibility no longer appear in the equations. This is the case for example for the mixing of miscible constituents with different densities. The application of second order closures can be made in the framework of finite volume methods which are direct extensions of those used for incompressible flows with constant density ([RUF 94], [GHA 96]).

17.13. Note on compressed turbulence

Compressed turbulence finds an important application in the study of flows in the cylinders of internal combustion engines. In this approximation of compressed turbulence, the density is supposed uniform, according to temperature and time and determined by the compression rate. Thus,

$$\rho = M/\mathscr{V} \ ,$$

M denoting the mass of fluid contained in the variable volume $\mathscr{V}$ and consequently:

$$\frac{dM}{dt} = \oiint_{\partial\mathscr{V}} \rho.(U - U_s)d\sigma \quad \text{or} \quad M = M_0 + \int_0^t dt \oiint_{\partial\mathscr{V}} \rho.(U - U_s)d\sigma \ ,$$

U_s denoting the velocity of the moving wall boundary in a coordinate system attached to the walls.

Supposing that the influence of acoustic waves on turbulence is negligible ([WU 85A and B]), the fluctuating field can be considered as incompressible with $u_{j,j} = 0$ for the velocity fluctuations, although interacting with compressed mean field.

In the case of a cylindrical volume with radius R and height $L(t)$, limited by a moving piston with area $s = \pi R^2$, $\mathcal{V} = \pi R^2 . L(t)$ and thus:

$$\frac{\rho}{\rho_0} = \frac{L_0 M_0 + \int_0^t \rho s (U - U_s) dt}{M_0 L(t)},$$

or after taking the derivative:

$$\frac{1}{\rho} \frac{d\rho}{dt} = \frac{\rho_0 L_0 s}{M_0 L(t)} U(t) - \frac{1}{L(t)} \frac{dL(t)}{dt},$$

which allows us to obtain the density.

The compression rate is given by: $S = div(\vec{U}) = -\frac{1}{\rho} \frac{d\rho}{dt}$ and the evolution of the density satisfies $\rho = \rho_0 e^{-\int_0^t S.dt}$.

A transformation of coordinates makes it possible to keep constant the number of discretization cells:

$$\begin{cases} T = t \\ \zeta = x/L(t) \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial x} = \dfrac{1}{L(t)} \dfrac{\partial}{\partial \zeta} \\[2mm] \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial T} - \dfrac{L'(t)}{L(t)} \zeta \dfrac{\partial}{\partial \zeta} \end{cases}, \quad L'(t) = U_s.$$

The set of the previous hypotheses allows us to use numerical techniques using a Poisson equation for pressure similar to the ones used in incompressible regime. In the turbulence equations, the terms involving the divergence of the mean field play an essential role (in particular the term P_{dil} introduced previously). The compression phenomenon in itself is often complicated by the occurrence of vortices

created by the injection flux ([LER 98]). The usual models, $k - \varepsilon$ and $R_{ij} - \varepsilon$, have, however, not yielded correct predictions of this type of flow with strong departures from equilibrium. Consequently, several works have resorted to the use of multiscale models ([WAT 91]). The combined effect of the compression and the vortex created at intake in the cylinder can be reasonably approximated ([JAK 00]) using an advanced low Reynolds number $R_{ij} - \varepsilon$ model. The contribution of direct numerical simulations of turbulence ([GUN 95], [JAK 00]) offers an interesting way to solve these flows with very complex behavior. The work by Hadzic I. *et al*. [HAD 01] on the study of turbulence submitted to cyclic strain in internal combustion engines, tends to show that second order closures are necessary, because all the eddy viscosity models cause the production of two much artificial energy during the whole cycle of operation.

This page intentionally left blank

Chapter 18

Multiple Scale Models

Equilibrium turbulence is based on the concepts of the Kolmogorov theory, which imply that the dissipation rate of energy is equal to the energy flux cascading from the large eddies towards the dissipative eddies. According to this view, the dissipation level is independent of the Reynolds number (if it is sufficiently high). Modifying the Reynolds number has the only consequence of changing the dissipation scale. Thus, the dissipation rate is entirely determined by the energy containing eddies.
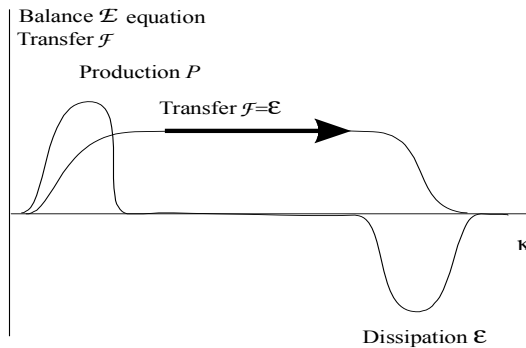
Figure 18.1. *The spectral "pipeline" of J.L. Lumley*: *spectral flux in equilibrium turbulence*

When the external conditions are changing, the problem becomes more complex and most of the traditional models use a modeled equation for ε in which the dominant terms represent the difference between two antagonistic effects that are the

energy cascade and the viscous sink. This equation is, however, largely empirical in its modeling because of our incomplete knowledge of the acting processes.

In reality, in an evolving turbulence, the true dissipation will not be exactly equal to the energy flux issuing from the energetic eddies, there will be some delay corresponding to the characteristic time scale for the big eddies to cascade and further give rise to the smaller dissipative eddies. This is what Lumley, 1996, (in [*GAT 96]) calls the "spectral pipeline" (Figures 18.1 and 18.2).
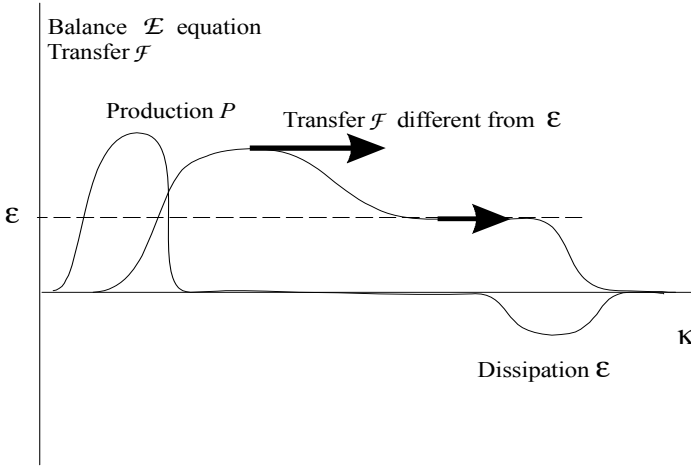
Figure 18.2. *The spectral "pipeline" of J.L. Lumley: spectral flux in non-equilibrium turbulence*

This picture is also simplified, because it is known that there are local interactions and non-local interactions. In reality, a small part of the energy transfer is directly made between more remote wavenumbers. The origin lies in the triad interactions that can be pointed out in the Navier-Stokes equations in spectral space:

$$\left(\frac{\partial}{\partial t} + \nu \kappa_j^2\right)\hat{u}_i = -i\kappa_j \Delta_{mi}\left(\hat{u}_m * \hat{u}_j\right)$$

$$= -i\kappa_j \Delta_{pi} \int \underbrace{\hat{u}_p(\vec{m})\hat{u}_j(\vec{q})\delta(\vec{m}+\vec{q}-\vec{\kappa})}_{(*)}\,d\vec{m}d\vec{q}\,,$$

(in the particular case $\overline{U}_j = 0$) and that can be recovered in the double correlation equations:

$$\left(\frac{\partial}{\partial t} + 2\nu\kappa_m^2\right)\overline{\hat{u}_i \hat{u}^*_j} = -i\kappa_p \Delta_{lj} \int \hat{u}_p(\vec{m})\hat{u}_i(\vec{q})\hat{u}_j(-\vec{\kappa})\delta(\vec{m}+\vec{q}-\vec{\kappa})d\vec{m}d\vec{q}$$

$$+ \text{term} \left(\vec{\kappa} \leftrightarrow -\vec{\kappa}, i \leftrightarrow j \right).$$

The Dirac factor shows that the wavevectors are only involved such that $\vec{\kappa} = \vec{m} + \vec{q}$, it is the triadic interaction which gives rise to local interactions if the triangle formed by the three vectors is almost an equilateral triangle and to non-local interactions if the triangle is very elongated or very flattened. Local interactions involve wavenumbers that are near whereas non-local interactions involve remote wavenumbers, skipping the intermediate range of wavenumbers. Thus, direct transfers towards immediately larger wavenumbers are strong, transfers towards more remote wavenumbers are weaker. There may also be an inverse transfer (backscatter) towards smaller wavenumbers (Figure 18.3).
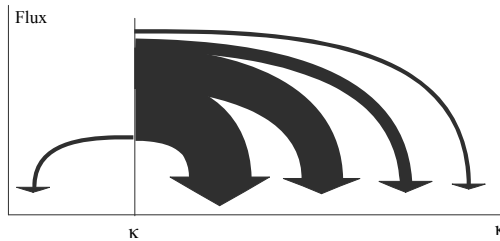
Figure 18.3. *Local and non-local interactions*

Turbulence models based on one point closures are particularly useful for studying and predicting non-homogenous turbulent shear flows, i.e. those encountered in industrial practice. However, these prediction methods, independently of their degree of sophistication, have not allowed the development of sufficiently universal tools that would apply to an extended range of flow types of real practice. If the study of complex turbulent flows led us to point out several limitations of the one point closures, there are also traditional shear flows in relatively simple geometries that are still posing modeling problems and paradoxes. Among these, we cited the example of the spreading rate anomaly in plane jets and circular jets, the paradox of turbulent flows with weak energy production or those with strong production, the skin friction in boundary layer in strong adverse pressure gradients. The limitations of traditional one point closures have various origins. First of all, they rely upon local hypotheses, whereas the large eddies play an important role in the turbulent transport process. The structure of these big eddies is dependent on the type of flow that is considered and an extended universality does not seem to be reachable if this fact is not taken into account correctly in modeling. The development techniques of one point closures as they have been systematized by Lumley show that strong hypotheses are necessary: choice of a level of closure, choice of known and unknown parameters, weak anisotropy and weak non-

homogenity hypotheses, doubtful hypothesis of isotropy of the small scales (a problem that has been reconsidered by Launder and Reynolds), and realizability conditions. Thus, even in the simplest case and the most academic case of homogenous turbulence, the complexity of turbulence interactions have been only partially and incompletely described. The usual one point closures also implicitly refer to the Kolmogorov theory of universal equilibrium by characterizing the whole energy cascade by a single scalar, which is the dissipation rate ε. The numerical constants appearing in the modeled equations are determined by reference to basic turbulent flows that are well known experimentally. For these various reasons, the models will be applicable only to a limited class of turbulent flows. These models have not succeeded in predicting correctly the flows that are complex due to their geometry (streamline curvature, configurations generating recirculations, presence of walls, etc.) or due to additional physical phenomena interacting with the turbulence field (rotation, gravity forces, etc.). On the practical point of view, among the weaknesses of one point closures, we shall emphasize the following: gradient turbulent diffusion hypothesis, eddy viscosity schemes, quasi-isotropic approximation for the rapid part of the pressure-strain correlation terms, the Rotta approximation for its non-linear part, modeling of source terms in the turbulence kinetic energy dissipation rate equation and lack of an account of the spectral character of turbulence.

Multiple scale modeling essentially tackles these two latter problems and also their consequences on the closure of the pressure-strain correlations. Almost all traditional turbulence models use a single time scale or a single length scale for describing the turbulent field. The precise definition of this time scale depends on the type of model considered: for example $T = (\partial U / \partial y)^{-1}$ is used in mixing length models whereas $T = k / \varepsilon$ is used in energy-dissipation k-ε models and the Reynolds stress transport models $R_{ij} - \varepsilon$. The same unique time scale is then used in the closure hypotheses in the model. The physical meaning of such a practice corresponds to an underlying hypothesis of universal spectrum with invariable shape. This means that the turbulence energy spectrum can only evolve by changing its global level of intensity or by rescaling its frequency scale. In reality, turbulence is characterized by an extended spectrum of fluctuation frequencies and the turbulent interactions are associated with the different parts of this evolving spectrum. The multiple scale models are a way to account for this physical reality in the framework of one point closures. The origin of this concept is a differential model by Schiestel R., [SCH 72A and 74A], [GOS 72 and 73] using two independent length scales obtained by solving two separate transport equations. The different turbulent processes were then described using different characteristic scales. This concept has been developed further and placed in the physical context of spectrum splitting in Launder B.E. and Schiestel R., [LAU 78]; Hanjalic K. *et al.*,

[HAN 79]. The model equations have been formulated afterwards in a more analytical way by deducing them from partial integration of the spectral equations in homogenous anisotropic turbulence, Schiestel R., [SCH 83 and 87]. The wavenumber intervals have been defined in such a way that their variations comply with the evolution of the energy spectrum in shape and size. In this framework, the closure hypotheses can be introduced on more reliable grounds and the source terms in the transport equations get a clearer meaning. The partial energies and the spectral fluxes are then used to build the different characteristic time scales and length scales. Thus, the different mechanisms such as return to isotropy, dissipation, turbulent diffusion, or eddy viscosity can be described by using different characteristic scales. Dimensionless grouping built on quantities such as partial energies and spectral fluxes can be viewed as non-equilibrium parameters which "measure" the departure from the equilibrium situation. The proposed model thus relies on the two point statistical equations and spectral closures. Most theories of this latter type are limited to homogenous turbulent flows and often also isotropic, Hinze J.O., 1975. The extension of spectral equations to anisotropic homogenous turbulence has developed, Jeandel D., Brison J.F., Mathieu J., [JEA 78] from the work of Craya A., [CRA 58]. The two point closures give a deeper and more refined insight into turbulent interaction mechanisms. This level of description is necessary for analyzing out of equilibrium flows in which the hypothesis of an instantaneous response of turbulent structures to the action of mean flow which created them is no longer valid (examples of a sudden disturbance applied to a turbulent flow, of an abrupt spatial variation, of the injection of orderly structures into a flow with prescribed frequencies, of pulsated forcing, etc.). These two point theories range from the first simpler closures such as the Kovasnay hypothesis or the Heisenberg hypothesis up to far more complex closures such as the direct interaction theory by Kraichnan or the EDQNM by Orszag.

In the approach presented in Schiestel R., [SCH 83] we have used the simplest hypothesis on inertial spectral transfer, the Kovasnay hypothesis, but the way remains open of course for devising more elaborate approaches.

The multiple scale concept leads to a new approach of several modeling problems in turbulence: determination of the dissipation rate of turbulence energy from transport equations, approximation of the pressure-strain correlations and the return to the isotropy process. Various applications have been considered, Gleize V. *et al.*, [GLE 96] have applied this type of model in particular in the case of non-equilibrium turbulent boundary layers, while Grégoire O. *et al.*, [GRE 97] consider non-equilibrium situations with very high density gradients.

This type of model has also been implemented by Cler A., [CLE 82] and then by Kim S.W., [KIM 90] in particular or Yamamoto M. [YAM 95] and by Aupoix B., [AUP 87] and by Wilcox D.C., [WIL 88B] in a somehow different form. We can

mention the work of Rubinstein R. related to this subject ([RUB 00]). The model by Wilcox D.C., [WIL 88B], uses, however, only one length scale equation and associates the spectral energy flux with the redistribution process. This cannot directly represent the spectral mechanisms (since the redistribution occurs without change in length scales). Liou W.W. *et al.*, [LIO 95] considered the case of compressible flows. Nagano Y. *et al.*, [NAG 97] rely on direct numerical simulations.

This concept of time delay in the energy cascade process together with that of history of turbulence has also led Lumley J.L. to propose a two scale scheme (cf. section 8.6 and also [*GAT 96] p.14).

In the split spectrum models, if the number of spectral slices is increased, we tend in the limit to a spectral closure model, the portioning corresponding then to a numerical discretization of the spectrum.

On this purpose we can mention the interesting approach developed by Parpais S. and Bertoglio J.P., [PAR 96] (cf. also [BES 96], [TOU 00], [TOU 02]) based on a non-homogenous extension of a spectral model. The transport equation of the energy spectrum is then written as:

$$\left(\frac{\partial}{\partial t} + U_i \partial_i \right) E(\kappa, \vec{x}) = P(\kappa, \vec{x}) + T(\kappa, \vec{x}) + D(\kappa, \vec{x}) + 2\nu \left(\frac{1}{4} E_{,jj}(\kappa, \vec{x}) - \kappa^2 E(\kappa, \vec{x}) \right).$$

The different terms in the right-hand side respectively represent: production, spectral transfer, non-homogenous transport, diffusion and viscous dissipation. The transfer term $T(\kappa, \vec{x})$ is obtained from EDQNM closure (cf. section 1.3.3.3). The other terms are obtained from the following hypotheses (with usual notations):

$$P(\kappa, \vec{x}) = -\varphi_{ij}(\kappa, \vec{x})U_{i,j}, \qquad \text{production term,}$$

$$\varphi_{ij}(\kappa, \vec{x}) = 2\left(E(\kappa, \vec{x})\frac{\delta_{ij}}{3} - \nu_{ts}(\kappa, \vec{x})S_{ij} \right), \qquad S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}),$$

approximation of the spectral tensor according to the energy spectrum $E(\kappa, \vec{x})$ and a spectral viscosity $\nu_{ts}(\kappa, \vec{x})$ in a form similar to [11.3] transposed to spectral space,

$$\nu_{ts}(\kappa, \vec{x}) = \frac{E(\kappa, \vec{x})\tau_s(\kappa, \vec{x})}{A_0 + A_\tau \sqrt{S_{ij}S_{ij} + \omega_{ij}\omega_{ij}}.\tau_s(\kappa, \vec{x})},$$

expression for the spectral viscosity,

$$\tau_s(\kappa,\vec{x}) = \left(\kappa^3 E(\kappa,\vec{x})\right)^{-1/2},$$

characteristic time scale (compare to [1.32]),

$$D(\kappa,\vec{x}) = \left[\left(\int_0^\infty \nu_{ts}(q,\vec{x})dq\right)\left(E(\kappa,\vec{x})\right)_{,i}\right]_{,i},$$

turbulent diffusion term,

$$R_{ij}(\vec{x}) = \int_0^\infty \varphi_{ij}(q,\vec{x})dq,$$

Reynolds stress tensor at point $\vec{x}$.

A spectral cut-off $\kappa_C(\vec{x})$ is also introduced in order to account for wall effects. We can consider globally this approach as a transported homogenous spectral model in which $E(\kappa,\vec{x})$ is the "tangent" spectrum at point $\vec{x}$.

18.1. Intuitive approach

18.1.1. *The single scale concept*

In the closure of the Reynolds stress transport equations, several hypotheses involve a length scale:

– the pressure-strain correlations

$$\Phi_{ij}^{(1)} = -c_1 \frac{\sqrt{k}}{L_\phi}\left(R_{ij} - \frac{2}{3}k\delta_{ij}\right), \tag{18.1}$$

(Rotta expression)

L_ϕ being the characteristic scale acting in this process;

– another important hypothesis is related to the dissipation rate

$$\varepsilon_{ij} = \frac{2}{3}\frac{k^{3/2}}{L_\varepsilon}\delta_{ij}, \tag{18.2}$$

(local isotropy hypothesis).

$L_{\mathcal{E}}$ is a dissipation scale given by contraction of [18.2]:

$$L_{\mathcal{E}} = \frac{k^{3/2}}{\varepsilon}$$

– length scales are also involved in the turbulent diffusion terms.

$$\overline{u_i q} = -c_q L_q \sqrt{k} \, \overline{Q}_{,i} \, .$$
[18.3]

Most of the usual turbulence models assume $L_\phi = L_{\mathcal{E}} = L_q$. Closure is thus achieved by an algebraic definition of the scale or by a transport equation for a quantity attached to this scale (notably the dissipation rate $\mathcal{E}$). These are single scale models. The turbulence field is then described using a single time scale or length scale.

From the physical point of view, $L_{\mathcal{E}}$ and $\mathcal{E}$ are attached to microturbulence whereas L_ϕ is associated with energy-containing eddies.

The hypothesis $L_\phi = L_{\mathcal{E}}$ is thus only acceptable in circumstances where the turbulence spectrum is in equilibrium, i.e. the energy flux through the spectrum is constant, it does not vary with the wavenumber.

The single scale models are thus based on an implicit hypothesis of spectral equilibrium.

18.1.2. *The multiple scale concept*

In reality, any rapid change in the production rate will cause imbalance between the different spectral zones and consequently variations in the energy fluxes through the spectrum. There is no reason for the scales L_ϕ and $L_{\mathcal{E}}$ to remain proportional.

Considering the mean energy spectrum equation (equation [5.9]) which will be written in abridged form:

$$\frac{\partial \mathcal{E}}{\partial t} = P + \mathcal{T} - 2\nu\kappa^2 \mathcal{E} \, ,$$

or $\quad \dfrac{\partial \mathcal{E}}{\partial t} = P - \dfrac{\partial \mathcal{F}}{\partial \kappa} - 2\nu\kappa^2\mathcal{E}$, with $\quad \mathcal{F} = -\displaystyle\int_0^\kappa \mathcal{T}(\kappa')d\kappa'$,
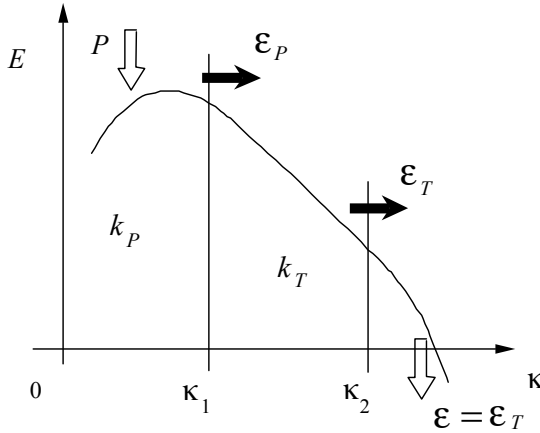
Figure 18.4. *Two level multiple scale model (two-scale model)*

The simplest basic scheme (cf. Figure 18.4) consists of introducing a splitting of the mean spectrum into three zones: production zone, transfer zone and dissipation zone.

We can suppose that production is only appreciable in the first zone and that dissipation is important only in the third zone. In the second zone only transfer takes place

– production zone, $\qquad \dfrac{\partial \mathcal{E}}{\partial t} = P - \dfrac{\partial \mathcal{F}}{\partial \kappa}$;

– transfer zone, $\qquad \dfrac{\partial \mathcal{E}}{\partial t} = -\dfrac{\partial \mathcal{F}}{\partial \kappa}$;

– dissipation zone, $\qquad \dfrac{\partial \mathcal{E}}{\partial t} = -\dfrac{\partial \mathcal{F}}{\partial \kappa} - 2\nu\kappa^2\mathcal{E}$.

We shall suppose also that the partial energy pertaining to the third zone is negligible (cf. Figure 18.5).
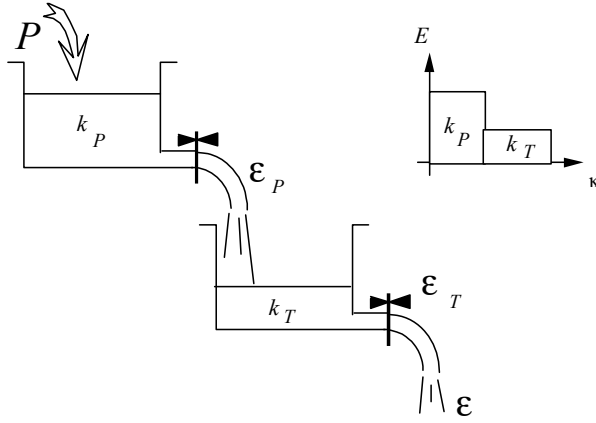
Figure 18.5. *Sketch of turbulence energy levels in spectral zones as a "hydraulic analogy"*

Partial energies and spectral fluxes are obtained from transport equations, Launder B.E. and Schiestel R., [LAU 78]:

$$\frac{dk_P}{dt} = P - F_P + Diff(k_P),$$

$$\frac{dk_T}{dt} = F_P - F_T + Diff(k_T),$$

$$\varepsilon = F_T,$$

$$\frac{dF_P}{dt} = C_{P1}\frac{PF_P}{k_P} - C_{P2}\frac{F_P^2}{k_P} + Diff(F_P),$$

$$\frac{dF_T}{dt} = C_{T1}\frac{F_P F_T}{k_T} - C_{T2}\frac{F_T^2}{k_T} + Diff(F_T), \qquad [18.4]$$

$C_{P1}, C_{P2}, C_{T1}, C_{T2}$ being numerical constants or functions of $\xi = k_P / k_T$ or $\eta = F_P / F_T$ (spectral shape parameters).

It appears that only the equation for F_P is directly sensitive to the action of mean flow through P. If coefficients $C_{P1}, C_{P2}, C_{T1}, C_{T2}$ were constant the equation for F_P would be entirely independent of the equation for F_T i.e. the large eddies would be independent of the smaller eddies. When these coefficients are considered according to the spectral energy distribution, this allows a feedback of information from large wavenumbers to smaller wavenumbers.

It would be possible to refine the description by adding thinner and more numerous spectral slices (the improvement compared to a single scale scheme must however be already noticeable on a simple three zone splitting).

18.1.3. *How can multiple scales be used?*

1) In a Reynolds stress transport model

Different scales are introduced to model on the one hand the pressure-strain correlations and on the other hand the viscous dissipation rate:

(*) Rotta hypothesis $\Phi_{ij}^{(1)} = -c_1 \dfrac{F_P}{k} \left(R_{ij} - \dfrac{2}{3} k \delta_{ij} \right)$,   [18.5]

related to the length scale $L_\phi = k^{3/2} / F_P$.

(**) dissipation terms $\varepsilon_{ij} = \dfrac{2}{3} F_T \delta_{ij}$,

related to the length scale $L_\varepsilon = k^{3/2} / F_T$.

2) In an eddy viscosity model

Different scales are introduced to model on the one hand the turbulence eddy viscosity and on the other hand the dissipation rate:

(*) turbulence eddy viscosity $\nu_t = c_\mu \dfrac{k^2}{F_P} = c_\mu L_\phi \sqrt{k}$,   [18.6]

related to the length scale $L_\phi = k^{3/2} / F_P$.

(**) dissipation scale $\varepsilon = \dfrac{k^{3/2}}{L_\varepsilon}$,

related to the scale $L_\varepsilon = k^{3/2} / F_T$.

It is possible to compare the expression for turbulence eddy viscosity to that obtained from algebraic modeling (cf. Chapter 11), indeed the hypothesis $\nu_t = c_\mu \dfrac{k^2}{F_P}$ can be formally written $\nu_t = c'_\mu \dfrac{k^2}{\varepsilon}$ as in the traditional k-ε model but with a variable coefficient $c'_\mu = c_\mu \dfrac{F_T}{F_P}$, that accounts for non-equilibrium.

18.1.4. *First applications to simple turbulent flows*

18.1.4.1. *Turbulence decay behind a grid*

It can be shown from equations [18.4] applied to the decay of isotropic turbulence behind a grid that:

$$\frac{k_P}{k_T} \to \frac{C_{P2} - C_{T2}}{C_{T2} - C_{T1}},$$ [18.7]

for *x* large, in the initial period of decay, k_P and k_T obey the same decay law:

$$k_P \approx x^{\frac{1}{1-C_{P2}}}, \qquad k_T \approx x^{\frac{1}{1-C_{P2}}}.$$

18.1.4.2. *Wall turbulence (constant shear region)*

It can be shown that $\dfrac{k_P}{k_T} \to \dfrac{C_{P2} - C_{P1}}{C_{T2} - C_{T1}}$, [18.8]

$$k_P = const., \quad k_T = const., \quad \varepsilon_P \sim 1/y, \quad \varepsilon_T \sim 1/y.$$

18.1.5. *First applications to turbulent free flows*

Within the approximation of thin shear flows:

$$\frac{dk_P}{dt} = -R_{12}\overline{U}_{1,2} - F_P + c_P \left(L\sqrt{k}\,k_{P,2}\right)_{,2},$$

$$\frac{dk_T}{dt} = F_T - F_P + c_T \left(L\sqrt{k}\,k_{T,2}\right)_{,2},$$

$$\frac{dF_P}{dt} = -C_{P1}\frac{F_P R_{12}}{k_P}\overline{U}_{1,2} - C_{P2}\frac{F_P^{\,2}}{k_P} + c_{P3}\left(L\sqrt{k}\,F_{P,2}\right)_{,2},$$

$$\frac{dF_T}{dt} = -C_{T1}\frac{F_P F_T}{k_T} - C_{T2}\frac{F_T^{\,2}}{k_T} + c_{T3}\left(L\sqrt{k}\,F_{T,2}\right)_{,2},$$ [18.9]

$$L = k^{3/2}/F_P,$$

$$C_{P1} = 0.5, \qquad\qquad C_{P2} = 1.41 - 1.55\chi,$$
$$C_{T1} = 0.84 - 0.58\chi, \qquad C_{T2} = 1.30 - 1.355\chi,$$

$$\chi = \frac{k_P - k_T}{k} = \frac{\xi - 1}{\xi + 1}.$$ [18.10]

The most interesting results in the prediction of free flows are related to the spreading rate in the similarity region (cf. Launder B.E., Schiestel R., [LAU 79B]).

The single scale k-ε model gives, contrary to experiments, a spreading rate for the round jet which is larger than for the plane jet. The two-scale model, however, allowed us to get for the round jet a spreading rate approximately 14% lower than the one for the plane jet.

Numerical predictions are also improved in the case of the plane wake.

18.2. Foundations of the method

18.2.1. *Multiple rank decomposition*

The decomposition of turbulent quantities into several ranks can be viewed as an extension of the Reynolds decomposition. For the velocity components, we can write:

$$U_j = \overline{U}_j + u_j^{(1)} + u_j^{(2)} + ... + u_j^{(n)},$$ [18.11]

where $\overline{U}_j$ is the ensemble average and $u_j^{(m)}$ are the components of partial fluctuating velocity defined by:

$$u_j^{(m)}(\vec{\xi}, t) = \int_{\kappa(m-1) < |\vec{\kappa}| \leq \kappa(m)} \hat{u}_j(\vec{\kappa}, t) e^{i\vec{\kappa} \cdot \vec{\xi}} d\vec{\kappa},$$ [18.12]

$\{\kappa(m)\}$ is a series of n wavenumbers forming a spectral partition and $\hat{u}_j(\vec{\kappa}, t)$ is the generalized Fourier transform of fluctuating velocities (whose mathematical conditions of existence will not be discussed here).

Decompositions such as [18.11] had been introduced by several authors (Liu J.T.C. and Merkine L., [LIU 76]; Bell T.L. and Nelkin M., [BEL 78]; Tchen C.M., [TCH 73] and also by Alper A. and Liu J.T.C., [ALP 78]; Reynolds W.C. and Hussain A.K.M.F., [REY 72] for somehow different purposes.

The multiple rank decomposition appears like a particular filtering operation. At the origin, the filtering method was introduced by Léonard for large eddy simulations of turbulence. If the Gaussian filter proved to be well suited in the case of turbulence large eddy simulations, the spectral cut-off filter will be preferred in the present context in view of the integration of spectral equations.

The splitting wavenumbers are defined in such a way that their values and their variations comply to the evolution of the mean energy spectrum. Otherwise, a partition using constant splitting wavenumbers would soon become inadequate because the range of fixed wavenumbers may become very different from the significant wavenumbers of the real flow.

It is convenient to define the truncating operator by:

$$\left\langle u_j \right\rangle^{(m)} = \sum_{p \leq m} u_j^{(p)} .$$

[18.13]

Deriving the definition relation:

$$\left\langle u_j \right\rangle^{(m)} = \int_{\left|\vec{\kappa}\right| \leq \kappa(m)} \hat{u}_j(\vec{\kappa}, t) e^{i\vec{\kappa}.\vec{\xi}} d\vec{\kappa} ,$$

[18.14]

We easily find:

$$\frac{d \left\langle u_j \right\rangle^{(m)}}{dt} = \left\langle \frac{du_j}{dt} \right\rangle^{(m)} + \frac{d\kappa(m)}{dt} \oiint_{\kappa^2 = \kappa(m)^2} \hat{u}_j(\vec{\kappa}, t) e^{i\vec{\kappa}.\vec{\xi}} dA(\kappa) .$$

[18.15]

It can be shown (Schiestel R., [SCH 83]) that the statistical correlation between two quantities of different ranks vanishes, but this is not true for higher order correlations.

In spectral space, the partial velocity components correspond to:

$$\widehat{u_j}^{(m)} = h_{m-1,m}(\kappa) \hat{u}_j(\vec{\kappa}, t) ,$$

[18.16]

where $h_{m-1,m}(\kappa)$ is the crenel function defined by:

$$\begin{cases} h = 1 & if \ \kappa \in \left[\kappa(m-1), \kappa(m) \right] \\ h = 0 & if \ \kappa \notin \left[\kappa(m-1), \kappa(m) \right] \end{cases}$$

The partial energies and the partial Reynolds stresses are defined by:

$$e^{(m)} = \int_{\kappa(m-1)}^{\kappa(m)} E(\kappa)d\kappa \qquad \text{and} \qquad R_{ij}^{(m)} = \int_{\kappa(m-1)}^{\kappa(m)} \varphi_{ij}(\kappa)d\kappa, \qquad [18.17]$$

with:

$$\varphi_{ij}(\kappa) = \oiint \Phi_{ij}(\vec{\kappa})dA(\kappa) \qquad \text{and} \qquad E(\kappa) = \frac{1}{2}\varphi_{jj}(\kappa), \qquad [18.18]$$

where $dA(\kappa)$ denotes the surface element on the sphere of radius $\kappa = |\vec{\kappa}|$ and $\Phi_{ij}(\vec{\kappa})$ is the three-dimensional spectral tensor for the double velocity correlations, $\varphi_{ij}(\kappa)$ being the one-dimensional tensor obtained from spherical averaging.

18.2.2. *Spectral dynamics equations*

The spectral dynamics of homogenous anisotropic turbulence has been studied by Jeandel D., Brison J.F. and Mathieu J., [JEA 78]. We consider here an incompressible fluid flow in the general case of non-homogenous and non-isotropic turbulence. The theory relies on the evolution equations of the dynamics of two point velocity correlations (Hinze J.O., 1975). We shall denote $u_i = u_i(\vec{x}_A)$, $u'_i = u_i(\vec{x}_B)$, $p = p(\vec{x}_A)$, $p' = p(\vec{x}_B)$ the fluctuating velocities and pressure at the points $A(\vec{x}_i)$ and $A(\vec{x'}_i)$.

$$\frac{\partial}{\partial t}\overline{u_i u'_j} + \overline{u_i u'_k}\frac{\partial \overline{U}'_j}{\partial x_k} + \overline{u'_j u_k}\frac{\partial \overline{U}_i}{\partial x_k} + \overline{U}_k\frac{\partial}{\partial x_k}\overline{u_i u'_j} + \overline{U}'_k\frac{\partial}{\partial x'_k}\overline{u_i u'_j} =$$

$$-\frac{\partial}{\partial x_k}\overline{u'_j u_i u_k} - \frac{\partial}{\partial x'_k}\overline{u_i u'_j u'_k} - \frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{pu'_j} - \frac{1}{\rho}\frac{\partial}{\partial x'_j}\overline{p'u_i}$$

$$+\nu\frac{\partial^2 \overline{u_i u'_j}}{\partial x_l \partial x_l} + \nu\frac{\partial^2 \overline{u_i u'_j}}{\partial x'_l \partial x'_l}. \qquad [18.19]$$

The distance between the two points and the position located midway are taken as new independent variables in the variable change:

$$\begin{cases} X_i = \dfrac{1}{2}(x_i + x'_i), \\ \xi_i = x'_i - x_i \end{cases} \qquad \begin{cases} \dfrac{\partial}{\partial x_i} = \dfrac{1}{2}\dfrac{\partial}{\partial X_i} - \dfrac{\partial}{\partial \xi_i}, \\ \dfrac{\partial}{\partial x'_i} = \dfrac{1}{2}\dfrac{\partial}{\partial X_i} + \dfrac{\partial}{\partial \xi_i} \end{cases} \qquad [18.20]$$

with $\dfrac{\partial^2}{\partial x_i{}^2} = \dfrac{1}{4}\dfrac{\partial^2}{\partial X_i{}^2} - \dfrac{\partial^2}{\partial \xi_i \partial X_i} + \dfrac{\partial^2}{\partial \xi_i{}^2}$, $\dfrac{\partial^2}{\partial x'_i{}^2} = \dfrac{1}{4}\dfrac{\partial^2}{\partial X_i{}^2} + \dfrac{\partial^2}{\partial \xi_i \partial X_i} + \dfrac{\partial^2}{\partial \xi_i{}^2}$,

and we denote:

$$\mathcal{R}_{Xij}(\xi_m) = \overline{u_{Xi} u'_{Xj}}, \quad u_{Xi} = u_i(X_m - \xi_m/2), \quad u'_{Xi} = u_i(X_m + \xi_m/2),$$
$$p_X = p(X_m - \xi_m/2), \quad p'_X = p(X_m + \xi_m/2). \qquad [18.21]$$

The evolution equation for $\mathcal{R}_{Xij}(X_j, \xi_m)$ is now deduced from [18.19], (Hinze J.O.,1975):

$$\underbrace{\frac{\partial \mathcal{R}_{Xij}}{\partial t} + \overline{U}_k(X_m)\frac{\partial \mathcal{R}_{Xij}}{\partial X_k}}_{(I)} + \underbrace{\mathcal{R}_{Xkj}\frac{\partial \overline{U}_i}{\partial X_k}(X_m) + \mathcal{R}_{Xik}\frac{\partial \overline{U}_j}{\partial X_k}(X_m)}_{(II)}$$

$$\underbrace{+\frac{1}{2}\frac{\partial}{\partial X_k}\left(\overline{u_{Xi}u'_{Xj}u_{Xk}} + \overline{u_{Xi}u'_{Xj}u'_{Xk}}\right) - \frac{\partial}{\partial \xi_k}\left(\overline{u_{Xi}u'_{Xj}u_{Xk}} - \overline{u_{Xi}u'_{Xj}u'_{Xk}}\right)}_{(III)}$$

$$= \underbrace{-\frac{1}{2\rho}\left(\frac{\partial}{\partial X_i}\overline{p_X u'_{Xj}} + \frac{\partial}{\partial X_j}\overline{p'_X u_{Xi}}\right) - \frac{1}{\rho}\left(\frac{\partial \overline{p_X u'_{Xj}}}{\partial \xi_i} - \frac{\partial \overline{p'_X u_{Xi}}}{\partial \xi_j}\right)}_{(IV)}$$

$$\underbrace{+\nu\left(\frac{1}{2}\frac{\partial^2 \overline{u_{Xi}u'_{Xj}}}{\partial X_l \partial X_l} + 2\frac{\partial^2 \overline{u_{Xi}u'_{Xj}}}{\partial \xi_l \partial \xi_l}\right)}_{(V)}$$

$$\underbrace{+\mathcal{R}_{Xik}\left[\frac{\partial \overline{U}_j}{\partial X_k}(X_p) - \frac{\partial \overline{U}_j}{\partial X_k}(X_p + \frac{\xi_p}{2})\right] + \mathcal{R}_{Xkj}\left[\frac{\partial \overline{U}_i}{\partial X_k}(X_p) - \frac{\partial \overline{U}_i}{\partial X_k}(X_p - \frac{\xi_p}{2})\right]}_{(VI)}$$

$$
+\underbrace{\frac{\partial \mathcal{R}_{Xij}}{\partial \xi_k}\left[\overline{U}_k(X_p - \frac{\xi_p}{2}) - \overline{U}_k(X_p + \frac{\xi_p}{2})\right]}_{(VIIa)}
$$

$$
+\underbrace{\frac{\partial \mathcal{R}_{Xij}}{\partial X_k}\left[\overline{U}_k(X_p) - \frac{1}{2}\left(\overline{U}_k(X_p + \frac{\xi_p}{2}) + \overline{U}_k(X_p - \frac{\xi_p}{2})\right)\right]}_{(VIIb)}.
\qquad [18.22]
$$

If we note that, in the one point Reynolds stress equation ([2.8]) the molecular diffusion term is $\nu R_{ij,mm}$ and not $(1/2)\nu R_{ij,mm}$, it may be beneficial to rewrite the term (V) in [18.22] in the form:

$$
(V) = \nu \frac{\partial^2 \overline{u_{Xi}u'_{Xj}}}{\partial X_l \partial X_l} + \left(2\nu\frac{\partial^2 \overline{u_{Xi}u'_{Xj}}}{\partial \xi_l \partial \xi_l} - \frac{1}{2}\nu\frac{\partial^2 \overline{u_{Xi}u'_{Xj}}}{\partial X_l \partial X_l}\right),
$$

and similarly the term (IV) will be rewritten as follows:

$$
(IV) = -\frac{1}{\rho}\left(\frac{\partial}{\partial X_i}\overline{p_X u'_{Xj}} + \frac{\partial}{\partial X_j}\overline{p'_X u_{Xi}}\right)
$$

$$
+\left[\frac{1}{2\rho}\left(\frac{\partial}{\partial X_i}\overline{p_X u'_{Xj}} + \frac{\partial}{\partial X_j}\overline{p'_X u_{Xi}}\right) - \frac{1}{\rho}\left(\frac{\partial \overline{p_X u'_{Xj}}}{\partial \xi_i} - \frac{\partial \overline{p'_X u_{Xi}}}{\partial \xi_j}\right)\right].
$$

This form allows a direct connection with the one point moment equation $\mathcal{R}_{Xij}(0)$. In particular, the non-homogenous terms will be similar to the corresponding terms in the Reynolds stress equation and will be thus treated in the same way.

The justification for the splitting of terms (IV) and (V) is based on the variable change [18.20]. Let us consider indeed, a function in two real variables $R(x,x')$ (extension to the three-dimensional case is trivial). The function in one variable $r(x) = R(x,x)$ has the following derivative:

$$
\frac{dr(x)}{dx} = \left(\frac{\partial R(x,x')}{\partial x}\right)_{x'=Cte}(x=x') + \left(\frac{\partial R(x,x')}{\partial x'}\right)_{x=Cte}(x=x') .
$$

The variable change: $\begin{cases} X = \dfrac{1}{2}(x + x') \\ \xi = x' - x \end{cases}$, $\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{1}{2}\dfrac{\partial}{\partial X} - \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial x'} = \dfrac{1}{2}\dfrac{\partial}{\partial X} + \dfrac{\partial}{\partial \xi} \end{cases}$,

$\dfrac{\partial^2}{\partial x^2} = \dfrac{1}{4}\dfrac{\partial^2}{\partial X^2} - \dfrac{\partial^2}{\partial \xi \partial X} + \dfrac{\partial^2}{\partial \xi^2}$, $\dfrac{\partial^2}{\partial x'^2} = \dfrac{1}{4}\dfrac{\partial^2}{\partial X^2} + \dfrac{\partial^2}{\partial \xi \partial X} + \dfrac{\partial^2}{\partial \xi^2}$, $\dfrac{\partial^2}{\partial x \partial x'} = \dfrac{1}{4}\dfrac{\partial^2}{\partial X^2} - \dfrac{\partial^2}{\partial \xi^2}$,

leads to $\dfrac{dr(x)}{dx} = \dfrac{1}{2}\left(\dfrac{\partial R}{\partial X}\right)_{\xi=Cte}(\xi = 0) + \dfrac{1}{2}\left(\dfrac{\partial R}{\partial X}\right)_{\xi=Cte}(\xi = 0)$ or $\dfrac{dr(x)}{dx} = \left(\dfrac{\partial R}{\partial X}\right)_{\xi=0}$.

We can show similarly that: $\dfrac{d^2 r(x)}{dx^2} = \left(\dfrac{\partial^2 R}{\partial X^2}\right)_{\xi=0}$.

In all cases, the derivative of the function in one variable $r(x)$ is equal to the partial derivative in X of the function $R(X, \xi)$ for $\xi = 0$.

In addition, the relation $\left(\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial x'}\right)^2_{(x=x')} R(x, x') = 4\left(\dfrac{\partial^2 R}{\partial \xi^2}\right)_{(\xi=0)}$, considered in the

particular case where R is formed as a product $R(x, x') = u(x)u(x')$, with

$r(x) = R(x, x) = u(x)^2$ leading to the relation $4\left(\dfrac{\partial^2 R}{\partial \xi^2}\right)_{(\xi=0)} = \dfrac{d^2 r(x)}{dx^2} - 4\left(\dfrac{du(x)}{dx}\right)^2$ used in

the following.

The last two terms of equation [18.22] have been approximated using a Taylor expansion for the mean velocities:

$$(VI) \approx \mathcal{R}_{Xkj}\dfrac{\xi_m}{2}\dfrac{\partial^2 \overline{U}_i}{\partial X_k \partial X_m}(X_p) - \mathcal{R}_{Xik}\dfrac{\xi_m}{2}\dfrac{\partial^2 \overline{U}_j}{\partial X_k \partial X_m}(X_p), \qquad [18.23]$$

$$(VII) \approx -\xi_m\dfrac{\partial \mathcal{R}_{Xij}}{\partial \xi_k}\dfrac{\partial \overline{U}_k}{\partial X_m}(X_h) - \dfrac{1}{8}\xi_m\xi_p\dfrac{\partial \mathcal{R}_{Xij}}{\partial X_k}\dfrac{\partial^2 \overline{U}_k}{\partial X_m \partial X_p}(X_h). \qquad [18.24]$$

The contribution (VI) will be neglected from now on, since its series expansion begins with second order terms. Indeed, if we assume that the non-homogenity remains moderate, the second derivatives of mean velocity will be neglected.

The transport equations for the one-dimensional spectral tensor are obtained from [18.22] by taking the Fourier transform followed by spherical mean on the sphere of radius κ. This method amounts to considering a "tangent" homogenous anisotropic turbulent field at the point X_k of the non-homogenous field.

The first step (Fourier transform) leads to the spectral tensor equations:

$$
\underbrace{\frac{\partial \Phi_{Xij}}{\partial t} + \overline{U}_k \frac{\partial \Phi_{Xij}}{\partial X_k}}_{(I)} + \underbrace{\Phi_{Xkj}\frac{\partial \overline{U}_i}{\partial X_k} + \Phi_{Xik}\frac{\partial \overline{U}_j}{\partial X_k}}_{(II)}
$$

$$
+\underbrace{\frac{1}{2}\frac{\partial}{\partial X_k}(\tau_{ik,j}+\tau_{i,kj})+i\kappa_k(\tau_{ik,j}-\tau_{i,kj})}_{(III)}
$$

$$
=\underbrace{-\frac{1}{2\rho}\left(\frac{\partial}{\partial X_i}\Sigma_{(p)j}+\frac{\partial}{\partial X_j}\Sigma_{i(p)}\right)+\frac{Q_{ij}}{\rho}}_{(IV)}
$$

$$
+\underbrace{\frac{1}{2}\nu\frac{\partial^2 \Phi_{Xij}}{\partial X_l \partial X_l}}_{(V)}+\mathscr{E}_{ij}+\underbrace{\frac{\partial \overline{U}_k}{\partial X_m}\left[\frac{\partial}{\partial \kappa_m}\left(\kappa_k \Phi_{Xij}\right)\right]}_{(VI)}. \qquad [18.25]
$$

We have used in this latter equation the following notations:

$$
\Phi_{Xij}=\widehat{\overline{u_i u'_j}}\,, \qquad \mathscr{E}_{ij}=2\nu\frac{\widehat{\overline{\partial^2 u_{Xi}u'_{Xj}}}}{\partial\xi_l\partial\xi_l}\,,
$$

$$
Q_{ij}=-\frac{\widehat{\overline{\partial p_X u'_{Xj}}}}{\partial\xi_i}+\frac{\widehat{\overline{\partial p'_X u_{Xi}}}}{\partial\xi_j}=-i\kappa_i\widehat{\overline{p_X u'_{Xj}}}+i\kappa_j\widehat{\overline{p'_X u_{Xi}}}\,,
$$

$$
\Sigma_{(p)j}=\widehat{\overline{p_X u'_{Xj}}}\,, \qquad \Sigma_{i(p)}=\widehat{\overline{p'_X u_{Xi}}}\,,
$$

$$
\tau_{i,jk}=\widehat{\overline{u_i u'_j u'_k}}\,, \qquad \tau_{ij,k}=\widehat{\overline{u_i u_j u'_k}}\,.
$$

The notation $\Sigma_{(p)j}$ for the pressure correlations reminds us of the presence of pressure in parentheses, but j is the only true tensorial index. The order in writing $(p)j$ or $i(p)$ indicates the position of the shifted point x'. In homogenous turbulence, these correlations could be expressed as functions of the velocity correlations (cf.

[SCH 83], and Chapter 5 and section 18.2.4 of the present book) by doing the contraction of the Φ_{Xij} equation by κ_i .

The derivation of equation [18.25] makes use of the various relations satisfied by the Fourier transforms, such as $\widehat{\dfrac{\partial \phi}{\partial \xi_j}} = i\kappa_j \hat{\phi}$, $\dfrac{\partial}{\partial \kappa_j} \hat{\phi} = -i\widehat{\xi_j \phi}$, $\widehat{\xi_j \phi} = i\dfrac{\partial}{\partial \kappa_j} \hat{\phi}$, $\kappa_j \hat{\phi} = -i\widehat{\dfrac{\partial \phi}{\partial \xi_j}}$ and also the mass conservation equation.

In addition, and as for equation [18.22], the term (V) will be rewritten as:

$$(V) = \frac{1}{2} v \frac{\partial^2 \Phi_{Xij}}{\partial X_l \partial X_l} + \mathscr{E}_{ij} = v \frac{\partial^2 \Phi_{Xij}}{\partial X_l \partial X_l} + \left(\mathscr{E}_{ij} - \frac{1}{2} v \frac{\partial^2 \Phi_{Xij}}{\partial X_l \partial X_l} \right),$$

and the term (IV) as:

$$(IV) = -\frac{1}{\rho} \left(\frac{\partial}{\partial X_i} \Sigma_{(p)j} + \frac{\partial}{\partial X_j} \Sigma_{i(p)} \right)$$

$$+ \left[\frac{1}{2\rho} \left(\frac{\partial}{\partial X_i} \Sigma_{(p)j} + \frac{\partial}{\partial X_j} \Sigma_{i(p)} \right) + \frac{Q_{ij}}{\rho} \right].$$

The second step applies the spherical mean defined by:

$$f^\Delta = \mathscr{M}_K(\hat{f}), \qquad \mathscr{M}_K(\hat{f}) = \oiint \hat{f}(\kappa) dA(\kappa),$$

i.e.,

$$f^\Delta(\kappa, t) = \frac{\kappa^2}{2\pi^2} \int_0^\infty \mathscr{M}_K(f) \frac{\sin(\kappa \xi)}{\kappa \xi} d\xi,$$

for deriving the equation for the one-dimensional tensor:

$$\frac{\partial \varphi_{ij}}{\partial t} + \overline{U}_k \frac{\partial \varphi_{ij}}{\partial X_k} + \varphi_{kj} \frac{\partial \overline{U}_i}{\partial X_k} + \varphi_{ik} \frac{\partial \overline{U}_j}{\partial X_k} + \theta_{ij} + \frac{1}{2} \frac{\partial}{\partial X_k} \left(\mathscr{T}_{ijk} + \mathscr{T}_{jik} \right) =$$

$$-\frac{\Pi_{ij}}{\rho} - \frac{1}{\rho} \left(\frac{\partial \mathscr{P}_j}{\partial X_i} + \frac{\partial \mathscr{P}_i}{\partial X_j} \right) + v \frac{\partial^2 \varphi_{ij}}{\partial X_l \partial X_l} - \mathscr{E}_{ij} + \frac{\partial \overline{U}_k}{\partial X_m} \zeta_{kimj}, \qquad [18.26]$$

with:

$$\varphi_{ij} = \left(\overline{u_i u'_j} \right)^{\Delta} ,$$

$$\mathcal{T}_{ijk} = \left(\overline{u_i u_k u'_j} \right)^{\Delta} = \left(\overline{u'_i u'_k u_j} \right)^{\Delta} ,$$

$$\theta_{ij} = \left[\frac{\partial}{\partial \xi_k} \left(\overline{u_i u'_k u'_j} - \overline{u_i u_k u'_j} \right) \right]^{\Delta} ,$$

$$\mathcal{P}_j = \left(\overline{pu'_j} \right)^{\Delta} = \left(\overline{p'u_j} \right)^{\Delta} ,$$

$$\Pi_{ij} = \frac{1}{\rho} \left(-\frac{\partial \overline{p_X u'_{Xj}}}{\partial \xi_i} + \frac{\partial \overline{p'_X u_{Xi}}}{\partial \xi_j} \right)^{\Delta} - \frac{1}{2\rho} \left(\frac{\partial \mathcal{P}_j}{\partial X_i} + \frac{\partial \mathcal{P}_i}{\partial X_j} \right) ,$$

$$\zeta_{kimj} = -\left(\xi_m \frac{\partial \mathcal{R}_{Xij}}{\partial \xi_k} \right)^{\Delta} ,$$

$$\mathcal{E}_{ij} = \frac{1}{2} \nu \frac{\partial^2 \varphi_{ij}}{\partial X_l \partial X_l} - 2\nu \left(\overline{\frac{\partial^2 u_{Xi} u'_{Xj}}{\partial \xi_l \partial \xi_l}} \right)^{\Delta} = \frac{1}{2} \nu \frac{\partial^2 \varphi_{ij}}{\partial X_l \partial X_l} + 2\nu \kappa_m^2 \varphi_{ij} .$$

We shall note the obvious properties:

$$\Pi_{jj} = 0 , \qquad\qquad \int_0^\infty \varphi_{ij} d\kappa = R_{ij} ,$$

$$\int_0^\infty \mathcal{E}_{ij} d\kappa = \varepsilon_{ij} , \qquad\qquad \int_0^\infty \mathcal{T}_{ijk} d\kappa = \overline{u_i u_j u_k} ,$$

$$\int_0^\infty \theta_{ij} d\kappa = 0 , \qquad\qquad \int_0^\infty \frac{\partial \overline{U}_k}{\partial X_m} \zeta_{kimj} d\kappa = 0 .$$

In homogenous turbulence the spectral dissipation tensor $\mathcal{E}_{ij}$ reduces to its usual expression $2\nu \kappa_m^2 \varphi_{ij}$. Still in homogenous turbulence we also have

$$Q_{ij} = \left(-i\kappa_i \Sigma_{(p)j} + i\kappa_j \Sigma_{i(p)} \right), \quad \frac{1}{\rho} \Sigma_{i(p)} = -2i \frac{\kappa_j}{\kappa^2} \Phi_{Xik} \frac{\partial \overline{U}_j}{\partial X_k} - \frac{\kappa_k \kappa_j}{\kappa^2} \tau_{i,kj} \quad ([18.31])$$

and $\Pi_{ij} = \mathcal{M}_{\|\kappa\|}(Q_{ij})$.

The integration with respect to κ of equation [18.26] allows us to recover exactly the one point Reynolds stress equation. Still in homogenous turbulence, we thus get:

$$\int_0^\infty \mathcal{E}_{ij}d\kappa = \int_0^\infty 2\nu\kappa^2 \varphi_{ij}(\kappa)d\kappa = \varepsilon_{ij} .$$

On the contrary, in the general case, in non-homogenous turbulence:

$$\int_0^\infty 2\nu\kappa^2 \varphi_{ij}(\kappa)d\kappa = \iiint_{\mathbb{R}^3} 2\nu\kappa^2 \Phi_{ij}(\vec{\kappa})d\vec{\kappa}$$

$$= -2\nu \left(\frac{\partial^2 \overline{u_i u'_j}}{\partial \xi_m \partial \xi_m} \right)_{(\xi=0)}$$

Considering the reversed variable change $\dfrac{\partial}{\partial \xi_m} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x'_m} - \dfrac{\partial}{\partial x_m} \right)$, we get:

$$\int_0^\infty 2\nu\kappa^2 \varphi_{ij}(\kappa)d\kappa = -2\nu\left[\frac{1}{4}\frac{\partial^2 \overline{u_i u'_j}}{\partial x'_m \partial x'_m} + \frac{1}{4}\frac{\partial^2 \overline{u_i u'_j}}{\partial x_m \partial x_m} - \frac{1}{2}\frac{\partial^2 \overline{u_i u'_j}}{\partial x_m \partial x'_m} \right]_{(x=x')}$$

$$= -\nu\left[\frac{1}{2}\overline{u_i u_{j,mm}} + \frac{1}{2}\overline{u_j u_{i,mm}} - \overline{u_{i,m}u_{j,m}} \right]$$

$$= -\nu\left[\frac{1}{2}\left(\overline{u_i u_j} \right)_{,mm} - 2\overline{u_{i,m}u_{j,m}} \right] = -\frac{\nu}{2}R_{ij,mm} + \varepsilon_{ij} .$$

We recover the fact that the integral includes a diffusive part, in accordance with the splitting of terms introduced in the two point equations.

The energy spectrum equation is obtained by tensorial contraction:

$$\frac{\partial E}{\partial t} + \overline{U}_k \frac{\partial E}{\partial X_k} + \varphi_{kj}\frac{\partial \overline{U}_j}{\partial X_k} + \mathcal{F} + \frac{\partial \mathcal{T}_k}{\partial X_k} = -\frac{1}{\rho}\left(\frac{\partial \mathcal{P}_j}{\partial X_j} \right) + \nu\frac{\partial^2 E}{\partial X_l \partial X_l} - \mathcal{E} , \quad [18.27]$$

with $E = \dfrac{1}{2}\varphi_{jj}$, $\mathcal{F} = \dfrac{1}{2}\left(\theta_{jj} - \dfrac{\partial \overline{U}_k}{\partial X_m}\zeta_{kjmj} \right)$,

$\mathcal{E} = \dfrac{1}{2}\mathcal{E}_{jj}$, $\mathcal{E} = \dfrac{1}{2}\nu\dfrac{\partial^2 E}{\partial X_l \partial X_l} + 2\nu\kappa^2 E$, $\mathcal{T}_k = \dfrac{1}{2}\mathcal{T}_{jjk}$.

18.2.3. *Transport equations for the partial Reynolds stresses*

The partial Reynolds stress equations for $R_{ij}{}^{(m)} = \overline{u_i u_j}^{(m)}$ are obtained from integration of the spectral equations at wavenumber intervals. In the calculation of the temporal derivatives it will be necessary to take into account relation [18.15] which gives, in the case of a one point quantity ($\xi = 0$) such as $\left\langle \overline{u_i u_j} \right\rangle^{(m)}$:

$$\frac{d \left\langle \overline{u_i u_j} \right\rangle^{(m)}}{dt} = \left\langle \frac{d \overline{u_i u_j}}{dt} \right\rangle^{(m)} + \varphi_{ij} \left[\kappa(m) \right] \frac{d \kappa(m)}{dt}. \qquad [18.28]$$
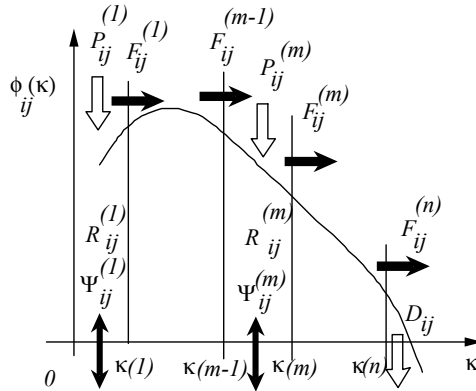
Figure 18.6. *Sketch of the split spectrum for the Reynolds stress components*

The final result can be written formally (cf. Figure 18.6):

$$\frac{dR_{ij}{}^{(m)}}{dt} = \underbrace{P_{ij}{}^{(m)}}_{(I)} + \underbrace{F_{ij}{}^{(m-1)} - F_{ij}{}^{(m)}}_{(II)} + \underbrace{\Psi_{ij}{}^{(m)}}_{(III)} + \underbrace{D_{ij}{}^{(m)}}_{(IV)} - \underbrace{\varepsilon_{ij}{}^{(m)}}_{(V)}. \qquad [18.29]$$

On the right-hand side of equation [18.29] appear several terms which are specified hereafter:

(I) production term:

$$P_{ij}{}^{(m)} = -R_{kj}{}^{(m)} \frac{\partial \overline{U}_i}{\partial X_k} - R_{ki}{}^{(m)} \frac{\partial \overline{U}_j}{\partial X_k},$$

(II) the turbulent transfer fluxes which include three contributions (the inertial transfer from the energy cascade, the rapid transfer due to straining of turbulence through the action of mean velocity gradients and the influence of variations in the splitting wavenumbers):

$$F_{ij}^{(m)} = \mathscr{F}_{Kij}^{(m)} + \mathscr{F}_{Uij}^{(m)} - \varphi_{ij}\left[\kappa(m)\right]\frac{d\kappa(m)}{dt},$$

$$\mathscr{F}_{Kij}^{(m)} = \int_0^{\kappa(m)} \theta_{ij}d\kappa, \quad \mathscr{F}_{Uij}^{(m)} = -\frac{\partial \overline{U}_k}{\partial X_m}\int_0^{\kappa(m)} \zeta_{kimj}d\kappa,$$

(III) the redistribution terms of the partial energies among the components of the Reynolds stress tensor through the effect of pressure-strain correlations:

$$\Psi_{ij}^{(m)} = -\frac{1}{\rho}\int_{\kappa(m-1)}^{\kappa(m)} \Pi_{ij}d\kappa,$$

(IV) the diffusion terms including the turbulent diffusion due to triple velocity correlations and to the pressure-velocity correlations together with the molecular diffusion:

$$D_{ij}^{(m)} = -\frac{\partial}{\partial X_k}T_{ijk}^{(m)} - \frac{1}{\rho}\left(\frac{\partial \mathscr{G}_j^{(m)}}{\partial X_i} + \frac{\partial \mathscr{G}_i^{(m)}}{\partial X_j}\right) + \nu\frac{\partial^2 R_{ij}^{(m)}}{\partial X_l \partial X_l},$$

$$T_{ijk}^{(m)} = \frac{1}{2}\int_{\kappa(m-1)}^{\kappa(m)}\left(\mathscr{T}_{ijk} + \mathscr{T}_{jik}\right)d\kappa,$$

$$\mathscr{G}_i^{(m)} = \int_{\kappa(m-1)}^{\kappa(m)} \mathscr{P}_i d\kappa,$$

(V) the viscous dissipation:

$$\varepsilon_{ij}^{(m)} = \int_{\kappa(m-1)}^{\kappa(m)} \mathscr{E}_{ij}d\kappa.$$

The tensorial contraction of equation [18.29] leads to the transport equations for the partial energies (cf. Figure 18.7):

$$\frac{de^{(m)}}{dt} = \underbrace{P^{(m)}}_{(I)} + \underbrace{F^{(m-1)} - F^{(m)}}_{(II)} + \underbrace{D^{(m)}}_{(III)} - \underbrace{\varepsilon^{(m)}}_{(IV)},$$ [18.30]

with $P^{(m)} = \dfrac{1}{2} P_{jj}^{(m)}$, $F^{(m)} = \dfrac{1}{2} F_{jj}^{(m)}$,

and also:

$$F^{(m)} = -\mathscr{F}_K^{(m)} - \mathscr{F}_U^{(m)} - E[\kappa(m)]\dfrac{d\kappa(m)}{dt},$$

$$D^{(m)} = \dfrac{1}{2} D_{jj}^{(m)},   \qquad \varepsilon^{(m)} = \dfrac{1}{2}\varepsilon_{jj}^{(m)}.$$

The last spectral zone $\kappa > \kappa(n)$ is supposed to contain no appreciable energy and we shall thus assume $F^{(n)} = \varepsilon^{(n+1)}$.
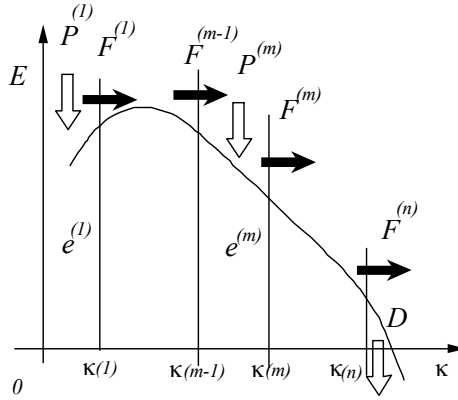
Figure 18.7. *Sketch of the split spectrum for the turbulence kinetic energy*

Let us mention that it is possible to derive the statistical equations by working directly in physical space as well (Schiestel R., [SCH 83]).

18.2.4. *The pressure correlations*

In the particular case of homogenous anisotropic turbulence, equation [18.25] can be multiplied with contraction by κ_j to give (taking into account that in homogenous turbulence $2\kappa_j Q_{ij} = i\kappa^2 \Sigma_{i(p)}$):

$$\dfrac{1}{\rho}\Sigma_{i(p)} = -2i\dfrac{\kappa_j}{\kappa^2}\Phi_{Xik}\dfrac{\partial \overline{U}_j}{\partial X_k} - \dfrac{\kappa_k \kappa_j}{\kappa^2}\tau_{i,kj}, \qquad\qquad [18.31]$$

and finally:

$$Q_{ij} = \frac{\kappa_m}{\kappa^2} \frac{\partial \overline{U}_m}{\partial X_k} \left(\kappa_i \Phi_{Xjk} + \kappa_j \Phi_{Xik} \right) - i \frac{\kappa_k \kappa_m}{2\kappa^2} \left(\kappa_i \tau_{j,km} + \kappa_j \tau_{i,km} \right),$$

$$\Psi_{ij}^{(m)} = 2 \int_{\kappa(m-1)}^{\kappa(m)} \mathcal{M}_\kappa (Q_{ij}) d\kappa. \qquad [18.32]$$

Consequently, it appears on the previous relations that $\Psi_{ij}^{(m)}$ contains two contributions. The first $\Psi_{ij}^{(m,1)}$ is the non-linear part which represents the action of turbulence on itself, and the second contribution $\Psi_{ij}^{(m,2)}$ corresponds to the linear part which describes the action of mean velocity gradients:

$$\Psi_{ij}^{(m,1)} = -i \int_{\kappa(m-1)<\kappa\leq\kappa(m)} \frac{\kappa_k \kappa_l}{\kappa^2} \left(\kappa_i \tau_{j,kl} + \kappa_j \tau_{i,kl} \right) d\vec{\kappa}, \qquad [18.33]$$

$$\Psi_{ij}^{(m,2)} = \left(a_{lj}^{ki(m)} + a_{li}^{kj(m)} \right) \frac{\partial \overline{U}_l}{\partial X_k},$$

$$a_{lj}^{ki(m)} = 2 \int_{\kappa(m-1)<\kappa\leq\kappa(m)} \frac{\kappa_l \kappa_j}{\kappa^2} \Phi_{Xik} d\vec{\kappa}. \qquad [18.34]$$

These latter coefficients satisfy several properties:

$$a_{jj}^{ki(m)} = 2R_{ik}^{(m)},$$
$$a_{lj}^{ki(m)} = a_{lj}^{ik(m)} = a_{jl}^{ki(m)},$$
$$a_{li}^{ki(m)} = 0. \qquad [18.35]$$

These relations are similar to those satisfied by the pressure-strain correlation in the case of the total Reynolds stresses (Launder B.E., Reece G.J. and Rodi W., [LAU 75C]) and thus suggest the following approximations:

$$\Psi_{ij}^{(m,1)} = -c_1^{(m)} \frac{F^{(m)}}{e^{(m)}} \left(R_{ij}^{(m)} - \frac{2}{3} e^{(m)} \delta_{ij} \right), \qquad [18.36]$$

$$\Psi_{ij}^{(m,2)} = -c_2^{(m)} \left(P_{ij}^{(m)} - \frac{2}{3} P^{(m)} \delta_{ij} \right), \qquad [18.37]$$

thus, the Rotta hypothesis and the hypothesis of isotropization of production are applied locally (within the spectral slice (m)). The overall behavior can be very different from the one obtained by applying these hypotheses to the total Reynolds stresses as it is done in traditional single scale models. For example, depending on the choice of initial conditions, the turbulence anisotropy may well increase during decay (cf. Schiestel R., [SCH 87A]).

In non-homogenous turbulent wall flows an additional wall term $\Psi_{ij}^{(m,3)}$ is to be added to the previous ones.

Direct numerical simulations [ROG 81A and B] made in homogenous anisotropic turbulent flows for various types of strain [LEE 85] have been used in order to shed light on the spectral behavior of the pressure-strain correlations [SCH 87B]. For this, the values of the pseudo-constants $C_2[\kappa]$ and $C_1[\kappa]$ are calculated from averaging in a spherical shell of radius κ, for the linear part $\Phi^L{}_{\alpha\beta}$ and the non-linear part $\Phi^N{}_{\alpha\beta}$ respectively.

$$C_2[\kappa] = \frac{-\Phi^L{}_{\alpha\beta}[\kappa,\kappa+\delta\kappa]}{P_{\alpha\beta}[\kappa,\kappa+\delta\kappa] - \frac{2}{3}P[\kappa,\kappa+\delta\kappa]\delta_{\alpha\beta}},$$

$$C_1[\kappa] = \frac{-\Phi^N{}_{\alpha\beta}[\kappa,\kappa+\delta\kappa]}{R_{\alpha\beta}[\kappa,\kappa+\delta\kappa] - \frac{2}{3}k[\kappa,\kappa+\delta\kappa]\delta_{\alpha\beta}}, \qquad \alpha,\beta \text{ fixed}$$

The case of homogenous shear suggests that C_1 increases with the wavenumber (Figure 18.8). This result justifies the hypothesis of increasing $c_1^{(m)}$ for the higher spectral slices. These same simulations do not show important variations for coefficient $c_2^{(m)}$ which can thus be approximately considered as constant (Figure 18.9).
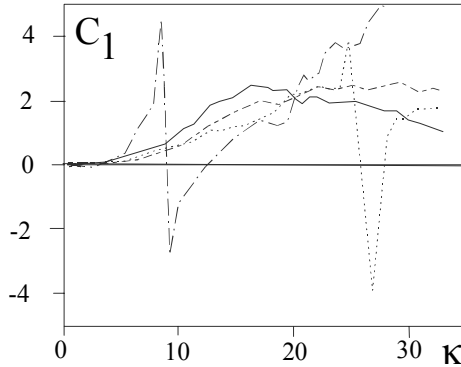
Figure 18.8. *Return to isotropy coefficient, slow part (homogenous shear)*
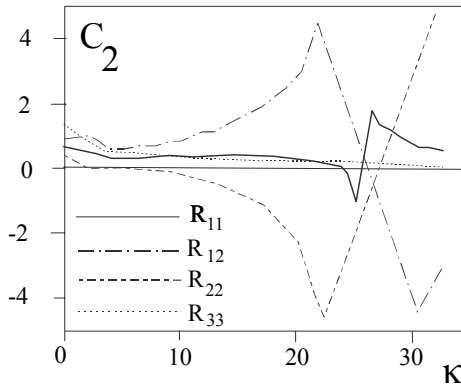
Figure 18.9. *Return to isotropy coefficient, rapid part (homogenous shear)*

Figure 18.10 shows the variations of $c_1^{(m)}$ in the case of the relaxation of homogenous turbulence after a complex strain (axisymmetric contraction and plane strain). The previous observations are still applicable. In this case, the numerical simulation shows an increase of anisotropy of the Reynolds stresses in the beginning of decay, even if the strain is suppressed. This paradoxical effect, can be qualitatively reproduced by multiple scale models when coefficients $c_1^{(m)}$ are such that their values are greater for the higher spectral slices and when the initial spectral distribution is such that the large-scale anisotropy and the small scale anisotropy almost balance (this is the case in the considered example). Thus, the return to isotropy at the level of the small scale is quicker (since $c_1^{(m)}$ is greater) and reveals the persistent anisotropy of the large-scales which was temporarily hidden. This behavior cannot be obtained using traditional one point single scale models [SCH 87A].
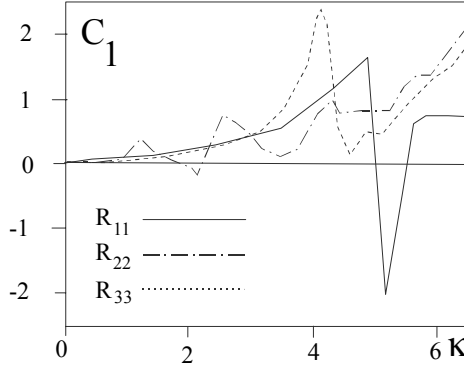
Figure 18.10. *Return to isotropy coefficient, slow part (relaxation after application of strain)*

18.2.5. *Closure problem*

It is necessary to estimate the spectral flux $\mathcal{F}^{(m)}$ composed of two parts $\mathcal{F}_K^{(m)} = \frac{1}{2}\mathcal{F}_{Kjj}^{(m)}$ and $\mathcal{F}_U^{(m)} = \frac{1}{2}\mathcal{F}_{Ujj}^{(m)}$.

A possible approach consists of using two point theory for estimating the non-linear transfer. The simplest hypothesis that can be considered can be for example the Kovasznay hypothesis:

$$\mathcal{F}_K^{(m)} = \gamma_K E\left[\kappa(m)\right]^{3/2} \kappa(m)^{5/2} . \tag{18.38}$$

The linear transfer is not easy to approximate simply. A physical discussion about this term is given in Hinze J.O., 1975 who clarifies its meaning:

$$\mathcal{F}_U^{(m)} = -\frac{1}{2}\frac{\partial \overline{U}_k}{\partial X_h}\int_{\kappa<\kappa(m)}\frac{\partial}{\partial\kappa_h}\left(\kappa_k \Phi_{Xjj}\right)d\vec{\kappa} = -\frac{1}{2}\frac{\partial \overline{U}_k}{\partial X_h}\oiint \kappa_k \Phi_{Xjj} n_h dA(\kappa) , \tag{18.39}$$

where $n_h = \kappa_h / \kappa$.

Expression [18.39] involves the three-dimensional spectrum Φ_{Xij} explicitly. It is thus necessary to relate the three-dimensional spectrum to the one-dimensional spectrum. The hypothesis introduced by Jeandel D., Brison J.F. and Mathieu J., [JEA 78] in spectral closures leads to the approximation (cf. Schiestel R., [SCH 83]):

$$\mathcal{F}_U^{(m)} = -\frac{\kappa(m)}{10}\varphi_{ql}\left[\kappa(m)\right]\frac{\partial \overline{U}_l}{\partial X_q}.$$ [18.40]

A hypothesis on $\mathcal{F}_{ij}^{(m)}$ would solve the closure problem for $F_{ij}^{(m)}$, however our knowledge on the spectral transfer tensor $\mathcal{F}_{ij}^{(m)}$ is rather scarce. Thus, we prefer a more empirical approximation obtained from a hypothesis such as:

$$\mathcal{F}_{ij}^{(m)} = \frac{\mathcal{F}^{(m)}}{E^{(m)}}\left[\alpha\varphi_{ij}^{(m)} + \frac{2}{3}(1-\alpha)E^{(m)}\delta_{ij}\right].$$

Taking into account the relations (cf. section 18.3):

$$F_{ij}^{(m)} = \mathcal{F}_{ij}^{(m)} - \varphi_{ij}\left[\kappa(m)\right]\frac{d\kappa(m)}{dt} \quad \text{and} \quad F^{(m)} = \mathcal{F}^{(m)} - E\left[\kappa(m)\right]\frac{d\kappa(m)}{dt},$$

from which we deduce $F_{ij}^{(m)} = \mathcal{F}_{ij}^{(m)} - \dfrac{\varphi_{ij}\left[\kappa(m)\right]}{E\left[\kappa(m)\right]}\left(\mathcal{F}^{(m)} - F^{(m)}\right),$

we get the following approximation:

$$F_{ij}^{(m)} \approx \frac{2}{3}\alpha\mathcal{F}^{(m)}\delta_{ij} - (1-\alpha)\frac{R_{ij}^{(m)}}{e^{(m)}}\mathcal{F}^{(m)} - \frac{R_{ij}^{(m)}}{e^{(m)}}\left(\mathcal{F}^{(m)} - F^{(m)}\right),$$

and if we assume $A^{(m)} = \alpha\dfrac{\mathcal{F}_{(m)}}{F_{(m)}}$, then:

$$F_{ij}^{(m)} = \frac{F^{(m)}}{e^{(m)}}\left[A^{(m)}R_{ij}^{(m)} + \frac{2}{3}(1-A^{(m)})e^{(m)}\delta_{ij}\right],$$ [18.41]

where $A^{(m)}$ are numerical constants which are dependent on the particular corresponding spectral slice (m). $A^{(m)}$ is assumed to take a value close to unity in the region of low wavenumbers and to go to zero in the region of large wavenumbers (tendency to isotropy of the microturbulence).

In reality, the fast transfer, while corresponding to a spectral energy transfer, causes at the same time an energy redistribution among the components of the Reynolds stress tensor. We can try to devise a more accurate account.

If we recall the expression for the tensorial flux:

$$\mathcal{F}_{ij}^{(m)} = \mathcal{F}_{Kij}^{(m)} + \mathcal{F}_{Uij}^{(m)}$$

$$\mathcal{F}_{ij}^{(m)} = \frac{\mathcal{F}_{U}^{(m)}}{E^{(m)}}\left[\alpha\varphi_{ij}^{(m)} + \frac{2}{3}(1-\alpha)E^{(m)}\delta_{ij}\right]$$

$$+ \frac{\mathcal{F}_{K}^{(m)}}{E^{(m)}}\left[\alpha\varphi_{ij}^{(m)} + \frac{2}{3}(1-\alpha)E^{(m)}\delta_{ij}\right] + \mathcal{B}_{ij},$$

where $\mathcal{B}_{ij} = \mathcal{F}_{Uij}^{(m)} - \dfrac{\mathcal{F}_{U}^{(m)}}{E^{(m)}}\left[\alpha\varphi_{ij}^{(m)} + \dfrac{2}{3}(1-\alpha)E^{(m)}\delta_{ij}\right],$

we thus get:

$$F_{ij}^{(m)} = \frac{F^{(m)}}{e^{(m)}}\left[A^{(m)}R_{ij}^{(m)} + \frac{2}{3}(1-A^{(m)})e^{(m)}\delta_{ij}\right] + B_{ij}^{(m)}$$

The expression for $B_{ij}^{(m)}$ can be approximated using the approach by Jeandel D., Brison J.F. and Mathieu J., [JEA 78] which leads to:

$$\mathcal{F}_{Uij} = \frac{\kappa}{20}\left[\varphi_{li}\left(\frac{\partial\overline{U}_l}{\partial X_j} + \frac{\partial\overline{U}_j}{\partial X_l}\right) + \varphi_{lj}\left(\frac{\partial\overline{U}_l}{\partial X_i} + \frac{\partial\overline{U}_i}{\partial X_l}\right)\right] - \frac{\kappa}{5}\varphi_{lm}\frac{\partial\overline{U}_l}{\partial X_m}\delta_{ij},$$

and finally:

$$\mathcal{B}_{ij}^{(m)} = \zeta^{(m)}\left\{\frac{1}{20}\left[R_{li}^{(m)}\left(\frac{\partial\overline{U}_l}{\partial X_j} + \frac{\partial\overline{U}_j}{\partial X_l}\right) + R_{lj}^{(m)}\left(\frac{\partial\overline{U}_l}{\partial X_i} + \frac{\partial\overline{U}_i}{\partial X_l}\right)\right]\right.$$

$$\left. + \frac{1}{5}R_{lp}^{(m)}\frac{\partial\overline{U}_l}{\partial X_p}\left(\frac{R_{ij}^{(m)}}{e^{(m)}} - \delta_{ij}\right)\right\}.$$

As a first approximation, we can choose $\zeta^{(1)} = 1$ and neglect $\zeta^{(m)}$ for $m>1$ since the fast transfer acts mainly at low wavenumbers.

18.2.6. *Wavenumber splitting*

In a turbulent flow in which the characteristic scales are evolving, it is not possible to use constant splitting wavenumbers, because the energy distribution would soon become either too concentrated or too spread out compared to the prescribed spectral intervals. Thus, the size in the evolving spectral slices has to be chosen in order to conform automatically to the energy distribution in wavenumbers. Practically, the wavenumber intervals are related to local parameters:

$$\kappa(m) - \kappa(m-1) = \alpha_m F^{(m)} / \left(e^{(m)}\right)^{3/2} , \qquad [18.42]$$

where α_m is a numerical constant and $\kappa(0) = 0$.

Relation [18.42] shows (Schiestel R., [SCH 87A]) that if α_m is constant, the ratio $\kappa(m)/\kappa(m-1)$ also remains constant, i.e. corresponding to a logarithmic distribution.

18.2.7. *The spectral flux equations*

From the expressions given in section 18.2.3 we deduce:

$$\frac{d\kappa(m)}{dt} = \frac{\mathscr{F}^{(m)} - F^{(m)}}{E[\kappa(m)]} , \quad \text{with} \quad \mathscr{F}^{(m)} = \mathscr{F}_K^{(m)} + \mathscr{F}_U^{(m)} . \qquad [18.43]$$

Using [18.42] we obtain from [18.43] the evolution equations for the fluxes:

$$\frac{dF^{(m)}}{dt} = \frac{3}{2} \frac{F^{(m)}}{e^{(m)}} \left(P^{(m)} + F^{(m-1)} - F^{(m)} - \varepsilon^{(m)}\right) + \frac{1}{\beta_m} \left(\frac{\mathscr{F}^{(m)}}{F^{(m)}} - 1\right) \frac{\left(F^{(m)}\right)^2}{e^{(m)}}$$

$$- \frac{1}{\beta_m} \frac{E[\kappa(m)]}{E[\kappa(m-1)]} \left(\frac{\mathscr{F}^{(m-1)}}{F^{(m-1)}} - 1\right) \frac{F^{(m)} F^{(m-1)}}{e^{(m)}} . \qquad [18.44]$$

Coefficient β_m is defined by $E[\kappa(m)] = \beta_m \dfrac{e^{(m)}}{\kappa(m) - \kappa(m-1)}$, it goes to unity when $\kappa(m) - \kappa(m-1) \to 0$. To have a general formalism in all cases, we will suppose that in equation [18.44] a fictitious $F^{(0)} = 0$ is used for $m = 0$.

In homogenous turbulence, the spectral flux equations can thus be written in the form:

$$\frac{dF^{(m)}}{dt} = A^{(m)} \frac{F^{(m)} P^{(m)}}{e^{(m)}} + C_1^{(m)} \frac{F^{(m)} F^{(m-1)}}{e^{(m)}} - C_2^{(m)} \frac{F^{(m)^2}}{e^{(m)}} - C_3^{(m)} \frac{F^{(m)} \varepsilon^{(m)}}{e^{(m)}},$$

$$A^{(m)} = 3/2,$$

$$C_1^{(m)} = \frac{3}{2} + \frac{1}{\beta_m} \frac{E[\kappa(m)]}{E[\kappa(m-1)]} \left(1 - \frac{\mathcal{F}^{(m-1)}}{F^{(m-1)}} \right),$$

$$C_2^{(m)} = \frac{3}{2} + \frac{1}{\beta_m} \left(1 - \frac{\mathcal{F}^{(m)}}{F^{(m)}} \right),$$

$$C_3^{(m)} = \frac{3}{2}. \qquad\qquad\qquad [18.45]$$

If it is supposed that the viscous dissipation is only effective in the last zone $[\kappa(n), +\infty[$, we can discard the $C_3^{(m)}$ term. In non-homogenous flows, additional terms for molecular and turbulent diffusion have to be included.

Two types of approach are thus possible. The first consists of determining coefficients $A^{(m)}, C_1^{(m)}, C_2^{(m)}, C_3^{(m)}$ from two point theories. This approach is developed in Schiestel R., [SCH 83 and 87]. However, the simple two point theories such as the Kovasnay hypothesis are very limited in the application domain (the Kovasnay hypothesis is nothing more than a pure dimensional relation) whereas the more advanced have a far more complex formulation. Another more practical approach consists of determining the coefficients in an empirical way by calibrating the model on traditional turbulent flows. It is this second approach which is developed in section 18.3, while the first approach was developed in Schiestel R., [SCH 83].

18.2.8. *A new look on the* k-ε *model*

The traditional k-ε model can be presented in a new light by considering a single spectral splitting. The underlying hypothesis in the one scale models is the existence of a self-similar energy spectrum with an invariable shape. Thus, the energy spectrum can be represented in the form:

$$E(\kappa) = f\left(\frac{\kappa}{\kappa_{ref}} \right) \frac{k}{\kappa_{ref}}, \qquad\qquad \kappa_{ref} = \varepsilon / k^{3/2},$$

where f is a function in the dimensionless variable κ/κ_{ref} representing the invariable spectrum shape with the normalizing condition: $\int_0^\infty f(\eta)d\eta = 1$. If $\kappa(1) = \alpha.\kappa_{ref}$ is located at the end of the inertial zone (using an appropriate choice for α), then $F^{(1)} \approx \varepsilon$ and equation [18.43] (in the case of a single splitting) yields:

$$\frac{d\kappa(1)}{dt} = \frac{\mathscr{F}^{(1)} - F^{(1)}}{E[\kappa(1)]},$$

and then, using the definition of $\kappa(1)$:

$$\frac{d\varepsilon}{dt} = \frac{3}{2}\frac{\varepsilon P}{k} - \frac{3}{2}\frac{\varepsilon^2}{k} + \frac{k^{3/2}}{\alpha.E[\kappa(1)]}\left(\mathscr{F}^{(1)} - \varepsilon\right).$$

By introducing, for example, the Kovasnay hypothesis [1.31] for estimating $\mathscr{F}^{(1)}$, we get:

$$\frac{d\varepsilon}{dt} = \underbrace{\frac{3}{2}}_{\tilde{C}_{\varepsilon 1}}\frac{\varepsilon P}{k} - \underbrace{\left[\frac{3}{2} + \frac{1 - \gamma_K f(\alpha)^{3/2}\alpha^{5/2}}{\alpha.f(\alpha)}\right]}_{C_{\varepsilon 2}}\frac{\varepsilon^2}{k}$$

Thus, the ε equation can be established by considering a variable splitting $\kappa(1)$ which automatically conforms to the variations of the energy spectrum (in size). Coefficient $C_{\varepsilon 2}$ is found to be dependent on spectral shape only. This approach also clearly sheds some light on the fact that in this model, the quantity ε is essentially viewed as a spectral flux, an interpretation which is physically more relevant than the viscous dissipation into heat.

The introduction of additional splitting wavenumbers allows us of course to obtain a more general behavior.

18.2.9. *Interpretation in physical space*

If we consider a two rank scheme, it is possible to make a decomposition of the fluctuating velocity into a micro-component and a macro-component:

$$u_i = v_i + w_i \quad \text{with} \quad v_i = \langle u_i \rangle .$$

From the equations for each of these components v_i and w_i, deduced from the Navier-Stokes equations, we can derive [SCH 83] the transport equations for the partial stresses:

$$\frac{dR_{ij}^{(1)}}{dt} = P_{ij}^{(1)} - \frac{\partial T_{ijm}^{(1)}}{\partial x_m} - \frac{1}{\rho}\frac{\partial \Pi_{ijm}^{(1)}}{\partial x_m} + \Phi_{ij}^{(1)} - \mathscr{F}_{ij}^{(1)} - \mathscr{H}_{ij}^{(1)} - \varepsilon_{ij}^{(1)} + \nu\frac{\partial^2 R_{ij}^{(1)}}{\partial x_m^2} ,$$

$$\frac{dR_{ij}^{(2)}}{dt} = P_{ij}^{(2)} - \frac{\partial T_{ijm}^{(2)}}{\partial x_m} - \frac{1}{\rho}\frac{\partial \Pi_{ijm}^{(2)}}{\partial x_m} + \Phi_{ij}^{(2)} + \mathscr{F}_{ij}^{(1)} + \mathscr{H}_{ij}^{(1)} - \varepsilon_{ij}^{(2)} + \nu\frac{\partial^2 R_{ij}^{(2)}}{\partial x_m^2} ,$$

and the equation for the total stresses:

$$\frac{dR_{ij}}{dt} = P_{ij} - \frac{\partial T_{ijm}}{\partial x_m} - \frac{1}{\rho}\frac{\partial \Pi_{ijm}}{\partial x_m} + \Phi_{ij} \quad \dotfill - \varepsilon_{ij} + \nu\frac{\partial^2 R_{ij}}{\partial x_m^2} .$$

The inertial transfer terms $\mathscr{F}_{ij}^{(1)} = -\langle w_m w_j \rangle \overline{\dfrac{\partial v_i}{\partial x_m}} - \langle w_m w_i \rangle \overline{\dfrac{\partial v_j}{\partial x_m}}$ due to the triple correlations together with the transfer $\mathscr{H}_{ij}^{(1)} = -\varphi_{ij}(\kappa_1)\dfrac{\partial \kappa_1}{\partial t}$ coming from the variations in the splitting wavenumber κ_1 (cf. [SCH 83] and [18.28]) are globally compensating in the total stresses equations. The total flux at the cut-off is $F_{ij}^{(1)} = \mathscr{F}_{ij}^{(1)} + \mathscr{H}_{ij}^{(1)}$. The other terms are expressed as follows:

$$P_{ij}^{(r)} = -R_{im}^{(r)}\frac{\partial U_j}{\partial x_m} - R_{jm}^{(r)}\frac{\partial U_i}{\partial x_m} , \quad r=1,2$$

$$T_{ijm}^{(1)} = \overline{\left(v_j w_i + v_i w_j + v_i v_j\right)u_m} \quad \text{and} \quad T_{ijm}^{(2)} = \overline{w_i w_j u_m} ,$$

$$\Pi_{ijm}^{(1)} = \overline{pv_j \delta_{im} + pv_i \delta_{jm}} \quad \text{and} \quad \Pi_{ijm}^{(2)} = \overline{pw_j \delta_{im} + pw_i \delta_{jm}} ,$$

$$\Phi_{ij}^{(1)} = \overline{\frac{p}{\rho}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)} \quad \text{and} \quad \Phi_{ij}^{(2)} = \overline{\frac{p}{\rho}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)} ,$$

$$\varepsilon_{ij}^{(1)} = \overline{\frac{\partial v_i}{\partial x_m}\frac{\partial v_j}{\partial x_m}} \quad \text{and} \quad \varepsilon_{ij}^{(2)} = \overline{\frac{\partial w_i}{\partial x_m}\frac{\partial w_j}{\partial x_m}} .$$

18.3. Practical formulations and extensions

Flows encountered in practice, often include low Reynolds number regions (for example, near a wall); moreover, the anisotropy of the flow field can play a more or less important role and simplifying hypotheses are sometimes desirable. We shall give in the present section various extended versions and simplified versions of the basic scheme in order to cover the main practical situations. These various versions are, however, not always definite models but rather suggestions for useful research directions for further applications.

18.3.1. *Reynolds stress multiple scale model with M levels and for high turbulence Reynolds numbers*

This is the basic prototype model.

<div align="center">

MRFM model

(Multiple scale model for the Reynolds stresses and the spectral fluxes in M levels)

</div>

$$\frac{dR_{ij}^{(m)}}{dt} = P_{ij}^{(m)} + \Phi_{ij}^{(m)} + F_{ij}^{(m-1)} - F_{ij}^{(m)} + c_s^{(m)} \left(\frac{kR_{pl}}{F^{(1)}} R_{ij,p}^{(m)} \right)_{,l} - \varepsilon_{ij}^{(m)} + B_{ij}^{(m)} \,,$$

$$\frac{de^{(m)}}{dt} = P^{(m)} + F^{(m-1)} - F^{(m)} + c_s^{(m)} \left(\frac{kR_{ij}}{F^{(1)}} e^{(m)}_{,i} \right)_{,j} - \varepsilon^{(m)} \,,$$

$$\varepsilon^{(m)} = 0 \quad \text{except} \quad \varepsilon^{(M+1)} = F^{(M)} \,,$$

$$\frac{dF^{(m)}}{dt} = C_{F0}^{(m)} \frac{P^{(m)} F^{(m)}}{e^{(m)}} + C_{F1}^{(m)} \frac{F^{(m-1)} F^{(m)}}{e^{(m)}}$$

$$-C_{F2}^{(m)} \frac{F^{(m)2}}{e^{(m)}} + C_{F3}^{(m)} \left(\frac{kR_{ij}}{F^{(1)}} F_{,i}^{(m)} \right)_{,j} \,.$$

In the usual case two levels are considered.

<div align="center">

MRF2 model

</div>

$$\frac{dR_{ij}^{(1)}}{dt} = P_{ij}^{(1)} + \Phi_{ij}^{(1)} + F_{ij}^{(0)} - F_{ij}^{(1)} + c_s^{(1)} \left(\frac{kR_{pl}}{F^{(1)}} R_{ij,p}^{(1)} \right)_{,l} \quad (\text{possibly } + B_{ij}^{(1)})$$

$$\frac{dR_{ij}^{(2)}}{dt} = P_{ij}^{(2)} + \Phi_{ij}^{(2)} + F_{ij}^{(1)} - F_{ij}^{(2)} + c_s^{(2)}\left(\frac{kR_{pl}}{F^{(1)}}R_{ij,p}^{(2)}\right)_{,l},$$

$$\frac{de^{(1)}}{dt} = P^{(1)} - F^{(1)} + c_s^{(1)}\left(\frac{kR_{ij}}{F^{(1)}}e^{(1)}_{,i}\right)_{,j},$$

$$\frac{de^{(2)}}{dt} = P^{(2)} + F^{(1)} - F^{(2)} + c_s^{(2)}\left(\frac{kR_{ij}}{F^{(1)}}e^{(2)}_{,i}\right)_{,j},$$

$$k = e^{(1)} + e^{(2)}, \qquad P^{(m)} = -R_{ij}^{(m)}\overline{U}_{i,j},$$

$$R_{ij} = R_{ij}^{(1)} + R_{ij}^{(2)}, \qquad \varepsilon = F^{(2)},$$

$$\frac{dF^{(1)}}{dt} = C_{F0}^{(1)}\frac{P^{(1)}F^{(1)}}{e^{(1)}} + C_{F1}^{(1)}\frac{F^{(0)}F^{(1)}}{e^{(1)}} - C_{F2}^{(1)}\frac{F^{(1)2}}{e^{(1)}} + C_{F3}^{(1)}\left(\frac{kR_{ij}}{F^{(1)}}F_{,i}^{(1)}\right)_{,j},$$

$$\frac{dF^{(2)}}{dt} = C_{F0}^{(2)}\frac{P^{(2)}F^{(2)}}{e^{(2)}} + C_{F1}^{(2)}\frac{F^{(1)}F^{(2)}}{e^{(2)}} - C_{F2}^{(2)}\frac{F^{(2)2}}{e^{(2)}} + C_{F3}^{(2)}\left(\frac{kR_{ij}}{F^{(1)}}F_{,i}^{(2)}\right)_{,j},$$

$$P_{ij}^{(m)} = -R_{ip}^{(m)}\overline{U}_{j,p} - R_{jp}^{(m)}\overline{U}_{i,p},$$

$$F_{ij}^{(m)} = \frac{F^{(m)}}{e^{(m)}}\left[A^{(m)}R_{ij}^{(m)} + \frac{2}{3}(1 - A^{(m)})e^{(m)}\delta_{ij}\right],$$

$$\Phi_{ij}^{(m)} = -c_1^{(m)}\frac{F^{(m)}}{e^{(m)}}\left(R_{ij}^{(m)} - \frac{2}{3}e^{(m)}\delta_{ij}\right) - c_2^{(m)}\left(P_{ij}^{(m)} - \frac{2}{3}P^{(m)}\delta_{ij}\right) + \Phi_{ij}^{(s)},$$

$$\Phi_{ij}^{(s,m)} = c_1'^{(m)}\frac{F^{(m)}}{e^{(m)}}\left(R_{lp}^{(m)}n_ln_p\delta_{ij} - \frac{3}{2}R_{il}^{(m)}n_ln_j - \frac{3}{2}R_{jl}^{(m)}n_ln_i\right)\frac{k^{3/2}}{c_lF^{(1)}\delta},$$

$$+ c_2'^{(m)}\left(\Phi_{lp}^{(m,2)}n_ln_p\delta_{ij} - \frac{3}{2}\Phi_{il}^{(m,2)}n_ln_j - \frac{3}{2}\Phi_{jl}^{(m,2)}n_ln_i\right)\frac{k^{3/2}}{c_lF^{(1)}\delta},$$

$$\mathcal{B}_{ij}^{(1)} = \zeta^{(1)}\left\{\frac{1}{20}\left[R_{li}^{(1)}\left(\frac{\partial\overline{U}_l}{\partial X_j} + \frac{\partial\overline{U}_j}{\partial X_l}\right) + R_{lj}^{(1)}\left(\frac{\partial\overline{U}_l}{\partial X_i} + \frac{\partial\overline{U}_i}{\partial X_l}\right)\right]\right.$$

$$+ \frac{1}{5}R_{lp}^{(1)}\frac{\partial\overline{U}_l}{\partial X_p}\left(\frac{R_{ij}^{(1)}}{e^{(1)}} - \delta_{ij}\right)\right\}, \qquad (\zeta^{(1)} = 1).$$

The numerical coefficients will be chosen by reference to experimental data in simple flows.

a) Grid turbulence

We consider the decay of turbulence in the initial period. Supposing that the ratios $r = e^{(2)}/e^{(1)}$ and $g = F^{(2)}/F^{(1)}$ go to an asymptotic limit, we deduce from the transport equations for $e^{(j)}$ and $F^{(j)}$ that a solution exists such that:

$$r \to r_G = \frac{C_{F2}^{(2)} - C_{F1}^{(2)}}{C_{F2}^{(1)} - C_{F2}^{(2)}}, \qquad g \to g_G = 1 + r_G, \qquad [18.46]$$

$C_{F2}^{(1)}$ controls the energy decay rate if $C_{F2}^{(2)} < C_{F2}^{(1)}$.

b) Homogenous shear flow

Although homogenous turbulence submitted to a constant shear does not seem to reach an asymptotic state, there is, however, an asymptotic spectral shape and thus asymptotic values for *r* and *g*.

From the model equations, we can easily derive:

$$r \to r_S = \frac{C_{F2}^{(2)} - C_{F1}^{(2)}}{C_{F2}^{(1)} - C_{F0}^{(1)}}, \qquad g \to g_G = 1, \qquad [18.47]$$

c) Local universal equilibrium spectrum with constant spectral flux

This academic case can exist only if:

$$C_{F1}^{(1)} = C_{F2}^{(1)} \quad \text{and} \quad C_{F1}^{(2)} = C_{F2}^{(2)}, \qquad [18.48]$$

thus $r \to r_E$.

To obtain this situation we can inject a constant flux $F^{(0)}$ at a given wavenumber whereas the shear production terms remain zero.

d) Logarithmic region of a turbulent boundary layer

In a logarithmic boundary layer, the Reynolds stresses are constant:

$$k = \alpha u_*^2, \qquad R_{12} = u_*^2, \qquad R_{22} = \beta_{22} u_*^2,$$

and we also have:

$$\frac{\partial \overline{U}}{\partial y} = \frac{u_*}{Ky}, \qquad P = \varepsilon = \frac{u_*^3}{Ky},$$

where K is the Karman constant. The model equations yield:

$$r \to r_{CL} = \frac{C_{F3}^{(1)}}{C_{F3}^{(2)}} \frac{C_{F2}^{(2)} - C_{F1}^{(2)}}{C_{F2}^{(1)} - C_{F0}^{(1)}}, \qquad g \to g_{CL} = 1. \qquad [18.49]$$

e) Choice of coefficients

$$c_s^{(1)} = c_s^{(2)} = 0.22 \qquad\qquad C_{F0}^{(1)} = 1.0 \qquad\qquad C_{F1}^{(1)} = 1.225$$

$$C_{F2}^{(1)} = 1.9 + 0.9\xi \qquad\qquad \xi = \frac{e^{(2)} - e^{(1)}}{e^{(2)} + e^{(1)}} \qquad\qquad C_{F3}^{(1)} = 0.2$$

$$C_{F0}^{(2)} = 1.0 \qquad\qquad C_{F1}^{(2)} = 1.0 \qquad\qquad C_{F2}^{(2)} = 1.45 + 0.6\xi$$

$$C_{F3}^{(2)} = 0.2 \qquad\qquad c_1^{(1)} = 1.20 \qquad\qquad c_1^{(2)} = 4.0$$

$$c_2^{(1)} = c_2^{(2)} = 0.6 \qquad\qquad A^{(1)} = 0.50 \qquad\qquad A^{(2)} = 0.$$

$$C_{F2}^{(m)} - C_{F1}^{(m)} = C_{F3}^{(m)} \beta_{22} \alpha \alpha_m K^2$$

$$\beta_{22} = \frac{R_{22}}{u_*^2}, \quad \alpha = \frac{k}{u_*^2}, \quad \alpha_m = \frac{e^{(m)}}{k} \text{ (logarithmic zone } \beta_{22} \approx 1., \alpha \approx 4.2)$$

$$c_1'^{(1)} = c_1'^{(2)} = 0.5, \qquad c_2'^{(1)} = c_2'^{(2)} = 0.3, \qquad c_l = 2.5$$

The previous numerical values lead to:

$$r_G = 1, \quad r_S = 1/3, \quad r_{CL} = 1/3, \quad r_E = 1/7.$$

In the case of homogenous shear the partial Reynolds stress equations can be used to derive the spectral anisotropy distributions according to the model coefficients.

$$\frac{R_{11}^{(1)}}{e^{(1)}} = \frac{2 - c_2 + s}{d_1}, \qquad \frac{R_{22}^{(1)}}{e^{(1)}} = \frac{R_{33}^{(1)}}{e^{(1)}} = \frac{0.5c_2 + s}{d_1},$$

$$s_1 = \frac{2}{3}\left(c_1^{(1)} - 1 + A^{(1)}\right), \qquad d_1 = c_1^{(1)} + A^{(1)}, \qquad c_2 = c_2^{(1)} = c_2^{(2)},$$

$$\frac{R_{11}^{(2)}}{e^{(2)}} = \frac{s_2 + h^{(11)}}{d_2}, \qquad \frac{R_{22}^{(2)}}{e^{(2)}} = \frac{s_2 + h^{(22)}}{d_2}, \qquad \frac{R_{33}^{(2)}}{e^{(2)}} = \frac{s_2 + h^{(33)}}{d_2},$$

$$s_2 = \frac{2}{3}\left(c_1^{(1)} - 1 + A^{(2)}\right), \qquad d_2 = c_1^{(2)} + A^{(2)},$$

$$h^{(ij)} = \frac{F^{(1)}}{F^{(2)}}\left[A^{(1)}\frac{R_{ij}^{(1)}}{e^{(1)}} + \frac{2}{3}\left(1 - A^{(1)}\right)\delta_{ij}\right],$$

$$P^{(1)} \approx F^{(1)}, \qquad P^{(2)} \approx 0, \qquad F^{(1)} \approx F^{(2)},$$

$$\left(\frac{R_{12}^{(1)}}{e^{(1)}}\right)^2 = \frac{1 - c_2}{c_1^{(1)} + A^{(1)}}\frac{R_{22}^{(1)}}{e^{(1)}}, \qquad \left(\frac{R_{12}^{(2)}}{e^{(2)}}\right)^2 = \frac{A^{(1)}}{c_1^{(2)} + A^{(2)}}\frac{R_{22}^{(1)}}{e^{(1)}}. \qquad [18.50]$$

The resulting numerical values are:

$$\frac{R_{11}^{(1)}}{e^{(1)}} = 1.10, \qquad \frac{R_{22}^{(1)}}{e^{(1)}} = 0.45, \qquad \frac{R_{33}^{(1)}}{e^{(1)}} = 0.45, \qquad \frac{R_{12}^{(1)}}{e^{(1)}} = 0.325,$$

$$\frac{R_{11}^{(2)}}{e^{(2)}} = 0.72, \qquad \frac{R_{22}^{(2)}}{e^{(2)}} = 0.64, \qquad \frac{R_{33}^{(2)}}{e^{(2)}} = 0.64, \qquad \frac{R_{12}^{(2)}}{e^{(2)}} = 0.04.$$

In the case of the logarithmic zone of the boundary layer on a flat plate, a similar calculation gives:

$$\frac{R_{22}^{(1)}}{e^{(1)}} = \frac{s + 0.5c_2 - c_2 c'_2}{d_1},$$

$$\frac{R_{11}^{(1)}}{e^{(1)}} = \frac{2 - c_2 + s + 0.5c_2 c'_2 + c'_1 R_{22}^{(1)} / e^{(1)}}{d_1},$$

$$\frac{R_{33}^{(1)}}{e^{(1)}} = \frac{s + 0.5c_2 + 0.5c_2 c'_2 + c'_1 R_{22}^{(1)} / e^{(1)}}{d_1},$$

$$s_1 = \frac{2}{3}\left(c_1^{(1)} - 1 + A^{(1)}\right), \quad d_1 = c_1^{(1)} + A^{(1)} + 2c'_1, \quad c_2 = c_2^{(1)} = c_2^{(2)},$$

$$\frac{R_{22}^{(2)}}{e^{(2)}} = \frac{s_2 + h^{(22)} - c_2 c'_2}{d_2},$$

$$\frac{R_{11}^{(2)}}{e^{(2)}} = \frac{s_2 + h^{(11)} + 0.5c_2 c'_2 + c'_1 R_{22}^{(2)} / e^{(2)}}{d_2},$$

$$\frac{R_{33}^{(2)}}{e^{(2)}} = \frac{s_2 + h^{(33)} + 0.5c_2 c'_2 + c'_1 R_{22}^{(2)} / e^{(2)}}{d_2},$$

$$s_2 = \frac{2}{3}\left(c_1^{(2)} - 1 + A^{(2)}\right), \qquad d_2 = c_1^{(2)} + A^{(2)} + 2c'_1,$$

$$h^{(ij)} = \frac{F^{(1)}}{F^{(2)}}\left[A^{(1)} \frac{R_{ij}^{(1)}}{e^{(1)}} + \frac{2}{3}\left(1 - A^{(1)}\right)\delta_{ij}\right],$$

$$\left(\frac{R_{12}^{(1)}}{e^{(1)}}\right)^2 = \frac{1 - c_2 + 1.5c_2 c'_2}{c_1^{(1)} + A^{(1)} - 1.5c'_1} \frac{R_{22}^{(1)}}{e^{(1)}}, \quad \left(\frac{R_{12}^{(2)}}{e^{(2)}}\right)^2 = \frac{A^{(1)}}{c_1^{(2)} + A^{(2)} - 1.5c'_1} \frac{R_{22}^{(1)}}{e^{(1)}}.$$

$$[18.51]$$

The resulting numerical values are:

$$\frac{R_{11}^{(1)}}{e^{(1)}} = 1.22, \qquad \frac{R_{22}^{(1)}}{e^{(1)}} = 0.22, \qquad \frac{R_{33}^{(1)}}{e^{(1)}} = 0.56, \qquad \frac{R_{12}^{(1)}}{e^{(1)}} = 0.39,$$

$$\frac{R_{11}^{(2)}}{e^{(2)}} = 0.82, \qquad \frac{R_{22}^{(2)}}{e^{(2)}} = 0.45, \qquad \frac{R_{33}^{(2)}}{e^{(2)}} = 0.73, \qquad \frac{R_{12}^{(2)}}{e^{(2)}} = 0.06.$$

f) Spectral anisotropy

The phenomenon of paradoxical "return to anisotropy" of the Reynolds stresses showed in the illustration given in section 18.2.4 for homogenous turbulence also occurs in the contraction experiment of Uberoi [UBE 56]. In this experiment, a grid generates a slightly anisotropic turbulence which is afterwards brought to isotropy through a small contraction, then an increase of the Reynolds stress anisotropy is

observed downstream in the straight duct after the contraction. The multiple scale approach can be given a possible explanation, if we assume that the initial anisotropy created by the grid is essentially concentrated in the big energy containing eddies. The contraction produces apparent isotropy but in reality the anisotropies of the large-scales and of the small scales are compensating each other. The return to isotropy being stronger in the region of small eddies, the anisotropy of the big eddies which was temporarily hidden then reappears (Figure 18.11 after [SCH 87B]).
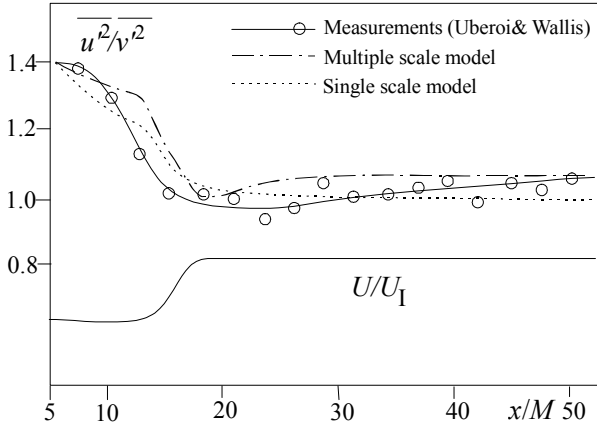
Figure 18.11. *Increase of anisotropy after an axisymmetric contraction*

18.3.2. *Expression for the spectral fluxes*

Equation [18.45] can also be written:

$$\frac{dF^{(m)}}{dt} = \varXi^{(m)} \frac{F^{(m)2}}{e^{(m)}} , \qquad [18.52]$$

if we define:

$$\varXi^{(m)} = \frac{3}{2}\frac{P^{(m)}}{F^{(m)}} + \left[\frac{3}{2} + \frac{1}{\beta_m}\frac{E\left[\kappa(m)\right]}{E\left[\kappa(m-1)\right]}\left(1 - \frac{\mathscr{F}^{(m-1)}}{F^{(m-1)}}\right)\right]\frac{F^{(m-1)}}{F^{(m)}}$$

$$- \left[\frac{3}{2} + \frac{1}{\beta_m}\left(1 - \frac{\mathscr{F}^{(m)}}{F^{(m)}}\right)\right] - \frac{3}{2}\frac{\varepsilon^{(m)}}{F^{(m)}} ,$$

and besides:

$$\Xi^{(m)} = C_{F0}^{(m)} \frac{P^{(m)}}{F^{(m)}} + C_{F1}^{(m)} \frac{F^{(m-1)}}{F^{(m)}} - C_{F2}^{(m)} - C_{F3}^{(m)} \frac{\varepsilon^{(m)}}{F^{(m)}} .$$

Identifying these two expressions leads to useful approximations for the spectral fluxes, which in the particular case of a two level model yield:

$$\frac{1}{\beta_1} \frac{\mathcal{F}^{(1)}}{F^{(1)}} = \frac{3}{2} + \frac{1}{\beta_1} - C_{F2}^{(1)} - \left(\frac{3}{2} - C_{F0}^{(1)} \right) \frac{P^{(1)}}{F^{(1)}} , \qquad [18.53]$$

$$\frac{1}{\beta_2} \frac{\mathcal{F}^{(2)}}{F^{(2)}} = \frac{3}{2} + \frac{1}{\beta_2} - C_{F2}^{(2)} - \left(\frac{3}{2} - C_{F0}^{(2)} \right) \frac{P^{(2)}}{F^{(2)}} - \left(\frac{3}{2} - C_{F1}^{(2)} \right) \frac{F^{(1)}}{F^{(2)}} , \qquad [18.54]$$

where coefficients $C_{Fj}^{(m)}$ have to be replaced by their empirical expression, according to partial energies.

18.3.3. *How can the splitting wavenumbers be obtained?*

Considering the definition relations for a two level model: $\kappa_1 = \alpha_1 \dfrac{F^{(1)}}{\left(e^{(1)} \right)^{3/2}}$,

$\kappa_2 - \kappa_1 = \alpha_2 \dfrac{F^{(2)}}{\left(e^{(2)} \right)^{3/2}}$ we just have to determine the numerical constants α_1 and

α_2. In single scale modeling, the wavenumber corresponding to the integral macroscale is of order $\kappa_1 = 0.1 \dfrac{\varepsilon}{k^{3/2}}$. We can infer from this value the order of magnitude of α_1 and α_2 ($\alpha_1 = 0.05$, $\alpha_2 = 0.05$) but more precise experimental comparisons would be necessary to get a better approximation.

18.3.4. *Simplified multiple scale model with M levels and for high turbulence Reynolds numbers*

<div align="center">

MKFM model

(Multiple scale model for kinetic energy and spectral fluxes in M levels)

</div>

$$\frac{de^{(m)}}{dt} = P^{(m)} + F^{(m-1)} - F^{(m)} + c_s^{(m)} \left(\frac{k^2}{F^{(1)}} e^{(m)}_{,j} \right)_{,j} - \varepsilon^{(m)},$$

$$\varepsilon^{(m)} = 0 \quad \text{except} \quad \varepsilon^{(M+1)} = F^{(M)},$$

$$\frac{dF^{(m)}}{dt} = C_{F0}^{(m)} \frac{P^{(m)} F^{(m)}}{e^{(m)}} + C_{F1}^{(m)} \frac{F^{(m-1)} F^{(m)}}{e^{(m)}}$$

$$-C_{F2}^{(m)} \frac{F^{(m)2}}{e^{(m)}} + C_{F3}^{(m)} \left(\frac{k^2}{F^{(1)}} F^{(m)}_{,j} \right)_{,j},$$

$$R_{ij}^{(m)} = \frac{2}{3} e^{(m)} \delta_{ij} - \nu_t^{(m)} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right), \qquad \nu_t^{(m)} = c_\mu^{(m)} \frac{k^2}{F^{(m)}},$$

$c_\mu^{(m)}$ strongly decreases at larger m.

Usual case for two levels:

<div align="center">

MKF2 model

</div>

The result obtained previously in the turbulent boundary layer on a flat plate using the Reynolds stress model give:

$$\frac{R_{12}^{(2)}}{R_{12}^{(1)}} = 0.15 \frac{e^{(2)}}{e^{(1)}}, \quad \frac{e^{(2)}}{e^{(1)}} = \frac{1}{3}, \quad k = e^{(1)} + e^{(2)}, \quad R_{12}^{(1)} + R_{12}^{(2)} = R_{12},$$

$$R_{12}^{(1)} = \frac{e^{(1)}}{e^{(1)} + 0.15 e^{(2)}} R_{12}, \quad R_{12}^{(2)} = \frac{0.15 e^{(2)}}{e^{(1)} + 0.15 e^{(2)}} R_{12},$$

Inspired by these results, we are led to the following formulation:

$$\frac{de^{(1)}}{dt} = P^{(1)} - F^{(1)} + C_s^{(1)} (\sigma_t e^{(1)}_{,j})_{,j} - \varepsilon^{(1)},$$

$$\frac{de^{(2)}}{dt} = P^{(2)} + F^{(1)} - F^{(2)} + C_s^{(2)}(\sigma_t e^{(2)}_{,j})_{,j} - \varepsilon^{(2)},$$

$$\frac{dF^{(1)}}{dt} = C_{F0}^{(1)}\frac{P^{(1)}F^{(1)}}{e^{(1)}} - C_{F2}^{(1)}\frac{F^{(1)2}}{e^{(1)}} + C_{F3}^{1}(\sigma_t \varepsilon^{(1)}_{,j})_{,j} - C_{F4}^{(1)}\frac{F^{(1)}\varepsilon^{(1)}}{e^{(1)}},$$

$$\frac{dF^{(2)}}{dt} = C_{F0}^{(2)}\frac{P^{(2)}F^{(2)}}{e^{(2)}} + C_{F1}^{(2)}\frac{F^{(1)}F^{(2)}}{e^{(2)}} - C_{F2}^{(2)}\frac{F^{(2)2}}{e^{(2)}}$$

$$+C_{F3}^{2}(\sigma_t \varepsilon^{(2)}_{,j})_{,j} - C_{F4}^{(2)}\frac{F^{(2)}\varepsilon^{(2)}}{e^{(2)}},$$

$$R_{ij}^{(m)} = \frac{2}{3}e^{(m)}.\delta_{ij} - v_t^{(m)}(U_{i,j} + U_{j,i}), \qquad P^{(m)} = v_t^{(m)}(U_{i,j} + U_{j,i}).U_{i,j}$$

$$\varepsilon^{(1)} = 0., \quad \varepsilon^{(2)} = F^{(2)}, \quad v_t^{(m)} = C_\mu^{(m)}\frac{k^2}{F^{(m)}} \quad (m=1,2), \quad v_t = v_t^{(1)} + v_t^{(2)}$$

$$C_\mu^{(1)} = C_{\mu\lim}\frac{e^{(1)}}{e^{(1)} + 0.15e^{(2)}}, \qquad\qquad C_\mu^{(2)} = C_{\mu\lim}\frac{0.15e^{(2)}}{e^{(1)} + 0.15e^{(2)}},$$

$$C_{\mu\lim} = 0.09, \qquad C_s^{(m)} = \frac{C_{\mu\lim}}{h_k^{(m)}}, \qquad\qquad C_{F3}^{(m)} = \frac{C_{\mu\lim}}{h_\varepsilon^{(m)}},$$

$$\sigma_t = \frac{e^{(1)}}{e^{(1)} + 0.15e^{(2)}}\frac{k^2}{F^{(1)}} + \frac{0.15e^{(2)}}{e^{(1)} + 0.15e^{(2)}}\frac{k^2}{F^{(2)}},$$

or also as a first approximation $\sigma_t = \dfrac{k^2}{F^{(1)}}$,

$h_k^{(1)} = 1.0$	$h_k^{(2)} = 1.0$	$h_F^{(1)} = 1.3$	$h_F^{(2)} = 1.3$
$C_{F0}^{(1)} = 1.60$	$C_{F0}^{(2)} = 0.$	$C_{F1}^{(1)} = 0.$	$C_{F1}^{(2)} = 1.70$
$C_{F2}^{(1)} = 1.90$	$C_{F2}^{(2)} = 1.80$	$C_{F4}^{(1)} = 1.90$	$C_{F4}^{(2)} = 1.80$

$C_{F1}^{(1)}$ is zero because the corresponding term does not appear in the equation for $k^{(1)}$ and $F^{(0)} \equiv 0$.

This type of model has been used for example for the calculation of unsteady jets ([MAT 01]).

18.3.5. *Multiple scale energy/flux model for high turbulence Reynolds numbers coupled with transport equations for the total Reynolds stresses*

RS-MKFM model
(Reynolds stress (RS) model
coupled with multiple scale kinetic energy/flux model for M levels)

$$\frac{de^{(m)}}{dt} = P^{(m)} + F^{(m-1)} - F^{(m)} + c_s^{(m)} \left(\frac{kR_{ij}}{F^{(1)}} e^{(m)}_{,i} \right)_{,j} - \varepsilon^{(m)},$$

$$\varepsilon^{(m)} = 0 \quad \text{except} \quad \varepsilon^{(M+1)} = F^{(M)},$$

$$\frac{dR_{ij}}{dt} = P_{ij} + \Phi_{ij} + c_s \left(\frac{kR_{pl}}{\varepsilon} R_{ij,p} \right)_{,l} - \varepsilon_{ij},$$

(total stresses)

$$\frac{dF^{(m)}}{dt} = C_{F0}^{(m)} \frac{P^{(m)}F^{(m)}}{e^{(m)}} + C_{F1}^{(m)} \frac{F^{(m-1)}F^{(m)}}{e^{(m)}} - C_{F2}^{(m)} \frac{F^{(m)^2}}{e^{(m)}} + C_{F3}^{(m)} \left(\frac{kR_{ij}}{F^{(1)}} F^{(m)}_{,i} \right)_{,j}.$$

In the production term $P^{(m)} = -R_{ij}^{(m)} \overline{U}_{i,j}$ it is necessary to approximate $R_{ij}^{(m)}$. With this aim let us introduce the anisotropy tensors:

$$a_{ij} = \frac{R_{ij} - \frac{2}{3}k\delta_{ij}}{k}, \qquad a_{ij}^{(m)} = \frac{R_{ij}^{(m)} - \frac{2}{3}e^{(m)}\delta_{ij}}{e^{(m)}}.$$

We can assume $a_{ij}^{(m)} = \alpha^{(m)} a_{ij}$,

where $\alpha^{(m)}$ strongly decreases at high values of m. Again inspired by the boundary layer solution, we get for the two level usual case the following.

RS-MKF2 model

$$R_{12}^{(1)} = \frac{e^{(1)}}{e^{(1)} + 0.15e^{(2)}} R_{12}, \qquad R_{12}^{(2)} = \frac{0.15e^{(2)}}{e^{(1)} + 0.15e^{(2)}} R_{12},$$

i.e. $$\alpha^{(1)} = \frac{k}{e^{(1)} + 0.15e^{(2)}}, \qquad \alpha^{(2)} = \frac{0.15k}{e^{(1)} + 0.15e^{(2)}},$$

$$R_{12}^{(1)} = \frac{2}{3}e^{(1)}\delta_{ij} + \frac{k.e^{(1)}}{e^{(1)}+0.15e^{(2)}}a_{ij}, \quad R_{12}^{(2)} = \frac{2}{3}e^{(2)}\delta_{ij} + \frac{0.15k.e^{(2)}}{e^{(1)}+0.15e^{(2)}}a_{ij}.$$

In the transport equations for the total stresses, it will be possible to introduce, as in section 18.1, coefficients (in particular for the Rotta term) which are functions of non-equilibrium parameters such as $e^{(1)}/e^{(2)}$ and $F^{(1)}/F^{(2)}$.

18.3.6. *Reynolds stress multiple scale model with M levels and for low turbulence Reynolds numbers*

The extension to low turbulence Reynolds numbers is necessary in view to consider real flows encountered in practice, nevertheless some empiricism is unavoidable. The models proposed hereafter are suggestions for guiding the development of extended formulations applicable to low Reynolds numbers, in particular for the near wall region. Further numerical tests for various types of flows that are experimentally well known are of course necessary before reaching a definitive formulation.

In the present case, the model coefficients are no longer constants but they become empirical functions of the turbulence Reynolds number. Moreover, it is important to emphasize that at low Reynolds numbers the spectral zones (production zone, transfer zone and dissipation zone) are no longer clearly separated and are more or less overlapping in the spectrum. Consequently, even if a wavenumber splitting is still possible (turbulent flow visualizations at low Reynolds number always show the coexistence of large eddies and small eddies), it will be necessary to include both the production, the transfer and the dissipation effects coexisting in each spectral slice (cf. Figure 18.12). Almost every single scale Reynolds stress transport model can be transposed into a multiple scale version (Hanjalic K. and Launder B.E. model [HAN 76], but also the model by Launder B.E. and Tselepidakis D.P. [LAU 91A] for example).

MRFLM model
(Multiple scale model for the Reynolds stresses
and the spectral fluxes for low Reynolds numbers and in M levels)

$$\frac{de^{(m)}}{dt} = P^{(m)} + F^{(m-1)} - F^{(m)} + c_s^{(m)}\left(\sigma_{ij}e^{(m)},_i\right)_{,j} - \varepsilon^{(m)} + \nu e_{,jj}^{(m)},$$

$$\sigma_{ij} = \frac{kR_{ij}}{F^{(1)}},$$

$$\overset{(m)}{\varepsilon} = \gamma\, F^{(m)} + \nu\, \frac{\overset{(m)}{e}_{,j}\, k_{,j}}{k}$$
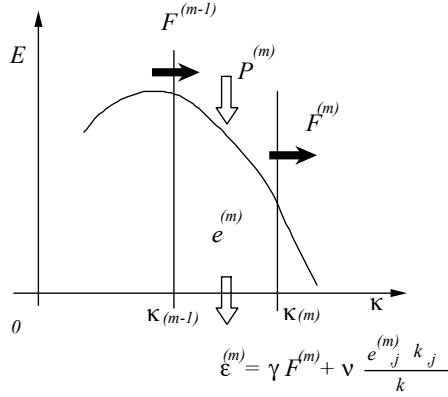
Figure 18.12. *Sketch of spectrum splitting for turbulence kinetic energy at low Reynolds numbers*

$$\varepsilon^{(m)} = \frac{f_T}{1-f_T}\, F^{(m)} + \nu \frac{e^{(m)}_{,j}\, k_{,j}}{2k}\,,$$

$$f_T = 0.3\exp(-\mathrm{Re}_t{}^2)\,,\qquad \gamma = \frac{f_T}{1-f_T}\,,\qquad \mathrm{Re}_t = \frac{k^2}{\nu\varepsilon}\,,$$

$$\frac{dR_{ij}^{(m)}}{dt} = P_{ij}^{(m)} + \Phi_{ij}^{(m)} + F_{ij}^{(m-1)} - F_{ij}^{(m)} + c_s^{(m)}\left(\sigma_{pl} R_{ij,p}^{(m)}\right)_{,l} - \varepsilon_{ij}^{(m)}\left(+B_{ij}^{(m)}\right) + \nu R_{ij,hh}^{(m)}\,,$$

$$\frac{dF^{(m)}}{dt} = C_{F0}^{(m)} \frac{P^{(m)} F^{(m)}}{e^{(m)}} + \Sigma^{(m)} + C_{F1}^{(m)} \frac{F^{(m-1)} F^{(m)}}{e^{(m)}}$$

$$-C_{F2}^{(m)} \frac{F^{(m)2}}{e^{(m)}} + C_{F3}^{(m)}\left(\sigma_{ij} F_{,i}^{(m)}\right)_{,j} - C_{F4}^{(m)} \frac{F^{(m)} \varepsilon^{(m)}}{e^{(m)}} + \nu F_{,hh}^{(m)}\,,$$

$$\Sigma^{(m)} = 2\nu c_\mu \frac{e^{(m)} k}{F^{(1)}} \overline{U}_{i,pl}\overline{U}_{i,pl} \quad\text{or}\quad \Sigma^{(m)} = 2\nu \frac{k^{(m)}}{F^{(1)}} \underbrace{R_{jq}.U_{i,jl}U_{i,ql}}_{R_{nn}\left(\partial^2 U / \partial n^2\right)^2}\,,$$

$$C_{F4}^{(m)} = C_{F2}^{(m)}\,,\qquad c_\mu = c_{\mu\lim}\exp\left[\frac{-3.4}{\left(1+\mathrm{Re}_t/50\right)^2}\right]\,,\qquad c_{\mu\lim} = 0.09$$

The usual case for two levels is as follows.

<div align="center">MRFL2 model</div>

$$\frac{de^{(1)}}{dt} = P^{(1)} - F^{(1)} + c_s^{(1)} \left(\sigma_{ij} e^{(1)}_{,i} \right)_{,j} - \varepsilon^{(1)} + \nu e^{(1)}_{,jj} \, ,$$

$$\frac{de^{(2)}}{dt} = P^{(2)} + F^{(1)} - F^{(2)} + c_s^{(2)} \left(\sigma_{ij} e^{(2)}_{,i} \right)_{,j} - \varepsilon^{(2)} + \nu e^{(2)}_{,jj} \, ,$$

$$\frac{dF^{(1)}}{dt} = C_{F0}^{(1)} \frac{P^{(1)} F^{(1)}}{e^{(1)}} + \varSigma^{(1)} + C_{F1}^{(1)} \frac{F^{(0)} F^{(1)}}{e^{(1)}} - C_{F2}^{(1)} \frac{F^{(1)^2}}{e^{(1)}}$$

$$+ C_{F3}^{(1)} \left(\sigma_{ij} F^{(1)}_{,i} \right)_{,j} - C_{F4}^{(1)} \frac{F^{(1)} \tilde{\varepsilon}^{(1)}}{e^{(1)}} - C_{F5}^{(1)} \left(\frac{\nu \tilde{\varepsilon}}{k} e^{(1)}_{,j} \right)_{,j} + \nu F^{(1)}_{,hh} \, ,$$

$$\frac{dF^{(2)}}{dt} = C_{F0}^{(2)} \frac{P^{(2)} F^{(2)}}{e^{(2)}} + \varSigma^{(2)} + C_{F1}^{(2)} \frac{F^{(1)} F^{(2)}}{e^{(2)}} - C_{F2}^{(2)} \frac{F^{(2)^2}}{e^{(2)}}$$

$$+ C_{F3}^{(2)} \left(\sigma_{ij} F^{(2)}_{,i} \right)_{,j} - C_{F4}^{(2)} \frac{F^{(2)} \tilde{\varepsilon}^{(2)}}{e^{(2)}} + C_{F5}^{(2)} \left(\frac{\nu \tilde{\varepsilon}}{k} e^{(2)}_{,i} \right)_{,j} + \nu F^{(2)}_{,hh} \, ,$$

$$k = e^{(1)} + e^{(2)} \, , \quad \sigma_{ij} = \frac{k R_{ij}}{F^{(1)}} \, , \quad P^{(m)} = -R_{ij}^{(m)} \overline{U}_{i,j} \, , \quad R_{ij} = R_{ij}^{(1)} + R_{ij}^{(2)} \, ,$$

$$\frac{dR_{ij}^{(1)}}{dt} = P_{ij}^{(1)} + \varPhi_{ij}^{(1)} - F_{ij}^{(1)} + c_s^{(1)} \left(\sigma_{pl} R_{ij,p}^{(1)} \right)_{,l} - \varepsilon_{ij}^{(1)} + \nu R_{ij,hh}^{(1)} \, ,$$

$$(\text{possibly} + B_{ij}^{(1)})$$

$$\frac{dR_{ij}^{(2)}}{dt} = P_{ij}^{(2)} + \varPhi_{ij}^{(2)} + F_{ij}^{(1)} - F_{ij}^{(2)} + c_s^{(2)} \left(\sigma_{pl} R_{ij,p}^{(2)} \right)_{,l} - \varepsilon_{ij}^{(2)} + \nu R_{ij,hh}^{(2)} \, ,$$

$$P_{ij}^{(m)} = -R_{ip}^{(m)} \overline{U}_{j,p} - R_{jp}^{(m)} \overline{U}_{i,p} \, ,$$

$$F_{ij}^{(m)} = \frac{F^{(m)}}{e^{(m)}} \left[A^{(m)} R_{ij}^{(m)} + \frac{2}{3} (1 - A^{(m)}) e^{(m)} \delta_{ij} \right] ,$$

$$A^{(m)} = 1 - f_S + A_{\lim}^{(m)} f_S \, , \quad f_S = \frac{\mathrm{Re}_t}{1 + \mathrm{Re}_t} \quad \text{or also} \quad f_S = 1 - \exp(-\mathrm{Re}_t^{\,2}) \, ,$$

$A_{\lim}^{(m)}$, corresponding value at high Reynolds number.

$$\Phi_{ij}^{(m)} = -c_1^{(m)} \frac{F^{(m)}}{e^{(m)}} \left(R_{ij}^{(m)} - \frac{2}{3} e^{(m)} \delta_{ij} \right) - c_2^{(m)} \left(P_{ij}^{(m)} - \frac{2}{3} P^{(m)} \delta_{ij} \right) + \Phi_{ij}^{(s)} ,$$

$$c_1^{(m)} = 1.1\left(1 - f_S\right) + c_{1\lim}^{(m)} f_S , \qquad c_2^{(m)} = c_{2\lim}^{(m)} ,$$

$c_{1\lim}^{(m)}$ and $c_{2\lim}^{(m)}$ being the corresponding values at high Reynolds number,

$$\varepsilon_{ij}^{(m)} = f_A \varepsilon_{ij}^{*(m)} + (1 - f_A) \left[A^{(m)} \frac{\varepsilon^{(m)} R_{ij}^{(m)}}{k^{(m)}} + \frac{2}{3} \left(1 - A^{(m)}\right) \varepsilon^{(m)} . \delta_{ij} \right] ,$$

$$\varepsilon_{ij}^{*(m)} = \frac{\varepsilon^{(m)}}{\varepsilon} \varepsilon_{ij}^{*} , \qquad f_A = \exp(-20A^2).\exp(-\mathrm{Re}_T^2/20) ,$$

A, flatness factor,

$$\varepsilon^{(m)} = f_0.F^{(m)} + \nu \frac{(e^{(m)})_{,j}(k)_{,j}}{2k} , \qquad f_0 = \frac{f_T}{1 - f_T} ,$$

$$\Phi_{ij}^{(s,m)} = c_1^{\prime(m)} \frac{F^{(m)}}{e^{(m)}} \left(R_{lp}^{(m)} n_l n_p \delta_{ij} - \frac{3}{2} R_{il}^{(m)} n_l n_j - \frac{3}{2} R_{jl}^{(m)} n_l n_i \right) \frac{k^{3/2}}{c_l F^{(1)} \delta}$$

$$+ c_2^{\prime(m)} \left(\Phi_{lp}^{(m,2)} n_l n_p \delta_{ij} - \frac{3}{2} \Phi_{il}^{(m,2)} n_l n_j - \frac{3}{2} \Phi_{jl}^{(m,2)} n_l n_i \right) \frac{k^{3/2}}{c_l F^{(1)} \delta} ,$$

$$\mathcal{B}_{ij}^{(1)} = \zeta^{(1)} \left\{ \frac{1}{20} \left[R_{li}^{(1)} \left(\frac{\partial \overline{U}_l}{\partial X_j} + \frac{\partial \overline{U}_j}{\partial X_l} \right) + R_{lj}^{(1)} \left(\frac{\partial \overline{U}_l}{\partial X_i} + \frac{\partial \overline{U}_i}{\partial X_l} \right) \right] \right.$$

$$\left. + \frac{1}{5} R_{lp}^{(1)} \frac{\partial \overline{U}_l}{\partial X_p} \left(\frac{R_{ij}^{(1)}}{e^{(1)}} - \delta_{ij} \right) \right\} , \quad \zeta^{(1)} = 1, \text{ assuming } F^{(m)} = 0 \text{ on a wall.}$$

Numerical constants:

$h_k^{(1)} = 1.0$	$h_k^{(2)} = 1.0$	$h_F^{(1)} = 1.3$	$h_F^{(2)} = 1.3$
$C_{F0}^{(1)} = 1.60$	$C_{F0}^{(2)} = 0.$	$C_{F1}^{(1)} = 0.$	$C_{F1}^{(2)} = 1.70$
$C_{F2}^{(1)} = 1.90$	$C_{F2}^{(2)} = 1.80$	$C_{F4}^{(1)} = 1.90$	$C_{F4}^{(2)} = 1.80$
$C_{F5}^{(1)} = C_{F5}^{(2)} = 0$ (other coefficients, see section 18.3.1)			

18.3.7. *Multiple scale energy/flux model with M levels and for low turbulence Reynolds numbers*

MKFLM model

(Multiple scale model for kinetic energy
and spectral fluxes for low Reynolds numbers and in M levels)

$$\frac{de^{(m)}}{dt} = P^{(m)} + F^{(m-1)} - F^{(m)} + c_s^{(m)} \left(\frac{k^2}{F^{(1)}} e^{(m)}_{,j} \right)_{,j} - \varepsilon^{(m)} + v e^{(m)}_{,jj} \,,$$

$$\varepsilon^{(m)} = \frac{f_T}{1 - f_T} F^{(m)} + v \frac{e^{(m)}_{,j} k_{,j}}{2k} \,, \quad f_T = 0.3 \exp(-\mathrm{Re}_t) \,,$$

$$R^{(m)}_{ij} = \frac{2}{3} e^{(m)} \delta_{ij} - v_t^{(m)} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right) \,, \quad v_t^{(m)} = c_\mu^{(m)} \frac{k^2}{F^{(m)}} \,,$$

$$c_\mu^{(m)} = c_{\mu\lim}^{(m)} \exp \left[\frac{-3.4}{\left(1 + \mathrm{Re}_t / 50\right)^2} \right] \,, \qquad \mathrm{Re}_t = k^2 / v\varepsilon \,,$$

$c_{\mu\lim}^{(m)}$ corresponding value at high Reynolds number,

$$\frac{dF^{(m)}}{dt} = C_{F0}^{(m)} \frac{P^{(m)} F^{(m)}}{e^{(m)}} + \Sigma^{(m)} + C_{F1}^{(m)} \frac{F^{(m-1)} F^{(m)}}{e^{(m)}}$$

$$- C_{F2}^{(m)} \frac{F^{(m)2}}{e^{(m)}} + C_{F3}^{(m)} \left(\frac{k^2}{F^{(1)}} F^{(m)}_{,j} \right)_{,j} - C_{F4}^{(m)} \frac{F^{(m)} \varepsilon^{(m)}}{e^{(m)}} + v F^{(m)}_{,hh} \,,$$

$$C_{F4}^{(m)} = C_{F2}^{(m)} \,, \qquad \Sigma^{(m)} = 2 v c_\mu \frac{e^{(m)} k}{F^{(1)}} \overline{U}_{i,pl} \overline{U}_{i,pl} \quad .$$

The usual case for two levels is as follows.

MKFL2 model

$$c_\mu^{(1)} = \frac{0.09 e^{(1)}}{e^{(1)} + 0.15 e^{(2)}} \exp \left[\frac{-3.4}{\left(1 + \mathrm{Re}_t / 50\right)^2} \right] \,,$$

$$c_\mu^{(2)} = \frac{0.09 \times 0.15 e^{(2)}}{e^{(1)} + 0.15 e^{(2)}} \exp\left[\frac{-3.4}{\left(1 + \mathrm{Re}_t / 50\right)^2}\right],$$

or in a first approximation:

$$c_\mu^{(1)} = 0.09 \exp\left[\frac{-3.4}{\left(1 + \mathrm{Re}_t / 50\right)^2}\right], \qquad c_\mu^{(2)} = 0 \quad .$$

The approximation $\varepsilon^{(m)} = \underbrace{\gamma F^{(m)}}_{\tilde{\varepsilon}^{(m)}} + \nu \dfrac{e_{,j}^{(m)} k_{,j}}{2k}$ leads to:

$$\varepsilon^{(1)} = \tilde{\varepsilon}^{(1)} + \nu \frac{e_{,j}^{(1)} k_{,j}}{2k}, \;\; \varepsilon^{(2)} = \tilde{\varepsilon}^{(2)} + \nu \frac{e_{,j}^{(2)} k_{,j}}{2k}, \;\; \varepsilon^{(3)} = \tilde{\varepsilon}^{(3)} = F^{(2)},$$

and we also note that $\varepsilon = \varepsilon^{(1)} + \varepsilon^{(2)} + \varepsilon^{(3)}$ and $\tilde{\varepsilon} = \tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)} + \tilde{\varepsilon}^{(3)}$.

Numerical coefficients:

$h_k^{(1)} = 1.0$	$h_k^{(2)} = 1.0$	$h_F^{(1)} = 1.3$	$h_F^{(2)} = 1.3$
$C_{F0}^{(1)} = 1.60$	$C_{F0}^{(2)} = 0.$	$C_{F1}^{(1)} = 0.$	$C_{F1}^{(2)} = 1.70$
$C_{F2}^{(1)} = 1.90$	$C_{F2}^{(2)} = 1.80$	$C_{F4}^{(1)} = 1.90$	$C_{F4}^{(2)} = 1.80$

18.3.8. *Multiple scale energy/flux model for low turbulence Reynolds numbers coupled with transport equations for the total Reynolds stresses*

RS-MKFLM model
(Reynolds stress (RS) model coupled with
multiple scale kinetic energy/flux model for low Reynolds numbers and in M levels)

This is a hybrid version including several levels for the kinetic energy but using the transport equations for the total stresses.

$$\frac{de^{(m)}}{dt} = P^{(m)} + F^{(m-1)} - F^{(m)} + c_s^{(m)} \left(\frac{k^2}{F^{(1)}} e^{(m)}_{,j}\right)_{,j} - \varepsilon^{(m)} + \nu e^{(m)}_{,jj},$$

$$\varepsilon^{(m)} = \frac{f_T}{1-f_T} F^{(m)} + v \frac{e_{,j}^{(m)} k_{,j}}{2k}, \quad f_T = 0.3\exp(-\text{Re}_t), \quad \text{Re}_t = k^2/v\varepsilon,$$

$$\frac{dR_{ij}}{dt} = P_{ij} + \Phi_{ij} + c_s \left(\frac{kR_{pl}}{\varepsilon} R_{ij,p} \right)_{,l} - \varepsilon_{ij} + v R_{ij,hh} \quad \text{(total stresses)},$$

$$\frac{dF^{(m)}}{dt} = C_{F0}^{(m)} \frac{P^{(m)} F^{(m)}}{e^{(m)}} + \Sigma^{(m)} + C_{F1}^{(m)} \frac{F^{(m-1)} F^{(m)}}{e^{(m)}}$$

$$-C_{F2}^{(m)} \frac{F^{(m)2}}{e^{(m)}} + C_{F3}^{(m)} \left(\frac{k^2}{F^{(1)}} F_{,j}^{(m)} \right)_{,j} - C_{F4}^{(m)} \frac{F^{(m)} \varepsilon^{(m)}}{e^{(m)}} + v F_{,hh}^{(m)},$$

$$\Sigma^{(m)} = 2v c_\mu \frac{e^{(m)} k}{F^{(1)}} \overline{U}_{i,pl} \overline{U}_{i,pl}, \quad C_{F4}^{(m)} = C_{F2}^{(m)},$$

$$c_\mu = c_{\mu\lim} \exp\left[\frac{-3.4}{(1+\text{Re}_t/50)^2} \right], \quad c_{\mu\lim} = 0.09,$$

with
$$a_{ij} = \frac{R_{ij} - \frac{2}{3} k \delta_{ij}}{k}, \quad a_{ij}^{(m)} = \frac{R_{ij}^{(m)} - \frac{2}{3} e^{(m)} \delta_{ij}}{e^{(m)}},$$

and a hypothesis such that $a_{ij}^{(m)} = \alpha^{(m)} a_{ij}$ and $\alpha^{(m)} = \frac{1}{1+\text{Re}_t} + \alpha_{\lim}^{(m)} \frac{\text{Re}_t}{1+\text{Re}_t}$ where

$\alpha_{\lim}^{(m)}$ is the value at high Reynolds numbers. This is a mean for approximating the partial stresses.

The usual case for two levels is as follows.

RS-MKFL2 model

The partial stresses are obtained from:

$$\alpha^{(1)} = \frac{k}{e^{(1)} + 0.1e^{(2)}} \cdot \frac{\text{Re}_T}{1+\text{Re}_T} + \frac{1}{1+\text{Re}_T},$$

$$\alpha^{(2)} = \frac{0.1k}{e^{(1)} + 0.1e^{(2)}} \cdot \frac{\mathrm{Re}_T}{1 + \mathrm{Re}_T} + \frac{1}{1 + \mathrm{Re}_T},$$

$$R_{ij}^{(1)} = \frac{2}{3}e^{(1)}\delta_{ij} + \alpha^{(1)}e^{(1)}a_{ij}, \qquad R_{ij}^{(2)} = \frac{2}{3}e^{(2)}\delta_{ij} + \alpha^{(2)}e^{(2)}a_{ij},$$

$$R_{ij} = \frac{2}{3}k\delta_{ij} + \left(\alpha^{(1)}e^{(1)} + \alpha^{(2)}e^{(2)}\right)a_{ij} = \frac{2}{3}k\delta_{ij} + ka_{ij}.$$

18.3.9. *Some typical applications*

The decay law of turbulence behind a grid can be different depending on the spectral initial distribution. In Figure 18.13 three cases have been considered using the energy/flux multiple scale model: an equilibrium initial spectrum, a spectrum with a local energy defect in the region of large eddies and a spectrum with a local peak of energy in the region of large eddies ([SCH 86]). It is found that to this peak will correspond a slower decay since the energy cascade is longer in this case. These results are in qualitative accordance with the work of C. Cambon [CAM 79] using an EDQNM model.
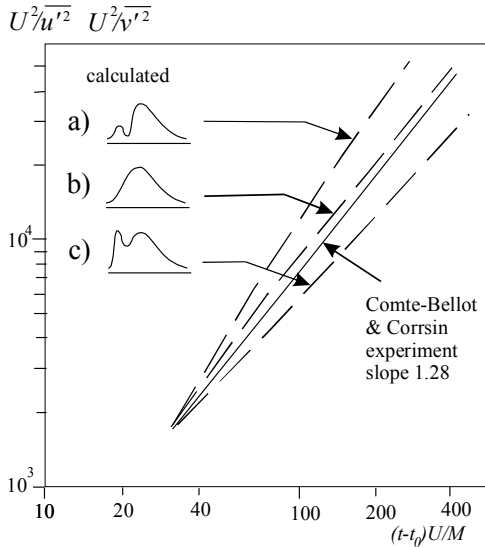
Figure 18.13. *Decay of turbulence from a perturbed initial energy spectrum*

In the case of contraction duct flow, studied with an energy/flux multiple scale model [HAN 79], the evolution of kinetic energy is faster (Figure 18.14) due to the effect of delays on the dissipation rate (Figure 18.15).
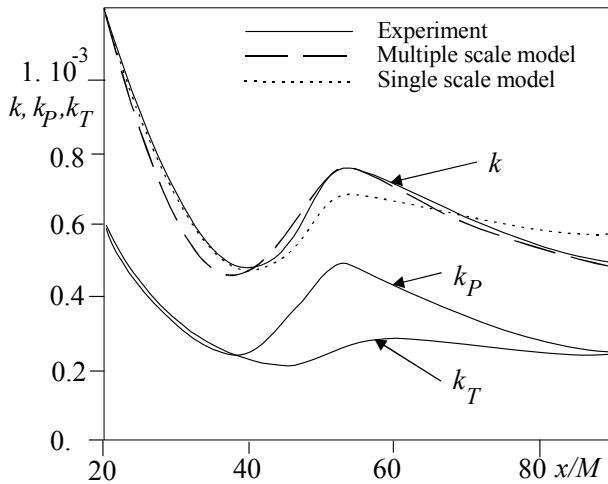
Figure 18.14. *Axisymmetric contraction in a duct (4:1 ratio), partial energies*

The multiple scale models have a practical interest in all turbulent flows with strong departures from equilibrium and among these, the turbulent unsteady (unsteadiness in the mean) flows ([MAT 01], [BRE 03]).
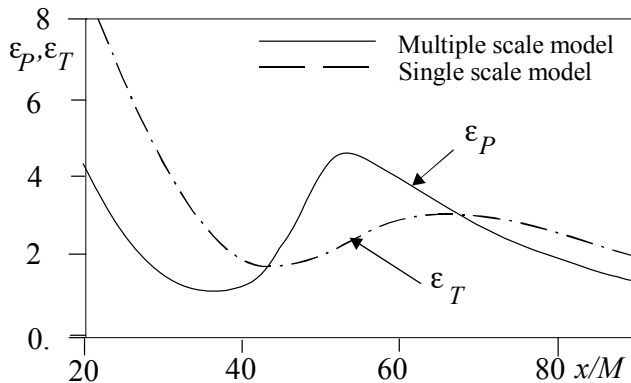
Figure 18.15. *Axisymmetric contraction in a duct (4:1 ratio), spectral fluxes*

18.3.10. *Note on the extension to variable density turbulent flows*

On the basis of mass weighted averaging (cf. Chapter 17) we can introduce a rank decomposition:

$$U_j = \widetilde{U}_j + u\,'_j^{(1)} + u\,'_j^{(2)} + \dots + u\,'_j^{(n)} ,$$

which is the analog of [18.11] with for example:

$$.u\,'_j^{(m)}(\vec{\xi},t;\rho) = \frac{1}{\rho} \int_{\kappa(m-1)<|\vec{\kappa}|\leq\kappa(m)} \widehat{\rho u_j}(\vec{\kappa},t)e^{i\vec{\kappa}\,\vec{\xi}} d\vec{\xi} .$$

In this case, the quantities $u\,'_j^{(m)}(\vec{\xi},t;\rho)$ form statistically independent ranks. Other decompositions are also possible such as:

$$u\,'_j^{(m)}(\vec{\xi},t;1) = \int_{\kappa(m-1)<|\vec{\kappa}|\leq\kappa(m)} \widehat{u_j}(\vec{\kappa},t)e^{i\vec{\kappa}\,\vec{\xi}} d\vec{\xi} .$$

In this case, the quantities $u\,'_j^{(m)}(\vec{\xi},t;1)$ form independent ranks. It is also possible to define the fluctuations (Grégoire O. *et al.*, [GRE 97, 99] and [SOU 02]) through:

$$u\,'_j^{(m)}(\vec{\xi},t;\sqrt{\rho}) = \frac{1}{\sqrt{\rho}} \int_{\kappa(m-1)<|\vec{\kappa}|\leq\kappa(m)} \widehat{\sqrt{\rho}u_j}(\vec{\kappa},t)e^{i\vec{\kappa}\,\vec{\xi}} d\vec{\xi} , \qquad [18.55]$$

the quantities $u\,'_j^{(m)}(\vec{\xi},t;\sqrt{\rho})$ forming time independent ranks, this has the advantage of leading to symmetric expressions for the kinetic energy of turbulence. The fluctuations are not centered, and the integrals have to be understood in the sense of generalized functions (Schwarz distributions).

Starting from this definition, it is possible to develop a formalism similar to the one already presented for turbulent flows with constant density. The same possibilities are also open for an approach in non-weighted variables.

18.4. Other multiple scale models: models using spectral weighted integration

This alternative is illustrated by the model by Stawiarsky K. and Hanjalic K. (cf. [STA 02], [CAD 04]) which is based on a weighted integration of the energy spectrum. The integrals under consideration have the following form:

$$\mathcal{T}_{ij}^{(m)} = \int_0^\infty \kappa^m \varphi_{ij}(\kappa) d\kappa \,.$$

Depending on the value of m, the different integrals $\mathcal{T}_{ij}^{(m)}$ will give more or less weight to the large and small eddies. This concept thus replaces the spectral splitting.

The spectral tensor is approximated on the basis of the relation:

$$\varphi_{ij}(\kappa) = f(\kappa)E(\kappa)\frac{R_{ij}}{k} + \left[1 - f(\kappa)\right]\frac{2}{3}E(\kappa)\delta_{ij} \,,$$

corresponding to the hypothesis (similar to [18.41]) that spectral anisotropy can be simply related to the global Reynolds stress anisotropy:

$$\frac{\varphi_{ij}(\kappa) - \frac{2}{3}E(\kappa)\delta_{ij}}{E(\kappa)} = f(\kappa)\frac{R_{ij} - \frac{2}{3}k\delta_{ij}}{k} \,.$$

The previous scheme is coupled with a hypothesis of simplified energy spectrum:

$$\begin{cases} E(\kappa) = B\kappa^4 & f = 1 & \text{for } \kappa < \kappa_L \\ E(\kappa) = C\varepsilon_P^{2/3}\kappa^{-5/3} & f = \left(\kappa/\kappa_L\right)^{-2/3} & \text{for } \kappa_L < \kappa < \kappa_\eta \\ E(\kappa) = 0 & f = 0 & \text{for } \kappa_\eta < \kappa \end{cases}$$

where κ_L denotes a characteristic wavenumber for the energy containing eddies and κ_η the Kolmogorov wavenumber. The parameters obtained for $m = -1$ and $m = 2$ yield two independent scales:

$$\frac{k^{5/2}}{\varepsilon_P} \sim \int_0^\infty \frac{E(\kappa)}{\kappa} d\kappa \text{ and } \varepsilon_T \sim \int_0^\infty \kappa^2 E(\kappa) d\kappa \,,$$

for which two transport equations are established. After rearranging the terms, the model can be written ([STA 02]):

$$\frac{dR_{ij}}{dt} = P_{ij} + \Phi_{ij} - \varepsilon_{Tij} + Diff(R_{ij}) \,,$$

$$\frac{d\varepsilon_P}{dt} = C_1^P \frac{\varepsilon_P}{k} P - C_2^P \frac{\varepsilon_P}{k} \varepsilon_P + Diff(\varepsilon_P),$$

$$\frac{d\varepsilon_T}{dt} = C_1^T \frac{\varepsilon_P}{k} \mathrm{Re}_t^{-1/2} P - C_2^T \frac{\varepsilon_T^2}{k} \mathrm{Re}_t^{1/2} \left(1 - \frac{\varepsilon_T}{\varepsilon_P}\right) + Diff(\varepsilon_T).$$

Since the model does not involve spectral splitting, the introduction of partial energies does not come naturally.

Chapter 19

Large Eddy Simulations

While the statistical approach remains a useful way in industrial practice or for environmental flows, it remains limited by the hypotheses on which modeling relies. Since the 1970s, the development of super-computers has allowed the advent of new simulation methods and in particular direct numerical simulations solving the full unsteady Navier-Stokes equations.

We can thus distinguish, in the prediction of turbulent flows, three levels of description:

– direct numerical simulation (DNS), solving all the turbulence scales using the Navier-Stokes equations;

– large eddy simulations (LES);

– statistical modeling (Chapters 6 to 18).

Before dealing with the large eddy simulations specifically, it is useful to present some methodological concepts for DNS of turbulence, because in many aspects they will also be relevant for the LES approach.

Direct numerical simulation (cf. section 1.2.1) has to solve all the turbulence scales from the large eddies down to the smallest Kolmogorov scales. They are based on a three-dimensional and unsteady solution of the Navier-Stokes equations. The size of the calculation domain must be sufficiently large compared to the integral turbulence scale and the step size of the mesh must be a fraction of the Kolmogorov scale. These requirements will limit the turbulence Reynolds number that can be reached in the simulation.

If the direct numerical simulations of turbulence are at the present time developed in relatively simple flows for fundamental studies on turbulence, they cannot be conceived with the aim of general numerical prediction of usual turbulent flows of practical interest, considering the huge amount of computer power that would be necessary. They allow us to generate very complete databases on the instantaneous turbulence field, and from this point of view they relate to the experimental approach, the measurement probe being replaced by the discretization grid point. The fundamental problem of turbulence remains, however, a problem of statistical nature, this emphasizing the importance of post-treatment of the databases generated by the simulation: correlations, spectra, identification of structures, Lagrangian tracers, etc. A review of DNS methods can be found in [MOI 98]. An important aspect is the identification of vortex structures produced by the simulation. The parameter $Q = \dfrac{1}{2}\left(\omega_{ij}\omega_{ij} - S_{ij}S_{ij}\right)$ introduced by Hunt J.C.R. allows us, through the study of Q-isosurfaces, to detect the regions in which the rotation balances the strain rate and that are possible envelopes of vortices. This criterion makes it possible to detect vortex structures more accurately than the pressure isosurfaces or the lines of curl ([DUB 00]).

DNS, even if they are limited in Reynolds number, may provide comparisons and test cases that are useful for statistical closures, in particular for testing hypotheses like the closures for the pressure correlations ([SPE 87C]) which are not directly within the reach of experiment.

These complete simulations in which all the eddying scales are resolved, are still limited by computer power, and thus they cannot be applied to practical industrial complex flows. From this point of view an intermediate approach is justified in which only the large eddies are resolved in the simulation whereas the fine grained turbulence is modeled. The LES can be viewed as a hybrid approach using partial simulation (large eddies) and partial modeling (small eddies). The idea is based on the observation that if the large eddies are in fact very different depending on the geometry of the flow, their structure varying strongly from one type of flow to another, on the contrary, the small eddies have a far more universal character. The method thus consists of calculating the three-dimensional and unsteady motion of the large eddies for particular realizations of the flow under consideration. The statistical properties are then obtained in a second step, as they had been done for signals obtained in laboratory measurements. It is then possible to get averages, in time, in space, in a plane or for realizations (by repeating the calculation with random initial conditions). The situation is almost like that of an experimentalist who would have at his disposal a hot wire at every mesh point!

Very roughly, modeling consists of performing a statistical treatment on the basic equations and then closing and solving numerically these equations whereas

the simulation uses the inverse approach: numerical solution of the basic equations and then statistical treatment of the databases that have been generated. The foundation of this type of approach lies in a simple idea: the big eddies produced by the mean flow are strongly dependent on it, they are anisotropic and they have a long lifetime, so they are difficult to model. On the contrary, the small eddies produced from inertial transfer have a more universal character and have a tendency to isotropy, their life duration is short and they are relatively more easily modeled. The numerical simulations at higher Reynolds numbers are thus based on a decomposition of the turbulent fluctuations which allows us to calculate the large energetic eddies and to model the smaller scales with more universal character.

Numerous authors have contributed to the development of these methods. After the pioneering work of Deardorff J.W., [DEA 70] inspired by the calculation methods in the atmospheric circulation (cf. also Redelsperger J.L. and Sommeria G., [RED 81]), the method has been extended to laboratory turbulent flows. We have thus observed, very roughly, two schools of thought: the team of Reynolds and Ferziger at Stanford University (California) whose works are pursued by Moin, Kim *et al.* (cf. Clark R.A. *et al.*, [CLA 79]; Ferziger J.H., [FER 77A]; Kim J. and Moin P., [KIM 79]; Moin P. and Kim J., [MOI 82]; Reynolds W.C., [REY 81]), and on the other hand the team of Schumann and collaborators in Germany (cf. Schumann U., [SCH 75]; Schumann U. and Patterson G.S., [SCH 78]). The approach developed in Stanford clearly distinguishes the pre-filtering of the equations from the method of numerical treatment. These two operations remains intimately connected to each other in the German approach.

We will note in fact that every finite difference numerical scheme induces implicit "filtering" of the calculated fluctuating quantities, but Kwak D., Reynolds W.C. and Ferziger J.H., [KWA 75] show that the simulation is improved when pre-filtering is carried out before discretization.

The method can of course be extended to the case of heat convection (cf., for example, Antonopoulos M. and Domis, [ANT 81], etc.).

In addition to actual LES methods, we have to mention the Very Large Eddy Simulations (VLES) in which the filtering only retains the largest big eddies. The inertial zone of the spectrum is then totally filtered and must be modeled. The theoretical justification is more difficult, but its practical interest is however undoubted for example in the approach of complex industrial flows with a particular difficulty.

The databases generated from the large eddy simulations, such as for the DNS, contain a large amount of information. This data can be treated statistically, thus yielding correlations and spectra, or be treated graphically, thus yielding

visualizations and graphical animations using colored tracers, useful for studying eddy interactions (cf. Lesieur, 1990). We again see that this approach closely resembles the experimental approach.

After the study of homogenous turbulence ([ROG 81A and B], [LEE 85]), one of the first turbulent wall flows to be studied through numerical simulation is the fully developed plane channel flow ([MOI 78], [MOI 82], [PIO 93], [MOS 99]), the flow over a backward facing step ([SIL 93], [LE 94], [LE 97]), the turbulent boundary layer ([SPA 88]). The LES methods are also progressively more and more used for studying complex flows in practical situations ([FUR 98]).

The application of the LES method to compressible flows can be performed on the basis of Favre filtering (cf. [PIO 99]), which is a direct transposition of the mass weighted Favre averaging.

19.1. The filters

The initial problem is what are the large eddies and how to define them? For this, most authors use a filtering approach applied to all the turbulent quantities (Léonard A., [LEO 74]).

19.1.1. *Definition*

Let ϕ be a turbulent quantity such as u_i or p for example, the filtered quantity is defined by:

$$\overline{\phi}(\vec{x},t) = \int_{\mathbb{R}^3} G(\vec{x},\vec{y};\varDelta)\phi(\vec{y},t)d\vec{y}, \tag{19.1}$$

G being the filter function at point $\vec{x}$.

The filter G exactly determines the part of the fluctuations which is included into the large eddies. The argument $\varDelta$ is a parameter which characterizes the filter width.

In the present chapter, we shall use the notation $\overline{(\phi)}$ for filtering and the notation $\langle\phi\rangle$ for statistical averaging.

The filter G must of course be normalized such that:

$$\int_{\mathbb{R}^3} G(\vec{x}, \vec{y}; \varDelta) d\vec{y} = 1, \quad \forall \vec{x}$$

In homogenous turbulence, a filtered quantity is not dependent on the position $\vec{x}$ considered and we can write:

$$\overline{\phi}(\vec{x}, t) = \int_{\mathbb{R}^3} G(\vec{x} - \vec{y}; \varDelta)\phi(\vec{y}, t) d\vec{y},$$

this is indeed a convolution product:

$$\overline{\phi} = G * \phi.$$

In spectral space, we thus deduce:

$$\hat{\overline{\phi}} = \hat{G}(\vec{\kappa}; \varDelta).\hat{\phi}.$$

19.1.2. *Properties of filters*

A turbulent quantity can always be decomposed into:

$$\phi = \overline{\phi} + \phi', \tag{19.2}$$

(instantaneous quantity = filtered part + small scale part)

$$\overline{\phi} = G * \phi \quad \text{and} \quad \phi' = \phi - G * \phi \quad \text{i.e.} \quad \phi' = (\delta - G) * \phi,$$

where δ is the Dirac distribution (generalized function).

The properties of a filter are however very different from those of traditional mean value. The filter does not satisfy the Reynolds rules and in particular:

$$\overline{\overline{\phi}} \neq \overline{\phi}, \quad \overline{\phi'} \neq 0, \quad \overline{\phi'} = \overline{\phi} - \overline{\overline{\phi}}.$$

This latter relation can be written more simply in spectral space:

$$\hat{G}(1 - \hat{G})\hat{\phi} = \hat{G}\hat{\phi} - \hat{G}\hat{G}\hat{\phi},$$

since

$$\hat{\hat{\phi}} = \widehat{G}\hat{\phi} \quad \text{and} \quad \widehat{\phi'} = (1 - \widehat{G})\hat{\phi}.$$

19.1.3. *Examples of filters*

1) The sharp cut-off filter

$$\widehat{G}(\vec{\kappa}) = \begin{cases} 0 & \text{if} \quad \kappa_i > \kappa_C \ \forall i \\ 1 & \text{otherwise} \end{cases}$$

Very simple in spectral space, this low pass filter has the drawback of leading to an oscillatory function in physical space:

$$G(\vec{\xi}) = \prod_{i=1}^{3} \frac{\sin \pi \xi_i / \Delta}{\pi \xi_i}, \quad \Delta = \pi / \kappa_C \quad \text{and} \quad \vec{\xi} = \vec{x} - \vec{y}.$$

2) The Top Hat filter

This is also called the Box filter, defined by:

$$G(\vec{\xi}) = \begin{cases} 1 / \Delta & \text{if} \quad |\vec{\xi}| < \Delta / 2 \\ 0 & \text{if} \quad |\vec{\xi}| \geq \Delta / 2 \end{cases}$$

The same drawback is now reported in spectral space, $\widehat{G}$ becoming oscillatory:

$$\widehat{G}(\vec{\kappa}) = \prod_{i=1}^{3} \frac{\sin \kappa_i \Delta / 2}{\kappa_i \Delta / 2}.$$

3) The Gaussian filter

This is defined by:

$$G(\vec{\xi}) = \left(\sqrt{\frac{\gamma}{\pi}} \frac{1}{\Delta} \right)^3 e^{-\gamma \frac{\xi^2}{\Delta^2}}, \quad \gamma = 6$$

$$\widehat{G}(\vec{\kappa}) = \exp\left(-\frac{\kappa^2 \varDelta^2}{4\gamma}\right).$$

This filter is smooth and progressive and thus will include a part (although very small) of the small eddies into the definition of resolved filtered scales. However, this filter can be used as well in physical space as in spectral space and it often gives better results.

The structure of strongly non-homogenous flows often requires us to use very different meshes in the three directions in space, thus bringing anisotropies in the three-dimensional mesh and consequently the characteristic length scale of subgrid-scale turbulence (often used in closure models) is more difficult to approximate.

19.1.4. *Filtering due to the discretization mesh*

Discretizing on a grid with a step size larger than the Kolmogorov scale will smooth out the higher order modes and consequently imply a filtering operation whose properties are generally not well known. We can consider that there are two types of approach according to whether the filtering operation is explicitly made (pre-filtering) or the numerical discretization grid also plays the role of the filter. In the first case, the explicit filter must have a width larger than the calculation step size and the closure model accounting for the non-resolved scales will be viewed as a subfilter-scale model (SFS). However, in the second case, the grid cell acts both as a numerical discretization cell and as a filter, the model is then properly a subgrid-scale model (SGS). In this latter case, the discretization and the filtering schemes are intimately connected. Some authors have also taken advantage of the numerical dissipation of the discretization scheme to substitute for the effect of the energy cascade and have thus carried on LES calculations "without a model". For example, an uncentered scheme of order 3 is often used ([BOR 92]) in the so-called "MILES (Monotone Integrated Large Eddy Simulation) method" in compressible flows and can give interesting practical results in the industrial domain.

19.1.5. *Differential filters*

The differential filters ([GER 86A and B]) are a natural extension of the convolution operator corresponding to filtering. In this way, the Gaussian filter can be considered as the integral kernel of the heat equation in the $\mathbb{R}^3$ space. In a bounded domain, the generalization of this concept leads us to solve an equation of the type:

$$\overline{u}_i - \Delta^2 \, \overline{u}_{i,jj} + \lambda_{,i} = u_i \quad \text{with} \quad \overline{u}_{i,i} = 0 \quad \text{in the domain } \Omega,$$

and appropriate boundary conditions on the border $\partial\Omega$, for example $\overline{u}_i = 0$ on a wall, λ can be viewed as a "pseudo-pressure". In the previous equation, u_i plays the role of a source term and the unknown is the filtered function $\overline{u}_i$. This method allows us to handle more easily filter widths $\Delta(\vec{x})$ that are variable.

19.2. The filtered Navier-Stokes equations

19.2.1. *In physical space*

For the Navier-Stokes equations applied to the instantaneous motion (the fluid flow is supposed incompressible)

$$\partial_t u_i + \left(u_i u_j \right)_{,j} = -\frac{1}{\rho} p_{,i} + \nu u_{i,jj} ,$$

$$u_{j,j} = 0 .$$

we operate filtering, with $u_i = \overline{u}_i + u'_i$ and $p = \overline{p} + p'$:

$$\partial_t \overline{u}_i + \left(\overline{u_i u_j} \right)_{,j} = -\frac{1}{\rho} \overline{p}_{,i} + \nu \overline{u}_{i,jj} ,$$

$$\overline{u}_{j,j} = 0 .$$

The filtered product of the instantaneous velocities can thus be written:

$$\overline{u_i u_j} = \overline{\left(\overline{u}_i + u'_i \right)\left(\overline{u}_j + u'_j \right)}$$
$$= \overline{u}_i \overline{u}_j + \underbrace{\left(\overline{\overline{u}_i \overline{u}_j} - \overline{u}_i \overline{u}_j \right)}_{L_{ij}} + \underbrace{\left(\overline{u'_i \overline{u}_j} + \overline{u'_j \overline{u}_i} + \overline{u'_i u'_j} \right)}_{R_{ij}} . \qquad [19.3]$$

where L_{ij} are the Leonard stresses and R_{ij} the subgrid-scale Reynolds stresses.

The Reynolds stresses include, in addition to the true subgrid-scale stress $\overline{u'_i u'_j}$, cross terms such as $\overline{u'_i \overline{u}_j}$. These terms can correspond either to energy

transfers from the simulated resolved scales to the modeled subgrid scales, this is a direct transfer, or to energy transfers from the subgrid scales towards the resolved large-scales, this is the inverse transfer or "backscatter".

The decomposition [19.3] remains almost formal, Speziale C.G., [SPE 85] having shown in particular that the Leonard term L_{ij} on the one hand and the cross term stresses $\left(\overline{u'_i \overline{u}_j} + \overline{u'_j \overline{u}_i}\right)$ on the other hand, considered separately, do not satisfy Galilean invariance, only their sum satisfies invariance. This observation could well pose problems for the subgrid-scale models in highly non-homogenous flows, but in fact it has only few consequences in usual practice (cf. also [VAS 98]).

Germano M., [GER 86] proposed another decomposition into three terms, each of them satisfying Galilean invariance:

$$\overline{u_i u_j} = \overline{u}_i \overline{u}_j + \underbrace{\left(\overline{\overline{u}_i \overline{u}_j} - \overline{\overline{u}}_i \overline{\overline{u}}_j\right)}_{L_{ij}} + \underbrace{\left(\overline{u'_i \overline{u}_j} + \overline{u'_j \overline{u}_i} - \overline{\overline{u'_i} \overline{u}_j} - \overline{\overline{u'_j} \overline{u}_i}\right)}_{C_{ij}} + \underbrace{\left(\overline{u'_i u'_j} - \overline{u'_i u'_j}\right)}_{R_{ij}}.$$

We can then derive the evolution equation for the filtered velocities:

$$v_i = \overline{u}_i, \quad \pi = \overline{p}/\rho,$$

$$\partial_t v_i + \left(v_i v_j\right)_{,j} = -\pi_{,i} - L_{ij,j} - R_{ij,j} + \nu v_{i,jj},$$

$$v_{j,j} = 0. \tag{19.4}$$

Note: the following notation $R_{ij} = \tau_{ij} + \dfrac{1}{3} R_{ll} \delta_{ij}$ is often used as is the modified pressure $\pi^* = \pi + \dfrac{1}{3} R_{ll}$.

Thus, the equation for v_i is written:

$$\partial_t v_i + \left(v_i v_j\right)_{,j} = -\pi^*_{,i} - L_{ij,j} - \tau_{ij,j} + \nu v_{i,jj}. \tag{19.5}$$

There is an alternative formulation:

$$\partial_t v_i + \left(\overline{v_i v_j}\right)_{,j} = -\pi *_{,i} - \tau_{ij,j} + v v_{i,jj} ,$$ [19.6]

in which the Leonard term is calculated explicitly.

The distinction between the two formulations is related to the implicit filtering concept [19.5] or to the explicit filtering concept [19.6]. In formulation [19.5], more commonly used, a part of filtering corresponding to the term $\left(v_i v_j\right)_{,j}$ (which, considered as a whole, is not necessarily a filtered quantity) is performed through the numerical algorithm.

19.2.2. *In spectral space*

The Fourier transform of the previous equation can be written in the homogenous case:

$$\partial_t \hat{v}_i + i\kappa_j \left(\hat{L}_{ij} + \hat{R}_{ij} + \widehat{v_i v_j}\right) = -i\kappa_i \hat{\pi} - v\kappa^2 \hat{v}_i ,$$

$$\kappa_j \hat{v}_j = 0 .$$

Thus, taking the divergence of the momentum equation, it is possible to express the pressure explicitly:

$$\hat{\pi} = -\frac{\kappa_i \kappa_j}{\kappa^2} \left(\hat{L}_{ij} + \hat{R}_{ij} + \widehat{v_i v_j}\right) .$$

Then, $\partial_t \hat{v}_i + i\kappa_j \Delta_{il} \left(\hat{L}_{lj} + \hat{R}_{lj} + \widehat{v_l v_j}\right) + v\kappa^2 \hat{v}_i = 0 ,$ [19.7]

with $\Delta_{il} = \delta_{il} - \dfrac{\kappa_i \kappa_l}{\kappa^2} .$

Let us consider in more detail the expression for the turbulent stresses:

$$\widehat{u_i u_j} = \hat{G}\left(\hat{u}_i * \hat{u}_j\right) = \widehat{u_i u_j} + \hat{L}_{ij} + \hat{R}_{ij} .$$ [19.8]

The terms involved in the decomposition in physical space and in Fourier space are:

Spectral space

$\widehat{G}\widehat{u}_i * \widehat{G}\widehat{u}_j,$

$(\widehat{G}-1)(\widehat{G}\widehat{u}_i * \widehat{G}\widehat{u}_j),$

$\widehat{G}\left[\widehat{G}\widehat{u}_i * (1-\widehat{G})\widehat{u}_j\right],$

$\widehat{G}\left[\widehat{G}\widehat{u}_j * (1-\widehat{G})\widehat{u}_i\right],$

$\widehat{G}\left[(1-\widehat{G})\widehat{u}_i * (1-\widehat{G})\widehat{u}_j\right],$

$\widehat{G}\left[\widehat{u}_i * \widehat{u}_j\right],$

Physical space

$\overline{\overline{u}_i \overline{u}_j}$ (resolved term)

L_{ij} (Leonard term)

$\overline{u'_j \overline{u}_i}$ (cross term)

$\overline{u'_i \overline{u}_j}$ (cross term)

$\overline{u'_i u'_j}$ (Reynolds stress term)

$\overline{u_i u_j}$ (non-linear term)

If we refer to the Navier-Stokes equation in Fourier space (cf. section 3.7) which clearly shows the triad interactions, we see that for a given $\vec{\kappa}$, the contributions to transfer come from $\vec{m}$ and $\vec{q}$ such that $\vec{\kappa} = \vec{m} + \vec{q}$. For a given value of the modulus κ, all the wavenumbers m and q for which there are three vectors $\vec{\kappa}$, $\vec{m}$ and $\vec{q}$ satisfying $\vec{\kappa} = \vec{m} + \vec{q}$, are located inside a semi-infinite "rectangle" (bounded by the straight lines: $\kappa = m + q$, $\kappa = m - q$, $\kappa = q - m$) and shown in Figure 19.1.
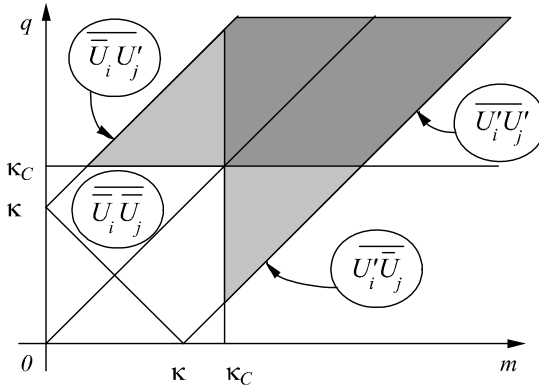
Figure 19.1. *Location of wavenumbers contributing to the subgrid-scale stresses*

In the particular case of the sharp cut-off filter, with a cut-off wavenumber κ_c, it is possible to distinguish clearly (cf. Aupoix B. and Cousteix J., [AUP 82]) the various contributions to the subgrid-scale fluxes (cf. Figure 19.1).

A connection can be made to equation [3.48], in the case where there is no mean velocity gradient, in order to single out the subgrid transfer term T_I :

$$\left(\frac{\partial}{\partial t}+\nu\kappa_j^2\right)\hat{u}_i = -i\kappa_j\Delta_{hi}\int_{\overrightarrow{m}<\kappa_c\,AND\overrightarrow{q}<\kappa_c}\hat{u}_h(\overrightarrow{m})\hat{u}_j(\overrightarrow{q})\delta(\overrightarrow{m}+\overrightarrow{q}-\overrightarrow{k})d\overrightarrow{m}d\overrightarrow{q}$$

$$\underbrace{-i\kappa_j\Delta_{hi}\int_{\overrightarrow{m}>\kappa_c\,OR\overrightarrow{q}>\kappa_c}\hat{u}_h(\overrightarrow{m})\hat{u}_j(\overrightarrow{q})\delta(\overrightarrow{m}+\overrightarrow{q}-\overrightarrow{k})d\overrightarrow{m}d\overrightarrow{q}}_{T_I}\,.$$

19.2.3. *Note about the Leonard term*

The first authors who worked in physical space proposed approximations for the Leonard term which were based on Taylor expansions. It is possible, indeed, by using a Taylor expansion about point $\overline{x}$ to get:

$$\overline{\overline{u_i}\overline{u}_j}(\overrightarrow{x},t) = \int G(\overrightarrow{x}-\overrightarrow{y})\overline{u}_i(\overrightarrow{y})\overline{u}_j(\overrightarrow{y})d\overrightarrow{y}$$

$$= \int\left[\overline{u}_i(\overrightarrow{x},t)+\xi_k\overline{u}_{i,k}+\frac{1}{2}\xi_k\xi_l\overline{u}_{i,kl}+\mathcal{O}(\xi^3)\right]\times$$

$$\left[\overline{u}_i(\overrightarrow{x},t)+\xi_k\overline{u}_{i,k}+\frac{1}{2}\xi_k\xi_l\overline{u}_{i,kl}+\mathcal{O}(\xi^3)\right]G(\overrightarrow{\xi})d\overrightarrow{\xi}\,, \qquad [19.9]$$

with $\overrightarrow{\xi}=\overrightarrow{x}-\overrightarrow{y}$,

and then after grouping the term of the product under the integrand:

$$\overline{\overline{u_i}\overline{u}_j} = \overline{u}_i\overline{u}_j+\left(\overline{u}_i\overline{u}_j\right)_{,kl}\int\xi_k\xi_l G(\overrightarrow{\xi})d\overrightarrow{\xi}\,.$$

For a Gaussian filter, we find:

$$\int\xi_k\xi_l G(\overrightarrow{\xi})d\overrightarrow{\xi} = \frac{\Delta^2}{24}\delta_{kl}\,,$$

which leads to the approximation $L_{ij} = \dfrac{\Delta^2}{24}\left(\overline{u}_i\overline{u}_j\right)_{,ll}+...$

However, considering that the Leonard term only involves resolved quantities, it is not impossible to make an exact calculation of this term. This can be done easily in spectral space, since:

$$\hat{L}_{ij} = \left(\hat{G} - 1\right)\widehat{\widehat{u_i u_j}} \ .$$

19.3. Subgrid-scale modeling

19.3.1. *Subgrid-scale models in physical space*

The simplest SGS models are based on a subgrid-scale turbulence eddy viscosity:

$$\tau_{ij} = -2\nu_t S_{ij} \,, \qquad\qquad S_{ij} = \frac{1}{2}\left(\nu_{i,j} + \nu_{j,i}\right),$$

ν_t will be a function of $\varDelta$ (filter width) and of the invariants of S_{ij} (strain rate tensor for the filtered field).

The well known Smagorinsky hypothesis, inspired by the mixing length hypothesis, can be written:

$$\nu_t = \left(C_s \varDelta\right)^2 \sqrt{2 S_{ij} S_{ij}} \ . \qquad\qquad\qquad [19.10]$$

The characteristic length scale of subgrid scale turbulence is indeed known and directly proportional to the filter width. Coefficient C_s is the Smagorinsky constant.

However, like in the mixing length model, the Smagorinsky constant C_s is generally not a true constant, but rather a function of filtering and of the type of flow. Moreover, near a wall in the viscous sublayer, the constant C_s must be decreased, and for this a Van Driest type coefficient can be used (cf. Chapter 13).

In case the turbulence Reynolds number becomes low, C_s can no longer be considered as a constant. Direct simulations carried out in a simple geometry (homogenous turbulence in a cube) and used for testing the subgrid-scale closures have shown (MacMillan O.J. and Ferziger J.H., [MAC 79]) that it is possible to use approximations of the type:

$$C_s = 0.128 \left(1 + \frac{24.5}{\mathrm{Re}_{sgs}}\right)^{-1}, \qquad\qquad [19.11]$$

where Re_{sgs} is the subgrid-scale Reynolds number based on Δ.

A variant of the Smagorinsky model has been proposed by Lilly D.K., [LIL 71]. This does not make use of the instantaneous filtered strain rate but its mean value:

$$\nu_t = (C_L \Delta)^2 \left\langle 2 S_{ij} S_{ij} \right\rangle^{1/2}. \qquad\qquad [19.12]$$

All these simple schemes assume that the subgrid-scale turbulence is in local equilibrium with the large-scale field and τ_{ij} is then determined by the big eddies. The energy flux brought by the eddies of resolved scales thus exactly balances the dissipation rate which occurs in the subgrid-scale range of turbulence. Indeed, the equilibrium energy balance in the subgrid-scale region is:

$$\nu_t S_{ij} S_{ij} = c \frac{k^{3/2}}{\Delta},$$

and using $\nu_t = c_1 \Delta \sqrt{k}$ it follows that:

$$k = \frac{c_1}{c} \Delta^2 S_{ij} S_{ij} \qquad \text{and} \qquad \nu_t = c_1 \left(\frac{c_1}{c}\right)^{1/2} \Delta^2 \sqrt{S_{ij} S_{ij}}.$$

It can be noted that in principle, the modeled approximation of τ_{ij} should be explicitly filtered, in order that all the terms in the filtered Navier-Stokes equations be effectively filtered, but this requirement is often omitted in practice.

19.3.2. Subgrid-scale models in spectral space

In homogenous turbulence, a relation of the type $\tau_{ij} = -2\nu_t S_{ij}$ becomes in Fourier space, with ν_t considered as constant:

$$\hat{\tau}_{ij} = -\nu_t i \left(\kappa_i \hat{v}_j + \kappa_j \hat{v}_i\right).$$

If we note that $i\kappa_j \Delta_{il} \hat{\tau}_{lj} = \nu_t \kappa^2 \hat{v}_i$, the filtered velocity equation can also be written (omitting the Leonard term):

$$\partial_t \hat{v}_i + i\kappa_j \Delta_{il} \left(\widehat{v_l v_j} \right) + \left(\nu + \nu_t \right) \kappa^2 \hat{v}_i = 0 \, .$$

The spectral eddy viscosity concept is based on the notion of non-local interactions (Lesieur M. and Schertzer D., [LES 78C]).

We first mention the model by Chollet J.P. and Lesieur M., [CHO 81] who, relying on analytical theories, propose:

$$\nu_t = B \left(\frac{E(\kappa_C)}{\kappa_C} \right)^{1/2} . \qquad\qquad [19.13]$$

This model involves the value of the mean energy spectrum at the cut-off κ_C. It is important to remark that, in this model, if $E(\kappa_C)$ vanishes, then the subgrid-scale viscosity also vanishes. Thus, in a transitional flow, as far as the energy has not reached to cut-off region, the viscosity will remain zero, and this is in accordance with the physics. In contrast, the Smagorinsky model in physical space does not retain this advantage, the subgrid-scale viscosity remaining non-zero even in the absence of turbulence energy, and consequently making impossible the calculation of transitional flows.

In spectral space, it is possible to make a more refined study of turbulence viscosity depending on the wavenumber. Indeed, the spectral theories suggest (Leslie D.C. and Quarini G.L., [LES 78B]) that the subgrid-scale interactions can be represented as the sum of two terms:

– a flux proportional to the local excitation (characterized by a viscosity ν_d);

– an inverse flux of energy from the small scales towards the large-scales (that can be represented formally by a "viscosity" ν_b of the inverse flux).

These authors show that ν_d reaches a constant value for small values of κ such as: $\nu_d(0) \to 0.292 \varepsilon^{1/3} \kappa_C^{-4/3}$ and that $\nu_d(\kappa)$ increases with κ until it undergoes a singularity in $(\kappa_C - \kappa)^{-2/3}$ at the cut-off. Coefficient ν_b is zero at the origin, also increases with κ and undergoes a singularity at the cut-off which exactly balances

the singularity of v_d. The apparent viscosity $v_n = v_d - v_b$ thus reaches a finite value at the cut-off:

$$v_n(\kappa_C) = 5.24 v_d(0).$$

This property can be connected to the fact that most of the interactions between large scales and small scales are concentrated near the cut-off (local interactions).

On this basis, several authors have devised, in spectral space, eddy viscosity hypotheses that are functions of the wavenumber, such as:

$$v_t(\kappa, \kappa_C) = v_e F(\kappa / \kappa_C) \quad \text{with} \quad v_e = B\left(\frac{E(\kappa_C)}{\kappa_C}\right)^{1/2}.$$

Let us mention the model by Basdevant C., Lesieur M. and Sadourny R., [BAS 78] (cf. also Dang K., [DAN 83]):

$$F = 1 + \lambda\left(\frac{\kappa}{\kappa_C}\right)^{2(\alpha-1)}, \qquad [19.14]$$

which produces a behavior which is in accordance with the previous description (cf. Figure 19.2).
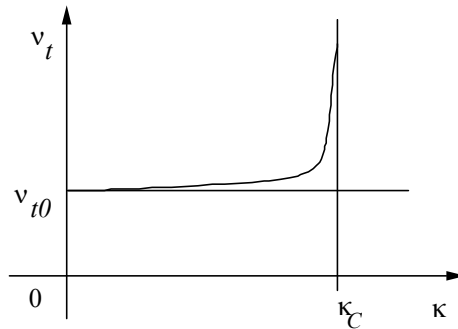
Figure 19.2. *Variation in the subgrid-scale viscosity as a function of the wavenumber*

Another formulation of the spectral dependency (Lamballais E. *et al.*, [LAM 97]) can be obtained from an exponential:

$$F = 1 + 34.5\exp\left[-3.03(\kappa_C/\kappa)\right], \quad \nu_e = 0.267F\left(\frac{\kappa}{\kappa_C}\right)\left(\frac{E(\kappa_C)}{\kappa_C}\right)^{1/2}.$$

The variations of ν_t according to κ suggest that the use of a Smagorinsky model reproducing the constant region ν_{t0} complemented by a hyperviscosity term (term involving a velocity derivative of even order greater than two and usually equal to four) may represent the reality almost correctly (cf. section 19.9).

In the case of an energy spectrum satisfying a κ^{-m} law with $m \neq 5/3$, Métais O. and Lesieur M., [MET 92] proposed the following adaptation:

$$\nu_t = \frac{1}{15a}\left(\frac{5-m}{1+m}\right)\sqrt{3-m}\left(\frac{E(\kappa_C)}{\kappa_C}\right)^{1/2}.$$

In this expression, the constant a is deduced from the EDQNM theory. This adaptation (Lamballais *et al.*, [LAM 97]) allows a dynamic behavior of the model in particular in transition regions and near the walls where the energy spectrum departs from the −5/3 law (cf. also [MET 92], [SIL 98], [LAM 97B]).

When ν_t is not a constant, the equation of motion is written:

$$\partial_t \hat{v}_i + i\kappa_j \Delta_{il}\left(\widehat{v_l v_j} - \widehat{\nu_t S_{lj}}\right) = 0$$

19.3.3. *Transport equation models*

1) Kinetic energy equation

We introduce the decomposition: $u_i = U_i + v_i + w_i$ and $p = P + \pi + \chi$,

$$U_i = \langle u_i \rangle, \quad v_i = \overline{u}_i - U_i, \quad w_i = u'_i,$$
$$P = \langle p \rangle, \quad \pi = \overline{p} - P, \quad \chi = p',$$

$$\left\langle \frac{u_i u_i}{2} \right\rangle = \frac{U_i U_i}{2} + \left\langle \frac{v_i v_i}{2} \right\rangle + \left\langle v_i w_i \right\rangle + \left\langle \frac{w_i w_i}{2} \right\rangle.$$

If the correlations between resolved and non-resolved components are neglected (these correlations are exactly zero in the particular case of a sharp filter), we see that the kinetic energy of turbulence is the sum of the kinetic energy of the large eddies and that of the small eddies.

$$k \approx \left\langle \frac{v_i v_i}{2} \right\rangle + \left\langle \frac{w_i w_i}{2} \right\rangle.$$

In a general way, we shall assume: $\left\langle v_i w_i \right\rangle$ negligible, $\left\langle u_i w_i \right\rangle \approx \left\langle w_i w_i \right\rangle$, etc.

The equations for v_i and w_i then read (with $R_{ij} = \overline{w_i w_j}$):

$$\partial_t v_i + U_j v_{i,j} = -U_{i,j} v_j - \left(v_i v_j \right)_{,j} + \left\langle v_i v_j \right\rangle_{,j} - \pi_{,i} - R_{ij,j} + \left\langle R_{ij} \right\rangle_{,j} + \nu v_{i,jj} \ ,$$

$$\partial_t w_i + U_j w_{i,j} = -U_{i,j} w_j - \left(w_i w_j \right)_{,j} - \chi_{,i} - \left(v_i w_j + v_j w_i \right)_{,j} + R_{ij,j} + \nu w_{i,jj} \ .$$

$$[19.15]$$

From these equations, we can deduce:

$$\partial_t \left\langle \frac{v_i v_i}{2} \right\rangle + U_j \left\langle \frac{v_i v_i}{2} \right\rangle_{,j} = -\left\langle v_i v_j \right\rangle U_{i,j} + \left\langle R_{ij} v_{i,j} \right\rangle + D^v_{j,j} - \nu \left\langle v_{i,j} v_{i,j} \right\rangle,$$

$$\partial_t \left\langle \frac{w_i w_i}{2} \right\rangle + U_j \left\langle \frac{w_i w_i}{2} \right\rangle_{,j} = -\left\langle w_i w_j \right\rangle U_{i,j} - \left\langle R_{ij} v_{i,j} \right\rangle + D^w_{j,j} - \nu \left\langle w_{i,j} w_{i,j} \right\rangle.$$

$$[19.16]$$

The notations $D^v_{j,j}$ and $D^w_{j,j}$ include altogether the turbulent diffusion terms (due to fluctuating velocities and fluctuating pressure) and the molecular diffusion.

The term $\left\langle R_{ij} v_{i,j} \right\rangle$ represents the energy flux through the cut-off. Adding corresponding members of these equations yields again the usual equation for

turbulence kinetic energy k. The last equation [19.16] can be related to equation [18.30] for $m = 2$, obtained from integration of equation [18.27].

2) Closures

Horiuti K. and Yoshizawa A., [HOR 85] have modeled the equation for $k_{sgs} = \dfrac{\overline{w_i w_i}}{2}$ introducing a closure of the following type:

$$\partial_t k_{sgs} + \overline{u}_j k_{sgs,j} = -\overline{w_i w_j}\,\overline{u}_{i,j} + Diff(k_{sgs}) - \nu\overline{w_{i,j} w_{i,j}} \,. \tag{19.17}$$

These authors are using hypotheses that are inspired by traditional one point closure models:

$$\overline{w_i w_j} = \frac{2}{3} k_{sgs}\delta_{ij} - \sigma\left(\overline{u}_{i,j} + \overline{u}_{j,i}\right),$$

$$\partial_t k_{sgs} + \overline{u}_j k_{sgs,j} = 2\sigma S_{ij} S_{ij} + \left[\left(\nu + c\Delta k_{sgs}^{1/2}\right) k_{sgs,j}\right]_{,j} - c\,\frac{k_{sgs}^{3/2}}{\Delta},$$

$$S_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} + \overline{u}_{j,i}\right), \qquad \sigma = c_\nu \Delta k_{sgs}^{1/2}, \qquad \Delta = \left(\Delta_x \Delta_y \Delta_z\right)^{1/3}.$$

The characteristic length scale being obtained from the filter width, it becomes useless in the present case to keep a modeled equation for the dissipation rate (it could become necessary, however, if the location of the cut-off was allowed to take any varying value).

In this type of modeling approach, the subgrid-scale viscosity is thus directly connected to the subgrid-scale kinetic energy k_{sgs} that creates this viscosity (cf. Aupoix B. and Cousteix J., [AUP 82]). The study of this type of one equation transport closure has been pursued in Dejoan A. *et al.* [DEJ 02].

Second order subgrid-scale transport models have been initially developed in the field of atmospheric turbulence (Deardorff J.W., [DEA 73]).

3) Spectral space

The energy spectrum equation in homogenous turbulence reads:

$$\frac{\partial E(\kappa)}{\partial t} + 2\nu\kappa^2 E(\kappa) = T(\kappa)\,, \tag{19.18}$$

$$T(\kappa) = T_E(\kappa) + T_I(\kappa) .$$

The term $T_E(\kappa)$ contains resolved scale contributions whereas $T_I(\kappa)$ denotes the transfer to the subgrid modeled scales (cf. Figure 19.3).
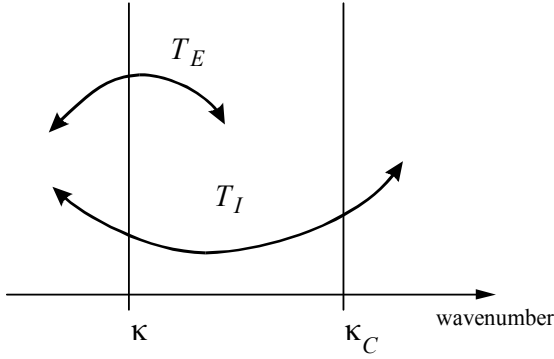
Figure 19.3. *Resolved and unresolved transfer terms in the turbulence energy spectrum*

The following relations also hold:

$$\left\langle \frac{v_i v_i}{2} \right\rangle = \int_0^{\kappa_c} E(\kappa)d\kappa ,$$

$$\left\langle R_{ij} v_{i,j} \right\rangle = \int_0^{\kappa_c} T_I(\kappa)d\kappa ,$$

$$\int_0^{\kappa_c} T_E(\kappa)d\kappa = 0 , \quad \int_0^{\infty} T(\kappa)d\kappa = 0 ,$$

$$\nu \left\langle \overline{u}_{i,j} \overline{u}_{i,j} \right\rangle = 2\nu \int_0^{\infty} \kappa^2 E(\kappa)d\kappa = \varepsilon .$$

In the case of a subgrid-scale eddy viscosity model:

$$\frac{\partial E(\kappa)}{\partial t} + 2\left(\nu + \nu_{sgs}\right)\kappa^2 E(\kappa) = T_E(\kappa) ,$$

$$T_I(\kappa) = 2\nu_{sgs}\kappa^2 E(\kappa) \quad \text{and} \quad \nu \left\langle \overline{v}_{i,j} \overline{v}_{i,j} \right\rangle = 2\nu_{sgs} \int_0^{\kappa_c} \kappa^2 E(\kappa)d\kappa .$$

This relation allows us to determine the constant B in the model by Chollet and Lesieur [19.13]. If it is supposed that the spectral cut-off is located in the inertial zone, implying $2\nu_{sgs}\int_0^{\kappa_C}\kappa^2 E(\kappa)d\kappa \approx \varepsilon$ and that the energy spectrum is the Kolmogorov spectrum $E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3}$, we find $B = \dfrac{2}{3}C^{-3/2}$.

19.4. Some notes on numerical methods

The question is to solve the filtered Navier-Stokes equations that are three-dimensional and unsteady. It is essential in this aim, in refined LES methods, to make use of highly accurate numerical methods that are free from numerical viscosity and numerical dissipation and satisfy at the level of the discretized equations good conservation properties (for energy, momentum in the $\nu = 0$ limit).

It is one of the favored domains of application of spectral methods (Orszag S.A., [ORS 70, 71]) and pseudo-spectral methods (Fourier expansions in the homogenous directions with periodic boundary conditions, Tchebycheff polynomial expansions in the non-homogenous directions bounded by walls). More traditional finite difference methods can also be used, using centered schemes. The time integration schemes have to be chosen according to their accuracy and their behavior on long integration times. Numerous authors have used the Adams-Bashforth scheme but there is a large variety of approaches (fractional step schemes). Mixed methods combining spectral and finite difference techniques are often used depending on the type of flow geometry to be considered (if containing both homogenous directions and non-homogenous directions).

In the case of homogenous turbulence, the choice of initial conditions is important since it determines the subsequent evolution of the turbulent field. It is possible to generate a pseudo-random velocity field satisfying a given energy spectrum (cf. section 19.5). In the case of a fully developed turbulent channel flow, the turbulence field tends to forget progressively the initial conditions and it will thus suffice to impose an initial disturbance and to carry on the calculation over a sufficiently long time, using periodic boundary conditions in the longitudinal direction. The eddies that are convected out of the domain are thus automatically transferred at the entry.

In the case of a domain with entry and exit cross-sections, the boundary conditions at inlet can be generated using a separate additional numerical simulation (for example, a channel flow with periodic boundary conditions), which is the best method since it provides a realistic fluctuating field at inlet but it is more costly in

calculation time (cf. for example [LI 00]). In the case of a boundary layer, it is possible to impose at inlet the fluctuating field obtained at the exit, provided an adequate change in scaling is made before transferring it (cf. for example [LUN 98]). It is also possible to use random generators, for example by imposing a random noise whose phases are adjusted in such a way that the correct values of the main statistical moments are recovered. This method remains however more artificial since it does not account for the structural properties ([LEE 92], [LE 94], [LE 97], [AKS 95]) and a very long distance is necessary for the statistical properties to reach equilibrium. To overcome this drawback, the method proposed in [WEN 03] allows us to synthesize an artificial "turbulence" from a three-dimensional random field. Linear transforms are then used to impose given values for the Reynolds stresses and the structure tensors. This technique has the advantage of preserving the structural properties of turbulence in space.

The exit boundary conditions must allow the turbulent eddies passing through the boundary to get out of the domain without appreciable deformation. The most common technique consists of assuming convective conditions on the fluctuating part of the velocity field. Each velocity component then satisfies the time dependent condition [LE 94]:

$$\frac{\partial u_i}{\partial t} + U_c \frac{\partial u_i}{\partial x} = 0 \,,$$

where U_c is the convection velocity which can be approximated by the statistical average of the fluid velocity at exit.

19.5. Simulation of homogenous flows

The simplest case is the one of decay of homogenous turbulence. The calculation can be carried in a cubic domain using for example spectral methods. The boundary conditions are corresponding to periodicity conditions.

We can work directly in Fourier space by solving:

$$\partial_t \hat{u}_i + \tilde{v}_e \kappa^2 \hat{u}_i = i\kappa_j \Delta_{im} \left(\widehat{v'_e u_{m,j}} + \widehat{v'_e u_{j,m}} - \widehat{u_m u_j} \right) , \qquad [19.19]$$

with $v_e = \tilde{v}_e + v'_e$

$$\frac{\hat{p}}{\rho} = \frac{\kappa_i \kappa_j}{\kappa^2} \left(\widehat{v'_e u_{i,j}} + \widehat{v'_e u_{j,i}} - \widehat{u_i u_j} \right).$$

If the Leonard term is also taken into account, we have to solve:

$$\partial_t \hat{u}_i + \tilde{v}_e \kappa^2 \hat{u}_i = i\kappa_j \mathit{\Delta}_{im} \left(\widehat{v'_e u_{m,j}} + \widehat{v'_e u_{j,m}} - \widehat{G u_m u_j} \right). \qquad [19.20]$$

The initial conditions are produced by a random velocity field with zero divergence. In practice, such an initial field can be obtained in the following way [ROY80], reasoning on the vector stream function:

– we draw lots of two real fields uniformly distributed on the sphere of radius unity $\widehat{\Psi}_{iR}$ and $\widehat{\Psi}_{iI}$ for $\kappa_1 \geq 0$;

– the vector stream function $\widehat{\Psi}_i$ is obtained from $\widehat{\Psi}_i = a\widehat{\Psi}_{iR} + ib\widehat{\Psi}_{iI}$, a and b being chosen in such a way that the velocity field $\hat{u}_i = i\kappa_j \varepsilon_{ijk} \widehat{\Psi}_k$ satisfies a given three-dimensional energy spectrum $E(\kappa)$.

For studying the uniform distortion of homogenous turbulence submitted to a mean velocity field $\bar{u}_i = \mathit{\Lambda}_{ij}(t)X_j$, Rogallo R.S., [ROG 81A and B] uses the coordinate transformation:

$$(X_i, T) \mapsto (\xi_i, t) \text{) such that } T = t \text{ and } X_i = F_{ij}\xi_j$$

with: $F_{ij} = \dfrac{\partial X_i}{\partial \xi_j}$, $\dot{F}_{ij} = \mathit{\Lambda}_{il} F_{lj}$ starting from $F_{ij}(t=0) = \delta_{ij}$ (Lagrangian description).

In this way, the mesh becomes distorted by the mean flow field and it is still possible to assume periodic boundary conditions. For example, in the particular case of homogenous shear, we will get the evolution sketched in Figure 19.4.

$t=0$ $t>0$
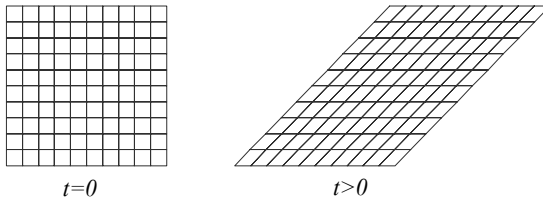
Figure 19.4. *Distorted mesh in homogenous turbulence submitted to constant mean shear*

After a certain period of time it is necessary to transfer the numerical solution onto a rectangular mesh.

Direct simulations of homogenous flows can be used as "numerical experiments" which allow fundamental studies on the structural properties of the turbulent field, coming to complement the data obtained from measurements (Lee M.J. and Reynolds W.C., [LEE 85]), and particularly useful for testing closure hypotheses in LES models or statistical models (for example [SAR 97]).

19.6. Simulation of non-homogenous turbulent wall flows

One of the first type of wall flows to be studied was the fully developed channel flow between two parallel plates (Deardorff J.W., [DEA 70], Moin P., Reynolds W.C. and Ferziger J.H., [MOI 78], Moin P. and Kim J., [MOI 82]).

The simulation of turbulence near a wall requires a particular treatment. From the numerical point of view, the mesh used by the previous authors is refined near the wall according to a hyperbolic tangent law (Oz direction) and a uniform mesh is kept in the directions Ox and Oy.

The space discretization is hybrid: finite differences in the direction Oz (non-homogenous) and spectral (Fourier expansions) in the directions Ox and Oy (homogenous). The time integration is based on a Crank-Nicolson scheme for the viscous terms and an Adams-Bashforth scheme for all the other terms. The Navier-Stokes equations are written and discretized in skew-symmetric form, which ensures good conservation properties whatever the discretization scheme.

From the modeling point of view, the closure hypotheses are based on the Smagorinsky model modified near the wall:

$$\sigma = \begin{cases} c_2 \dfrac{\ell^4}{\nu} \left(2 S_{ij} S_{ij} \right) & z \leq z_C \\[2mm] \left(c_s \ell \right)^2 \left(2 S_{ij} S_{ij} \right)^{1/2} & z > z_C \end{cases}$$

with $\tau_{ij} = -2\sigma S_{ij}$, $S_{ij} = \dfrac{1}{2} \left(\overline{u}_{i,j} + \overline{u}_{j,i} \right).$

The modification for $z \le z_C$ is similar to a Van Driest type correction factor at low Reynolds numbers. z_C corresponds to the distance from the wall for which the mean value of $< \sigma >$ in the plane (x,y) is the same for the two models.

The characteristic length ℓ is defined by:

$$\ell = \left[\min\left(\varDelta_1, \ell'\right) \times \min\left(\varDelta_2, \ell'\right) \times \min\left(h_3, \ell'\right) \right]^{1/3},$$

$$\ell' = \begin{cases} 0.1\delta & z > 0.1\delta / K \\ Kz & z \le 0.1\delta / K \end{cases}$$

In the expression for ℓ, $\varDelta_1$ and $\varDelta_2$ are the widths of the Gaussian filters in the directions x and y. In the direction Oz no explicit filtering is made, we can say that the finite difference scheme performs an implicit filtering determined by h_3 the space step along Oz (with variable step size).

When $\varDelta \to \infty$ the characteristic length scale of turbulence ℓ goes to an upper limit bound $\ell = Kz$ (the mixing length) which corresponds to an overall modeling of the whole turbulence spectrum.

Moin P. and Kim J., [MOI 82] use an improved version (inspired by the work of Schumann U., [SCH 75]) which involves two viscosity coefficients:

$$\tau_{ij} = -2\nu_t \left(S_{ij} - \left\langle S_{ij} \right\rangle \right) - 2\nu_t^* \left\langle S_{ij} \right\rangle.$$

In this expression:

$$\nu_t = \left(c_S \varDelta\right)^2 \left[2\left(S_{ij} - \left\langle S_{ij} \right\rangle\right)^2 \right]^{1/2}, \quad \text{with} \quad \varDelta = \sqrt[3]{\varDelta_1 \varDelta_2 \varDelta_3} \left[1 - \exp\left(-\frac{z^+}{A^+}\right) \right]$$

stands for the subgrid-scale eddy viscosity similar to the one used in homogenous turbulence and:

$$\nu_t^* = c\left(D\varDelta_2\right)^2 \left[2\left\langle S_{ij} \right\rangle^2 \right]^{1/2}, \quad \text{with} \quad D = 1 - \exp\left(-\frac{z^{+2}}{A^{+2}}\right)$$

represents a non-homogenous contribution which accounts for the "streaks", longitudinal eddies near the wall which cannot be entirely resolved if the transverse mesh is not sufficiently refined (cf. Kline *et al.*, [KLI 67]).

We can mention that more complex geometries, with the aim of approximating industrial flow situations which have been considered (for example by Baron F., [BAR 82]).

More accurate methods are used at present (Tchebycheff polynomial expansions in the non-homogenous directions) and extended to the boundary layers.

Piomelli U. *et al.*, [PIO 88] show that the filter and the subgrid-scale model must be chosen in order to achieve the best compatibility and for this they have to be based on the same characteristic scales. Thus, the Gaussian filter which embodies a wide range of scales is better suited to the mixed model by Bardina (cf. section 19.9).

The detailed full calculation of the viscous region near the wall is always very costly in computer time because of the very refined mesh near the wall. For this reason, in more practical applications, less detailed and more economical techniques are very useful. The simulation is then performed without solving the whole flow field.

The wall boundary conditions, when the mesh is not sufficiently refined for describing the viscous sublayer region, fall into two main types ([PIO 02]). The first approach is based on the equilibrium laws. For example Schumann U. [SCH 75] assumes a connection between the wall stresses and the flow velocity:

$$\tau^{(w)}{}_{12} = \frac{\left\langle \tau^{(w)} \right\rangle}{\left\langle \bar{u}(y) \right\rangle} \bar{u}(y), \quad \tau^{(w)}{}_{13} = \nu \frac{\overline{w}(y)}{y},$$

direction 2 being the y direction normal to the wall. The mean stress at the wall $\left\langle \tau^{(w)} \right\rangle$ can be obtained by assuming that the logarithmic law is verified by the mean velocity $\left\langle \bar{u}(y) \right\rangle$ at a distance y from the wall. The stress $\tau^{(w)}{}_{12}$ thus fluctuates about this value, whereas in $\tau^{(w)}{}_{13}$ a linear profile is assumed for $\overline{w}(y)$.

In order to link the wall friction to the velocity in the flow field, Werner H. and Wengle H., [WER 89,91] assume that the velocity in the wall layer is proportional to the instantaneous wall friction. The coefficient of proportionality is deduced from a

power law in 1/7 for the assumed mean velocity profile. The instantaneous velocity components parallel to the wall are supposed to be well in phase with the skin friction at the wall and the instantaneous velocity profile is assumed to be given by:

$$
\begin{cases}
u^+(y) = y^+ & \text{if} \quad y^+ < 11.8 \\
u^+(y) = 8.3\left(y^+\right)^{1/7} & \text{if} \quad y^+ \geq 11.8
\end{cases}
$$

The values of the velocity components are then linked to the values of the wall friction by performing the integration of the given velocity profile across the interval going from the wall to the first discretization cell.

Piomelli U. *et al.*, [PIO 89] have also proposed improvements in these wall relations, by introducing a shift between the wall friction and the instantaneous velocity at the first discretization point near the wall, this hypothesis being suggested by the phenomenology of wall turbulence. Piomelli U. [PIO 02] accounts in this way for the tilt in the elongated structures, by introducing the shift:

$$
\tau^{(w)}{}_{12} = \frac{\left\langle \tau^{(w)} \right\rangle}{\left\langle \overline{u(x,y)} \right\rangle} \overline{u}(x+\delta x, y), \quad \tau^{(w)}{}_{23} = \frac{\left\langle \tau^{(w)} \right\rangle}{\left\langle \overline{u(x,y)} \right\rangle} \overline{w}(x+\delta x, y),
$$

where δx is of order $\delta x = y.\mathrm{cotg}\,(8°)$ to $\delta x = y.\mathrm{cotg}\,(13°)$ depending on the distance from the wall. Another suggestion by the same authors ([PIO 89]) consists of linking the wall friction to the normal velocities near the wall, emphasizing in this way the importance of vertical motions in the production process.

The second group of methods makes use of a zonal approach. A first type of method uses two wall layers with two different calculation meshes ([CAB 00]) and a second type of method uses a single calculation grid and thus can be related to zonal hybrid methods RANS/LES (cf. section 19.14.1) and to the DES by Spalart (cf. section 19.14.2).

The use of refined grids near the wall generally leads to discretization cells that are very flat, and consequently modeling errors follow due to the anisotropy in the mesh. They are often neglected but there are approximate methods to correct these errors ([SCO 93, SCO 97]).

One of the main limitations of the usual viscosity hypothesis is its local character. The extension to non-local formulations ([HAM 05]) may allow a better account of the properties of momentum transfer.

19.7. Estimate of subgrid-scale energy

A large eddy numerical simulation using a Smagorinsky subgrid-scale model will provide the resolved components of the fluctuating field and therefore only part of the turbulence energy. Various approximate approaches exist for estimating the global mean turbulence quantities.

Let us consider for example the case of decay of homogenous turbulence. For the total turbulent field, we can write:

$$\frac{\partial k}{\partial t} = -\varepsilon , \quad \varepsilon \approx \frac{k^{3/2}}{L} .$$

For the filtered field:

$$\frac{\partial k_f}{\partial t} = -\varepsilon_f , \quad \varepsilon_f \approx \frac{k_f^{3/2}}{L_f}$$

where ε_f stands for the energy flux from the large resolved eddies to the small modeled eddies. Since the filtered field contains the same large eddies as the whole field, we can suppose that $L \approx L_f$. Consequently,

$$\varepsilon \approx \varepsilon_f \left(\frac{k}{k_f} \right)^{3/2} .$$

For the subgrid-scale field $\dfrac{\partial k_{sgs}}{\partial t} = -(\varepsilon - \varepsilon_f)$.

If we assume that viscous dissipation occurs mainly in the subgrid-scale region, Δ being the characteristic scale:

$$\varepsilon \approx \frac{k_{sgs}^{3/2}}{\Delta} .$$

Thus, $k_{sgs} \approx \left(\Delta \varepsilon_f \right)^{2/3} \dfrac{k}{k_f}$.

Bardina, Ferziger and Reynolds ([BAR 83] and Gatski, Hussaini, Lumley, 1996, p. 121) are thus led to write:

$$k_{sgs} \approx c_f \left(2\Delta\varepsilon_f \right)^{2/3} \frac{k}{k_f} ,$$

where c_f is a numerical coefficient which is dependent on the particular filter in use ($c_f \approx 1.04$ for the Gaussian filter).

$$k = k_f + k_{sgs} \implies k_{sgs} = \frac{c_f \left(2\Delta\varepsilon_f \right)^{2/3} k_f}{k_f - c_f \left(2\Delta\varepsilon_f \right)^{2/3}} ,$$

and finally
$$k = \frac{k_f{}^2}{k_f - c_f \left(2\Delta\varepsilon_f \right)^{2/3}} , \qquad [19.21]$$

where ε_f is given by the simulation.

We can also refer on this subject to the study by Meyers J. and Baelmans M. [MEY 04].

19.8. Variable filters

Variable filters in space or time bring important complexities in the form of the equations and their treatment. In the case of a time variation, we would have indeed:

$$\frac{\partial \bar{\phi}}{\partial t} = \beta(\phi) + \overline{\frac{\partial \phi}{\partial t}} \qquad \text{with} \quad \beta(\phi) = \frac{\partial G}{\partial t} * \phi .$$

When the filters vary in space, as is the case for applications to non-homogenous flows, a commutation error occurs, due to the fact that the filtering operation does not commute with the derivative. We can, however, derive variable filters that can commute with the derivative up to a certain fixed order (cf. Van Der Ven H., [VDV 95], Ghosal S. and Moin P. [GHO 95B]).

19.9. Advanced subgrid-scale models

19.9.1. *Scale similarity models*

The scale similarity model introduced by Bardina J., Ferziger J.H. and Reynolds W.C., ([BAR 80], [*COLL 84A]) relies on the idea that most of the interactions between large scales and small scales are concentrated near the cut-off.

If a progressive filter $\bar{\varphi}$ is used, we can thus note that the twice filtered field $\bar{\bar{\varphi}}$ is richer in large eddies than the $\bar{\varphi}$ field. We are thus led to define $\varphi^* = \bar{\varphi} - \bar{\bar{\varphi}}$ which is supposed to be representative of the eddies with a characteristic scale of order of the cut-off wavenumber (i.e., the smallest among the resolved large-scales).

The model, applied to formulation [19.6], thus assumes:

$$\tau_{ij} = c_B \left(\bar{u}_i \bar{u}_j - \bar{\bar{u}}_i \bar{\bar{u}}_j \right), \tag{19.22}$$

which is expressing the hypothesis that the subgrid-scale fluctuating field has a structure similar to the one of the part of the filtered field corresponding to the largest wavenumbers. The constant c_B is of order unity. In practice, this model is proved to be weakly dissipative, thus it is in combination with the Smagorinsky model that this scheme is the most efficient (mixed model). In particular, it allows the inverse energy cascade from the small scales down to the large-scales, an interesting property to account for "backscatter". It is the so-called mixed model:

$$\tau_{ij} = c'_B \left(\bar{u}_i \bar{u}_j - \bar{\bar{u}}_i \bar{\bar{u}}_j \right) - 2\nu_t S_{ij}. \tag{19.23}$$

The Bardina J. model can also be interpreted using the three term decomposition of Germano M. (cf. section 19.2.1); indeed:

$$L_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j \qquad\qquad C_{ij} = \bar{u}_i \bar{u}_j - \bar{\bar{u}}_i \bar{\bar{u}}_j$$

$$\mathcal{L}_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{\bar{u}}_i \bar{\bar{u}}_j \qquad \mathcal{C}_{ij} = 0 \qquad \mathcal{R}_{ij} = R_{ij}$$

thus verifies $L_{ij} + C_{ij} + R_{ij} = \mathcal{L}_{ij} + \mathcal{C}_{ij} + \mathcal{R}_{ij}$.

This type of decomposition is the basis of the model by Horiuti K., [HOR 97] who relying on the concepts of Germano M., [GER 86,92] approximates the various centered moments in the form of a generalized similarity model:

$$\overline{fg} - \overline{f}\,\overline{g} = C\left(\overline{\overline{f}\,\overline{g}} - \overline{\overline{f}}\,\overline{\overline{g}}\right),$$

afterwards associated with a dynamic procedure (cf. section 19.9.2).

The scale similarity model is efficient for describing the local transfers in the region near the cut-off, while the viscosity models give a better description of non-local transfers. We thus understand the relevance of mixed formulations. We shall mention in this direction the contribution of Krajnovic S., Davidson L. [KRA 02] who introduce a mixed formulation using the scale similarity model and a viscosity model with a transport equation for the subgrid-scale kinetic energy.

19.9.2. *The dynamic model*

The model by Germano M. *et al.*, [GER 90] allows a dynamic estimate of the Smagorinsky "constant" from the simulation itself. This is performed by using a double filtering process. This technique has been systematized in Germano M., [GER 93]. More precisely, let R_{ij} be the subgrid-scale stresses acting with the real filter denoted $\overline{(\phi)}$, and R_{ij}^{*} the stresses acting with a test filter denoted $\widetilde{(\phi)}$, it is thus possible to determine the resolved part containing the largest subgrid scales:

$$G_{ij} = R_{ij}^{*} - R_{ij}.$$

If we apply the real filter to the Navier-Stokes equation,

$$\partial_t u_i + \left(u_i u_j\right)_{,j} = -\pi_{,i} + \nu u_{i,jj},$$

we get $\overline{[NS]}$:

$$\partial_t \overline{u}_i + \left(\overline{u}_i \overline{u}_j\right)_{,j} = -\overline{\pi}_{,i} - T_{ij,j} + \nu \overline{u}_{i,jj}, \qquad\qquad [19.24]$$

where $T_{ij} = R_{ij} + L_{ij}$ and applying the filter $\overline{\overline{G}} = \widetilde{G} * \overline{G}$, we get $\left[\widetilde{\overline{NS}}\right]$:

$$\partial_t \widetilde{\overline{u}}_i + \left(\widetilde{\overline{u}}_i \widetilde{\overline{u}}_j\right)_{,j} = -\widetilde{\overline{\pi}}_{,i} - T^*_{ij,j} + \nu \widetilde{\overline{u}}_{i,jj} .$$ [19.25]

Applying the test filter on equation [19.24], it follows:

$$\partial_t \widetilde{\overline{u}}_i + \left(\widetilde{\overline{u}_i \overline{u}_j}\right)_{,j} = -\widetilde{\overline{\pi}}_{,i} - \widetilde{T}_{ij,j} + \nu \widetilde{\overline{u}}_{i,jj} ,$$ [19.26]

with $T_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$, $\quad T^*_{ij} = \widetilde{\overline{u_i u_j}} - \widetilde{\overline{u}}_i \widetilde{\overline{u}}_j$, $\quad \widetilde{T}_{ij} = \widetilde{\overline{u_i u_j}} - \widetilde{\overline{u}_i \overline{u}_j}$.

Identifying equations [19.25] and [19.26] leads to the Germano identity:

$$T^*_{ij} - \widetilde{T}_{ij} = \widetilde{\overline{u}_i \overline{u}_j} - \widetilde{\overline{u}}_i \widetilde{\overline{u}}_j .$$ [19.27]

It appears that this quantity denoted G_{ij} can be obtained from the simulation itself:

$$G_{ij} = \widetilde{\overline{u}_i \overline{u}_j} - \widetilde{\overline{u}}_i \widetilde{\overline{u}}_j .$$

Thus, by assuming that each of these tensors can be expressed using a similar model, it is possible to deduce from [19.27] the value of the constant appearing in this model.
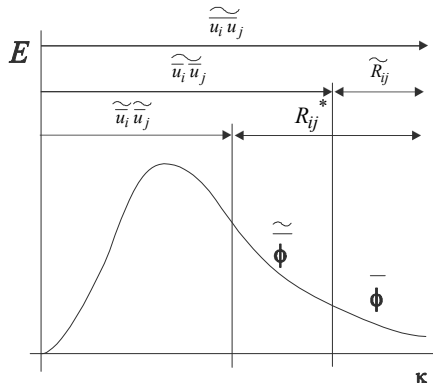
Figure 19.5. *Double filtering in the Germano dynamic model*

In the framework of the Smagorinsky model:

$$G_{ij} = -c_G \underbrace{\left(\widetilde{\overline{\Delta}}^2 \left| \widetilde{\overline{S}} \right| \widetilde{\overline{S}}_{ij} - \overline{\Delta}^2 \widetilde{\left| \overline{S} \right| \overline{S}_{ij}} \right)}_{H_{ij}}. \qquad [19.28]$$

The filter widths $\overline{\Delta}$ and $\widetilde{\overline{\Delta}}$ pertain to $\overline{(\phi)}$ and $\widetilde{(\phi)}$ respectively.

Making tensorial contraction by S_{ij}, we derive:

$$c_G = -\frac{G_{ij} S_{ij}}{H_{ij} S_{ij}}. \qquad [19.29]$$

This expression, also used in the averaged form:

$$c_G = -\frac{\langle G_{ij} S_{ij} \rangle}{\langle H_{ij} S_{ij} \rangle},$$

allows us to calculate coefficient c_G and to reach in this way a better generality level of the model. This averaged formulation avoids some problems encountered in the previous formulation of c_G whose sign can change rapidly and in which the denominator can occasionally become very small and produce instabilities. The method has also been further improved by Lilly D.K., [LIL 92] who proposed to calculate c_G in the least square sense, leading to:

$$c_G = -\frac{G_{ij} H_{ij}}{H_{ij} H_{ij}}.$$

The "constant" c_G vanishes every time G_{ij} also vanishes, whereas such was not the case for the original Smagorinsky model, this property is valuable in the calculation of transitional flows. We can also note that the spectral flux which is proportional to $\langle G_{ij} \overline{S}_{ij} \rangle$ can have any sign, positive or negative, i.e. making possible the inverse flux of energy (backscatter).

The dynamic approach, in its principle, provides a modeling technique of general application. This technique can thus be applied to models other than the

Smagorinsky viscosity model, such as the mixed model (cf. Zang Y. *et al.*, [ZAN 93]), it can even be adapted to the case of transport equation models (Ghosal *et al.*, [GHO 95A]). These latter works ([GHO 95A]) and also the works by U. Piomelli and J. Liu [PIO 95] lead to a rational extension of the method to the case of non-homogenous flows.

The similarity and the dynamic models are thus approaching the subgrid scales from the smallest of the resolved scales only, which makes them more able to apply to turbulent flows departing from equilibrium ([PIO 97]).

19.9.3. *The structure function model*

In order to consider applications in physical space, Métais O. and Lesieur M., [MET 92] (cf. also [*LES 05]) proposed an extension of the spectral eddy viscosity concept to highly intermittent turbulent fields, based on the structure functions (cf. Chapter 1 section 4.2). The structure function (SF) model is the extension to physical space of the spectral model using spectral turbulence eddy viscosity. The hypothesis (deduced from [19.13]) is then established in physical space by resorting to the dualism between Fourier modes in spectral space and velocity differences in physical space, to get finally:

$$v_t(\vec{x},t) = 0.105 c_K^{-3/2} \Delta . \overline{D}_\Delta^{1/2} \quad \text{(SF)} \qquad\qquad [19.30]$$

with $\quad \overline{D}_\Delta = \left\langle \left\| \vec{u}(\vec{x},t) - \vec{u}(\vec{x}+\vec{r},t) \right\|^2 \right\rangle_{\|\vec{r}\|=\Delta}$ and c_K is the Kolmogorov constant.

The subgrid-scale stresses are obtained from:

$$T_{ij} = \overline{u}_i\,\overline{u}_j - \overline{u_i u_j} = 2 v_t^{SF}(\vec{x},\delta x,t)\overline{S}_{ij} + \frac{1}{3} T_{mm}\delta_{ij} .$$

The derivation of [19.30] is based on the formula $\overline{D}_\Delta = 2\overline{\vec{u}(\vec{x})^2} - 2\overline{\vec{u}(\vec{x})\vec{u}(\vec{x}+\vec{r})}$ deduced from the definition of $\overline{D}_\Delta$.

By using the relation $R_{jj}(r) = 2\int_0^\infty E(\kappa)\dfrac{\sin(\kappa r)}{\kappa r}\,d\kappa$ (cf. section 1.3.1.4), we get:

$$\overline{D}_\Delta = 4\int_0^{\kappa_c} E(\kappa)\left(1 - \frac{\sin \kappa r}{\kappa r}\right) d\kappa ,$$

the integration interval being limited to $[0,\kappa_c]$ since only the resolved part of the velocity field has to be considered. For a Kolmogorov spectrum, it can then be shown that $v_t(\vec{x},t) = 0.105 c_K^{-3/2} \Delta \overline{D}_\Delta^{1/2}$ ([MET 92]).

When $\Delta \to 0$, the differences can be replaced by derivatives in the structure function and we can show ([LES 96]) that:

$$v_t^{SF} \to 0.777 (C_S \Delta)^2 \sqrt{2 S_{ij} S_{ij} + \omega_{ij} \omega_{ij}} \ .$$

This expression, compared to the Smagorinsky model, leads to a less dissipative and less diffusive behavior when the flow is weakly rotational (low curl).

From the practical point of view $\overline{D}_\Delta$ is calculated from a local mean of quadratic velocity differences between $\vec{x}$ and the six neighboring discretization points. In the presence of quasi two-dimensional big eddies, this model proved to be too dissipative. To overcome this drawback Ducros F. *et al.*, [DUC 96] apply a high pass filtering on $\overline{D}_\Delta$ in order to cancel out the low frequency fluctuations. This high pass filter is performed using a Laplacian discretized to second order. The spectrum associated with this filter, including discretization, is like $\widetilde{E(\kappa)} = 40^3 \left(\dfrac{\kappa}{\kappa_c} \right)^9 E(\kappa)$

([DUC 96]). The filtered structure function (FSF) model then reads:

$$\tilde{v}_t(\vec{x},t) = 0.0014 c_{\widetilde{K}}^{-3/2} \Delta \widetilde{\overline{D}}_\Delta^{1/2}, \quad \text{(FSF)}$$

where $\widetilde{\overline{D}}_\Delta$ is the filtered structure function.

Another version, the Selective Structure Function (SSF) relies on the idea of suppressing viscosity in the regions where the flow is not sufficiently three-dimensional (cf. [LES 96], [*LES 05]).

For this, the mean value on six points surrounding point $\vec{x}$ is denoted:

$$\overrightarrow{\Omega^*}(\vec{x},t) = \frac{\displaystyle\sum_{i=1}^{6} \overrightarrow{\Omega(\overrightarrow{x_{(i)}},t)} / d(\vec{x}, \overrightarrow{x_{(i)}})}{\displaystyle\sum_{i=1}^{6} 1 / d(\vec{x}, \overrightarrow{x_{(i)}})} ,$$

where d is the distance between $\vec{x}$ and $\overrightarrow{x_{(i)}}$, allowing us the calculate the angle:

$$\alpha(\vec{x},t) = \arccos\left(\frac{\overline{\vec{\Omega}(\vec{x},t)}\overline{\vec{\Omega}^{*}}(\vec{x},t)}{\left\|\overline{\vec{\Omega}(\vec{x},t)}\right\|\left\|\overline{\vec{\Omega}^{*}(\vec{x},t)}\right\|}\right).$$

If $\alpha < 20°$ the turbulence viscosity is then suppressed.

The value of the constant 0.105 is, however, increased in order that the spatial mean of viscosity is not changed, and thus:

$$\nu_t(\vec{x},t) = 0.172\,\Phi.c_K{}^{-3/2}\Delta.\overline{D}_\Delta{}^{1/2}\,, \qquad \Phi = \begin{cases} 1 & if\ \alpha \geq 20° \\ 0 & if\ \alpha < 20° \end{cases} \quad \text{(SSF)}$$

A modification (MSSF, "Modified Selective Structure Function") leads to a less dissipative behavior in the wall region [ACK 01].

The model introduces a dependency of the limiting angle α on the filter width:

$$\Phi = \begin{cases} 1 & if\ \alpha \geq \alpha_c \\ 0 & if\ \alpha < \alpha_c \end{cases}, \qquad \alpha_c = \begin{cases} 23\left(\dfrac{\kappa_c}{\kappa_M}\right)^{-0.4} & if\ \dfrac{\kappa_c}{\kappa_M} \leq 10 \\[2ex] 9 & if\ \dfrac{\kappa_c}{\kappa_M} > 10 \end{cases},$$

where κ_M corresponds to the maximum in the spectrum and κ_c the spectral cut-off of the filter. Its expression is then:

$$\nu_t(\vec{x},t) = C_{MSSF}\Phi.c_K{}^{-3/2}\Delta.\overline{D}_\Delta{}^{1/2}\,, \qquad \text{(MSSF)}$$

with $C_{MSSF} = 0.142$ in order to get $\left\langle 2\nu_t^{SF}\,\overline{S}_{ij}\,\overline{S}_{ij}\right\rangle = \left\langle 2\nu_t^{MSSF}\,\overline{S}_{ij}\,\overline{S}_{ij}\right\rangle$.

The formulation in physical space allows us to consider applications in non-homogenous shear flows ([LES 97]) whereas the spectral formulation is almost pertaining to homogenous turbulence.

19.9.4. *Hyperviscosity*

If in the spectral model (section 19.3.2) we use a function F in the form:

$$F = 1 + A \left(\frac{\kappa}{\kappa_C} \right)^{2n} ,$$

with $2n = 3.7 \approx 4$ and A being determined from the energy conservation ($\int_0^{\kappa_c} 2\nu_t \kappa^2 E d\kappa = \varepsilon$), Lamballais E. *et al.*, [LAM 97] show that the equivalent in physical space of the spectral model with a local peak is a hyperviscosity model which reads for the (i) component of the equation of motion:

$$0.661 \left(2\nu_t S_{ij} \right)_{,j} + 0.014 \Delta^4 \nu_t \overline{u}_{i,jjllmm} .$$

In the case of the dynamic model, it is necessary to multiply this operator by
$$\frac{\sqrt{15}}{5} \frac{5-m}{1+m} \sqrt{3-m} .$$

19.9.5. *Inverse flux (backscatter) and stochastic models*

One of the difficult problems in subgrid-scale modeling is the problem of inverse transfer ("backscatter"). The traditional Smagorinsky model implies that the spectral transfer between resolved scales and subgrid scales would always be positive, it thus cannot account for the inverse transfer phenomenon. In LES, the inverse transfer mainly acts in the spectral region of the local peak in the effective viscosity near the cut-off κ_C. Härtel C. *et al.*, [HAR 94] have studied the energy transfers between resolved scales and subgrid scales referring to direct numerical simulations of turbulence (so-called "*a priori*" tests) for testing subgrid-scale models. They show that the inverse transfer is enhanced by the coherent structures near the wall such as the "bursts" (Kline S.J. *et al.*, [KLI 67]). If we consider a three level decomposition as in section 19.3.3 for the subgrid-scale stresses:

$$u_i = U_i + v_i + w_i , \quad u'_i = v_i + w_i , \quad \overline{u'_i u'_j} = \overline{v_i v_j} + \overline{v_i w_j} + \overline{w_i v_j} + \overline{w_i w_j} ,$$
$$\overline{u'_i u'_j} = \overline{v_i v_j} + \overline{w_i w_j} \text{ (cross term stresses are neglected)}, \quad R_{ij} = \overline{w_i w_j} ,$$

the subgrid-scale flux can be decomposed into two contributions:

$$-\left\langle \overline{w_i w_j}\, u'_{i,j} \right\rangle = -\left\langle \overline{w_i w_j}\, U_{i,j} \right\rangle - \left\langle \overline{w_i w_j}\, v_{i,j} \right\rangle .$$

The first term on the right-hand side corresponds to the flux produced by the mean velocity gradient acting in shear flows whereas the second term is produced by the fluctuating velocity gradients and remains the only one present in homogenous turbulence. It is this latter term which strongly contributes to the inverse flux.

Several models allow us, to some extent, to account for the effects of inverse transfer. The Bardina model, the mixed model (from its scale similarity term), can produce inverse transfers. The model by Germano M. *et al.*, [GER 90] can also produce inverse transfers. Indeed, the quantity $G_{ij} H_{ij}$ (related to the "dissipation" of the resolved scales) which is involved in the calculation of c_G may become negative for some moments in time.

Other alternatives, again in subgrid-scale closures, can also produce inverse transfer; these are the stochastic models. Leith C.E., in [LEI 90] soon suggested to add to the Smagorinsky model, a stochastic inverse flux as an extra term deduced from a random vector potential. The right-hand side of the resolved equation of motion thus contains the following terms:

$$+\left(v_t S_{ij}\right)_{,j} + a_i$$

with $v_t = (c_S \ell)^2 S$ and $S = \sqrt{2 S_{ij} S_{ij}}$,

with also $a_i = \varepsilon_{ijk} \phi_{,k}$ and $\phi_j = c_b \left| S \delta t \right|^{3/2} \left(\frac{1}{\delta t}\right)^2 g$,

where g is a Gaussian random number obtained from drawing lots for g in a set with zero mean and standard deviation unity, ℓ the length scale and δt the time step.

Chasnov J.R., [CHA 91] more directly refers to stochastic turbulence models. The stochastic models are model equations whose exact solution in the mean coincides with the spectral equation models with an analytical transfer term $T(\kappa, t)$. In particular, the EDQNM approximation of the Navier-Stokes equations can be used as a starting point for the development of subgrid-scale models. This equation takes the form:

$$\frac{d\hat{u}_i(\vec{\kappa},t)}{dt} + \left(\nu + \eta(\kappa,t)\right)\kappa^2 \hat{u}_i(\vec{\kappa},t) = f_i(\vec{\kappa},t) \, ,$$

where $f_i(\vec{\kappa},t)$ is a random forcing term not correlated in time and where the term $\eta(\kappa,t)$ can be connected to $E(\vec{\kappa},t)$, the energy spectrum. The subgrid-scale model is obtained by restricting the previous equation to the interval $\kappa < \kappa_C$.

The Chasnov model is then composed of two terms: a subgrid-scale viscosity (which comes in addition to the molecular viscosity) and a stochastic force uncorrelated in time (random force in the Navier-Stokes equations).

The work by Bertoglio J-P., [BER 86] using the Langevin stochastic equation, also relates to this category of models. Piomelli U. *et al.*, [PIO 91] emphasize in particular the importance of inverse transfer in transitional flows and show that the "backscatter" rate is dependent on the type of filtering that is used.

The model by Schumann U., [SCH 95A] supplements the Smagorinsky model by adding a realizable random tensor obtained from the quadratic terms of a random Gaussian velocity. Thus,

$$\overline{u'_i u'_j} = -\nu_t \overline{S}_{ij} + \frac{2}{3}k_{sgs}\delta_{ij} + \mathcal{R}_{ij} \, ,$$

where $\mathcal{R}_{ij}$ is the stochastic contribution of zero mean $\left\langle \mathcal{R}_{ij} \right\rangle = 0$.

The fluctuating stresses are modeled as if they were produced by fluctuating velocity vectors defined by:

$$\mathcal{R}_{ij} = \gamma\left(v_i v_j - \frac{2}{3}e\delta_{ij} \right) \quad , \quad e = \frac{v_j v_j}{2}, \quad 0 \leq \gamma \leq 1, \tag{19.31}$$

and this ensures realizability.

The fluctuating velocities are built from a random field Z_i, which forms a white noise in space and which is correlated in time relative to the scale τ_V . Thus,

$$\left\langle Z_i \right\rangle = 0 \, ,$$

$$\left\langle Z_i(\vec{x},t) Z_j(\vec{x}',t')\right\rangle = \delta_{ij}\delta_{\overline{X}\,\overline{X}'} e^{\dfrac{|t-t'|}{\tau}},$$

$$v_i = \left(\frac{2e}{3}\right)^{1/2} Z_i .$$

Moreover, in incompressible flows, v_i must satisfy the zero divergence condition, which is practically obtained after correction by the gradient of a potential.

19.9.6. *Variational multiscale formulation*

Hughes *et al.* [HUG 01A and B] made large eddy simulations using projection on a variational basis of Legendre polynomials instead of filtering. It is in fact a type of spectral method. This decomposition allows us to separate the different scales of eddies. The subgrid-scale modeling can then be based on information relative to only part of the resolved scales: the smallest scales among the resolved scales. This approach is similar to the spectral eddy viscosity concept as in Chollet and Lesieur, [CHO 81].

The expansions in the variational technique are based on Fourier series (homogenous directions) or Legendre polynomials (non-homogenous directions) and the Galerkin method is used ([*CAN 88]).

In the variational multiscale method ([HUG 01A and B]), the solution (resolved velocity) is decomposed into a large-scale component $\overline{u}_i^{(L)}$ and a small scale component $\overline{u}_i^{(S)}$:

$$u_i = \overline{u}_i^{(L)} + \overline{u}_i^{(S)} ,$$

and modeling is restricted to the small scales:

$$T_{ij} = \nu^{(S)}{}_T\left(\overline{u}^{(S)}{}_{i,j} + \overline{u}^{(S)}{}_{j,i}\right),$$

with two possibilities:

$$\nu^{(S)}{}_T = \frac{1}{2}\left(C'_S \, \Delta'\right)^2 \left|\overline{u}^{(S)}{}_{i,j} + \overline{u}^{(S)}{}_{j,i}\right| \quad (\text{"small-small" model}),$$

$$\nu^{(S)}{}_T = \frac{1}{2}\left(C'_S \, \Delta'\right)^2 \left|\overline{u}^{(L)}{}_{i,j} + \overline{u}^{(L)}{}_{j,i}\right| \quad (\text{"large-small" model}).$$

In all cases, the model is only applied to the small resolved scales. The viscosity in the "small-small" model only detects the gradients of fluctuations, like in the FSF model by Ducros *et al.* in which a high pass filter is applied on the velocities before calculating the gradients.

Before closing this section, let us mention other types of approach using analytical theories for deriving a subgrid-scale model. For example, Yoshizawa A. ([YOS 82]) uses the direct interaction theory for developing a subgrid-scale model. Also, Aupoix B., [AUP 86, 87] proposes a subgrid-scale closure in homogenous turbulence based on the EDQNM theory.

19.10. Flows undergoing laminar-turbulent transition

Subgrid-scale models which are based on the dynamics at high turbulence Reynolds numbers cannot be applied in general to transitional or relaminarizing flows. For example, the traditional Smagorinsky model gives a non-zero subgrid-scale viscosity even if the flow is laminar! On the contrary, the dynamic Smagorinsky model no longer fails in this case.

The structure function model is also too dissipative for predicting transition. The new improved versions of this model allow better behavior. In the selective structure function model (David, [DAV 93]) the viscosity is canceled when the flow is not sufficiently three-dimensional. The three-dimensionality is measured using the angle between vorticity at a given point and the averaged vorticity in the six surrounding points and then a test is made. This improvement can be transposed to models other than the structure function model. In the filtered structure function model [DUC 96] already discussed in section 19.9.3, the high pass filter applied to the resolved filtered velocities allows the suppression of the low frequency oscillations in the energy spectrum and thus the reduction of the dissipative character of the model.

19.11. Other transport equation models

It is possible to devise more complex subgrid-scale models by being inspired by traditional statistical closures of higher order and adapting them to subgrid closure. The form of the equations is formally similar, apart from the characteristic length scale which is given in LES by the filter width and there is generally no need for a length scale transport equation. However, when important departure from equilibrium happens in the small scale region, it can be understood that this practice would have to be reconsidered.

From the decomposition:

$$u_i = U_i + v_i + w_i \quad \text{with} \quad U_i = \langle u_i \rangle, \quad v_i = \overline{u}_i - U_i, \quad w_i = u'_i,$$

Goutorbe T. and Laurence D. [GOU 94] have proposed a model which links the subgrid-scale stresses $R_{ij}^{sgs} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$ to the Reynolds stresses of the resolved fluctuations $R_{ij}^E = \overline{\overline{u}_i \overline{u}_j} - U_i U_j$, using a similarity relation:

$$R_{jj}^{sgs} = f_1 R_{jj}^E,$$

$$a_{ij}^{sgs} = f_2 a_{ij}^E + f_3 \left(a_{il}^E a_{lj}^E - \frac{1}{3} a_{lm}^E a_{lm}^E \delta_{ij} \right),$$

$$a_{ij}^E = \left(R_{ij}^E - \frac{1}{3} R_{ll}^E \delta_{ij} \right) \bigg/ k^E, \qquad k^E = \frac{1}{2} R_{jj}^E,$$

$$a_{ij}^{sgs} = \left(R_{ij}^{sgs} - \frac{1}{3} R_{ll}^{sgs} \delta_{ij} \right) \bigg/ k^{sgs}, \qquad k^{sgs} = \frac{1}{2} R_{jj}^{sgs},$$

coefficients $f_2 = 1$ and $f_3 = 0$ simply being given as a first approximation whereas f_1 is deduced from a simplified $-(5/3)$ spectrum law. These authors then propose to determine f_1 from the solution of a transport equation for k^{sgs}. This model has, in principle, the advantage of being able to account for the "backscatter" phenomenon.

The turbulent flows usually studied using large eddy numerical simulations are generally flows in relatively simple geometries and they require the use of supercomputers. The extension to industrial type flows cannot be considered currently as a common practice. Among the remaining open problems, there is of course, the central problem of subgrid-scale closure, but also other more technical problems such as the determination of efficient and realistic boundary conditions at inlet and exit sections. In practice, periodic boundary conditions are used every time

the corresponding direction is considered as homogenous. However, in more general cases, the problem is more difficult to handle. On the fundamental aspects of the theory, the filtering operation cannot be simply viewed as conditional statistical averaging. From this fact, originates the more complex operational properties of the filtering operator.

Although different in spirit, it is possible to relate to simulation methods, the so-called "semi-deterministic" modeling (cf. section 16.9 and Lebret Y., Vandromme D. and Ha Minh H., [LEB 91]) which is based on the decomposition of any instantaneous quantity into a mean part, organized part and random part (the splitting is made in terms of organized/random and no longer in terms of large/small scales). However, the present practical formulation of these models is not fundamentally different from the formulation used in large eddy simulations, so the question of how to perform the splitting between organized part and random part remains an open problem.

19.12. Approximate deconvolution methods

The filtering operation $\overline{u_i} = \mathscr{F}(u_i)$ in physical space being obtained through a convolution operator, or a product in Fourier space, the inverse operation called "defiltering" or "deconvolution" $u_i = \mathscr{D}(\overline{u_i})$ would allow us in principle to solve the closure problem through $\overline{u_i u_j} \approx \overline{\mathscr{D}(\overline{u_i})\mathscr{D}(\overline{u_j})}$. It must be emphasized, however, that the exact inversion $u_i = \mathscr{D}(\overline{u_i})$ is an illusion, even more that the filter is decreasing more rapidly to zero when κ increases. Even if the information is still contained into $\overline{u_i}$, the convolution cannot be reversed in a stable way because it would be similar to divide by a very small quantity, and the information cannot be extracted accurately. The operation becomes impossible if the filter vanishes. It results from this remark that the method requires some approximation. The deconvolution models try to recover in an approximate way a part of the information which was lost during filtering.

In the case of a Gaussian filter with a width $\varDelta$ ([CLA 79],[*GEU 01], [*GEU 04], [*BER 05]), the operator $\overline{u_i} = \mathscr{G}_\varDelta * u_i$ or $\hat{\overline{u}}_i = \widehat{\mathscr{G}}_\varDelta \hat{u}_i$ is formally reversed into $\hat{u}_i = \widehat{\mathscr{G}}_\varDelta^{-1} \hat{\overline{u}}_i$. Using Taylor expansions leads to:

$$\widehat{\mathscr{G}}_\varDelta(\vec{\kappa}) = 1 - \frac{\varDelta^2}{4\gamma}|\kappa|^2 + \mathscr{O}(\varDelta^4) \quad \text{and} \quad \widehat{\mathscr{G}}_\varDelta^{-1}(\vec{\kappa}) = \left(1 - \frac{\varDelta^2}{4\gamma}|\kappa|^2\right)^{-1} + \mathscr{O}(\varDelta^4)$$

or $\widehat{\mathcal{G}}_{\Delta}^{-1}(\vec{\kappa}) = 1 + \dfrac{\Delta^2}{4\gamma}|\kappa|^2 + \mathcal{O}(\Delta^4)$.

and thus $\hat{u}_i = \dfrac{1}{\widehat{\mathcal{G}}}\hat{\bar{u}}_i$ and $\hat{u}'_i = \left(\dfrac{1}{\widehat{\mathcal{G}}} - 1\right)\hat{\bar{u}}_i$, with $\widehat{\overline{u_i u_j}} = \widehat{\mathcal{G}}(\vec{\kappa})\hat{\bar{u}}_i * \hat{\bar{u}}_j$ and similar relations for cross terms.

From the expression $\overline{u_i u_j} = \overline{\bar{u}_i \bar{u}_j} + \overline{\bar{u}_i u'_j} + \overline{u'_i \bar{u}_j} + \overline{u'_i u'_j}$, it is possible to approximate each term and after using Fourier transform:

$$\overline{\bar{u}_i \bar{u}_j} = \bar{u}_i \bar{u}_j + \dfrac{\Delta^2}{4\gamma}\left(\bar{u}_i \bar{u}_j\right)_{,mm} + \mathcal{O}(\Delta^4),$$

$$\overline{\bar{u}_i u'_j} = -\dfrac{\Delta^2}{4\gamma}\bar{u}_i \bar{u}_{j,mm} + \mathcal{O}(\Delta^4),$$

$$\overline{\bar{u}_j u'_i} = -\dfrac{\Delta^2}{4\gamma}\bar{u}_j \bar{u}_{i,mm} + \mathcal{O}(\Delta^4),$$

$$\overline{u'_i u'_j} = \mathcal{O}(\Delta^4),$$

taking into account the identity $h_{,m} g_{,m} = \dfrac{1}{2}\left[\left(hg\right)_{,mm} - h g_{,mm} - g h_{,mm}\right]$ we thus derive the gradient model (or Clark model):

$$\overline{u_i u_j} - \bar{u}_i \bar{u}_j = \dfrac{\Delta^2}{2\gamma}\bar{u}_{i,m} \bar{u}_{j,m} + \mathcal{O}(\Delta^4).$$

The underlying philosophy here is completely different from that in subgrid-scale closure modeling. The deconvolution methods rely on a numerical analysis approach whereas the subgrid-scale closure models rely on physical phenomena.

Another more advanced form, the rational model, uses a Padé approximant for approximating the Gaussian filter, thus retaining the smoothing property of the filtered function:

$$\widehat{\mathcal{G}}_{\Delta}(\vec{\kappa}) = \dfrac{1}{1 + \dfrac{\Delta^2}{4\gamma}|\kappa|^2} + \mathcal{O}(\Delta^4),$$

and consequently $u_i = \left(Id - \dfrac{\Delta^2}{4\gamma} \nabla^2 \right) \bar{u}_i + \mathcal{O}(\Delta^4)$, the corresponding subgrid scale

model is more complex ([*BER 05], [ILI 03]):

$$\overline{u_i u_j} - \bar{u}_i \bar{u}_j = \frac{\Delta^2}{2\gamma} \mathcal{A}(\bar{u}_{i,m} \bar{u}_{j,m}),$$

where $H = \mathcal{A}(\phi)$ stands for the solution of the equation $\phi = \left(Id - \dfrac{\Delta^2}{4\gamma} \nabla^2 \right) H$

symbolically denoted using the pseudo-operator $\mathcal{A}(\phi) = \left(Id - \dfrac{\Delta^2}{4\gamma} \nabla^2 \right)^{-1} \phi$.

We can note from the preceding relations that the contribution $\overline{u'_i u'_j}$ is not taken into account whereas this term is essential from the physical point of view. We will thus notice that the Clark model is similar to the terms obtained in expansion [19.9] for approximating the Leonard term. This observation supports a mixed formulation such as rational model + viscosity model. This combination indeed allows better stability and better behavior of the global model. Numerous variants of these methods have been developed ([GEU 97], [DOM 97], [DOM 99], [STO 99], [*GEU 01], [*GEU 04], [*BER 05]).

The gradient model can be related to the incremental model introduced by Brun and Friedrich [BRU 99], which is derived from the Taylor expansion of filtered quantities according to non-filtered quantities (using the same technique as in section 19.2.3 but applied to the filtered velocity). It reads:

$$R_{SGSij} = (2b - a^2)\Delta x_m u_{i,m} \Delta x_l u_{j,l} + \mathcal{O}(\Delta x_m \Delta x_l),$$

where a and b are related to the filter shape and where Δx_m is the filter width in the direction m. A first order discretization leads to the model:

$$R_{SGSij} \approx (2b - a^2)\delta u_i \delta u_j,$$

in which appears the velocity increment linked to the filter width. This model is more efficient when it is combined with the diffusive properties of a viscosity based model (the structure model for example [BRU 02]).

19.13. Simulations based on POD or wavelets

These examples deal with methods that are significantly different from the large eddy simulation, aiming at simulating the coherent structures (cf. section 16.9).

19.13.1. *Methods based on POD*

The proper orthogonal decomposition (POD) introduced by Lumley ([*GAT 96], Chap.1) allows to define and to extract the coherent structures from the turbulent field ([*LUM 70], [LUM 81], [AUB 91], [MAN 93]). The turbulent field is expanded according to spatial deterministic functions with random coefficients in time. The spatial eigenfunctions are obtained by solving an associated eigenvalue problem.

In the basic formulation ([*LUM 70], [LUM 81]), the velocity field is expressed as an expansion about the eigenmodes:

$$u_i(\vec{x},t) = \sum_{i=1}^{N} a_n(t)\varphi_i^n(\vec{x}) .$$

[19.32]

The eigenmodes form an orthogonal basis such that:

$$\left\langle \varphi_i^n \varphi_i^m \right\rangle = \int_{\mathbb{R}^3} \varphi_i^n(\vec{x})\varphi_i^m(\vec{x})d\vec{x} = \delta_{nm} .$$

The associated eigenvalue problem is:

$$\int R_{ij}(\vec{x},\vec{x}')\varphi_j^n(\vec{x}')d\vec{x}' = \lambda^{(n)}\varphi_i^n(\vec{x}) \quad \text{with}$$

$$R_{ij}(\vec{x},\vec{x}') = \overline{u_i(\vec{x},t)u_j(\vec{x}',t)} = \int_T u_i(\vec{x},t)u_j(\vec{x}',t)dt , \; T \text{ being a sufficiently long period}$$

of integration. It can be shown that $\lambda^{(n)} = \overline{a_n(t)^2}$ and $\left\langle \varphi_j^n u_j \right\rangle = a_n(t) .$

In the biorthogonal decomposition formulation ([AUB 91], [MAN 93]), the expansion is performed symmetrically in space and time:

$$u_i(\vec{x},t) = \sum_{i=1}^{N} \alpha_n \underbrace{\psi_n(t)}_{chronos} \underbrace{\varphi_i^n(\vec{x})}_{topos} ,$$

[19.33]

with $\left\langle \varphi_i^n \varphi_i^m \right\rangle = \delta_{nm}$ and $\overline{\psi_n \psi_m} = \delta_{nm}$.

The eigenvalues are given by $\lambda^{(n)} = \overline{\alpha_n^2}$.

Due   to   the   symmetry   in   the   formulation,   the   operator $Q(t,t') = \int_{\mathbb{R}^3} u_j(\vec{x},t) u_j(\vec{x},t') d\vec{x}$ has the same eigenvalues as $R_{ij}(\vec{x},\vec{x}')$ and the associated eigenmodes are the functions $\psi_n(t)$. Moreover, it can be easily shown that $\left\langle \varphi_j^n u_j \right\rangle = \alpha_n \psi_n(t)$ and $\overline{\psi_n u_i} = \alpha_n \varphi_i^n(\vec{x})$.

If the Navier-Stokes equation $\mathcal{N}_i = 0$ is projected onto the orthogonal basis [19.32] (cf. [*GAT 96], Chapter 1), performing the scalar product:

$$\left\langle \varphi_i^n \mathcal{N}_i \right\rangle = 0 ,$$
[19.34]

we get a system of differential equations:

$$\dot{a}_m(t) = B_{mp} a_p(t) + N.L. \text{ where } NL \text{ stands for the non-linear terms.}$$
[19.35]

After filtering in order to separate the resolved modes from the non-resolved modes, a model derived from the Heisenberg model is introduced to account for the non-resolved modes. The application (cf. [*GAT 96], Chapter 1) to wall turbulence allowed us to show evidence of an intermittency phenomenon which is characteristic of wall turbulence.

19.13.2. *Methods based on wavelets*

The work by Farge M. *et al.* ([FAR 99], [FAR 01A and B]), using wavelet bases (cf. section 3.8) in order to exhibit coherent structures (cf. section 16.9) leads to a method for simulating so-called "coherent vortex simulation" (CVS) turbulent structures. The vorticity is then projected onto an orthogonal wavelet basis. This scheme is based on the hypothesis that the coherent structures are responsible for the non-Gaussianity of the vorticity PDF. The aim is thus to eliminate the coefficients in the expansion of vorticity which have a Gaussian PDF. A coherent structure is thus defined as the non-Gaussian part of the vorticity field. In practice, the coherence filter selects the coefficients which are higher than a given threshold, thus allowing the decomposition of the vorticity field into two orthogonal components, one being coherent and the other non-coherent:

$$\omega = \omega_C + \omega_I \; .$$

The Gaussianity is tested against the skewness and flatness coefficients and iterations can be made for determining the adequate value of the threshold.

Considering that the non-coherent part is not related to the concept of scale of the eddies, it cannot correspond to the viscous dissipation. The modeling of the non-coherent part should rather express the reorganization of the random background turbulence through the effect of convection of coherent structures. Farge *et al.* [FAR 01] proposes for example to solve a convection-diffusion equation for ω_I.

19.14. Hybrid methods

These methods combine the technique of large eddy simulations "LES" and the statistical models "RANS". The aim of hybrid models is to improve numerical prediction of complex flows in engineering applications by introducing LES approaches in combination with traditional statistical approaches using relatively coarse meshes. They thus lead to economical approaches that are useful for practical applications.

In the wall region in which prevail a high shear and a strong anisotropy along with generally non-uniform discretization grids, the approach reduces to the RANS model whereas in the external zones the approach switches to a LES type model. Most of these models rely on a zonal approach, some regions of the flow are calculated using simulation, other regions are calculated from statistical modeling, depending on the specific dynamic properties of the type of turbulent flow under consideration. Also, depending on the physical problems to be studied, some regions of the flow may require a more refined description of the turbulent eddy interactions using finer grids together with a LES type treatment, whereas other regions that are less complex can be calculated from statistical models. The hybrid methods consider this mixed approach.

19.14.1. *Zonal hybrid methods*

The first hybrid methods were introduced with a practical aim, in order to model efficiently the wall layer which is then continued using an LES approach in the external region of the flow. They mainly provided boundary conditions for the LES calculation LES ([BAL 96], [CAB 00]). Davidson L. *et al.* [DAV 01] have introduced a two equation statistical model for describing the near wall region and a one equation model as a subgrid-scale model in the external region. The RANS

region near the wall is described using a low Reynolds number k-ω type model whereas the LES region makes use of the subgrid-scale model by Yoshizawa (similar to [HOR 85]). The separating boundary between the RANS and the LES regions is located at a fixed distance y_f from the wall. If we mark out with the index $j = j_m$ the discretization cell located immediately below the line y_f, the imposed connecting conditions can be written:

$$j = j_m: \qquad \frac{\partial k}{\partial y} = \frac{\partial \omega}{\partial y} = 0 \,,$$

$$j = j_{m+1}: \qquad v_{sgs}(j_{m+1}) = v_t(j_m) \Rightarrow k_{sgs}(j_m) = \left(\frac{v_t}{C_k \Delta} \right)^2_{j_m} \,,$$

where C_k is the constant involved in $v_{sgs} = C_k \Delta \sqrt{k_{sgs}}$.

This approach allowed a good prediction of the velocity profiles in a plane channel. However, some shortcomings still appear near the connection line between the RANS/LES regions, this is the main problem inherent to this type of approach.

Batten P. *et al.* [BAT 01] have shed some light on the problems posed by the hybrid approach and in particular the difficulties encountered at the RANS/LES interface. The turbulence energy transfers between resolved scales and unresolved scales through the interface are not fully accounted for. It is thus suggested to introduce a stochastic reconstruction of the resolved turbulent field when flowing through the interface.

A zonal RANS/LES approach has also been developed by Hamba F. [HAM 01] and applied to the case of a plane channel flow, the near wall region being solved using LES whereas the standard statistical k-ε model is used far from the wall on a one-dimensional grid. The approaches of [DAV 01] and [HAM 01] seem to be apparently antinomic but they proceed from the same general concept. We can think that the choice of the zone which is described more accurately using LES is dependent on the answers that are needed on the physical point of view, and the regions that need to be explored in more detail.

The major problem in zonal hybrid methods remains the difficulty to connect the two regions that are calculated using different models. Hamba F. [HAM 03] has introduced a new scheme with additional filtering and succeeded in reducing the discontinuities and irregularities at the interface.

19.14.2. *The DES (detached eddy simulation) method*

The detached eddy simulation (DES) method introduced by Spalart *et al.*, 1997 ([SPA 97]) is a hybrid model of RANS/LES type whose essential characteristic is the modification of the turbulence length scale ℓ usually involved in statistical closures and now depending on the size of the discretization cell:

$$\ell_{DES} = \min\left(\ell_{RANS}, C_{DES}.\Delta\right),$$ [19.36]

where Δ is the grid step size $\Delta = \max\left(\delta x, \delta y, \delta z\right)$ and C_{DES} a numerical parameter which is dependent on the particular choice of model. In turbulent wall flows, the length scale ℓ_{RANS} increases with the distance from the wall, it thus follows that near a wall:

$$\ell_{DES} = \ell_{RANS} \ll C_{DES}\Delta,$$ [19.37]

and the DES then exactly performs like the associated RANS model. Far from the wall,

$$\ell_{DES} = C_{DES}\Delta \ll \ell_{RANS},$$ [19.38]

the model becomes a subgrid-scale model and the DES performs like a LES. The DES approach was originally conceived for the calculation of boundary layer flows in aerodynamics, with the juxtaposition of a thin boundary layer to a large separated region in which the model allows us to predict detached eddies, justifying the denomination DES. The model which is most commonly used in this approach is the Spalart-Allmaras model ([SPA 92 and 94], cf. section 12.1.3), however, other RANS models can be transposed along the same principles. The approach is still zonal, but considering that the same basic model is used both in the RANS zone and the LES zone, the transition between the two zones is made without discontinuity.

In the case of the Spalart-Allmaras model, the system of equations becomes:

$$\frac{d\tilde{v}}{dt} = C_{b1}\tilde{v}\tilde{S} + \left[\frac{\left(v+\tilde{v}\right)}{\sigma}\tilde{v}_{,j}\right]_{,j} + C_{b2}\frac{\tilde{v}_{,j}\tilde{v}_{,j}}{\sigma} - C_{w1}f_w\frac{\tilde{v}^2}{d^2},$$ [19.39]

$$d = \min(y, C_{DES}\Delta), \quad \tilde{S} = S + \frac{\tilde{v}}{\mathrm{K}^2 d^2}f_{v2}(\chi), \quad S = \sqrt{2.\omega_{ij}\omega_{ij}},$$

$$\nu_t = \tilde{\nu} f_{v1}, \quad \tilde{\nu} = \nu + \nu_t, \quad \chi = \frac{\tilde{\nu}}{\nu}, \quad f_{v1} = \frac{\chi^3}{\chi^3 + C_{v1}^3}, \quad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}},$$

$$f_w = g\left(\frac{1 + C_{w3}^6}{g^6 + C_{w3}^6}\right)^{1/6}, \quad g = r + C_{w2}(r^6 - r), \quad r = \frac{\tilde{\nu}}{\tilde{S}K^2 d^2},$$

with $\quad C_{b1} = 0.1355, \quad C_{b2} = 0.622, \quad C_{w2} = 0.3, \quad C_{w3} = 2,$

$$\sigma = 2/3, \quad C_{w1} = \frac{C_{b1}}{K^2} + \frac{1 + C_{b2}}{\sigma}, \quad C_{v1} = 7.1, \quad C_{DES} = 0.65.$$

This model has been extensively used in engineering fields and applications to aerodynamics. We can mention in particular the applications to the plane channel flow (Nikitin *et al.*, [NIK 00]), the flow around and past a cylinder (Travin *et al.*, [TRA 99]) and the flow around an airfoil (Shur M. *et al.* [SHU 99]), but also industrial applications ([KAP 03] using an alternative version with $f_{v3} \neq 1$; cf. section 12.1.3).

The method has been extended afterwards for two equation models ([BUS 01], [HAM 04]). The extension proposed by Bush R.H. and Mani Mori [BUS 01] is based on the SST model by Menter (cf. section 11.2.7.2) in which the following modifications are made:

the variable ω is replaced by $\omega_B = \max\left(\omega, \sqrt{k}/C_{DES}\Delta\right),$

with $\Delta = \max\left(\delta x, \delta y, \delta z, u \delta t, \sqrt{k}\delta t\right).$

As in the statistical model by Menter, the DES version is composed of two branches, one of which comes from the $k\text{-}\omega$ formulation and the other coming from the $k\text{-}\varepsilon$ formulation, combined using the weighting function F_1. The value of C_{DES} is then obtained from ([TRA 02]):

$$C_{DES} = (1 - F_1)C_{DES}^{k-\varepsilon} + F_1 C_{DES}^{k-\omega}, \quad C_{DES}^{k-\varepsilon} = 0.61, \quad C_{DES}^{k-\omega} = 0.78.$$

The model has been applied to the flow over an open cavity by Hamed A. *et al.* [HAM 04] who highlighted the better robustness of the two equation model compared to the one equation model.

Piomelli U. *et al.* [PIO 03] studied the interaction between the internal and the external regions in RANS/LES methods which still have the drawback of producing an irregular and non-physical transition at their interface. These errors due to discontinuities between the RANS and LES levels of description are also noticeable in the DES method. Piomelli U. *et al.* [PIO 03] show that the introduction of a stochastic term in the momentum equation ("backscatter") is a possible way for improvement.

19.14.3. *Non-zonal hybrid methods*

In this kind of approach, the same model is used in the whole flow domain with a continuous modification of the model, allowing a coupling solution without discontinuity. This approach leads us to address the issue of a continuous formulation allowing us to unify the RANS and LES formalisms ([GER 99]).

We give below a description of the partially integrated transport modeling method "PITM" introduced in [DEJ 01] and [SCH 05].

The PITM approach

The approach is based on the technique of partial spectral integration (cf. Chapter 18). The splitting in turbulence energy spectrum sketched in Figure 19.6 introduces the cut-off wavenumber κ_c corresponding to the filter width in the LES filtering operator. In a first step, we consider fully developed turbulence at a high Reynolds number.
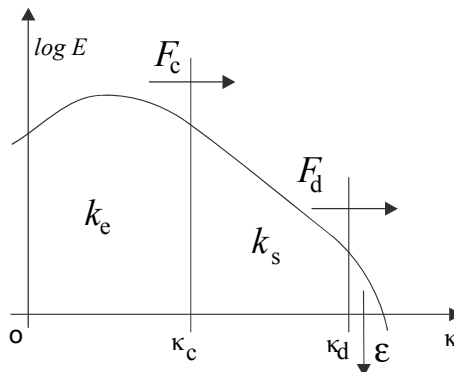
Figure 19.6. *Sketch of filtering by spectral splitting*

As in the multiple scale approach (cf. Chapter 18), a wavenumber κ_d is introduced, such that:

$$\kappa_d - \kappa_c = \xi \frac{\varepsilon}{k_s^{3/2}}, \qquad k_s = \int_{\kappa_c}^{\kappa_d} E(\kappa) d\kappa \, . \tag{19.40}$$

The wavenumber κ_d is chosen sufficiently large in order to include the inertial region and in order that the spectral energy beyond κ_d is negligible. The value of the numerical coefficient ξ is chosen such that this condition is fulfilled, this will allow us to avoid infinite integration bounds and to suppose that the flux through the splitting at wavenumber κ_d is equal to the dissipation rate.

The equation for the partial kinetic energy in the interval $[\kappa_c, \infty[$ reads:

$$\frac{dk_s}{dt} = P[\kappa_c, \kappa_d] + F(\kappa_c) - F(\kappa_d) \, , \tag{19.41}$$

with $F(\kappa_c) = \mathscr{T}_c - E_c \dfrac{d\kappa_c}{dt}$ and $F(\kappa_d) = \varepsilon$.

This relation expresses the total transfer through the variable cut-off κ_c as the sum of the local spectral flux and the energy exchanged due to the variation in the splitting wavenumber, $F_c = \mathscr{T}_c - E_c \dfrac{d\kappa_c}{dt}$. At the wavenumber κ_d, the preceding hypotheses lead to $F_c = \mathscr{T}_c = \varepsilon$, the turbulence Reynolds number being assumed to be large.

Thus, reporting these relations in equation [19.40] and taking the derivative, we get:

$$\frac{d}{dt}\left(\xi \frac{\varepsilon}{k_s^{3/2}}\right) = \frac{\mathscr{T}_d - \varepsilon}{E_d} - \frac{\mathscr{T}_c - F_c}{E_c} \, , \tag{19.42}$$

and consequently:

$$\frac{d\varepsilon}{dt} = \frac{3}{2}\frac{\varepsilon}{k_s}(P[\kappa_c, \kappa_d] + F_c - \varepsilon) - \frac{k_s}{(\kappa_d - \kappa_c) E_d}\left[\left(1 - \frac{\mathscr{T}_d}{\varepsilon}\right) - \frac{F_c}{\varepsilon}\frac{E_d}{E_c}\left(1 - \frac{\mathscr{T}_c}{F_c}\right)\right]\frac{\varepsilon^2}{k_s} \, .$$

If we assume $E_d \ll E_c$ and $\kappa_d \gg \kappa_c$, it then follows:

$$\frac{d\varepsilon}{dt} = \underbrace{\frac{3}{2}\frac{\varepsilon}{k_s}}_{C_{s1}}(P[\kappa_c, \kappa_d] + F_c) - \underbrace{\left[\frac{3}{2} - \frac{k_s}{\kappa_d E_d}\left(\frac{\mathscr{F}_d}{\varepsilon} - 1\right)\right]}_{C_{s2}}\frac{\varepsilon^2}{k_s}.$$ [19.43]

When κ_c goes to zero, we recover the usual equation in statistical closures with:

$$C_{\varepsilon 1} = \frac{3}{2} \quad \text{and} \quad C_{\varepsilon 2} = \frac{3}{2} - \frac{k}{\kappa_d E_d}\left(\frac{\mathscr{F}_d}{\varepsilon} - 1\right).$$ [19.44]

From the physical point of view, the value of ε, which corresponds to the energy flux of the cascade, must remain the same whatever the location of the spectral splitting. If we compare [19.43] to [19.44] it can be deduced that:

$$C_{s1} = C_{\varepsilon 1} = \frac{3}{2} \quad \text{and} \quad C_{s2} = \frac{3}{2} + \frac{k_s}{k}\left(C_{\varepsilon 2} - \frac{3}{2}\right).$$ [19.45]

Note: if we assume that C_{s2} is an empirical linear function in P/ε, then it would be possible to readjust the C_{s1} coefficient to a value somehow different from 3/2 as in references [SCH 05],[CHA 05B].

Determination of variable coefficients

The question is now to determine the ratio k_s / k according to the location of the cut-off. This calibration can be made by considering the decay of turbulence behind a grid in the initial period. We consider for this a simplified initial energy spectrum similar to those introduced by Von Karman (cf. [*HIN 75]) defined by:

$$E(\kappa) = \frac{\chi \varepsilon^{2/3} \kappa^{\mu}}{\left[\left(\frac{\chi \varepsilon^{2/3}}{C}\right)^{\frac{m-1}{m+\mu}} + \kappa^{m-1}\right]^{\frac{m+\mu}{m-1}}},$$

with $m=5/3$, $\chi = 1.5$ (Kolmogorov constant).

At the origin the spectrum behaves like $E(\kappa) \approx C\kappa^{\mu}$ (cf. Lesieur M., [*LES 90]), C being a constant (hypothesis of permanence of very big eddies).

The total energy under the spectrum is obtained from integration:

$$k = \int_0^{\infty} E(\kappa)d\kappa = \frac{1}{1+\mu}\left(C^{\frac{2}{3(\mu+1)}} \chi\varepsilon^{2/3} \right)^{\frac{3(\mu+1)}{3\mu+5}},$$

whereas the equation for decay reduces to $\dfrac{dk}{dt} = -\varepsilon$.

We then deduce from the previous equations: $\dfrac{d\varepsilon}{dt} = -\underbrace{\left[\frac{3\mu+5}{2(\mu+1)} \right]}_{C_{\varepsilon 2}}\frac{\varepsilon^2}{k}$.

The usual value $C_{\mu} = 1.92$ is recovered for $\mu = 1.4$.

If we introduce the dimensionless quantities: $\eta = \dfrac{\kappa}{\kappa_{ref}}$, $E^*(\kappa) = \dfrac{\kappa_{ref}}{k}E(\kappa)$ with $\kappa_{ref} = \varepsilon/k^{3/2}$, we get:

$$E^*(\eta) = \frac{\chi\eta^{\mu}}{\left(\dfrac{\chi}{1+\mu} + \eta^{2/3} \right)^{\frac{3\mu+5}{2}}},$$

$$H^*(\eta) = \int_0^{\eta} E^*(\eta')d\eta' = \left(\frac{\chi\eta^{-2/3}}{1+\mu} + 1 \right)^{-\frac{3(\mu+1)}{2}}, \qquad \frac{k_s}{k} = 1 - H^*(\eta_c),$$

or $\qquad \dfrac{k_s}{k} = 1 - \left(\dfrac{\eta_c^{2/3}}{\dfrac{\chi}{1+\mu} + \eta_c^{2/3}} \right)^{\frac{3(\mu+1)}{2}}, \qquad \left.\dfrac{k_s}{k}\right|_{\eta_c \to \infty} \approx \frac{3}{2}\chi\eta_c^{-2/3}.$ [19.46]

This behavior is in agreement with the work of Rubinstein and Zhou ([RUB 96], [RUB 01]) who derive the ε equation from integration of analytical models.

Relation [19.46], introduced in the subgrid-scale model equation [19.43], allows us to sensitize to model to the filter width. Thus, coefficient C_{s2} acts like a dynamic parameter which controls the spectral distribution towards which the turbulence is relaxing.

Practical formulations

Simplified formulations can be considered in practice, for example:

$$C_{s1} = 1.5, \qquad C_{s2} = 1.5 + 0.42 \frac{1}{1 + \beta \left(\dfrac{L_{RANS}}{L_{LES}} \right)^{2/3}},$$

$$L_{RANS} = \frac{k_{total}^{3/2}}{\varepsilon} = \frac{\left(k_{resolved} + k_s\right)^{3/2}}{\varepsilon}, \quad L_{LES} = \frac{1}{\kappa_c} = \frac{1}{\pi}\left(\delta x \delta y \delta z\right)^{1/3},$$

the ratio $\dfrac{L_{RANS}}{L_{LES}}$ being proportional to η_c. Equation [19.46] shows that $\beta \approx 2/3\chi \approx 0.44$. This approximation is only valid for sufficiently high values of η_c. Other approximations, likely to be valid on a wider range can be considered, for example:

$$H^* = \left[1 + \left(\frac{3\chi}{2}\right)^{-\frac{1}{\gamma}} \eta^{\frac{2}{3\gamma}} \right]^{-\gamma}, \qquad E^* = \frac{\left(\dfrac{3\chi}{2}\right)^{-\left(\frac{1}{\gamma}+1\right)} \gamma \eta^{\frac{2}{3\gamma}-1}}{\left[1 + \left(\dfrac{3\chi}{2}\right)^{-\frac{1}{\gamma}} \eta^{\frac{2}{3\gamma}} \right]^{\gamma+1}},$$

parameter γ is controlling the exponent in the power law at the origin of the spectrum ($1/3 \le \gamma \le 1/6$) and χ is the Kolmogorov constant.

Subgrid-scale viscosity

The subgrid-scale turbulence eddy viscosity can be estimated using the Heisenberg hypothesis (cf. [*HIN 75]):

$$\nu_{sgs} = C_{Kc} \int_{\kappa}^{\infty} \kappa^{-3/2} E(\kappa)^{1/2} d\kappa,$$

and assuming a Komogoroff spectrum $\nu_{sgs} = \dfrac{3}{4} C_{Kc} \chi^{1/2} \dfrac{k^2}{\varepsilon} \eta_c^{-4/3}$.

If we note that $\dfrac{k_s}{k} \approx \dfrac{3}{2} \chi \eta_c^{-2/3}$, then we find the equivalent formulation to use in the subgrid-scale model $\nu_{sgs} = \dfrac{1}{3} C_{Kc} \chi^{-3/2} \dfrac{k_s^2}{\varepsilon}$. In order to ensure the compatibility with the formulation of the statistical k-ε model when $\kappa \to 0$, the following expression emerges:

$$\nu_{sgs} = C_\mu \frac{k_s^2}{\varepsilon}, \quad C_\mu = 0.09. \tag{19.47}$$

Extension to non-homogenous flows

In non-homogenous flows the turbulent diffusion terms have to be accounted for in the model:

$$\begin{cases} \dfrac{dk_s}{dt} = F_c - \varepsilon + \mathscr{D}_{ksgs} \\[2mm] \dfrac{d\varepsilon}{dt} = C_{1s} \dfrac{F_c \varepsilon}{k_s} - C_{2s} \dfrac{\varepsilon^2}{k_s} + \mathscr{D}_{\varepsilon sgs} \end{cases}$$

with $\mathscr{D}_{ksgs} = \left(\dfrac{\nu_{sgs}}{\sigma_{ksgs}} (k_s)_{,j} \right)_{,j}$, $\mathscr{D}_{\varepsilon sgs} = \left(\dfrac{\nu_{sgs}}{\sigma_{\varepsilon sgs}} (\varepsilon)_{,j} \right)_{,j}$,

$$F_c = \nu_{sgs} \overline{U}_{i,j} \left(\overline{U}_{i,j} + \overline{U}_{j,i} \right).$$

In order to compare with the statistical k-ε model, we should proceed to the averaging of the previous equations. With this aim, we consider the decomposition

$U_j = \underbrace{\langle U_j \rangle + u'_{(e)j}}_{\overline{U}_j} + u'_{(i)j}$ which contains the statistical mean $\langle U_j \rangle$, the resolved

part of the fluctuation $u'_{(e)j}$ and the modeled part $u'_{(i)j}$. Also, $k_s = \langle k_s \rangle + k'$:

$$
\left\{
\begin{aligned}
& \frac{\partial \langle k_s \rangle}{\partial t} + \langle U_j \rangle \langle k_s \rangle_{,j} = \langle F_c \rangle - \langle \varepsilon \rangle + \left(\frac{\nu_{sgs}}{\sigma_{ksgs}} \langle k_s \rangle_{,j} \right)_{,j} - \left\langle u'_{(e)j} k' \right\rangle_{,j} \\
& \frac{\partial \langle \varepsilon \rangle}{\partial t} + \langle U_j \rangle \langle \varepsilon \rangle_{,j} = C_{1s} \frac{\langle F_c \rangle \langle \varepsilon \rangle}{\langle k_s \rangle} - C_{2s} \frac{\langle \varepsilon \rangle^2}{\langle k_s \rangle} + \left(\frac{\nu_{sgs}}{\sigma_{\varepsilon sgs}} \langle \varepsilon \rangle_{,j} \right)_{,j} - \left\langle u'_{(e)j} \varepsilon' \right\rangle_{,j}
\end{aligned}
\right.
$$

with approximations such as $\langle F_c \varepsilon \rangle \simeq \langle F_c \rangle \langle \varepsilon \rangle$. The equations clearly bring out the turbulent diffusion due to resolved scales and turbulent diffusion due to non-resolved scales. In the case of the logarithmic boundary layer, the dissipation equation yields:

$$
(C_{s2} - C_{s1}) \frac{\langle \varepsilon \rangle^2}{\langle k_s \rangle} = C_\mu \frac{k^2}{\sigma_{\varepsilon sgs}} \left(\frac{\langle \varepsilon \rangle_{,j}}{\langle \varepsilon \rangle} \right)_{,j}.
$$

Taking into account $(C_{s2} - C_{s1}) \dfrac{k}{k_s} = C_{\varepsilon 2} - C_{\varepsilon 1}$ and $\left(\dfrac{\langle \varepsilon \rangle_{,j}}{\langle \varepsilon \rangle} \right)_{,j} = \dfrac{1}{y^2}$, we find

$\sigma_{\varepsilon sgs} = 1.3$. For compatibility with the limiting statistical model, we must also have $\sigma_{ksgs} = 1.0$.

Extension to low Reynolds numbers

Low Reynolds number terms can be introduced in an empirical way by extending the formulation of the statistical model by Jones and Launder (cf. [JON 72]):

$$
\frac{d\tilde{\varepsilon}}{dt} = C_{1s} \frac{F_c \tilde{\varepsilon}}{k_s} + \left(\frac{\nu_{sgs}}{\sigma_{\varepsilon sgs}} (\tilde{\varepsilon})_{,j} \right)_{,j} - \left(C_{1s} + (C_{2s} - C_{1s}) f_{\varepsilon sgs} \right) \frac{\tilde{\varepsilon}^2}{k_s}
$$

$$
+ 2\nu \nu_{sgs} \left(\overline{U}_{j,mp} \right)^2,
$$

$$\varepsilon = \tilde{\varepsilon} + 2\nu \left(\sqrt{k_s} \right)_{,j}^{2}, \quad f_{\varepsilon sgs} = 1.0 - \exp\left(-R_{sgs}^{2} \right),$$

$$\nu_{sgs} = 0.09 \exp\left[\frac{-3.4}{\left(1 + R_{sgs} / 50 \right)^{2}} \right] \frac{k_s^{2}}{\tilde{\varepsilon}}.$$

The term $f_{\varepsilon sgs}$ is such that the limiting statistical model is practically recovered when $\kappa_c \to 0$.

Thus, the model varies continuously from the statistical closure for $\kappa_c = 0$ up to the DNS when $\kappa_c \to \infty$. However, when κ_c is very small, the use of turbulence eddy viscosity may become questionable and an account for "backscatter" appears necessary. On the contrary for $\kappa_c \to \infty$, $C_{s2} \to C_{s1}$, in practice, turbulence cannot maintain and it is generally not necessary to impose $C_\mu \to 0$. We will note that the model is of course no longer necessary for a calculation approaching the DNS.

From the theoretical point of view, the concept of tangent homogenous space [CHA 07] allows a natural extension to the study of non-homogenous turbulence. Integration over the whole tangent space is indeed identical to the statistical mean and moreover it is possible to use any value for the filter width, disconnected from the grid size.

This approach has been applied with success by Hanjalic K. *et al.*, 2004 [HAN 04] in their continuous hybrid approach (with "seamless coupling") and also by Kenjeres S. and Hanjalic K. [KEN 05].

Subgrid-scale PITM closure for the Reynolds stresses

The same method can be transposed to the case of a second order closure ([CHA 05A and B]). The subgrid-scale Reynolds stresses equation can be written in the form:

$$\frac{\partial R_{(s)ij}}{\partial t} + \overline{\overline{U}_h R_{(s)ij,h}} = P_{(s)ij} - \varepsilon_{(s)ij} + \varPhi_{(s)ij} + \mathscr{D}_{(s)ij},$$

with: $P_{(s)ij} = -R_{(s)im}\overline{U}_{j,m} - R_{(s)jm}\overline{U}_{i,m},$

$\varepsilon_{(s)ij} = 2\nu\overline{u'_{i,m}\, u'_{j,m}},$

$$\Phi_{(s)ij} = \overline{\frac{p'}{\rho}\left(u'_{i,j}+u'_{j,i}\right)},$$

$$\mathscr{D}_{(s)ij} = -\overline{\left(u'_i u'_j u'_m\right)}_{,m} + \overline{u'_j R_{(s)ik,k}} + \overline{u'_i R_{(s)jk,k}}$$

$$-\overline{\left(\frac{p'}{\rho}u'_j\right)}_{,i} - \overline{\left(\frac{p'}{\rho}u'_i\right)}_{,j} + \nu R_{(s)ij,mm}.$$

The closure hypotheses are chosen such that the statistical model by Launder B.E. and Shima N. (cf. section 13.6.3 and [LAU 89B]) is recovered at the limit $\kappa_c \to 0$. These closure hypotheses are:

$$\varepsilon_{(s)ij} = \frac{2}{3}\varepsilon\delta_{ij},$$

$$\Phi^{(1)}_{(s)ij} = -c_{sgs1}\frac{\varepsilon}{k_s}\left(R_{(s)ij} - \frac{2}{3}k_s\delta_{ij}\right),$$

$$\Phi^{(2)}_{(s)ij} = -c_{sgs2}\left(P_{(s)ij} - \frac{2}{3}P_s\delta_{ij}\right),$$

$$\Phi^{(w)}_{(s)ij} = -c^w_{sgs1}\frac{\varepsilon}{k_s}\left(R_{(s)kl}n_k n_l\delta_{ij} - \frac{3}{2}R_{(s)ki}n_k n_j - \frac{3}{2}R_{(s)kj}n_k n_i\right)f_w,$$

$$+c^w_{sgs2}\left(\Phi^{(2)}_{(s)kl}n_k n_l\delta_{ij} - \frac{3}{2}\Phi^{(2)}_{(s)ik}n_k n_j - \frac{3}{2}\Phi^{(2)}_{(s)jk}n_k n_i\right)f_w,$$

$$\mathscr{D}_{(s)ij} = \left(\nu R_{(s)ij,k} + c_s\frac{k_s R_{(s)kl}}{\varepsilon}R_{(s)ij,l}\right)_{,k}.$$

Coefficient c_{sgs1} is considered according to η_c in order to convey the physical idea of a return to isotropy faster in the range of high wavenumbers:

$$c_{sgs1} = \frac{1+\alpha_\eta\eta_c^2}{1+\eta_c^2}c_1, \quad \text{with} \quad \lim_{\eta_c\to 0}c_{sgs1}(\eta_c) = c_1,$$

$$c_1 = 1+2.58A.II^{1/4}\left\{1-\exp\left[-(R_t/150)^2\right]\right\},$$

$$c_{sgs2} = c_2 = 3/4A^{1/2}, \quad a_{(s)ij} = \left(R_{(s)ij} - \frac{2}{3}k_s\delta_{ij}\right)/k_s,$$

$$c^w_{sgs1} = c^w_1 = -\frac{2}{3}c_1 + \frac{5}{3}, \quad c^w_{sgs2} = c^w_2 = \max\left(\frac{2}{3}c_2 - \frac{1}{6},0\right),$$

$$f_w = 0.4 k_s^{3/2} / \varepsilon y \,.$$

The dissipation rate equation is similar to the one described previously (apart from the turbulent diffusion term which includes a tensorial diffusion coefficient). Thus, the following practical formulation can be used:

$$c_{sgs1} = \frac{1 + \alpha_{\mathscr{N}} \cdot \mathscr{N}^2}{1 + \mathscr{N}^2} c_1 \quad \text{and} \quad C_{s2} = C_{\varepsilon 1} + \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{1 + \beta \cdot \mathscr{N}^{2/3}} \,, \quad \mathscr{N} = \frac{L_{RANS}}{L_{LES}} \,.$$

Applications of this model have been made for the fully developed plane channel flow and in a channel with wall injection ([CHA 05B]).

The numerical methods used in this type of hybrid model for calculating the filtered velocities are similar to the methods used for standard LES. However, the numerical schemes for the subfilter transport equations may be different, often of lower order for improving stability (see Plate 20.40).

The "PANS" method

Among the non-zonal approaches also appears the method proposed by Girimaji S. and Abdol-Hamid K.S. [GIR 05] who introduce a "partial mean" or a filtering of the equations of motion, so-called "PANS" (partially averaged Navier-Stokes) relying on the fact that the form of the statistical equations is no longer dependent on the particular filter when these equations are expressed in terms of cumulants (cf. [GER 92]). From this fact, the subfilter stresses have the same characteristics as the statistical Reynolds stresses and the same modeling strategies can be used. The equations are then established by introducing modeled fractions in the basic functions of the model, i.e. $f_k = k_{SFS} / k$ and $f_\varepsilon = \varepsilon_{SFS} / \varepsilon$ in a two equation model ([GIR 05]).

This page intentionally left blank

Chapter 20

Synopsis on Numerical Methods

The turbulence models presented previously are intended to be implemented in computational fluid dynamic codes that allow the numerical prediction of turbulent flows. If the turbulence modeling aspect is in principle relatively distinct from the numerical technique used for solving the system of equations, specific numerical techniques have been developed to suit one point statistical modeling, and others for numerical simulations of turbulence.

The favorite techniques in one point modeling are finite volume methods, finite difference or finite elements. Finite volume methods in particular have been extensively used considering their robustness for solving equations with complex source terms and also considering their global conservative properties for the numerical schemes.

In the case of direct numerical simulations or large eddy simulations of turbulence, highly accurate methods are necessary, they must also satisfy good conservation properties (momentum, energy, etc. for their in viscid form) and good behavior over long periods of integration.

We shall give in this present chapter, a very schematic overview of the most usual numerical techniques for solving the governing equations of the flow numerically. Generally, the equations to solve have the form of a convection-diffusion equation with source and sink terms:

$$\frac{\partial}{\partial t}\left(\rho\Phi\right)+\left(\rho u_j\Phi\right)_{,j}=\left(\sigma_\Phi\Phi_{,j}\right)_{,j}+\rho S_\Phi \qquad [20.1]$$

which express the balance in a control volume:

> *time variation + convective transport = diffusion*
> *+ volumic source/sink terms and surfacic effects*   [20.2]

20.1. Numerical techniques

Numerical techniques will be only summed up in recapitulative plates; the interested reader will find a detailed presentation in the numerical analysis books cited in the bibliography.

The discretization methods allow us to replace the continuous partial differential equations by numerical equations. Among the oldest methods, the finite difference methods are generally based on the approximation of a function by its Taylor expansion about a discretization point (plate 20.1). More accurate higher order approximations are obtained either from an increase of the number of discretization points in the expression of derivatives or from using implicit relations (Hermitian schemes, plate 20.2). Polynomial interpolation is sometimes used, but it is still necessary to use Taylor expansions for determining the truncation error. Uncentered approximation schemes (plate 20.3) generally have lesser order of approximation but may have stabilizing properties (upwind scheme). Nevertheless they present the drawback of introducing false diffusion whose origin is purely numerical and which must be controlled if we want the error thus introduced to remain limited. Time integration schemes generally involve two or three levels, implicit schemes are generally more stable than explicit schemes. Let us also mention the prediction-correction schemes (plate 20.5). In the case of complex differential operators, the fractional step method ("splitting") allows us to discretize successively the q operators on a fraction $\delta t/q$ of the time step (plate 20.6). A particularly important application of fractional steps is made in the alternate direction schemes (ADI) used for solving the convection-diffusion equations in several variables. Multigrid methods are at present developing, they are using discretization grids in space ranging from very refined meshes down to coarser and coarser meshes and the numerical solutions are transferred from one of these different meshes to another in order to speed up convergence. Semi-Lagrangian methods may be interesting and useful for treating advective terms ([XU 01]).

The fundamental theoretical bases of discretization methods can be established on the three concepts that are stability, consistency and convergence (plate 20.8). The Lax theorem for linear systems (which can be extended to quasi-linear systems) gives a connection between these properties. The usual method for analyzing the properties of a numerical scheme is the Neumann method based of Fourier series

expansions (plate 20.10). These expansions also introduce the concepts of dissipation and dispersion of a numerical scheme.

Among the techniques for approximating the solutions, the weighted residual methods in their different forms remain the more general approach. The spectral methods and the finite element methods for example are particular cases of this approach.

The calculation of flows in complex geometries can be tackled either by using finite element methods, or by using finite difference or finite volume methods on curvilinear meshes (Amsden A.A. and Hirt C.W., [AMS 73]; Thompson J.F. *et al.*, [THO 74]).

Specific finite difference methods have been developed for solving the momentum equations in primitive variables. For incompressible flows, the main difficulty lies in the pressure-velocity coupling through the continuity equation. One of the oldest methods in steady flows is the artificial compressibility method which introduces a pressure equation with a fictitious evolution term. Projection methods and also the MAC methods (Welch J.E. *et al.*, [WEL 66]), SMAC methods (Amsden A.A. and Harlow F.H., [AMS 70]) and related methods use a Poisson equation for pressure and can be applied in unsteady flows. Compressible flows can be tackled using specific methods (Peyret R. and Viviand H., [PEY 75]). More accurate Hermitian schemes can also be introduced in the previous methods (Bontoux P., [BON 78]). The more recent development of spectral methods (Fourier series expansions and Tchebycheff polynomial expansions) has led to efficient highly accurate methods. The spectral elements technique allows us to extend these methods to more complex geometries other than the parallelepipedic domain. Finite volume methods are based on the concept of integration of the equations on the numerical discretization cell. They are often limited to low orders (order one or two) but the numerical schemes are chosen in order to yield approximations that remain physically realistic even on very coarse meshes (Gosman A.D. *et al.*, [GOS 69]; Gosman A.D. *et al.*, [GOS 77]; Patankar S.V., 1980). Let us note that this condition is not necessarily fulfilled in finite difference methods, even for high order methods: a finite difference method reputed to be highly accurate may become poor or even unusable on a grid which is too coarse. Several algorithms for pressure-velocity coupling have been developed in finite volume methods (the SIMPLE algorithm and its related algorithms such as SIMPLER, SIMPLEC, PISO, etc.), which are inspired by finite difference methods and use them as staggered grids for the velocity components. Staggered grids allow us to eliminate the parasitic modes in the pressure fields. However, the SIMPLE algorithm can be extended to the case of a mesh with collocated variables (Rhie C.M. and Chow W.L., [RHI 83]). Hybrid convection-diffusion schemes have been largely used in practice because they allow a good stability while reducing the decentering. In order to improve the accuracy,

Leonard B.P., [LEO 79] introduced the QUICK scheme based on a quadratic interpolation with decentering. Another method has been developed by Leonard B.P., [LEO 88] ((SHARP) Simple High Accuracy Resolution Program) inspired by the limiter method (like the TVD methods), it allows the use of more accurate higher order schemes by removing overshoots.

The previous methods, mainly developed for incompressible flows, can be extended to the case of weakly compressible flows (cf. Patankar S.V., 1980; Ferziger J.H. and Peric, 1996). In the case of highly compressible flows, it is necessary to make use of specific methods. we can mention in particular the MacCormack methods (1969), the Beam and Warming methods (1978), and also the methods with limiters (Total Variation Diminishing (TVD) schemes) and the methods of multigrid acceleration. We have to mention the various high order reconstruction schemes allowing us to interpolate a function from its mean values in each finite volume cell, among them the MUSCL (Monotone Upstream Scheme for Conservation Laws) scheme which is widely used in numerical fluid mechanics ([VLE 79]).

The thin shear flow approximation that can be applied to usual boundary layer flows on a wall and also to flows such as jets and wakes, leads to parabolic equations which can be solved using specific simplified techniques (Patankar S.V. and Spalding D.B., 1970; Jeandel D. and Mathieu J., [JEA 72B]). The method by Patankar S.V. and Spalding D.B., [PAT 72] can be applied to three-dimensional flows that are parabolic in one direction of space and elliptic in the cross-section.

In numerical simulations of turbulence, whether it is a DNS or a LES, highly accurate methods with good conservation properties, are generally necessary. Spectral methods and pseudo-spectral methods (Orszag S.A., [ORS 69]) are particularly appropriate in the homogenous directions in space with periodic boundary conditions. In the non-homogenous directions (for example in the presence of a wall), we can use high precision finite difference methods such as the Hermitian methods (cf. Lele S.K., [LEL 92]), or at least second order methods with centered schemes. Spectral methods in Tchebycheff expansions can also be used. The time integration schemes will also be chosen considering their precision and free from numerical dissipation.

20.2. Plates

The remainder of this, intentionally schematic, chapter will be limited to some short indications serving as a guide for practical implementation of models. The reader can refer to basic books and articles on numerical methods, some of which are Fletcher C.A.J., [*FLE 88]; Patankar S.V., [*PAT 80]; Peyret R. and Taylor

T.D., [*PEY 83]; Richtmyer R.D. and Morton K.W., [*RIC 67]; Roache P., [*ROA 72]; Zienkiewicz O., [*ZIE 71]; Patankar S.V. and Spalding D.B., [*PAT 70], and also [PAT 67]; Dhatt G. and Touzot G., [*DHA 81]; Chapman D.R., [CHA 79]; Fortin M., Peyret R. and Temam R., [FOR 71]; Leschziner M.A., [LES 89]; Leschziner M.A. and Drikakis D. [LES 02].

Plate 20.1. *Diagram on discretization numerical techniques*
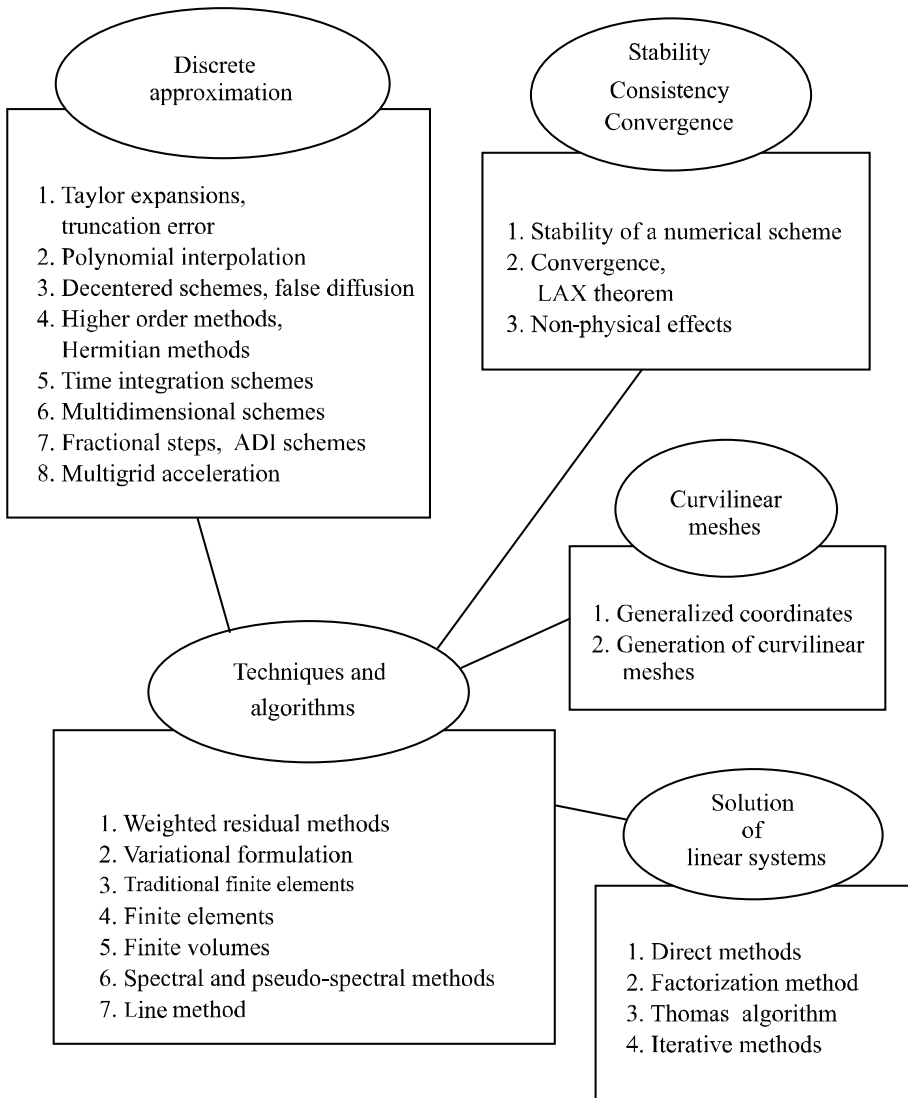
Plate 20.2. *Diagram on methods for solving the momentum equations of fluid flow*
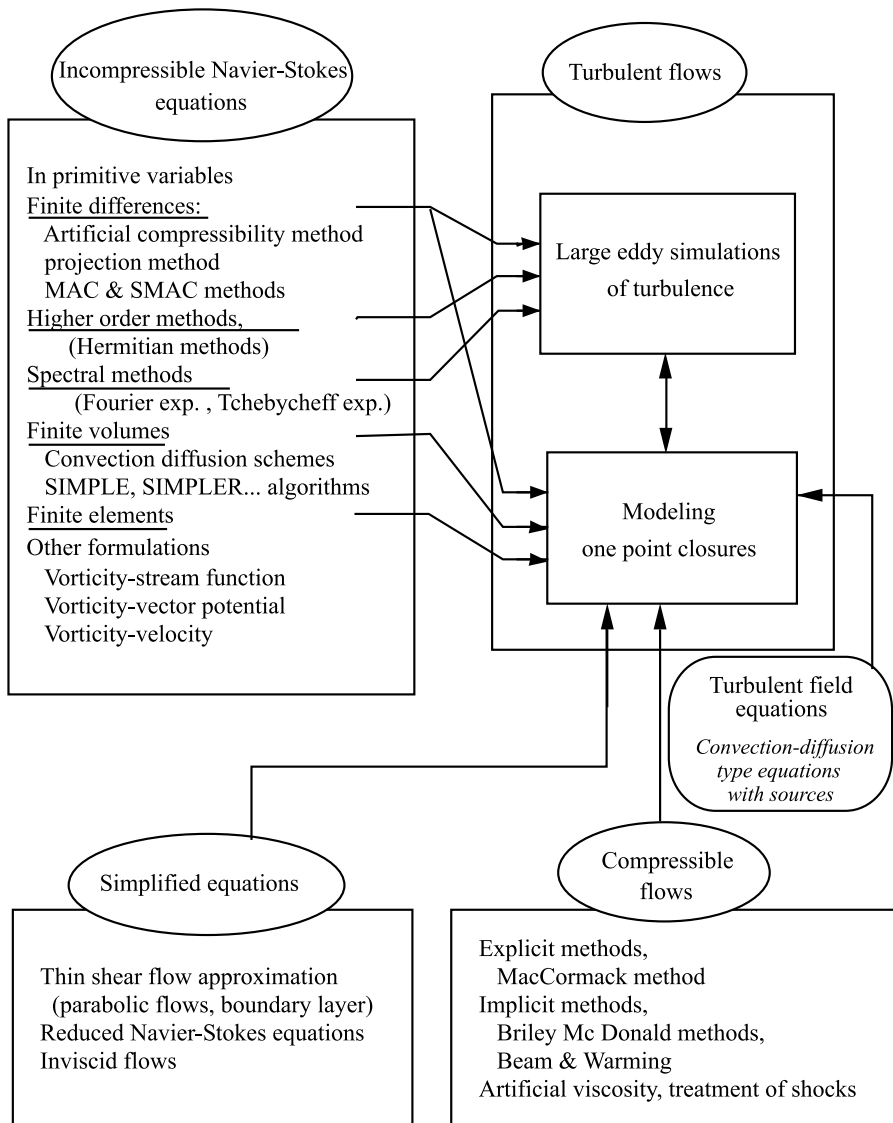
Plate 20.3. *Discrete approximation*

One-dimensional schemes

$$U_{i-1} \qquad U_i \qquad U_{i+1} \qquad U_{i+2}$$

$$h$$
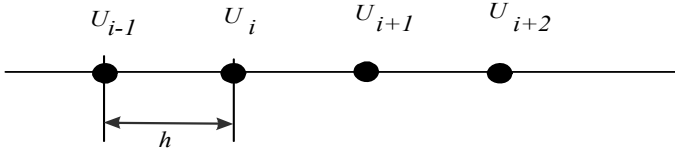
$$\frac{du}{dx}(x = x_i) = \left(u_x\right)_i + \mathscr{O}(h^n), \qquad \frac{d^2u}{dx^2}(x = x_i) = \left(u_{xx}\right)_i + \mathscr{O}(h^p)$$

Discretizing using Taylor expansions

$$\begin{cases} u_{i+1} = u_i + h\left(\dfrac{du}{dx}\right)_i + \dfrac{h^2}{2!}\left(\dfrac{d^2u}{dx^2}\right)_i + \mathscr{O}(h^3) \\ u_{i-1} = u_i - h\left(\dfrac{du}{dx}\right)_i + \dfrac{h^2}{2!}\left(\dfrac{d^2u}{dx^2}\right)_i + \mathscr{O}(h^3) \end{cases} \Rightarrow \left(\dfrac{du}{dx}\right)_i = \underbrace{\dfrac{u_{i+1} - u_{i-1}}{2h}}_{u_{xi}} + \underbrace{\mathscr{O}(h^2)}_{\substack{\text{truncation} \\ \text{error}}}$$

Examples:

$$\left(\frac{du}{dx}\right)_i = \frac{1}{12h}(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}) + \mathscr{O}(h^4)$$

$$\left(\frac{d^2u}{dx^2}\right)_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \mathscr{O}(h^2)$$

$$\left(\frac{d^2u}{dx^2}\right)_i = \frac{1}{12h^2}(-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}) + \mathscr{O}(h^4)$$

Uncentered schemes

"Backward" scheme and "forward" scheme

$$\left(\frac{du}{dx}\right)_i = \frac{u_{i+1} - u_i}{h} + \mathscr{O}(h), \qquad \left(\frac{du}{dx}\right)_i = \frac{u_i - u_{i-1}}{h} + \mathscr{O}(h)$$

$$\left(\frac{du}{dx}\right)_i = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2h} + \mathscr{O}(h^2)$$

Plate 20.4. *Discrete approximation in several dimensions*

Taylor expansions in several variables are used:

$$u(M_1) = u(M_0) + \delta_{M_0} + \frac{1}{2!}\delta^{(2)}{}_{M_0} + ... \qquad \delta \equiv h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y} + l\frac{\partial u}{\partial z}$$

Examples of a calculation "molecule":

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{ij} = \frac{1}{4h^2}\left\{\begin{array}{ccc} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{array}\right\} + \mathcal{O}(h^2)$$

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{ij} = \frac{1}{h^2}\left\{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{array}\right\} + \mathcal{O}(h^2)$$

Plate 20.5. *Polynomial interpolation*
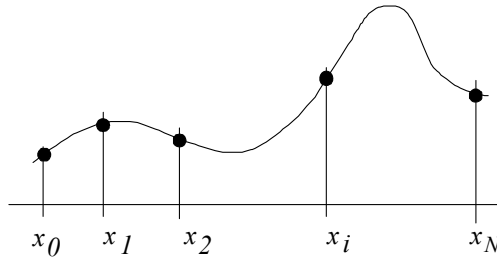
$$P(x_i) = u_i, \qquad P(x) = \sum_{i=0}^{N} L_i(x)u_i,$$

$$L_i(x) = \frac{\prod_{j}(x - x_j)}{\prod_{\substack{j=0...N \\ j \neq i}}(x_i - x_j)} \quad (*)$$

(*) Lagrange polynomial of degree N

Plate 20.6. *False diffusion*

Behavior in one dimension
Truncation error that behaves like numerical diffusion

$$\frac{\partial \Phi}{\partial t} + A \frac{\partial \Phi}{\partial x} = 0, \quad A > 0$$

Decentered discretization "upwind":

$$\frac{\Phi_i^{n+1} - \Phi_i^n}{\delta t} + A \frac{\Phi_i^n - \Phi_{i-1}^n}{h} = 0, \quad \mathcal{E} = \mathcal{O}(\delta t) + \mathcal{O}(h)$$

$$\Rightarrow \underbrace{\frac{\Phi_i^{n+1} - \Phi_i^n}{\delta t} + A \frac{\Phi_{i+1}^n - \Phi_{i-1}^n}{2h}}_{\mathcal{E} = \mathcal{O}(\delta t) + \mathcal{O}(h^2)} = \frac{Ah}{2} \frac{\Phi_{i+1}^n + \Phi_{i-1}^n - 2\Phi_i^n}{h^2}$$

$Ah/2$ artificial viscosity coefficient
Equivalent to second order discretization of the equation:

$$\frac{\partial \Phi}{\partial t} + A \frac{\partial \Phi}{\partial x} = \frac{Ah}{2} \frac{\partial^2 \Phi}{\partial x^2}$$

Plate 20.7. *Higher order methods, Hermitian relations*
(cf. [LEL 92], [BON 78], [*PEY 83])

Closure is achieved using implicit relations:

$$H_i = \sum_{j=-1}^{+1} \left(a_j f_{i+j} + b_j f'_{i+j} + c_j f''_{i+j} \right) = 0$$

Identifying the Taylor expansion at point (i) allows us to express the coefficients according to 5 parameters.

Symbolic notation: $H_i [\alpha, \beta, \gamma, \rho, \theta] = 0$

Examples:

$$f'_{i+1} + 4 f'_i + f'_{i-1} = \frac{3}{h} (f_{i+1} - f_{i-1}) + \mathcal{O}(h^4)$$

$$f''_{i+1} + 10 f''_i + f''_{i-1} = \frac{12}{h^2} (f_{i+1} - 2f_i + f_{i-1}) + \mathcal{O}(h^4)$$

$$f''_i = 2 \frac{f_{i+1} + f_{i-1} - 2f_i}{h^2} - \frac{f'_{i+1} - f'_{i-1}}{2h} + \mathcal{O}(h^4)$$

Relation at the boundary: $2h \left(2 f'_2 + f'_1 \right) = f_3 + 4f_2 - 5f_1 + \mathcal{O}(h^3)$

Plate 20.8. *Time schemes*

$$\frac{\partial u}{\partial t} = G(u,x,y,t)$$

in G all the spatial derivatives have been replaced by finite differences

$$u_j^{n+1} = u_j^{n-p} + \int_{(n-p)\delta t}^{(n+1)\delta t} G\, dt$$

Approximate integration using:

$$u_j^{n-p} + \delta t\left(a_0 G_j^{n+1} + a_1 G_j^n + \ldots + a_{q+1} G_j^{n-q}\right), \qquad q \geq p$$

if $a_0 = 0$ explicit scheme (extrapolation)

if $a_0 \neq 0$ implicit scheme (interpolation)

Usual example:

forwards $\qquad\qquad\qquad u_j^{n+1} = u_j^n + \delta t.G_j^n$

backwards $\qquad\qquad\qquad u_j^{n+1} = u_j^n + \delta t.G_j^{n+1}$

trapezoidal $\qquad\qquad\qquad u_j^{n+1} = u_j^n + \frac{\delta t}{2}.\left(G_j^n + G_j^{n+1}\right)$

leap-frog $\qquad\qquad\qquad u_j^{n+1} = u_j^{n-1} + 2\delta t.G_j^{n+1}$

Adam-Bashforth $\qquad\qquad u_j^{n+1} = u_j^n + \delta t.\left(\frac{3}{2}G_j^n - \frac{1}{2}G_j^{n-1}\right)$

Plate 20.9. *Predictor-corrector schemes*

u_j^* is obtained from the explicit method and is then used as an initial value for the implicit method.

Example: Matsuno scheme

$$u_j^* = u_j^n + \delta t.G_j^n$$
$$u_j^{n+1} = u_j^n + \frac{\delta t}{2}.\left[G_j^n + G_j^{n+1}(u_j^*)\right]$$

Plate 20.10. *Time schemes, two level scheme*

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = u_{XXj}^n + u_{YYj}^n$$

Example:
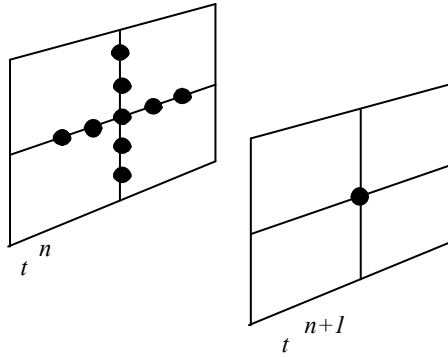
accuracy $\mathcal{O}(h^4) + \mathcal{O}(\delta t)$

Two level implicit scheme:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = \frac{1}{2}\left(u_{XXj}^n + u_{YYj}^n + u_{XXj}^{n+1} + u_{YYj}^{n+1}\right)$$

Example:
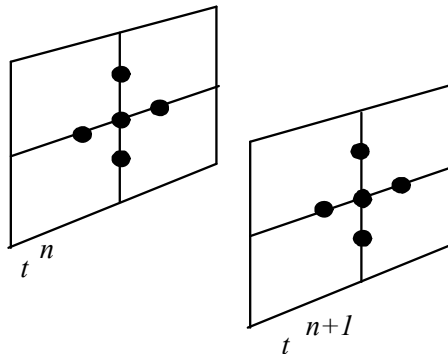
accuracy $\mathcal{O}(h^2) + \mathcal{O}(\delta t^2)$

Plate 20.11. *Fractional step method*

$$\frac{\partial u}{\partial t} = G, \quad G = G_1 + G_2 + ... + G_q$$

G is successively replaced by qG_1, qG_2, etc. during a fraction $\delta t/Q$ of the time step δt. Thus, it is possible, for example, to replace a complex multidimensional problem with a succession of simpler one-dimensional problems.

$$\begin{cases} \dfrac{\partial u}{\partial t}_{(\delta t/q)} = q.G_1 \\[2mm] \dfrac{\partial u}{\partial t}_{(\delta t/q)} = q.G_2 \\[2mm] \text{........................} \\[2mm] \dfrac{\partial u}{\partial t}_{(\delta t/q)} = q.G_q \end{cases}$$

Plate 20.12. *ADI method*

$$\frac{u_j^* - u_j^n}{\delta t/2} = u_{XXj}^n + u_{YYj}^*, \qquad \frac{u_j^{n+1} - u_j^*}{\delta t/2} = u_{XXj}^{n+1} + u_{YYj}^*$$

Example:
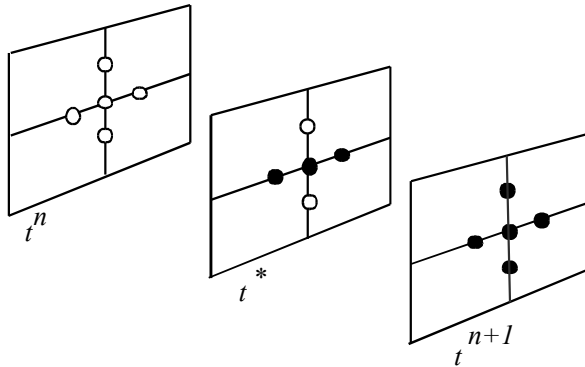
$$B_{Yij} u_{i\,j+1}^* + A_{Yij} u_{i\,j}^* + C_{Yij} u_{i\,j-1}^* = D_{Yij}$$

$$B_{Yij} u_{i+1\,j}^{n+1} + A_{Yij} u_{i\,j}^{n+1} + C_{Yij} u_{i-1\,j}^{n+1} = D_{Yij}$$

Plate 20.13. *Stability and convergence*

$\mathscr{L}$ linear partial differential operator
$\mathscr{L}_d$ approximate discrete operator
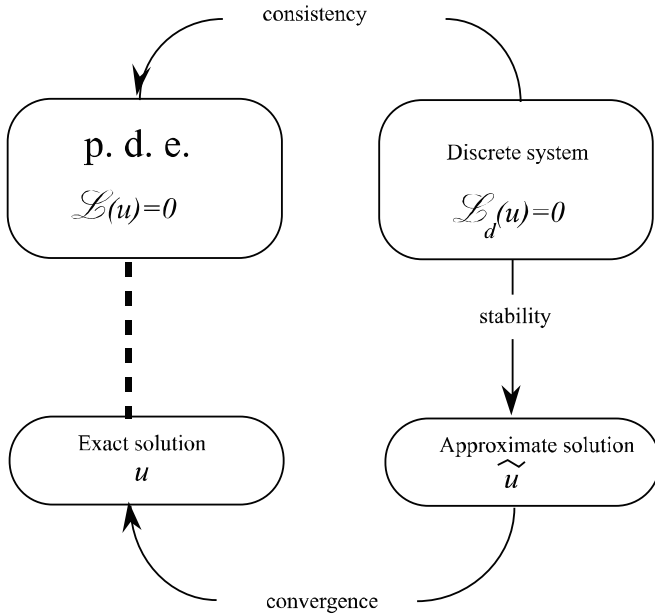
Consistency: $\left\| \mathscr{L}_d u - \mathscr{L} u \right\| \to 0 \; dt \to 0 \, , h \to 0$

Truncation error: $\left\| \mathscr{L}_d u - \mathscr{L} u \right\| = \mathscr{O}(\delta t^p , h^q ...)$

Stability: $u^{n+1} = \mathscr{C}^n (\delta t, h, ...) u^n$

$\mathscr{C}^n (\delta t, h, ...)$ remains bounded for $0 < \delta t < \tau$, $0 \le n\delta t \le T$

Convergence: $\left\| \mathscr{C}^n (\delta t, h, ...) u^0 - u \right\| \to 0 \; n \to +\infty \, , \delta t \to 0 \begin{cases} 0 \le t \le T \\ T = n.\delta t \end{cases} T$ fixed

LAX theorem: Consistency + Stability = Convergence

Plate 20.14. *Stability analysis, Von Neumann method*

The Von Neumann method studies the amplification of a perturbation expressed as a Fourier series.

$$U_{i-1} \qquad U_i \qquad U_{i+1} \qquad U_{i+2}$$

$$\delta x$$
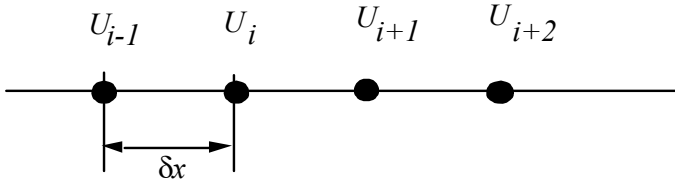
$$u_j^n = U^n.e^{i\,kj\delta x} \Rightarrow U^{n+1} = G.U^n$$

$$\overbrace{u^{n+1} = G(\delta t, \delta x,...)u^n} \quad \text{with} \quad i = \sqrt{-1}$$

U^n is the amplitude at the instant n of the particular component whose wavenumber is k. δx is the discretization step.

Neumann stability condition: $\lambda_{\max}(G) \leq 1$
where $\lambda_{\max}(G)$ is the maximum eigenvalue

In the case of one dimension: $\lambda \leq 1$, λ amplification factor

Plate 20.15. *Non-physical effects*

Dissipation and dispersion of a numerical scheme
$$U^{n+1} = G.U^n, \qquad G = r.e^{i.\delta\varphi}$$

If $|r| = 1$ non-dissipative scheme
(no decay of the amplitude).

If $C = \dfrac{\delta\varphi}{k.\delta t}$ is independent of k, the scheme is non-dispersive
(the wave celerity is identical at all wavenumbers).

Truncation errors
Example of numerical diffusion.

Plate 20.16. *The finite volume concept*

Finite volume cell $\Delta\Omega$
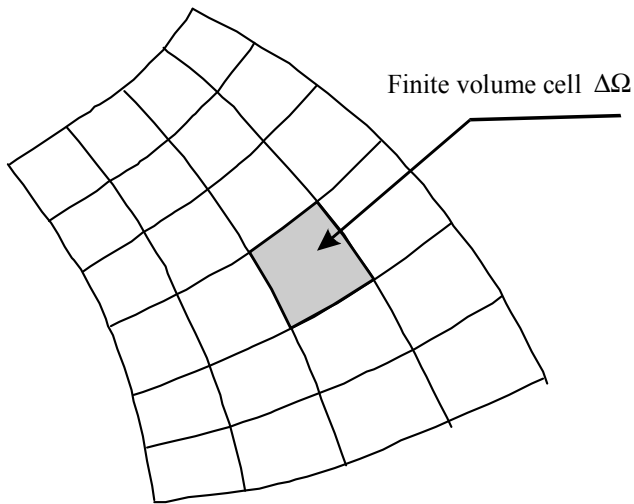
1) Integration of the laws of mechanics over each cell

$$\begin{cases} \text{Equation} \quad \mathscr{L}(\varPhi) = 0 \\ + \text{Boundary conditions} \end{cases}$$
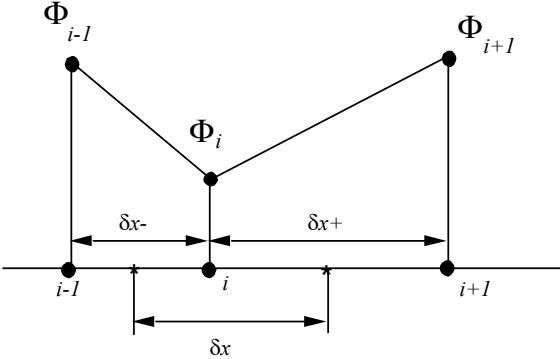
$$\mathscr{L}^*[\varPhi] = \iiint_{\Delta\Omega} \mathscr{L}[\varPhi]\,d\Omega$$

2) Example for the convective terms

When written in the form of a flux on the frontier, the conservation properties are directly satisfied.

$$\iiint_{\Delta\Omega} \vec{U}\vec{\nabla}\varPhi\,d\Omega = \oiint_{\partial\Omega} \vec{U}\varPhi\vec{n}\,dS$$

Plate 20.17. *Finite volume discretization: basic principles*

$$\frac{d}{dx}\left(\sigma\frac{d\Phi}{dx}\right) + S = 0$$

$$\left(\sigma\frac{d\Phi}{dx}\right)_+ - \left(\sigma\frac{d\Phi}{dx}\right)_- + \int_-^+ S.dx = 0$$

$$\sigma_+\frac{\Phi_{i+1}-\Phi_i}{\delta x_+} - \sigma_-\frac{\Phi_i-\Phi_{i-1}}{\delta x_-} + \overline{S}.\delta x = 0$$

$$a_i\Phi_i = a_{i+1}\Phi_{i+1} + a_{i-1}\Phi_{i-1} + b$$

General case: $a_i\Phi_i = \displaystyle\sum_{\text{neighboring points}} a_p\Phi_p + b$

4 basic rules (S.V. Patankar):

1- Compatibility at the boundaries of control volumes

2- Positive coefficients a_i

3- Linearization of source terms with negative slope
$$S = S_C + S_P\Phi, \quad S_C = \max(S,0), \quad S_P = \min(S,0)/\Phi$$

4- $a_i = \displaystyle\sum_{\text{neighboring points}} a_p$

Plate 20.18. *Convection-diffusion problem: centered scheme*

$$\begin{cases} \dfrac{d}{dx}(\rho u \Phi) = \dfrac{d}{dx}\left(\sigma \dfrac{d\Phi}{dx}\right) \\[2mm] \dfrac{d}{dx}(\rho u) = 0 \end{cases}$$

$$(\rho u \Phi)_+ - (\rho u \Phi)_- = \left(\sigma \dfrac{d\Phi}{dx}\right)_+ - \left(\sigma \dfrac{d\Phi}{dx}\right)_-$$

$$\Phi_+ = \dfrac{\Phi_{i+1} + \Phi_i}{2} \ , \ \Phi_- = \dfrac{\Phi_{i-1} + \Phi_i}{2}$$

$$\tfrac{1}{2}(\rho u)_+ (\Phi_{i+1} + \Phi_i) - \tfrac{1}{2}(\rho u)_- (\Phi_{i-1} + \Phi_i) = \sigma_+ \dfrac{(\Phi_{i+1} - \Phi_i)}{\delta x_+} - \sigma_- \dfrac{(\Phi_i - \Phi_{i-1})}{\delta x_-}$$

$$F = \rho u \ , \ D = \sigma / \delta x$$

$$a_i \Phi_i = a_{i+1} \Phi_{i+1} + a_{i-1} \Phi_{i-1}$$

$$a_{i+1} = D_+ - F_+ / 2$$

$$a_{i-1} = D_- + F_- / 2$$

$$a_i = a_{i+1} + a_{i-1} \text{ since } F_+ - F_- = 0$$

Plate 20.19. *Convection-diffusion problem: upwind scheme*

$$(F\Phi)_+ - (F\Phi)_- = \left(\sigma \dfrac{d\Phi}{dx}\right)_+ - \left(\sigma \dfrac{d\Phi}{dx}\right)_-$$

$$\Phi_+ = \Phi_i \text{ if } F_+ \geq 0$$

$$\Phi_+ = \Phi_{i+1} \text{ if } F_+ < 0$$

$$F_+ \Phi_+ = \Phi_i \left[\!\left[F_+, 0 \right]\!\right] - \Phi_{i+1} \left[\!\left[-F_+, 0 \right]\!\right]$$

$$F_- \Phi_- = -\Phi_i \left[\!\left[-F_-, 0 \right]\!\right] + \Phi_{i-1} \left[\!\left[F_-, 0 \right]\!\right]$$

where $\left[\!\left[a, b \right]\!\right]$ denotes the maximum of *a* and *b*

$$F = \rho u \ , \ D = \sigma / \delta x$$

$$a_i \Phi_i = a_{i+1} \Phi_{i+1} + a_{i-1} \Phi_{i-1}$$

$$a_{i+1} = D_+ + \left[\!\left[-F_+, 0 \right]\!\right] , \ a_{i-1} = D_- + \left[\!\left[F_-, 0 \right]\!\right]$$

$$a_i = a_{i+1} + a_{i-1} \text{ since } \left[\!\left[F, 0 \right]\!\right] - \left[\!\left[-F, 0 \right]\!\right] \equiv F$$

Plate 20.20. *Convection-diffusion problem*

Exact solution of the 1D problem and exponential scheme

$$
\left\{
\begin{aligned}
&\frac{d}{dx}(\rho u \Phi) = \frac{d}{dx}\left(\sigma \frac{d\Phi}{dx}\right) \\
&\quad\frac{d}{dx}(\rho u) = 0
\end{aligned}
\right.
$$

$$
\frac{F - F_i}{F_{i+1} - F_i} = \frac{\exp(Px/\delta x_+) - 1}{\exp(P) - 1} \,, \quad P = \frac{F}{D}
$$

$$
F = \rho u \,, \quad D = \sigma/\delta x
$$

$$
a_i \Phi_i = a_{i+1}\Phi_{i+1} + a_{i-1}\Phi_{i-1}
$$

$$
a_{i+1} = \frac{F_+}{e^{F_+/D_+} - 1} \,, \quad a_{i-1} = \frac{F_- e^{F_-/D_-}}{e^{F_-/D_-} - 1}
$$

$$
a_i = a_{i+1} + a_{i-1}
$$

Plate 20.21. *Convection-diffusion problem*

Hybrid scheme

$$
P_+ \leq -2 \quad a_{i+1} = -F_+
$$

$$
-2 \leq P_+ \leq +2 \quad a_{i+1} = D_+ - F_+/2
$$

$$
P_+ \geq 2 \quad a_{i+1} = 0
$$

$$
F = \rho u \,, \quad D = \sigma/\delta x
$$

$$
a_i \Phi_i = a_{i+1}\Phi_{i+1} + a_{i-1}\Phi_{i-1}
$$

$$
a_{i+1} = \llbracket -F_+, D_+ - F_+/2, 0 \rrbracket
$$

$$
a_{i-1} = \llbracket F_-, D_- + F_-/2, 0 \rrbracket
$$

$$
a_i = a_{i+1} + a_{i-1}
$$

Plate 20.22. *Convection-diffusion problem*

General formulation (Patankar S.V., 1980)

$$J = \rho u \Phi - \sigma d\Phi / dx$$

$$dJ / dx = 0, \qquad J_+ - J_- = 0$$

$$J(x) = F.\Phi - D \frac{d\Phi}{d(x/\delta x)}, \qquad \Phi_+ = \alpha \Phi_i + (1-\alpha)\Phi_{i+1}$$

$$\left[\frac{d\Phi}{d(x/\delta x)} \right]_+ = \beta(\Phi_{i+1} - \Phi_i), \qquad \alpha, \beta(P)$$

$$\frac{J_+}{D} = \underbrace{\left(P\alpha + \beta\right)}_{B(P)} \Phi_i - \underbrace{\left(P\alpha + \beta - P\right)}_{A(P)} \Phi_{i+1}$$

Properties:

$$B = A + P \qquad A(-P) = B(P) \qquad B(-P) = A(P)$$

$$\text{If} \qquad P > 0 \qquad A(P) = A(|P|)$$

$$\text{If} \qquad P < 0 \qquad A(P) = B(P) - P = A(|P|) - P$$

$$\forall P, \quad A(P) = A(|P|) + [\![-P, 0]\!] \quad \text{and} \quad B(P) = A(|P|) + [\![+P, 0]\!]$$

$$F = \rho u \qquad D = \sigma / \delta x \qquad P = F / D$$

$$a_i \Phi_i = a_{i+1}\Phi_{i+1} + a_{i-1}\Phi_{i-1}$$

$$a_{i+1} = D_+ A(|P_+|) + [\![-F_+, 0]\!], \qquad a_{i-1} = D_- A(|P_-|) + [\![F_-, 0]\!]$$

$$a_i = a_{i+1} + a_{i-1}$$

Examples of schemes: $\qquad\qquad\qquad A(|P|)$

Centered:	$1 - 0.5|P|$
Upwind:	1
Hybrid:	$[\![0, 1 - 0.5|P|]\!]$
Exponential:	$|P|/(e^{|P|} - 1)$
Power law:	$[\![0, (1 - 0.1|P|)^5]\!]$

The numerical diffusion problem:

Other schemes: $\qquad\qquad\qquad$ QUICK (Quadratic Upwind)

Plate 20.23. *Incompressible flow*

Navier-Stokes equations in primitive variables

$$\begin{cases} \dfrac{\partial V}{\partial t} + C(V) + \nabla p = \nu \nabla^2 V \\ \qquad\quad \nabla V = 0 \end{cases}$$

$C(V) = \nabla.(VV)$ or $C(V) = (V.\nabla)V$

Typical problem: V, p? in Ω,

V given on $\partial \Omega$ and $V = V_0$ given on Ω at time $t = 0$

$V_{\partial \Omega}$ must satisfy $\int_{\partial \Omega} V_{\partial \Omega} N.dS = 0$ and V_0 must satisfy $\nabla.V_0 = 0$

Plate 20.24. *Incompressible flows*

Artificial compressibility method (Chorin)

$$\begin{cases} C(V) + \nabla p = \nu \nabla^2 V \\ \quad\ \nabla V = 0 \end{cases}$$

which is a pseudo-unsteady method

$$\begin{cases} \dfrac{\partial V}{\partial t} + C(V) + \nabla p = \nu \nabla^2 V \\ \qquad \dfrac{\partial p}{\partial t} + c^2 \nabla V = 0 \end{cases}$$

Interpretation: behavior law $p = c^2 \rho$

(c dimensioned numerical constant)

<center>**Plate 20.25.** *Artificial compressibility method*</center>

Staggered grids
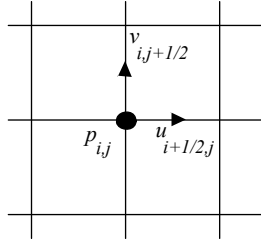
The equation for u is discretized at point $(i+1/2, j)$, and the pressure gradient is obtained from $\left(p_{i+1\,j} - p_{i\,j} \right)/\delta x$.

Likewise, the equation for v is discretized at point $(i,j+1/2)$ and the pressure gradient is obtained from $\left(p_{i\,j+1} - p_{i\,j} \right)/\delta y$

Boundary conditions
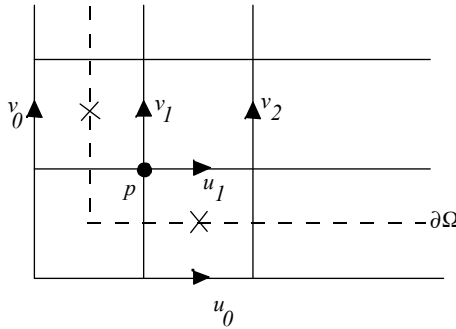
p is not defined on $\partial\Omega$ and there is no condition to specify conditions for the velocities:

    – reflection technique $v_0 = 2v_{\partial\Omega} - v_1$

    – more elaborate technique (Peyret R. and Taylor T.D., 1983)

$$v_0 = \frac{1}{3}(v_2 - 6v_1 + 8v_{\partial\Omega}) \text{ for calculating } \partial v/\partial x$$

$$v_0 = -\frac{1}{5}(v_3 - 5v_2 + 15v_1 - 16v_{\partial\Omega}) \text{ for calculating } \partial^2 v/\partial x^2$$

Plate 20.26. *Unsteady Navier-Stokes equations*

Prototypes
MAC method (Harlow and Welsh, [HAR 65], [WEL 66])
Projection method (Chorin, [CHO 67])

Projection method
(cf. Peyret R. and Taylor T.D., 1983)
Based on an iterative scheme similar to the fractional step method

$$\frac{V^* - V^n}{\delta t} + C(V^n) - \nu \nabla^2 V^n = 0 \tag{1}$$

$$\frac{V^{n+1} - V^*}{\delta t} + \nabla p^{n+1} = 0 \tag{2}$$

$$\nabla . V^{n+1} = 0 \tag{3}$$

$$(2) \& (3) \Rightarrow \nabla^2 p^{n+1} = \frac{1}{\delta t} \nabla V^* \tag{4}$$

The condition on p at the boundary is obtained from projecting equation [2] on the external normal vector to the boundary $\partial \Omega$.

$$\left(\frac{\partial p}{\partial N}\right)_{\partial \Omega}^{n+1} = -\frac{1}{\delta t} \left(V_{\partial \Omega}^{n+1} - V_{\partial \Omega}^*\right).N$$

It can be shown that the result is independent of the choice of $V_{\partial \Omega}^*$.

We then choose $V_{\partial \Omega}^* = V_{\partial \Omega}^{n+1}$

and thus $\dfrac{\partial p}{\partial N} = 0$

Plate 20.27. *Unsteady Navier-Stokes equations*

MAC method

$$\begin{cases} \dfrac{1}{\delta t}\left(u_{i+1/2,j}^{n+1}-u_{i+1/2,j}^{n}\right)+a_{i+1/2,j}^{n}+\delta_x p_{i+1/2,j}^{n+1}-v\delta_+^2 u_{i+1/2,j}^{n}=0 \\[2mm] \dfrac{1}{\delta t}\left(v_{i,j+1/2}^{n+1}-v_{i,j+1/2}^{n}\right)+b_{i,j+1/2}^{n}+\delta_y p_{i,j+1/2}^{n+1}-v\delta_+^2 v_{i,j+1/2}^{n}=0 \\[2mm] \qquad\qquad D=\delta_x u_{i,j}^{n+1}+\delta_y v_{i,j}^{n+1}=0 \end{cases}$$

along with the definitions:

$$\begin{cases} \delta_x f_{i,j}=\dfrac{1}{\delta x}\left(f_{i+1/2,j}-f_{i-1/2,j}\right) \\[2mm] \delta_y f_{i,j}=\dfrac{1}{\delta y}\left(f_{i,j+1/2}-f_{i,j-1/2}\right)f_{i,j} \\[2mm] \quad \delta_+^2 f_{i,j}=\left(\delta_x\delta_x+\delta_y\delta_y\right) \end{cases}$$

$$\delta_x(\text{eq. }u)+\delta_y(\text{eq. }v)=0\Rightarrow \delta_+^2 p=-\frac{\partial D}{\partial t}-Q+v\delta_+^2 D$$

$$\frac{\partial D}{\partial t}=-\frac{1}{\delta t}D_{i,j}^n, \qquad Q=\delta_x a_{i,j}^n+\delta_y b_{i,j}^n$$

Boundary conditions

The projection of the momentum equation on the normal yields:

$$\frac{1}{\delta x}\left(p_{1,m}^{n+1}-p_{0,m}^{n+1}\right)=-\frac{1}{\delta t}\left(u_{1/2,m}^{n+1}-u_{1/2,m}^n\right)-a_{1/2,m}^n+v\delta_+^2 u_{1/2,m}^n$$

Choosing: $a_{1/2,m}^n-v\delta_+^2 u_{1/2,m}^n=-\dfrac{u_{\partial\Omega}^{n+1}-u_{\partial\Omega}^n}{\delta t}$, corresponding to a particular choice

of $u_{1/2,m}$ we then recover the zero pressure gradient condition at the boundary.

Plate 20.28. *Unsteady Navier-Stokes equations*

Implicit methods

$$\begin{cases} \dfrac{V^{n+1}-V^n}{\delta t}+\dfrac{3C^n-C^{n-1}}{2}+\nabla p^{n+1/2}-\nu\dfrac{\nabla^2 V^n+\nabla^2 V^{n+1}}{2}=0 \\[2mm] \nabla.V^{n+1}=0 \end{cases}$$

System solved using fractional step technique

$$\begin{cases} \dfrac{V^*-V^n}{\delta t}+\dfrac{3C^n-C^{n-1}}{2}-\nu\dfrac{\nabla^2 V^n}{2}=0 \\[2mm] \dfrac{V^{n+1}-V^*}{\delta t}+\nabla p^{n+1/2}-\nu\dfrac{\nabla^2 V^{n+1}}{2}=0 \\[2mm] \nabla.V^{n+1}=0 \end{cases}$$

$$\text{[1], [2], [3]}$$

$$\nabla\left[\text{eq.[2]}\right]\Rightarrow\nabla^2 p^{n+1/2}=\frac{1}{\delta t}\nabla V^* \qquad\qquad\qquad\text{[4]}$$

The method consists of:

1) solving [1] explicitly

2) solving [2] and [4] simultaneously using for [2] the boundary conditions: $V=V_{\partial\Omega}$

and for [4] the Neumann conditions:

$$\left(\frac{\partial p}{\partial N}\right)_{\partial\Omega}^{n+1/2}=-\frac{1}{\delta t}\left(V^{n+1}-V^*-\underbrace{\nu\frac{\nabla^2 V^{n+1}}{2}}_{\text{coupling}}\right)N$$

Plate 20.29. *Unsteady Navier-Stokes equations,*
fractional step method of Kim J. and Moin P. [KIM 85]

N-S equations: $\quad \dfrac{\partial u_i}{\partial t} + \dfrac{\partial}{\partial x_i} u_i u_j = -\dfrac{\partial p}{\partial x_i} + \dfrac{1}{\text{Re}} \dfrac{\partial}{\partial x_j} \dfrac{\partial}{\partial x_j} u_i, \qquad \dfrac{\partial u_i}{\partial x_i} = 0$

Discretized equations on staggered grids:

$$\frac{\hat{u}_i - u_i^n}{\Delta t} = \frac{1}{2}\left(3H_i^n - H_i^{n-1}\right) + \frac{1}{2}\frac{1}{\text{Re}}\left(\frac{\delta^2}{\delta x_1^2} + \frac{\delta^2}{\delta x_2^2} + \frac{\delta^2}{\delta x_3^2}\right)\left(\hat{u}_i + u_i^n\right)$$

$$\frac{u_i^{n+1} - \hat{u}_i}{\Delta t} = -G(\phi^{n+1})$$

with $\qquad D(u_i^{n+1}) = 0, \qquad H_i = -\dfrac{\delta}{\delta x_j} u_i u_j, \qquad p = \phi + \dfrac{\Delta t}{2\,\text{Re}}\nabla^2\phi$

Approximate factorization with $\mathcal{O}(\Delta t^3)$ accuracy

$$(1 - A_1)(1 - A_2)(1 - A_3)\left(\hat{u}_i - u_i^n\right) = \frac{\Delta t}{2}\left(3H_i^n - H_i^{n-1}\right)$$

$$+2(A_1 + A_2 + A_3)u_i^n$$

Poisson equation for ϕ,

$$\left(\frac{\delta^2}{\delta x_1^2} + \frac{\delta^2}{\delta x_2^2} + \frac{\delta^2}{\delta x_3^2}\right)\phi^{n+1}(i, j, k) = \frac{1}{\Delta t} D\hat{u} = Q(i, j, k)$$

solved using direct spectral method.

Step $n+1$: $\qquad\qquad\qquad u_i^{n+1} = \hat{u}_i - \Delta t G(\phi^{n+1})$

Boundary conditions at a wall: $\hat{u}_i = u_i^{n+1} + \Delta t.\dfrac{\partial \phi^n}{\partial x}$, to order $\mathcal{O}(\Delta t^2)$

Plate 20.30. *SMAC method*

1) The momentum equation is solved using an approximate pressure field p^*:

$$\frac{\widetilde{V}^{n+1} - V^n}{\delta t} + C(V^n) + \nabla p^* - \nu \nabla^2 V^n = 0 \qquad [1]$$

2) We note that the vorticity is exact even if p^* is not exact (indeed the equation for vorticity does not involve pressure).

Thus the correction to give on V must be a gradient correction:
$$V = \widetilde{V} - \nabla \varphi \qquad [2]$$

3) $\nabla V = 0 \Rightarrow \Delta \varphi = \nabla V$ \qquad [3]

4) Interpretation of φ:

$$[1] \text{ and } [2] \Rightarrow \frac{\widetilde{V}^{n+1}}{\delta t} + \nabla p^* \equiv \frac{V^{n+1}}{\delta t} + \nabla \left(p^* + \frac{\varphi}{\delta t} \right)$$

$p^* + \dfrac{\varphi}{\delta t}$ is the true pressure $p^{n+1} = p^* + \dfrac{\varphi}{\delta t}$ \qquad [4]

In practice we solve [1] and then [3], afterwards V is corrected using [2] and p^* is corrected using [4]

Plate 20.31. *Navier-Stokes equations, other formulations*

2D flows:

   * Vorticity-stream function

3D flows:

   *Vorticity-vector potential

   *Vorticity-velocities

Plate 20.32. *Finite volumes: mesh for primitive variables*

Integration cells
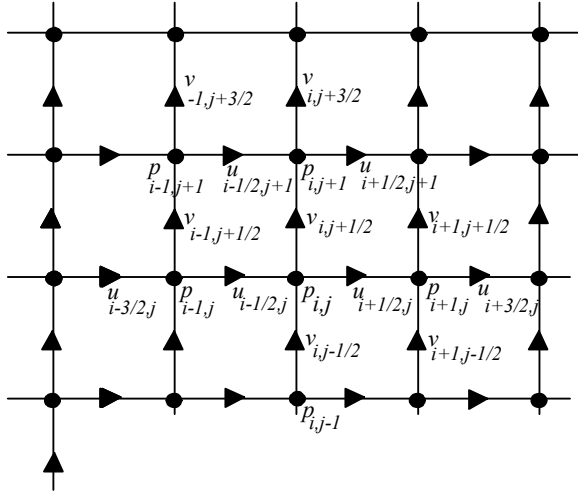
Boundary cells
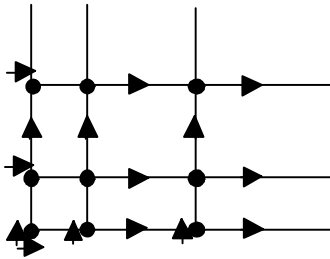
In general, the finite volume method is applied from integrating the equations on basic volumes of quadrilaterals with arbitrary shape. It is then possible to consider complex geometries without using equations written in curvilinear coordinates (Peyret R. and Taylor T.D., 1983).

Plate 20.33. *Finite volumes: pressure-velocity coupling SIMPLE (Semi Implicit Method for Pressure Linked Equations) algorithm*

1) An approximate pressure field is given p^*.

2) The momentum equations are solved to yield approximate velocities u^* and v^*:

$$a^{(u)}_{i+1/2,j} u^*_{i+1/2,j} = \sum_{p=[i+1/2\pm1,j],[i+1/2,j\pm1]} \left(a^{(u)}_p u^*_p\right) + b^{(u)}_{i+1/2,j} + g^{(u)}_{i+1/2,j}(p^*_{i,j} - p^*_{i+1,j})$$

$$a^{(v)}_{i,j+1/2} v^*_{i,j+1/2} = \sum_{p=[i\pm1,j+1/2],[i,j+1/2\pm1]} \left(a^{(v)}_p v^*_p\right) + b^{(v)}_{i,j+1/2} + g^{(v)}_{i,j+1/2}(p^*_{i,j} - p^*_{i,j+1})$$

$b^{(u)}$ and $b^{(v)}$ are source terms containing possible volumic forces and also additional diffusive terms when viscosity is variable.

3) Solution of the pressure correction equation:

$$p = p^* + p', \quad u = u^* + u', \quad v = v^* + v'$$

$$u_{i+1/2,j} = u^*_{i+1/2,j} + \frac{g^{(u)}_{i+1/2,j}}{a^{(u)}_{i+1/2,j}}(p'_{i,j} - p'_{i+1,j})$$

$$v_{i,j+1/2} = v^*_{i,j+1/2} + \frac{g^{(v)}_{i,j+1/2}}{a^{(v)}_{i,j+1/2}}(p'_{i,j} - p'_{i,j+1})$$

Continuity equation $\Rightarrow \left(u_{i+1/2,j} - u_{i-1/2,j}\right)\delta y + \left(v_{i,j+1/2} - v_{i,j-1/2}\right)\delta x = 0$

$$\alpha_{i,j} p'_{i,j} = \sum_{q=[i\pm1,j],[i,j\pm1]} \left(\alpha_q p'_q\right) + \beta_{i,j}$$

$$\alpha_{i+1,j} = g^{(u)}_{i+1/2,j}\delta y / a^{(u)}_{i+1/2,j}, \quad \alpha_{i-1,j} = g^{(u)}_{i-1/2,j}\delta y / a^{(u)}_{i-1/2,j}$$

$$\alpha_{i,j+1} = g^{(v)}_{i,j+1/2}\delta x / a^{(v)}_{i,j+1/2}, \quad \alpha_{i,j-1} = g^{(v)}_{i,j-1/2}\delta x / a^{(v)}_{i,j-1/2}$$

$$\beta = \left(u^*_{i+1/2,j} - u^*_{i-1/2,j}\right)\delta y + \left(v^*_{i,j+1/2} - v^*_{i,j-1/2}\right)\delta x$$

4) Correction for pressure and for velocity components.

5) Solution of other possible coupled equations.

6) p is a new improved approximation for p^*, loop on 2) until convergence.

Plate 20.34. *SIMPLER (SIMPLE revised) algorithm*

The *x* momentum equation can be written:

$$u_{i+1/2,j} = \underbrace{\frac{\sum a_q u_q + b}{a_{i+1/2,j}}}_{\hat{u}i+1/2,j} + g_{i+1/2,j}(p_{i,j} - p_{i+1,j})$$

The equation for velocities can thus be written symbolically:

$$u_{i+1/2,j} = \hat{u}_{i+1/2,j} + g_{i+1/2,j}(p_{i,j} - p_{i+1,j})$$
$$v_{i,j+1/2} = \hat{v}_{i,j+1/2} + g_{i,j+1/2}(p_{i,j} - p_{i,j+1})$$

from which follows a pressure equation:

$$a_{i,j}p_{i,j} = a_{i+1,j}p_{i+1,j} + a_{i-1,j}p_{i-1,j} + a_{i,j+1}p_{i,j+1} + a_{i,j-1}p_{i,j-1} + b$$
$$b = \left(\hat{u}_{i-1/2,j} - \hat{u}_{i+1/2,j}\right)\delta y - \left(\hat{v}_{i,j-1/2} - \hat{v}_{i,j+1/2}\right)\delta x$$

Successive steps in the algorithm:

1) An approximate velocity field is given.

2) Calculation of coefficients $a^{(u)}$, $a^{(v)}$, and also $\hat{u}$ and $\hat{v}$.

3) Solution of the pressure equation: p^*.

4) Solution of the momentum equations: u^* and v^*.

5) Solution of the equation for p'.

6) Correction of velocities (but not for pressure).

7) Solution of other possible coupled equations.

8) Loop on 2) until convergence.

The velocities are thus corrected in the same way as in the SIMPLE algorithm, whereas the pressure is obtained in a different way using another equation, and this improves the efficiency of the algorithm.

Plate 20.35. *Unsteady 2D problem*

$$\frac{\partial}{\partial t}(\rho\Phi) + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = S$$

$$J_x = \rho u\Phi - \sigma\frac{\partial\Phi}{\partial x}, \quad J_y = \rho v\Phi - \sigma\frac{\partial\Phi}{\partial y}$$
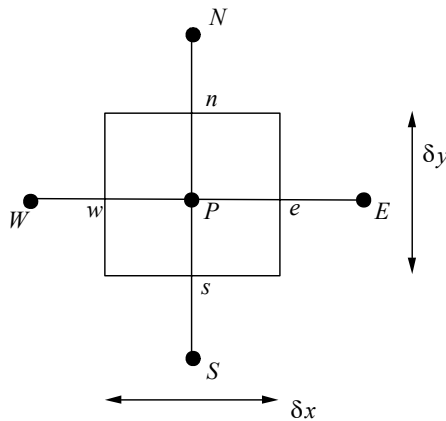
$$\int_w^e \int_s^n \int_t^{t+\delta t} [\text{Eq.}]\, dx\, dy\, dt$$

$$\left(\rho_P\Phi_P - \rho_P^0\Phi_P^0\right)\delta x\delta y + \left(\tilde{J}_e - \tilde{J}_w\right)\delta y\delta t + \left(\tilde{J}_n - \tilde{J}_s\right)\delta x\delta t = \left(S_C + S_P\Phi_P\right)\delta x\delta y\delta t$$

$$\widetilde{\Phi} = \frac{1}{\delta t}\int_t^{t+\delta t}\Phi\, dt = \left\{f\Phi + (1-f)\Phi^0\right\}$$

$$\tilde{J}_e = \int_t^{t+\delta t} dt \int_{(e)} J_x\, dy \text{ , etc.}$$

$$0 < f \leq 1 \qquad\qquad f = 0 \text{ explicit scheme}$$
$$f = 1 \text{ implicit scheme}$$
$$f = 1/2 \text{ Crank-Nicolson scheme}$$

$$\widetilde{\Phi}_e = \alpha\widetilde{\Phi}_P + (1-\alpha)\widetilde{\Phi}_E$$

$$\left(\frac{\partial\widetilde{\Phi}}{\partial x}\right)_e = \beta(\widetilde{\Phi}_E - \widetilde{\Phi}_P)$$

Plate 20.36. *Unsteady 2D problem (continued)*

$$\frac{J_e}{D_e \delta t} = B(P)\widetilde{\Phi}_P - A(P)\widetilde{\Phi}_E$$

$$\left(\rho_P \Phi_P - \rho_P^0 \Phi_P^0\right)\delta x \delta y + fDB\Phi_P \delta t - fDA\Phi_E \delta t +$$

$$(1-f)DB\Phi_P \delta t - (1-f)DA\Phi_E \delta t + \ldots = \left(S_C + S_P \Phi_P\right)\delta x \delta y \delta t \qquad [1]$$

$$\left(\rho_P - \rho_P^0\right)\delta x \delta y + \left(F_e - F_w + F_n - F_s\right)\delta t = 0 \qquad [2]$$

$$[1] - [2] \times \Phi_P \Rightarrow$$

$$\left(\Phi_P - \Phi_P^0\right)\rho_P^0 \delta x \delta y + \ldots - \left(F_e - F_w + F_n - F_s\right)\Phi_P \delta t = \left(S_C + S_P \Phi_P\right)\delta x \delta y \delta t$$

$$a_P \Phi_P = \sum_{N,E,W,S} a_q \Phi_q + b\Phi_P^0 - c$$

$$a_E = f\, D_e\, A(|P_e|) + f\,[\![-F_e, 0]\!]$$

$$a_W = f\, D_w\, A(|P_w|) + f\,[\![F_w, 0]\!]$$

$$a_N = f\, D_n\, A(|P_n|) + f\,[\![-F_n, 0]\!]$$

$$a_S = f\, D_s\, A(|P_s|) + f\,[\![F_s, 0]\!]$$

$$a_P = \rho_P^0 \frac{\delta x \delta y}{\delta t} + f\sum_{N,E,W,S} D.A(|P|) + (f-1)\left(F_e - F_w + F_n - F_s\right) - S_P \delta x \delta y$$

$$b = \rho_P^0 \frac{\delta x \delta y}{\delta t} + (1-f)\sum_{N,E,W,S} D.A(|P|) + (1-f)\left(F_e - F_w + F_n - F_s\right) - S_P \delta x \delta y$$

$$c = S_C \delta x \delta y$$

Plate 20.37. *Unsteady and variable density flows*

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$(\rho_P - \rho_P^0)\delta x \delta y + \left[(\rho u)_e - (\rho u)_w\right]\delta t \delta y + \left[(\rho v)_n - (\rho v)_s\right]\delta t \delta x = 0$$

$$u_e = u_e^* + g_e(p'_P - p'_E), \qquad v_n = v_n^* + g_n(p'_P - p'_N)$$

$$a_P p'_P = \sum_{N,E,W,S} a_q p'_q + b$$

$$b = (\rho_P^0 - \rho_P)\delta x \delta y - \left[\left(\rho u^*\right)_e - \left(\rho u^*\right)_w\right]\delta t \delta y - \left[\left(\rho v^*\right)_n - \left(\rho v^*\right)_s\right]\delta t \delta x$$

Plate 20.38. *Solution of linear systems*

Direct methods

$$A.X = B \quad \Rightarrow \quad X = A^{-1}.B$$

Gauss method (elimination)

$$\Rightarrow \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix} X = B^*$$

Factorization method

$$A = LU, \ LUX = B, \begin{cases} LY = B \\ UX = Y \end{cases}, \ L = \begin{bmatrix} \times & & \\ \times & \times & \\ \times & \times & \times \end{bmatrix}, U = \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{bmatrix}$$

Iterative methods

$$MX^{n+1} = B - (A - M)X^n$$

Point Jacobi method
M = diagonal of A

Point Gauss-Seidel method
M = lower triangular extracted from A

Block Jacobi method

$$M = \begin{bmatrix} [\ ] & & \\ & [\ ] & \\ & & [\ ] \end{bmatrix}$$

Block Gauss-Seidel method

$$M = \begin{bmatrix} [\ ] & & \\ [\ ] & [\ ] & \\ [\ ] & [\ ] & [\ ] \end{bmatrix}$$

Relaxation method

$$x_{i\,j}^{n+1} = \omega \tilde{x}_{i\,j}^{n+1} + (1-\omega)x_{i\,j}^{n}$$

where x is the result of the usual iterative method after one iteration,

$$\omega < 1 \text{ under-relaxation}, \quad 2 > \omega > 1 \text{ over-relaxation}$$

Plate 20.39. *Thomas algorithm*

Particular case of the Gauss method applied to tridiagonal matrices

$$\left[\begin{array}{c} \diagdown \\ A \end{array}\right]\left[X\right]=\left[B\right]$$

$$\begin{cases} \beta_1 x_1 + \gamma_1 x_2 = b_1 \\ \alpha_i x_{i-1} + \beta_i x_i + \gamma_i x_{i+1} = b_i \qquad i = 2,\ldots,n-1 \\ \alpha_n x_{n-1} + \beta_n x_n = b_n \end{cases} \qquad [1]$$

$$\left[\begin{array}{c} \diagdown \end{array}\right]\left[X\right]=\left[F\right]$$

$$\begin{cases} x_n = F_n \\ x_i = -E_i x_{i+1} + F_i \end{cases}, \quad i = n-1,\ldots,1 \qquad [2]$$

1) Elimination

$$E_1 = \frac{\gamma_1}{\beta_1}, F_1 = \frac{b_1}{\beta_1}$$

$$E_i = \frac{\gamma_i}{\beta_i - \alpha_i E_{i-1}}, \quad F_i = \frac{b_i - \alpha_i F_{i-1}}{\beta_i - \alpha_i E_{i-1}}, \quad i = 2,\ldots,n-1$$

$$F_n = \frac{b_n - \alpha_n F_{n-1}}{\beta_n - \alpha_n E_{n-1}}$$

2) Back solution
    using [2]

Plate 20.40.

Source terms

Source terms are linearized (negative slope; cf. Plate 20.17).

Non-linearities

Non-linearities are solved iteratively.
Under-relaxation parameters are usually necessary.

Convergence control

Uniform decay of residuals.
Quantities calculated at monitor locations go to an asymptotic value.

Treatment of second order closure models

The momentum equations lose their diffusive character since the Reynolds stresses appear as source terms. Specific numerical stabilization techniques are necessary, for example the use of a fourth shifted grid for the shear stresses and the introduction of fictitious effective viscosity coefficients (Pope and Whitelaw [POP 76], Huang and Leschziner [HUA 85]).

This stabilization procedure can also be extended to the case of non-staggered grids (Obi S. *et al.* [OBI 89]).

Moreover, it is often useful in order to have good stability properties, to solve the redundant equation for kinetic energy coupled with the dissipation equation numerically.

Thus, we solve inside the main iteration

$$
\begin{cases}
\dfrac{dk^*}{dt} = f(P, k^*, \varepsilon) \\[2mm]
\dfrac{d\varepsilon}{dt} = g(P, k^*, \varepsilon)
\end{cases}
$$

and then:

$$
\frac{dR_{ij}}{dt} = Prod - C \frac{\varepsilon}{k^*}(R_{ij} - \frac{2}{3}k^*\delta_{ij}) - Diss. + Diff.
$$

$$k = \frac{1}{2}(R_{11} + R_{22} + R_{33})$$

We verify that after convergence $k = k^*$.
Sub-iterations can be used for convergence.

Numerical codes for the calculation of laminar and turbulent flows

TEACH (developed at Imperial College, London)
TEAM (developed at the University of Manchester)
Commercial CFD codes for engineering

Calculation of rotating turbulent flows (Coriolis forces)

Solving the Reynolds stress transport equations using block solution allows a better stability of the calculation (block tridiagonal algorithm).

Solving multiple scale models

It is preferable to solve all the equations relative to slice (1) and afterwards solve the equations relative to slice (2).

In the case of the hybrid model with partial energies and total Reynolds stresses, it is necessary to ensure $R_{11} + R_{22} + R_{33} = k^{(1)} + k^{(2)}$. To do this, the Rotta term will be favorably discretized in the form:

$$-c_1 \frac{\varepsilon}{k^{(1)} + k^{(2)}} \left[R_{ij} - \frac{2}{3}(k^{(1)} + k^{(2)})\delta_{ij} \right].$$

An under-relaxation such as $\alpha \left[\left(k^{(1)} + k^{(2)}\right) - \frac{1}{2}(R_{11} + R_{22} + R_{33}) \right]$ may even be used if necessary in the equations for the normal stresses.

Plate 20.41. *Thin shear flows*

$$\begin{cases} u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=-\dfrac{1}{\rho}\dfrac{dp}{dx}+\dfrac{\partial}{\partial y}\left(\sigma\dfrac{\partial u}{\partial y}\right) \\[2mm] \dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=0 \end{cases}$$

Von Mises transformation
and Patankar and Spalding method

$$\begin{Bmatrix} x \\ y \end{Bmatrix}\Rightarrow\begin{Bmatrix} X \\ \psi \end{Bmatrix}, \quad X=x,\quad \frac{\partial \psi}{\partial y}=u,\quad \frac{\partial \psi}{\partial x}=-v$$

$$u\frac{\partial u}{\partial X}=-\frac{1}{\rho}\frac{dp}{dX}+u\frac{\partial}{\partial \psi}\left(\sigma u\frac{\partial u}{\partial \psi}\right)$$
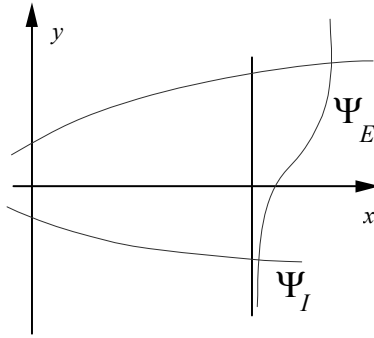
$$\begin{Bmatrix} X \\ \psi \end{Bmatrix}\Rightarrow\begin{Bmatrix} \chi \\ \omega \end{Bmatrix},\quad \chi=X,\quad \omega=\frac{\psi-\psi_I(X)}{\psi_E(X)-\psi_I(X)}$$

$$\frac{\partial u}{\partial \chi}+(a+b\omega)\frac{\partial u}{\partial \omega}=\frac{\partial}{\partial \omega}\left(c\frac{\partial u}{\partial \omega}\right)+d$$

$$a=-\frac{\psi_I}{\Delta\psi},\quad b=-\frac{\Delta\psi'}{\Delta\psi},\quad \Delta\psi=\psi_E-\psi_I$$

$$c=\frac{\sigma u}{\Delta\psi^2},\quad d=-\frac{1}{\rho u}\frac{dp}{d\chi}$$

+ similar equations for other quantities Φ

Plate 20.42. *Patankar and Spalding method*
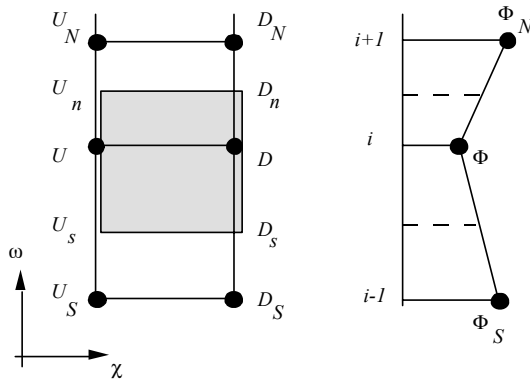
$$\frac{1}{\delta x}\left[\int_s^n \Phi_D d\omega - \int_s^n \Phi_U d\omega\right] + \left[(a+b\omega)\Phi\right]\big|_s^n - b\int_s^n \Phi d\omega = \left[c\frac{\partial \Phi}{\partial \omega}\right]\bigg|_s^n + \int_s^n (d).d\omega$$

Introducing a hybrid type scheme for the convection-diffusion terms

$$\Rightarrow \quad \Phi_P = A\Phi_N + B\Phi_S + C$$

Tridiagonal recurrence
Calculation of y:

$$y = \Delta\psi \int_0^\omega \frac{d\omega}{u}$$

Plate 20.43. *Patankar and Spalding method*

Case of free flows

$$\psi'_I = \frac{2\sigma_{2.5}}{y_3 - y_2}\left(\frac{u_3 - u_2}{\delta u^*}\right)^n$$

where δu^* is a fraction of the total velocity difference in a cross-section of the flow

(fictitious point)
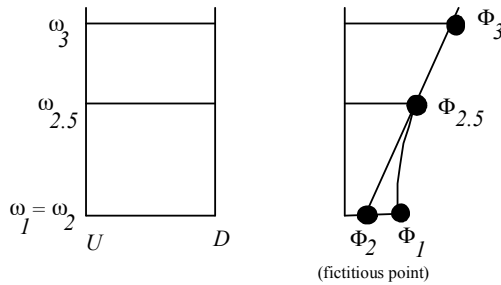
Case of confined flows
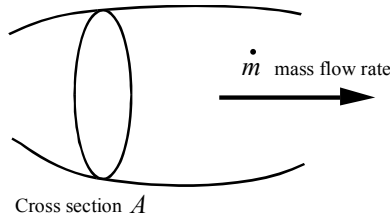
Cross section A

$$\dot{m} = A\tilde{u} = \int u dS \quad, \qquad \frac{dA}{A} = -\frac{d\tilde{u}}{\tilde{u}}$$

the momentum equation $\Rightarrow$ $Adp + F'dx + \dot{m}d\tilde{u} = 0$, $\bar{u} = \int u^2 dS / \int u dS$

if the shape of the velocity profile remains unchanged $\tilde{u} = k\bar{u}$

$$\frac{dp}{dx} = -\frac{F'}{A} + \frac{\dot{m}\bar{u}}{A^2}\frac{dA}{dx}, \qquad \frac{dA}{dx} = \frac{A_D - A_U}{x_D - x_U}$$

A_U, upstream effective cross-section and A_D, desired downstream cross-section.

Plate 20.44. *Methods for numerical simulations of turbulence, DNS and LES*

Highly accurate methods are necessary, with good conservation properties (energy conservation, momentum conservation verified by the in viscid discretized equations). In practice, two groups of methods are used for spatial discretization: high order finite difference methods and spectral or pseudo-spectral methods (cf. [*CAN 88], [*BER 92]). Time schemes must also give sufficient accuracy on long periods of integration.

High order finite difference methods

In traditional high order finite difference methods, the derivatives are expressed in an explicit way from expansions of the unknown function using more than three mesh points. In Hermitian methods or compact methods, the derivatives are taken into account as additional unknowns obtained from subsidiary equations (cf. Plate 20.7).

Application to the Navier-Stokes equations can be associated with the projection method (cf. [MER 02], [ELS 79], [FRI 01]).

Spectral and pseudo-spectral methods

Spectral methods are based on Fourier series expansions (homogenous case) or on Tchebycheff polynomial expansions (non-homogenous case in the presence of walls). The method is thus global, contrary to traditional finite differences which use local approximations. This is the reason these methods reach a high accuracy degree. There are three main approaches ([*GOT 77], [*PEY 83], [*CAN 88]) mentioned in the next plate for solving the generic equation:

$$\frac{\partial u(x,t)}{\partial t} = L(x,t)u(x,t) + f(x,t) \ x \in \Omega \,,$$
$$B(x)u(x,t) = 0 \ x \in \partial\Omega \,, \qquad u(x,0) = g(x) \ \text{to} \ t{=}0$$

These are the Galerkin approximation, the Tau method and the collocation method.

Plate 20.45. *Spectral methods for numerical simulations*

In the Galerkin approximation, the solution is sought in the form of an expansion $u(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x)$, the basis functions $\phi_n(x)$ satisfying the boundary conditions. The modes are obtained from the Galerkin equations $\frac{d}{dt}(\phi_n, u) = (\phi_n, Lu) + (\phi_n, f)$ in which the scalar product is defined by $(\phi, \psi) = \int_{\Omega} \phi\psi \, dx$.

In the "Tau" approach, the Galerkin equations are only considered for $n = 1, N - k$, the system being closed by supplementing the k boundary conditions $\sum_{n=1}^{N} a_n B\phi_n = 0$, the basis functions do not necessarily satisfy the boundary conditions.

The collocation method (or pseudo-spectral method) uses N collocation points in the domain and the projection operator is such that $P^{(N)}u = \sum_{n=1}^{N} a_n\phi_n(x)$. The collocation is then characterized by $P^{(N)}u(x_i) = u(x_i)$ ($i=1,N$). This is equivalent to writing the collocation equations into the form $\frac{d}{dt}(\delta(x - x_i), u) = (\delta(x - x_i), Lu) + (\delta(x - x_i), f)$ where δ is the Dirac distribution. The points x_i are chosen as the extrema of the basis functions (Tchebycheff polynomials or Legendre polynomials). In practice, what is sought is the set of values of the function at the collocation points. It is thus possible to work directly in physical space (pseudo-spectral method) and calculate the derivatives in the transformed space. For example, in the case of Fourier space $\partial u / \partial x_j = \mathcal{F}^{-1}\{-i\kappa_j \mathcal{F}(u)\}$.

Passages from spectral space to physical space and reciprocally are made using fast Fourier transforms (FFT). Non-linear terms introduce aliasing errors that can be reduced using appropriate techniques ([PAT 71]).

Plate 20.46. *Spectral methods and Navier-Stokes equations*

Navier-Stokes equations in spectral space

Homogenous turbulent flows can be studied using Fourier expansions (Orszag S.A., [ORS 69, 70, 71, 72B]). In homogenous turbulence, pressure can be eliminated using the mass conservation equation in Fourier space.

In the non-homogenous case, a basis of orthogonal polynomials is appropriate (Tchebycheff polynomials or Legendre polynomials, [HUG 01B]). The application of spectral approximations to the non-homogenous Navier-Stokes equations in pressure-velocity formulation can also be done using projection methods (cf. [RAS 02]) with Tchebycheff expansions. We are led to the solution of a Poisson equation for pressure (or two equations).

High order time schemes

A first category of schemes involves several time steps, like the Adams-Bashforth scheme. The most widely used is the hybrid Adams-Bashforth/Crank-Nicolson scheme. In the Navier-Stokes equations, a semi-implicit approach can be retained, the linear terms are then treated implicitly (Crank-Nicolson scheme) whereas the non-linear terms are treated explicitly (Adams-Bashforth scheme). A second category of schemes uses a single time step, it comprises the explicit Kutta-Runge methods which can also be used in semi-implicit formulations (cf. [*HAI 87], [*HAI 91]).

Plate 20.47. *A Hermitian finite difference method for solving the Navier-Stokes equations ([SCH 95B], [VIA 95], [VED 03])*

The Skew-symmetric form ([ZAN 90]) of the N-S equations is used:

$$\frac{\partial U}{\partial t} = -\frac{1}{2} U.\nabla U - \frac{1}{2}\nabla(U.U) - \nabla p + \frac{1}{Re}\Delta U , \qquad \nabla.U = 0$$

Coordinate transformation for variable meshes:

If ξ = X or Z (uniform transformed plane) and ζ = x or z (physical plane)

$\zeta = \zeta(\xi)$ defined analytically $\quad g_\xi = \dfrac{d\xi}{d\zeta} \; h_\xi = \dfrac{d^2\xi}{d\zeta^2}$

Discretized equation: $\quad \dfrac{1}{\Delta t}\left(U^{n+1} - U^n\right) = -\dfrac{1}{2}\left(3H^n - H^{n-1}\right) - Gp^{n+1}$

$$+ \frac{1}{2Re}\left(g_X^2 \delta_{XX}^2 + g_Z^2 \delta_{ZZ}^2\right)\left(U^{n+1} + U^n\right), \quad H = \frac{1}{2}(\nabla U.U + U.\nabla U) + A ,$$

(A = diffusive terms not included in the Crank-Nicolson part), $\quad D.U^{n+1} = 0$

Hermitian relations (Plate 20.7)

Fractional step scheme:

$$\frac{1}{\Delta t}\left(\hat{U} - U^n\right) = H^{n,n-1} - Gp^n + \frac{1}{2Re}L\left(\hat{U} + U^n\right), \qquad \frac{1}{\Delta t}(U^{n+1} - \widehat{U}) = -G\Phi$$

Poisson equation: $\quad D.G\Phi = \dfrac{1}{\Delta t}D.\hat{U} , \qquad p^{n+1} = p^n + \Phi + \dfrac{\Delta t}{2Re}L\Phi$

Solution procedure using secondary iterations:

$$\frac{1}{\Delta t}\left(U^{n+1,m} - U^n\right) = \Xi^{n,n-1} - Gp^{n+1,m-1} + \frac{1}{2Re}L\left(U^{n+1,m-1} + U^n\right)$$

$$U^{n+1,m+1} = U^{n+1,m} - \Delta t G\Phi , \qquad p^{n+1,m+1} = p^{n+1,m} + \Phi$$

$$D.G\Phi = \frac{1}{\Delta t}D.U^{n+1,m} \quad \text{where} \quad \left(\delta_{xx}^2 + \delta_{zz}^2 + d_y^2\right)\Phi = \frac{1}{\Delta t}D.U^{n+1,m}$$

after Fourier transform: $\left(\delta_{xx}^2 + \delta_{zz}^2 - k_y^2\right)\mathscr{T}_y(\Phi) = \mathscr{T}_y\left(\dfrac{1}{\Delta t}D.U^{n+1,m}\right),$

equation solved using an efficient digitization technique.

Note that the accuracy of order 4 is recovered after convergence since thus $\Phi \to 0$.

Boundary conditions $(\partial\Phi/\partial n)_{\partial\Omega} = 0$ (given velocity at the boundaries).

Compatibility condition $\dfrac{1}{\Delta t}\displaystyle\int_{\partial\Omega} U.n = \int_{\partial\Omega} G\Phi.n = 0$

Plate 20.48. *A Hermitian finite difference method for solving the Navier-Stokes equations, extension to LES*

$$\frac{\partial \overline{U}}{\partial t} = -\frac{1}{2}\left[\overline{U}.\nabla \overline{U} + \nabla\left(\overline{U}.\overline{U}\right)\right] - \nabla \overline{P} + \frac{1}{\mathrm{Re}}\Delta \overline{U} - \nabla T - \nabla L , \quad \nabla.\overline{U} = 0$$

$$T = R - \frac{2}{3}kI , \qquad R = \overline{U'\otimes \overline{U}} + \overline{\overline{U}\otimes U'} + \overline{U'\otimes U'} , \qquad L = \overline{\overline{U}.\overline{U}} - \overline{U}.\overline{U}$$

Smagorinsky model or one equation subgrid-scale transport model ([VIA 01], [DEJ 01], [MER 01], [WEN 03], [DEJ 02]).

Boundary conditions at a wall:

$$u_{i,N_z} = \left(-6u_{i,N_z-1} + u_{i,N_z-2}\right)\big/3 , \qquad u_{i,1} = \left(-6u_{i,2} + u_{i,3}\right)\big/3$$

Exit boundary conditions (frozen turbulence): $f = \langle f\rangle + f^{\#}$

$$\frac{\partial \langle w\rangle_Y}{\partial x} = \frac{\partial \langle v\rangle_Y}{\partial x} = 0 , \quad \langle \Phi\rangle_Y = 0 \quad \text{(mean part)}$$

$$\frac{\partial U^{\#}}{\partial t} + U_c.\nabla U^{\#} = 0 , \quad \frac{\partial \Phi^{\#}}{\partial x} = 0 \quad \text{(fluctuating part)}$$

U_c convection velocity (mean velocity at exit).

This page intentionally left blank

Exercises

No. 1 – We consider the term $S = \overline{U}_{i,j} D_{ij}$ with $D_{ij} = \nu \overline{u_{m,i} u_{m,j}}$ which appears as a source term in the ε equation. /a/ Determine the order of magnitude of S with respect to the velocity scale $u = \sqrt{k}$ and the length scale ℓ which characterize the turbulent field. /b/ In the framework of second order closures, obtain an approximation for D_{ij} using the invariant modeling method of J.L. Lumley in homogenous turbulence to second order in anisotropy. /c/ Suggest a simple approximation for D_{ij} at high turbulence Reynolds numbers referring to physical bases. What are the consequences on S?

Indications: /a/ $\mathcal{O}(S) = u^4 / \ell^2$, /b/ $D_{ij} = \varepsilon \left[(\frac{1}{3} + \alpha II) \delta_{ij} + \beta a_{ij} - 3\alpha a_{im} a_{mj} \right]$ taking into account $D_{jj} = 0$, /c/ $D_{ij} = \frac{\varepsilon}{3} \delta_{ij}$, $S \to 0$.

No. 2 – We consider the two equation model in xOy axes:

$$\frac{dk}{dt} = \frac{Z}{\sqrt{k}} \left(\frac{\partial U}{\partial y} \right)^2 - c_0 \frac{k^{5/2}}{Z} + \frac{\partial}{\partial y} \left(\frac{Z}{\sigma_1 \sqrt{k}} \frac{\partial k}{\partial y} \right),$$

c_0, σ_1 numerical constants,

$$\frac{dZ}{dt} = C_{Z1} \frac{Z^2}{k\sqrt{k}} \left(\frac{\partial U}{\partial y} \right)^2 - C_{Z2} k\sqrt{k} + \frac{\partial}{\partial y} \left(\frac{Z}{\sigma_2 \sqrt{k}} \frac{\partial Z}{\partial y} \right),$$

C_{Z1}, C_{Z2}, σ_2 numerical constants, U, longitudinal mean velocity, k, kinetic energy.

/a/ What is the physical meaning of the quantity Z? Give the turbulence macro-length scale according to Z and k? /b/ We consider the case of decay of turbulence behind a grid. Supposing that k decays according to a power law x^α, $\alpha = -1.1$ according to the distance behind the grid, deduce the value of c_0 / C_{Z2}. /c/ We consider now the case of the logarithmic region in a turbulent boundary layer. Let u_* be the friction velocity and K the Karman constant, determine k / u_*^2 and ν_t, deduce an expression for C_{Z1} according to the other constants of the model and of K. /d/ We consider the case of an homogenous turbulent flow submitted to uniform shear with $\partial U / \partial y = Cte$, show that $\tau / \rho k$ is constant, τ being the turbulent shear stress. Values of c_0, C_{Z1}, C_{Z2} if $\sigma_1 \approx \sigma_2 \approx 1$ and $(\tau / \rho k)_{\text{measured}} \approx 0.3$.

Indications: /a/ $\ell = Z / k$, /b/ $c_0 / C_{Z2} = \alpha / (1.5\alpha + 1) = 1.7$, /c/ $k / u_*^2 = c_0^{-1/2}$,

$$\nu_t = u_* K y, \qquad C_{Z1} = \frac{C_{Z2}}{c_0} - \frac{K^2}{\sigma_2 c_0^{1/2}}, \qquad \text{/d/} \qquad \tau / \rho k = u_*^2 / k = c_0^{1/2}, \qquad c_0 = 0.09,$$

$C_{Z1} = 0.06$, $C_{Z2} = 0.053$.

No. 3 – We consider the turbulent flux $D_j = \overline{\varepsilon' u_j}$ in which u_j stands for the j component of the fluctuating velocity and ε' the fluctuation in turbulent kinetic energy dissipation rate. In the framework of Reynolds stress transport modeling, D_j is considered as a functional of known quantities R_{ij} and ε. Give an approximation of D_j using the invariant modeling method of J.L. Lumley, to first order in non-homogenity, /a/ at first to first order in anisotropy, /b/ then to zero order in anisotropy. /c/ Would it be possible to get a similar approximation directly from phenomenological reasoning?

Indications: /a/ $\overline{\varepsilon' u_j} = k(\alpha_0 \delta_{jp} + \alpha_1 a_{jp}) k_{,p} + \dfrac{k^2}{\varepsilon} (\alpha_2 \delta_{jp} + \alpha_3 a_{jp}) \varepsilon_{,p} + \alpha_4 k^2 a_{jm,m}$,

/b/ $\overline{\varepsilon' u_j} = \alpha_0 k k_{,j} + \alpha_2 \dfrac{k^2}{\varepsilon} \varepsilon_{,j}$, /c/ gradient approximation $\overline{\varepsilon' u_j} = -(\nu_t / \text{Pr}_\varepsilon) \varepsilon_{,j}$.

No. 4 – With the aim of studying free turbulent shear flows, we develop an algebraic modeling of the Reynolds stresses deduced from the following transport model:

$$\frac{dR_{ij}}{dt} = P_{ij} + D_{ij} - c_1 \frac{\varepsilon}{k} b_{ij} - \gamma (P_{ij} - \frac{2}{3} P\delta_{ij}) - \frac{2}{3} \varepsilon \delta_{ij}$$ (model and notations of Chapter 6,

version RS2). The algebraic model will be based on the hypothesis:

$$T_{ij} = T \left[(1+\alpha) \frac{R_{ij}}{k} - \frac{2}{3} \alpha \delta_{ij} \right], \quad T_{ij} = \frac{dR_{ij}}{dt} - D_{ij}, \quad T = \frac{dk}{dt} - D$$, where α is a numerical

constant. /a/ Write in the general case the algebraic system of equations allowing the calculation of the Reynolds stresses. /b/ Let us suppose that the thin shear flow approximations hold, the main direction of flow velocity is along Ox. Give the expressions for R_{22}, R_{12} and $\nu_t = C_\mu(\lambda) k^2 / \varepsilon$, $\lambda = P / \varepsilon$.

Indications: /a/ $$\frac{R_{ij} - \frac{2}{3} k \delta_{ij}}{k} = \Lambda \frac{P_{ij} - \frac{2}{3} P\delta_{ij}}{\varepsilon}, \qquad \Lambda = \frac{1-\gamma}{c_1 + (1+\alpha)(\frac{P}{\varepsilon} - 1)},$$ /b/

$$R_{22} = \frac{2}{3}\left(1 - \Lambda \frac{P}{\varepsilon}\right), \quad R_{12} = -\Lambda \frac{R_{22} U_{1,2}}{\varepsilon}, \quad C_\mu = \frac{2}{3} \frac{1-\gamma}{c_1 + (1+\alpha)(\frac{P}{\varepsilon}-1)} \left[1 - \frac{(1-\gamma)\frac{P}{\varepsilon}}{c_1 + (1+\alpha)(\frac{P}{\varepsilon}-1)} \right].$$

No. 5 – We consider homogenous turbulence and we study the decay of turbulence generated by a grid. We suppose that the energy spectrum keeps a constant shape given by the following mathematical form:

$$E(\kappa, t) = \alpha \frac{\left(\kappa / \kappa_M\right)^m}{\left[1 + \left(\kappa / \kappa_M\right)^{m+1}\right]^{(3m+5)/(3m+3)}}$$

Deduce an evolution equation for ε.

Indications: cf. section 5.10.

No. 6 – We consider the pressure-gradient of a passive scalar correlation term γ.

$\Phi_{\gamma j} = \overline{\dfrac{p}{\rho} \gamma_{,j}}$ which appears in the transport equations for the turbulent fluxes of a

scalar. /a/ Give the order of magnitude of $\Phi_{\gamma j}$ according to the characteristic scales of the turbulent field. /b/ In the framework of second order closures, we study the case of decay of homogenous anisotropic turbulence (without mean velocity gradient). Thus, the correlation $\Phi_{\gamma j}$ can be considered as a functional R_{ij}, $F_{\gamma j}$,

$g = \overline{\gamma^2}$ and ε. Give an approximation using the invariant modeling method of Lumley (approximation in homogenous turbulence limited to second order in anisotropy). /c/ Same question for the correlation $\varepsilon_{ij}^{\gamma} = \sigma \overline{\gamma_{,i} \gamma_{,j}}$.

Indications: /a/ $\mathscr{O}(\Phi_{\gamma j}) = u^2 \gamma / \ell$,

/b/ $\Phi_{\gamma j} = \dfrac{\varepsilon}{k}\Big[(\alpha + \beta II + \chi B)F_{\gamma j} + a_{mj}F_{\gamma m} + a_{mj}a_{mh}F_{\gamma h}\Big]$, $II = a_{ij}a_{ij}$,

$B = F_{\gamma m}F_{\gamma m}/kg$. /c/ $\varepsilon_{ij}^{\gamma} = (f_0^0 + f_0^1 II + f_0^2 III + f_0^3 B + f_0^4 a_{hm}F_{\gamma h}F_{\gamma m}/kg)\delta_{ij}$

$+(f_1^0 + f_1^1 II + f_0^2 B)a_{ij} + f_2^0 a_{im}a_{mj} + f_3^0 F_{\gamma i}F_{\gamma j} + f_3^1 a_{im}F_{\gamma m}F_{\gamma j} + f_3^2 a_{jm}F_{\gamma i}F_{\gamma m}$

No. 7 – We consider a second order turbulence model in homogenous flow:

$\dfrac{dR_{ij}}{dt} = P_{ij} - c_1 \dfrac{\varepsilon}{k}(R_{ij} - \dfrac{2}{3}k\delta_{ij}) - \gamma(P_{ij} - \dfrac{2}{3}P\delta_{ij}) - \dfrac{2}{3}\varepsilon\delta_{ij}$ (1), $\dfrac{d\varepsilon}{dt} = C_{\varepsilon 1}\dfrac{P\varepsilon}{k} - C_{\varepsilon 2}\dfrac{\varepsilon^2}{k}$

(2). /a/ Deduce from (1), the modeled equation for the deviator $b_{ij} = R_{ij} - \dfrac{2}{3}k\delta_{ij}$ of

the Reynolds stress tensor. /b/ After passing through a distorting channel which allows us to generate an homogenous anisotropic turbulence field, the flow reaches a straight channel with constant cross-section in which the mean velocity is uniform. We then study the decay of the initial anisotropy in the uniform flow $U = Cte$.

Particularize the equation for b_{ij} to this case and show that there is a solution in

the form $b_{ij} = b_{ij0}x^{\mu}$. The experiment gives $\mu = -1.65$, deduce the numerical value

of the constant c_1. /c/ We consider now, the case of homogenous anisotropic turbulence with constant mean velocity gradient $U_1 = \Lambda_{12}y$, where Λ_{12} is the only non-zero gradient. Write the equation for the invariant $H = b_{ij}b_{ij}$. /d/ We suppose

that the present flow, reaches an asymptotic equilibrium state and that dH/dt can be assumed to be zero. Deduce an approximation for H according to k, ε, P_{ij} and b_{ij}.

From this result, suggest a method for determining the constant γ. /e/ Write the evolution equations for the invariants II and III with $a_{ij} = (R_{ij} - 2/3.k\delta_{ij})/2k$ and deduce the trajectory followed in the realizability diagram of Lumley by the representative point of the turbulence state. /f/ The experiment gives $R_{11}/k = 1$, $R_{22}/k = 0.4$, $R_{33}/k = 0.6$ and $R_{12}/k = 0.3$, deduce γ.

Indications: /a/ (The time evolution d/dt is likened to the longitudinal space evolution $U d/dx$ while still keeping the homogenity hypothesis in the equations)

$$\frac{db_{ij}}{dt} = (1-\gamma)(P_{ij} - \frac{2}{3}P\delta_{ij}) - c_1\frac{\varepsilon}{k}b_{ij}, \quad /b/ \quad \frac{db_{ij}}{dt} = -c_1\frac{\varepsilon}{k}b_{ij}, \quad k = k_0 x^\alpha, \varepsilon = \varepsilon_0 x^\beta,$$

$$b_{ij} = b_{ij0}x^\mu, \qquad \alpha = -1.1, \qquad \beta = \alpha - 1, \qquad c_1 = \mu/\alpha \approx 1.5.$$

/c/ $\dfrac{dH}{dt} = 2b_{ij}\dfrac{db_{ij}}{dt} = 2(1-\gamma)P_{ij}b_{ij} - 2c_1\dfrac{\varepsilon}{k}H$,/d/ $H = \dfrac{1-\gamma}{c_1}\dfrac{k}{\varepsilon}P_{ij}b_{ij}$ or $II = \dfrac{1-\gamma}{4c_1}\dfrac{P_{ij}b_{ij}}{\varepsilon.k}$.

/e/ $II = H/4k^2$, $III = b_{ij}b_{jm}b_{mi}/8k^3$, $\dfrac{dII}{dt} = 2(c_1-1)\dfrac{\varepsilon}{k}II$, $\dfrac{dIII}{dt} = 3(c_1-1)\dfrac{\varepsilon}{k}III$,

$\mathscr{I}_{3a}/\mathscr{I}_{2a}^{3/2} = \text{Constant}$, /f/ $II \approx 0.09$, $P \approx \varepsilon$, $\dfrac{P_{ij}b_{ij}}{4c_1P.k} \approx 0.243$, $\gamma \approx 0.62$.

No. 8 – We study the flow in a distorting channel, in which an homogenous isotropic grid turbulence is submitted to sudden distortion during a very short time δt. This distorting channel allows us to apply during the short time interval δt a mean velocity gradient given by: $U_{i,j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{bmatrix}$.

/a/ How is the equation for the deviator of the Reynolds stresses b_{ij} in this case written? /b/ What are the values of b_{11}, b_{22}, b_{33} at the exit of the distorting channel according to δt, a and k?

Indications: /a/ $\dfrac{db_{ij}}{dt} = (1-\gamma)(P_{ij} - \dfrac{2}{3}P\delta_{ij})$,   (rapid   distortion),   /b/ $b_{11} = 0$,

$b_{22} = \dfrac{4}{3}(1-\gamma)ka.\delta t$, $b_{33} = -\dfrac{4}{3}(1-\gamma)ka.\delta t$.

No. 9 – We consider the transport of two chemical species, A and B, whose mass fraction is given at each instant of time by the following instantaneous evolution equations: $\dfrac{dM_A}{dt} = \left(\lambda_A M_{A,j}\right)_{,j} + R_A$, $\quad \dfrac{dM_B}{dt} = \left(\lambda_B M_{B,j}\right)_{,j} + R_B$, $\quad M = \overline{M} + m$ stands for the instantaneous mass fraction (mean value + fluctuation) and R stands for the production rate of the species A and B by chemical reaction. We shall suppose that there is a correlation between the chemical reaction rates r_A and r_B (although the nature of this mixing is not considered here).

/a/ What is the evolution equation for the fluctuating mass fraction m_A? /b/ Deduce an equation for the transport of the correlation $\overline{m_A m_B}$. /c/ Suggest simple approximations for evaluating these unknown correlations in terms of known quantities ($\lambda_A = \lambda_B$).

Indications: /a/ $\dfrac{\partial m_A}{\partial t} + u_j m_{A,j} + u_j \overline{M}_{A,j} + \overline{U}_j m_{A,j} - \overline{u_j m_{A,j}} = \left(\lambda m_{A,j}\right)_{,j} + r_A$,

/b/ $\dfrac{\partial \overline{m_A m_B}}{\partial t} + \overline{U}_j \left(\overline{m_A m_B}\right)_{,j} = -\overline{u_j m_B}.\overline{M}_{A,j} - \overline{u_j m_A}.\overline{M}_{B,j} - \left(\overline{u_j m_A m_B}\right)_{,j}$

$+ \lambda \left(\overline{m_A m_B}\right)_{,jj} - 2\lambda \overline{m_{A,j} m_{B,j}} + \overline{m_B r_A} + \overline{m_A r_B}$, $\quad$ /c/ $\quad \overline{u_j m_A m_B} = -\nu_t \left(\overline{m_A m_B}\right)_{,j}$ or

$\overline{u_j m_A m_B} = -c \dfrac{k R_{jm}}{\varepsilon} \left(\overline{m_A m_B}\right)_{,m}$, $\overline{m_{A,j} m_{B,j}} = \alpha \dfrac{\varepsilon}{k} \overline{m_A m_B}$.

No. 10 – We consider the correlation $H_{imj} = \overline{u_i u_{m,j}}$. /a/ What is the order of magnitude of H? /b/ Give an approximation using the method by Lumley at order 1 in anisotropy and at order 1 in non-homogenity. /c/ What about H_{iij} and H_{ijj}? /d/ Give an approximation at order zero in anisotropy and at order 1 in non-homogenity taking into account the relations obtained previously.

Indications: /a/ $\mathcal{O}(H) = u^2 / \ell$, /b/ $H_{imj} = A a_{ij,m} + B a_{im,j} + C a_{jm,i}$

$+ A_1 \delta_{ij} k_{,m} + B_1 \delta_{im} k_{,j} + C_1 \delta_{jm} k_{,i} + A_2 a_{ij} k_{,m} + B_2 a_{im} k_{,j} + C_2 a_{jm} k_{,i}$

$+ A_3 \delta_{ij} \varepsilon_{,m} + B_3 \delta_{im} \varepsilon_{,j} + C_3 \delta_{jm} \varepsilon_{,i} + A_4 a_{ij} \varepsilon_{,m} + B_4 a_{im} \varepsilon_{,j} + C_4 a_{jm} \varepsilon_{,i}$,

$A, B, C \propto k$, $A_1, B_1, C_1 \propto$ constants, $A_2, B_2, C_2 \propto$ constants, $A_3, B_3, C_3 \propto k/\varepsilon$,

$A_4, B_4, C_4 \propto k/\varepsilon$, /c/ $H_{iij} = 2 k_{,j}$, $H_{ijj} = 0$,

/d/ $H_{imj} = \alpha_1 k_{,m} \delta_{ij} + \dfrac{3-\alpha_1}{4} k_{,j} \delta_{im} - \dfrac{1+\alpha_1}{4} k_{,i} \delta_{jm} + \alpha_2 \varepsilon_{,m} \delta_{ij} - \dfrac{\alpha_2}{4} \varepsilon_{,j} \delta_{im} - \dfrac{\alpha_2}{4} \varepsilon_{,i} \delta_{jm}$

No. 11 – We consider the following two equation turbulence model in Cartesian axes xOy, applied to turbulent thin shear flows:

$$\frac{dk}{dt} = \ell\sqrt{k}\left(\frac{\partial U}{\partial y}\right)^2 - c_D \frac{k^{3/2}}{\ell} + \frac{\partial}{\partial y}\left(\frac{\ell\sqrt{k}}{\sigma_1}\frac{\partial k}{\partial y}\right),$$

$$\frac{d(k\ell)}{dt} = Cl1\ell^2\sqrt{k}\left(\frac{\partial U}{\partial y}\right)^2 - C_2 k^{3/2} + \frac{\partial}{\partial y}\left(\frac{\ell\sqrt{k}}{\sigma_2}\frac{\partial(k\ell)}{\partial y}\right),$$

where c_D, C_1, C_2, σ_1, σ_2 are numerical constants.

/a/ Give the expression for the turbulent shear stress τ, the turbulence viscosity ν_t and the dissipation rate ε of kinetic energy according to k and ℓ and their physical justification. /b/ We consider the case of decay of turbulence behind a grid. The experiment shows that k decreases approximatively according to power law x^α (x denoting the distance behind the grid and $\alpha = -1$). Deduce the value of C_2/c_D. /c/ We consider at present the case of the logarithmic region of a turbulent boundary layer. The friction velocity is denoted u_* and the Karman constant is denoted K . Determine k/u_*^2 and ν_t. /d/ Using the previous results, derive from the model, the expression for the constant C_1 according to the other constants of the model.

Indications: /a/ $\tau = -\rho\ell\sqrt{k}\left(\frac{\partial U}{\partial y}\right)$, $\nu_t = \ell\sqrt{k}$, /b/ $C_2/c_D = \dfrac{\alpha}{1+3\alpha/2} = 0.5$, /c/

$k/u_*^2 = c_D^{-1/2}$, $\nu_t = u_* K y$, /d/ $C_1 = \dfrac{C_2}{c_D} - \dfrac{K^2}{\sigma_2 c_D^{1/2}}$.

No. 12 – With the aim of studying turbulent free shear flows, we develop an algebraic modeling of the Reynolds stresses derived from the following transport model: $\dfrac{dR_{ij}}{dt} = P_{ij} + D_{ij} - c_1 \dfrac{\varepsilon}{k}(R_{ij} - \frac{2}{3}k\delta_{ij}) - \gamma(P_{ij} - \frac{2}{3}P\delta_{ij}) - \frac{2}{3}\varepsilon\delta_{ij}$ (Launder, Reece and Rodi model, simplified version RS2, notations Chapter 6).

The fluid is supposed to be incompressible and the turbulence is fully developed. The algebraic modeling is based on the hypothesis: $\dfrac{dR_{ij}}{dt} - D_{ij} = \dfrac{2}{3}\left(\dfrac{dk}{dt} - D\right)\delta_{ij}$.

/a/ Derive in the general case the system of algebraic equations allowing us to calculate the Reynolds stresses. /b/ We assume the approximations of thin shear flows, the main velocity direction of the flow is along Ox_1. Give the expressions for the components of the Reynolds stress tensor.

Show that in this case it is possible to introduce a turbulence viscosity coefficient $\nu_t = C_\mu k^2 / \varepsilon$ where C_μ is a function of $\lambda = P/\varepsilon$.

Indications: /a/ $(1-\gamma)\dfrac{P_{ij} - \dfrac{2}{3}P\delta_{ij}}{\varepsilon} = c_1 \dfrac{R_{ij} - \dfrac{2}{3}k\delta_{ij}}{k}$,   /b/ $R_{11} = \dfrac{2}{3}k\left(1 + 2\dfrac{1-\gamma}{c_1}\lambda\right)$,

$R_{22} = R_{33} = \dfrac{2}{3}k\left(1 - \dfrac{1-\gamma}{c_1}\lambda\right)$, $R_{12} = -C_\mu \dfrac{k^2}{\varepsilon}U_{1,2}$, $C_\mu = \dfrac{2}{3}(1-\gamma)\dfrac{c_1 - (1-\gamma)\lambda}{c_1^2}$.

No. 13 – We consider the fourth order tensor $T_{ijkl} = \overline{\nu u_{i,j} u_{k,l}}$. /a/ Show that in homogenous turbulence $T_{ijkl} = -\nu\left(\dfrac{\partial^2 Q_{ik}}{\partial \xi_j \partial \xi_l}\right)_{\vec{\xi}=0}$ where Q_{ik} denotes the two point double velocity correlation $Q_{ik} = \overline{u_i(\vec{x} = \vec{X}).u_k(\vec{y} = \vec{X} + \vec{\xi})}$. Then deduce the symmetry properties of T_{ijkl}. /b/ What about T_{ijij}, T_{jjkl}, T_{ijll} ? /c/ In the framework of Reynolds stress closures T_{ijkl} can be considered as a functional of the known quantities $R_{ij} = (Q_{ij})_{\vec{\xi}=0}$ and ε. Give an approximation of T_{ijkl} using the invariant modeling method of J.L. Lumley in homogenous turbulence limited to first order in anisotropy. /d/ From the previous result, give the approximation of T_{ijkl} in homogenous isotropic turbulence. /e/ Derive the approximation of T_{ijkl} in homogenous isotropic turbulence using this time the traditional relations for the two point double velocity correlations $Q_{ik}(\vec{\xi})$ in homogenous isotropic turbulence and the expansions of usual correlations f and g (longitudinal and transverse). Compare. /f/ From the approximation obtained in section /c/, derive the value of T_{ijji}. Was it possible to get this result directly from the definition of T_{ijkl} given at the beginning?

Indications: /a/ $\begin{cases} x = X \\ y = X + \xi \end{cases}$, $\begin{cases} \partial/\partial x = \partial/\partial X - \partial/\partial \xi \\ \partial/\partial y = \partial/\partial \xi \end{cases}$,   /b/ $T_{ijij} = \varepsilon$, $T_{jjkl} = T_{ijll} = 0$,

/c/ $T_{ijkl} = \varepsilon\left[\left(\dfrac{2}{15}\delta_{ik}\delta_{jl} - \dfrac{1}{30}\delta_{ij}\delta_{kl} - \dfrac{1}{30}\delta_{il}\delta_{jk}\right)\right]$

$$+c\left(a_{ij}\delta_{kl}+a_{il}\delta_{jk}+a_{kl}\delta_{ij}+a_{jk}\delta_{il}-5a_{jl}\delta_{ik}\right)+c'\left(a_{ik}\delta_{jl}-a_{jl}\delta_{ik}\right)\Big],$$

/d/ $T_{ijkl}=\left(\dfrac{2}{15}\delta_{ik}\delta_{jl}-\dfrac{1}{30}\delta_{ij}\delta_{kl}-\dfrac{1}{30}\delta_{il}\delta_{jk}\right)\varepsilon,$

/e/ $Q_{ik}(\vec{\xi})=u_0^{\,2}\left(\dfrac{f(\xi)-g(\xi)}{\xi^2}\xi_i\xi_k+g(\xi)\delta_{ik}\right),$ $f=1-\xi^2/2\lambda^2,\ g=1-\xi^2/\lambda^2,$

$\varepsilon=15\nu u_0^{\,2}/\lambda^2$, same result as previously, /f/ $T_{ijji}=0$, $T_{ijji}=\nu R_{ij,ij}$ is zero in homogenous turbulence.

No. 14 – /a/ Derive an approximation of the correlation $D_j=\dfrac{1}{2}\overline{u_ju_mu_m}$ using the method of invariant modeling of J.L. Lumley (approximation at first order in non-homogenity and order 1 in anisotropy). /b/ Is it possible to get some of the terms appearing in this approximation directly from phenomenological reasoning?

Indications:

/a/ $D_j=\alpha_0\dfrac{k^2}{\varepsilon}k_{,j}+\alpha_1\dfrac{k^3}{\varepsilon^2}\varepsilon_{,j}+\alpha_2\dfrac{k^2}{\varepsilon}a_{jm}k_{,m}+\alpha_3\dfrac{k^3}{\varepsilon^2}a_{jm}\varepsilon_{,m}+\alpha_4k^{3/2}a_{jm,m},$

/b/ gradient hypothesis $D_j=-c\dfrac{kR_{jm}}{\varepsilon}k_{,m}$ (terms in α_0 and α_2).

No. 15 – We consider an incompressible turbulent plane jet developing in stagnant surroundings. The flow is considered at sufficiently large distance from the nozzle for the self-similarity conditions to be satisfied. The momentum equations written in the thin shear flow approximations are:

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(\nu_t\frac{\partial U}{\partial y}\right),\qquad \frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0$$

/a/ Determine the boundary conditions of the problem. /b/ Derive the equation satisfied by the stream function Ψ. /c/ A similarity variable is sought in the form $\eta=y/\delta(x)$ where $\delta(x)$ is the flow thickness. To do this, we introduce the dimensionless stream function f defined by: $\Psi=-U_e(x)\delta(x)f(\eta)$ where $U_e(x)$ is the mean flow velocity on the axis of the jet. It is supposed that all quantities are self-similar, in particular, the shear stress can be written: $\tau/\rho=U_e^{\,2}(x).g(\eta)$. /d/

After performing the variable change $(x, y) \to (X, \eta)$ with $X = x$ and $\eta = y/\delta(x)$, derive the equation for f and show that the similarity conditions can be written as:

$\dfrac{1}{U_e}\dfrac{d(U_e\delta)}{dx} = C_1$ and $\dfrac{d\delta}{dx} = C_2$. /e/ Show using integration of the momentum

equation that $U_e^2\delta(x) = C_3$. Express C_2 according to C_1. /f/ Using the Görtler hypothesis on turbulence eddy viscosity $v_t = \chi\delta(x)U_e(x)$ where χ is a numerical constant, express g according to f. The equation for f can then be solved analytically and it is possible to express U/U_e according to η.

Indications:

/a/ $V = 0$ and $\partial U/\partial y = 0$ for $y=0$ and $U = 0$ for $y \to \infty$, /b/ $U = \partial\Psi/\partial y$, $V = -\partial\Psi/\partial x$,

$\dfrac{\partial\Psi}{\partial y}\dfrac{\partial^2\Psi}{\partial x\partial y} - \dfrac{\partial\Psi}{\partial x}\dfrac{\partial^2\Psi}{\partial y^2} = \dfrac{1}{\rho}\dfrac{\partial\tau}{\partial y}$, /c/ $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial X} - \dfrac{\eta}{\delta}\dfrac{d\delta}{dx}\dfrac{\partial}{\partial\eta}$, $\dfrac{\partial}{\partial y} = \dfrac{1}{\delta}\dfrac{\partial}{\partial\eta}$, $\dfrac{\partial^2}{\partial y^2} = \dfrac{1}{\delta^2}\dfrac{\partial^2}{\partial\eta^2}$,

/d/ $\dfrac{1}{U_e}\dfrac{d(U_e\delta)}{dx}\left(f'^2 - f.f''\right) - \dfrac{d\delta}{dx}f'^2 - g' = 0$, separation of variables,

/e/ $\rho\displaystyle\int_{-\infty}^{+\infty}U^2\,dy = J$, J constant momentum flux,

$\rho U_e^2\delta\displaystyle\int_{-\infty}^{+\infty}f'^2\,d\eta = J \Rightarrow U_e^2\delta(x) = C_3$, $\delta = C_2 x$, $U_e = \sqrt{\dfrac{C_3}{C_2 x}}$, $C_2 = 2C_1$,

/f/ $g(\eta) = \chi f''(\eta)$, $C_1(ff'' + f'^2) + \chi f''' = 0$,

$f = 0$, $f' = 1$ for $\eta = 0$ and $f' = 0$ for $\eta \to \infty$,

$C_1 ff' + \chi f'' = C_0$, $C_0 = 0$, $\dfrac{C_1}{2\chi}f^2 + f' = 1$, $f = \sqrt{\dfrac{2\chi}{C_1}}Th\sqrt{\dfrac{C_1}{2\chi}}\eta$,

$\dfrac{U}{U_e} = f' = \dfrac{1}{Ch^2\sqrt{\dfrac{C_1}{2\chi}}\eta}$,

If we define δ such that $\eta = y/\delta = 1$ when $U = 0.5U_e$, $\sqrt{C_1/2\chi} \simeq 0.8815$.

The experiment gives $d\delta/dx = C_2 = 0.1$, $\Rightarrow \chi \simeq 0.0321$.

No. 16 – We consider a square channel with a generating grid of turbulence mounted at the inlet followed by a sudden contraction. We suppose that the grid creates homogenous and isotropic turbulence (for this, the observer is located far

from the walls, near the center of the channel). /a/ In the contraction region, we apply the Reynolds stress model (notations Chapter 6):

$$\frac{dR_{ij}}{dt} = P_{ij} - c_1 \frac{\varepsilon}{k}(R_{ij} - \frac{2}{3}k\delta_{ij}) - \gamma(P_{ij} - \frac{2}{3}P\delta_{ij}) - \frac{2}{3}\varepsilon\delta_{ij}, \quad (1)$$

$$\frac{d\varepsilon}{dt} = C_{\varepsilon 1}\frac{P\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k}, \quad (2)$$

The contraction is supposed to be rapid; what is the simplified form of the transport equation for R_{ij} applicable in this case ?

/b/ We suppose that the sudden contraction acts on the initial isotropic turbulence during a very short time δt. Express P_{ij} and P. /c/ Show that the Reynolds stress tensor after contraction R_{ij}^+ is given by $R_{ij}^+ = R_{ij}^0 + H_{ij}\delta t$ where R_{ij}^0 is the Reynolds stress tensor before contraction. Give the explicit expression of H_{ij} according to k and the mean velocity gradient $U_{i,j}$ only.

Apply to the particular case of the axisymmetric contraction defined by:

$$\left[U_{i,j}\right] = \begin{bmatrix} b & 0 & 0 \\ 0 & -b/2 & 0 \\ 0 & 0 & -b/2 \end{bmatrix}, \quad (b = \text{constant})$$

in order to obtain R_{11}^+, R_{22}^+, R_{33}^+ and k^+.

Indications: /a/ $\dfrac{dR_{ij}}{dt} = P_{ij} - \gamma(P_{ij} - \dfrac{2}{3}P\delta_{ij})$, /b/ $P_{ij} = -\dfrac{2}{3}k(U_{i,j} + U_{j,i})$, $P=0$,

/c/ $H_{ij} = -\dfrac{2}{3}(1-\gamma)k(U_{i,j} + U_{j,i})$, $R_{11}^+ = \dfrac{2k^0}{3}[1 - 2(1-\gamma)b.\delta t]$,

$R_{22}^+ = R_{33}^+ = \dfrac{2k^0}{3}[1 + (1-\gamma)b.\delta t]$, $k^+ = k^0$.

No. 17 – We consider the two equation turbulence model (k-ω) in a Cartesian frame:

$$\frac{dk}{dt} = \frac{k}{\omega}(U_{i,j} + U_{j,i})U_{i,j} - \beta k\omega + \left(\frac{k}{\sigma_1\omega}k_{,j}\right)_{,j}$$

$$\frac{d\omega}{dt} = C_{\omega1}(U_{i,j} + U_{j,i})U_{i,j} - C_{\omega2}\omega^2 + \left(\frac{k}{\sigma_2\omega}\omega_{,j}\right)_{,j}$$

where β, $C_{\omega1}$, $C_{\omega2}$, σ_1 and σ_2 are numerical constants.

/a/ Write the equations in xOy axes with the boundary layer approximation. /b/ For calibrating the model, determine the numerical constants by referring on the one hand to the case of decay of turbulence behind a grid (the experiment shows that k decreases according to a power law $x^{-1.1}$, x denoting the distance behind the grid), and on the other hand to the case of the logarithmic region of a turbulent boundary layer (the experiment shows that $\tau/\rho k \approx 0.3$). We shall assume $\sigma_1 = 1$ and $\sigma_2 = 1.3$.

Indications:

/a/ $\dfrac{dk}{dt} = \dfrac{k}{\omega}\left(\dfrac{\partial U}{\partial y}\right)^2 - \beta k\omega + \dfrac{\partial}{\partial y}\left(\dfrac{k}{\sigma_1\omega}\dfrac{\partial k}{\partial y}\right)$, $\dfrac{d\omega}{dt} = C_{\omega1}\left(\dfrac{\partial U}{\partial y}\right)^2 - C_{\omega2}\omega^2 + \dfrac{\partial}{\partial y}\left(\dfrac{k}{\sigma_2\omega}\dfrac{\partial\omega}{\partial y}\right)$

/b/ Grid turbulence: $\beta/C_{\omega2} = 1.1$, boundary layer: $k/u_*^2 = \beta^{-1/2}$, $\tau/\rho = -u_*^2$,

$\nu_t = k/\omega$, $C_{\omega1} = \dfrac{C_{\omega2}}{\beta} - \dfrac{K^2}{\sigma_2\beta^{1/2}}$.

No. 18 – /a/ Give the equation for the fluctuating passive scalar in homogenous turbulence in physical space and in Fourier space. /b/ Express the linear part $\varphi^{(2)}_{\gamma i}(\vec{\kappa})$ of the pressure-gradient of the scalar $\widehat{u_i(\vec{\kappa})\hat{\gamma}(-\vec{\kappa})} = \varphi_{\gamma i}(\vec{\kappa})$ in Fourier space and in physical space, /c/ We assume $\varphi^{(2)}_{\gamma i}(\vec{\kappa}) = \dfrac{a}{4\pi\kappa^2}\phi^{(2)}_{\gamma i}(\kappa) + \dfrac{b}{4\pi\kappa^4}\kappa_i\kappa_m\phi^{(2)}_{\gamma m}(\kappa)$, determine a and b. /d/ Derive the corresponding approximation for $\Phi^{(2)}_{\gamma i}$ and compare to the quasi-isotropic approximation in Chapter 9.

Indications:

/a/ $\dfrac{\partial\gamma}{\partial t} + \overline{U}_j\gamma_{,j} = -u_j\overline{\Gamma}_{,j} - u_j\gamma_{,j} + \sigma\gamma_{,jj}$,

$\left(\dfrac{\partial}{\partial t} + \sigma\kappa^2\right)\hat{\gamma} + i\kappa_j\overline{U}^0_j\hat{\gamma} - \kappa_j\Lambda_{jm}\dfrac{\partial\hat{\gamma}}{\partial\kappa_m} = -\widehat{u_j}\overline{\Gamma}_{,j} - i\kappa_j\left(\hat{u}_j * \hat{\gamma}\right)$

/b/ From $\dfrac{d\varphi_{\gamma i}(\vec{\kappa})}{dt} = \overline{\dfrac{d\hat{u}_i(\vec{\kappa})}{dt}\hat{\gamma}(-\vec{\kappa})} + \overline{\hat{u}_i(\vec{\kappa})\dfrac{d\hat{\gamma}(-\vec{\kappa})}{dt}}$, only the first term gives a

contribution coming from $\left(\dfrac{\partial}{\partial t} + \nu\kappa_j^2\right)\hat{u}_i = -2\dfrac{\kappa_i\kappa_h}{\kappa_m^2}\Lambda_{hj}\hat{u}_j + \ldots$, or

$\phi^{(2)}{}_{\gamma i} = 2\dfrac{\kappa_i\kappa_h}{\kappa_m^2}\Lambda_{hj}\varphi_{\gamma j}$, $\Phi^{(2)}{}_{\gamma i} = \int_{\mathbb{R}^3}\phi^{(2)}{}_{\gamma i}d\mathscr{V} = 2\Lambda_{hj}\int_{\mathbb{R}^3}\dfrac{\kappa_i\kappa_h}{\kappa^2}\varphi^{(2)}{}_{\gamma j}d\mathscr{V}$,

/c/ $\varphi^{(2)}{}_{\gamma i}(\vec{\kappa})\kappa_i = 0 \Rightarrow a + b = 0$ and $F_{\gamma i} = \iiint_{\mathbb{R}^3}\varphi_{\gamma i}(\vec{\kappa})d\kappa \Rightarrow a + \dfrac{1}{3}b = 1$, since

$F_{\gamma i} = \int_0^\infty \phi_{\gamma m}(\kappa)d\kappa\int_{S(\kappa)}\left(a\dfrac{\delta_{im}}{4\pi\kappa^2} + b\dfrac{\kappa_i\kappa_m}{4\pi\kappa^4}\right)dS(\kappa) = F_{\gamma m}\int_{S(\kappa)}\left(a + \dfrac{1}{3}b\right)\delta_{im}dS(\kappa)$

$a = 3/2$, $b = -3/2$, /d/ $\Phi^{(2)}{}_{\gamma i} = 2\Lambda_{hj}\int_{\mathbb{R}^3}\dfrac{\kappa_i\kappa_h}{\kappa^2}\left(\dfrac{a}{4\pi\kappa^2}\phi_{\gamma j} + \dfrac{b}{4\pi\kappa^4}\kappa_j\kappa_m\phi_{\gamma m}\right)d\mathscr{V}$,

$\Phi^{(2)}{}_{\gamma i} = 2\Lambda_{hj}\int_0^\infty \phi_{\gamma m}d\kappa\int_{S(\kappa)}\dfrac{3}{8\pi\kappa^2}\left(\dfrac{\kappa_i\kappa_h}{\kappa^2}\delta_{jm} - \dfrac{\kappa_i\kappa_h\kappa_j\kappa_m}{\kappa^4}\right)dS(\kappa)$,

$\Phi^{(2)}{}_{\gamma i} = \Lambda_{hj}F_{\gamma m}\left(\delta_{ih}\delta_{jm} - \dfrac{1}{5}\left(\delta_{ih}\delta_{jm} + \delta_{ij}\delta_{hm} + \delta_{im}\delta_{hj}\right)\right) = 0.8\Lambda_{ij}F_{\gamma j} - 0.2\Lambda_{ji}F_{\gamma j}$

(use formulae in section 5.3).

No. 19 – We consider the non-linear model by Speziale (section 11.4.1), derive the components of the Reynolds stresses in the case of the logarithmic region of a boundary layer.

Indications:

$\Lambda = U_{1,2}$ is the only velocity gradient which is non-zero,

$S_{ij}S_{ij} = \Lambda^2/2$, $\overset{0}{D}S_{hh} = -\Lambda^2$,

$R_{12} = -\nu_t\Lambda$, $\nu_t = c_\mu k^2/\varepsilon$, $\nu_t\Lambda^2 = \varepsilon$, $k\Lambda/\varepsilon = c_\mu^{-1/2}$,

$\dfrac{R_{11}}{k} = \dfrac{2}{3} + 4c_\mu\left(-\dfrac{c_D}{12} + \dfrac{2c_E}{3}\right)$, $\dfrac{R_{22}}{k} = \dfrac{2}{3} + 4c_\mu\left(-\dfrac{c_D}{12} - \dfrac{c_E}{3}\right)$,

$\dfrac{R_{33}}{k} = \dfrac{2}{3} + 4c_\mu\left(+\dfrac{c_D}{6} - \dfrac{c_E}{3}\right)$, $\dfrac{R_{12}}{k} = -c_\mu^{1/2}$

No. 20 – The k-ε model is applied to a turbulent homogenous shear flow. The mean velocity gradient is constant $U_{1,2} = \Lambda$. /a/ Show that there is an exponential solution of type $k = k_0 e^{\lambda t}$, $\varepsilon = \varepsilon_0 e^{\lambda t}$, $R_{ij} = R_{0ij} e^{\lambda t}$ and that P/ε goes to a constant. /b/ Express λ according to ε/k.

Indications:

/a/ $\dot{k} = \lambda k$, $\dot{\varepsilon} = \lambda \varepsilon \Rightarrow (C_{\varepsilon 2} - 1)\dfrac{\varepsilon_0}{k_0} = (C_{\varepsilon 1} - 1) c_\mu \dfrac{k_0}{\varepsilon_0} \Lambda^2$, $P = c_\mu \dfrac{k^2}{\varepsilon} \Lambda^2$ implies

$\dfrac{P}{\varepsilon} \to \dfrac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1}$, /b/ $\lambda = c_\mu \dfrac{k_0}{\varepsilon_0} \Lambda^2 - \dfrac{\varepsilon_0}{k_0}$ and $\dfrac{\lambda}{C_{\varepsilon 2}} = \dfrac{C_{\varepsilon 1}}{C_{\varepsilon 2}} c_\mu \dfrac{k_0}{\varepsilon_0} \Lambda^2 - \dfrac{\varepsilon_0}{k_0}$,

$\Rightarrow \lambda = \dfrac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon 2} - 1} \cdot \dfrac{P}{k}$ and $\lambda = \dfrac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon 1} - 1} \cdot \dfrac{\varepsilon}{k}$

No. 21 – Determine the realizability condition of the simplified second order model RS2 (Chapter 6).

Indication: $c_1 \geq 1 - \gamma \dfrac{P}{\varepsilon}$.

No. 22 – Establish the Reynolds equation in general tensorial form (non-Cartesian tensors).

Indications: $\dfrac{\partial \overline{U}^i}{\partial t} + \overline{U}^m \overline{U}^i{}_{,m} = -g^{im} \dfrac{\overline{P}_{,m}}{\rho} - \left(\overline{u^m u^i}\right)_{,m} + \nu g^{rm} \overline{U}^i{}_{,rm} + 2\varepsilon^{irm} \Omega_r \overline{U}_m$

No. 23 – Establish the Navier-Stokes equations in cylindrical coordinates on an orthonormal local frame.

Indications: (cf. mnemotechnic method of section 3.3)

$$\partial_t U_r + U_r \partial_r U_r + \dfrac{U_\theta}{r} \partial_\theta U_r + U_z \partial_z U_r - \dfrac{U_\theta^2}{r} = -\partial_r \dfrac{p}{\rho}$$

$$+ \dfrac{1}{r} \partial_r \left(r T_{rr}\right) + \dfrac{1}{r} \partial_\theta T_{r\theta} + \partial_z T_{rz} - \dfrac{T_{\theta\theta}}{r}$$

$$\partial_t U_\theta + U_r \partial_r U_\theta + \dfrac{U_\theta}{r} \partial_\theta U_{r\theta} + U_z \partial_z U_\theta - \dfrac{U_r U_\theta}{r} = -\dfrac{1}{r} \partial_\theta \dfrac{p}{\rho}$$

$$+\frac{1}{r^2}\partial_r\left(r^2 T_{\theta r}\right)+\frac{1}{r}\partial_\theta T_{\theta\theta}+\partial_z T_{\theta z}$$

$$\partial_t U_z +U_r\partial_r U_z +\frac{U_\theta}{r}\partial_\theta U_z +U_z\partial_z U_z = -\partial_z\frac{p}{\rho}$$

$$+\frac{1}{r}\partial_r\left(r T_{zr}\right)+\frac{1}{r}\partial_\theta T_{z\theta}+\partial_z T_{zz}$$

$$\partial_r\left(r U_r\right)+\partial_\theta U_\theta +r\partial_z U_z = 0$$

$$T_{rr}=2\nu\partial_r U_r ,\ \ T_{\theta\theta}=2\nu\left(\frac{\partial_\theta U_\theta}{r}+\frac{U_r}{r}\right),\ \ T_{zz}=2\nu\partial_z U_z ,$$

$$T_{r\theta}=\nu\left(\frac{1}{r}\partial_\theta U_r +\partial_r U_\theta -\frac{U_\theta}{r}\right),\ T_{z\theta}=\nu\left(\frac{1}{r}\partial_\theta U_z +\partial_z U_\theta\right),$$

$$T_{rz}=\nu\left(\partial_r U_z +\partial_z U_r\right).$$

-oOo-

This page intentionally left blank

Bibliography

General reference books

[*ABR 63] ABRAMOVICH G.N., *The Theory of Turbulent Jets*, The MIT press, Cambridge, Massachusetts, 1963.

[*AME 77] AMES W.F., *Numerical Methods for Partial Differential Equations*, Academic Press and Thomas Nelson and Son, 1977.

[*BAI 03] BAILLY Ch., COMTE-BELLOT G., *Turbulence*, Sciences et Techniques de l'Ingénieur, CNRS Editions, 2003.

[*BAS 67] BASS J., *Éléments de calcul des probabilités théoriques et appliqués*, Masson et Cie ed., Paris, 1967.

[*BAT 56] BATCHELOR G.K., *The Theory of Homogenous Turbulence*, Cambridge University Press, 1956.

[*BER 84] BERGE P., POMMEAU Y., VIDAL CH., *L'ordre dans le chaos*, Hermann, 1984.

[*BER 92] BERNARDI C. and MADAY Y., *Approximations spectrales de problèmes aux limites elliptiques*, Springer-Verlag France, Paris, 1992.

[*BER 05] BERSELLI L.C., ILIESCU T., LAYTON W.J., *Mathematics of Large Eddy Simulation of Turbulent Flows*, Springer, 2005.

[*BLA 53] BLANC-LAPIERRE A., FORTET R., *Théorie des fonctions aléatoires et application à divers phénomènes de fluctuations*, Masson et Cie ed., Paris, 1953.

[*BRA 81] BRADSHAW P., CEBECI T. and WHITELAW J.H., *Engineering calculation methods for turbulent flows*, Academic Press, 1981.

[*BRI 60] BRILLOUIN L., *Les tenseurs en mécanique et en élasticité. Cours de Physique théorique*, Masson et Cie, Paris, 1960.

[*CAN 88] CANUTO C., HUSSAINI M.Y., QUARTERONI A., ZANG T.A., *Spectral methods in fluid dynamics*, Springer-Verlag, 1988

[*CEB 74] CEBECI T. and SMITH A.M.O., *Analysis of Turbulent Boundary Layers*, Ser. in Appl. Meth. Math. and Mech., vol. XV, Academic Press, 1974.

[*CHA 00] CHASSAING P., *Turbulence en mécanique des fluides*, Cépaduès, 2000.

[*CHO 68] CHOQUET-BRUHAT Y., *Géométrie différentielle et systèmes extérieurs*, Dunod, 1968.

[*COU 66] COURANT R. and HILBERT D., *Methods of mathematical physics*, vol. I and II, Interscience Publishers and John Wiley, 1966.

[*COU 88] COUSTEIX J., *Aérodynamique: turbulence et couche limite*, Cépaduès, 1988.

[*DAU 84] DAUTREY R., LIONS J.L., *Analyse mathématiques et calcul numérique pour les sciences et les techniques*, 9 volumes, Masson, Paris, 1984.

[*DEI 98] DEISSLER R. C., *Turbulent Fluid Motion*, Ed. Taylor and Francis, Combustion, An International Series, 1998.

[*DHA 81] DHATT G. and TOUZOT G., *Une présentation de la méthode des éléments finis*, Les Presses de l'Université de Laval (Quebec) and Maloine S.A., 1981.

[*DUR 01] DURBIN P. A., PETTERSSON REIF B. A., *Statistical Theory and Modeling for Turbulent Flows*, Wiley, 2001.

[*FAV 76] FAVRE A., KOVASZNAY L.S.G., DUMAS R., GAVIGLIO J., COANTIC M., *La turbulence en Mécanique des Fluides. Bases théoriques et expérimentales*, Paris, Gauthier-Villars, 1976.

[*FAV 88] FAVRE A., GUITTON H., GUITTON J., LICHNEROWICZ A. and WOLFF E., *De la causalité à la finalité: a propos de la turbulence*, Maloine, 1988.

[*FER 96] FERZIGER J.H. and PERIC M., *Computational Methods for Fluid Dynamics*, Springer, 1996.

[*FLE 88] FLETCHER C.A.J., *Computational Methods for Fluid Dynamics*, vol. I and II, Springer-Verlag, 1988.

[*FRI 62] FRIEDLANDER J.K. and TOPPER L., *Turbulence. Classic Papers on Statistical Theory*, Interscience Publishers, London, 1962.

[*FRI 95] FRISCH U., *Turbulence*, *the Legacy of A.N. Kolmogorov*, Cambridge University Press, 1995.

[*GAL 02] GALLAVOTTI G., *Foundations of Fluid Dynamics*, Springer, 2002.

[*GAT 96] GATSKI T.B., HUSSAINI M.Y. and LUMLEY J.L. (ed.), *Simulation and modeling of turbulent flows*, ICASE/LaRC Series in computational science and engineering, Oxford University Press, 1996.

[*GEU 01] GEURTS B.J. (ed.), *Modern Simulation Strategies for Turbulent Flow*, Edwards, 2001.

[*GEU 04] GEURTS B.J., *Elements of Direct and large-Eddy Simulation*, Edwards, 2004.

[*GOT 77] GOTTLIEB D., ORSZAG S. A., *Numerical analysis of spectral methods: theory and applications*, SIAM (Soc. for Industrial and Appl. Math.), Philadelphia, 1977.

[*HAI 87] HAIRER E., NORSETT S.P., WANNER G., *Solving Ordinary Differential Equations*, *I-Non Stiff Problems*, Springer, Berlin, 1987.

[*HAI 91] HAIRER E., WANNER G., *Solving Ordinary Differential Equations*, *II-Stiff and Differential-algebraic Problems*, Springer, Berlin, 1991.

[*HIL 00] HILBORN R.C., *Chaos and Non-linear Dynamics*, Oxford University Press, 2000.

[*HIN 75] HINZE J.O., *Turbulence*, McGraw Hill Book Company, London, 2nd ed., 1975.

[*JOS 76] JOSEPH D.D., *Stability of fluid motions*, Springer Tracts in Natural Philosophy, 1976.

[*JOU 01] JOU D., CASAS-VÀZQUEZ J., LEBON G., *Extended Irreversible Thermodynamics*, Springer, 2001.

[*KRE 67] KREITH F., *Transmission de la chaleur et thermodynamique*, Masson et Cie, Paris, 1967.

[*LAU 72] LAUNDER B.E., SPALDING D.B., *Mathematical models of turbulence*, Academic Press, London, 1972.

[*LAU 02] LAUNDER B.E., SANDHAM N.D. (Eds.), *Closure Strategies for Turbulent and Transitional Flows*, Cambridge University Press, 2002.

[*LEG 71] LEGRAS J., *Méthodes et techniques de l'analyse numérique*, Dunod, 1971.

[*LES 73] LESLIE D.C., *Developments in the Theory of Turbulence*, Clarendon Press, Oxford, 1973.

[*LES 90] LESIEUR M., *Turbulence in Fluids*, Kluwer Acad. Publi. Groups, The Netherlands, (2nd ed .), 1990.

[*LES 94] LESIEUR M., *La Turbulence*, EDP Sciences, Presses Universitaires de Grenoble, Collection Grenoble Sciences, 1994.

[*LES 05] LESIEUR M., MÉTAIS O., COMTE P., *Large Eddy Simulations of Turbulence*, Cambridge University Press, 2005.

[*LIC 67] LICHNEROWICZ A., *Calcul tensoriel*, Colin, 1967.

[*LUM 70] LUMLEY J.L., *Stochastic Tools in Turbulence*, Academic Press, 1970.

[*MAT 00] MATHIEU J., SCOTT J., *An Introduction to Turbulent Flow*, Cambridge University Press, 2000.

[*MCA 58] MC ADAMS W.H., *Heat Transmission*, McGraw Hill B.C., 1958.

[*MCC 96] MC COMB W.D., *The Physics of Fluid Turbulence*, Clarendon Press, Oxford Engineering Science Series 25, Oxford Science Publ., 1996.

[*MET 97] METAIS O. and FERZIGER J.L. (ed.), *New Tools in Turbulence Modeling. Les Houches School*, Springer, Les Editions de Physique, 1997.

[*MIR 80] MIRA C. and GUMOWSKI I., *Dynamique chaotique*, Cépaduès, 1980.

[*MOH 94] MOHAMMADI B. and PIRONNEAU O., *Analysis of the k-epsilon Turbulence Model*, P.G. Ciarlet, J.L. Lions (ed.), Research in applied mathematics, Wiley, Masson, 1994.

[*MON 71] MONIN A.S., YAGLOM A.M., *Statistical Fluid Mechanics*, vol. I and II, The MIT Press, Cambridge, Massachusetts, 1971.

[*PAN 71] PANCHEV S., *Random Functions and Turbulence*, Int. Series of Monographs in Natural Philosophy, Pergamon Press, Oxford, 1971.

[*PAT 70] PATANKAR S.V. and SPALDING D.B., *Heat and Mass Transfer in Boundary Layers*, *A General Calculation Procedure*, Intertext Books, London, 1970.

[*PAT 80] PATANKAR S.V., *Numerical Heat Transfer and Fluid Flow*, Series in computational methods in mechanics and thermal sciences, Hemisphere Publishing Corporation and McGraw-Hill B.C., 1980.

[*PEY 83] PEYRET R. and TAYLOR T.D., *Computational Methods for Fluid Flow*, Springer-Verlag, 1983.

[*PIQ 99] PIQUET J., *Turbulent Flows*, *Models and Physics*, Springer, 1999.

[*POP 00] POPE S.B., *Turbulent Flows*, Cambridge University Press, 2000.

[*RIC 67] RICHTMYER R.D. and MORTON K.W., *Difference Methods for Initial Value Problems*, Interscience Publishers, 1967.

[*ROA 72] ROACHE P., *Computational Fluid Dynamics*, Hermosa Publishers, Albuquerque, 1972.

[*SCH 68] SCHLICHTING H., *Boundary Layer Theory*, McGraw Hill Book Company, London, 1968.

[*SCH 84] SCHWARTZ L., *Théorie des distributions*, Hermann, Paris, 1984.

[*SPE 71] SPENCER A.J.M., *Theory of Invariants*. In Continuum Physics, Eringen A.C. (ed.), Academic Press, New York, 1971

[*SWI 81] SWINNEY H.L., GOLLUB J.P., *Hydrodynamic Instabilities and Transition to Turbulence*, Springer-Verlag, 1981

[*TEN 72] TENNEKES H. and LUMLEY J.L., *A First Course in Turbulence*, The MIT Press, Cambridge, Massachusetts, 1972

[*TOW 56] TOWNSEND A.A., *The Structure of Turbulent Shear Flow*, Cambridge University Press, 1956

[*WIL 93] WILCOX D.C., *Turbulence Modeling for CFD*, DCW Industries Inc. ed., La Canada, California, USA, 1993

[*ZIE 71] ZIENKIEWICZ O.C., *Finite Element Method in Science and Engineering*, McGraw Hill, London, 1971

[*COLL 84A] *La simulation des modèles de turbulence et leurs applications*, *Simulation of turbulence models and their applications*, Ecole d'Eté d'Analyse Numérique CEA.EDF.INRIA, Volume 1 (Mathieu J., Jeandel D., Launder B.E., Reynolds W.C., Rodi W.) and Volume 2 (Launder B.E., Reynolds W.C., Rodi W., Mathieu J., Jeandel D.) Collection de la Direction des Études et Recherches EDF, Eyrolles, 1984.

[*COLL 78A] *Numerical Methods in Laminar and Turbulent Flows*, Taylor C., Morgan K., Brebbia C.A. (ed.), Proc. 1st Int. Conf. Univ. Coll. Swansea, Pentech Press, 1978.

[*COLL 69] *Proceedings Computation of Turbulent Boundary Layers*, AFOVR – IFP Stanford Conference (1968). Stanford University Press, California, 1969.

[*COLL 84B] *Proceedings Stanford Conf. on Complex Turbulent Flows. Comparison of computations and experiments*. AFOSR-HTTM, September 14–18, 1981, Stanford University, California, USA, 1981.

[*COLL 78B] *Turbulence – Topics in Applied Physics*, Vol. 12, P. Bradshaw (ed.), Springer-Verlag, 2nd ed., 1978.

[*COLL 89] *Whither Turbulence? Turbulence at the Crossroads*, J.L. Lumley (ed.), Proc. Ithaca NY, Lectures Notes in Physics, no. 357, Springer-Verlag, 1989.

Articles in journals

[ABE 94] ABE K., KONDOH T. and NAGANO Y., "A new turbulence model for predicting fluid flow and heat transfer in separating and reattaching flows-I. Flow field calculations", *Int. J. Heat Mass Transfer*, 37(1), 1994, p. 139–151.

[ABE 95] ABE K., KONDOH T. and NAGANO Y., "A new turbulence model for predicting fluid flow and heat transfer in separating and reattaching flows-II. Thermal field calculations", *Int. J. Heat Mass Transfer*, 38(8), 1995, p. 1467–1481.

[ABE 96] ABE K., KONDOH T., NAGANO Y., "A two-equation heat transfer model reflecting second moment closures for wall and free turbulent flows", *Int. J. Heat Fluid Flow*, 17, 1996, p. 228–237.

[ABE 97] ABE K., KONDOH T., NAGANO Y., "On Reynolds-stress expressions and near-wall scaling parameters for predicting wall and homogenous turbulent shear flows", *Int. J. Heat Fluid Flow*, 18(3), 1997, p. 266–282.

[ACK 01] ACKERMANN C., MÉTAIS O., "A modified selective structure function subgrid-scale model", *Journal of Turbulence*, 2, e-journal, 011, 2001, (http://jot.iop.org).

[ACT 80] ACTON E., "A modeling of large eddies in an axisymmetric jet", *J. Fluid Mech.*, 98(1), 1980, p.1–31.

[AGO 50] AGOSTINI L. and BASS J., "Les Théories de la turbulence", *Publ. Sci. Tech. Min. Air*, no. 237 a, 1950.

[AKS 95] AKSELVOLL K., MOIN P., "Large-eddy simulation of turbulent confined coannular jets and turbulent flow over a backward facing step", Thermosciences Div., *Dept. Mech. Engng. Stanford Univ. Stanford Report* TF-63, Stanford, CA-94395, USA, 1995.

[ALP 78] ALPER A. and LIU J.T.C., "On the interaction between large-scale structure and fine grained turbulence in a free shear flow. The development of spatial interactions in the mean", *Proc. Roy. Soc. London A*, 359, no. 1699, 1978, p. 497–523.

[AMA 83] AMANO R.S., JENSEN M.K. and GOEL P., "A numerical and experimental investigation of turbulent heat transport downstream from an abrupt pipe expansion", *J. of Heat Transfer*,105, 1983, p. 862–872.

[AMA 84] AMANO R.S., "Development of a turbulence near wall model and its applications to separated and reattached flows", *Num. Heat Transfer*, 17, 1984, p. 59–75.

[AMS 70] AMSDEN A.A. and HARLOW F.H., "The SMAC Method", *Los Alamos Scient. Lab. Report*, Univ. of California, LA-4370, 1970.

[AMS 73] AMSDEN A.A. and HIRT C.W., "A simple scheme for generating general curvilinear grids", *J. Comp. Phys.*, 11, 1973, p. 348.

[AND 77] ANDRÉ J.C., DE MOOR G., LACARRERE P., THIERRY G. and DU VACHAT R., "The clipping approximation for inhomogenous turbulence", *Proc. 1st Symp. on Turb. Shear Flows*, Penn. State, Ed. Durst F. *et al.*, 1977, p.4/39–4/46.

[ANS 84] ANSELMET F., GAGNE Y., HOPFINGER E.J., ANTONIA R.A., "High-order velocity structure functions in turbulent shear flows", *J. Fluid Mech.*, 140, 1984, p.63–89.

[ANT 81] ANTONOPOULOS M. and DOMIS, "Large eddy simulation of a passive scalar in isotropic turbulence", *J. Fluid Mech.*, 104, 1981, p. 55–80.

[APS 98] APSLEY D.D., LESCHZINER M.A., "A new low-Reynolds number non-linear two-equation turbulence model for complex flows", *Int. J. Heat Fluid Flow*, 19, 1998, p. 209–222.

[AUB 91] AUBRY N., "On the hidden beauty of the proper orthogonal decomposition", *J. Theor. and Comput. Fluid Dynamics*, 2, 1991, p. 339–352.

[AUP 82] AUPOIX B. and COUSTEIX J., "Modèles simples de tension de sous-maille en turbulence homogène isotrope", *La Recherche Aérospatiale*, no. 1982–4, 1982, p. 273–283.

[AUP 86] AUPOIX B., "Subgrid scale models for homogenous anisotropic turbulence". *Notes on Numerical Fluid Mechanics*, Vol. 15, Direct and Large Eddy Simulations of Turbulence, Proc. Euromech Coll. 199, Schumann U. and Friedrich R. (ed.), Vieweg, 1986, p. 37–66.

[AUP 87] AUPOIX B., "Application de modèles dans l'espace spectral à d'autres niveaux de fermeture en turbulence homogène", PhD Thesis, Lyon I University, 1987.

[AUP 92] AUPOIX B., "Modèles de turbulence en couche limite supersonique", *Revue Sc. et Tech. de la Défense*, 2nd trimester 1992, p. 71–88.

[BAK 74] BAKER R.J. and LAUNDER B.E., "The turbulent boundary layer with foreign gas injection", *Int. J. Heat Mass Transfer*, 17, 1974, p. 275.

[BAL 78] BALDWIN B.S. and LOMAX H., "Thin layer approximation and algebraic model for separated turbulent flows", *A.I.A.A. Paper 78-257*, 1978. 16th Aerosp. Sc. Meeting. Huntsville, Alabama, 1978.

[BAL 90] BALDWIN B.S., BARTH T.J., "A one-equation turbulence transport model for high Reynolds number wall-bounded flows", *NASA Tech. Memo.*, Vol. 102847, 1990.

[BAL 96] BALARAS E., BENOCCI C. and PIOMELLI U., "Two-layer approximate boundary conditions for large eddy simulations", *A.I.A.A. J.*, 34(6), 1996, p. 1111–1119.

[BAR 80] BARDINA J., FERZIGER J.H. and REYNOLDS W.C., "Improved subgrid scale models for large eddy simulations". *A.I.A.A. Paper 80-1357*, 1980.

[BAR 82] BARON F., "Macrosimulations tridimensionnelles d'écoulements turbulents cisaillés", Thesis, Paris VI University, 1982.

[BAR 83] BARDINA J., FERZIGER J.H. and REYNOLDS W.C., "Improved turbulence models based on large eddy simulation of homogenous incompressible turbulent flows". *Dept. Mech. Engng. Rep.*, TF-19, Stanford University, USA, 1983.

[BAR 85] BARDINA J., FERZIGER J.H. and ROGALLO R.S., "Effect of rotation on isotropic turbulence: computation and modeling", *J. Fluid Mech.*, 154, 1985, p. 321–336.

[BAR 97] BARENGHI C.F., SAMUELS D.C., BAUER G.H., DONNELLY R.J., "Superfluid vortex lines in a model of turbulent flow", *Phys. of Fluids*, 9(9), 1997, p. 2631–2643.

[BAS 62] BASS J., "Principes d'une théorie non probabiliste de la turbulence". *Mécanique de la Turbulence*, Editions du CNRS, no. 108, 1962.

[BAS 78] BASDEVANT C., LESIEUR M. and SADOURNY R., "Subgrid-scale modeling of enstrophy transfer in two-dimensional turbulence", *J. Atmos. Sciences.*, 35, 1978, p. 1028.

[BAT 48A] BATCHELOR G.K. and TOWNSEND A.A., "Decay of turbulence in the initial period", *Proc. Roy. Soc.*, 193A, 1948, p. 539.

[BAT 48B] BATCHELOR G.K. and TOWNSEND A.A., "Decay of turbulence in the final period", *Proc. Roy. Soc.* 194A, 1948, p. 527.

[BAT 54] BATCHELOR G.K. and PROUDMAN I., "The effect of rapid distortion of a fluid in turbulent motion", *J. Mech. and Applied Math.*, VII.(1), 1954, p. 83–103.

[BAT 01] BATTEN P., GOLDBERG U.C., PALANISWAMY S. and CHAKRAVARTHY S.R., "Hybrid RANS/LES: Spatial resolution and energy-transfer issues", in E. Lindborg *et al.* (eds.), *Turbulence and Shear Flow Phenomena TSFP-2* (Proc. Int. Symp., Stockholm, Sweden, 27-29 June 2001), KTH Stockholm, Sweden, Vol. 2, p.159–164.

[BEG 78] BEGUIER C., DEKEYSER I. and LAUNDER B.E., "Ratio of scalar and velocity dissipation time scales in shear flow turbulence", *Phys. of Fluids*, 21(3), 1978, p. 307–310.

[BEH 96] BEHNIA M., PARNEIX S. and DURBIN P., "Simulation of jet impingement heat transfer with the k-ε-v2 model", Center for Turbulence Research. *Annual Research Briefs 1996*, Stanford University, 1996, p. 3–16.

[BEL78] BELL T.L. and NELKIN M., "Time dependant scaling relations and a cascade model of turbulence", *J. Fluid Mech.*, 88(2), 1978, p. 369–391.

[BER 78A] BERTOGLIO J.P., CHARNAY G., GENCE J.N. and MATHIEU J., "Calcul d'une turbulence homogène soumise à une rotation en bloc et à un cisaillement", *C.R. Acad. Sci. sér. A*, 286, no. 20, 1978, p. 957–959.

[BER 78B] BERTOGLIO J.P., CHARNAY G., GENCE J.N. and MATHIEU J., "Effet d'une rotation en bloc sur les spectres des corrélations doubles d'une turbulence homogène soumise à des gradients de vitesse", *C.R. Acad. Sci. sér. B*, 286, 1978, p. 167–170.

[BER 79] BERTOGLIO J.P., CAMBON C., JEANDEL D. and MATHIEU J., "Comparaison directe entre une modélisation directionnelle du tenseur spectral des corrélations doubles et une solution analytique dans le cas d'une turbulence homogène anisotrope", *C.R. Acad. Sci. sér. A*, 288, no. 8, 1979, p. 473–476.

[BER 86] BERTOGLIO J.P., Étude d'une turbulence anisotrope: modélisation de sous-maille et approche statistique, PhD Thesis, Lyon University, 1986.

[BER 92] BERNARD P.S. and SPEZIALE C.G., "Bounded energy states in homogenous turbulent shear flows – An alternative view", *J. of Fluid Engng.*, Trans. ASME, 114, 1992, p. 29–39.

[BER 95] BEREZIN Y.A. and TROFIMOV V.M., "A model of non-equilibrium turbulence with an asymmetric stress. Application to the problems of thermal convection", *Continuum Mech. Thermodyn.*, 7, 1995, p. 415–437.

[BES 96] BESNARD D.C., HARLOW F.H., RAUENZAHN R.M., ZEMACH C., "Spectral transport model for inhomogenous turbulence", *J. Theor. and Comput. Fluid Dynamics*, 8, 1996, p. 1–35.

[BIL 75] BILGER R.W., "A note on Favre averaging in variable density flows", *J. Combust. Science and Technology*, 11, 1975, p. 215–217.

[BIS 02] BISSET D.K., HUNT J.C.R., ROGERS M.M., "The turbulent/non-turbulent interface bounding a far wake", *J. Fluid Mech.*, 451, 2002, p. 383–410.

[BON 78] BONTOUX P., Contribution à l'étude des écoulements visqueux en milieu confiné. Analyse et optimisation de méthodes numériques de haute précision. PhD Thesis, Aix-Marseille II University, 1978.

[BOR 92] BORIS J.P., GRINSTEIN F.F., ORAN E.S., KOLBE R. L., "New insights into large-eddy simulation", *Fluid Dyn. Res.*, 10, 1992, p. 199–228.

[BOS 96] BOSCH G. and RODI W., "Simulation of vortex shedding past a square cylinder near a wall", *Int. J. Heat Fluid Flow*, 17(3), 1996, p. 267–275.

[BRA 67] BRADSHAW P., FERRISS D.H. and ATWELL N.P., "Calculation of boundary layer development using the turbulent energy equation", *J. Fluid Mech.*, 28, 1967, p. 593.

[BRA 71] BRADSHAW P., "Calculation of three dimensional turbulent boundary layers", *J. Fluid Mech.*, 46, 1971, p. 417.

[BRA 73] BRADSHAW P., "Effect of streamline curvature on turbulent flow". *Agard.*, no. 169, 1973.

[BRA 75] BRADSHAW P., "Review of complex turbulent flows", *J. of Fluid Engng.*, Trans. ASME I., 97, 1975, p. 146–154.

[BRE 02] BREDBERG J., PENG S-H., DAVIDSON L. "An improved k-ω turbulence model applied to recirculating flows", *Int. J. Heat Fluid Flow*, 23, 2002, p. 731–743.

[BRE 03] BREMHORST K., CRAFT T.J., LAUNDER B.E., "Two-time-scale turbulence modeling of a fully-pulsed axisymmetric air jet", in N. Kasagi *et al.* (eds.), *Turbulence and Shear Flow Phenomena TSFP-3* (Proc. Int. Symp., Sendai, Japan, 27-29 June 2003), Sendai, Japan, 6 p.

[BRO 74] BROWN G.L. and ROSHKO A., "On density effects and large structures in turbulent mixing layers", *J. Fluid Mech.*, 64, 1974, p. 775–816.

[BRU 99] BRUN C., FRIEDRICH R., "A-priori tests of SGS stress models in fully developed pipe flow and a new local formulation", in P.R. Voke *et al.* (eds.), *Direct and Large-Eddy Simulation III*, Kluwer Academic Publishers, 1999, p. 249–262.

[BRU 02] BRUN C., FRIEDRICH R., DA SILVA C.B., MÉTAIS O., "A new mixed model based on the velocity structure function", Proc. Euromech Coll. 412, Munich, Germany, 4-6 October 2000, in Friedrich R. and Rodi W. (eds), *Advances in LES of Complex Flows*, Kluwer Academic Publishers, 2002, p. 49–64.

[BUI 77] BUILTJES P.J.H., Memory effects in turbulent flows, PhD Thesis, Delft, 1977.

[BUS 01] BUSH R.H. and MANI MORI, "A two-equation large eddy stress model for high sub-grid shear", *31st AIAA Computational Fluid Dynamics Conference*, *AIAA-2001-2561*, 2001.

[BYG 81] BYGGSTOYL S. and KOLLMANN W., "Closure model for intermittent turbulent flows", *Int. J. Heat Mass Transfer*, 24(11), 1981, p. 1811–1822.

[CAB 00] CABOT W. and MOIN P., "Approximate wall boundary conditions in the large eddy simulation of high Reynolds number flows", *Flow, Turbulence and Combustion*, 63, 2000, p. 269–291.

[CAD 95] CADIOU A. and PIQUET J., "Modélisation du tenseur des corrélations pression-déformation par une équation de transport", *C.R. Acad. Sci. sér. IIb*, 321, 1995, p. 323–330.

[CAD 96] CADIOU A., Contribution à l'étude de modèles de turbulence au second ordre, PhD Thesis, Nantes University et ECN, 1996.

[CAD 04] CADIOU A., HANJALIC K., STAWIARSKY S., "A two-scale second-moment closure based on weighted spectrum integration", *J. Theor. and Comput. Fluid Dynamics*, 18(1), 2004, p. 1–26.

[CAM 78] CAMBON C., JEANDEL D. and MATHIEU J., "Modélisation directe des tenseurs spectraux $\Phi_{ij}(\kappa, t)$ et $t_{ij}(\kappa, t)$ dans le cas d'un champ turbulent homogène", *C.R. Acad. Sci. sér. B*, 286, no. 20, 1978, p. 261–264.

[CAM 79] CAMBON C., Modélisation spectrale en turbulence homogène anisotrope, PhD Thesis, Claude Bernard University, Lyon, 1979.

[CAM 81] CAMBON C., JEANDEL D. and MATHIEU J., "Spectral modeling of homogenous non-isotropic turbulence", *J. Fluid Mech.*, 104, 1981, p. 247–262.

[CAM 82] CAMBON C., Étude spectrale d'un champ turbulent incompressible soumis à des effets couplés de déformation et de rotation imposés extérieurement, PhD Thesis, Lyon I University, 1982.

[CAM 89] CAMBON C. and JACQUIN L., "Spectral approach to non-isotropic turbulence subjected to rotation", *J. Fluid Mech.*, 202, 1989, p. 295–317.

[CAM 92A] CAMBON C., JACQUIN L. and LUBRANO J.L., "Toward a new Reynolds stress model for rotating turbulent flows", *Phys. of Fluids* A, 4(4), 1992, p. 812–824.

[CAM 92B] CAMBON C., MAO Y. and JEANDEL D., "On the application of time dependent scaling to the modeling of turbulence undergoing compression", *Eur. J. Mech. B/Fluids*,11(6), 1992, p. 683–703.

[CAM 93] CAMBON C., COLEMAN G.N. and MANSOUR N.N., "Rapid distortion analysis and direct simulation of compressible homogenous turbulence at finite Mach number", *J. Fluid Mech.*, 257, 1993, p. 641–665.

[CAM 97] CAMBON C., MANSOUR N.N., GODEFERD F.S., "Energy transfer in rotating turbulence", *J. Fluid Mech.*, 337, 1997, p. 303–332.

[CAM 04] CAMBON C., RUBINSTEIN R., GODEFERD F.S., "Advances in wave turbulence: rapidly rotating flows", *New J. Phys.* (e-journal), 6, 2004, 073.

[CAN 81] CANTWELL B.J., "Organized motion in turbulent flow", *Ann. Review Fluid Mech.*,13, 1981, p. 457–515.

[CAN 96] CANUTO V.M. and DUBOVIKOV M.S., "A dynamic model for turbulence: I – General formulation, II – Shear driven flows, III – Numerical results", *Phys. of Fluids*, 8(2), 1996, pp. 571, 587, 599.

[CAN 97A] CANUTO V.M. and DUBOVIKOV M.S., "A dynamic model for turbulence: V – The effect of rotation", *Phys. of Fluids*, 9(7), 1997, p. 2132.

[CAN 97B] CANUTO V.M., DUBOVIKOV M.S. and DIENSTFREY A., "A dynamic model for turbulence: IV – Buoyancy driven flows", *Phys. of Fluids*, 9(7), 1997, p. 2118.

[CAN 97C] CANUTO V.M., DUBOVIKOV M.S. and WIELAARD M., "A dynamic model for turbulence: VI – Two-dimensional turbulence", *Phys. of Fluids*, 9(7), 1997, p. 2141.

[CAT 99] CATRIS S., Étude des contraintes et qualification de modèles à viscosité turbulente, PhD Thesis, École Nat. Sup. de l'Aéronaut. et de l'Espace, 1999.

[CAZ 94] CAZALBOU J.B., SPALART P.R. and BRADSHAW P., "On the behavior of two-equation models at the edge of a turbulent region", *Phys. of Fluids*, 6(5), 1994, p. 1797–1804.

[CAZ 02] CAZALBOU J. B. and CHASSAING P., "The structure of the solution obtained with Reynolds-stress-transport models at the free-stream edges of turbulent flows", *Phys. of Fluids*, 14(2), 2002, p. 597–611.

[CEB 70] CEBECI T., SMITH A.M.O. and MOSINSKI G., "Solution of the incompressible turbulent boundary layer equations with heat transfer", *J. of Heat Transfer*, 92, 1970, p. 133.

[CEB 73] CEBECI T., "A model for eddy conductivity and turbulent Prandtl number", *J. of Heat Transfer*, 95, 1973, p. 227–236.

[CHA 70] CHAMPAGNE F.H., HARRIS V.G. and CORRSIN S., "Experiments on nearly homogenous turbulent shear flows", *J. Fluid Mech.*, 41, 1970, p. 81.

[CHA 79] CHAPMAN D.R., "Computational aerodynamics development and outlook", *A.I.A.A. J.*, 17, no. 12, 1979, p. 1293–1313.

[CHA 81] CHAUVE M.P. and SCHIESTEL R. "Experimental and numerical studies of the turbulent shear flow in a pipe with a weakly wavy wall", *Third Symp. on Turbulent Shear Flows*, University of Calif. Davis, 1981.

[CHA 85] CHASSAING P., "Une alternative à la formulation des équations du mouvement turbulent d'un fluide à masse volumique variable", *J. Mécanique Théorique et Appliquée*, 4(3), 1985, p. 375.

[CHA 91] CHASNOV J.R., "Simulation of the Kolmogorov inertial subrange using an improved subgrid model", *Phys. of Fluids* A, 3(1), 1991, p. 188–200.

[CHA 94] CHASSAING P., HARRAN G. and JOLY L., "Density fluctuation correlations in free turbulent binary mixing", *J. Fluid Mech.*, 279, 1994, p. 239–278.

[CHA 02] CHAOUAT B., SCHIESTEL R., "Reynolds stress transport modeling for steady and unsteady channel flow with wall injection", *Journal of Turbulence* (e-journal), 3 (021), 2002.

[CHA 05A] CHAOUAT B., SCHIESTEL R., "A new subgrid-scale tress transport model and its application to LES of evolving complex turbulent flows", *4th Int. Symp. on Turbulence and Shear Flow Phenomena*, Proc., J.A.C. Humphrey, T.B. Gatski, J.K. Eaton, R. Friedrich, N. Kasagi, M.A. Leschziner (ed.), Williamsburg, VA USA, 27-29 June, 2005, Vol. III, p. 1061–1066.

[CHA 05B] CHAOUAT B., SCHIESTEL R., "A new partially integrated transport model for subgrid-scale stresses and dissipation rate for turbulent developing flows", *Phys. of Fluids*, 17(6), published online 3 June 2005.

[CHA 06] CHAOUAT B., "Reynolds stress transport modeling for high lift airfoil flows", *A.I.A.A. J.*, 44(10), 2006, p. 2390–2403.

[CHA 07] CHAOUAT B., SCHIESTEL R., "From single-scale turbulence models to multiple scale and subgrid-scale models by Fourier transform", *J. Theor. and Comput. Fluid Dynamics* (to appear), Springer-Verlag DOI 10.1007/s00162-007-0044-3, 2007, 29 pages.

[CHE 88] CHEN H.C. and PATEL V.C., "Near wall turbulence models for complex flows including separation", *A.I.A.A. J.*,26(6), 1988, p. 641.

[CHE 94] CHEN W., LIEN F.S. and LESCHZINER M.A., "Computational modeling of turbulent flow in turbomachine passage with low Re two-equation models", In *Comput. Fluid Dynamics 94*, S. Wagner, E.H. Hirschel, J. Périaux, and R. Piva (ed.), John Wiley and Sons, Chichester, 1994, p. 517–524.

[CHI 80] CHIENG C.C. and LAUNDER B.E., "On the calculation of turbulent heat transport downstream from an abrupt pipe expansion", *Num. Heat Transfer*, 3, 1980, p. 189.

[CHI 82] CHIEN K.Y. "Predictions of channel and boundary layer flows with a low-Reynolds number turbulence model", *A.I.A.A. J.*,20, 1982, p. 33.

[CHO 45A] CHOU P.Y., "On velocity correlations and the solutions of the equations of turbulent fluctuations", *Quart. of Appl. Math.*, 3, 1945, p. 38.

[CHO 45B] CHOU P.Y., "Pressure flow of a turbulent fluid between two infinite parallel planes", *Quart. of Appl. Math.*, 3, 1945, p. 198.

[CHO 67] CHORIN A.J., "A numerical method for solving incompressible viscous flow problems", *J. Comp. Phys.*, 2, 1967, p. 12.

[CHO 73] CHORIN A.J., "Numerical study of slightly viscous flow", *J. Fluid Mech.*, 57(4), 1973, p. 785–796.

[CHO 81] CHOLLET J.P. and LESIEUR M., "Parameterization of small scales of three-dimensional isotropic turbulence utilizing spectral closures", *J. Atmos. Sciences.*, 38, 1981, p. 2747.

[CHO 89A] CHOMIAK J., "Mathematical modeling of turbulent reactive flows", *Turbulence Annual Int. J.*, Technical University of Czestochowa,1, 1989, p. 9–34.

[CHO 89B] CHOLLET J.P. and METAIS O., "Predictability of three dimensional turbulence in large eddy simulations", *European J. of Mech.*, 89, no. 8(6), 1989, p. 523–548.

[CHO 92] CHO J.R. and CHUNG M.K., "A k-ε−γ equation turbulence model", *J. Fluid Mech.*, 237, 1992, p. 301–322.

[CHU 70] CHUE S.H. and MC DONALD, "Application of a new effective viscosity model to turbulent plane Couette flow", *A.I.A.A. J.*, 8, 1970, p. 2076.

[CIO 89] CIOFALO M., COLLINS M.W., "k–ε predictions of heat transfer in turbulent recirculating flows using an improved wall treatment", *Num. Heat Transfer*, 15(3), 1989, p. 21–47.

[CLA 56] CLAUSER F.H., "The turbulent boundary layer". *Advances in Applied Mechanics*, Vol. IV, Academic Press, 1956, p. 1–51.

[CLA 79] CLARK R.A., FERZIGER J.H. and REYNOLDS W.C., "Evaluation of subgrid-scale models using an accurately simulated turbulent flow", *J. Fluid Mech.*, 91(1), 1979, p. 1–16.

[CLA 88] CLAVIN P., LALLEMAND P., POMEAU Y. and SEARBY G., "Simulation of free boundaries in flow systems by lattice-gas models", *J. Fluid Mech.*, 188, 1988, p. 437–464.

[CLE 82] CLER A., Modélisation de la turbulence dans le cadre des fermetures en un point. Calcul d'échelles de temps et de longueur. PhD Thesis, ENSAE Toulouse, 1982.

[COL 91] COLEMAN G.N. and MANSOUR N.N., "Simulation and modeling of homogenous compressible turbulence under isotropic mean compression", *Proc. 8th Symp. on Turb. Shear Flows*, Münich, 1991, 21/3/1-6.

[COM 65] COMTE-BELLOT G., "Écoulement turbulent entre deux parois parallèles", *Publ. Sci. Tech. Min. Air*, no. 419, Paris, 1965.

[COM 66] COMTE-BELLOT G. and CORRSIN S., "The use of a contraction to improve the isotropy of grid generated turbulence", *J. Fluid Mech.*, 25(4), 1966, p.657.

[COR 54] CORRSIN S. and KISTLER A.L., "The free-stream boundary of a turbulent flow". *NACA-TN* 3133, 1954.

[COR 62] CORRSIN S., "Theories of turbulent dispersion", *Mécanique de la Turbulence*, Ed. du CNRS, no. 108, 1962, p. 27.

[COR 74] CORRSIN S., "Limitation of gradient transport models in random walks and in turbulence", *Adv. in Geophysics*, 18A, 1974, p. 25–71.

[COT 93] COTTON M.A. and ISMAEL J.O., "Development of a two-equation turbulence model with reference to strain parameter", *Proc. 5th Int. Symp. on Refined Flow Modeling and Turbulence measurements*, Presse de l'Ecole Nationale des Ponts et Chaussées, 1993, p. 117–124.

[COT 98] COTTON M.A., ISMAEL J.O., "A strain parameter turbulence model and its application to homogenous and thin shear flows", *Int. J. Heat Fluid Flow*, 19, 1998, p. 326–337.

[COU 72A] COURSEAU P. and LOISEAU M., "Les corrélations de vitesse-pression dans une turbulence homogène associée à une déformation plane", *C.R. Acad. Sci. sér. A*, 274, no. 22, 1972, p. 1582–1585.

[COU 72B] COURSEAU P. and LOISEAU M. "Le tenseur Reynolds dans une turbulence homogène associée à une déformation plane". *C.R. Acad. Sci. sér. A*, 274, no. 23, 1972, p. 1655–1658.

[COU 72C] COUSTEIX J., QUEMARD C. and MICHEL R., "Application d'un schéma amélioré de longueur de mélange à l'étude des couches limites turbulentes tridimensionnelles", *AGARD Conference Proc. CP 93 on Turbulent Shear Flows, London*, 1972.

[COU 78] COURSEAU P. and LOISEAU M., "Contribution à l'analyse de la turbulence homogène anisotrope". *J. Mécanique*, 17(2), 1978, p. 245–297.

[COU 81A] COUSTEIX J. "Modélisation de la turbulence et méthodes de calcul de couches limites", *Lectures Series 1981-1 on Separated Flows in Turbomachinery Components*. January 12-16, 1981, Rapport Technique OA 43/2259 AYD. DERAT 27/5004 DY (Onera).

[COU 81B] COUSTEIX J. and AUPOIX B., "Modélisation des équations aux tensions de Reynolds dans un repère en rotation", *La Rech. Aérospatiale*, no. 4, 1981, p. 275–285.

[CRA 58] CRAYA A., "Contribution à l'analyse de la turbulence associée à des vitesses moyennes", *Publ. Sci. Tech. Min. Air*, no. 345. 1958.

[CRA 89] CRAFT T.J., FU S., LAUNDER B.E. and TSELEPIDAKIS D.P., "Developments in modeling the turbulent second-order pressure correlations", *UMIST Mech. Engng. Dept. Report*, TFD/89/1, 1989.

[CRA 91A] CRAFT T.J., Second moment modeling of turbulent scalar transport, PhD Thesis, University of Manchester, UK, 1991, (also appeared under the tile: UMIST Internal Report, Mechanical Engineering Department, TFD/91/3).

[CRA 91B] CRAFT T.J. and LAUNDER B.E., "Computation of impinging flows using a second-moment closure", *8th Int. Symp. on Turb. Shear Flows*, Munich, 8/5/1-6, 1991.

[CRA 92] CRAFT T.J. and LAUNDER B.E. "New wall-reflection model applied to the turbulent impinging jet", *A.I.A.A. J.*, 30, 1992, p. 2970.

[CRA 93] CRAFT T.J., LAUNDER B.E. and SUGA K., "Extending the applicability of eddy viscosity models through the use of deformation invariants and non-linear elements", *5th IAHR Conference on Refined Flow Modeling and Turbulence Measurements*, Presse Ponts et Chaussées, Paris, 1993, p. 125–132.

[CRA 95A] CRAFT T.J., LAUNDER B.E., "Improvements in near-wall Reynolds stress modeling for complex flow geometries", *Proc. 10th Symp. on Turb. Shear Flows*, Penn. State Univ., 1995, p. 20/25-20/30.

[CRA 95B] CRAFT T.J., LAUNDER B.E. and SUGA K., "A non-linear eddy viscosity model including sensitivity to stress anisotropy", *Proc. 10th Symp. on Turbulent Shear Flows*, Penn. State Univ., vol. 3, 1995, p. 23/19-23/24.

[CRA 96A] CRAFT T.J., INCE N.Z. and LAUNDER B.E., "Recent developments in second moment closure for buoyancy affected flows", *Dyn. of Atm. and Oceans*, 23, 1996, p. 99–114.

[CRA 96B] CRAFT T.J. and LAUNDER B.E., "A Reynolds stress closure designed for complex geometries", *Int. J. Heat Fluid Flow*, 17(3), 1996, p. 245–254.

[CRA 96C] CRAFT T.J., LAUNDER B.E. and SUGA K., "Development and application of a cubic eddy-viscosity model of turbulence", *Int. J. Heat Fluid Flow*, 17(2), 1996, p. 108–115.

[CRA 97A] CRAFT T.J., "Computation of separating and reattaching flows using a low Reynolds number second moment closure", *Proc. 11th Symp. on Turbulent Shear Flows*, Grenoble, France, 3, 1997, p. 30/9–30/24.

[CRA 97B] CRAFT T.J., LAUNDER B.E. and SUGA K., "Prediction of turbulent transitional phenomena with a non-linear eddy-viscosity model", *Int. J. Heat Fluid Flow*,18, 1997, p. 15–28.

[CRA 98] CRAFT T.J., "Developments in a low-Reynolds-number second-moment closure and its application to separating and reattaching flows", *Int. J. Heat Fluid Flow*, 19, 1998, p. 541–548.

[CRA 01] CRAFT T.J., GANT S.E., IACOVIDES H., LAUNDER B.E., "Development and application of a numerical wall-function strategy for complex near-wall flows", in *ECCOMAS CFD 2001 Conf.*, Swansea, 2001.

[CRA 01B] CRAFT T.J. and LAUNDER B.E., "Principles and Performance of TCL-Based Second-Moment Closures", *Flow Turbulence and Combustion*, 66, 2001, p. 355–372.

[CRA 02] CRAFT T.J., GERASIMOV A.V., IACOVIDES H., LAUNDER B.E., "Progress in the generalization of wall-function treatments", *Int. J. Heat Fluid Flow*, 23, 2002, p. 148–160.

[CRA 04] CRAFT T.J., GERASIMOV A.V., IACOVIDES H., KIDGER J.W., LAUNDER B.E., "The negatively buoyant turbulent wall jet: performance of alternative options in RANS modeling", *Int. J. Heat Fluid Flow*, 25, 2004, p. 809–823.

[CRO 68A] CROW J., "Turbulent Rayleigh shear flow", *J. Fluid Mech.*, 32(1), 1968, p. 113.

[CRO 68B] CROW J., "Viscoelastic properties of fine-grained incompressible turbulence", *J. Fluid Mech.*, 33(1), 1968, p. 1–20.

[CSA 66] CSANADY G.T., "Generalisation of the mixing-length argument in turbulent diffusion", *J. of Atm. Sci.*, 23(6), 1966, p. 667–671.

[DAL 70] DALY B.J. and HARLOW F.H., "Transport equations in turbulence", *Phys. of Fluids*, 23(11), 1970, p. 2634.

[DAL 74] DALY B.J., "A numerical study of turbulence transitions in convective flows", *J. Fluid Mech.*,64(1), 1974, p. 129–165.

[DAN 83] DANG K., "Evaluation of simple subgrid scale models for the numerical simulation of homogenous isotropic and anisotropic turbulence", *A.I.A.A. Paper* 83-1692/TP-ONERA 1983-69, 1983, p. 1–8.

[DAV 93] DAVID E., Modélisation des écoulements compressibles et hypersoniques: une approche instationnaire, PhD Thesis, Inst. Nat. Polytech., Grenoble, 1993.

[DAV 61] DAVIDOV B.I., "On the statistical dynamics of an incompressible fluid", *Dokl. AMSSSR* 136, 1961, p.47.

[DAV 01] DAVIDSON L. and PENG S.H., "A hybrid LES-RANS model based on a one-equation SGS model and a two-equation k-ω model", in E. Lindborg *et al.* (eds.), *Turbulence and Shear Flow Phenomena TSFP–2* (Proc. Int. Symp., Stockholm, Sweden, 27-29 June 2001), KTH Stockholm, Sweden, Vol. 2, p. 175–180.

[DEA 70] DEARDORFF J.W., "A numerical study of three-dimensional turbulent channel flow at large Reynolds Numbers", *J. Fluid Mech.*, 41(2), 1970, p. 453.

[DEA 73] DEARDORFF J.W., "The use of subgrid transport equations in a three-dimensional model of atmospheric turbulence", *J. of Fluid Engng.*, Trans. ASME, 95, 1973, p. 429–438.

[DEI 79] DEISSLER R.G., "Decay of homogenous turbulence from a given state at higher Reynolds number", *Phys. of Fluids*, 22(10), 1979, p. 1852–1856.

[DEJ 01] DEJOAN A., SCHIESTEL R., "Large eddy simulations of non-equilibrium pulsed turbulent flow using transport equations subgrid scale model", in E. Lindborg *et al.* (eds) *Turbulence and Shear Flow Phenomena 2* (Proc. Int. Symp., Stockholm, Sweden, 27-29 June 2001), KTH Stockholm, Sweden, 2001, p. 341–346.

[DEJ 02] DEJOAN A., SCHIESTEL R., "LES of unsteady turbulence via a one-equation subgrid-scale transport model", *Int. J. Heat Fluid Flow*, 23, 2002, p. 398–412.

[DEK 82] DEKEYSER I., Étude d'un jet plan dissymétrique chauffé en régime turbulent incompressible. PhD Thesis, University of Aix-Marseille II, 1982.

[DOM 97] DOMARADZKI J.A., SAIKI E.M., "A sugbrid scale model based on the estimation of unresolved scales of turbulence", *Phys. of Fluids*, 9(7), 1997, p. 2148–2164.

[DOM 99] DOMARADZKI J.A., LOH K-C., "The sugbrid scale estimation model in the physical space representation", *Phys. of Fluids*, 11(8), 1999, p. 2330–2342.

[DON 69] DONALDSON C. DU P., "A computer study of an analytical model of boundary layer transition", *A.I.A.A. J.*, 7(1), 1969, p. 271.

[DON 71] DONALDSON C. DU P., "A progress report on an attempt to construct an invariant model of turbulent shear flows", *AGARD Conference Proc. C.P. 93 on Turbulent Shear Flows*, London, September 1971.

[DON 72] DONALDSON C. DU P., SULLIVAN R.D. and ROSENBAUM H., "A theoretical study of the generation of atmospheric-clear air turbulence", *A.I.A.A. J*, 10(2), 1972, p. 162.

[DOP 77] DOPAZO C., "On conditioned averages for intermittent turbulent flows", *J. Fluid Mech.*, 81, 1977, p.433–438.

[DOP 94] DOPAZO C., "Recents developments in the pdf methods", in *Turbulent Reacting Flows*, Libby P.A. and Williams F.A. (ed.), Academic Press, 1994, p. 375–474.

[DUB 00] DUBIEF Y., DELCAYRE F., "On coherent-vortex identification in turbulence", *J. of Turbulence*, e-journal, 1, 2000, 011(http://jot.iop.org).

[DUC 96] DUCROS F., COMTE P. and LESIEUR M., "Large eddy simulation of transition to turbulence in a boundary layer developing spatially over a flat plate", *J. Fluid Mech.*, 326, 1996, p. 1–36.

[DUH 81] DUHAMEL P., "A detailed derivation of conditioned equations for intermittent turbulent flows", *Letters in Heat Mass Transfer*, 8(6), 1981, p. 491–502.

[DUR 91A] DURBIN P.A., "Near wall turbulence modeling without damping function", *J. Theor. and Comput. Fluid Dynamics*, 3, 1991, p. 1–13.

[DUR 91B] DURBIN P.A. and SPEZIALE C.G., "Local anisotropy in strained turbulence at high Reynolds numbers", *J. Fluid Engng.*, ASME, 113, 1991, p. 707–709.

[DUR 92] DURBIN P.A. and ZEMAN O. "Rapid distortion theory for homogenous compressed turbulence with application to modeling", *J. Fluid Mech.*, 242, 1992, p. 349–370.

[DUR 93] DURBIN P.A., "A Reynolds stress model for near wall turbulence", *J. Fluid Mech.*, 249, 1993, p. 465–498.

[DUR 94A] DURBIN P.A. and SPEZIALE C.G., "Realizability of second moment closure via stochastic analysis", *J. Fluid Mech.*, 280, 1994, p. 395–407.

[DUR 94B] DURBIN P.A., MANSOUR N.N., YANG Z., "Eddy viscosity transport model for turbulent flow", *Phys. of Fluids*, 6(2), February, 1994, p. 1007–1015.

[DUR 95] DURBIN P.A., "Separated flow computations with a k-ε-υ2 model", *A.I.A.A. J.*, 33(4), 1995, p. 659–664.

[DUR 96] DURBIN P.A., "On the $k−\varepsilon$ stagnation point anomaly", *Int. J. Heat Fluid Flow*, 17(1), Tech. Note, 1996, p. 89–90.

[DUR 99] DURBIN P.A., PETTERSSON REIF B.A., "On algebraic second moment models", *Flow Turbulence and Combustion*, 63, 199, p. 23–37.

[DUT 81] DUTOYA D. and MICHARD P., "A program for calculating boundary layers along compressor and turbine blades", *Numerical methods in Heat Transfer*, J. Wiley and Sons R.W. Lewis, K. Morgan, O.C. Zienkiewicz (ed.), 1981.

[DUV 77] DU VACHAT R., "Realizability inequalities in turbulent flows", *Phys. of Fluids*, 20, April, 1977, p. 551–556.

[DUV 78] DU VACHAT R., "Inégalités de réalisabilité pour des paramètres statistiques, application à un écoulement turbulent", *La Météorologie*, VI Series, no. 13, June 1978, p. 5–15.

[ELB 91] ELBAZ A.M., "Calculation of supersonic free turbulent mixing layers", *UMIST Report TFD/91/4*, Manchester, 1991.

[ELE 96] ELENA L., SCHIESTEL R., "Turbulence modeling of rotating confined flows", *Int. J. Heat Fluid Flow*, 17(3), 1996, p. 283–289.

[ELS 79] ELSAESSER E. and PEYRET R., "Méthodes hermitiennes pour la résolution numérique des équations de Navier-Stokes", in *Méthodes Numériques dans les Sciences de l'Ingénieur*, E. Absi, R. Glowinsky (ed.), Dunod, Paris, 1979, p. 249–258.

[ESC 66] ESCUDIER M.P., "The distribution of mixing-length in turbulent flows near walls", *Imp. Coll. Heat Transfer Section Rep.* TWF/TN/1. 1966.

[FAR 85] FARSHCHI M., KOLLMANN W., DIBBLE R.W. and SCHEFER R.W., "Second order closure for turbulent diffusion flows", *5th Symp. on Turbulent shear Flows.*, Cornell University, 1985, p 10–19.

[FAR 92] FARGE M., "Wavelet transforms and their Applications to turbulence", *Ann. Rev. of Fluid Mech.*, vol. 24, p. 395–457.

[FAR 99] FARGE M., SCHNEIDER K., KEVLAHAN N., "Non-Gaussianity and coherent vortex simulation for two-dimensional turbulence using unadaptive orthogonal wavelet basis", *Phys. of Fluids*, 11(8), 1999, p. 2187–2201.

[FAR 01A] FARGE M., PELLEGRINO G., SCHNEIDER K., "Coherent vortex extraction in 3D turbulent flows using orthogonal wavelets", *Phys. Review Letters*, 87(5), 2001, p. 054501-1–4.

[FAR 01B] FARGE M., SCHNEIDER K., "Coherent Vortex Simulation (CVS), A Semi-Deterministic Turbulence Model Using Wavelets", *Flow, Turbulence and Combustion*, 66, 2001, p. 393–426.

[FAV 89] FAVRE A., "Formulation of the statistical equations of turbulent flows with variable density", *Studies in Turbulence*, T.B. Gatski, S. Sarkar, C.G. Speziale (ed.), Springer-Verlag, 1989, p. 324.

[FAV 91] FAVRE A., "Corrélations spatio-temporelles, déterminisme et chaos", *Coll. Int. "La Correlation", Toulouse, Nov. 1991, Acad. Sci. et Acad. Nationale de l'Air et de l'Espace*, Cepadues ed., 1991, p. 77–106.

[FEI 81] FEIEREISEN W.J., REYNOLDS W.C. and FERZIGER J.H., "Numerical simulation of a compressible homogenous turbulent shear flow". *Dept. Mech. Engng. Rep.* TF-13, Stanford University, USA, 1981.

[FER 77A] FERZIGER J.H., "Large eddy numerical simulation of turbulent flows", *A.I.A.A. J.*, 15, no. 9, 1977, p. 1261–1267.

[FER 77B] FERZIGER J.H. and MACMILLAN O.J., "Testing of turbulence models by exact numerical solution of the Navier-Stokes equations", *NEAR T.R.* 155, 1977.

[FOR 71] FORTIN M., PEYRET R. and TEMAM R., "Résolution numérique des équations de Navier-Stokes pour un fluide incompressible", *J. Mécanique*, 10(3), 1971, p. 357.

[FRI 01] FRIEDRICH R., HÜTTL T.J., MANHART M., WAGNER C., "Direct numerical simulation of incompressible turbulent flows", *Computers and Fluids*, 30, 2001, p. 555–579.

[FU 87A] FU S., LAUNDER B.E., LESCHZINER M.A., "Modeling strongly swirling recirculating jet flows with Reynolds stress transport closure", *6th Int. Symp. of Turbulent Shear Flows*, September 7–9, 1987, Toulouse, France.

[FU 87B] FU S., LAUNDER B.E. and TSELEPIDAKIS D.P., "Accommodating the effects of high strain rates in modeling the pressure strain correlation", *UMIST Mech. Engn. Dept. Report* TFD/87/5. 1987.

[FU 88] FU S., Computational modeling of turbulent swirling flows with second moment closures, PhD Thesis, Faculty of Technology, University of Manchester, 1988.

[FU 97] FU S., RUNG T. and THIELE F., "Realizability of non-linear stress strain relationships for Reynolds stress closures", *Proc. 11th Symp. on Turbulent Shear Flows*, Grenoble, France, vol. 2, 1997, p. 13/1–13/6.

[FU 97B] FU S., WANG C., "Second moment closure modeling of turbulence in a non-inertial frame", *Fluid Dyn. Research*, 20, 1997, p. 43–65.

[FUJ 00] FUJIWARA H., MATSUO Y., ARAKAWA C., "A turbulence model for the pressure-strain correlation term accounting for compressible effects", *Int. J. Heat Fluid Flow*, 21, 2000, p. 354–358.

[FUL 72] FULACHIER L., Contribution à l'étude des analogies des champs dynamique et thermique dans une couche limite turbulente, effet de l'aspiration, PhD Thesis, University of Provence. 1972.

[FUR 98], FUREBY C., "Towards large eddy simulations of flows in complex geometries", *29th AIAA Fluid Dyn. Conf.*, June 15-18, Albuquerque, NM, AIAA-98-2806.

[GAL 81] GALLAGHER B., MAGAARD L. and GUTTELING E., "Closure for velocity pressure gradient correlations in turbulent shear flows", *Phys. of Fluids*, 24(9), 1981, p. 1605–1610.

[GAT 93] GATSKI T.B. and SPEZIALE C.G., "On explicit algebraic stress models for complex turbulent flows", *J. Fluid Mech.*, 254, 1993, p. 59–78.

[GAW 70] GAWAIN T.H. and PRITTCHETT J.W., "A unified heuristic model of fluid turbulence", *J. Comp. Phys.*, 5, 1970, p. 383.

[GEL 74] GELCI R., "Calcul numérique de certains écoulements par la théorie des tourbillons", *La Houille Blanche*, 6, 1974, p. 415.

[GEN 77A] GENCE J.N., Turbulence homogène associée à des effets de gravité, PhD Thesis, University of Claude Bernard Lyon I., 1977.

[GEN 77B] GENCE J.N., ANGEL Y. and MATHIEU, "Evolution des parties linéaires des corrélations mettant en jeu la pression dans une turbulence homogène associée à un cisaillement et un effet de gravité", *C.R. Acad. Sci. sér. B*, 285, no. 1, 1977, p. 5–8, 1977.

[GEN 78] GENCE J.N., ANGEL Y. and MATHIEU J., "Partie linéaire des corrélations mettant en jeu la pression dans une turbulence homogène associée à un cisaillement et un effet de gravité", *J. Mécanique*, 17(3), 1978, p. 329–357.

[GEN 79A] GENCE J.N., Action de deux déformations pures planes successives sur une turbulence isotrope, PhD Thesis, University of Claude Bernard Lyon, 6–7-1979.

[GEN 79B] GENCE J.N. and MATHIEU J., "On the application of successive plane strains to grid generated turbulence", *J. Fluid Mech.*, 93(3), 1979, p. 501–514.

[GEN 79C] GENCE J.N. and MATHIEU J., "Action d'une déformation plane pure sur une turbulence homogène dans laquelle les axes principaux du tenseur de Reynolds ne coïncident pas avec ceux de la déformation", *C.R. Acad. Sci. sér. B*, 288, no. 2, 1979, p. 29–32.

[GEN 80] GENCE J.N. and MATHIEU J., "The return to isotropy of an homogenous turbulence having been submitted to two successive plane strains", *J. Fluid Mech.*, 101(3), 1980, p. 555–566.

[GER 86] GERMANO M., "A proposal for a redefinition of the turbulent stresses in the filtered Navier-Stokes equations", *Phys. of Fluids*, 29(7), 1986, p. 2323–2324.

[GER 86A] GERMANO M., "Differential filters for the large eddy numerical simulation of turbulent flows", *Phys. of Fluids*, 29(6), 1986, p. 1755–1757.

[GER 86B] GERMANO M., "Differential filters of elliptic type", *Phys. of Fluids*, 29(6), 1986, p. 1757–1758.

[GER 90] GERMANO M., PIOMELLI U., MOIN P. and CABOT W., "A dynamic subgrid-scale eddy viscosity model", *Proc. 1990 Summer Program CTR* (Center of Turbulence Research, Stanford Univ./NASA Ames), 1990.

[GER 91] GERMANO M., PIOMELLI U., MOIN P. and CABOT W., "A dynamic subgrid-scale eddy viscosity model", *Phys. of Fluids* A, 3(7), 1991, p. 1760–1765.

[GER 92] GERMANO M., "Turbulence: the filtering approach", *J. Fluid Mech.*, 238, 1992, p. 325–336.

[GER 93] GERMANO M., "Multilevel subgrid modeling", *Proc. 5th Int. Symp. on Refined Flow Modeling and Turbulence Measurements*, Presse de l'Ecole Nationale des Ponts et Chaussées, 1993, p. 81–88.

[GER 99] GERMANO M., "From RANS to DNS: towards a bridging model", in Direct and Large-Eddy Simulation III, *Proc. of the Isaac Newton Inst. Symp./ERCOFTAC Workshop*, Cambridge, UK, 12-14 May 1999, P.R. Voke, N.D. Sandham, L. Kleiser (ed.), Kluwer Acad. Publ., 1999, p. 225–236.

[GER 01] GEROLYMOS G.A., VALLET I., "Wall-normal-free near wall Reynolds stress closure for 3D compressible separated flows", *A.I.A.A. J.*, 39, 2001, p. 1833–1842.

[GER 02] GEROLYMOS G.A., VALLET I., "Wall-normal-free Reynolds-stress model for rotating flows applied to turbomachinery", *A.I.A.A. J.*, 40, 2001, p. 199–208.

[GER 04] GEROLYMOS G.A., SAURET E., VALLET I., "Contribution to single-point closure Reynolds-stress modeling of inhomogenous flow", *J. Theor. and Comput. Fluid Dynamics*, 17, 2004, p. 407–431.

[GES 77] GESSNER F.B. and EMERY A.F., "A length-scale for developing turbulent flows in a rectangular duct", *J. of Fluid Engng.*, 99(2), 1977, p. 347–356.

[GEU 97] GEURTS B.J., "Inverse modeling for large eddy simulation", *Phys. of Fluids*, 9(12), 1997, p. 3585–3587.

[GHA 96] GHARBI A., RUFFIN E., ANSELMET F., SCHIESTEL R., "Numerical modeling of variable density turbulent jets", *Int. J. Heat Mass Transfer*, 39(9), 1996, p. 1865–1882.

[GHO 95A] GHOSAL S., LUND T.S., MOIN P. and AKSELVOLL K., "A dynamic localization model for large eddy simulation of turbulent flows", *J. Fluid Mech.*, 286, 1995, p. 229–255.

[GHO 95B] GHOSAL S., MOIN P., "The basic equations for Large Eddy Simulation of turbulent flows in complex geometry", *J. Comp. Phys.*, Vol. 118, 1995, p. 24–37.

[GIB 76] GIBSON M.M. and LAUNDER B.E., "On the calculation of horizontal turbulent, free shear flows under gravitational influence", *J. of Heat Transfer*, Trans. ASME, 1976, p. 81–87.

[GIB 78A] GIBSON M.M., "An algebraic stress and heat flux model for turbulent shear flow with streamline curvature", *Int. J. Heat Mass Transfer*, 21(12), 1978, p. 1609–1617.

[GIB 78B] GIBSON M.M. and LAUNDER B.E., "Ground effects on pressure fluctuations in the atmospheric boundary layer", *J. Fluid Mech.*, 86(3), 1978, p. 491–511.

[GIB 78C] GIBSON M.M., SPALDING D.B. and ZINSER W., "Boundary layer calculations using the Hassid-Poreh one-equation energy model", *Letters in Heat and Mass Transfer*, 5(2), 1978, p. 73–80.

[GIB 79A] GIBSON M.M., "Prediction of curved free shear layers with a Reynolds-stress model of turbulence", *Symp. on Turb. Shear Flows*, 2, London 2.6–2.11, 1979.

[GIB 79B] GIBSON M.M., Mc GUIRK J.J., Jones W.P. and Whitelaw J.H., "Mathematical modeling of turbulent flows", *Short Post Exp. Course. Imp. Coll. Mech. and Chem. Engng. Dept.*, 20/22 June 1979.

[GIB 81A] GIBSON M.M., JONES W.P. and YOUNIS B.A., "Calculation of turbulent boundary on curved surfaces", *Phys. of Fluids*, 24(3), 1981, p. 386–395.

[GIB 81B] GIBSON M.M. and RODI W., "A Reynolds stress closure model for turbulence applied to the calculation of a highly curved mixing layer", *J. Fluid Mech.*, 103, 1981, p. 161–162.

[GIB 82] GIBSON M.M. and YOUNIS B.A., "Modeling the curved turbulent wall jet", *A.I.A.A. J.*, 20(12), 1982, p. 1707.

[GIB 87] GIBSON M.M., JONES W.P. and KANELLOPOULOS V.E., "Turbulent temperature mixing layer: measurement and modeling", *Proc. 6th Symp. on Turbulent Shear Flows*, Toulouse, France, 1987.

[GIR 04] GIRIMAJI S., "A new perspective on realizability of turbulence models", *J. Fluid Mech.*, 512, 2004, p. 191–210.

[GIR 05] GIRIMAJI S.S., ABDOL-HAMID K.S., "Partially-averaged Navier-Stokes model for turbulence: implementation and validation", *43rd AIAA Aerospace Science Meeting and Exhibit*, January 10-13, 2005, Reno, Nevada, Paper 2005-0502, 14 p.

[GLE 96] GLEIZE V., SCHIESTEL R. and COUAILLIER V., "Multiple scale modeling of turbulent nonequilibrium boundary layer flows", *Phys. of Fluids*, 8(10), 1996, p. 2716–2732.

[GLU 71] GLUSHKO G.S., "Quelques particularités des écoulements turbulents de fluides incompressibles avec mutations transversales", *Mécanique des Liquides et des Gaz*, 4, *Isv. Acad. Sc.*, URSS, 1971, p. 126.

[GLU 72] GLUSHKO G.S., "Differential equation for the scale of turbulence and calculation of the turbulent boundary layer at a flat plate", *Fluid Mech. Soviet Research*, 1(2), 1972, p. 155.

[GOL 91] GOLDBERG U.C., "Derivation and testing of a one-equation model based on two time scales", *A.I.A.A. J.*, 29(8), 1991, p. 1337–1340.

[GOS 61] GOSSE J., "Sur l'écoulement turbulent des fluides", *C.R. Acad. Sci.*, 253, no. 1, 1961, p. 61.

[GOS 69] GOSMAN A.D., PUN W.M., RUNCHAL A.K., SPALDING D.B. and WOLFSHTEIN M., "Heat and Mass Transfer in recirculating flows", *Academic Press*, London, New-York., 1969.

[GOS 72] GOSSE J., SCHIESTEL R., "Modèle descriptif du mouvement turbulent d'un fluide isovolume", *C.R. Acad. Sci. Sér. A*, 275, 1972, p. 471–474.

[GOS 73] GOSSE J., SCHIESTEL R. and ROBIN M.G., "A new model of turbulence applied to prediction of liquid metal heat transfer", *ASME-AICHE Heat Transfer Conference*, Atlanta Ga., August 5–8, 1973.

[GOS 75] GOSSE J., SCHIESTEL R., "Estimation numérique de la convection turbulente dans les espaces annulaires", *Int. J. Heat Mass Transfer*, 18(6), 1975, p. 743.

[GOS 77] GOSMAN A.D., KHALIL E.E. and WHITELAW J.H., "The calculation of two-dimensional turbulent recirculating flows", *Symp. on Turbulent Shear Flows*. Penn State, 1977.

[GOS 78A] GOSMAN A.D. and RAPLEY C., "A prediction method for fully developed flow through non-circular passages". *Proc. 1st Int. Conf. on Num. Meth. in Laminar and Turbulent flows*. Univ. Swansea, 1978, p. 271.

[GOS 78B] GOSSE J., SCHIESTEL R., "Thermal convection in the wavy tubes of a heat exchanger", *Int. Chem. Engng.*, 18(1), 1978, p. 1–7.

[GOS 79] GOSSE J., SCHIESTEL R., "Convection forcée turbulence dans les faisceaux de tubes en attaque longitudinale", *Revue Générale de Thermique*, XVIII, no. 206, 1979, p. 71.

[GOU 94] GOUTORBE T. and LAURENCE D., "Modeling anisotropy and backscatter effects in the subgrid scale stress tensor", in AGARD Conf. Proc. 551, *Application of Direct and Large Eddy simulation to Transition and Turbulence*, 1994, p. 16/1–16/7.

[GRE 97] GRÉGOIRE O., SOUFFLAND D., GAUTHIER S. and SCHIESTEL R., "Vers une modélisation multiéchelle des mélanges turbulents compressibles en présence de forts gradients de masse volumique", *C.R. Acad. Sci.*, IIb, 325, 1997, p. 631–634.

[GRE 99] GRÉGOIRE O., SOUFFLAND D., GAUTHIER S., SCHIESTEL R., "A two-time-scale Turbulence model for compressible flows: Turbulence dominated by mean deformation interaction", *Phys. of Fluids*, 11(12), 1999, p. 3793–3807.

[GRU 05] GRUNDESTAM O., WALLIN S., JOHANSSON A.V., "An explicit algebraic Reynolds stress model based on a non-linear pressure strain rate model", *Int. J. Heat Fluid Flow*, 26(5), 2005, p. 732–745.

[GUE 99] GUÉZENGAR D., FRANCESCATTO J., GUILLARD H., DUSSAUGE J-P., "Variations on a k-ε model for supesonic boundary layers", *Eur. J. Mech. B/Fluids*, 18, 1999, p. 713–738.

[GUN 95] GÜNTSCH E. and FRIEDRICH R., "Compression of initially isotropic turbulence in a cylinder at low Reynolds number", *Proc. 10th Symp. on Turb. Shear Flows*, Penn. State, 1995, p. 29/19–29/24.

[HAD 01] HADZIC I., HANJALIC K. and LAURENCE D., "Modeling the response of turbulence subjected to cyclic irrotational strain", *Phys. of Fluids*, 2001, 13(6), p. 1739–1747.

[HAL 90] HALLBÄCK M., GROTH J., JOHANSSON A.V., "An algebraic model for nonisotropic turbulent dissipation rate in Reynolds stress closures", *Phys. of Fluids A*, 2(10), 1990, p. 1859–1866.

[HAM 78] HAMINH H. and CHASSAING P., "Restructuration d'écoulements turbulents", *J. Mécanique*, 17(3), 1978, p. 359–386.

[HAM 81] HAMINH H., LAUNDER B.E. and MAC INNES H., "A new approach to the analysis of turbulent mixing in variable density flows", *3rd Int. Symp. on Turbulent Shear Flows*, Davis USA, Calif., 1981.

[HAM 99] HAMINH H., "La modélisation statistique de la turbulence, ses capacités et ses limitations", *C.R. Acad. Sci.*, *IIb*, 327, 1999, p. 343–358.

[HAM 01] HAMBA F., "An attempt to combine large eddy simulation with the k-epsilon model in a channel flow calculation", *J. Theor. and Comput. Fluid Dynamics*, 14(5), p. 323–336.

[HAM 03] HAMBA F., "A hybrid RANS/LES simulation of turbulent channel flow", *J. Theor. and Comput. Fluid Dynamics*, 16(5), p. 387–403.

[HAM 04] HAMES A., BASU D. and DAS K., "Assessment of hybrid turbulence models for unsteady high speed separated flow predictions", *42nd AIAA Aerospace Sciences Meeting and Exhibit,*, *AIAA-2004-0684*, January 5-8, 2004, Reno, NV, USA, 2004.

[HAM 05] HAMBA F., "Non-local analysis of the Reynolds stress in turbulent shear flow", *Phys. of Fluids*, 17, 2005, 115 102, 9 pages.

[HAN 72] HANJALIC K. and LAUNDER B.E., "A Reynolds stress model of turbulence and its application to thin shear flow", *J. Fluid Mech.*, 52(4), 1972, p. 609.

[HAN 74] HANJALIC K., "Prediction of turbulent flow in annular ducts with differential transport model of turbulence", *Wärme und Stoffübertragung*, 7(2), 1974, p. 71–78.

[HAN 76] HANJALIC K. and LAUNDER B.E., "Contribution towards a Reynolds stress closure for low Reynolds number turbulence", *J. Fluid Mech.*, 74, 1976, p. 593–610.

[HAN 79] HANJALIC K., LAUNDER B.E. and SCHIESTEL R., "Multiple time-scale concepts in turbulence transport modeling", *2nd Int. Symp. on Turb. Shear Flows.* Impr. Coll. London, July 2–4, 1979.

[HAN 80] HANJALIC K. and LAUNDER B.E., "Sensitizing the dissipation equation to irrotational strain", *J. of Fluid Engng.*, Trans. ASME, 102, 1980, p. 34–40.

[HAN 85] HANJALIC K. and STOSIK N., "Hysteresis of turbulent stresses in wall flows subjected to periodic disturbances", *Proc. 4th Symp. on Turbulent Shear Flows*, (ed.) Bradbury L.S.G. *et al.*, Springer, 1985, p. 287–300.

[HAN 94] HANJALIC K., "Advanced turbulence closure models: a view of current status and future prospects", *Int. J. Heat Fluid Flow*, 15(3), 1994, p. 178–203.

[HAN 97] HANJALIC K., JAKIRLIC S., HADZIC I., "Expanding the limits of "equilibrium" second moment turbulence closures", *Fluid Dynamics Research*, 20, 1997, p. 25–41.

[HAN 02] HANJALIC K., "One-point closure models for buoyancy driven turbulent flows", *Ann. Rev. Fluid Mech.*, 2002, 34, p. 321–347.

[HAN 04] HANJALIC K., HADZIABDIC M., TEMMERMAN L. and LESCHZINER M., "Merging LES and RANS strategies: zonal or seamless coupling?", in *Direct and Large Eddy Simulations no. V*, *Proc. of the Fifth Int. ERCOFTAC Workshop on Direct and Large Eddy Simulation*, held at Munich University of Technology, Germany, Aug. 27-29, Ed. by R. Friedrich, B. Geurts and O. Métais, ERCOFTAC Series vol. 9, Kluwer Academic, Dordrecht, 2004, p. 451–464.

[HAN 04B] HANJALIC K., POPOVAC M., HADZIABDIC M., "A robust near-wall elliptic relaxation eddy-viscosity turbulence model for CFD", *Int. J. Heat Fluid Flow*, 25, 2004, p. 1047–1051.

[HAR 65] HARLOW F.H. and WELSH J.E., "Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface", *Phys. of Fluids*, 8, 1967, p. 2182–2189.

[HAR 67] HARLOW F.H. and NAKAYAMA P.I., "Turbulence transport equations", *Phys. of Fluids*, 10(11), 1967, p. 2323.

[HAR 68A] HARLOW F.H., "Transport of anisotropic or low-intensity turbulence", *Los Alamos Scientific Laboratory Report* LA–3947, 1968.

[HAR 68B] HARLOW F.H. and NAKAYAMA P.I., "Transport of turbulence energy decay rate", *Los Alamos Scientific Laboratory Report*, LA–3854, 1968.

[HAR 69] HARLOW F.H. and HIRT C.W., "Generalized transport theory of anisotropic turbulence", *Los Alamos Scientific Laboratory Report*, LA–4086, 1969.

[HAR 70] HARSHA P.T. and LEE S.C., "Correlation between turbulent shear and turbulent kinetic energy", *A.I.A.A. J.*, 8(8), 1970, p. 1508.

[HAR 94] HÄRTEL C., KLEISER L., UNGER F. and FRIEDRICH R., "Subgrid scale energy transfer in the near wall region of turbulent flows", *Phys. of Fluids*, 6(9), 1994, p. 3130–3143.

[HAS 78] HASSID S. and POREH M., "A turbulent energy dissipation model for flows with drag reduction", *J. of Fluid Engng.*, 100, 1978, p. 107.

[HAW 86] HAWORTH D.C. and POPE S.B., "A generalized Langevin model for turbulent flows", *Phys. of Fluids*, 29(2), 1986, p. 387–405.

[HIN 74] HINZE J.O., SONNENBERG R.E. and BUILTJES P.J.H., "Memory effect in a turbulent boundary layer flow due to a relatively strong axial variation of the mean velocity gradient", *Appl. Sci. Res.*, 29, Jan., 1974, p. 1–13.

[HIR 69] HIRT C.W., "Generalized turbulence transport equations", *Int. Seminar of the Int. Centre for Heat and Mass Transfer*, Herceg Novi, Yugoslavia, September 1969.

[HOF 75] HOFFMAN G., "Improved form of the low-Reynolds number k–ε turbulence model", *Phys. of Fluids*, 18, 1975, p. 309–312.

[HOR 85] HORIUTI K. and YOSHIZAWA A., "Large eddy simulation of turbulent channel flows by 1-equation model", *Proc. Euromech Coll. 199*, *Direct and large eddy simulation of turbulence*, (ed.) U. Schumann and R. Friedrich, Vieweg, 1985, p. 119.

[HOR 97] HORIUTI K., "Subgrid-scale modeling based on the generalized scale similarity models", *Proc. 11th Symp. on Turbulent Shear Flows*, Grenoble, France, vol. 2, 1997, p. 16/7–16/12.

[HUA 85] HUANG P.G. and LESCHZINER M.A., "Stabilization of recirculating flow computations performed with second moment closures and third order discretization", *5th Int. Symp. on Turb. Shear Flows*., Cornell Univ., August 7-9/1985.

[HUA 96] HUANG Y.N., RAJAGOPAL K.R., "On a generalized non-linear k-ε model for turbulence that models relaxation effects", *J. Theor. and Comput. Fluid Dynamics*, 8, 1996, p. 275–288.

[HUA 01] HUANG Y.N., DURST F., "A note on thermodynamic restriction on turbulence modeling", *Int. J. Heat Fluid Flow*, 22, 2001, p. 495–499.

[HUF 73] HUFFMAN G.D., JONES G.D. and BRODKEY R.S., "Optimization of turbulence models by means of a logical search algorithm", *Appl. Sci. Res.*, 27(5), 1973, p. 321–334.

[HUG 01A] HUGHES T.J.R., MAZZEI L., OBERAI A.A. and WRAY A.A., "The multiscale formulation of large eddy simulation: the decay of homogenous isotropic turbulence", *Phys. of Fluids*, 13(2), 2001, p. 505–512.

[HUG 01B] HUGHES T.J.R., OBERAI A.A. and MAZZEI L., "Large eddy simulation of turbulent channel flow by the variational multiscale method", *Phys. of Fluids*, 13(6), 2001, p. 1784–1799.

[HUN 99] HUNT J.C.R., DURBIN P.A., "Perturbed vortical layers and shear sheltering", *Fluid Dyn. Research*, 24, 1999, p. 375–404.

[HWA 97] HWANG C.B. and LIN C.A., "Proposal of a Low-Reynolds Number two-equation model to predict dynamic and thermal field", *Proc. 11th Symp. on Turbulent Shear Flows*, Grenoble, France, vol. 2, 1997, p. P2/95–P2/100.

[IAC 93] IACOVIDES H. and LI H., "Near wall turbulence modeling of developing flow through curved ducts and channels", *Proc. 5th Int. Symp. on Refined Flow Modeling and Turbulence measurements*, Presses de l'Ecole Nationale des Ponts et Chaussées, 1993, p. 141–148.

[IAC 95] IACOVIDES H., LAUNDER B. E., "Computational fluid dynamics applied to internal gas-turbine blade cooling: a review", *Int. J. Heat Fluid Flow*, 16, 1995, p. 454–470.

[IAC 97] IACOVIDES H. and RAISEE M., "Computation of flow and heat transfer in 2D rib roughened passages", in *Proc. 2nd Int. Symp. on Turbulence*, *Heat and Mass Transfer*, Hanjalic K., Peeters T.W.J. (eds.), Delft, 1997.

[ILI 03] ILIESCU T, FISCHER P.F., "Large eddy simulation of turbulent channel flows by the rational large eddy simulation model", *Phys. of Fluids*, 15(10), 2003, p. 3036–3047.

[INO 72] INOUYE M., MARVIN J.G. and SHEAFFER Y.S., "Turbulent wake calculations with an eddy viscosity model", *A.I.A.A. J.*, 10(2), 1972, p. 216.

[JAC 90] JACQUIN L., LEUCHTER O., CAMBON C. and MATHIEU J., "Homogenous turbulence in the presence of rotation", *J. Fluid Mech.*, 220, 1990, p. 1–52.

[JAK 95] JAKIRLIC S. and HANJALIC K., "A second moment closure for non-equilibrium separating high and low Re-number flows", *Proc. 10th Symp. on Turbulent Shear Flows*, Penn. State Univ., vol. 3, 1995, p. 23/25–23/30.

[JAK 00] JAKIRLIC S., VOLKERT J., PASCAL H., HANJALIC K., TROPEA C., "DNS, experimental and modeling study of axially compressed in-cylinder swirling flow", *Int. J. Heat Fluid Flow*, 21, 2000, p. 627–639.

[JAK 02] JAKIRLIC S., HANJALIC K., "A new approach to modeling near-wall turbulence energy and stress dissipation", *J. Fluid Mech.*, 459, 2002, p. 139–166.

[JAY 69] JAYATILLEKE C.L.V., "The influence of Prandtl number and surface roughness on the resistance of the laminar sub-layer to momentum and heat transfer", *Prog. in Heat Mass Transfer*, vol. 1, 1969, p. 193.

[JEA 72A] JEANDEL D., Méthode de calcul de la couche limite turbulente utilisant une équation de comportement du coefficient de Boussinesq, PhD Thesis, University of Claude Bernard Lyon, 1972.

[JEA 72B] JEANDEL D. and MATHIEU J., "Méthode de résolution des équations aux dérivées partielles de la couche limite", *C.R. Acad. Sci. sér. A*, 274, no. 21, 1972, p. 1510.

[JEA 73] JEANDEL D. and MATHIEU J., "Méthode de calcul de la couche limite turbulente utilisant une équation de comportement du coefficient de Boussinesq", *J. Mécanique*, 12, 1973, p. 629.

[JEA 76] JEANDEL D., Une approche phénoménologique des écoulements turbulents inhomogènes, PhD Thesis, University of Claude Bernard Lyon, 1976.

[JEA 77] JEANDEL D. and MATHIEU J., "Modélisation spectrale du champ turbulent homogène", *C.R. Acad. Sci. sér. B*, 285, no. 2, 1977, p. 33–36.

[JEA 78] JEANDEL D., BRISON J.F. and MATHIEU J., "Modeling methods in physical and spectral space", *Phys. of Fluids*, 21(2), 1978, p. 169–182.

[JIS 79] JISCHA M. and RIEKE H.B., "About the prediction of turbulent Prandtl and Schmidt numbers from modeled transport equations", *Int. J. Heat Mass Transfer*, 22, 1979, p. 1547–1555.

[JOH 82] JOHNSON R.W. and LAUNDER B.E., "Discussion of 'on the calculation of the turbulent heat transport downstream from an abrupt pipe expansion'", *Num. Heat Transfer*, 5, 1982, p. 493–496.

[JOH 94] JOHANSSON A.V. and HALLBÄCK M., "Modeling of rapid pressure strain in Reynolds stress closures", *J. Fluid Mech.*, 269, 1994, p. 143–168.

[JON 72] JONES W.P. and LAUNDER B.E., "The prediction of laminarisation with a two-equations model of turbulence", *Int. J. Heat Mass Transfer*, 15, 1972, p. 301.

[JON 73] JONES W.P. and LAUNDER B.E., "The calculation of low-Reynolds number phenomena with a two-equations model of turbulence", *Int. J. Heat Mass Transfer*, 16, 1973, p. 1119.

[JON 88] JONES W.P., MUSONGE P., "Closure of the Reynolds stress and scalar flux equations", *Phys. of Fluids*, 31(12), 1988, p. 3589–3604.

[JON 98] JONGEN T. and GATSKI T.B., "General explicit algebraic stress relations and best approximation for three-dimensional flows", *Int. J. Eng. Sci.*, 36, 1998, p. 739–763.

[KAP 03] KAPADIA S., ROY S. and WURTZLER K., "Detached eddy simulation over a reference Ahmed car model", *41st AIAA Aerospace Sciences Meeting and Exhibit, AIAA-2003-0857*, January 6-9, 2003, Reno, NV, USA, 2003.

[KAS 94A] KASAGI N. and SHIKAZONO N., "Contribution of direct numerical simulations to understanding and modeling turbulent transport", *Proc. Osborne Reynolds Centenary Symposium*, 24 May 1994, University of Manchester, Institute for Science and Technology, 1994.

[KAS 94B] KASSINOS S.C. and REYNOLDS W.C., "A structure based model for the rapid distortion of homogenous turbulence", *Thermosciences Div.*, *Dept. Mech Engng. Report* TF-61, Stanford University, USA, 1994.

[KAS 00] KASSINOS S.C., LANGER C.A., HAIRE S.L., REYNOLDS W.C., "Structure-based turbulence modeling for wall bounded flows", *Int. J. Heat Fluid Flow*, 21, 2000, p. 599–605.

[KAS 01] KASSINOS S. C., REYNOLDS W. C., ROGERS M. M., "One-point turbulence structure tensors", *J. Fluid Mech.*, 428, p. 213–248.

[KAT 93] KATO M. and LAUNDER B.E., "The modeling of turbulent flow around a stationary and vibrating square cylinder", *Proc. 9th Symp. on Turbulent Shear Flows*, Kyoto, Japan, vol. 9, 1993, p. 10.4/1–10.4/6.

[KAW 95] KAWAMURA H. and WAKAO Y., "Modeling of scalar transport in a turbulent channel flow consistent with the linearity principle", *Proc. 10th Symp. on Turbulent Shear Flows*, Penn. State University, vol. 3, 1995, p. P3/1–P3/6.

[KAY 75] KAYS W.M. and MOFFAT R.J., "Studies in Convection", 1, B.E. Launder (ed.), 1975, p. 223.

[KEB 85] KEBEDE W., LAUNDER B.E. and YOUNIS B.A., "Large amplitude periodic flow: a second moment closure study", *Proc. 5th Symp. on Turbulent Shear Flows*, Cornell University, USA, 1985.

[KEL 24] KELLER L.V., FRIEDMAN A.A., "Differential Gleichung für die turbulente Bewegung einer kompressiblen Flüssigkeit", *Proc. 1st Int. Congr. Appl. Mech.*, Delft, 1924, p. 395–405.

[KEN 95] KENJERES S. and HANJALIC K., "On the prediction of thermal convection in horizontal slender enclosures", *Proc. 10th Symp. on Turbulent Shear Flows*, Penn. State University, vol. 3, 1995, p. P3/25–P3/30.

[KEN 05] KENJERES S., HANJALIC K., "LES, T-RANS and Hybrid simulations of thermal convection at high Ra numbers", *Engineering Turbulence Modeling and Experiments 6*, Proc. ERCOFTAC Int. Symp. on Engineering Turbulence Modeling and Measurements, ETMM6, Sardinia, Italy, 23-25 May 2005, W. Rodi and M. Mulas (ed.), Elsevier, p. 369–378.

[KES 63] KESTIN J. and RICHARDSON P.D., "Heat transfer across turbulent incompressible boundary layers", *Int. J. Heat Mass Transfer*, 6, 1963, p. 147.

[KID 89] KIDA S. and HUNT J.C.R., "Interaction between different scales of turbulence over short times", *J. Fluid Mech.*, 201, 1989, p. 411–445.

[KIM 79] KIM J. and MOIN P., "Large eddy simulation of turbulent channel flow. Illiac IV Calculation", *Moffatt USA/Ames Center. AGRED Symp.*, The Hague, The Netherlands, September 24–27, 1979.

[KIM 85] KIM J., MOIN P., "Application of a fractional-step method to incompressible Navier-Stokes equations", *J. Comp. Phys*, 59(2), 1985, p. 308–323.

[KIM 90] KIM S.W., "Near-wall turbulence model and its application to fully developed turbulent channel and pipe flows", *Num. Heat Transfer* B, 17, 1990, p. 101–122.

[KIM 95] KIM S.K. and CHUNG M.K., "Roles of pressure transport and intermittency for computation of turbulent free shear flows", *Int. J. Heat Fluid Flow*, 16(3), 1995, p. 194–201.

[KLE 54] KLEBANOFF P.S., "Characteristics of turbulence in a boundary layer with zero pressure gradient", *NACA TN Report* 1247, 1954.

[KLE 67] KLEINSTEIN G., "Generalized law of the wall and eddy viscosity model for wall boundary layers", *A.I.A.A. J.*, 5(8), 1967, p. 1402–1407.

[KLE 71] KLEINSTEIN G., "Eddy viscosity model for turbulent pipe flow", *A.I.A.A. J.*, 9(8), 1971, p. 1626.

[KLI 67] KLINE S.J., REYNOLDS W.C., SCHRAUB F.A. and RUNSTADLER P.W., "The structure of turbulent boundary layers", *J. Fluid Mech.*, 30, 1967, p. 741–773.

[KOK 00] KOK J.C., "Resolving the dependence on freestream values for the k-ω turbulence model", *A.I.A.A. J.*, 38(7), 2000, p. 1292–1295.

[KOL 41A] KOLMOGOROV A.N., "The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers", *Dokl. Akad. Nauk SSSR.*, 30, 1941, p. 9–13.

[KOL 41B] KOLMOGOROV A.N., "On degeneration of isotropic turbulence in an incompressible viscous liquid", *Dokl. Akad. Nauk SSSR.*, 31, 1941, p. 538–540.

[KOL 41C] KOLMOGOROV A.N., "Dissipation of energy in locally isotropic turbulence", *Dokl. Akad. Nauk SSSR.*, 32, 1941, p.16–18.

[KOL 62] KOLMOGOROV A.N., "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number", *J. Fluid Mech.*, 13, 1962, p.82–85.

[KOL 68] KOLOVANDINE B.A., "Transfer of scalar substance in turbulent shear flows", *Int. Summer School on Heat. and Mass Transfer*, Herceg-Novi, Yugoslavia, September 1968.

[KOL 69] KOLOVANDINE B.A. and VATUTINE I.A., "On the statistical theory of non-uniform turbulence", *Progress in Heat and Mass Transfer*, Vol. 4, Pergamon Press, London. 1969.

[KOL 70] KOLOVANDINE B.A., "Scalar substance transfer in non-homogenous turbulence", *4th Int. Heat Transfer Conference*, Paris-Versailles, 1970, C.T. 2.6.

[KOL 72] KOLOVANDINE B.A. and VATUTINE I.A., "Statistical transfer theory in non-homogenous turbulence", *Int. J. Heat Mass Transfer*, 15(12), 1972, p. 2371.

[KOO 74] KOOSINLIN M.L. and LOCKWOOD F.C., "The prediction of axisymmetric turbulent swirling boundary layers", *A.I.A.A. J.*, 12(4), 1974, p. 547.

[KRA 59] KRAICHNAN R., "The structure of isotropic turbulence at very high Reynolds numbers", *J. Fluid Mech.*, 5(4), 1959, p. 497–543.

[KRA 02] KRAJNOVIC S., DAVIDSON L., "A mixed one-equation subgrid model for large-eddy simulation", *Int. J. Heat Fluid Flow*, 23, 2002, p. 413–425.

[KWA 75] KWAK D., REYNOLDS W.C. and FERZIGER J.H., "Three dimensional time dependant computation of turbulent flows", *Stanford Univ. Calif. Rep.* N.T.F. 5, *NASA* Ngr 05–020–622, 1975.

[LAI 90] LAI Y.G., SO R.M.C., "On near wall turbulent flow modeling", *J. Fluid Mech.*, 221, 1990, p. 641–673.

[LAM 81] LAM C.K.G. and BREMHORST K., "A modified form of the k–ε model for predicting wall turbulence", *J. of Fluid Engng.*, 103(3), 1981, p. 456–460.

[LAM 97] LAMBALLAIS E., LESIEUR M. and SILVESTRINI J.H., "Spectral dynamic model for LES of free and wall shear flows", *Proc. 11th Symp. on Turbulent Shear Flows*, Grenoble, France, vol. 3, 1997, p. 28/7–28/12.

[LAM 97B] LAMBALLAIS E., MÉTAIS O. and LESIEUR M., "Spectral-dynamic model for large eddy simulations of turbulent rotating channel flow", *Theor. Comp. Fluid Dyn.*, 12, 1997, p. 149–177.

[LAN 67] LANDAHL M.T., "A wave-guide model for turbulent shear flow", *J. Fluid Mech.*, 29, 1967, p. 441.

[LAP 95] LAPORTA A., Etude spectrale et modélisation de la turbulence inhomogène, PhD Thesis, Ecole Centrale de Lyon, France, 1995.

[LAU 69A] LAUNDER B.E. and JONES W.P., "A note on Bradshaw's hypothesis for laminarization", *ASME Paper* 69-HT-12, 1969.

[LAU 69B] LAUNDER B.E. and JONES W.P., "Sink flow turbulent boundary layers", *J. Fluid Mech.*, 38, 1969, p. 817.

[LAU 73A] LAUNDER B.E., PRIDDIN C.H., "A comparison of some proposals for the mixing length near a wall", *Int. J. Heat Mass Transfer*, 16(3), 1973, p. 700–702.

[LAU 73B] LAUNDER B.E. and YING W.M., "Prediction of flow and heat transfer in ducts of square cross-section", *Proc. Inst. of Mech. Engng.*, Vol. 187, Paper 37/73, 1973, DI47-149, p. 455.

[LAU 74A] LAUNDER B.E. and SHARMA B.I., "Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc", *Letters in Heat and Mass Transfer*, 1(2), 1974, p. 131–138.

[LAU 74B] LAUNDER B.E. and SPALDING D.B., "The numerical computation of turbulent flows", *Computer methods in applied mechanics and energy*, Vol. 3, 1974, p. 269–289.

[LAU 75A] LAUNDER B.E., "Prediction methods for turbulent flows", *IVK Lectures Series 76*, March 3–7–1975, *"Progress in the modeling of turbulent transport"*, 1975.

[LAU 75B] LAUNDER B.E., "On the effect a gravitational field on the turbulent transport of heat and momentum", *J. Fluid Mech.*, 67(3), 1975, p. 569–581.

[LAU 75C] LAUNDER B.E., REECE G.J. and RODI W., "Progress in the development of a Reynolds stress turbulence closure", *J. Fluid Mech.*, 68(3), 1975, p. 537–566.

[LAU 76] LAUNDER B.E., "Heat and Mass Transport", *Topics in applied physics: Turbulence*, Chapter 6, Vol. 12, Turbulence, P. Bradshaw (ed.), Springer-Verlag, 1976, p. 231–287.

[LAU 77] LAUNDER B.E., PRIDDIN C.H. and SHARMA B.I., "The calculation of turbulent boundary layers on spinning and curved surfaces", *J. of Fluid Engng.*, Trans. ASME, March 1977, p. 231.

[LAU 78] LAUNDER B.E. and SCHIESTEL R., "Sur l'utilisation d'échelles temporelles multiples en modélisation des écoulements turbulents", *C.R. Acad. Sci. sér. A*, 286, 1978, p. 709–712.

[LAU 79A] LAUNDER B.E. and SAMARAWEERA D.S.A., "Application of a second moment turbulence closure to Heat and Mass Transport in thin shear flows, I-Two-dimensional transport.", *Int. J. Heat Mass Tranfer*, 22(12), 1979, p. 1631–1643.

[LAU 79B] LAUNDER B.E. and SCHIESTEL R., "Application d'un modèle de turbulence à échelles multiples au calcul d'écoulements libres turbulents", *C.R. Acad. Sci. sér. B*, 288, 1979, p. 127.

[LAU 82] LAUNDER B.E., "A generalized algebraic stress-transport hypothesis", *A.I.A.A. J.*, 20(3), 1982, p. 436–437.

[LAU 83] LAUNDER B.E. and REYNOLDS W.C., "Asymptotic near-wall stress dissipation rates in a turbulent flow", *Phys. of Fluids*, 26, 1983, p. 1157.

[LAU 86] LAUNDER B.E., "Low Reynolds number turbulence near walls", *UMIST Internal Report* TFD/86/4, University of Manchester, 1986.

[LAU 89A] LAUNDER B.E., "Second moment closure: present and future?", *Int. J. Heat Fluid Flow*, 10(4), 1989, p. 282–300.

[LAU 89B] LAUNDER B.E., SHIMA N., "Second moment closure for the near wall sublayer: development and application", *A.I.A.A. J.*, 27(10), 1989, p. 1319–1325.

[LAU 91A] LAUNDER B.E., TSELEPIDAKIS D.P., "Directions in second-moment modeling of near wall turbulence 29th Aerospace Sciences Meeting", January 7–10, Reno/Nevada, *A.I.A.A. paper* 91–0219, 1991.

[LAU 91B] LAUNDER B.E. and TSELEPIDAKIS D.P., "Progress and paradoxes in modeling near-wall turbulence", *Proc. 8th Symp. on Turbulent Shear Flows*, Munich, Germany, vol. 2, 1991, p. 29/1/1–29/1/6.

[LAU 94] LAUNDER B.E. and LI S.P., "On the elimination of wall-topography parameters from second moment closures", *Phys. of Fluids*, 6, 1994, p. 999–1006.

[LAW 74] LAWN C.J., "The use of an eddy viscosity model to predict the heat transfer and pressure drop performance of roughened surfaces", *Int. J. Heat Mass Transfer*, 17, 1974, p. 421.

[LE 94] LE H., MOIN P., "Direct numerical simulation of turbulent flow over a backward-facing step", *Stanford Univ. Report TF-58*, Dept. of Mech. Engng., 1994.

[LE 97] LE H., MOIN P. and KIM J., "Direct numerical simulation of turbulent flow over a backward-facing step", *J. Fluid Mech.*, 330, 1997, p. 349–374.

[LEB 91] LEBRET Y., VANDROMME D. and HAMINH H., "Structure and modeling in strongly sheared turbulent compressible flow", *8th Int. Symp. on Turb. Shear Flows*, Munich, 1991, 8/4/1-6.

[LEE 70] LEE S.C. and AUILER J.E., "Theory of two-dimensional turbulent wakes", *A.I.A.A. J.*, 8(10), 1970, p. 1876.

[LEE 85] LEE M.J. and REYNOLDS W.C., "Numerical experiments on the structure of homogenous turbulence", *Stanford Univ. Report TF-24*, Dept. of Mech. Engng., 1985.

[LEE 90] LEE M.J., "A contribution toward rational modeling of the pressure strain rate correlation", *Phys. of Fluids A*, 2, 1990, p. 630–633.

[LEE 92] LEE S., LELE S.K., MOIN P., "Simulation of spatially evolving turbulence and applicability of Taylor's hypothesis in compressible flow", *Phys. of Fluids* A, Vol. 4, 1992, p.1521–1530.

[LEI 90] LEITH C.E., "Stochastic backscatter in a subgrid scale model: plane shear layer". *Phys. of Fluids* A, Vol. 2, no. 3, 1990, p. 297–299.

[LEL 92] LELE S.K., "Compact finite difference schemes with spectral like resolution", *J. Comp. Phys.*, 103, 1992, p. 16–42.

[LEO 74] LEONARD A., "On the energy cascade in large eddy simulations of turbulent fluids flows", *Adv. in Geophysics*, 18A, 1974, p. 237.

[LEO 79] LEONARD B.P., "A stable and accurate convective modeling procedure based on quadratic upstream interpolation", *Computer Methods in Applied Mechanics and Engineering*, 19, 1979, p. 59–98.

[LEO 88] LEONARD B.P., "Simple high accuracy resolution program for convective modeling of discontinuities", *Int. J. Num. Meth. in Fluids*, 8, 1988, p. 1291–1318.

[LEP 93] LE PENVEN L. and SERRE G., "Homogenous turbulence submitted to compression: experimental study and modeling", *Proc. 9th Symp. on Turbulent Shear Flows*, Kyoto, Japan, 1993, p. 59–78.

[LER 98] LE ROY O., LE PENVEN L., "Compression of a turbulent vortex flow", *Int. J. Heat Fluid Flow*, 19, 1998, p. 533–540.

[LES 78A] LESIEUR M., Turbulence homogène et isotrope, University courses on Fluid Mechanics, INPG, University of Grenoble, 1978.

[LES 78B] LESLIE D.C. and QUARINI G.L., "The application of turbulence theory to the formulation of subgrid modeling procedures", *J. Fluid Mech.*, 91(1), 1978, p. 65–92.

[LES 78C] LESIEUR M. and SCHERTZER D., "Amortissement auto-similaire d'une turbulence à grand nombre de Reynolds", *J. de Mécanique*, 17(4), 1978, p. 609.

[LES 80] LESLIE D.C., "Analysis of a strongly sheared nearly homogenous turbulent shear flow", *J. Fluid Mech.*, 98(2), 1980, p. 435.

[LES 89] LESCHZINER M.A., "Modeling turbulent recirculating flows by finite volume methods – current status and future directions", *Int. J. Heat Fluid Flow*, 10(3), 1989, p. 186–202.

[LES 90] LESIEUR M., "Turbulence et structures cohérentes dans les fluides", *"Non-linear partial differential equations and their applications"*, Collège de France, Seminar 1989-90, *Pitman Research Notes in Mathematics*, 11, Longman, H.Brezis and J.L. Lions (ed.) 1990.

[LES 96] LESIEUR M., MÉTAIS O., "New trends in large-eddy simulations of turbulence", *Ann. Rev. Fluid Mech*., 28, 1996, p. 45–82.

[LES 97] LESIEUR M., COMTE P., LAMBALLAIS E., MÉTAIS O. and SILVESTRINI J., "Large-eddy simulations of shear flows", *J. Engineering Mathematics*, 32, 1997, p. 195–215.

[LES 02] LESCHZINER M.A., DRIKAKIS D., "Turbulence modeling and turbulent-flow computation in aeronautics", *The Aeronautical J.*, July 2002, p. 349–384.

[LI 00] LI N., BALARAS E., PIOMELLI U., "Inflow conditions for large eddy simulations of mixing layers", *Phys. of Fluids*, 12(4), 2000, p. 935–938.

[LIB 75] LIBBY P.A., "On the prediction of intermittent flows", *J. Fluid Mech.*, 68(2), 1975, p. 273–295.

[LIL 71] LILLY D.K., "Numerical simulation of developing and decaying two-dimensional turbulence", *J. Fluid Mech.*, 45(2), 1971, p. 395.

[LIL 92] LILLY D.K., "A proposed modification of the Germano subgrid scale closure model", *Phys. of Fluids A*, 4(3), 1992, p. 633-635.

[LIN 73] LIN S.C. and LIN S.C., "Study of strong temperature mixing in subsonic grid turbulence", *Phys. of Fluids*, 16(10) 1973, p. 1587–1598.

[LIN 80] LIN A. and WOLFSTEIN M., "Tensorial volume of turbulence", *Phys. of Fluids*, 23(3), 1980, p. 644–646.

[LIO 95] LIOU W.W., SHIH T.S. and DUNCAN B.S., "A multiple scale model for compressible turbulent flows", *Phys. of Fluids*, 7(3), 1995, p. 658–666.

[LIU 76] LIU J.T.C. and MERKINE L., "On the interaction between large-scale structure and fine grained turbulence in a free shear flow, The development of temporal interactions in the mean. ", *Proc. Roy. Soc.*, London A, no. 352, no. 1669, 1976, p. 213–247.

[LOI 73] LOISEAU M., Evolution d'une turbulence homogène soumise à un cisaillement, PhD thesis, University of Lyon, 1973.

[LUM 70] LUMLEY J.L., "Toward a turbulent constitutive relation", *J. Fluid Mech.*, 41(2), 1970, p. 413.

[LUM 74] LUMLEY J.L. and KHAJEH–NOURI B., "Computational modeling of turbulent transport", *Adv. in Geophysics.*, 18A, 1974, p. 169–192.

[LUM 75A] LUMLEY J.L., "Modeling turbulent flux of passive scalar quantities in inhomogenous flows", *Phys. of Fluids*, 18(6), 1975, p. 619–621.

[LUM 75B] LUMLEY J.L., "The pressure strain correlation", *Phys. of Fluids*, 18(6), 1975, p. 750–751.

[LUM 75C] LUMLEY J.L., "Prediction methods for turbulent flows", *IVK Lecture Series 76*, March 3–7, 1975.

[LUM 77] LUMLEY J.L. and NEWMAN G.R., "The return to isotropy of homogenous turbulence", *J. Fluid Mech.*, 82(1), 1977, p. 161–178.

[LUM 78A] LUMLEY J.L., "Computational modeling of turbulent flows", *Adv. in Appl. Mech.*, 18, 1978, p. 123–176.

[LUM 78B] LUMLEY J.L., ZEMAN O. and SIESS J., "The influence of buoyancy on turbulent transport", *J. Fluid Mech.*, 84(3), 1978, p. 581–597.

[LUM 80] LUMLEY J.L., "Second order modelling of turbulent flows", in *Prediction Methods for Turbulent Flows*, W. Kollmann (ed.), 1980, p.1–31.

[LUM 81] LUMLEY J.L., "Coherent structures in turbulence", in *Transition and turbulence*, Ed. R.E. Meyer, Acad. Press, New York, 1981, p. 215–245.

[LUM 83] LUMLEY J.L., "Turbulence modeling", *J. of Applied Mech.*, Trans. ASME, 50, 1983, p. 1097–1103.

[LUN 67] LUNDGREN T.S., "Distribution functions in the statistical theory of turbulence", *Phys. of Fluids*, 10(5), 1967, p. 969–975.

[LUN 69] LUNDGREN T.S., "Model equation for nonhomogenous turbulence", *Phys. of Fluids*, 12(1), 1969, p. 485–497.

[LUN 71] LUNDGREN T.S., "Turbulent pipe and channel flow", *Phys. of Fluids*, 14(2), 1971, p. 225–230.

[LUN 73] LUNDGREN T.S. and WANG F.C., "Eddy viscosity models for free turbulent flows", *Phys. of Fluids*, 16(2), 1973, p. 174–178.

[LUN 98] LUND T. S., WU X., SQUIRES K. D., "Generation of inflow data for spatially-developing boundary layer simulations", *J. Comput. Phys.*, 140, 1998, p. 233.

[MCI 85] MACINNES J., Turbulence modeling of flows with non-uniform density, PhD Thesis, UMIST, Manchester, 1985.

[MAC 79] MACMILLAN O.J. and FERZIGER J.H., "Direct tests of subgrid scale models", *A.I.A.A. J.*, 17, 1979, p. 1340.

[MAG 93] MAGNAUDET J., "Modeling of inhomogenous turbulence in the absence of mean velocity gradients", *J. of Applied Sci. Res.*, 51, 1993, p. 525–531.

[MAL 56] MALKUS W.V.R., "Outline of a theory of turbulent shear flow", *J. Fluid Mech.*, 1(2), 1956, p. 521.

[MAN 89] MANSOUR N.N., KIM J., MOIN P., "Near wall k-ε turbulent modeling", *A.I.A.A. J.*, 27(8), 1989, p. 1068–1073.

[MAN 91] MANSOUR N.N., SHIH T.H. and REYNOLDS W.C., "The effects of rotation on initially anisotropic homogenous flows", *Phys. of Fluids* A, 3(10), 1991, p. 2421–2425.

[MAN 93] MANHART M. and WENGLE H., "A spatiotemporal decomposition of a fully inhomogenous turbulent flow field", *J. Theor. and Comput. Fluid Dynamics*, 5, 1993, p. 223–242.

[MAN 94] MANTEL T. and BORGHI R., "A new model of premixed wrinkled flame propagation based on a scalar dissipation equation", *J. Combustion and Flame*, 96, 1994, p. 443–457.

[MAN 99] MANCEAU R., Modélisation de la turbulence. Prise en compte de l'influence des parois par relaxation elliptique, PhD Thesis, University of Nantes, 1999.

[MAN 01] MANCEAU R., WANG M. and LAURENCE D., " Inhomogenity and anisotropy effects on the redistribution term in Reynolds-averaged Navier-Stokes modeling", *J. Fluid Mech.*, 438, 2001, p. 307–338.

[MAR 70] MARECHAL J., Contribution à l'étude de la déformation plane de la turbulence, PhD Thesis, University of Grenoble, 1970.

[MAR 72] MARECHAL J., "Étude expérimentale de la déformation plane d'une turbulence homogène", *J. Mécanique*, 11, 1972, p. 263–294.

[MAT 61] MATHIEU J., "Contribution à l'étude aérothermique d'un jet plan évoluant en présence d'une paroi", *Publ. Sci. Tech. Min. Air*, no. 374, 1961.

[MAT 71] MATHIEU J., "Remarks on turbulent flows with free stream boundaries", *Von Karman Institute for Fluid Dynamics*, *Lecture Series 36*, Turbulent Jet Flows, 1971.

[MAT 74] MATHIEU J., GENCE J.N., SCHON J.P. and JEANDEL D., "Tenseur spectral des corrélations vitesse-température pour une turbulence homogène associée à des champs moyens de vitesse et de température à gradients constants", *C.R. Acad. Sci. sér. A*, 283, no. 7, 1974, p. 521–523.

[MAT 75] MATHIEU J. and JEANDEL D., "Prediction methods for turbulent flows", *IVK Lectures Series 76*, March 3-7 1975.

[MAT 92] MATHIEU J., "La turbulence: les méthodes spectrales", *Revue Scient. et Tech. de la Défense*, 2nd trimester 1992, p. 89–96.

[MAT 01] MATAOUI A., SCHIESTEL R., SALEM A., "Flow regimes of interaction of a turbulent plane jet into a rectangular cavity: experimental approach and numerical modeling", *Flow, Turbulence and Combustion*, Kluwer, 67(4), 2001, p. 267–304.

[MAX 82] MAXEY M.R. "Distortion of turbulence in flows with parallel streamlines", *J. Fluid Mech.*, 124, 1982, p. 261–282.

[MEI 72] MEI J. and SQUIRE W., "A simple eddy viscosity model for turbulent pipe and channel flow", *A.I.A.A. J.*, 10, 1972, p. 350.

[MEL 73] MELLOR G.L. and HERRING H.J., "A survey of the mean turbulent field closure models", *A.I.A.A. J.*, 11, no. 5, 1973, p. 590–595.

[MEN 92] MENTER F.R., "Influence of freestream values on k-ω turbulence model predictions", *A.I.A.A. J.*, 30(6), 1992, p. 1657–1659.

[MEN 94] MENTER F.R., "Two-equation eddy viscosity turbulence models for engineering applications", *A.I.A.A. J.*, 32(8), 1994, p. 1598–1605.

[MER 01] MERI A., WENGLE H., SCHIESTEL R., "DNS and LES of a backward-facing step flow using 2nd and 4th order spatial discretization and LES of the spatial development of mixing of turbulent streams with non-equilibrium inflow conditions". E.H. Hirschel *et al.* (ed.), Deutsche Forschungsgemeinschaft, Centre National de la Recherche Scientifique, *Numerical Flow Simulation II*, *"Notes on Numerical Fluid Mechanics"*, Springer-Verlag Series,75, 2001, p. 268–287.

[MER 02] MERI A., WENGLE H., "DNS and LES of turbulent backward facing step flow using 2nd and 4th order discretization", Proc. Euromech Coll. 412, Munich, Germany, 4-6 October 2000, in Friedrich R. and Rodi W. (eds), *Advances in LES of Complex Flows*, Kluwer Academic Publishers, 2002, p. 99–114.

[MET 92] MÉTAIS O. and LESIEUR M., "Spectral large-eddy simulation of isotropic and stably stratified turbulence", *J. Fluid Mech.*, 239, 1992, p. 157–194.

[MEY 04] MEYERS J., BAELMANS M., "Determination of subfilter energy in large-eddy simulations", *J. of Turbulence*, 5, 2004, (http://jot.iop.org), 026, 17 p.

[MIN 91] MINER E.W., SWEAN T.F., HANDLER R.A. and LEIGHTON R.I., "Examination of wall damping for the *k-ε* turbulence model using direct simulations of turbulent channel flow", *Int. J. Num. Meth. Fluids*, 12, 1991, p. 609–624.

[MON 65] MONIN A.S., "On the symmetry properties of turbulence in the surface layer of air", *Izvest. Atmos. Oceanic Phys.*, 1, 1965, p. 45.

[MOI 78] MOIN P., REYNOLDS W.C. and FERZIGER J.H., "Large eddy simulation of incompressible turbulent channel flow", *Stanford Univ. Calif. Rep.* N.T.F. 12, NASA NgR o5–020–622, 1978.

[MOI 82] MOIN P. and KIM J., "Numerical investigation of turbulent channel flow", *J. Fluid Mech.*, 118, 1982, p. 341–377.

[MOI 98] MOIN P., MAHESH K., "Direct Numerical Simulation: a tool in turbulence research", *Ann. Rev. Fluid. Mech.*, 30, 1998, p. 539–578.

[MOR 74] MOREL T. and TORDA T.P., "Calculation of free turbulent mixing by the interaction approach", *A.I.A.A. J.*, 12(4), 1974, p. 533–540.

[MOR 88] MORSE A.P., "Numerical prediction of turbulent flow in rotating cavities", *J. Turbomachinery*, 110, 1988, p. 202.

[MOS 99] MOSER R.D., KIM J. and MANSOUR N.N., "Direct numerical simulation of turbulent channel flow up to $Re_\tau = 590$", *Phys. of Fluids*, 11(4), 1999, p. 943–945.

[MUR 96] MURAKAMI S., KATO S., CHIKAMOTO T., LAURENCE D. and BLAY D., "New low-Reynolds number k-ε model including damping effect due to buoyancy in a stratified flow field", *Int. J. Heat Mass Transfer*, 39(16), 1996, p. 3483–3496.

[NAG 90] NAGANO Y. and TAGAWA M., "An improved k-ε model for boundary layer flows", *J. of Fluid Engng.*, Trans. ASME, 112, 1990, p. 33–39.

[NAG 97] NAGANO Y., KONDOH M. and SHIMADA M., "Multiple time scale turbulence model for wall and homogenous shear flows based on direct numerical simulations", *Int. J. Heat Fluid Flow*, 18(4), 1997, p. 346–359.

[NAG 02] NAGANO Y., HATTORI H., "An improved turbulence model for rotating shear flows", *J. of Turbulence*, 3, 2002, (http://jot.iop.org), 006, 14 p.

[NAO 73] NAOT D., SHAVIT A. and WOLFSTEIN M., "Two-point correlation model and the redistribution of Reynolds stresses", *Phys. of Fluids*, 16(6), 1973, p. 738.

[NAO 74] NAOT D., "Numerical calculation of Reynolds stresses in a square duct with secondary flow", *Wärme und Stoffübertragung*, 7(3), 1974, p. 151.

[NAO 77] NAOT D., "Local equilibrium solutions for a stress transport turbulence model", *Phys. of Fluids*, 20 April 1977, p. 546–550.

[NAO 78] NAOT D., "Rapid distortion solutions for a stress transport turbulence model in contracting flow", *Phys. of Fluids*, 21(5), 1978, p. 752–756.

[NAS 69] NASH J.F., "The calculation of three dimensional turbulent boundary layers in incompressible flow", *J. Fluid Mech.*, 37, 1969, p. 625.

[NEE 69A] NEE W. and KOVASZNAY L.S.G., "Simple phenomenological theory of turbulent shear flows", *Phys. of Fluids*, 12(3), 1969, p. 473.

[NEE 69B] NEE W. and KOVASZNAY L.S.G., "The calculation of the incompressible turbulent boundary layer by a simple theory", *Proc. Comput. of turbulent Boundary Layers*, AF OVR-IFP Conf., Stanford, Calif.,1969.

[NEV 45] NEVZGLJADOV V., "A phenomenological theory of turbulence", *J. of Physics*, 9(3), 1945, p. 235.

[NEW 81] NEWMANN G.R., LAUNDER B.E. and LUMLEY J.L., "Modeling the behavior of homogenous scalar turbulence", *J. Fluid Mech.*, 111, 1981, p. 217–232.

[NG   72] NG K.H. and SPALDING D.B., "Turbulence model for boundary Layers near walls", *Phys. of Fluids*, 15, 1972, p. 20.

[NG 76] NG K.H. and SPALDING D.B., "Predictions of two-dimensional boundary layers on smooth walls with a two-equations model of turbulence", *Int. J. Heat Mass Transfer*, 19(10), 1976, p. 1161–1172.

[NIH 67] NIHOUL J.C.J., "Description hamiltonienne des phénomènes irréversibles stationnaires et éléments d'une théorie de la turbulence", *J. Mécanique*, 6(2), 1967, p. 171.

[NIK 00] NIKITIN N.V., NICOUD F., WASISTHO B., SQUIRE K. D. and SPALART P.R., "An approach to wall modeling in large-eddy simulations", *Phys. of Fluids* 12(7), 2000, p. 1629–1632.

[NIS 86] NISIZIMA S. and YOSHIZAWA A., "Turbulent channel and Couette flow using an anisotropic k-ε model", *A.I.A.A. J.*, 25(3), 1986, p. 414–420.

[NIK 72] NIKITIN J.K., "Generalization of the semi-empirical theory of turbulence to flows near rough surfaces with various types of roughness", *Fluid Mech. Soviet Research*, 1(3), 1972, p. 44–52.

[OBE 97] OBERLACK M. "Non-isotropic dissipation in non-homogenous turbulence", *J. Fluid Mech.*, 350, 1997, p. 351–374.

[OBI 89] OBI S., PERIC M., SCHEUERER G., "A finite volume calculation procedure for turbulent flows with second order closure and collocated variable arrangement", *Proc. 7th Symp. on Turbulent Shear Flows*, Stanford Univ., USA, 1989, Vol. 2, p. 17.4.1–17.4.6.

[ORS 69] ORSZAG S.A., "Numerical methods for the simulation of turbulence", *Phys. of Fluids*, *Suppl. II*, 1969, p. 11–250.

[ORS 70] ORSZAG S.A., "Analytical theories of turbulence", *J. Fluid Mech.*, 41(2), 1970, p. 363.

[ORS 71] ORSZAG S.A., "Numerical simulation of incompressible flows within simple boundaries accuracy", *J. Fluid Mech.*, 49(1), 1971, p. 75–112.

[ORS 72] ORSZAG S.A. and PATTERSON G.S., "Numerical simulation of turbulence", *Lectures Notes in Physics*, Vol. 12, Springer-Verlag, 1972, p. 127–147.

[ORS 72B] ORSZAG S.A., PATTERSON G.S., "Numerical Simulation of Three-dimensional Homogenous isotropic turbulence", *Phys. Review Letters*, 28(2), 1972, p. 76–79.

[ORS 74] ORSZAG S.A. and PAO Y.H., "Numerical computation of turbulent shear flows", *Adv. in Geophysics*, 18 A, 1974, p. 225–236.

[PAN 93A] PANCHAPAKESAN N.R. and LUMLEY J.L., "Turbulence measurements in axisymmetric jets of air and helium. Part 1 Air Jet", *J. Fluid Mech.*, 246, 1993, p. 197–223.

[PAN 93B] PANCHAPAKESAN N.R. and LUMLEY J.L., "Turbulence measurements in axisymmetric jets of air and helium. Part 2 Helium Jet", *J. Fluid Mech.*, 246, 1993, p. 125–247.

[PAR 96] PARPAIS S. and BERTOGLIO J-P., "A spectral closure for inhomogenous turbulence applied to turbulent confined flows", in *Adv. in Turbulence VI*, Gavrilakis S., Machiels S. and Monkewitz P.A. (ed.), Kluwer Academics, 1996, p. 75–76.

[PAR 97] PARK T.S., SUNG H.J., "A new low-Reynolds number k-ε-fμ model for predictions involving multiple surfaces", *Fluid Dynamics Research*, 20, 1997, p. 97–113.

[PAR 98] PARNEIX S., DURBIN P.A., BEHNIA M., "Computation of 3D turbulent boundary layers using the υ2 –f model", *Flow Turbulence and Combustion*, 10, 1998, p. 19–46.

[PAT 67] PATANKAR S.V. and SPALDING D.B., "A finite difference procedure for solving the equations of the two-dimensional boundary layer", *Int. J. Heat Mass Transfer*, 10, 1967, p. 1389.

[PAT 71] PATTERSON G.S., ORSZAG S. A., "Spectral calculations of isotropic turbulence: efficient removal of aliasing interactions", *Phys. of Fluids*, 14, 1971, p. 2538–2541.

[PAT 72] PATANKAR S.V. and SPALDING D.B., "A calculation procedure for heat, mass and momentum transfer in three dimensional parabolic flows", *Int. J. Heat Mass Transfer*, 15(10), 1972, p. 1787.

[PAT 75] PATANKAR S.V., PRATAP V.S. and SPALDING D.B., "Prediction of turbulent flow in curved pipes", *J. Fluid Mech.*, 67(3), 1975, p. 583–595.

[PAT 85] PATEL V.C., RODI W., SCHEUERER G., "Turbulence models for near wall low Reynolds number flows: a review", *A.I.A.A. J.*, 23(9), 1985, p. 1308–1319.

[PAV 02] PAVAGEAU M., REY C., "Observation of volume variation effects in turbulent free convection", *International Journal of Heat and Mass Transfer*, 45(1), 2002, p. 181–192.

[PEE 92] PEETERS T.W.J. and HENKES R.A.W.M., "The Reynolds stress model of turbulence applied to the natural convection boundary layer along a heated vertical plate", *Int. J. Heat Mass Transfer*, 35(2), 1992, p. 403–420.

[PEN 87] PENNISI S., TROVATO M., "On the irreducibility of Prof. G.F. Smith's representations for isotropic functions", *Int. J. Engng. Sci.*, 25(8), 1987, p. 1059–1065.

[PER 87] PERRY A.E., "Turbulence modeling using coherent structures in wakes, plane mixing layers and wall turbulence", in *Perspectives in Turbulence*, Meier H.U. and Bradshaw P. (ed.), Springer-Verlag, 1987, p. 115–153.

[PET 99] PETTERSSON REIF B.A., DURBIN P.A., OOI A., "Modeling rotational effects in eddy-viscosity closures", *Int. J. Heat Fluid Flow*, 20, 1999, p. 563–573.

[PEY 75] PEYRET R. and VIVIAND H., "Computation of viscous compressible flows based on the Navier-Stokes equations", *AGARD*-AG-212, Agardograph, September 1975.

[PHI 67] PHILLIPS O.M., "The maintenance of Reynolds stress in turbulent shear flow", *J. Fluid Mech.*, 21(1), 1967, p. 131.

[PIN 97] PINHEIRO J., GONZALEZ M. and BORGHI R., "A two-scale model for a passive scalar field undergoing a spherical compression", *Proc. 11ᵗʰ Symp. on Turbulent Shear Flows*, Grenoble, France, vol. 1, 1997, p. 7/18–7/23.

[PIO 88] PIOMELLI U., MOIN P. and FERZIGER J.H. "Model consistency in large eddy simulation of turbulent channel flows", *Phys. of Fluids*, 31(7), 1988, p. 1884–1891.

[PIO 89] PIOMELLI U., FERZIGER J.H., MOIN P. and KIM J., "New approximate boundary conditions for large eddy simulations of wall bounded flows", *Phys. of Fluids A*, 1, 1989, p. 1061.

[PIO 91] PIOMELLI U., CABOT W.H., MOIN P. and LEE S., "Subgrid scale backscatter in turbulent and transition flows", *Phys. of Fluids A*, 3(7), 1991, p. 1766–1771.

[PIO 93] PIOMELLI U., "High Reynolds number calculations using the dynamic subgrid-scale stress model", *Phys. of Fluids A*, 5(6), 1993, p. 1484–1490.

[PIO 95] PIOMELLI U. and LIU J., "Large eddy simulation of rotating channel flows using a localized dynamic model", *Phys. of Fluids*, 7(4), 1995, p. 839–848.

[PIO 97] PIOMELLI U., COLEMAN G. N., KIM J., "On the effects of nonequilibrium on the subgrid-scale stresses", *Phys. of Fluids*, 9(9), 1997, p. 2740–2748.

[PIO 99] PIOMELLI U., "Large-eddy simulation: achievements and challenges", *Progress in Aerospace Sciences*, 35, 1999, p. 335–362.

[PIO 02] PIOMELLI U., BALARAS E., "Wall-layer models for large-eddy simulations", *Ann. Rev. Fluid Mech.*, 34, 2002, p. 349–374.

[PIO 03] PIOMELLI U., BALARAS E., PASINATO H., SQUIRES K.D., SPALART PH.R., "The inner-outer interface in large eddy simulations with wall-layer models", *Int. J. Heat Fluid Flow*, 24, 2003, p. 538–550.

[PLE 72] PLETCHER R.H., "Calculation methods for compressible turbulent boundary flows with heat transfer", *A.I.A.A. J.*, 10(3), 1972, p. 245.

[PON 05] PONCET S., CHAUVE M-P., SCHIESTEL R., "Batchelor versus Stewartson flow structures in a rotor-stator cavity with throughflow", *Physics of Fluids*, 17(7), published online, 13 July 2005.

[POP 75] POPE S.B., "A more general effective viscosity hypothesis", *J. Fluid Mech.*, 72(2), 1975, p. 331–340.

[POP 76] POPE S.B. and WHITELAW J.H., "The calculation of near wake flows", *J. Fluid Mech.*, 73(1), 1976, p. 9–32.

[POP 78A] POPE S.B., "An explanation of the turbulent round jet/plane jet anomaly", *A.I.A.A. J.*, 16(3), 1978, p. 279–281.

[POP 78B] POPE S.B., "The calculation of turbulent recirculating flows in general orthogonal coordinates", *J. Comp. Phys.*, 26, 1978, p. 197.

[POP 83] POPE S.B., "Consistent modeling of scalars in turbulent flows", *Phys. of Fluids*, 26(2), 1983, p. 404–408.

[POP 85] POPE S.B., "Pdf methods for turbulent reactive flows", *J. Prog. Energy Combust. Sci.*, 11, 1985, p. 119–192.

[POP 94A] POPE S.B., "Lagrangian Pdf methods for turbulent flows", *Annual Rev. Fluid Mech.*, 26, 1994 p. 23–63.

[POP 94B] POPE S.B., "On the relationship between stochastic Lagrangian models of turbulence and second-moment closures", *Phys. of Fluids*, 6(2), 1994, p. 973–985.

[PRA 76] PRATAP V.S. and SPALDING D.B., "Fluid flow and heat transfer in three dimensional duct flows", *Int. J. Heat Mass Transfer*, 19, 1976.

[PRO 61] PROUDMAN I., "On Kraichnan's theory of turbulence", *Mécanique de la Turbulence.* CNRS Editions, no. 108, 1961.

[PRU 72] PRUDNIKOV A.G., SAGALOVICH V.N. and YUKINA E.P., "Diffusion and vortex models of the turbulent jet", *Fluid Mech. Soviet Research*, 1(5), 1972, p. 87.

[PUN 77] PUN W.M. and SPALDING D.B., "A general computer program for two-dimensional elliptic flows", *Imp. Coll. HTS Report* HTS/76/2, London, 1977.

[RAJ 99] RAJAGOPAL K.R., "Remarks on: On a generalized non-linear k-ε model and the use of extended thermodynamics in turbulence", *J. Theor. and Comput. Fluid Dynamics*, 8, 1999, p. 167–169.

[RAN 01] RANDRIAMAMPIANINA A., SCHIESTEL R., WILSON M., "Spatio-temporal behavior in an enclosed corotating disk pair", *J. Fluid Mech.*, 434, 2001, p.39–64.

[RAN 04] RANDRIAMAMPIANINA A., SCHIESTEL R., WILSON M., "The turbulent flow in an enclosed corotating disk pair: axisymmetric numerical simulation and Reynolds stress modeling", *Int. J. Heat Fluid Flow*, 25, 2004, p. 897–914.

[RAS 02] RASPO I., HUGUES S., SERRE E., RANDRIAMAMPIANINA A., BONTOUX P., "A spectral projection method for the simulation of complex three-dimensional rotating flows", *Computers and Fluids*, 31, 2002, p. 745–767.

[RED 81] REDELSPERGER J.L. and SOMMERIA G., "Méthode de représentation de la turbulence d'échelle inférieure à la maille pour un modèle tridimensionnel de convection nuageuse", *Boundary Layer Meteorology*, 21(4), 1981, p. 509–530.

[REY 72] REYNOLDS W.C. and HUSSAIN A.K.M.F., "The mechanics of an organized wave in turbulent shear flow", *J. Fluid Mech.*, 54, 1972, p. 263–288.

[REY 75A] REYNOLDS A.J., "The prediction of turbulent Prandtl and Schmidt numbers", *Int. J. Heat Mass Transfer*, 18, 1975, p. 1055.

[REY 75B] REYNOLDS A.J. and TUCKER H.J., "The distortion of turbulence by general uniform irrotational strain", *J. Fluid Mech.*, 68(4), 1975, p. 673–693.

[REY 76A] REY C., CLAUDE R., SCHON J.P. and MATHIEU J., "Modèle de transfert spectral pour les petites structures des champs turbulents scalaires et cinématiques", *C.R. Acad. Sci. Sér. B*, 283, no. 11, 1976, p. 293–296.

[REY 76B] REY C., GENCE J.N., SCHON J.P. and MATHIEU J., "Étude de l'analogie entre les champs thermique et dynamique en turbulence homogène et isotrope dont les nombres de Reynolds et de Péclet sont petits dans l'espace des nombres d'onde", *C.R. Acad. Sci. sér. B*, 282, no. 1, 1976, p. 59–62.

[REY 76C] REYNOLDS A.J., "The variation of turbulent Prandtl number and Schmidt number in wakes and jets", *Int. J. Heat Mass Transfer*, 19(7), 1976, p. 757.

[REY 76D] REYNOLDS W.C., "Computation of turbulent flows", *Annual Review of Fluid Mech.*, 8, 1976 p 183.

[REY 78] REY C. and MATHIEU J., "Étude spectrale des mécanismes d'échange entre champ cinématique et champ scalaire", *Int. J. Heat Mass Transfer*, 21(8), 1978, p.1009–1018.

[REY 81] REYNOLDS W.C., "Simulation of turbulent shear flows: what can we do and what have we learned?", *7th Biennial Symposium on turbulence*, Rolla, Missouri, September 21, 1981.

[REY 85] REY C., "Mise en évidence des effets de variation de la masse volumique dans les équations aux corrélations des champs turbulents de vitesse et de température". Journée d'étude "Écoulements turbulents à masse volumique variable", IMST, 86/10 report, 1985, p. 85–95.

[REY 87] REYNOLDS W.C., "Fundamental of turbulence for turbulence modeling and simulation", *AGARD* Report no. 755, 1987.

[REY 90] REY C. and ROSANT J.M., "Influence of density variations on small turbulent structures of temperature in strongly heated flows", *Proc. Conf. Int. sur le Transfert de Chaleur*, Jerusalem, 19-24 August 1990, in "Heat Transfer 1990", Hetsroni G. (ed.), Hemisphere Pub. Corp., vol. 2, 1990, p. 405–409.

[REY 91] REY C., BLAQUART B. and BOUDJEMAA A., "Experiments on free convection turbulence", *Experimental Heat Transfer*, *Fluid Mechanics and Thermodynamics*, Keffer J.F., Shah R.K. and Ganic E.N. (ed.), Elsevier Science Publ. Co. Inc., 1991, p. 703–707.

[REY 94] REYNOLDS W.C. and KASSINOS S.C., "A one point model for the evolution of the Reynolds stress and structure tensors in rapidly deformed homogenous turbulence", *Proc. Osborne Reynolds Centenary Symposium*, 24 May 1994, University of Manchester Institute of Technology, UK, 1994.

[REY 00] REY C., "Analyse des effets de variation de volume des gaz dans les équations générales de bilan", *Int. J. Heat Mass Transfer*, 43, 2000, p. 4311–4326.

[REY 02] REYNOLDS W.C., LANGER C.A., KASSINOS S.C., "Structure and scales in turbulence modeling", *Phys. of Fluids*, 14(7), 2002, p.2485–2492.

[RHE 96] RHEE G.H. and SUNG H.J., "A non-linear low Reynolds number k-ε model for turbulent separated and reattaching flows – II Thermal field computations", *Int. J. Heat Mass Transfer*, 39(16), 1996, p. 3465–3474.

[RHI 83] RHIE C.M. and CHOW W.L., "A numerical study of the turbulent flow past an isolated airfoil with trailing edge separation", *A.I.A.A. J.*, 21, 1983, p. 1525–1532.

[RIS 93] RISTORCELLI J.R. and LUMLEY J.L., "A second order simulation of the Czochralski crystal growth melt: the buoyantly driven flow", *J. of Cryst. Growth*, 129, 1993, p. 249–265.

[RIS 94] RISTORCELLI J.R., LUMLEY J.L. and ABID R., "A rapid pressure correlation representation consistent with the Taylor-Proudman theorem materially frame indifferent in the 2D limit", *ICASE Report* no. 94-1, NASA Contractor Report 191591, Langley Res. Center, 1994.

[RIS 95] RISTORCELLI J.R., LUMLEY J.L. and ABID R., "A rapid pressure correlation representation consistent with the Taylor-Proudman theorem materially frame indifferent in the 2D limit", *J. Fluid Mech.*, 292, 1995, p. 111–152.

[ROB 39] ROBERTSON H.P., "Invariant theory of isotropic turbulence", *Proc. Camb. Phil. Soc.*, 36, 1939, p. 209–223.

[ROD 70] RODI W. and SPALDING D.B., "A two parameter model of turbulence and its application to free jets", *Wärme und Stoffübertragung*, 3, 1970, p. 85.

[ROD 72A] RODI W., The prediction of free turbulent boundary layer by use of a two equation model of turbulence, PhD Thesis, Imp. Coll. London. 1972.

[ROD 72B] RODI W., "Influence of buoyancy and rotation on equations for the turbulent length scale", *2ⁿᵈ Symp. on Turbulent Shear Flows*, July 2-4, 1979, Imp. Coll. London, 1972.

[ROD 76] RODI W., "A new algebraic relation for calculating the Reynolds stresses", *ZAMM*, *56*, 1976, p. 219–221.

[ROD 80] RODI W., "Turbulence models and their application in hydraulics", *Conf. IARH-Section on fundamentals of Division II. Exp. and Math. Fluid Dynamics*, Delft. 1980.

[ROD 82] RODI W., "Examples of turbulence models for incompressible flows", *A.I.A.A. J.*, 20(7), 1982, p. 872–879.

[ROG 81A] ROGALLO R.S., "Numerical experiments in homogenous turbulence", *NASA Tech. Memorandum 81315 A8677*, 1981.

[ROG 81B] ROGALLO R.S., "Numerical experiments in homogenous turbulence", *Proc. Conf. on Computation of Complex Turbulent Flows*, Stanford Univ. Calif, 1981.

[ROT 51A] ROTTA J.C., "Statistiche Theorie nichthomogener Turbulenz I", *Zeitschrift für Physik*, 129, 1951, p. 547.

[ROT 51B] ROTTA J.C., "Statistiche Theorie nichthomogener Turbulenz II", *Zeitschrift für Physik* 131, 1951, p. 51.

[ROT 72] ROTTA J.C., "Turbulent shear layer prediction on the basis of the transport equations for the Reynolds stresses", *Appl. Mechanics. Proc. 13ᵗʰ Int. Congress of Theor. and Appl. Mech. Moscow*, August 1972.

[ROT 77] ROTTA J.C., "A family of turbulence models for three dimensional thin shear flows", *Proc. Symp. on Turbulent Shear Flows. Penn State paper 10-27/10-34*, 1977.

[ROT 79] ROTTA J.C., "On the effect of the pressure strain correlation on the three dimensional turbulent boundary layer", *Proc. Symp. on Turbulent Shear Flows*, *London paper 10-26/10-30*, 1979.

[ROY 80] ROY Ph., "Résolution des équations de Navier-Stokes par un schéma de haute précision en espace et en temps", *La Recherche Aérospatiale*, no. 6, 1980, p. 373–385.

[RUB 76] RUBESIN M.W., "A one-equation model of turbulence for use with the compressible Navier-Stokes equations", *NASA Tech. Memo.* TMX-73-128, 1976.

[RUB 92] RUBINSTEIN R. and BARTON J.M., "Renormalization group analysis of the Reynolds stress transport equation". *Phys. of Fluids* A, Vol. 4, no. 8, 1992, p. 1759–1766.

[RUB 96] RUBINSTEIN R. and ZHOU Y.E., "Analytical theory of the destruction terms in dissipation rate transport equations", *Phys. of Fluids*, 8(11), 1996, p. 3172–3178.

[RUB 00] RUBINSTEIN R., "Formulation of a two-scale model of turbulence", *ICASE Report no. 2000-5*, NASA/CR-2000-209853, 2000.

[RUB 01] RUBINSTEIN R. and ZHOU Y.E., "Schiestel's Derivation of the epsilon equation and two equations modeling of rotating turbulence", NASA/CR-2001-211060, *ICASE Report no. 2001-24*, 2001, p. 1–6.

[RUE 71] RUELLE D. and TAKENS F., "On the nature of turbulence", *Communications Math. Phys.*, 20, 1971, p. 167–192.

[RUE 91] RUELLE D., "The turbulent fluid as a dynamic system", in *New Perspectives in Turbulence*, Sirovich L. (ed.), Springer, Germany, 1991, p. 123–138.

[RUF 94] RUFFIN E., SCHIESTEL R., ANSELMET F., AMIELH M., FULACHIER L., "Investigation of characteristic scales in variable density turbulent jets using a second-order model", *Phys. of Fluids A*, 6(8), 1994, p. 2785–2799.

[SAD 94] SADDOUGHI S.G. and VEERAVALLI SV., "Local isotropy in turbulent boundary layers at high Reynolds number", *J. Fluid Mech.*, 268, 1994, p. 333–372.

[SAF 70] SAFFMAN P.G., "A model for inhomogenous turbulent flow", *Proc. Roy. Soc. London A 317*, 1970, p. 417.

[SAF 74] SAFFMAN P.G. and WILCOX D.C., "Turbulence model prediction for turbulent boundary layers", *A.I.A.A. J.*, 12(4), 1974, p. 541–546.

[SAF 76] SAFFMAN P.G., "Development of a complete model for the calculation of turbulent shear flows", *Proc. Symp. on Turbulence and Dynamic Systems*, Duke Univ., Durham, USA, 1976.

[SAM 78] SAMARAWEERA D.S.A., Turbulent heat transport in two and three dimensional temperature fields, PhD Thesis, Imp. Coll. London, 1978.

[SAR 90] SARKAR S. and SPEZIALE C.G., "A simple non-linear model for the return to isotropy in turbulence", *Phys. of Fluids* A, 2, 1990, p. 84–93.

[SAR 91] SARKAR S. and BALAKRISHNAN L., "Application of a Reynolds stress turbulence model to the compressible shear layer", *A.I.A.A. J.*, 29, 1991, p. 743–749.

[SAR 92] SARKAR S., "The pressure dilatation correlation in compressible flows", *Phys. of Fluids* A, 4, 1992, p. 2674–2682.

[SAR 97] SARKAR A. and SO R.M.C., "A critical evaluation of near-wall two-equation models against direct numerical simulation data", *Int. J. Heat and Fluid Flow*, 18(2), 1997, p. 197–208.

[SCH 72A] SCHIESTEL R. and GOSSE J., "Prévision numérique du phénomène de transition pour un écoulement entre deux plans parallèles", *C.R. Acad. Sci. Sér. A*, 275, 1972, p. 1371–1374.

[SCH 72B] SCHONAUER W., "An eddy viscosity based on the second principal invariant of the deformation tensor", *AGARD Conf. Proc. CP 93 on Turbulent Shear Flows*, 1972.

[SCH 74A] SCHIESTEL R., Sur un nouveau modèle de turbulence appliqué aux transferts de quantité de mouvement et de chaleur, PhD Thesis, University of Nancy I., 1974.

[SCH 74B] SCHIESTEL R. and GOSSE J., "Prévision numérique de la convection turbulente dans les espaces annulaires", *C.R. Acad. Sci. Sér. B*, 279, 1974, p. 543–546.

[SCH 75] SCHUMANN U., "Subgrid scale model for finite difference simulations of turbulent flows in plane channels and annuli", *J. Comp. Phys.*, 18, 1975, p 376–404.

[SCH 77] SCHUMANN U., "Realizability of Reynolds stress turbulence models", *Phys. of Fluids*, 20, March, 1977, p. 721–725.

[SCH 78] SCHUMANN U. and PATTERSON G.S., "Numerical study of the return of axisymmetric turbulence to isotropy", *J. Fluid Mech.*, 88(4), 1978, p. 711.

[SCH 82] SCHIESTEL R. and CHAUVE M.P., "Influence d'ondulations de faible amplitude sur la convection turbulente dans un écoulement en conduite axisymétrique: expérience et prévision numérique", *Int. J. Heat Mass Transfer*, 25(12), 1982, p. 1895–1904.

[SCH 83] SCHIESTEL R., "Sur le concept d'échelles multiples en modélisation des écoulements turbulents", *J. de Mécanique Théor. et Appl.*, Part. I, 2(3), p. 417, Part. II, 2(4), 1983, p. 601.

[SCH 86] SCHIESTEL R., "Sur la modélisation des écoulements turbulents en non-équilibre spectral.", *C.R. Acad. Sci. Sér. II*, 302, no. 11, 1986, p. 727–730.

[SCH 87A] SCHIESTEL R., "Multiple time scale modeling of turbulent flows in one point closures", *Phys. of Fluids*, 30 (3), 1987, p. 722.

[SCH 87B] SCHIESTEL R., "Study of one-dimensional spectral dynamic equations of the Reynolds stresses in homogenous anisotropic turbulence: application to split-spectrum modeling", *Proceedings of the 1987 Summer Program "Studying Turbulence using Numerical Simulation Databases", Center for Turbulence Research. Report CTR S87*, December 1987.

[SCH 95A] SCHUMANN U., "Stochastic backscatter of turbulence energy and scalar variance by random subgrid scale fluxes", *Proc. Roy. Soc. London A*, 451, 1995, p. 293–318.

[SCH 95B] SCHIESTEL R., VIAZZO S., "A Hermitian-Fourier numerical method for solving the incompressible Navier-Stokes equations", *International Journal of Computers and Fluids*, 24(6), 1995, p. 739–752.

[SCH 97] SCHIESTEL R. and ELENA L., "Modeling of anisotropic turbulence in rapid rotation", *J. Aerospace Science and Technology*, 7, 1997, p. 441–451.

[SCH 05] SCHIESTEL R., DEJOAN A., "Towards a new partially integrated transport model for coarse grid and unsteady turbulent flow simulations", *J. Theor. and Comput. Fluid Dynamics*, 18(6), 2005, p. 443–468.

[SCH 05B] SCHIESTEL R., "Modèle de turbulence générique à deux équations de transport", *IRPHE note*, 2005(1), 2005.

[SCO 93] SCOTTI A., MENEVEAU C. and LILLY D.K., "Generalized Smagorinsky model for non isotropic grids", *Phys. of Fluids* A, 5(9), 1993, p. 2306–2308.

[SCO 97] SCOTTI A., MENEVEAU C. and FATICA M., "Dynamic Smagorinsky model on anistropic grids", *Phys. of Fluids*, 9(6), 1997, p. 1856–1858.

[SHA 82] SHARMA R.N. and PATANKAR S.V., "Numerical computation of wall jet flows", *Int. J. Heat Mass Transfer*, 25(11), 1982, p. 1709.

[SHA 92] SHAO L. Étude d'une couche de mélange turbulente non cisaillée par simulation des grandes échelles, PhD Thesis, École Centrale Lyon, 1992.

[SHA 94] SHAO L. BERTOGLIO J-P., MICHARD M., "Large –eddy simulation of the interaction between two distinct turbulent velocity scales", *74th Fluid Dyn. Symp. on Applications of Direct and Large Eddy Simulation to Transition and Turbulence*, Chania, Crete, Greece, 1994.

[SHI 73] SHIR C.C., "A preliminary numerical study of atmospheric turbulent flows in the idealized planetary boundary layer", *J. Atm. Sci.*, 30, 1973, p. 1327.

[SHI 85A] SHIH T.H. and LUMLEY J.L., "Modeling of pressure correlation terms in Reynolds stress and scalar flux equations", *Sibley School of Mech. Aero. Engng. Cornell Univ. Report* FDA-85-3, 1985.

[SHI 85B] SHIH T.H., LUMLEY J.L. and CHEN J.Y., "Second order modeling of boundary free turbulent shear flows with a new form of pressure correlation", *Sibley School of Mech. Aero. Engng.*, *Cornell Univ. Report FDA 85/7*, 1985.

[SHI 87] SHIH T.H., LUMLEY J.L. and JANICKA J., "Second order modelling of a variable density mixing layer", *J. Fluid Mech.*, 180, 1987, p. 93–116.

[SHI 90] SHIH T.H., REYNOLDS W.C. and MANSOUR N.N., "A spectrum model for weakly anisotropic turbulence", *Phys. of Fluids* A, 2(8), 1990, p. 1500–1502.

[SHI 91] SHIMA N., "Prediction of three-dimensional turbulent boundary layers using a second moment closure", *8th Int. Symp. on Turb. Shear Flows*, Munich, 1991, 8/2/1-6.

[SHI 93] SHIH T.H. and LUMLEY J.L., "Critical comparison of second order closures with direct numerical simulations of homogenous turbulence", *A.I.A.A. J.*, 31(4), 1993, p. 663–670.

[SHI 95] SHIH T.H., LIOU W.W., SHABBIR A., YANG Z. and ZHU J., "A new k-ε eddy viscosity model for high Reynolds number turbulent flows", *Computers Fluids*, 24(3), 1995, p. 227–238.

[SHI 97] SHIMA N., "Low Reynolds number second moment closure without reflection redistribution terms", *Proc. 11th Symp. on Turbulent Shear Flows*, Grenoble, France, Vol. 1, 1997, p. 7/12–7/17.

[SHI 97B] SHIH T-H., "Some developments in computational modeling of turbulent flows", *Fluid Dyn. Research*, 20, 1997, p. 67–96.

[SHI 98] SHIMA N., "Low Reynolds number second moment closure without wall reflection redistribution terms", *Int. J. Heat Fluid Flow*, 19, 1998, p. 549–555.

[SHU 99] SHUR M., SPALART P.R., STRELETS M. and TRAVIN A., "Detached eddy simulation of an airfoil at high angle of attack", *Proc. Fourth International Symposium on Engineering Turbulence Modeling and Measurements*, May 24-26, W. Rodi and D. Laurence (ed.), Elsevier, Amsterdam, 1999, Corsica, France, 1999, p.669–678.

[SIE 75] SIESS J., Étude d'après la méthode de J.L. Lumley de la pénétration de la turbulence dans un milieu stratifié, PhD Thesis, University of Aix-Marseille II, 1975.

[SIL 93] SILVEIRA NETO A., GRAND D., MÉTAIS O. and LESIEUR M., "A numerical investigation of the coherent vortices in turbulence behind a backward facing step", *J. Fluid Mech.*, 256, 1993, p.1–25.

[SIL 98] SILVESTRINI J.H., LAMBALLAIS E., LESIEUR M., "Spectral-dynamic model for LES of free and wall shear flows", *Int. J. Heat Fluid Flow*, 19, 1998, p. 492–504.

[SIM 70] SIMPSON R.L., WHITTEN D.G. and MOFFATT R.J., "An experimental study of the turbulent Prandtl number of air with injection and suction", *Int. J. Heat Mass Transfer*, 13(1), 1970, p. 125.

[SMI 71] SMITH G.F., "On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors", *Int. J. Engng. Sci.*, 9(10), 1971, p. 899–916.

[SMI 90] SMITH B.R., "The k-$k\ell$ turbulence and wall layer model for compressible flows", *AIAA paper*, 90-1483, *21st Fluid Dynamics*, *plasma dynamics and lazers Conf.*, Seattle, WA, 1990.

[SMI 92] SMITH L.M. and REYNOLDS W.C., "On the Yakhot-Orszag renormalization group method for deriving turbulence statistics and models", *Phys. of Fluids* A, 4(2), 1992, p. 364–390.

[SMI 94] SMITH B.R., "A near wall model for the k-ℓ two-equation turbulence model", *AIAA paper*, 94-2386, *25th Fluid Dynamics Conf.*, Colorado, CO, 1994.

[SO 91] SO R.M.C., LAI Y.G., ZHANG H.S. and HWANG B.C., "Second order near wall turbulence closures: a review", *A.I.A.A. J.*, 29(11), 1991, p. 1819–1835.

[SO 95] SO R.M.C., SOMMER T.P., "An explicit algebraic heat flux model for the temperature field", *Int. J. Heat Mass Transfer*, 39, 1995, p. 455–465.

[SO 02] SO R.M C., VIMALA P., JIN L.H., ZHAO C.Y., GATSKI T.B., "Accounting for buoyancy effects in the explicit algebraic stress model: homogenous turbulent shear flows", *Theor. Comput. Fluid Dyn.*, 2002, 15, p. 283–302.

[SO 04] SO R.M.C., JIN L.H., GATSKI T.B., "An explicit algebraic Reynolds stress and heat flux model for incompressible turbulence: Part I Non-isothermal flow", *J. Theor. and Comput. Fluid Dynamics*, 17, 2004, p. 351–376.

[SOU 02] SOUFFLAND D., GRÉGOIRE O., GAUTHIER S., SCHIESTEL R., "A two time scale model for turbulent Mixing Flows Induced by Rayleigh-Taylor and Richmyer-Meshkov Instabilities", *Flow*, *Turbulence and Combustion*, Kluwer, 69, 2002, p. 123–160.

[SPA 61] SPALDING D.B., "A single formula for the law of the wall", *J. Appl. Mech.*, 28, 1961, p. 455–457.

[SPA 69] SPALDING D.B., "The calculation of the length scale of turbulence in some shear flows remote from walls", *Progress in Heat an Mass Transfer*, Vol. 2, Pergamon Press, London, 1969.

[SPA 71] SPALDING D.B., "Concentration fluctuations in a round turbulent free jet", *J. Chem. Engng. Sci.*, 26, 1971, p. 95.

[SPA 72] SPALDING D.B., "A two equation model of turbulence", *VDI–Forsch–Heft*, 549, 1972, p. 5.

[SPA 73] SPALDING D.B. and PRIDDIN C.H., "A comparison of some proposals for the mixing length near a wall", *Int. J. Heat Mass Transfer*, 16, 1973, p. 700–702.

[SPA 88] SPALART, P.R. "Direct simulation of a turbulent boundary layer up to R=1410", *J. Fluid Mech.*, 187, 1988, p. 61–98.

[SPA 92] SPALART P.R. and ALLMARAS S.R., "A one equation turbulence model for aerodynamic flows", *A.I.A.A. Paper* 92-0439, *30th Aerospace Sciences Meeting and Exhibit*, Reno, NV, 1992.

[SPA 94] SPALART P.R. and ALLMARAS S.R., "A One Equation Turbulence Model for Aerodynamic Flows", *La Recherche Aerospatiale*, no. 1, 1994, p. 5–21.

[SPA 97] SPALART P.R., JOU W.H., STRELETS M. and ALLMARAS S.R., "Comments on the Feasibility of LES for Wings, and on a Hybrid RANS/LES Approach", *1st AFOSR International Conference on DNS/LES, August 4-8, 1997, Ruston, Louisiana.* in *Advances in DNS/LES*, Liu, C., and Liu, Z. (ed.), Greyden Press, Columbus, Ohio, 1997.

[SPE 59] SPENCER A.J.M. and RIVLIN R.S., "The theory of matrix polynomials and its application to the mechanics of isotropic continua.", *Arch. Rat. Mech. Anal.*, 2, 1959, p. 309–336.

[SPE 60] SPENCER A.J.M. and RIVLIN R.S., "Further results in the theory of matrix polynomials.", *Arch. Rat. Mech. Anal.*, 4, 1960, p. 214–230.

[SPE 79] SPEZIALE C.G., "Invariance of turbulent closure models.", *Phys. of Fluids.*, 22(6), 1979, p. 1033–1037.

[SPE 80] SPEZIALE C.G., "Closure relations for the pressure-strain correlation of turbulence", *Phys. of Fluids*, 23(3), 1980, p. 459–463.

[SPE 85] SPEZIALE C.G., "Galilean invariance of subgrid scale stress models in the large eddy simulation of turbulence", *J. Fluid Mech.*, 156, 1985, p. 55–62.

[SPE 87A] SPEZIALE C.G., "Second-order closure models for rotating turbulent flows", *Quart. Appl. Math.*, XLV(4), 1987, p. 721–733.

[SPE 87B] SPEZIALE C.G., "On non-linear k-l and k-ε models of turbulence", *J. Fluid Mech.*, 178, 1987, p. 459–475.

[SPE 87C] SPEZIALE C.G., GATSKI T.B., SARKAR S., "On testing models for the pressure-strain correlation of turbulence using direct simulations", *Phys. of Fluids* A, 4(12), 1992, p. 2887–2899.

[SPE 89] SPEZIALE C.G., "Turbulence modeling in non-inertial frames of reference", *J. Theor. and Comput. Fluid Dynamics*, 1(1), 1989, p. 3–19.

[SPE 91A] SPEZIALE C.G., "Analytical methods for the development of Reynolds stress closures in turbulence", *Ann. Rev. Fluid Mech.*, 23, 1991, p. 107–157.

[SPE 91B] SPEZIALE C.G., SARKAR S. and GATSKI T.B., "Modeling the pressure strain correlation of turbulence: an invariant dynamic system approach", *J. Fluid Mech.*, 227, 1991, p. 245–272.

[SPE 92A] SPEZIALE C.G. and GATSKI T.B., "Modeling anisotropies in the dissipation rate of turbulence", *Bull. Amer. Phys. Soc.*, 37, 1992, p. 1799.

[SPE 92B] SPEZIALE C.G., BERNARD P.S., "The energy decay in self-preserving isotropic turbulence revisited", *J. Fluid Mech.*, 241, 1992, p. 645–667.

[SPE 94A] SPEZIALE C.G., ABID R. and DURBIN P.A., "New results on the realizability of Reynolds stress turbulence closures", *J. Sci. Comput.*, 9(4), 1994, p. 369–403.

[SPE 94B] SPEZIALE C.G. and GATSKI T.B., "Assessment of second order closure models in turbulent shear flows", *A.I.A.A. J.*, 32(10), 1994, p. 2113–2115.

[SPE 96] SPEZIALE C.G., XU X.H., "Towards the development of second order closure models for non-equilibrium turbulent flows" *Int. J. Heat Fluid Flow*, 17, 1996, p. 238–244.

[SPE 97] SPEZIALE C.G. and GATSKI T. B., "Analysis and modeling of anisotropies in the dissipation rate of turbulence", *J. Fluid Mech.*, 344, 1997, p. 155–180.

[SPE 99] SPEZIALE C.G., "On a generalized non-linear k-ε model and the use of extended thermodynamics in turbulence", *J. Theor. and Comput. Fluid Dynamics*, 13, 1999, p. 161–166.

[SPI 72] SPIEGEL E.A., "Qualitative model for turbulent intermittency", *Phys. of Fluids*, 15(8), 1972, p. 1372.

[STA 02] STAWIARSKY K. and HANJALIC K., "A two-scale second-moment one-point turbulence closure", *Engineering Turbulence Modeling and Experiments 5*, W. Rodi, N. Fueyo (ed.), Elsevier, 2002, p.97–106.

[STO 99] STOLZ S., ADAMS N.A., "An approximate deconvolution procedure for large-eddy simulation", *Phys. of Fluids*, 11(7), 1999, p. 1699–1701.

[SUG 97] SUGA K., "Non-linear eddy viscosity modeling with a transport equation for Lumley's stress flatness parameter", *Proc. 11th Symp. on Turbulent Shear Flows*, Grenoble, France, vol. 2, 1997, p. 13/18–13/23.

[SUG 01] SUGA K., NAGAOKA M., HORINOUCHI N., ABE K., KONDO Y., "Application of a three-equation cubic eddy viscosity model to 3-D turbulent flows by the unstructured grid method", *Int. J. Heat Fluid Flow*, 22, 2001, p. 259–271.

[SUG 04] SUGA K., "Improvement of second moment closure for turbulent obstacle flow and heat transfer", *Int. J. Heat Fluid Flow*, 25, 2004, p. 776–784.

[SUG 05] SUGA K., CRAFT T.J., IACOVIDES H., "Extending an analytical wall-function for turbulent flows over rough walls", *Engineering Turbulence Modeling and Experiments 6*, Proc. ERCOFTAC Int. Symp. on Engineering Turbulence Modeling and Measurements, ETMM6, Sardinia, Italy, 23-25 May 2005, W. Rodi and M. Mulas (eds.), Elsevier, p. 157–166.

[TAG 91] TAGAWA M., NAGANO Y. and TSUJI T., "Turbulence model for the dissipation components of Reynolds stresses", *8th Int. Symp. on Turb. Shear Flows*, Munich, 1991, 8/3/1-6.

[TAN 82] TANAKA H., KAWAMURA H., TATENO A. and HATAMIYA S., "Effect of laminarization and retransition on heat transfer for Low Reynolds number flow through a converging to constant area duct", *J. of Heat Transfer*, 104, 1982, p. 363.

[TAU 91] TAULBEE D. and VAN OSDOL J., "Modeling turbulent compressible flows: the mass fluctuating velocity and squared density", *A.I.A.A. Paper* 91-0524, 29th Aerospace Sciences Meeting, Reno, NV, 1991, p. 1–9.

[TAU 93] TAULBEE D.B., SONNENMEIER J.R. and WALL K.M., "Application of a new non-linear stress-strain model to axisymmetric swirling flows", *Proc. 2nd. Int. Symp. on Engng Turbulence Modeling and Measurement*, Rodi W. and Martelli F. (ed.), 1993, Elsevier.

[TCH 73] TCHEN C.M., "Repeated cascade theory of homogenous turbulence", *Phys. of Fluids*, 16(1), 1973, pp 13–30.

[THA 92] THANGAM S. and SPEZIALE C.G., "Turbulent flow past a backward facing step: a critical evaluation of two-equation models", *A.I.A.A. J.*, 30(5), 1992, p. 1314–1320.

[THO 74] THOMPSON J.F., THAMES F.C. and MASTIN C.W., "Automatic numerical generation of body fitted curvilinear coordinate system for field containing any number of arbitrary two-dimensional bodies", *J. Comp. Phys.*, 15, 1974, p. 299.

[TOU 00] TOUIL H., BERTOGLIO J-P., PARPAIS S., "A spectral closure applied to anisotropic inhomogenous turbulence", *8th ETC EUROMECH Conference*, Barcelona, in *Advances in Turbulence VIII*, C. Dopazo (ed.), CIMNE, Spain, 2000, p. 689.

[TOU 02] TOUIL H., BERTOGLIO J-P., SHAO L., "The decay of turbulence in a bounded domain", *J. of Turbulence*, 3, 2002, 049, (http://jot.iop.org), 12 p.

[TOW 54] TOWNSEND A.A., "The distortion of homogenous turbulence", *Quart. J. Mech. and Appl. Math.*, 7, 1954, p. 104.

[TOW 80] TOWNSEND A.A., "The response of sheared turbulence to additional distortion", *J. Fluid Mech.*, 98(1), 1980, p. 171–191.

[TRA 99] TRAVIN A., SHUR M., STRELETS M. and SPALART P. R., "Detached eddy simulations past a circular cylinder", *Flow*, *Turbulence and Combustion*, 63, 1999, p. 269–291.

[TRA 02] TRAVIN A., SHUR M., STRELETS M. and SPALART P.R., "Physical and numerical upgrades in the detached eddy simulations of complex turbulent flows", in *Advances in LES of Complex Flows*, R. Friedrich and W. Rodi (ed.), Kluwer Acad. Publishers, Fluid Mechanics and its Applications, 2002, p. 239–254.

[TRU 72] TRUSOV B.G., "The establishment of a relationship between the intensity value and the dissipation of energy of turbulence", *Fluid Mech. Soviet Research*, 1(5), 1972, p. 164.

[TUC 68] TUCKER M.J. and REYNOLDS A.J., "The distortion of turbulence by irrotational plane strain", *J. Fluid Mech.*, 32 (4), 1968, p. 657.

[UBE 56] UBEROI M.S., "Effects of wind-tunnel contraction on free stream turbulence", *J. of the Aeronautical Sciences*, 23, 1956, p. 754.

[UBE 66] UBEROI M.S. and WALLIS S., "Small axisymmetric contraction of grid turbulence", *J. Fluid Mech.*, 24 (3), 1966, p. 539.

[VAN 83] VANDROMME D., HA-MINH H., VIEGAS J.R., RUBESIN M.W. and KOLLMANN W., "Second order closure for the calculation of compressible wall bounded flows with an implicit Navier-Stokes solver", *4th Int. Symp. on Turbulent Shear Flows*, Karlsruhe, Germany, 1983.

[VAS 98] VASILYEV O.V., LUND T.S., MOIN P., "A General class of commutative filters for LES in Complex Geometries", *J. of Computational Physics*, 146, 1998, p. 82–104.

[VDR 51] VAN DRIEST E.R., "Turbulent boundary layer in compressible fluids", *J. of the Aeronautical Sciences*, 18(3), 1951, p. 145–160.

[VDR 56] VAN DRIEST E.R., "On turbulent flow near a wall", *J. Aero. Sci.*, 23, 1956, p. 1007–1011.

[VDV 95] VAN DER VEN H., "A family of large eddy simulation (LES) filters with non uniform filter widths", *Phys. of Fluids*, 7(5), 1995, p. 1171–1172.

[VED 03] VÉDY E., VIAZZO S., SCHIESTEL R., "A high-order finite difference method for incompressible fluid turbulence simulations", *Int. J. for Numerical methods in fluids*, 42, 2003, p. 1155–1188.

[VIA 95] VIAZZO S., SCHIESTEL R., "Simulation des grandes échelles turbulentes en canal à l'aide d'un schéma hermitien", *C.R. Acad. Sci.*, Paris, 321, Ser. IIb, 1995, p. 225–232.

[VIA 01] VIAZZO S., DEJOAN A., SCHIESTEL R., "Spectral features of wall-pressure fluctuations in turbulent wall flows with and without perturbations using LES", *Int. J. Heat Fluid Flow*, 22, 2001, p. 39–52.

[VIE 83] VIEGAS J.R. and RUBESIN M.W., "Wall function boundary conditions in the solution of the Navier-Stokes equations for complex compressible flows", *A.I.A.A. Paper* 83-1964, 1983.

[VIE 85] VIEGAS J.R., RUBESIN M.W. and HORSTMAN C.C., "On the use of wall functions as boundary conditions for two-dimensional separated compressible flows", *A.I.A.A. Paper* no. 85-0180, 1985.

[VIL 99] VILLERMAUX E., INNOCENTI C., "On the geometry of turbulent mixing", *J. Fluid Mech.*, 393, 1999, p. 123–147.

[VLE 79] VAN LEER B., "Towards the ultimate conservation scheme, V–A second order sequel to Godunov's method", *J. Comp. Phys.*, 32, 1979, p. 101–136.

[WAS 73] WASSEL A.T. and CATTON Y., "Calculation of turbulent boundary layers flat plates with different phenomenological theories of turbulence and variable turbulent Prandtl number", *Int. J. Heat Mass Transfer*, 16, 1973, p. 1547–1563.

[WAT 91] WATKINS A.P., LEA C.J., GUL M.Z., Development of advanced turbulence models for the calculation of flows in internal combustion engine cylinders", in *Proc. of Institution of Mechanical Engineers International Conf. on Computers in Engine Technology*, Cambridge, 1991, C430/025, p. 7–20.

[WEB 64] WEBSTER C.A.G., "An experimental study of turbulence in a density-stratified shear flow", *J. Fluid Mech.*, 19(2), 1964, p. 221.

[WEI 81] WEINSTOCK J., "Theory of pressure strain rate correlation for Reynolds stress turbulence closures, Pt 1 – Off diagonal elements", *J. Fluid Mech.*, 105, April, 1981, p. 369–396.

[WEI 82] WEINSTOCK J., "Theory of pressure strain rate correlation for Reynolds stress turbulence closures, Pt 2 – Diagonal elements", *J. Fluid Mech.*, 116, Mach, 1982, p. 1–29.

[WEL 66] WELCH J.E., HARLOW F., SHANNON J.P. and DALY B.J., "The MAC method, a computing technique for solving viscous incompressible transient fluid flow problems in evolving free surfaces", *Los Alamos Rep.* LA3425., 1966.

[WEN 03] WENGLE H., SCHIESTEL R., BEFENO I., MERI A., "Large-eddy simulations of the spatial development of a shearless turbulence mixing layer", E.H. Hirschel and Dr. E. Heinrich (ed.), *Numerical Flow Simulation III*, *Series: Notes on Numerical Fluid Mechanics and Multidisciplinarity Design (NNFM)*, 82, Springer, 2003, p.271–286.

[WER 89] WERNER H. and WENGLE H., "Large eddy simulation of turbulent flow over a square rib in a channel", *Proc. 7th Symp. on Turbulent Shear Flows*, Stanford University, USA, 1989.

[WER 91] WERNER H. and WENGLE H., "Large eddy simulation of turbulent flow over and around a cube in a plate channel", *Proc. 8th Symp. on Turbulent Shear Flows*, Munich, Germany, 1991, p. 19/4/1–19/4/6, (Also in *Selected Papers from the 8th Symposium on Turbulent Shear Flows*, F. Durst, R. Friedrich, B.E. Launder, U. Schumann, J. H. Whitelaw (ed.), New York, Springer, 1993, p. 155–168)

[WES 71] WESSELING P. and LINDHOUT J.P.F., "A calculation method for three dimensional incompressible turbulent boundary layers", *AGARD Conf. Proc. CP 93 on Turbulent Shear Flows*, London, September 1971.

[WIL 75] WILLMARTH W.W., "The structure of turbulence in boundary layers", *Adv. in Appl. Mech.*, 15, 1975, p. 159–253.

[WIL 80] WILCOX D.C. and RUBESIN W.M., "Progress in turbulence modeling for complex flow fields including effects of compressibility", *NASA Tech. Paper* no. 1517, 1980.

[WIL 88A] WILCOX D.C., "Reassessment of the scale-determining equation for advanced turbulence models", *A.I.A.A. J.*, 26 (11), 1988, p. 1299–1310.

[WIL 88B] WILCOX D.C., "Multiscale model for turbulent flows", *A.I.A.A. J.*, 26(11), 1988, p. 1311–1320.

[WIL 93] WILCOX D.C., "A two-equation turbulence model for wall-bounded and free shear flows", *24ᵗʰ AIAA Fluids Dynamics Conference*, Orlando, Florida, USA, 1993.

[WIZ 96] WIZMAN V., LAURENCE D., KANNICHE M., DURBIN P. and DEMUREN A., "Modeling near-wall effects in second moment closures by elliptic relaxation", *Int. J. Heat Fluid Flow*, 17(3), 1996, p. 255–266.

[WOL 69] WOLFSHTEIN M., "The velocity and temperature distribution in one-dimensional flow with turbulence augmentation and pressure gradient", *Int. J. Heat Mass Transfer*, 12, 1969, p. 301.

[WOL 70] WOLFSHTEIN M., "On the length scale of turbulence equation", *Israel J. of Technology*, 8, 1–2, 1970, p. 87.

[WU 85A] WU C.T., FERZIGER J.H., CHAPMAN D.R., "Simulation and modeling of homogenous, compressed turbulence", *5ᵗʰ Int. Symp. on Turbulent Shear Flows*, Ithaca, NY, USA, 1985.

[WU 85B] WU C.T., FERZIGER J.H., CHAPMAN D.R., "Simulation and modeling of homogenous, compressed turbulence", *Thermosciences Div.*, *Dept. Mech Engng. Report* TF-21, Stanford University, USA, 1985.

[WYN 74] WYNGAARD J.C. and COTE O.R., "The evolution of a convective planetary boundary layer-A higher order closure model study", *Boundary Layer Meteorology*, 7, 1974, p. 289.

[XU 01] XU D., KARNIADAKIS G.E., "A semi-Lagrangian High-Order method for Navier-Stokes equations", *J. Comput. Physics*, 172, 2001, p. 658–684.

[YAK 86] YAKHOT V. and ORSZAG S.A., "Renormalization group analysis of turbulence – 1- Basic theory", *J. of Scientific Computing*, 1(1), 1986, p. 3–51.

[YAK 92] YAKHOT V., ORSZAG S.A., THANGAM S., GATSKI T.B. and SPEZIALE C.G., "Development of turbulence models for shear flows by a double expansion technique", *Phys. of Fluids* A, 4(7), 1992, p. 1510–1520.

[YAM 95] YAMAMOTO M., "Reynolds stress model in homogenous anisotropic turbulence", *Int. J. Heat Fluid Flow*, 16, 1995, p. 417–428.

[YAP 87] YAP C., Turbulent heat and momentum transfer in recirculating and impinging flows, PhD Thesis, University of Manchester and UMIST, Dept. of Mech. Engng. Report TFD/87/1, 1987.

[YAS 83] YASSOUR Y. and WOLFSHTEIN M., "A spectral model for the calculation of homogenous turbulent flows", *4ᵗʰ Int. Symp. on Turb. Shear Flows*, Karlsruhe, 9.23–9.29. 1983.

[YEN 72] YEN J.T., "Kinetic theory of turbulent flow", *Phys. of Fluids*, 15(10), 1972, p. 1728.

[YOS 82] YOSHIZAWA A., "A statistically derived subgrid model for the large eddy simulations of turbulence", *Phys. of Fluids*, 25, 1982, p. 1532–1538.

[YOS 84] YOSHIZAWA A., "Statistical analysis of the deviation of the Reynolds stress from its eddy-viscosity representation", *Phys. of Fluids*, 27(6), 1984, p. 1377–1387.

[YOS 88] YOSHIZAWA A., "Statistical modeling of passive scalar diffusion in turbulent shear flows", *J. Fluid Mech.*, 195, 1988, p. 541–555.

[ZAN 90] ZANG T.A., "Spectral methods for simulation of transition and turbulence", in *Spectral and Higher Order Methods for Partial Differential Equations*, Proc. ICOSAHOM89 Conf., Como, Italy, 1989, C. Canuto and A. Quarteroni (ed.), North-Holland, 1990, p. 209–221.

[ZAN 93] ZANG Y., STREET R.L. and KOSEFF J.R., "A dynamic mixed subgrid scale model and its application to turbulent recirculating flows", *Phys. of Fluids* A, 5(12), 1993, p. 3186–3196.

[ZEM 76] ZEMAN O. and LUMLEY J.L., "Modeling buoyancy driven mixed layers", *J. of Atm. Sci.*, 33, 1976, p. 1974–1988.

[ZEM 77] ZEMAN O. and LUMLEY J.L., "Buoyancy effects in entraining turbulent boundary layers: a second order closure study", *Proc. Symp. on Turbulent Shear Flows. I*, 1977, p. 6–21/6–27.

[ZEM 90] ZEMAN O., "Dilatation dissipation: the concept and application on modeling compressible turbulence", *Phys. of Fluids* A, 2, 1990, p. 178–188.

[ZEM 91] ZEMAN O., "Compressible turbulence subjected to shear and rapid compression", *Proc. 8th Symp. on Turbulent Shear Flows*, Munich, Germany, 1991, p. 21/4/1–21/4/6.

[ZEM 91B] ZEMAN O., "On the decay of compressible turbulence", *Phys. of Fluids* A, 3, 1991, p. 951–955.

This page intentionally left blank

Nomenclature

$\boldsymbol{a}_{ij}$	anisotropy tensor $b_{ij}/2k$
a_{ij}	anisotropy tensor b_{ij}/k (other definition)
b_{ij}	deviator of the Reynolds stress tensor $R_{ij}-(2/3)k\delta_{ij}$
$\boldsymbol{E}$	constant in the logarithmic velocity law
$E(\kappa)$	turbulence energy density spectrum
g	acceleration of gravity
G_{ij}	production of the Reynolds stresses from buoyancy forces
G	production of turbulence energy from buoyancy forces
k	turbulence kinetic energy
ℓ	turbulence macro-length scale $k^{3/2}/\varepsilon$
P_{ij}	production rate of the Reynolds stresses from shear
P	production rate of turbulence energy from shear
P	instantaneous pressure
p	fluctuating pressure
$\overline{P}$	mean pressure
Pr	Prandtl number
Pr_t	turbulence Prandtl number
q	half-variance of a transported scalar
Re	Reynolds number of the flow
Re_t	turbulence Reynolds number $\ell\sqrt{k}/\nu$

Ri	gradient Richardson number
Rf	flux Richardson number
R_{ij}	Reynolds stress components
S_{ij}	strain rate tensor $(U_{i,j} + U_{j,i})/2$
t	time
U_i	instantaneous velocity
U'_i, u_i	fluctuating velocity
$\overline{U}_i$	mean velocity
u_*	friction velocity $\sqrt{\tau_P/\rho}$
x_i	point Cartesian coordinates
II^a	second invariant of anisotropy $a_{ij}a_{ij}$
	($II^a = a_{ij}a_{ij} = II^a/4$)
III^a	third invariant of anisotropy $a_{ij}a_{jk}a_{ki}$
	($III^a = a_{ij}a_{jk}a_{ki} = III^a/8$)
γ	fluctuation of a transported scalar
Γ	instantaneous transported scalar (temperature, concentration, etc.)
$\overline{\Gamma}$	mean transported scalar
δ_{ij}	unit Kronecker tensor
ε	turbulence kinetic energy dissipation rate $\nu\overline{u_{i,j}u_{i,j}}$
ε_{ij}	dissipation rate tensor of the Reynolds stresses $\nu\overline{u_{i,m}u_{j,m}}$
ε_{ijk}	alternate pseudo-tensor
ε_γ	scalar variance dissipation rate $\sigma\overline{\gamma_{,j}\gamma_{,j}}$
η	Kolmogorov length scale
κ	wavenumber
K	Karman constant
λ	Taylor microscale
Λ_{ij}	constant mean velocity gradient $\partial\overline{U}_i/\partial x_j$ in HAT

μ	dynamic viscosity of the fluid
ν	kinematic viscosity of the fluid
ρ	fluid density
ρ'	fluctuating density
$\overline{\rho}$	mean density of the fluid
σ	molecular diffusivity of the transported scalar
τ	Kolmogorov time scale
τ_P	skin friction
$\varphi_{ij}(\vec{\kappa})$, $\Gamma_{ij}(\vec{\kappa})$	spectral tensor of double velocity correlations
$\phi_{ij}(\kappa)$	one-dimensional spectral density of double velocity correlations (spherical average with radius κ)
Φ_{ij}	pressure-strain correlation tensor
Ω	rotation rate
ω_{ij}	rotation tensor $(U_{i,j} - U_{j,i})/2$

This page intentionally left blank

Index