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Sedimentation of spheres at small Reynolds number

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The effect of fluid inertia on the settling of spheres in a viscous incompressible fluid is studied in the limit of small Reynolds number. The kinetic energy of flow depends on the positions of the spheres, and gives rise to forces on the spheres. In the dilute limit it suffices to study the corresponding pair interaction. The interaction is calculated from the Stokes flow for two spheres settling between plane walls in the point particle limit. The dissipative interaction between a pair of spheres is calculated from the Proudman–Pearson [I. Proudman and J. R. A. Pearson, J. Fluid Mech. **2**, 237 (1957)] solution of the Navier–Stokes equations for flow about a sphere in unbounded geometry. The combination of kinetic and dissipative interaction gives rise to a repulsive force of range of the order of the sphere diameter divided by the Reynolds number. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1905583]

I. INTRODUCTION

The long range of hydrodynamic interactions between particles settling in a viscous fluid at low Reynolds number poses severe theoretical problems even in the case of dilute suspensions. It was pointed out by Caflisch and Luke[1] that the variance of particle velocity fluctuations diverges with the size of the system if the particle distribution is assumed random. In their calculation the hydrodynamic interaction is approximated by the Oseen tensor,[2] which decays with the inverse distance between particles. It has been suggested that a form of hydrodynamic screening is necessary to keep the variance finite.[3] The divergence is not found in experiment,[4–8] and long-distance correlations are observed consistent with hydrodynamic screening. Although a tentative description of the statistics of sedimentation in dilute suspensions has been formulated,[9] there is no explanation for the dynamical buildup of correlations. Two particles interacting via the Oseen tensor settle without change of relative position, so that apparently dynamical processes involving at least three particles are responsible for the correlations.

In the following we explore a different route. It was pointed out by Vasseur and Cox[10] that the inertial correction to the hydrodynamic pair interaction has an extremely long range. The authors were primarily interested in the effects near the walls of the system, but they also found a considerable difference in the mean migration velocity of a pair of spherical particles from that of an isolated sphere for separations of the order of the distance from the wall. We show in Sec. VI that their result can be understood and generalized on the basis of the Proudman–Pearson solution for flow past a sphere in unbounded fluid at small Reynolds number.[11] This solution shows that the inertial correction to the flow pattern about a sphere remains nearly constant over distances of the order of the Oseen screening length $\lambda = 2a/\mathrm{Re}$, where a is the radius of the sphere, and Re the Reynolds number. The typical value of Re in experiments is 10^{-4}, so that one may expect significant pair effects in the bulk of the system, even though the amplitude of the inertial correction is small. The theory of inertial corrections was reviewed recently by Feuillebois.[12]

Vasseur and Cox[10] calculated the migration velocity of a single particle near a wall from the perturbed flow velocity caused by the wall and acting on the particle. A similar calculation was performed for a pair. We argue in the following that in general when considering a change of configuration in quasisteady hydrodynamics one must take account of the work required to change the kinetic energy of flow. For Stokes flow about a single sphere in unbounded fluid the kinetic energy of flow diverges, but in a bounded system it is finite. In general the energy depends on the position of the sphere, and hence results in a force. For a pair of particles the dependence of the energy of flow on positions results in a kinetic interaction. We calculate the interaction explicitly to first order in Re for two point particles between plane walls. In that case the total kinetic energy of flow is finite. The interaction turns out to have a long range, and provides a correction to the dynamics of a pair of comparable magnitude to that of the flow perturbation. The calculation is rather technical, and is presented in Secs. IV and V and the Appendix.

II. KINETIC FORCE ON A SINGLE SPHERE

We consider a sphere of radius a and mass m, immersed in a viscous incompressible fluid of shear viscosity η and mass density ρ, centered at position $\boldsymbol{R}(t)$ at time t. The sphere and the fluid are confined to volume Ω with boundary $\Sigma(\Omega)$. The fluid velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ and the pressure $p(\boldsymbol{r},t)$ are assumed to satisfy the Navier–Stokes equations

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right] = \eta\nabla^2\boldsymbol{v} - \nabla p,$$
$$\nabla\cdot\boldsymbol{v} = 0. \tag{2.1}$$

The fluid velocity satisfies stick boundary conditions at $\Sigma(\Omega)$ and at the surface of the sphere. For $t<0$ sphere and fluid are

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assumed to be at rest. From time $t=0$ the sphere is subjected to an external force $\boldsymbol{K}$, acting at the center, and applied steadily. Since no torque is applied, the sphere is rotating freely. After an initial transient the position of the sphere and the flow field of the fluid vary slowly in time. On the slow time scale the time derivative on the left-hand side of Eq. (2.1) may be neglected, and the fluid equations of motion may be approximated by the steady-state equations

$$-\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\eta\nabla^2\boldsymbol{v}-\nabla p=0,$$
$$\nabla\cdot\boldsymbol{v}=0. \tag{2.2}$$

In principle, these equations, together with the stick boundary conditions, may be solved for prescribed translational velocity $\boldsymbol{U}$ and rotational velocity $\boldsymbol{\Omega}$ of the sphere. This yields a flow pattern $\boldsymbol{v}(\boldsymbol{r};\boldsymbol{R},\boldsymbol{U},\boldsymbol{\Omega})$ and pressure $p(\boldsymbol{r};\boldsymbol{R},\boldsymbol{U},\boldsymbol{\Omega})$ depending parametrically on the position $\boldsymbol{R}$ and the two velocities $\boldsymbol{U}$ and $\boldsymbol{\Omega}$. The hydrodynamic force $\boldsymbol{K}_{\text{hyd}}$ exerted by the fluid on the sphere may be calculated from the integral of the normal component of the stress tensor over a spherical surface S located in the fluid and enclosing the sphere,

$$\boldsymbol{K}_{\text{hyd}}=\int_S\boldsymbol{\sigma}_{\text{hyd}}\cdot\boldsymbol{n}dS. \tag{2.3}$$

The hydrodynamic stress tensor is given by

$$\boldsymbol{\sigma}_{\text{hyd}}=-\rho\boldsymbol{v}\boldsymbol{v}-p\mathbf{1}+\boldsymbol{\sigma}_v, \tag{2.4}$$

where the viscous part $\boldsymbol{\sigma}_v$ has Cartesian components

$$\sigma_{v\alpha\beta}=\eta(\partial_\alpha v_\beta+\partial_\beta v_\alpha). \tag{2.5}$$

Similar to Eq. (2.3), the hydrodynamic torque exerted by the fluid on the sphere is given by

$$\boldsymbol{T}_{\text{hyd}}=\int_S(\boldsymbol{r}-\boldsymbol{R})\times(\boldsymbol{\sigma}_{\text{hyd}}\cdot\boldsymbol{n})dS. \tag{2.6}$$

The requirement that this vanishes determines the angular velocity $\boldsymbol{\Omega}$ for given $\boldsymbol{R}$ and $\boldsymbol{U}$. After substitution of $\boldsymbol{\Omega}(\boldsymbol{R},\boldsymbol{U})$ the angular velocity no longer appears as an independent variable, and we write the flow pattern simply as $\boldsymbol{v}(\boldsymbol{r};\boldsymbol{R},\boldsymbol{U})$.

The hydrodynamic force can be expressed as the sum of a potential force, deriving from the energy of the fluid,

$$\boldsymbol{K}_T(\boldsymbol{R},\boldsymbol{U})=-\frac{\partial\mathcal{E}}{\partial\boldsymbol{R}}, \tag{2.7}$$

and a dissipative force, deriving from the dissipation function,

$$\boldsymbol{K}_d(\boldsymbol{R},\boldsymbol{U})=-\frac{\partial\mathcal{D}}{\partial\boldsymbol{U}}. \tag{2.8}$$

The energy of the fluid is given by

$$\mathcal{E}(\boldsymbol{R},\boldsymbol{U})=\mathcal{U}+\frac{1}{2}\rho\int_\Omega\boldsymbol{v}(\boldsymbol{r};\boldsymbol{R},\boldsymbol{U})^2d\boldsymbol{r}, \tag{2.9}$$

where $\mathcal{U}$ is the internal energy. On the slow time scale heat conduction is fast, and the temperature and internal energy density may be regarded as constant throughout the fluid. Hence the internal energy $\mathcal{U}$ does not contribute to the effective force in Eq. (2.7). The potential force is determined by the kinetic energy of flow, and we call $\boldsymbol{K}_T$ the kinetic force. The rate of heat production is[13]

$$\dot{Q}(\boldsymbol{R},\boldsymbol{U})=\frac{1}{2\eta}\int_\Omega\boldsymbol{\sigma}_v(\boldsymbol{r};\boldsymbol{R},\boldsymbol{U}):\boldsymbol{\sigma}_v(\boldsymbol{r};\boldsymbol{R},\boldsymbol{U})d\boldsymbol{r}. \tag{2.10}$$

This is related to the dissipation function $\mathcal{D}(\boldsymbol{R},\boldsymbol{U})$ by

$$\dot{Q}(\boldsymbol{R},\boldsymbol{U})=\boldsymbol{U}\cdot\frac{\partial\mathcal{D}}{\partial\boldsymbol{U}}. \tag{2.11}$$

The hydrodynamic force is the sum $\boldsymbol{K}_{\text{hyd}}=\boldsymbol{K}_T+\boldsymbol{K}_d$, and the total balance of forces reads

$$\boldsymbol{K}+\boldsymbol{K}_T(\boldsymbol{R},\boldsymbol{U})+\boldsymbol{K}_d(\boldsymbol{R},\boldsymbol{U})=0. \tag{2.12}$$

In principle, for a given position $\boldsymbol{R}$, this equation may be solved for the translational velocity $\boldsymbol{U}(\boldsymbol{R})$. The equation of motion

$$\frac{d\boldsymbol{R}}{dt}=\boldsymbol{U}(\boldsymbol{R}) \tag{2.13}$$

determines the time evolution of the whole system on the slow time scale.

If the applied force $\boldsymbol{K}$ is small, then the fluid is only slightly perturbed from equilibrium, and the nonlinear equations (2.2) may be replaced by the linear Stokes equations

$$\eta\nabla^2\boldsymbol{v}-\nabla p=0,$$
$$\nabla\cdot\boldsymbol{v}=0. \tag{2.14}$$

In this limit the hydrodynamic force is linear in the sphere velocity

$$\boldsymbol{K}^{(0)}_{\text{hyd}}(\boldsymbol{R},\boldsymbol{U})=-\boldsymbol{\zeta}(\boldsymbol{R})\cdot\boldsymbol{U}, \tag{2.15}$$

with friction tensor $\boldsymbol{\zeta}(\boldsymbol{R})$, and is completely dissipative. The corresponding Stokes flow pattern is also linear in $\boldsymbol{U}$, and may be expressed as

$$\boldsymbol{v}^{(0)}(\boldsymbol{r};\boldsymbol{R},\boldsymbol{U})=\mathsf{F}_0(\boldsymbol{r},\boldsymbol{R})\cdot\boldsymbol{U}, \tag{2.16}$$

with a tensor $\mathsf{F}_0(\boldsymbol{r},\boldsymbol{R})$ determined by the shape of the volume Ω and the size and location of the sphere. For small radius a the tensor becomes identical with $6\pi\eta a\mathsf{G}(\boldsymbol{r},\boldsymbol{R})$, where $\mathsf{G}(\boldsymbol{r},\boldsymbol{R})$ is the Green tensor of the Stokes equations for the chosen geometry. From Eq. (2.12) one finds by the use of Eq. (2.15)

$$\boldsymbol{U}^{(0)}=\boldsymbol{\mu}(\boldsymbol{R})\cdot\boldsymbol{K}, \tag{2.17}$$

where $\boldsymbol{\mu}(\boldsymbol{R})=\boldsymbol{\zeta}(\boldsymbol{R})^{-1}$ is the Stokes mobility tensor. Substitution into Eq. (2.13) yields the equation of Stokesian dynamics.

Substituting Eq. (2.16) into Eq. (2.9) one finds from Eq. (2.7) that the kinetic force is quadratic in $\boldsymbol{U}$. The corresponding correction to Stokesian dynamics is to be compared with a correction to the dissipative force, calculated to the same order in the particle Reynolds number

$$\mathrm{Re} = \frac{\rho K}{6\pi\eta^2}. \tag{2.18}$$

Here we have used the typical sphere velocity $U_0 = K/(6\pi\eta a)$. To first order in the Reynolds number the hydrodynamic force may be expressed as

$$\boldsymbol{K}_{\mathrm{hyd}} = \boldsymbol{K}_{\mathrm{hyd}}^{(0)} + \boldsymbol{K}_T^{(1)} + \boldsymbol{K}_d^{(1)} + O(\mathrm{Re}^2), \tag{2.19}$$

where $\boldsymbol{K}_T^{(1)}$ is the first-order kinetic force, and $\boldsymbol{K}_d^{(1)}$ is the first-order dissipative force. In bounded geometry Whitehead's paradox[14] does not apply and we can expect a regular expansion in the Reynolds number to be valid.[15] Substituting into Eq. (2.12) and using Eq. (2.15) we find to first order

$$\boldsymbol{U} = \boldsymbol{\mu}(\boldsymbol{R}) \cdot [\boldsymbol{K} + \boldsymbol{K}_T^{(1)} + \boldsymbol{K}_d^{(1)}] + O(\mathrm{Re}^2). \tag{2.20}$$

In the first-order terms on the right-hand side the velocity $\boldsymbol{U}$ is replaced by the Stokes value $\boldsymbol{U}^{(0)}$.

It may be worthwhile to take a thermodynamic point of view. Consider a quasisteady state situation with particular force $\boldsymbol{K}$, position $\boldsymbol{R}$, corresponding velocity $\boldsymbol{U}$, and flow field $\boldsymbol{v}(\boldsymbol{r})$. Change the force slowly in such a way that the sphere returns to the same position with the same force after a long time, so that also the sphere velocity and the flow field are the same at the end of the cycle. At all times during the cycle a positive amount of heat is produced, but the total kinetic energy of flow may increase or decrease, and finally returns to its initial value. Thus there is a clear distinction between the kinetic and the dissipative forces.

III. MANY SPHERES

Next we consider N identical spheres of radius a centered at positions $(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_N)$ and immersed in the viscous incompressible fluid. The whole system is confined to volume Ω with boundary $\Sigma(\Omega)$. The fluid velocity satisfies the stick boundary conditions at $\Sigma(\Omega)$ and at the surface of each sphere. We use the shorthand notation $\mathsf{X} = (\boldsymbol{R}_1, \ldots, \boldsymbol{R}_N)$. For given translational velocities $(\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N)$ and rotational velocities $(\boldsymbol{\Omega}_1, \ldots \boldsymbol{\Omega}_N)$ the steady-state equations (2.2) in combination with stick boundary conditions have a unique solution.[13] We denote $\mathsf{U} = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N)$, $\Omega = (\boldsymbol{\Omega}_1, \ldots, \boldsymbol{\Omega}_N)$, and similarly $\mathsf{K} = (\boldsymbol{K}_1, \ldots, \boldsymbol{K}_N)$. The linear relation between velocities and forces in the Stokes regime is expressed by

$$\mathsf{U}^{(0)} = \boldsymbol{\mu}^T(\mathsf{X}) \cdot \mathsf{K}, \quad \Omega^{(0)} = \boldsymbol{\mu}^R(\mathsf{X}) \cdot \mathsf{K}, \tag{3.1}$$

with translational mobility matrix $\boldsymbol{\mu}^T(\mathsf{X})$ and rotational mobility matrix $\boldsymbol{\mu}^R(\mathsf{X})$.

The total energy of the fluid

$$\mathcal{E}_N(\mathsf{X}, \mathsf{U}) = \mathcal{U} + \frac{1}{2}\rho \int_\Omega \boldsymbol{v}^2(\boldsymbol{r}; \mathsf{X}, \mathsf{U}) d\boldsymbol{r} \tag{3.2}$$

depends on configuration and therefore yields an effective force on each of the spheres

$$\boldsymbol{K}_{Tj} = -\frac{\partial \mathcal{E}_N(\mathsf{X}, \mathsf{U})}{\partial \boldsymbol{R}_j}, \quad j = (1, \ldots, N). \tag{3.3}$$

We call $\boldsymbol{K}_{Tj}$ the kinetic force acting on sphere j, and denote $\mathsf{K}_T = (\boldsymbol{K}_{T1}, \ldots, \boldsymbol{K}_{TN})$.

To first order in the particle Reynolds number the particle equations of motion read

$$\frac{d\mathsf{X}}{dt} = \boldsymbol{\mu}^T(\mathsf{X}) \cdot [\mathsf{K} + \mathsf{K}_T^{(1)} + \mathsf{K}_d^{(1)}], \tag{3.4}$$

in analogy to Eq. (2.20). These determine the rate of change of configuration due to the sum of forces.

Since the spheres are identical, the energy $\mathcal{E}_N(\mathsf{X}, \mathsf{U})$ of the fluid is a symmetric function of the particle labels, and may be decomposed into a sum of many-body contributions by the use of a cluster expansion. We write

$$\mathcal{E}_N(\mathcal{N}) = \sum_{\mathcal{M} \subset \mathcal{N}} \Phi_M(\mathcal{M}), \tag{3.5}$$

where $\mathcal{N}$ is a set of N labels and the sum is over all subsets $\mathcal{M}$ of $\mathcal{N}$. Here M is the number of labels in subset $\mathcal{M}$. The inverse of the rule (3.5) is

$$\Phi_N(\mathcal{N}) = \sum_{\mathcal{M} \subset \mathcal{N}} (-1)^{N-M} \mathcal{E}_M(\mathcal{M}). \tag{3.6}$$

In particular, the one-body contribution

$$\Phi_1(\boldsymbol{R}, \boldsymbol{U}) = \mathcal{E}_1(\boldsymbol{R}, \boldsymbol{U}) - \mathcal{U} \tag{3.7}$$

can be found from the solution of the steady-state problem for a single sphere, and the two-body contribution

$$\begin{aligned}\Phi_2(\boldsymbol{R}_1, \boldsymbol{U}_1, \boldsymbol{R}_2, \boldsymbol{U}_2) &= \mathcal{E}_2(\boldsymbol{R}_1, \boldsymbol{U}_1, \boldsymbol{R}_2, \boldsymbol{U}_2) - \mathcal{E}_1(\boldsymbol{R}_1, \boldsymbol{U}_1) \\ &\quad - \mathcal{E}_1(\boldsymbol{R}_2, \boldsymbol{U}_2) + \mathcal{U}\end{aligned} \tag{3.8}$$

can be found from the solution of the steady-state problem for a pair of spheres. The two-body contribution clearly is due to interference of the flow fields of the two spheres.

The situation simplifies in the dilute limit. Then the solution of the N-sphere problem can be approximated by

$$\boldsymbol{v}(\boldsymbol{r}; \mathsf{X}, \mathsf{U}) = \sum_{j=1}^{N} \boldsymbol{V}(\boldsymbol{r}; \boldsymbol{R}_j, \boldsymbol{U}_j), \tag{3.9}$$

where $\boldsymbol{V}(\boldsymbol{r}; \boldsymbol{R}, \boldsymbol{U})$ is the solution of the steady-state Navier–Stokes equations (2.2) for a single sphere in volume Ω with stick boundary condition for the fluid velocity on $\Sigma(\Omega)$. With the approximation (3.9) all many-body contributions $\Phi_M(\mathcal{M})$ of order $M > 2$ vanish identically. The one-body potential $\Phi_1(\boldsymbol{R})$ can be expressed as

$$\Phi_1(\boldsymbol{R}, \boldsymbol{U}) = \frac{1}{2}\rho \int_\Omega \boldsymbol{V}(\boldsymbol{r}; \boldsymbol{R}, \boldsymbol{U})^2 d\boldsymbol{r}, \tag{3.10}$$

and the two-body potential $\Phi_2(\boldsymbol{R}_1, \boldsymbol{U}_1, \boldsymbol{R}_2, \boldsymbol{U}_2)$ as

$$\Phi_2(\boldsymbol{R}_1, \boldsymbol{U}_1, \boldsymbol{R}_2, \boldsymbol{U}_2) = \rho \int_\Omega \boldsymbol{V}(\boldsymbol{r}; \boldsymbol{R}_1, \boldsymbol{U}_1) \cdot \boldsymbol{V}(\boldsymbol{r}; \boldsymbol{R}_2, \boldsymbol{U}_2) d\boldsymbol{r}. \tag{3.11}$$

To lowest order in the Reynolds number $\boldsymbol{V}(\boldsymbol{r}; \boldsymbol{R}, \boldsymbol{U})$ is linear in $\boldsymbol{U}$, and takes the form Eq. (2.16). Therefore, to first order in the Reynolds number the one-body potential can be expressed as

$$\Phi_1^{(1)}(\boldsymbol{R},\boldsymbol{U}) = \tfrac{1}{2}\rho \boldsymbol{U}\cdot \mathsf{S}_1(\boldsymbol{R})\cdot \boldsymbol{U}, \tag{3.12}$$

with tensor

$$\mathsf{S}_1(\boldsymbol{R}) = \int_\Omega \mathsf{F}_0(\boldsymbol{R},\boldsymbol{r})\cdot \mathsf{F}_0(\boldsymbol{r},\boldsymbol{R})d\boldsymbol{r}. \tag{3.13}$$

Here we have used the symmetry relation

$$F_{0\alpha\beta}(\boldsymbol{r},\boldsymbol{R}) = F_{0\beta\alpha}(\boldsymbol{R},\boldsymbol{r}) \tag{3.14}$$

on account of Lorentz reciprocity.[2] The two-body potential is found to be

$$\Phi_2^{(1)}(\boldsymbol{R}_1,\boldsymbol{U}_1,\boldsymbol{R}_2,\boldsymbol{U}_2) = \rho \boldsymbol{U}_1\cdot \mathsf{S}_2(\boldsymbol{R}_1,\boldsymbol{R}_2)\cdot \boldsymbol{U}_2, \tag{3.15}$$

with tensor

$$\mathsf{S}_2(\boldsymbol{R}_1,\boldsymbol{R}_2) = \int_\Omega \mathsf{F}_0(\boldsymbol{R}_1,\boldsymbol{r})\cdot \mathsf{F}_0(\boldsymbol{r},\boldsymbol{R}_2)d\boldsymbol{r}. \tag{3.16}$$

The two tensors in Eqs. (3.13) and (3.16) are therefore simply related by

$$\mathsf{S}_1(\boldsymbol{R}) = \mathsf{S}_2(\boldsymbol{R},\boldsymbol{R}). \tag{3.17}$$

For small radius a the tensor $\mathsf{F}_0(\boldsymbol{R}_1,\boldsymbol{r})$ may be approximated by $6\pi\eta a$ times the Green tensor, as noted below Eq. (2.16). Moreover, for a given force $\boldsymbol{K}$ the velocities in Eqs. (3.12) and (3.15) may be replaced by $\boldsymbol{U}_0=\boldsymbol{K}/6\pi\eta a$, so that it is sufficient to calculate the tensors S_1 and S_2.

For infinite fluid the Green tensor is identical with the Oseen tensor $\mathsf{T}_0(\boldsymbol{r}-\boldsymbol{R})$ with the well-known expression

$$\mathsf{T}_0(\boldsymbol{r}) = \frac{1}{8\pi\eta}\frac{\mathbf{1}+\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r}, \tag{3.18}$$

where $\hat{\boldsymbol{r}}=\boldsymbol{r}/r$. In that case the integrals in Eqs. (3.13) and (3.16) diverge because of the slow decay with distance of the Oseen tensor. For bounded geometry the integrals exist, and the kinetic force is well defined. In the following we consider, in particular, the case of planar geometry.

IV. PLANAR GEOMETRY

In this section we show that the two-body potential $\Phi_2^{(1)}(\boldsymbol{R}_1,\boldsymbol{U}_1,\boldsymbol{R}_2,\boldsymbol{U}_2)$, defined in general by Eq. (3.15), can be evaluated for the case of planar geometry and in point approximation. We consider the volume Ω bounded by two parallel planes, defined in Cartesian coordinates by $z=-W$ and $z=W$. Thus the origin midway between the planes, and the x and y axes are parallel to the planes. We consider small sphere radius, and approximate the tensor $\mathsf{F}_0(\boldsymbol{R}_1,\boldsymbol{r})$ by $6\pi\eta a$ times the Green tensor. The Green tensor $\mathsf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ for this geometry has been derived by Liron and Mochon[16] by the method of images. The solution was recently cast in a different form by Jones.[7] We have based our calculation on the solution presented by Jones.[17] His work allows us to express the potential $\Phi_2^{(1)}$ as a one-dimensional integral that can be evaluated to any desired accuracy.

By translational symmetry the tensor $\mathsf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ can be expressed as $\mathsf{G}(\boldsymbol{s}_1-\boldsymbol{s}_2,z_1,z_2)$ with two-dimensional position vectors $\boldsymbol{s}_1=(x_1,y_1)$ and $\boldsymbol{s}_2=(x_2,y_2)$. The distance $s=|\boldsymbol{s}_1-\boldsymbol{s}_2|$ runs from zero to infinity. The crucial observation is that the Green tensor $\mathsf{G}(\boldsymbol{s}_1-\boldsymbol{s}_2,z_1,z_2)$ as a function of s decays as $1/s^2$ at large distance, as has been shown by Liron and Mochon.[16] Hence the integral for the tensor $\mathsf{S}_2(\boldsymbol{R}_1,\boldsymbol{R}_2)$ defined in Eq. (3.16) has a finite value.

The tensor $\mathsf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ can be expressed as a two-dimensional Fourier integral

$$\mathsf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \int d\boldsymbol{q}\, e^{i\boldsymbol{q}\cdot(\boldsymbol{s}_1-\boldsymbol{s}_2)}\hat{\mathsf{G}}(\boldsymbol{q},z_1,z_2), \tag{4.1}$$

with a two-dimensional wave vector $\boldsymbol{q}=(q_x,q_y)$. The Fourier transform $\hat{\mathsf{G}}(\boldsymbol{q},z_1,z_2)$ reads in dyadic form

$$\begin{aligned}\hat{\mathsf{G}}(\boldsymbol{q},z_1,z_2) = \frac{1}{16\pi^2\eta q}[&t_{nn}(q,z_1,z_2)\boldsymbol{e}_z\boldsymbol{e}_z + it_{np}(q,z_1,z_2)\boldsymbol{e}_z\hat{\boldsymbol{q}}\\ &+ it_{pn}(q,z_1,z_2)\hat{\boldsymbol{q}}\boldsymbol{e}_z - t_{pp}(q,z_1,z_2)\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\\ &+ r_{pp}(q,z_1,z_2)(\mathbf{1}-\boldsymbol{e}_z\boldsymbol{e}_z)],\end{aligned} \tag{4.2}$$

with real scalar functions $t_{nn},\ldots,r_{pp}$. The explicit expression for these functions is given by Jones.[17] Substituting the expression Eq. (4.1) into Eq. (3.16), using the symmetry Eq. (3.14), and performing the integration over x, y we obtain

$$\begin{aligned}\mathsf{S}_2(\boldsymbol{R}_1,\boldsymbol{R}_2) = 4\pi^2\zeta_0^2\int d\boldsymbol{q}\, e^{i\boldsymbol{q}\cdot(\boldsymbol{S}_1-\boldsymbol{S}_2)}\int_{-W}^{W}&\hat{\mathsf{G}}(\boldsymbol{q},Z_1,z)\\ &\cdot\hat{\mathsf{G}}(\boldsymbol{q},z,Z_2)dz,\end{aligned} \tag{4.3}$$

with $\zeta_0=6\pi\eta a$ and in obvious notation.

The various components of the tensor $\mathsf{S}_2(\boldsymbol{R}_1,\boldsymbol{R}_2)$ must be considered separately. The zz component of the integrand in Eq. (4.3) is given by

$$\begin{aligned}&\boldsymbol{e}_z\cdot\hat{\mathsf{G}}(\boldsymbol{q},Z_1,z)\cdot\hat{\mathsf{G}}(\boldsymbol{q},z,Z_2)\cdot\boldsymbol{e}_z\\ &\quad=\frac{1}{256\pi^4\eta^2q^2}[t_{nn}(q,Z_1,z)t_{nn}(q,z,Z_2)\\ &\qquad - t_{np}(q,Z_1,z)t_{pn}(q,z,Z_2)].\end{aligned} \tag{4.4}$$

The xx component of the integrand is somewhat more complicated, and is given by

$$\begin{aligned}&\boldsymbol{e}_x\cdot\hat{\mathsf{G}}(\boldsymbol{q},Z_1,z)\cdot\hat{\mathsf{G}}(\boldsymbol{q},z,Z_2)\cdot\boldsymbol{e}_x\\ &\quad=\frac{1}{256\pi^4\eta^2q^2}\{[t_{pp}(q,Z_1,z)t_{pp}(q,z,Z_2)\\ &\qquad - t_{pn}(q,Z_1,z)t_{np}(q,z,Z_2) - t_{pp}(q,Z_1,z)r_{pp}(q,z,Z_2)\\ &\qquad - r_{pp}(q,Z_1,z)t_{pp}(q,z,Z_2)]\cos^2\varphi\\ &\qquad + r_{pp}(q,Z_1,z)r_{pp}(q,z,Z_2)\},\end{aligned} \tag{4.5}$$

where φ is the angle between the vector $\boldsymbol{q}$ and the x axis. The yy component of the integrand in Eq. (4.3) is given by the same expression, but with $\cos^2\varphi$ replaced by $\sin^2\varphi$. The off-diagonal elements of the integrand are given by similar expressions.

The integral over φ in Eq. (4.3) can be performed by the use of the Bessel function generating function[18]

$$\exp[iu\cos\varphi] = \sum_{n=-\infty}^{\infty} i^n J_n(u)\exp[in\varphi]. \tag{4.6}$$

It turns out that the integral over z can also be performed explicitly, so that we are left with a single quadrature over wave number q. The final expression for the element $S_{2zz}(\boldsymbol{R}_1,\boldsymbol{R}_2)$ reads

$$S_{2zz}(\boldsymbol{R}_1,\boldsymbol{R}_2) = \frac{W\zeta_0^2}{32\pi\eta^2}\int_0^\infty J_0(qS)F(q,Z_1,Z_2)/q\,dq, \tag{4.7}$$

where $S=|\boldsymbol{S}_1-\boldsymbol{S}_2|$. The final factor $1/q$ is convenient. The explicit expression for the dimensionless function $F(q,Z_1,Z_2)$ is given in the Appendix.

The other elements of the tensor $\mathsf{S}_2(\boldsymbol{R}_1,\boldsymbol{R}_2)$ can be calculated in similar fashion. It is convenient to choose the x axis in the direction of the vector $\boldsymbol{S}_2-\boldsymbol{S}_1$. The final expression for the element $S_{2xx}(\boldsymbol{R}_1,\boldsymbol{R}_2)$ reads

$$S_{2xx}(\boldsymbol{R}_1,\boldsymbol{R}_2) = \frac{W\zeta_0^2}{32\pi\eta^2}\int_0^\infty [J_0(qS)G(q,Z_1,Z_2) - J_2(qS)H(q,Z_1,Z_2)]/q\,dq. \tag{4.8}$$

The expressions for the functions $G(q,Z_1,Z_2)$ and $H(q,Z_1,Z_2)$ are given in the Appendix. With the same notation the final expression for the element $S_{2yy}(\boldsymbol{R}_1,\boldsymbol{R}_2)$ reads

$$S_{2yy}(\boldsymbol{R}_1,\boldsymbol{R}_2) = \frac{W\zeta_0^2}{32\pi\eta^2}\int_0^\infty [J_0(qS)G(q,Z_1,Z_2) + J_2(qS)H(q,Z_1,Z_2)]/q\,dq. \tag{4.9}$$

The xy and yx components of the tensor can be seen to vanish after integration over φ. The yz and zy components of the tensor vanish for the same reason.

Finally, we consider the xz and zx components of the tensor. The xz component is given by

$$S_{2xz}(\boldsymbol{R}_1,\boldsymbol{R}_2) = \frac{W\zeta_0^2}{32\pi\eta^2}\int_0^\infty J_1(qS)C_{xz}(q,Z_1,Z_2)/q\,dq, \tag{4.10}$$

and the zx component is given by the same expression with subscripts x and z interchanged. The expressions for the functions $C_{xz}(q,Z_1,Z_2)$ and $C_{zx}(q,Z_1,Z_2)$ are given in the Appendix. We write the elements of the tensor $\mathsf{S}_2(\boldsymbol{R}_1,\boldsymbol{R}_2)$ as

$$S_{2\alpha\beta}(\boldsymbol{R}_1,\boldsymbol{R}_2) = \frac{W\zeta_0^2}{32\pi\eta^2}\phi_{2\alpha\beta}(s,z_1,z_2), \tag{4.11}$$

with dimensionless variables $s=S/W$ and $z_i=Z_i/W$ and with dimensionless functions $\phi_{2\alpha\beta}(s,z_1,z_2)$ given by the integrals in Eqs. (4.7)–(4.10).

By the use of Eq. (2.17) we can write the pair interaction in Eq. (3.15) with $\boldsymbol{U}_1=\boldsymbol{U}_2=\boldsymbol{K}/6\pi\eta a$ as

$$\begin{aligned}\Phi_2^{(1)}(\boldsymbol{R}_1,\boldsymbol{R}_2) = \tfrac{3}{16}\,\mathrm{Re}\,KW\{&\phi_{2xx}(s,z_1,z_2)\hat{K}_x^2 \\ &+ \phi_{2yy}(s,z_1,z_2)\hat{K}_y^2 + \phi_{2zz}(s,z_1,z_2)\hat{K}_z^2 \\ &+ [\phi_{2xz}(s,z_1,z_2) + \phi_{2zx}(s,z_1,z_2)]\hat{K}_x\hat{K}_z\}.\end{aligned} \tag{4.12}$$

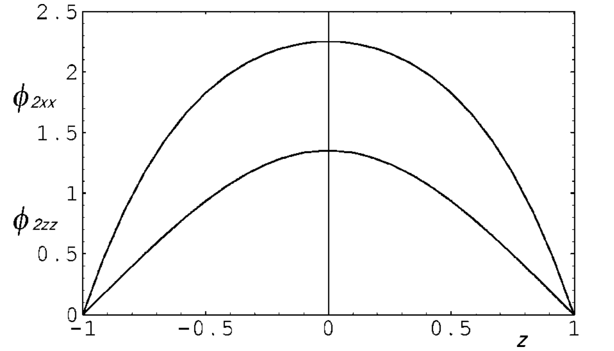

FIG. 1. Plot of the potentials $\phi_{2xx}(0,z,z)$ (top curve) and $\phi_{2zz}(0,z,z)$ (bottom curve), defined in Eq. (4.13), as functions of $z=Z/W$.

The interaction is of course translationally invariant in the directions parallel to the walls.

By the use of Eq. (3.17) we find for the single-particle kinetic potential defined in Eq. (3.12)

$$\Phi_1^{(1)}(\boldsymbol{R}) = \tfrac{3}{32}\,\mathrm{Re}\,KW[\phi_{2xx}(0,z,z)(\hat{K}_x^2+\hat{K}_y^2) + \phi_{2zz}(0,z,z)\hat{K}_z^2]. \tag{4.13}$$

The potential depends only on $z=Z/W$. From the definition Eq. (3.7) it is positive, and vanishes at the walls, because at the walls no flow can be generated due to the stick boundary condition. The potential is of course symmetric with respect to the midplane, and must be expected to decay monotonically towards the walls. The values at the midplane are given by $\phi_{2xx}(0,0,0)$ and $\phi_{2zz}(0,0,0)$, and can be found by quadrature from Eqs. (4.7) and (4.8). We find

$$\phi_{2xx}(0,0,0) = \tfrac{9}{4}, \quad \phi_{2zz}(0,0,0) = 1.351. \tag{4.14}$$

The first value is exact. The second value was found by numerical integration.

In Fig. 1 we plot the two dimensionless potentials $\phi_{2xx}(0,z,z)$ and $\phi_{2zz}(0,z,z)$ as functions of z. The behavior is parabolic at the center. The kinetic force drives a particle from the middle towards the walls. The kinetic force is opposed to the lift velocity of Vasseur and Cox,[10] showing that the dissipative force $\boldsymbol{K}_d^{(1)}$ in Eq. (2.20) is directed away from the nearest wall and dominates.

V. ASYMPTOTIC BEHAVIOR OF TWO-BODY POTENTIAL

In this section we consider the long- and short-distance behaviors of the kinetic interaction potential. It is clear from the above equations that the interaction depends on the direction of the applied force with respect to the walls. However, the short-distance behavior of the kinetic force is independent of the orientation with respect to the walls. For fixed location of the pair the kinetic interaction force has a well-defined limiting value as the distance $2W$ tends to infinity.

The behavior of the dimensionless potentials $\phi_{2\alpha\beta}(s,z_1,z_2)$, given by Eqs. (4.7)–(4.10), for large distance s is determined by the behavior of the functions F, G, H, C_{xz}, and C_{zx} for small wave number q. The short-distance behavior is determined by the behavior of these functions for large wave number q.

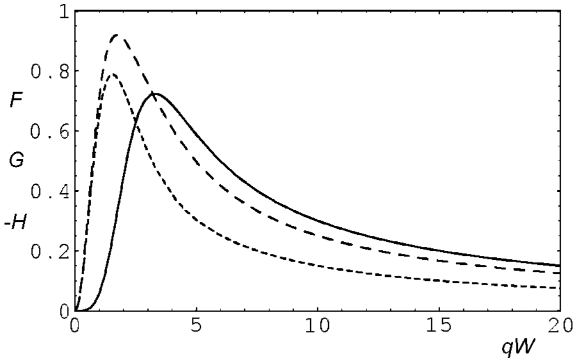

FIG. 2. Plot of the functions $F(q,0,0)$ (solid curve), $G(q,0,0)$ (long dashes), and $-H(q,0,0)$ (short dashes) as functions of dimensionless wave number qW.

The explicit expressions for the functions F, G, H, C_{xz}, and C_{zx} given in the Appendix are quite complicated, but the behavior simplifies for small and large q. In Fig. 2 we plot $F(q,0,0)$, $G(q,0,0)$, and $H(q,0,0)$ as functions of dimensionless wave number $k=qW$. The functions satisfy the symmetry relations

$$F(q,Z_1,Z_2) = F(q,Z_2,Z_1) = F(q,-Z_2,-Z_1),$$

$$G(q,Z_1,Z_2) = G(q,Z_2,Z_1) = G(q,-Z_2,-Z_1),$$

$$H(q,Z_1,Z_2) = H(q,Z_2,Z_1) = H(q,-Z_2,-Z_1), \tag{5.1}$$

$$C_{xz}(q,Z_1,Z_2) = C_{xz}(q,Z_2,Z_1) = C_{zx}(q,-Z_2,-Z_1),$$

$$C_{zx}(q,Z_1,Z_2) = C_{zx}(q,Z_2,Z_1) = C_{xz}(q,-Z_2,-Z_1).$$

The functions F, G, and H are even in q,

$$F(q,Z_1,Z_2) = F(-q,Z_1,Z_2),$$

$$G(q,Z_1,Z_2) = G(-q,Z_1,Z_2), \tag{5.2}$$

$$H(q,Z_1,Z_2) = H(-q,Z_1,Z_2),$$

and are analytic at $q=0$, so that their series expansion in powers of q contains only even powers. The functions C_{xz} and C_{zx} are odd in q,

$$C_{xz}(q,Z_1,Z_2) = -C_{xz}(-q,Z_1,Z_2),$$
$$C_{zx}(q,Z_1,Z_2) = -C_{zx}(-q,Z_1,Z_2), \tag{5.3}$$

and their series expansion in powers of q contains only odd powers. The first terms of the expansion are

$$F(q,Z_1,Z_2) = f_4(z_1,z_2)q^4W^4 + O(q^6),$$

$$G(q,Z_1,Z_2) = g_2(z_1,z_2)q^2W^2 + O(q^4),$$

$$H(q,Z_1,Z_2) = h_2(z_1,z_2)q^2W^2 + O(q^4), \tag{5.4}$$

$$C_{xz}(q,Z_1,Z_2) = c_{3xz}(z_1,z_2)q^3W^3 + O(q^5),$$

$$C_{zx}(q,Z_1,Z_2) = c_{3zx}(z_1,z_2)q^3W^3 + O(q^5),$$

with polynomials for $Z_1 < Z_2$,

$$f_4(z_1,z_2) = \tfrac{1}{30}(1+z_1)^2(1-z_2)^2(2-4z_1+4z_2+z_1^2+z_2^2 - 6z_1z_2 + 2z_1^3 - 2z_2^3 - 2z_1^2z_2 + 2z_1z_2^2 + z_1^3z_2 + z_1z_2^3),$$

$$g_2(z_1,z_2) = \tfrac{1}{60}(1+z_1)(1-z_2)[82(1-z_1+z_2) - 65(z_1^2+z_2^2) + 78z_1z_2 - 15(z_1^3 - z_2^3 - z_1^2z_2 + z_1z_2^2 + z_1^3z_2 + z_1z_2^3)],$$

$$h_2(z_1,z_2) = \tfrac{-1}{20}(1-z_1^2)(1-z_2^2)(26-5z_1^2-5z_2^2), \tag{5.5}$$

$$c_{3xz}(z_1,z_2) = \tfrac{1}{30}(1+z_1)(1-z_2)^2[-10z_1 + 2z_2 + 10z_1^2 + 4z_2^2 - 22z_1z_2 - 5z_1^2z_2 + 6z_1z_2^2 + 10z_1^3 - 3z_2^3 + 5z_1^3z_2 + 3z_1z_2^3],$$

$$c_{3zx}(z_1,z_2) = \tfrac{1}{30}(1+z_1)^2(1-z_2)[-2z_1 + 10z_2 + 4z_1^2 + 10z_2^2 - 22z_1z_2 - 6z_1^2z_2 + 5z_1z_2^2 + 3z_1^3 - 10z_2^3 + 3z_1^3z_2 + 5z_1z_2^3],$$

where again $z_j = Z_j/W$. For $Z_1 > Z_2$ one must interchange z_1 and z_2 on the right-hand side.

The behavior of the interaction at large distance S follows by comparison with some Bessel function integrals with integrands that have similar dependence on q for small q. We note that for positive α

$$\int_0^\infty J_0(ks)\frac{k\alpha^3}{(k^2+\alpha^2)^{3/2}}dk = \alpha^2 e^{-\alpha s}. \tag{5.6}$$

This shows exponential decay at a large distance. A second integral is

$$\int_0^\infty J_0(ks)\frac{k^3\alpha^3}{(k^2+\alpha^2)^{3/2}}dk = \alpha^3\frac{1-\alpha s}{s}e^{-\alpha s}. \tag{5.7}$$

This also shows exponential decay at a large distance. The third integral to consider is

$$\int_0^\infty J_2(ks)\frac{k\alpha^3}{(k^2+\alpha^2)^{3/2}}dk = \frac{2}{s^2} - \frac{1}{s^2}(2+2\alpha s - \alpha^2 s^2)e^{-\alpha s}. \tag{5.8}$$

This decays with a power law. We conclude that the dimensionless potential $\phi_{2zz}(s,z_1,z_2)$ decays exponentially at large s, but that $\phi_{2xx}(s,z_1,z_2)$ decays as

$$\phi_{2xx}(s,z_1,z_2) \approx -2h_2(z_1,z_2)/s^2 \quad \text{as } s \to \infty. \tag{5.9}$$

It follows from Eq. (5.5) that the numerator is positive. The potential $\phi_{2yy}(s,z_1,z_2)$ decays in the same way, but with numerator $+2h_2(z_1,z_2)$. Finally, we consider the integral

$$\int_0^\infty J_1(ks)\frac{k^2\alpha^3}{(k^2+\alpha^2)^{3/2}}dk = \alpha^3 \exp[-\alpha s]. \tag{5.10}$$

This shows that the functions $\phi_{2xz}(s,z_1,z_2)$ and $\phi_{2zx}(s,z_1,z_2)$ decay exponentially at large distance s.

Next we consider the behavior at small distance S. At large q the function $F(q,Z_1,Z_2)$ behaves as

$$F(q,Z_1,Z_2) \approx F_{as}(k,z), \tag{5.11}$$

with variables $k=qW$ and $z=(Z_2-Z_1)/W$, and function

$$F_{as}(k,z) = (3+3k|z|+k^2z^2)\frac{\exp[-k|z|]}{k}. \tag{5.12}$$

The function $G(q,Z_1,Z_2)$ behaves as

$$G(q,Z_1,Z_2) \approx G_{as}(k,z), \tag{5.13}$$

with

$$G_{as}(k,z) = (5+5k|z|-k^2z^2)\frac{\exp[-k|z|]}{2k}, \tag{5.14}$$

and the function $H(q,Z_1,Z_2)$ behaves as

$$H(q,Z_1,Z_2) \approx H_{as}(k,z), \tag{5.15}$$

with

$$H_{as}(k,z) = -(3+3k|z|+k^2z^2)\frac{\exp[-k|z|]}{2k}. \tag{5.16}$$

The functions $C_{xz}(q,Z_1,Z_2)$ and $C_{zx}(q,Z_1,Z_2)$ behave for large q as

$$C_{xz}(q,Z_1,Z_2) \approx C_{zx}(q,Z_1,Z_2) \approx C_{as}(k,z), \tag{5.17}$$

with

$$C_{as}(k,z) = |z|(1+k|z|)\exp[-k|z|]. \tag{5.18}$$

These expressions show that for nonvanishing values of $|Z_1-Z_2|$ the integrals in Eqs. (4.7)–(4.10) converge exponentially. For small S the integrals therefore equal their value at $S=0$ plus a term of order S^2 that one may evaluate by expanding the Bessel functions to second order in S, and performing the resulting integral.

The behavior is more subtle when both S and $|Z_1-Z_2|$ are small. We find the behavior of the potentials $\phi_{2\alpha\beta}(s,z_1,z_2)$ for small s and small $|z_1-z_2|$ by comparison with Bessel function integrals. We note that

$$\int_0^\infty J_0(ks)\left[F_{as}(k,z)-\frac{3}{k}\right]/k\,dk = 3s - \frac{3s^2+2z^2}{\sqrt{s^2+z^2}}. \tag{5.19}$$

Here we have subtracted the term $3/k$ in the integrand to make the integral convergent at $k=0$. The second integral is

$$\int_0^\infty J_0(ks)\left[G_{as}(k,z)-\frac{5}{2k}\right]/k\,dk = \frac{5}{2}s - \frac{5s^2+6z^2}{2\sqrt{s^2+z^2}}. \tag{5.20}$$

The third integral is

$$\int_0^\infty J_2(ks)H_{as}(k,z)/k\,dk = \frac{-s^2}{2\sqrt{s^2+z^2}}. \tag{5.21}$$

The subtracted term in Eqs. (5.19) and (5.20) may be taken into account by consideration of the integral

$$\int_0^\infty J_0(ks)\frac{1}{k^2+\varepsilon^2}dk = \frac{\pi}{2\varepsilon}[I_0(\varepsilon s) - \boldsymbol{L}_0(\varepsilon s)], \tag{5.22}$$

where $I_0(x)$ is a modified Bessel function, and $\boldsymbol{L}_0(x)$ is a modified Struve function.[18] The right-hand side of Eq. (5.22) has the expansion for small s

$$\frac{\pi}{2\varepsilon}[I_0(\varepsilon s) - \boldsymbol{L}_0(\varepsilon s)] = \frac{\pi}{2\varepsilon} - s + O(s^2). \tag{5.23}$$

The fourth integral to consider is

$$\int_0^\infty J_1(ks)C_{as}(k,z)/k\,dk = \frac{s|z|}{\sqrt{s^2+z^2}}. \tag{5.24}$$

From the above expressions we conclude that at fixed z_1 and for small s and small $|z_1-z_2|$ the dimensionless interaction potential $\phi_{2zz}(s,z_1,z_2)$ behaves as

$$\phi_{2zz}(s,z_1,z_2) \approx \phi_{2zz}(0,z_1,z_1) - \frac{3s^2+2(z_1-z_2)^2}{\sqrt{s^2+(z_1-z_2)^2}}, \tag{5.25}$$

the interaction potential $\phi_{2xx}(s,z_1,z_2)$ behaves as

$$\phi_{2xx}(s,z_1,z_2) \approx \phi_{2xx}(0,z_1,z_1) - \frac{2s^2+3(z_1-z_2)^2}{\sqrt{s^2+(z_1-z_2)^2}}, \tag{5.26}$$

the interaction potential $\phi_{2yy}(s,z_1,z_2)$ behaves as

$$\phi_{2yy}(s,z_1,z_2) \approx \phi_{2xx}(0,z_1,z_1) - \frac{3s^2+3(z_1-z_2)^2}{\sqrt{s^2+(z_1-z_2)^2}}, \tag{5.27}$$

and the interaction potentials $\phi_{2xz}(s,z_1,z_2)$ and $\phi_{2zx}(s,z_1,z_2)$ behave as

$$\phi_{2xz}(s,z_1,z_2) \approx \phi_{2zx}(s,z_1,z_2) \approx \frac{s|z_1-z_2|}{\sqrt{s^2+(z_1-z_2)^2}}. \tag{5.28}$$

The analysis is confirmed by numerical calculation of the potentials from Eqs. (4.7)–(4.10).

Finally, we study the behavior of the kinetic force on particles located near the midplane in the limit where the walls become infinitely distant. We consider first the one-body potential $\Phi_1^{(1)}(\boldsymbol{R})$, given by Eq. (4.13), at a fixed height Z in the limit $W\to\infty$. Since the potential depends only on $z=Z/W$, we may effectively replace Z by 0. Though the potential at $Z=0$ diverges in proportion to W, it also becomes very flat. By Taylor expansion it varies as $AW+BZ^2/W$ for small Z, where A and B are constants. Hence the force on a particle at a fixed height Z due to the one-body potential vanishes in the limit $W\to\infty$.

A similar argument holds for a pair of particles. One needs to consider only the limiting behavior given by Eqs.

(5.25)–(5.28). The first term in Eqs. (5.25)–(5.27) has a Taylor expansion in the variable Z_1 of the same type as considered above. Hence this term may effectively be omitted from consideration. The remaining terms are linear in $s=S/W$ and $z_1-z_2=(Z_1-Z_2)/W$. In the limit $W\to\infty$ these terms lead via Eq. (4.12) to a translation-invariant interaction potential $V_T^{(1)}(\boldsymbol{R}_2-\boldsymbol{R}_1)$ of the form

$$V_T^{(1)}(\boldsymbol{r}) = \frac{3}{16}\,\mathrm{Re}\,K\left[-3r+\frac{(\boldsymbol{r}\cdot\hat{\boldsymbol{K}})^2}{r}\right]. \tag{5.29}$$

The corresponding pair interaction force $\boldsymbol{K}_T^{(1)}(\boldsymbol{R}_2-\boldsymbol{R}_1)$ acting from particle 1 on particle 2 is given by

$$\boldsymbol{K}_T^{(1)}(\boldsymbol{r}) = \frac{3}{16}\,\mathrm{Re}\,K\{[3+(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})^2]\hat{\boldsymbol{r}}-2(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})\hat{\boldsymbol{K}}\}. \tag{5.30}$$

Hence the force is repulsive and independent of distance. The direction of the force is determined by both the distance vector and the direction of the applied force. The strength of the interaction is proportional to K^2.

Remarkably, the kinetic interaction potential and force in the limit of distant walls become independent of the orientation of the applied force $\boldsymbol{K}$ with respect to the walls. Only the orientation with respect to the relative distance vector is relevant. Presumably the same is true for other shapes of the volume Ω. In other words, our calculation suggests that the kinetic interaction force has a well-defined thermodynamic limit.

We emphasize that the results derived here describe only the limiting behavior for interparticle distances much smaller than the distance between the walls. For distances of the order of the width W the force decays to zero, and the complete results derived above must be used.

VI. TWO-PARTICLE DYNAMICS

In this section we study the dynamics of two equal-sized spheres of radius a in infinite fluid, taking account of the kinetic pair interaction found in the preceding section. We must attempt to incorporate also the dissipative force.

As a starting point we consider the solution of the steady-state Navier–Stokes equations to first order in the Reynolds number Re for a single sphere with stick boundary conditions, as found by Proudman and Pearson.[11] We choose the z axis in the direction of the applied force $\boldsymbol{K}$, take the origin at the center of the sphere, and use polar coordinates (r,θ,φ). The sphere moves with velocity U in the z direction. The axisymmetric solution of the Navier–Stokes equations may be expressed with the aid of a Stokes stream function $\psi(r,\theta)$, such that[19]

$$v_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\quad v_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{6.1}$$

Then the condition of incompressibility is satisfied. The Navier–Stokes equations are equivalent to

$$\eta E^2(E^2\psi)=\frac{\rho}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}+2\cot\theta\frac{\partial\psi}{\partial r}-\frac{2}{r}\frac{\partial\psi}{\partial\theta}\right)E^2\psi, \tag{6.2}$$

where E^2 denotes the linear differential operator

$$E^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{6.3}$$

At short distances the solution of Eq. (6.2) is calculated to first order in the Reynolds number as

$$\psi_S(r,\theta)=\psi_S^{(0)}(r,\theta)+\psi_S^{(1)}(r,\theta), \tag{6.4}$$

with zero-order Stokes solution

$$\psi_S^{(0)}(r,\theta)=\frac{1}{4}U\left(3ar-\frac{a^3}{r}\right)\sin^2\theta, \tag{6.5}$$

and first-order term

$$\psi_S^{(1)}(r,\theta)=-\frac{3}{32}\,\mathrm{Re}\,U\frac{(r-a)^2}{ar^2}[ar(a+2r)+(a^2+ar+2r^2)\cos\theta]\sin^2\theta. \tag{6.6}$$

At large distances the solution is approximated by a solution of the Oseen equations obtained by replacement of the inertial term $\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ in the Navier–Stokes equations by $\rho\boldsymbol{U}\cdot\nabla\boldsymbol{v}$. In terms of the stream function this leads to the equation

$$\eta E^2(E^2\psi)=\frac{\rho U}{r^2\sin\theta}\left[\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]E^2\psi. \tag{6.7}$$

The linear operator in square brackets commutes with E^2. A solution of Eq. (6.7) is

$$\psi_L(r,\theta)=\frac{3}{2}\frac{Ua^2}{\mathrm{Re}}(1-\cos\theta)\left(1-\exp\left[-(1+\cos\theta)\frac{r}{\lambda}\right]\right), \tag{6.8}$$

with screening length

$$\lambda=2a/\mathrm{Re}. \tag{6.9}$$

The prefactor is chosen such that to lowest order in Re the corresponding flow agrees with the flow $\mathsf{T}_0(\boldsymbol{r})\cdot\boldsymbol{K}$ of a point particle in the Stokes regime. The solution $\psi_S^{(1)}$ in Eq. (6.6) is constructed such that when $\psi_L(r,\theta)$ is expanded in powers of the Reynolds number, the approximation $\psi_L^{(0)}+\psi_L^{(1)}$ matches with $\psi_S(r'/\mathrm{Re},\theta)$ when expanded in powers of Re at fixed $r'=\mathrm{Re}\,r$.

From the solution $\psi_S(r,\theta)$, given by Eq. (6.4), one finds for the hydrodynamic force acting on the sphere to first order in the Reynolds number

$$\boldsymbol{K}_{\mathrm{hyd}}=-6\pi\eta a\left(1+\tfrac{3}{8}\mathrm{Re}\right)\boldsymbol{U}. \tag{6.10}$$

At the sphere surface the hydrodynamic stress tensor can be approximated by the viscous stress tensor given by Eq. (2.5) on account of the stick boundary condition. The result Eq. (6.10) is identical to that found by Oseen,[20,21] though the stream function $\psi_L(r,\theta)$ given by Eq. (6.8) differs from that corresponding to Oseen's velocity field by a solution of the homogeneous equation $E^2\psi=0$.

The dissipation function corresponding to the solution $\psi_S(r,\theta)$ diverges when the integral in Eq. (2.10) is performed over all space. Expanding the integrand to first order in Re and using a cutoff radius $b=a/\beta$ one obtains

$$\mathcal{D}_{Sb}=3\pi\eta a\left(1+\tfrac{3}{4}\,\mathrm{Re}\right)\left(1-\tfrac{3}{2}\beta+\beta^3-\tfrac{1}{2}\,\beta^5\right)U^2+O(\mathrm{Re}^2). \tag{6.11}$$

The dissipation function corresponding to the solution $\psi_L(r,\theta)$ with the integral calculated for the outer space $b<r<\infty$ reads to first order in Re

$$\mathcal{D}_{Lb}=3\pi\eta a\left(1+\tfrac{3}{8}\,\mathrm{Re}\right)\left(\tfrac{3}{2}\beta-\tfrac{3}{8}\,\mathrm{Re}\right)U^2+O(\mathrm{Re}^2). \tag{6.12}$$

One therefore finds

$$\mathcal{D}_{Sb}+\mathcal{D}_{Lb}=3\pi\eta a\left(1+\tfrac{3}{8}\,\mathrm{Re}\right)U^2+O(\mathrm{Re}^2), \tag{6.13}$$

in agreement with Eq. (6.10), provided that β is chosen of order Re. We choose $\beta=\frac{1}{4}\mathrm{Re}$ corresponding to cutoff radius $b=4a/\mathrm{Re}=2\lambda$. For this cutoff $\mathcal{D}_{Lb}$ is of order Re^2.

We write the flow in the form

$$\boldsymbol{V}_P(\boldsymbol{r})=\boldsymbol{V}_S(\boldsymbol{r})\Theta(2\lambda-r)+\boldsymbol{V}_L(\boldsymbol{r})\Theta(r-2\lambda), \tag{6.14}$$

where $\Theta(x)$ is the Heaviside step function. Here $\boldsymbol{V}_S(\boldsymbol{r})$ is calculated from the stream function $\psi_S(\boldsymbol{r})$, and $\boldsymbol{V}_L(\boldsymbol{r})$ is calculated from the stream function $\psi_L(\boldsymbol{r})$. By construction $\boldsymbol{V}_L(\boldsymbol{r})=\mathsf{T}_0(\boldsymbol{r})\cdot\boldsymbol{K}+O(\mathrm{Re})$. The flow given by Eq. (6.14) has a slight discontinuity at $r=2\lambda$. It is worth noting that the flow velocity in the Proudman–Pearson solution still has a long range, falling off as $1/r^2$ at large distance, faster than the $1/r$ behavior of the Stokes solution. The dissipation falls off as $1/r^6$, and occurs mostly within a few diameters about the sphere.

The dissipation function of a sphere of velocity $\boldsymbol{U}$ placed in a uniform stream of velocity $\boldsymbol{V}$ is given by

$$\mathcal{D}=\tfrac{1}{2}\zeta(\boldsymbol{U}-\boldsymbol{V})^2, \tag{6.15}$$

with friction coefficient $\zeta=6\pi\eta a\left(1+\frac{3}{8}\mathrm{Re}\right)$ to first order in Re. On account of the localization of dissipation a good approximation to the dissipation function of two distant spheres in quiescent fluid is given by

$$\mathcal{D}=\tfrac{1}{2}\zeta(\boldsymbol{U}_1-\boldsymbol{V}_{12})^2+\tfrac{1}{2}\zeta(\boldsymbol{U}_2-\boldsymbol{V}_{21})^2, \tag{6.16}$$

where $\boldsymbol{V}_{12}$ is the velocity at the center $\boldsymbol{R}_1$ of the first sphere generated by the second one. Since the spheres are distant, reflections may be neglected, and this may be approximated by

$$\boldsymbol{V}_{12}=\boldsymbol{V}_P(\boldsymbol{R}_1-\boldsymbol{R}_2). \tag{6.17}$$

According to Eq. (2.8) the dissipative force acting on the first sphere is given by

$$\boldsymbol{K}_{d1}=-\zeta[\boldsymbol{U}_1-\boldsymbol{V}_P(\boldsymbol{R}_1-\boldsymbol{R}_2)], \tag{6.18}$$

and similarly for sphere 2.

As a next step we extrapolate the interaction potential given by Eq. (5.29) to higher order in the Reynolds number by assuming shielding with the same screening length $\lambda=2a/\mathrm{Re}$. This leads to the expression

$$V_T(\boldsymbol{r})=\frac{9}{8}Ka\exp\left[-\frac{r}{\lambda}+\frac{1}{3}\frac{(\boldsymbol{r}\cdot\hat{\boldsymbol{K}})^2}{\lambda r}\right]. \tag{6.19}$$

The corresponding interaction force is given by

$$\boldsymbol{K}_T(\boldsymbol{r})=\frac{3}{16}\,\mathrm{Re}\,K\{[3+(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})^2]\hat{\boldsymbol{r}}-2(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})\hat{\boldsymbol{K}}\}\times\exp\left[-\frac{r}{\lambda}+\frac{1}{3}\frac{(\boldsymbol{r}\cdot\hat{\boldsymbol{K}})^2}{\lambda r}\right]. \tag{6.20}$$

The extrapolation implies that the interaction is repulsive over a distance of order λ, but then decays to zero. Since the screening length is large in the limit of small Reynolds number, we may expect significant effects, even though the amplitude of the interaction is proportional to Re.

The dissipative force acting on each of the spheres may now be expressed as

$$\begin{aligned}\boldsymbol{K}_{d1}&=-\zeta\boldsymbol{U}_1+\zeta\boldsymbol{V}_{12}=-\boldsymbol{K}-\boldsymbol{K}_{T12},\\ \boldsymbol{K}_{d2}&=-\zeta\boldsymbol{U}_2+\zeta\boldsymbol{V}_{21}=-\boldsymbol{K}-\boldsymbol{K}_{T21},\end{aligned} \tag{6.21}$$

in an obvious notation. Solving for the velocities $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ we therefore find for a pair of distant spheres the dynamical equations

$$\begin{aligned}\frac{d\boldsymbol{R}_1}{dt}&=\mu\boldsymbol{K}+\boldsymbol{V}_P(\boldsymbol{R}_1-\boldsymbol{R}_2)+\mu\boldsymbol{K}_T(\boldsymbol{R}_1-\boldsymbol{R}_2),\\ \frac{d\boldsymbol{R}_2}{dt}&=\mu\boldsymbol{K}+\boldsymbol{V}_P(\boldsymbol{R}_2-\boldsymbol{R}_1)+\mu\boldsymbol{K}_T(\boldsymbol{R}_2-\boldsymbol{R}_1),\end{aligned} \tag{6.22}$$

where $\mu=1/\zeta$ is the mobility of a sphere. In the limit of zero Reynolds number these equations reduce to the equations of Stokesian dynamics.

To first order in Re the second term in Eqs. (6.22) agrees with the calculation of Vasseur and Cox[10] for the configuration considered by them, as shown in their Fig. 5. They consider two spheres with center-to-center vector $\boldsymbol{R}_2-\boldsymbol{R}_1$ at right angles to the applied force $\boldsymbol{K}$. According to their calculation each sphere moves away from the other one with velocity component $v_l=\frac{3}{16}\mathrm{Re}\ U_0$ along the line of centers, where $U_0=K/6\pi\eta a$ is the Stokes velocity. This agrees precisely with the constant term in the Proudman–Pearson solution, as given by $\boldsymbol{V}_S(\boldsymbol{r})$ in Eq. (6.14). The last term in Eqs. (6.22) adds to this the velocity $v_T=\frac{9}{16}\mathrm{Re}\ U_0$, so that the sum is four times as large as the value predicted by Vasseur and Cox. It would be of interest to compare the two theoretical predictions of the velocity with experiment. Vasseur and Cox calculate the flow velocity experienced by each of the spheres, but in identifying this with the change of velocity of the sphere implicitly assume that no force acts on it besides the applied force $\boldsymbol{K}$. This takes no account of the force due to the difference of flow kinetic energy for two neighboring configurations. The kinetic force is mediated by the stress tensor, as calculated from the Proudman–Pearson flow pattern with velocities given by Eq. (6.22).

The kinetic interaction force $\boldsymbol{K}_T(\boldsymbol{r})$ has the symmetry

$$\boldsymbol{K}_T(-\boldsymbol{r}) = -\boldsymbol{K}_T(\boldsymbol{r}), \tag{6.23}$$

but this symmetry is not shared by the velocity field $\boldsymbol{V}_P(\boldsymbol{r})$. The rate of change of the relative distance vector $\boldsymbol{r}=\boldsymbol{R}_1-\boldsymbol{R}_2$ is given by

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{V}_a(\boldsymbol{r}) + 2\mu \boldsymbol{K}_T(\boldsymbol{r}), \tag{6.24}$$

with vector

$$\boldsymbol{V}_a(\boldsymbol{r}) = \boldsymbol{V}_P(\boldsymbol{r}) - \boldsymbol{V}_P(-\boldsymbol{r}). \tag{6.25}$$

This vector clearly has the symmetry

$$\boldsymbol{V}_a(-\boldsymbol{r}) = -\boldsymbol{V}_a(\boldsymbol{r}). \tag{6.26}$$

The divergence of the field vanishes,

$$\nabla \cdot \boldsymbol{V}_a(\boldsymbol{r}) = 0, \tag{6.27}$$

since $\nabla \cdot \boldsymbol{V}_P(\boldsymbol{r})=0$, but for the interaction force $\boldsymbol{K}_T^{(1)}(\boldsymbol{r})$ the divergence is positive. The explicit expression for the vector field for distances less than 2λ is

$$\begin{aligned}\boldsymbol{V}_a(\boldsymbol{r}) = \tfrac{3}{16}\,\mathrm{Re}\,\mu K\frac{r-a}{r^4}&[(2r^3 - ar^2 - a^3)\hat{\boldsymbol{r}} \\ &- (2r^3 - 4ar^2 - a^2 r - 5a^3)(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})^2\hat{\boldsymbol{r}} \\ &- (4r^3 + ar^2 + a^2 r + 2a^3)(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})\hat{\boldsymbol{K}}].\end{aligned} \tag{6.28}$$

For distances much larger than a, but less than 2λ, this may be approximated by

$$\boldsymbol{V}_a(\boldsymbol{r}) \approx \tfrac{3}{8}\,\mathrm{Re}\,\mu K[\hat{\boldsymbol{r}} - (\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})^2\hat{\boldsymbol{r}} - 2(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{K}})\hat{\boldsymbol{K}}]. \tag{6.29}$$

It is not consistent to employ the full detail of Eq. (6.28) since we have neglected similar terms in the calculation of the force $\boldsymbol{K}_T(\boldsymbol{r})$. The sum of terms in Eq. (6.24) therefore becomes

$$\boldsymbol{V}_a(\boldsymbol{r}) + 2\mu\boldsymbol{K}_T(\boldsymbol{r}) \approx \tfrac{3}{2}\,\mathrm{Re}\,\mu K(\boldsymbol{1} - \hat{\boldsymbol{K}}\hat{\boldsymbol{K}})\cdot\hat{\boldsymbol{r}}. \tag{6.30}$$

The equation implies repulsion independent of distance and perpendicular to the applied force $\boldsymbol{K}$. The interaction force by itself would lead to a steady decline of potential energy $V_T(\boldsymbol{r})$, but the scalar product $\boldsymbol{K}_T(\boldsymbol{r})\cdot\boldsymbol{V}_a(\boldsymbol{r})$ takes both signs as $\boldsymbol{r}$ varies. Both terms in Eq. (6.24) are at most of first order in Re, and are of comparable magnitude. The term $\boldsymbol{V}_a(\boldsymbol{r})$ may be regarded as the solenoidal part of the interaction, and the term $2\mu\boldsymbol{K}_T(\boldsymbol{r})$ as the longitudinal part. The time scale of the motion described by Eq. (6.24) is in inverse proportion to Re. The relative vector tends to orient in the direction perpendicular to the applied force in the course of time. This agrees with the experimental observations of Jayaweera *et al.*[22] For the dynamics described by Eq. (6.30) the relative distance r increases monotonically in time, but this is not the case if the kinetic force is omitted.

VII. DISCUSSION

We have calculated the inertial correction to the hydrodynamic interaction between two settling spherical particles to first order in the Reynolds number and in point approximation. As a result we find that the interaction is repulsive, has a long range, and depends on the angle between the relative vector of positions and the applied force. The approximate expression is given by Eqs. (6.24) and (6.30). The interaction is weak for small Reynolds number, but has excessively long range, and can therefore have an appreciable influence in a sedimenting suspension. The interaction has the effect of turning the relative distance vector perpendicular to the applied force, and therefore tends to create an anisotropic pair distribution.

The theory of sedimentation of a many-particle suspension with account of the inertial correction remains to be elaborated. On account of the long range one must expect collective effects, and a nontrivial pair distribution function even for dilute suspensions, as in the Debye–Hückel theory of electrolyte solutions. The simple relative motion given by Eq. (6.30), describing repulsion independent of distance in the direction perpendicular to the applied force, leads to an interesting model of a sedimenting suspension. The repulsion must be counteracted by confining walls perpendicular to the applied force.

APPENDIX: INTERACTION FUNCTIONS

In this Appendix we provide the explicit expressions of the functions F, G, H, C_{xz}, and C_{zx} occurring in Eqs. (4.7)–(4.10). We use dimensionless variables $k=qW$ and $z_i=Z_i/W$, and the abbreviations

$$\xi = \exp k, \quad \xi_1 = \exp kz_1, \quad \xi_2 = \exp kz_2. \tag{A1}$$

In the following expressions it is assumed that $z_2 > z_1$. For $z_2 < z_1$ the variables z_1 and z_2 must be interchanged.

For $z_2 > z_1$ the function $F(k,z_1,z_2)$ may be expressed as

$$F(k,z_1,z_2) = \frac{N_F(k,z_1,z_2)}{D_F(k,z_1,z_2)}, \tag{A2}$$

with numerator of the form

$$\begin{aligned}N_F(k,z_1,z_2) = N_{F0} + N_{F1}k + N_{F2}k^2 + N_{F3}k^3 + N_{F4}k^4 \\ + N_{F5}k^5 + N_{F6}k^6,\end{aligned} \tag{A3}$$

and denominator given by

$$D_F(k,z_1,z_2) = k\xi_1\xi_2(1 + 4k\xi^2 - \xi^4)^2(1 - 4k\xi^2 - \xi^4)^2. \tag{A4}$$

The coefficients in the numerator are given by

$$N_{F0} = 3(1-\xi^4)^3(1-\xi^2\xi_1^2)(\xi^2-\xi_2^2),$$

$$N_{F1} = -3(1-\xi^4)^2[(\xi^8\xi_1^2 + \xi_2^2)(z_1 - z_2) - \xi^4(\xi_1^2 + \xi_2^2)(4 + z_1 - z_2) + \xi^2(1 + \xi^4\xi_1^2\xi_2^2)(2 - z_1 - z_2) + \xi^2(\xi^4 + \xi_1^2\xi_2^2)(2 + z_1 + z_2)],$$

$$N_{F2} = (-1+\xi^4)[(\xi^{12}\xi_1^2+\xi_2^2)(z_1-z_2)^2+\xi^4(\xi_1^2+\xi^4\xi_2^2)(24+12z_1-12z_2+z_1^2-2z_1z_2+z_2^2)-2\xi^4(\xi^4\xi_1^2+\xi_2^2)$$
$$\times(36+6z_1-6z_2+z_1^2-2z_1z_2+z_2^2)+2\xi^6(1+\xi_1^2\xi_2^2)(30+z_1^2+4z_1z_2+z_2^2)-\xi^2(1+\xi^8\xi_1^2\xi_2^2)$$
$$\times(6-6z_1-6z_2+z_1^2+4z_1z_2+z_2^2)-\xi^2(\xi^8+\xi_1^2\xi_2^2)(6+6z_1+6z_2+z_1^2+4z_1z_2+z_2^2)], \tag{A5}$$

$$N_{F3} = 2\xi^2[(1+\xi^{12}\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(-2+z_1+z_2)-(\xi^{12}+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2+z_1+z_2)+2\xi^2(\xi^8\xi_1^2+\xi_2^2)$$
$$\times(8+18z_1-18z_2+z_1^2-2z_1z_2+z_2^2)-4\xi^6(\xi_1^2+\xi_2^2)(32+12z_1-12z_2+z_1^2-2z_1z_2+z_2^2)+2\xi^2(\xi_1^2+\xi^8\xi_2^2)$$
$$\times(8+6z_1-6z_2+z_1^2-2z_1z_2+z_2^2)+\xi^4(1+\xi^4\xi_1^2\xi_2^2)(50-33z_1-33z_2+z_1^2+4z_1z_2+z_2^2-3z_1^2z_2-3z_1z_2^2)+\xi^4(\xi^4+\xi_1^2\xi_2^2)$$
$$\times(50+33z_1+33z_2+z_1^2+4z_1z_2+z_2^2+3z_1^2z_2+3z_1z_2^2)],$$

$$N_{F4} = 8\xi^4[(\xi_1^2-\xi^8\xi_2^2)(1+z_1)(1-z_2)(-4-z_1+z_2)-2\xi^2(1-\xi^4\xi_1^2\xi_2^2)(2-6z_1-6z_2+z_1^2+8z_1z_2+z_2^2)+2\xi^2(\xi^4-\xi_1^2\xi_2^2)$$
$$\times(2+6z_1+6z_2+z_1^2+8z_1z_2+z_2^2)+2\xi^4(\xi_1^2-\xi_2^2)(24+17z_1-17z_2+2z_1^2-4z_1z_2+2z_2^2-z_1^2z_2+z_1z_2^2)+(\xi^8\xi_1^2-\xi_2^2)$$
$$\times(4-5z_1+5z_2-3z_1^2+2z_1z_2-3z_2^2+z_1^2z_2-z_1z_2^2)],$$

$$N_{F5} = -32\xi^6[(\xi^4+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2-z_1-z_2)+(1+\xi^4\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(2+z_1+z_2)+2\xi^2(\xi_1^2+\xi_2^2)(4+6z_1-6z_2$$
$$+z_1^2-6z_1z_2+z_2^2)],$$

$$N_{F6} = +128\xi^8(\xi_1^2-\xi_2^2)(1+z_1)(1-z_2)(z_1-z_2).$$

For $z_2 > z_1$ the function $G(k,z_1,z_2)$ may be expressed as

$$G(k,z_1,z_2) = \frac{N_G(k,z_1,z_2)}{D_G(k,z_1,z_2)}, \tag{A6}$$

with numerator of the form

$$N_G(k,z_1,z_2) = N_{G0}+N_{G1}k+N_{G2}k^2+N_{G3}k^3+N_{G4}k^4+N_{G5}k^5+N_{G6}k^6, \tag{A7}$$

and denominator given by

$$D_G(k,z_1,z_2) = 2k\xi_1\xi_2(1-\xi^4)^2(1+4k\xi^2-\xi^4)^2(1-4k\xi^2-\xi^4)^2. \tag{A8}$$

The coefficients in the numerator are given by

$$N_{G0} = \tfrac{5}{3}(1-\xi^4)^2 N_{F0}, \quad N_{G1} = \tfrac{5}{3}(1-\xi^4)^2 N_{F1}, \tag{A9}$$

$$N_{G2} = (1-\xi^4)^3[(\xi^{12}\xi_1^2+\xi_2^2)(z_1-z_2)^2+\xi^4(\xi_1^2+\xi^4\xi_2^2)(24+12z_1-12z_2+z_1^2-2z_1z_2+z_2^2)+2\xi^4(\xi^4\xi_1^2+\xi_2^2)$$
$$\times(92-6z_1+6z_2-z_1^2+2z_1z_2-z_2^2)-2\xi^6(1+\xi_1^2\xi_2^2)(98-z_1^2-4z_1z_2-z_2^2)-\xi^2(1+\xi^8\xi_1^2\xi_2^2)$$
$$\times(6-6z_1-6z_2+z_1^2+4z_1z_2+z_2^2)-\xi^2(\xi^8+\xi_1^2\xi_2^2)(6+6z_1+6z_2+z_1^2+4z_1z_2+z_2^2)],$$

$$N_{G3} = 2\xi^2(1-\xi^4)^2[(1+\xi^{12}\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(-2+z_1+z_2)-(\xi^{12}+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2+z_1+z_2)+2\xi^2(\xi^8\xi_1^2+\xi_2^2)$$
$$\times(12+30z_1-30z_2+z_1^2-6z_1z_2+z_2^2)-4\xi^6(\xi_1^2+\xi_2^2)(52+20z_1-20z_2+z_1^2-6z_1z_2+z_2^2)+2\xi^2(\xi_1^2+\xi^8\xi_2^2)$$
$$\times(12+10z_1-10z_2+z_1^2-6z_1z_2+z_2^2)+\xi^4(1+\xi^4\xi_1^2\xi_2^2)(82-49z_1-49z_2+z_1^2+4z_1z_2+z_2^2-3z_1^2z_2-3z_1z_2^2)$$
$$+\xi^4(\xi^4+\xi_1^2\xi_2^2)(82+49z_1+49z_2+z_1^2+4z_1z_2+z_2^2+3z_1^2z_2+3z_1z_2^2)],$$

$$N_{G4} = 8\xi^4(1-\xi^4)[(\xi_1^2+\xi^{12}\xi_2^2)(1+z_1)(1-z_2)(4+z_1-z_2)+4\xi^6(1+\xi_1^2\xi_2^2)(38-z_1^2-z_2^2)-2\xi^2(\xi^8+\xi_1^2\xi_2^2)(6+2z_1+2z_2-z_1^2$$
$$-z_2^2)-2\xi^2(1+\xi^8\xi_1^2\xi_2^2)(6-2z_1-2z_2-z_1^2-z_2^2)+\xi^4(\xi_1^2+\xi^4\xi_2^2)(12-7z_1+7z_2-5z_1^2+14z_1z_2-5z_2^2+3z_1^2z_2-3z_1z_2^2)$$
$$+(\xi^{12}\xi_1^2+\xi_2^2)(4-5z_1+5z_2-3z_1^2+2z_1z_2-3z_2^2+z_1^2z_2-z_1z_2^2)-\xi^4(\xi^4\xi_1^2+\xi_2^2)(148-7z_1+7z_2-7z_1^2+10z_1z_2-7z_2^2$$
$$+3z_1^2z_2-3z_1z_2^2)],$$

$$N_{G5} = -32\xi^6[(\xi^{12}+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2-z_1-z_2)+(1+\xi^{12}\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(2+z_1+z_2)+2\xi^2(\xi^8\xi_1^2+\xi_2^2)(z_1-z_2)$$
$$\times(18+z_1-z_2)+2\xi^2(\xi_1^2+\xi^8\xi_2^2)(z_1-z_2)(2+z_1-z_2)+4\xi^6(\xi_1^2+\xi_2^2)(32+10z_1-10z_2+z_1^2-2z_1z_2+z_2^2)+\xi^4(1+\xi^4\xi_1^2\xi_2^2)$$
$$\times(62-29z_1-29z_2+z_1^2+z_2^2-3z_1^2z_2-3z_1z_2^2)+\xi^4(\xi^4+\xi_1^2\xi_2^2)(62+29z_1+29z_2+z_1^2+z_2^2+3z_1^2z_2+3z_1z_2^2)],$$

$$N_{G6} = -(1-\xi^4)^2 N_{F6}.$$

For $z_2 > z_1$ the function $H(k,z_1,z_2)$ may be expressed as

$$H(k,z_1,z_2) = \frac{N_H(k,z_1,z_2)}{D_H(k,z_1,z_2)}, \tag{A10}$$

with numerator of the form

$$N_H(k,z_1,z_2) = N_{H0} + N_{H1}k + N_{H2}k^2 + N_{H3}k^3 + N_{H4}k^4 + N_{H5}k^5 + N_{H6}N^6, \tag{A11}$$

and denominator given by

$$D_H(k,z_1,z_2) = D_G(k,z_1,z_2). \tag{A12}$$

The coefficients in the numerator are given by

$$N_{H0} = -(1-\xi^4)^2 N_{F0}, \quad N_{H1} = -(1-\xi^4)^2 N_{F1},$$

$$N_{H2} = -(1-\xi^4)^2 N_{F2},$$

$$\begin{aligned} N_{H3} = {} & 2\xi^2(1-\xi^4)^2[(1+\xi^{12}\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(-2+z_1+z_2) - (\xi^{12}+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2+z_1+z_2) + 2\xi^2(\xi_1^2+\xi^8\xi_2^2) \\ & \times(12+10z_1-10z_2+z_1^2-6z_1z_2+z_2^2) + 4\xi^6(\xi_1^2+\xi_2^2)(76+12z_1-12z_2-z_1^2+6z_1z_2-z_2^2) + 2\xi^2(\xi^8\xi_1^2+\xi_2^2) \\ & \times(12-34z_1+34z_2+z_1^2-6z_1z_2+z_2^2) - \xi^4(1+\xi^4\xi_1^2\xi_2^2)(174-79z_1-79z_2-z_1^2-4z_1z_2-z_2^2+3z_1^2z_2+3z_1z_2^2) \\ & -\xi^4(\xi^4+\xi_1^2\xi_2^2)(174+79z_1+79z_2-z_1^2-4z_1z_2-z_2^2-3z_1^2z_2-3z_1z_2^2)], \end{aligned} \tag{A13}$$

$$\begin{aligned} N_{H4} = {} & 8\xi^4(1-\xi^4)[(\xi_1^2+\xi^{12}\xi_2^2)(1+z_1)(1-z_2)(4+z_1-z_2) - 4\xi^6(1+\xi_1^2\xi_2^2)(26+z_1^2+z_2^2) - 2\xi^2(\xi^8+\xi_1^2\xi_2^2)(6+2z_1+2z_2-z_1^2 \\ & -z_2^2) - 2\xi^2(1+\xi^8\xi_1^2\xi_2^2)(6-2z_1-2z_2-z_1^2-z_2^2) + \xi^4(\xi_1^2+\xi^4\xi_2^2)(12-7z_1+7z_2-5z_1^2+14z_1z_2-5z_2^2+3z_1^2z_2-3z_1z_2^2) \\ & +(\xi^{12}\xi_1^2+\xi_2^2)(4-5z_1+5z_2-3z_1^2+2z_1z_2-3z_2^2+z_1^2z_2-z_1z_2^2) + \xi^4(\xi^4\xi_1^2+\xi_2^2)(108+7z_1-7z_2+7z_1^2-10z_1z_2+7z_2^2 \\ & -3z_1^2z_2+3z_1z_2^2) \;, \end{aligned}$$

$$\begin{aligned} N_{H5} = {} & -32\xi^6[(\xi^{12}+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2-z_1-z_2) + (1+\xi^{12}\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(2+z_1+z_2) - 2\xi^2(\xi^8\xi_1^2+\xi_2^2)(z_1-z_2) \\ & \times(14-z_1+z_2) + 2\xi^2(\xi_1^2+\xi^8\xi_2^2)(z_1-z_2)(2+z_1-z_2) + 4\xi^6(\xi_1^2+\xi_2^2)(32+6z_1-6z_2-z_1^2+2z_1z_2-z_2^2) - \xi^4(1+\xi^4\xi_1^2\xi_2^2) \\ & \times(66-35z_1-35z_2-z_1^2-z_2^2+3z_1^2z_2+3z_1z_2^2) - \xi^4(\xi^4+\xi_1^2\xi_2^2)(66+35z_1+35z_2-z_1^2-z_2^2-3z_1^2z_2-3z_1z_2^2)], \end{aligned}$$

$$N_{H6} = -(1-\xi^4)^2 N_{F6}.$$

For $z_2 > z_1$ the function $C_{xz}(k,z_1,z_2)$ may be expressed as

$$C_{xz}(k,z_1,z_2) = \frac{N_{xz}(k,z_1,z_2)}{D_C(k,z_1,z_2)}, \tag{A14}$$

with numerator of the form

$$N_{xz}(k,z_1,z_2) = N_{xz0} + N_{xz1}k + N_{xz2}k^2 + N_{xz3}k^3 + N_{xz4}k^4 + N_{xz5}k^5, \tag{A15}$$

and denominator given by

$$D_C(k,z_1,z_2) = \xi_1\xi_2(1+4k\xi^2-\xi^4)^2(1-4k\xi^2-\xi^4)^2. \tag{A16}$$

The coefficients in the numerator are given by

$$N_{xz0} = \tfrac{1}{3}(z_2-z_1)N_{F0}, \quad N_{xz1} = \tfrac{1}{3}(z_2-z_1)N_{F1},$$

$$\begin{aligned} N_{xz2} = {} & -2\xi^2(1-\xi^4)[(1+\xi^8\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(-2+z_1+z_2) + (\xi^8+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2+z_1+z_2) + 2\xi^2(\xi_1^2+\xi^4\xi_2^2) \\ & \times(2-2z_1-6z_2-z_1^2-2z_1z_2+z_2^2) - 2\xi^2(\xi^4\xi_1^2+\xi_2^2)(2-6z_1+14z_2-z_1^2-2z_1z_2+z_2^2) - 2\xi^4(1+\xi_1^2\xi_2^2)(7z_1-17z_2+z_1^2z_2 \\ & +z_1z_2^2)], \end{aligned}$$

$$\begin{aligned} N_{xz3} = {} & 8\xi^4[(\xi_1^2+\xi^8\xi_2^2)(1+z_1)(1-z_2)(-4-z_1+z_2) + 2\xi^2(1+\xi^4\xi_1^2\xi_2^2)(4-2z_1-6z_2+z_1^2+4z_1z_2-z_2^2) - 2\xi^2(\xi^4+\xi_1^2\xi_2^2) \\ & \times(4+2z_1+6z_2+z_1^2+4z_1z_2-z_2^2) + 2\xi^4(\xi_1^2+\xi_2^2)(9z_1+7z_2+2z_1^2-4z_1z_2+2z_2^2-z_1^2z_2+z_1z_2^2) + (\xi^8\xi_1^2+\xi_2^2) \\ & \times(4-5z_1+5z_2-3z_1^2+2z_1z_2-3z_2^2+z_1^2z_2-z_1z_2^2)], \end{aligned}$$

$$N_{xz4} = -32\xi^6[(\xi^4 - \xi_1^2\xi_2^2)(1+z_1)(1+z_2)(-2+z_1+z_2) - (1-\xi^4\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(2+z_1+z_2) + 2\xi^2(\xi_1^2-\xi_2^2)(2+2z_1-2z_2 + z_1^2 - 2z_1z_2 - z_2^2)], \tag{A17}$$

$$N_{xz5} = 128\xi^8(\xi_1^2+\xi_2^2)(1+z_1)(1-z_2)(z_1-z_2).$$

For $z_2 > z_1$ the function $C_{zx}(k,z_1,z_2)$ may be expressed as

$$C_{zx}(k,z_1,z_2) = \frac{N_{zx}(k,z_1,z_2)}{D_C(k,z_1,z_2)}, \tag{A18}$$

with numerator of the form

$$N_{zx}(k,z_1,z_2) = N_{zx0} + N_{zx1}k + N_{zx2}k^2 + N_{zx3}k^3 + N_{zx4}k^4 + N_{zx5}k^5, \tag{A19}$$

and the same denominator as in Eq. (A16). The coefficients in the numerator are given by

$$N_{zx0} = N_{xz0}, \quad N_{zx1} = N_{xz1},$$

$$N_{zx2} = 2\xi^2(1-\xi^4)[(1+\xi^8\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(-2+z_1+z_2) + (\xi^8+\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(2+z_1+z_2) - 2\xi^2(\xi_1^2+\xi^4\xi_2^2)(2+6z_1 + 2z_2 + z_1^2 - 2z_1z_2 - z_2^2) + 2\xi^2(\xi^4\xi_1^2+\xi_2^2)(2-14z_1+6z_2+z_1^2-2z_1z_2-z_2^2) + 2\xi^4(1+\xi_1^2\xi_2^2)(17z_1-7z_2-z_1^2z_2-z_1z_2^2)],$$

$$N_{zx3} = 8\xi^4[(\xi_1^2+\xi^8\xi_2^2)(1+z_1)(1-z_2)(-4-z_1+z_2) - 2\xi^2(1+\xi^4\xi_1^2\xi_2^2)(4-6z_1-2z_2-z_1^2+4z_1z_2+z_2^2) + 2\xi^2(\xi^4+\xi_1^2\xi_2^2)(4+6z_1 + 2z_2 - z_1^2 + 4z_1z_2 + z_2^2) - 2\xi^4(\xi_1^2+\xi_2^2)(7z_1+9z_2-2z_1^2+4z_1z_2-2z_2^2+z_1^2z_2-z_1z_2^2) + (\xi^8\xi_1^2+\xi_2^2)(4-5z_1+5z_2-3z_1^2 + 2z_1z_2 - 3z_2^2 + z_1^2z_2 - z_1z_2^2)], \tag{A20}$$

$$N_{xz4} = 32\xi^6[(\xi^4-\xi_1^2\xi_2^2)(1+z_1)(1+z_2)(-2+z_1+z_2) - (1-\xi^4\xi_1^2\xi_2^2)(1-z_1)(1-z_2)(2+z_1+z_2) - 2\xi^2(\xi_1^2-\xi_2^2)(2+z_1-2z_2 - z_1^2 - 2z_1z_2 + z_2^2)],$$

$$N_{zx5} = N_{xz5}.$$

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