1.21 A summary of the main points

- The turbulence-model approach is the only practical way of treating turbulent flows.
- The task of optimising the empirical inputs increases rapidly with the number of differential equations.
- The employment of a more complex model is warranted only if the increase of universality is commensurate with the extra cost.

The box above summarises the main points from this first lecture.

Details of the turbulent motion are too small-scale in character to be directly amenable to analytical treatment; and, in any case, it is not the details, but the time-averaged consequences which interest the engineer. He wants to know how the turbulent motion will affect the transport of heat and momentum, the mixing of chemical species or the rate of progress of a combustion process; and he wants reasonable accuracy of prediction at low cost. So he will adopt a more elaborate model of turbulence only if it possesses decisive advantages over the simpler alternatives available to him.

The originator of a turbulence model also wants to quit at the simplest possible level; for, the task of optimising the constants and functions in the model increases very rapidly with the number of equations.

Ultimately, as we have remarked, the best choice of model will depend on the flow to be predicted. It is our hope that, by the time the last lecture is reached, the reader will not only have learned much about available models of turbulence but will also be able to select the one most appropriate to the flows which interest him.

Lecture 2

The Mixing-Length Hypothesis for the Transfer of Momentum

2.1 Preliminary remarks

"The simpler the better" is a good rule for an engineer in search of a theory. He is interested in truth of course; but it is the truth of the artist, which consists in emphasizing essentials and stimulating the imagination, not in accumulating detail without regard to relevance. In this lecture therefore we deal with the simplest turbulence model. It is the earliest; and, with only a few modifications to its original form, it can be the one which the engineer should use.

What most people learn, from cursory readings of the fluid-mechanics literature, is that the mixing-length hypothesis (MLH) is an outmoded historical exhibit, long superseded by the statistical theory of turbulence. This notion is only partially true. It is as though fluids were first held to be continua, and the science of fluid dynamics was built on that basis, with the aid of empirical laws like Newton's law of viscous action; and then later we learned that there are spaces between the molecules, and began to found a theory of viscosity on the mechanics of collision. The new knowledge does not invalidate the old; and we employ it not for its greater sophistication but only, if at all, when it works better than the simpler model.

2.1 (continued)

- Nature of mixing-length hypothesis.
- Free turbulent flows and flows near walls.
- The van Driest hypothesis and its modifications.
- Discussion.

The panel shows how the lecture will be broken up. It will take only a few minutes to explain the MLH, but much more time to supply the individual items of information about it, and to exemplify its implications.

Free turbulent flows will be distinguished from those near walls. The former comprise jets, wakes and plumes; the latter include flows on aerofoils, wall jets, flows in diffusers with and without separation, and many other phenomena. The free turbulent flows are manifestations of what one might call "pure" turbulence; the laminar properties of the fluid have no influence at all, and the flow pattern is independent of Reynolds number.

The presence of a wall has two influences. The first is that its rigidity limits the eddying motion: the sizes of the eddies must diminish as the wall is approached. Then the reduced eddy size near the wall means that laminar effects can at last become important there; the nearer is the wall, the greater these effects. This is what we shall discuss in the context of the van Driest hypothesis.

2.2 Nature of the hypothesis. Part 1.

- Originator: Prandtl (1925).

  Starting point: kinetic theory of gases, which gives:

  $\mu \simeq 1/3\ \rho\ \ell_{\text{free path}}\ V_{\text{molecular}}$.

- First part of hypothesis: $\frac{\tau}{\partial u/\partial y} \equiv \mu_t = \rho\ \ell_m\ V_t$.

  [ℓ_m ≡ mixing length; V_t ≡ turbulence velocity]

Like many valuable novelties in fluid dynamics, the MLH was invented by Ludwig Prandtl (1925). Just how he came to the idea we cannot know; but his paper suggests that the kinetic theory of gases was in his mind, as an analogy. One result of this theory is the first equation in the box: the viscosity of the fluid is proportional to density, times mean free path, times a random velocity.

The MLH has two parts, which are often run together. We separate them here, because one can then see how the MLH might have been arrived at; and one can also recognise the possibility of adopting only the first part, and replacing the second by something more accurate.

The idea that Newton's viscosity law applied to turbulent flow, i.e. that τ and $\partial u/\partial y$ might vanish together, had already been proposed by Boussinesq (1877). Prandtl adopted this idea, and supposed that this "turbulent viscosity", μ_t, might be formed from a product of the density, a length, and a random velocity. The analogy with the kinetic theory of gases is clear; but, in turbulent flow, the length (called "mixing length") and the random velocity, must be expected to vary from place to place, and to have values influenced by the particular pattern and velocity of the mean flow.

2.3 Nature of hypothesis. Part 2

- Second part of hypothesis: $V_t \equiv \ell_m \, |\partial u/\partial y|$.
- Consequences: $\tau \equiv \mu_t \frac{\partial u}{\partial y} = \rho \, \ell_m^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial u}{\partial y}$; $\mu_t = \rho \, (\frac{\tau}{\rho})^{\frac{1}{2}} \, \ell_m$.
- Question: what determines ℓ_m?

With the hypothesis in the form on the previous page, there are two quantities to be ascribed values at each point: ℓ_m and V_t. Prandtl therefore introduced the second part of the hypothesis, so as to reduce the number of unknowns to one.

The idea is: the magnitude of the fluctuations in lateral velocity is proportional to that of those in the longitudinal velocity; and these are proportional to the distance from which material is fetched, times the mean-velocity gradient. Since at this point there is no additional meaning to be ascribed to ℓ_m, we might just as well take the proportionality constant as unity. This is what Prandtl did.

The first entry under "consequences" is what most people understand by the Prandtl mixing-length hypothesis. However, it is evident that one need not adopt the whole of it; or not all the time. This remark will prove to be important later.

There is still the question of how ℓ_m is to be determined. Prandtl went on to propose that ℓ_m was proportional to the distance from the nearest wall. This is an additional feature that some might regard as essential to Prandtl's hypothesis; but it will be useful to us to recognise that it is not: one can calculate ℓ_m in any way one pleases.

2.4 Values of mixing length for some free turbulent flows

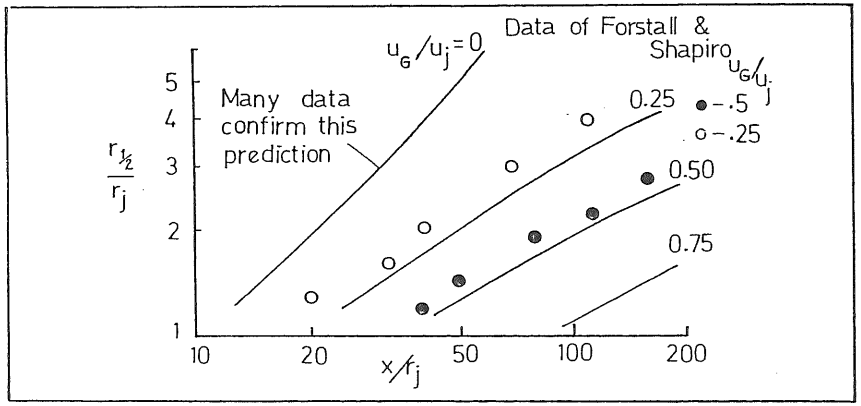
- Plane mixing layer: $\ell_m/\delta = 0.07$ [$\delta \equiv$ layer width].
- Plane jet in stagnant surroundings: $\ell_m/\delta = 0.09$ [$\delta \equiv$ width of $\frac{1}{2}$ jet].
- "Fan" jet in stagnant surroundings: $\ell_m/\delta = 0.125$ [$\delta \equiv$ width of $\frac{1}{2}$ jet].
- "Round" jet in stagnant surroundings: $\ell_m/\delta = 0.075$ [$\delta \equiv$ width of $\frac{1}{2}$ jet].

Here are some answers to the question about ℓ_m. They relate to four different kinds of free turbulent flow, and give the ratio of the mixing length to the width of the turbulent region, δ. Because this width increases along the length of the flow, so does ℓ_m; but this quantity varies remarkably little with distance across the flow. The variation can often be ignored.

The concept of width is not quite definite, because no edge of the turbulent region can be precisely located. Interpretation of the formulae can proceed on the assumption that they refer to the point at which the fluid velocity equals 1% of the maximum velocity difference across the layer. "Width of $\frac{1}{2}$ jet" means distance from the symmetry axis to this 1% point.

The list is not exhaustive. One could also supply ℓ_m/δ values for: the plane and round jet in surroundings moving with the jet, for the plane and round buoyant plumes, and for the plane and round wakes. However, there is not much more information. From one point of view the most significant thing about this lecture is what is not said, because the information is not available.

2.5 Round-jet predictions: effect of free-stream velocity



We cannot support all our assertions by demonstrations. However, here is just one illustration of the agreement that can be achieved between predictions and experiments. The curves represent the shapes ($r_{\frac{1}{2}}$ versus x) for axisymmetrical jets in stagnant surroundings ($u_G/u_J = 0$) and for various ratios of surrounding-fluid velocity u_G to injected-fluid velocity u_J. $r_{\frac{1}{2}}$ is the radius at the point where the velocity has half the axial value; x is the distance from the jet orifice.

The curve on the left fits the experimental data. The mixing-length constant 0.075 was chosen to ensure this, so we can base no claims on the agreement. However, the other curves were computed with the same value of the constant; so agreement with experiment signifies that a single constant has more than single-experiment validity.

The agreement for the two velocity ratios for which data are available is indeed quite good; but there is a tendency for the actual rates of spread to exceed the predicted ones. This may be the consequence of (neglected) free-stream turbulence. The computations, by the way, were performed with the aid of the Patankar-Spalding (1970) computer program for two-dimensional boundary-layer flows.

2.6 Mixing-length distribution in boundary layers near a single wall

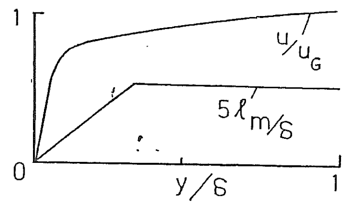
The Escudier formula:

$$\frac{y}{\delta} < \frac{\lambda}{\kappa}, \quad \frac{\ell_m}{\delta} = \kappa \frac{y}{\delta}$$

$$\frac{y}{\delta} > \frac{\lambda}{\kappa}, \quad \frac{\ell_m}{\delta} = \lambda$$

$\kappa \simeq .41$

$\lambda \simeq .09$



Note: This formula needs modification very near the wall.

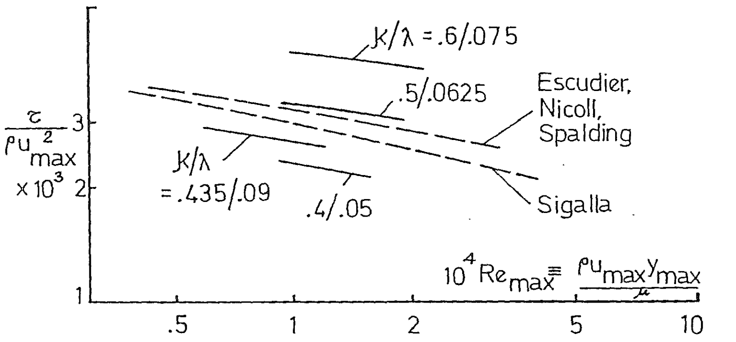
We now turn to boundary layers which lie along a single wall. Escudier (1966) has analysed a large amount of experimental data, and concluded that, except in the semi-laminar region close to the wall, the mixing-length distribution usually has the form represented by the sketch and formula. Other authors have come to similar conclusions, although somewhat different algebraic forms have been employed.

Once again, δ requires definition. It can be taken as the distance from the wall to the point where the time-average velocity differs from the free-stream velocity by ±1% of the maximum velocity difference occurring at that x-station.

The formula is valid for variable-density (e.g. high-Mach-number) flows as well as uniform-density ones; and also for velocity profiles with maxima (e.g. wall jets). We shall be presenting some examples of the agreement between prediction and experiment, but with $\kappa = .435$, not .41.

Further details, and references to other literature, can be found in the book by Patankar and Spalding (1970).

2.13 Wall jet in stagnant surroundings: drag on wall



This box throws light on this question. The ordinate is the stress on the wall, and the abscissa is the Reynolds number of the jet; the latter is based upon the maximum velocity of the cross-section, u_{max}, and the distance of the point where it occurs from the wall, y_{max}.

On the graph, the experimental data are represented by the two broken lines. They cannot of course both be correct; so the distance between them can be taken as a measure of the disagreements that can be found between ostensibly identical circumstances. The curves represent the predictions of wall shear which correspond to the various $\kappa \sim \lambda$ combinations. The differences between them are very large indeed; and we notice that our favoured .6/.075 is not at all satisfactory.

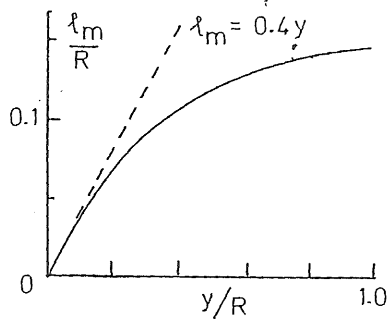
The pair .435/.09 gives values that are not too bad. Perhaps we should, after all, continue to work with it. The importance of having a perfect velocity profile prediction, we may decide, is not paramount.

What can be concluded from all this is that we can get fairly good predictions from a single pair of constants; but if we want perfection, we shall probably have to use a more elaborate model of turbulence than the mixing-length one.

2.14 Mixing-length distribution for turbulent pipe flow

Nikuradse's formula:

$$\frac{\ell_m}{R} = 0.14 - 0.08(1-y/R)^2 - 0.06(1-y/R)^4$$



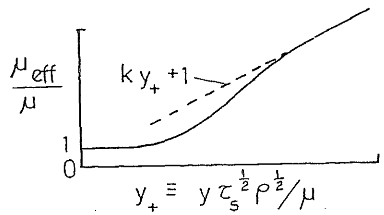
There now follow three figures which concern boundary layers in ducts. For turbulent pipe flow, under conditions of full development of the velocity profile, there is a well-known prescription of the mixing-length distribution. This figure gives the formula for the distribution, and also a graphical representation. One sees that it looks like a rounded version of that which has been used for boundary layers near one wall.

There are quantitative similarities and differences. A similarity is that the slope is 0.4 near the wall; a difference is that the maximum is appreciably larger, namely 0.14 instead of 0.09. Very close to the wall, the mixing length has to be modified to account for the laminar effects. However we shall slur over this for the time being, just as we did for flows on one wall. The semi-laminar region will be dealt with shortly (page 40).

2.17 The mixing-length near a wall; van Driest's (1956) hypothesis

- Formula for μ_{eff}: $\mu_{eff} = \mu + \rho\, \ell_m^{\;2} \left|\frac{\partial u}{\partial y}\right|$.
- Formula for ℓ_m: $\ell_m = \kappa y \left[1 - \exp\left(-\frac{y\tau_s^{\frac{1}{2}}\rho^{\frac{1}{2}}}{A\,\mu}\right)\right]$

  with $A = 26.0$.
- Resulting distribution:



We now turn to the treatment of the semi-laminar region close to a smooth wall. We confine attention to the van Driest (1956) hypothesis and its modifications. There are other approaches, e.g. that of Deissler (1955); but van Driest's starting point allows more varied excursions, and is the most popular.

The first formula shows that the effective viscosity, when both laminar and turbulent contributions play their parts, is given by the simple sum of the two viscosities. However, the second formula shows that the mixing length is not just κy: it involves also the multiplier in the square bracket. This has the effect of lowering ℓ_m when the argument of the exponential is close to zero.

When the argument is large (and negative) however, the exponential term is very small; then $\ell_m = \kappa y$ simply. The result can be expressed by way of the curve on the graph. The abscissa is y_+, which, when divided by $-A$, is the argument of the exponential.

2.18 Some modifications of the van Driest hypothesis

- Patankar-Spalding (1970): argument $= -y_+\, \tau_+^{\frac{1}{2}}/A$.
- Launder-Jones (1969): argument $= -y_+\, \tau_+/A$.
- Cebeci (1970): argument $= -\frac{y_+}{A}\left[\frac{p_+}{m_+}\left(1-e^{11.8m^+}\right)+e^{11.8m^+}\right]^{\frac{1}{2}}$.
- Pletcher (1970): argument $= -(y_+/A)(\tau_{ref}/\tau_s)^{\frac{1}{2}}$.

  Where $\tau_+ \equiv \tau/\tau_s$; $m_+ \equiv \dot{m}''/(\rho\tau_s)^{\frac{1}{2}}$; $p_+ \equiv (\mu/\rho^{\frac{1}{2}}\tau_s^{\;3/2})\cdot\frac{dp}{dx}$.

  Note when streamwise transport is negligible,

  $\tau = \tau_s + \dot{m}''u + y\; dp/dx$.

Various people have proposed modifications to the van Driest formula. Sometimes these modifications are expressed by making A a function rather than a constant; sometimes A is held constant and further multipliers are affixed. The distinction is formal only.

Patankar and Spalding (1970) proposed that the local shear stress should appear, not that at the wall. It seemed reasonable, and worked fairly well for moderate variations of τ in the semi-laminar layer. Then Launder and Jones (1969) went one better, multiplying by a further $\tau_+^{\frac{1}{2}}$. This improved the agreement with experiment in the cases they examined. Cebeci (1970), and Pletcher (1970), have made further proposals; and others have been made by Kays, Moffat and Thielbahr (1970) and by Powell and Strong (1970) among many others.

A comprehensive study needs to be made. It should be noted by the way, that for a Couette flow there are two agencies affecting τ_+; pressure gradient and mass transfer, so there is quite an extensive set of combinations to be tried.

2.19 Couette-flow analysis of near-wall region

- Differential equation: $du_+/dy_+ = \tau_+/\mu_+$

  with $\tau_+ \equiv 1 + m_+u_+ + p_+y_+$.

- Solution: $u_+ = \int_0^{y_+} (\tau_+/\mu_+)dy_+ = u_+\{y_+, p_+, m_+\}$.

- Hence drag law: $s = s\{Re, F, M\}$, where:

  $s \equiv \frac{\tau_s}{\rho u^2} = u_+^{-2}$; $\quad Re \equiv \frac{\rho y u}{\mu} = y_+u_+$;

  $M \equiv \frac{\dot{m}''}{\rho u} = m_+u_+^{-1}$; $\quad F \equiv \frac{y}{\rho u^2}\frac{dp}{dx} = p_+y_+u_+^{-2}$.

We shall now give an analysis of some of the implications of the van Driest hypothesis and its modifications, for the special case of a Couette flow. Further details can be found in the book by Patankar and Spalding (1970). Of course, we need not pay special attention to this case; for we can always solve the parabolic boundary-layer equations all the way to the wall. However, it is educationally instructive to do so; and the results can be employed, as "wall laws", to shorten computation by bridging over the semi-laminar region.

The differential equation is shown at the top right; it may be recognised as a non-dimensionalised version of the last equation in the previous box. Its solution is the $u_+ \sim y_+$ profile, with m_+ and p_+ as parameters, the non-dimensional mass-transfer and pressure-gradient characteristics.

From the $u_+ \sim y_+$ profile, one can deduce a "drag law" for the friction coefficient s, in terms of quantities measured at the two faces of the Couette-flow layer. This is represented in general form in the box. Of course the s-function, which has three arguments, can always be obtained by numerical integration, the results being expressed as a table of numbers. In

2.20 High Reynolds numbers: the simplest solution

- Condition: $m_+ = p_+ = 0$, so that $\tau_+ = 1$.
- $u_+ \sim y_+$ relation:

  Since $\mu_+ \to \kappa y_+$ as $y_+ \to \infty$,

  $u_+ \to \kappa^{-1} \ln(Ey_+)$ $\quad$ ($E \equiv$ constant).
- Resulting drag law:

  $s = \kappa^2[\ln(E\ Re\ s^{\frac{1}{2}})]^{-2}$.
- Value of $E \simeq 9.0$ for smooth wall with $\kappa \simeq .4$.

We first consider the case in which the shear stress is uniform across the layer, and the value of y_+ for the outer face lies beyond that for which "damping" is significant (i.e. the Reynolds number is large). In this case, as is well known, the $u_+ \sim y_+$ relation is a logarithmic one, the constant E being deducible from one integration. The resulting drag law is shown on the slide. It is known to work very well for the conditions which we have indicated.

From the point of view of convenience, it is to be regretted that the formula is not an explicit one; for s appears on the right as well as the left of the equals sign. However, at large Re the influence of $s^{\frac{1}{2}}$ in the argument of the logarithm is slight; therefore substitution of a guessed value on the right leads to a good approximation for the whole expression. If this approximation is now substituted in its turn, the new value obtained for s is very good indeed.

When either m_+ or p_+ is not equal to zero, other formulae are valid. These are shown in the next box.