Chapter 6

Fuzzy Set Theory

Problems in the real world quite often turn out to be complex owing to an element of uncertainty either in the parameters which define the problem or in the situations in which the problem occurs.

Although probability theory has been an age old and effective tool to handle uncertainty, it can be applied only to situations whose characteristics are based on random processes, that is, processes in which the occurrence of events is strictly determined by chance. However, in reality, there turn out to be problems, a large class of them whose uncertainty is characterized by a nonrandom process. Here, the uncertainty may arise due to partial information about the problem, or due to information which is not fully reliable, or due to inherent imprecision in the language with which the problem is defined, or due to receipt of information from more than one source about the problem which is conflicting.

It is in such situations that *fuzzy set theory* exhibits immense potential for effective solving of the uncertainty in the problem. *Fuzziness* means 'vagueness'. Fuzzy set theory is an excellent mathematical tool to handle the uncertainty arising due to vagueness. Understanding human speech and recognizing handwritten characters are some common instances where fuzziness manifests.

It was Lotfi A. Zadeh who propounded the fuzzy set theory in his seminal paper (Zadeh, 1965). Since then, a lot of theoretical developments have taken place in this field. It is however, the Japanese who seem to have fully exploited the potential of fuzzy sets by commercializing the technology. More than 2000 patents have been acquired by the Japanese in the application of the technique and the area spans a wide spectrum, from consumer products and electronic instruments to automobile and traffic monitoring systems.

6.1 FUZZY VERSUS CRISP

Consider the query, "Is water colourless?" The answer to this is a definite *Yes/True*, or *No/False*, as warranted by the situation. If "yes"/"true" is accorded a value of 1 and "no"/"false" is accorded a value of 0, this statement results in a 0/1 type of situation. Such a logic which demands a binary (0/1) type of handling is termed *crisp* in the domain of fuzzy set theory. Thus, statements such as "Temperature is 32°C", "The running time of the program is 4 seconds" are examples of crisp situations.

On the other hand, consider the statement, "Is Ram honest?" The answer to this query need not be a definite "yes" or "no". Considering the degree to which one knows Ram, a variety of answers spanning a range, such as "extremely honest", "extremely dishonest", "honest at times", "very honest" could be generated. If, for instance, "extremely honest" were to be accorded a value of 1, at the high end of the spectrum of values, "extremely dishonest" a value of 0 at the low end of the spectrum, then, "honest at times" and "very honest" could be assigned values of 0.4 and 0.85 respectively. The situation is therefore so fluid that it can accept values between 0 and 1, in contrast to the earlier one which was either a 0 or 1. Such a situation is termed *fuzzy*. Figure 6.1 shows a simple diagram to illustrate fuzzy and crisp situations.

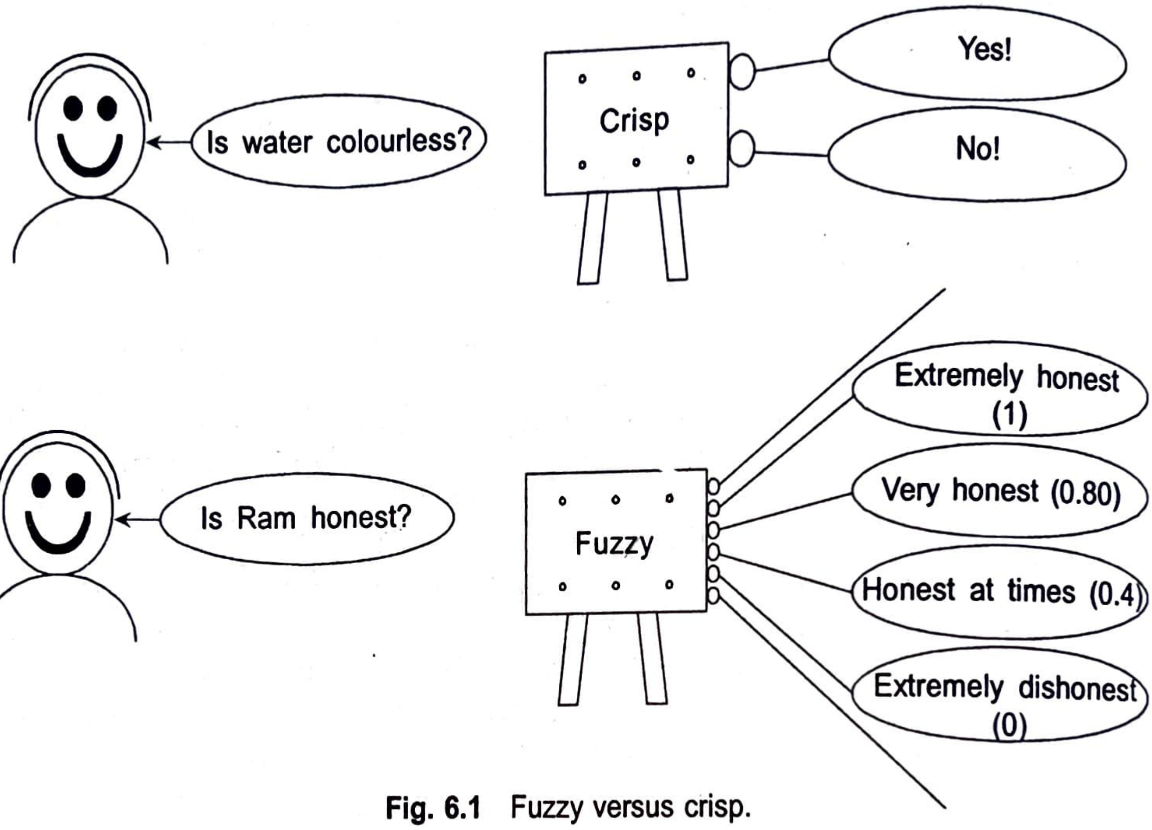

Fig. 6.1 Fuzzy versus crisp.

Classical set theory also termed *crisp set theory* and propounded by George Cantor is fundamental to the study of fuzzy sets. Just as Boolean logic had its roots in the theory of crisp sets, fuzzy logic has its roots in the theory of fuzzy sets (refer Fig. 6.1).

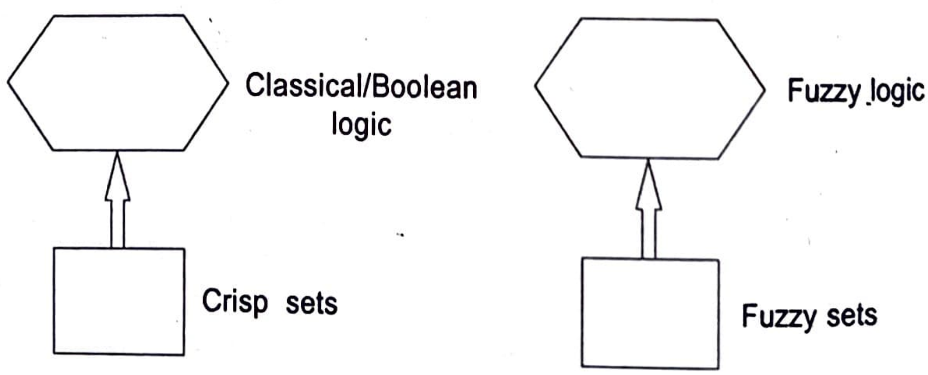

Fig. 6.2 Crisp sets and fuzzy sets.

We now briefly review crisp sets and its operations before a discussion on fuzzy sets is undertaken.

6.2 CRISP SETS

1 *Universe of discourse*

The *universe of discourse* or *universal set* is the set which, with reference to a particular context, contains all possible elements having the same characteristics and from which sets can be formed. The universal set is denoted by E.

Example

(i) The universal set of all numbers in Euclidean space.

(ii) The universal set of all students in a university.

2 *Set*

A set is a *well defined* collection of objects. Here, well defined means the object either belongs to or does not belong to the set (observe the "crispness" in the definition).

A set in certain contexts may be associated with its universal set from which it is derived.

Given a set A whose objects are $a_1, a_2, a_3,..., a_n$, we write A as $A = \{a_1, a_2,..., a_n\}$. Here, $a_1, a_2,..., a_n$ are called the *members* of the set. Such a form of representing a set is known as *list form*.

Example

$A = \{\text{Gandhi, Bose, Nehru}\}$

$B = \{\text{Swan, Peacock, Dove}\}$

A set may also be defined based on the properties the members have to satisfy. In such a case, a set A is defined as

$$A = \{x \mid P(x)\} \tag{6.1}$$

Here, $P(x)$ stands for the property P to be satisfied by the member x. This is read as 'A is the set of all X such that $P(x)$ is satisfied'.

Example

$A = \{x \mid x \text{ is an odd number}\}$

$B = \{y \mid y > 0 \text{ and } y \bmod 5 = 0\}$

3 *Venn diagram*

Venn diagrams are pictorial representations to denote a set. Given a set A defined over a universal set E, the Venn diagram for A and E is as shown in Fig. 6.3.

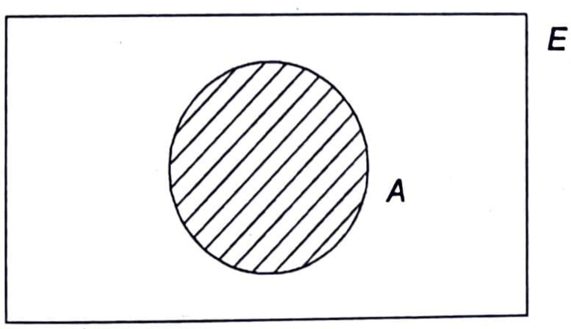

Fig. 6.3 Venn diagram of a set A.

Example

In Fig. 6.3, if E represents the set of university students then A may represent the set of female students.

4 *Membership*

An element x is said to be a member of a set A if x belongs to the set A. The membership is indicated by '$\in$' and is pronounced "belongs to". Thus, $x \in A$ means x belongs to A and $x \notin A$ means x *does not belong to A*.

Example

Given $A = \{4, 5, 6, 7, 8, 10\}$, for $x = 3$ and $y = 4$, we have $x \notin A$ and $y \in A$

Here, observe that each element either belongs to or does not belong to a set. The concept of membership is definite and therefore crisp (1—belongs to, 0—does not belong to). In contrast, as we shall see later, a fuzzy set accommodates membership values which are not only 0 or 1 but *anything* between 0 and 1.

5 *Cardinality*

The number of elements in a set is called its cardinality. Cardinality of a set A is denoted as $n(A)$ or $|A|$ or $\#A$.

Example

If $A = \{4, 5, 6, 7\}$ then $|A| = 4$

6 *Family of sets*

A set whose members are sets themselves, is referred to as a family of sets.

Example

$A = \{\{1, 3, 5\}, \{2, 4, 6\}, \{5, 10\}\}$ is a set whose members are the sets $\{1, 3, 5\}$, $\{2, 4, 6\}$, and $\{5, 10\}$.

7 *Null Set/Empty Set*

A set is said to be a *null set* or *empty set* if it has no members. A null set is indicated as $\emptyset$ or $\{\}$ and indicates an impossible event. Also, $|\emptyset| = 0$.

Example

The set of all prime ministers who are below 15 years of age.

8 *Singleton Set*

A set with a single element is called a *singleton set*. A singleton set has cardinality of 1.

Example

If $A = \{a\}$, then $|A| = 1$

9 *Subset*

Given sets A and B defined over E the universal set, A is said to be a *subset* of B if A is fully contained in B, that is, every element of A is in B.

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Denoted as $A \subset B$, we say that A is a subset of B, or A is a *proper subset* of B. On the other hand, if A is contained in or equivalent to that of B then we denote the subset relation as $A \subseteq B$. In such a case, A is called the *improper* subset of B.

10 *Superset*

Given sets A and B on E the universal set, A is said to be a *superset* of B if every element of B is contained in A.

Denoted as $A \supset B$, we say A is a superset of B or A contains B. If A contains B and is equivalent to B, then we denote it as $A \supseteq B$.

Example

Let $A = \{3, 4\}$ $B = \{3, 4, 5\}$ and $C = \{4, 5, 3\}$

Here, $A \subset B$, and $B \supset A$
$C \subseteq B$, and $B \supseteq C$

11 *Power set*

A *power set* of a set A is the set of all possible subsets that are derivable from A including null set.

A power set is indicated as $P(A)$ and has cardinality of $|P(A)| = 2^{|A|}$

Example

Let $A = \{3, 4, 6, 7\}$

$$P(A) = \{\{3\}, \{4\}, \{6\}, \{7\}, \{3, 4\}, \{4, 6\}, \{6, 7\}, \{3, 7\}, \{3, 6\}, \{4, 7\}, \{3, 4, 6\}, \{4, 6, 7\}, \{3, 6, 7\}, \{3, 4, 7\}, \{3, 4, 6, 7\}, \emptyset\}$$

Here, $|A| = 4$ and $|P(A)| = 2^4 = 16$.

6.2.1 Operations on Crisp Sets

1 *Union* ($\cup$)

The union of two sets A and B ($A \cup B$) is the set of all elements that belong to A or B or both.

$$A \cup B = \{x / x \in A \text{ or } x \in B\} \quad (6.2)$$

Example

Given $A = \{a, b, c, 1, 2\}$ and $B = \{1, 2, 3, a, c\}$, we get $A \cup B = \{a, b, c, 1, 2, 3\}$

Figure 6.4 illustrates the Venn diagram representation for $A \cup B$

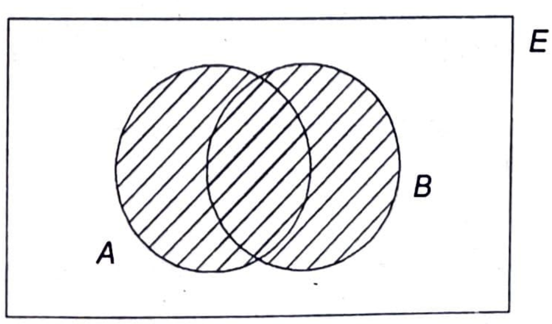


Fig. 6.4 Venn diagram for $A \cup B$.

2 Intersection ($\cap$)

The intersection of two sets A and B ($A \cap B$) is the set of all elements that belong to A and B

$$A \cap B = \{x/x \in A \text{ and } x \in B\} \tag{6.3}$$

Any two sets which have $A \cap B = \varnothing$ are called *Disjoint Sets.*

Example

Given $A = \{a, b, c, 1, 2\}$ and $B = \{1, 2, 3, a, c\}$, we get $A \cap B = \{a, c, 1, 2\}$

Figure 6.5 illustrates the Venn diagram for $A \cap B$

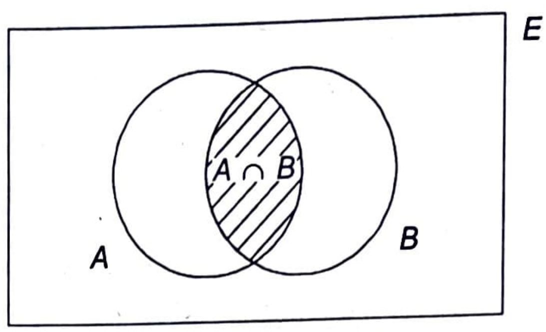


Fig. 6.5 Venn diagram for $A \cap B$.

3 Complement (^{c})

The complement of a set $A\,(\bar{A}\,|\,A^c)$ is the set of all elements which are in E but not in A.

$$A^c = \{x/x \notin A,\ x \in E\} \tag{6.4}$$

Example

Given $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{5, 4, 3\}$, we get $A^c = \{1, 2, 6, 7\}$

Figure 6.6 illustrates the Venn diagram for A^c.

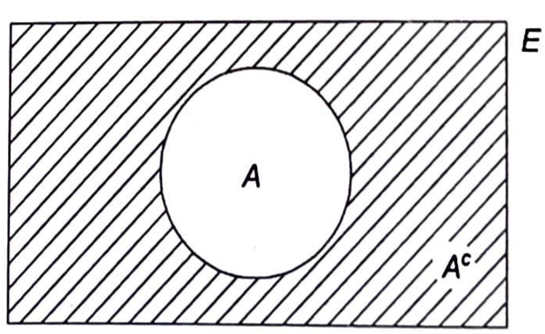


Fig. 6.6 Venn diagram for A^c.

4 Difference (–)

The difference of the set A and B is $A - B$, the set of all elements which are in A but not in B.

$$A - B = \{x \mid x \in A \text{ and } x \notin B\} \tag{6.5}$$

Example

Given $A = \{a, b, c, d, e\}$ and $B = \{b, d\}$, we get $A - B = \{a, c, e\}$

Figure 6.7 illustrates the Venn diagram for $A - B$.

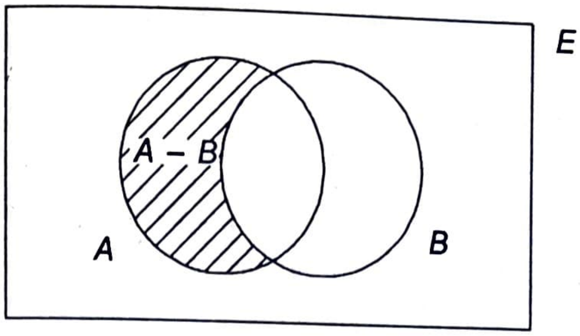


Fig. 6.7 Venn diagram for $A - B$.

6.2.2 Properties of Crisp Sets

The following properties of sets are important for further manipulation of sets.

Commutativity:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A \tag{6.6}$$

Associativity:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C) \tag{6.7}$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{6.8}$$

Idempotence:

$$A \cup A = A$$

$$A \cap A = A \tag{6.9}$$

Identity:

$$A \cup \varnothing = A$$

$$A \cap E = A$$

$$A \cap \varnothing = \varnothing$$

$$A \cup E = E \tag{6.10}$$

Law of Absorption:

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A \tag{6.11}$$

Transitivity: If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$ (6.12)

Involution:

$$(A^c)^c = A \tag{6.13}$$

Law of the Excluded Middle:

$$A \cup A^c = E \tag{6.14}$$

Law of Contradiction:

$$A \cap A^c = \varnothing \tag{6.15}$$

De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c \tag{6.16}$$

All the properties could be verified by means of Venn diagrams.

Example 6.1

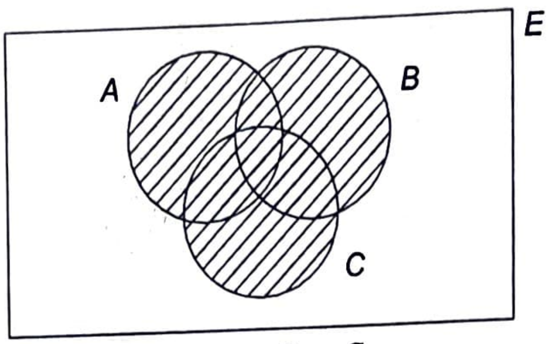
Given three sets A, B, and C. Prove De Morgan's laws using Venn diagrams.

Solution

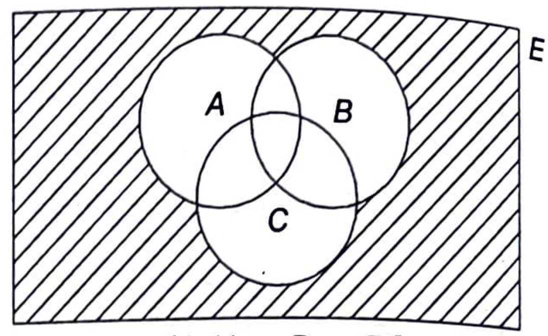
To prove De Morgan's laws, we need to show that

(i) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

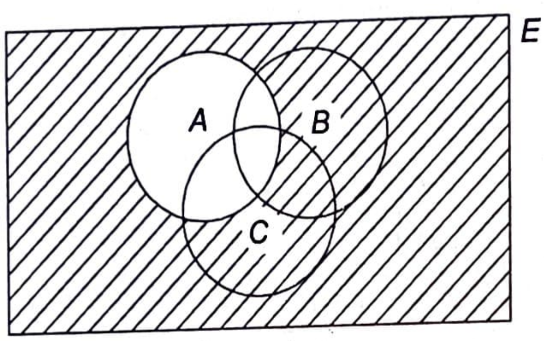
(ii) $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$



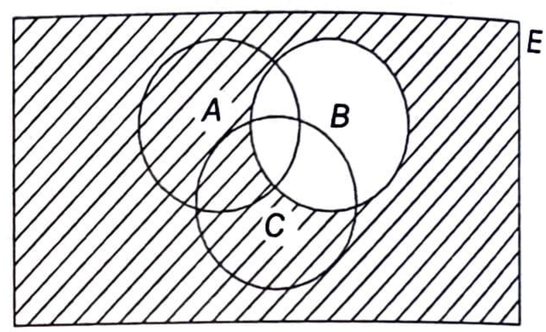
(a) $A \cup B \cup C$



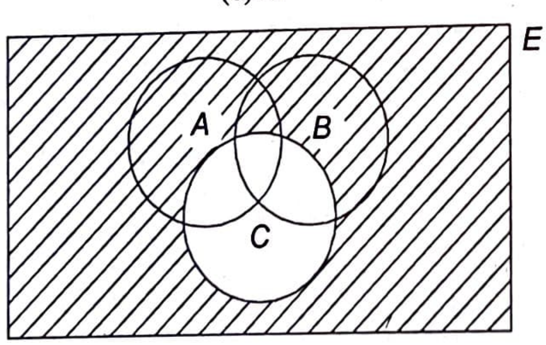
(b) $(A \cup B \cup C)^c$



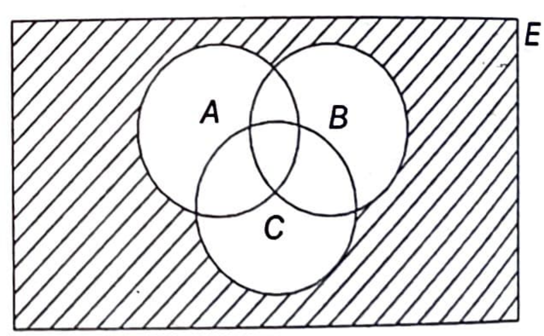
(c) A^c



(d) B^c

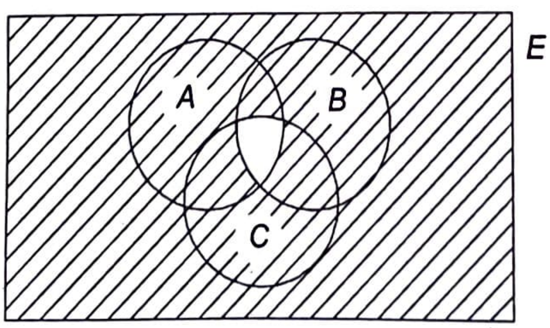


(e) C^c

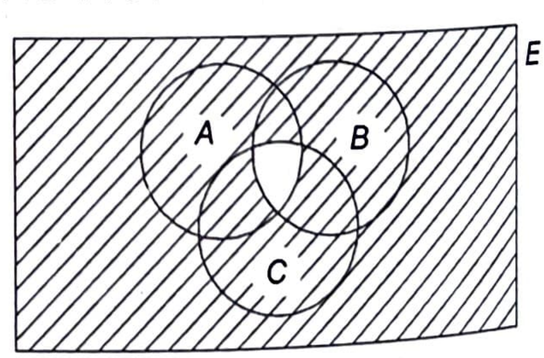


(f) $A^c \cap B^c \cap C^c$

(i) Here, it can be seen that $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$.



(g) $(A \cap B \cap C)^c$



(h) $(A^c \cup B^c \cup C^c)$

(ii) Figures (g) to (h) show that $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$.

Example 6.2

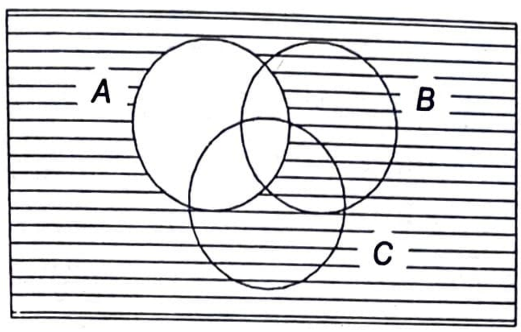
Let the sets A, B, C, and E be given as follows:

E = all students enrolled in the university cricket club.

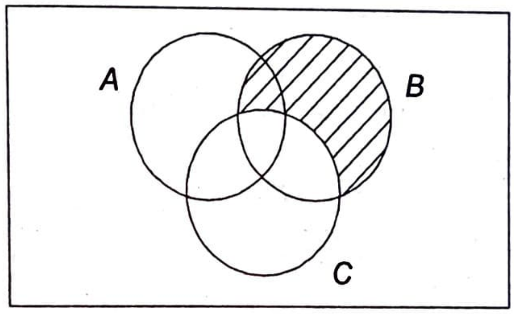
A = male students, B = bowlers, and C = batsmen.

Draw individual Venn diagrams to illustrate (a) female students (b) bowlers who are not batsmen (c) female students who can both bowl and bat.

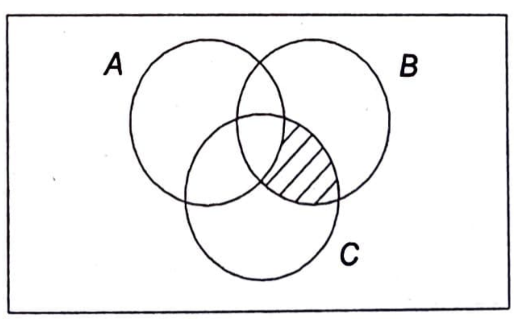
Solution



(a) Female students



(b) Bowlers who are not batsmen



(c) Female students who can both bowl and bat

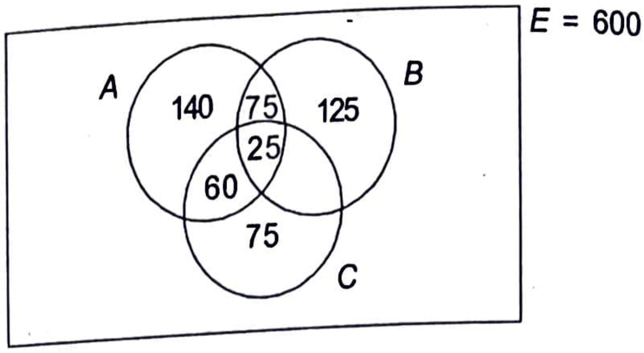
Example 6.3

In Example 6.2, assume that $|E| = 600$, $|A| = 300$, $|B| = 225$, $|C| = 160$. Also, let the number of male students who are bowlers ($A \cap B$) be 100, 25 of whom are batsmen too ($A \cap B \cap C$), and the total number of male students who are batsmen ($A \cap C$) be 85.

Determine the number of students who are: (i) Females, (ii) Not bowlers, (iii) Not batsmen, (iv) Female students who can bowl.

Solution

From the given data, the Venn diagram obtained is as follows:



(i) No. of female students $|A^c| = |E| - |A| = 600 - 300 = 300$

(ii) No. of students who are not bowlers $|B^c| = |E| - |B| = 600 - 225 = 375$

(iii) No. of students who are not batsmen $|C^c| = |E| - |C| = 600 - 160 = 440$

(iv) No. of female students who can bowl $|A^c \cap B| = 125$ (from the Venn diagram)

6.2.3 Partition and Covering

Partition

A *partition* on A is defined to be a set of non-empty subsets A_i, each of which is pairwise disjoint and whose union yields the original set A.

Partition on A indicated as $\Pi(A)$, is therefore

(i) $A_i \cap A_j = \varnothing$ for each pair $(i, j) \in I,\ i \neq j$ (6.17)

(ii) $\bigcup_{i \in I} A_i = A$

The members A_i of the partition are known as blocks (refer Fig. 6.8).

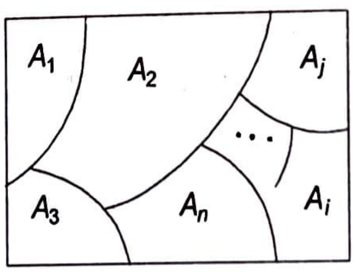


Fig. 6.8 Partition of set A.

Example

Given $A = \{a, b, c, d, e\}$, $A_1 = \{a, b\}$, $A_2 = \{c, d\}$ and $A_3 = \{e\}$, which gives

$$A_1 \cap A_2 = \varnothing,\ A_1 \cap A_3 = \varnothing,\ A_2 \cap A_3 = \varnothing$$

Also,

$$A_1 \cup A_2 \cup A_3 = A = \{a, b, c, d, e\}$$

Hence, $\{A_1, A_2, A_3\}$, is a partition on A.

Covering

A *covering* on A is defined to be a set of non-empty subsets A_i, whose union yields the original set A. The non-empty subsets need not be disjoint (Refer Fig. 6.9).

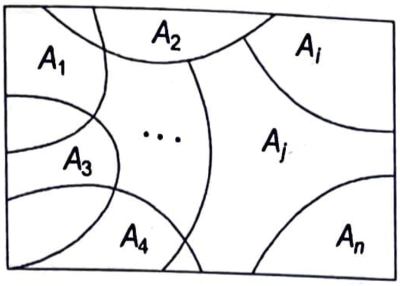


Fig. 6.9 Covering of set A.

Example

Given $A = \{a, b, c, d, e\}$, $A_1 = \{a, b\}$, $A_2 = \{b, c, d\}$, and $A_3 = \{d, e\}$. This gives

$$A_1 \cap A_2 = \{b\}$$

$$A_1 \cap A_3 = \varnothing$$

$$A_2 \cap A_3 = \{d\}$$

Also, $$A_1 \cup A_2 \cup A_3 = \{a, b, c, d, e\} = A$$

Hence, $\{A_1, A_2, A_3\}$ is a covering on A.

Rule of Addition

Given a partition on A where A_i, $i = 1, 2, \ldots, n$ are its non-empty subsets then,

$$|A| = |\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| \tag{6.18}$$

Example

Given $A = \{a, b, c, d, e\}$, $A_1 = \{a, b\}$, $A_2 = \{c, d\}$, $A_3 = \{e\}$, $|A| = 5$, and $\Sigma_{i=1}^{3}|A_i| = 2 + 2 + 1 = 5$

Rule of Inclusion and Exclusion

Rule of addition is not applicable on the covering of set A, especially if the subsets are not pairwise disjoint. In such a case, the rule of inclusion and exclusion is applied.

Example

Given A to be a covering of n sets $A_1, A_2, \ldots, A_n$,

for $n = 2$, $$|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \tag{6.19}$$

for $n = 3$, $$|A| = |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| \tag{6.20}$$

Generalizing,

$$|A| = |\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} |A_i \cap A_j|$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ i \neq j \neq k}}^{n} |A_i \cap A_j \cap A_k| \ldots (-i)^{n+1} \left| \bigcap_{i=1}^{n} A_i \right| \tag{6.21}$$

Example 6.4

Given $|E| = 100$, where E indicates a set of students who have chosen subjects from different streams in the computer science discipline, it is found that 32 study subjects chosen from the Computer Networks (CN) stream, 20 from the Multimedia Technology (MMT) stream, and 45 from the Systems Software (SS) stream. Also, 15 study subjects from both CN and SS streams, 7 from both MMT and SS streams, and 30 do not study any subjects chosen from either of the three streams.

Find the number of students who study subjects belonging to all three streams.

Solution

Let A, B, C indicate students who study subjects chosen from CN, MMT, and SS streams respectively. The problem is to find $|A \cap B \cap C|$.

The no. of students who do not study any subject chosen from either of the three streams = 30.

i.e. $$|A^c \cap B^c \cap C^c| = 30$$

$\Rightarrow$ $$|(A \cup B \cup C)^c| = 30 \quad \text{(using De Morgan's laws)}$$

$\Rightarrow$ $$|E| - |A \cup B \cup C| = 30$$

$\Rightarrow$ $$|A \cup B \cup C| = |E| - 30$$

$$= 100 - 30 = 70$$

From the principle of inclusion and exclusion,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$\Rightarrow$ $$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |A \cap C|$$

$$= 70 - 32 - 20 - 45 + 15 + 7 + 10$$

$$= 5$$

Hence, the no. of students who study subjects chosen from all the three streams is 5.

6.3 FUZZY SETS

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Fuzzy sets support a flexible sense of membership of elements to a set. While in crisp set theory, an element either belongs to or does not belong to a set, in fuzzy set theory many degrees of membership (between 0 and 1) are allowed. Thus, a membership function $\mu_A(x)$ is associated with a

Krijanandh

fuzzy set $\tilde{A}$ such that the function maps every element of the universe of discourse X (or the reference set) to the interval [0, 1].

Formally, the mapping is written as $\mu_{\tilde{A}}(x) : X \to [0, 1]$

A *fuzzy set* is defined as follows:

If X is a universe of discourse and x is a particular element of X, then a fuzzy set A defined on X may be written as a collection of ordered pairs

$$A = \{(x, \mu_{\tilde{A}}(x)), x \in X\} \tag{6.23}$$

where each pair $(x, \mu_{\tilde{A}}(x))$ is called a singleton. In crisp sets, $\mu_{\tilde{A}}(x)$ is dropped.

An alternative definition which indicates a fuzzy set as a union of all $\mu_{\tilde{A}}(x)/x$ singletons is given by

$$A = \sum_{x_i \in X} \mu_{\tilde{A}}(x_i)/x_i \quad \text{in the discrete case} \tag{6.24}$$

and

$$A = \int_X \mu_{\tilde{A}}(x)/x \quad \text{in the continuous case} \tag{6.25}$$

Here, the summation and integration signs indicate the union of all $\mu_{\tilde{A}}(x)/x$ singletons.

Example

Let $X = \{g_1, g_2, g_3, g_4, g_5\}$ be the reference set of students. Let $\tilde{A}$ be the fuzzy set of "smart" students, where "smart" is a fuzzy linguistic term.

$$\tilde{A} = \{(g_1, 0.4)\,(g_2, 0.5)\,(g_3, 1)\,(g_4, 0.9)\,(g_5, 0.8)\}$$

Here $\tilde{A}$ indicates that the smartness of g_1 is 0.4, g_2 is 0.5 and so on when graded over a scale of 0–1.

Though fuzzy sets model vagueness, it needs to be realized that the definition of the sets varies according to the context in which it is used. Thus, the fuzzy linguistic term "tall" could have one kind of fuzzy set while referring to the height of a building and another kind of fuzzy set while referring to the height of human beings.

6.3.1 Membership Function

The membership function values need not always be described by discrete values. Quite often, these turn out to be as described by a continuous function.

The fuzzy membership function for the fuzzy linguistic term "cool" relating to temperature may turn out to be as illustrated in Fig. 6.10.

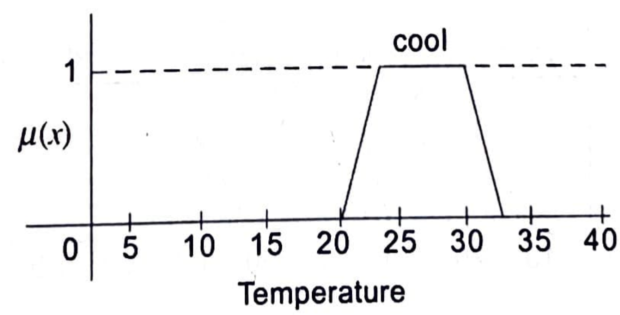


Fig. 6.10 Continuous membership function for "cool".

A membership function can also be given mathematically as

$$\mu_{\tilde{A}}(x) = \frac{1}{(1+x)^2}$$

The graph is as shown in Fig. 6.11.

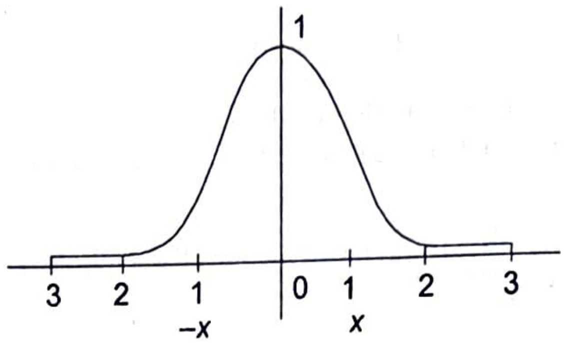


Fig. 6.11 Continuous membership function dictated by a mathematical function.

Different shapes of membership functions exist. The shapes could be triangular, trapezoidal, curved or their variations as shown in Fig. 6.12.

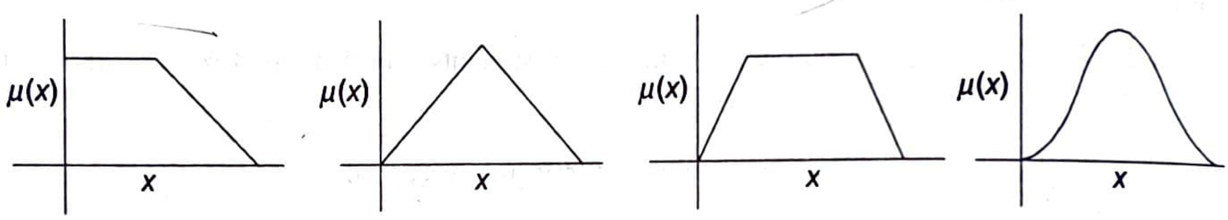


Fig. 6.12 Different shapes of membership function graphs.

Example

Consider the set of people in the following age groups

0–10	40–50
10–20	50–60
20–30	60–70
30–40	70 and above

The fuzzy sets "young", "middle-aged", and "old" are represented by the membership function graphs as illustrated in Fig. 6.13.

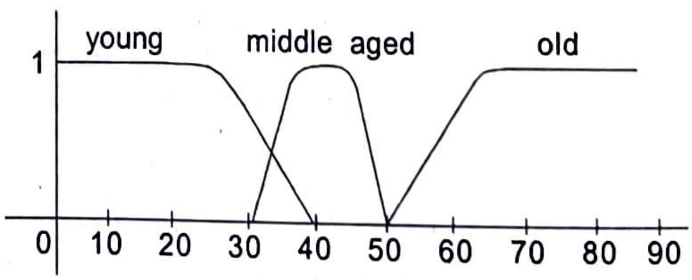


Fig. 6.13 Example of fuzzy sets expressing "young", "middle-aged", and "old".

6.3.2 Basic Fuzzy Set Operations

Given X to be the universe of discourse and $\tilde{A}$ and $\tilde{B}$ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ as their respective membership functions, the basic fuzzy set operations are as follows:

Union

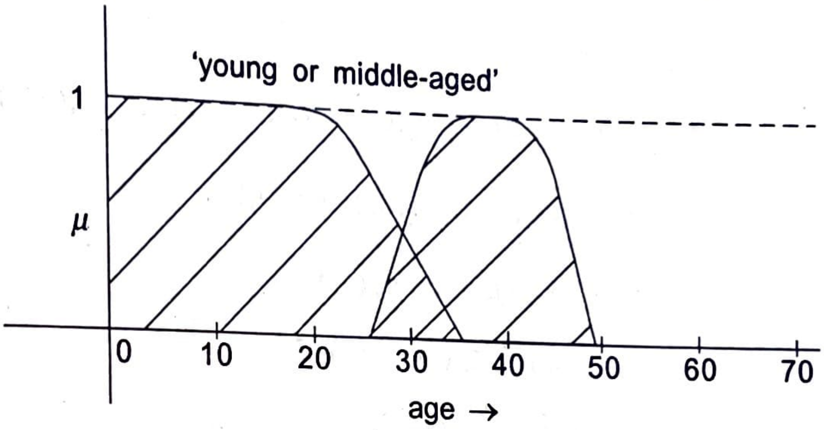
The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cup \tilde{B}$ also on X with a membership function defined as

$$\boxed{\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))} \tag{6.26}$$

Example

Let $\tilde{A}$ be the fuzzy set of young people and $\tilde{B}$ be the fuzzy set of middle-aged people as illustrated in Fig. 6.13. Now $\tilde{A} \cup \tilde{B}$, the fuzzy set of "young or middle-aged" will be given by

$\tilde{A} \cup \tilde{B}$:



In its discrete form, for x_1, x_2, x_3

if
$$A = \{(x_1,0.5), (x_2,0.7), (x_3,0)\} \text{ and } \tilde{B} = \{(x_1,0.8), (x_2,0.2), (x_3,1)\}$$

$$\tilde{A} \cup \tilde{B} = \{(x_1,0.8), (x_2,0.7), (x_3,1)\}$$

since,
$$\mu_{\tilde{A} \cup \tilde{B}}(x_1) = \max(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(x_1))$$
$$= \max(0.5, 0.8)$$
$$= 0.8$$

$$\mu_{\tilde{A} \cup \tilde{B}}(x_2) = \max(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2)) = \max(0.2,0.7) = 0.7$$

$$\mu_{\tilde{A} \cup \tilde{B}}(x_3) = \max(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3)) = \max(0,1) = 1$$

Intersection

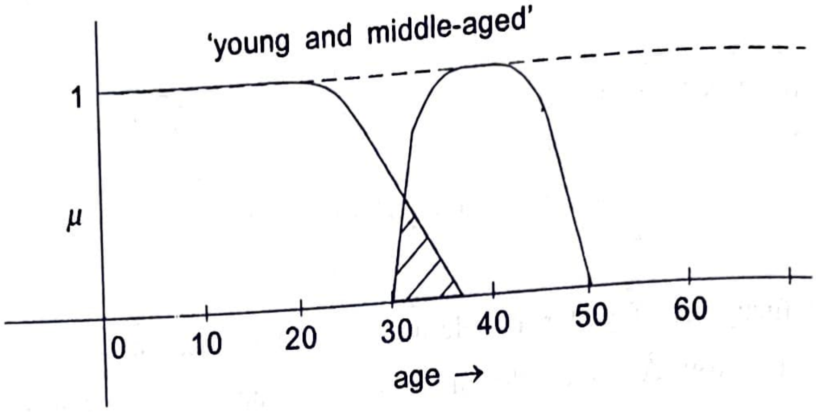
The intersection of fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cap \tilde{B}$ with membership function defined as

$$\boxed{\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))} \tag{6.27}$$

Example

For $\tilde{A}$ and $\tilde{B}$ defined as "young" and "middle-aged" as illustrated in previous examples.

$\tilde{A} \cap \tilde{B}$:



In its discrete form, for x_1, x_2, x_3

if $$\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\} \text{ and } \tilde{B} = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$$

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}$$

since,
$$\begin{aligned}
\mu_{\tilde{A} \cap \tilde{B}}(x_1) &= \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(x_1)) \\
&= \min(0.5, 0.8) \\
&= 0.5 \\
\mu_{\tilde{A} \cap \tilde{B}}(x_2) &= \min(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2)) \\
&= \min(0.7, 0.2) \\
&= 0.2 \\
\mu_{\tilde{A} \cap \tilde{B}}(x_3) &= \min(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3)) \\
&= \min(0, 1) \\
&= 0
\end{aligned}$$

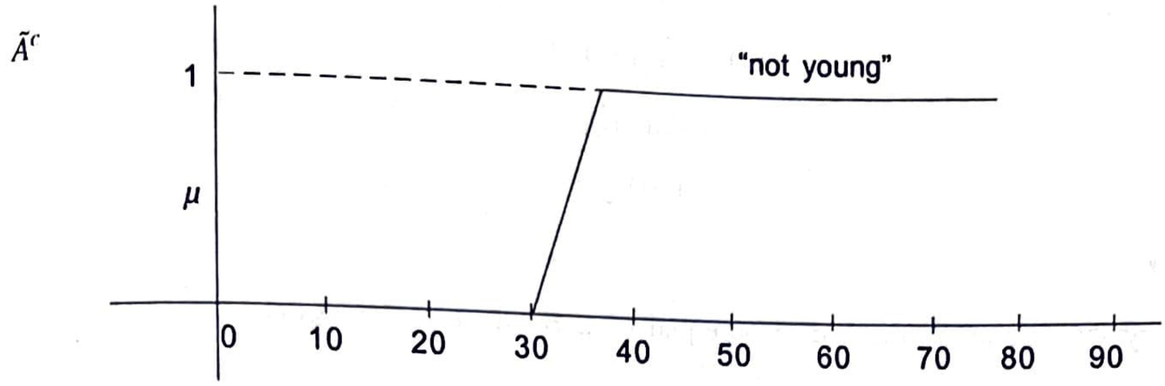
Complement

The complement of a fuzzy set $\tilde{A}$ is a new fuzzy set $\tilde{A}$ with a membership function

$$\boxed{\mu_{\bar{\tilde{A}}}(x) = 1 - \mu_{\tilde{A}}(x)} \tag{6.28}$$

Example

For the fuzzy set $\tilde{A}$ defined as "young" the complement "not young" is given by $\tilde{A}^c$. In its discrete form, for x_1, x_2, and x_3



if $\tilde{A} = \{(x_1, 0.5)\ (x_2, 0.7)\ (x_3, 0)\}$

then, $\tilde{A}^c = \{(x_1, 0.5)\ (x_2, 0.3)\ (x_3, 1)\}$

since,

$$\mu_{\tilde{A}^c}(x_1) = 1 - \mu_{\tilde{A}}(x_1)$$
$$= 1 - 0.5$$
$$= 0.5$$

$$\mu_{\tilde{A}^c}(x_2) = 1 - \mu_{\tilde{A}}(x_2)$$
$$= 1 - 0.7$$
$$= 0.3$$

$$\mu_{\tilde{A}^c}(x_3) = 1 - \mu_{\tilde{A}}(x_3)$$
$$= 1 - 0$$
$$= 1$$

Other operations are,

Product of two fuzzy sets

The product of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cdot \tilde{B}$ whose membership function is defined as

$$\mu_{\tilde{A}\cdot\tilde{B}}(x) = \mu_{\tilde{A}}(x)\, \mu_{\tilde{B}}(x) \tag{6.29}$$

Example

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$

$$\tilde{B} = \{(x_1, 0.4)\ (x_2, 0), (x_3, 0.1)\}$$

$$\tilde{A} \cdot \tilde{B} = \{(x_1, 0.08)\ (x_2, 0)\ (x_3, 0.04)\}$$

Since

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_1) = \mu_{\tilde{A}}(x_1)\cdot\mu_{\tilde{A}}(x_2)$$
$$= 0.2 \cdot 0.4 = 0.08$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_2) = \mu_{\tilde{A}}(x_2)\cdot\mu_{\tilde{A}}(x_2)$$
$$= 0.8 \cdot 0 = 0$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_3) = \mu_{\tilde{A}}(x_3)\cdot\mu_{\tilde{A}}(x_3)$$
$$= 0.4 \cdot 0.1$$
$$= 0.04$$

Equality

Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are said to be equal ($\tilde{A} = \tilde{B}$) if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ (6.30)

Example

$$\tilde{A} = \{(x_1,0.2)(x_2,0.8)\}$$
$$\tilde{B} = \{(x_1,0.6)(x_2,0.8)\}$$
$$\tilde{C} = \{(x_1,0.2)(x_2,0.8)\}$$
$$\tilde{A} \neq \tilde{B}$$

since $\mu_{\tilde{A}}(x_1) \neq \mu_{\tilde{B}}(x_1)$ although

$$\mu_{\tilde{A}}(x_2) = \mu_{\tilde{B}}(x_2)$$

but $\tilde{A} = \tilde{C}$

since $\mu_{\tilde{A}}(x_1) = \mu_{\tilde{C}}(x_1) = 0.2$

and $\mu_{\tilde{A}}(x_2) = \mu_{\tilde{C}}(x_2) = 0.8$

Product of a fuzzy set with a crisp number

Multiplying a fuzzy set $\tilde{A}$ by a crisp number a results in a new fuzzy set product a. $\tilde{A}$ with the membership function

$$\mu_{a\cdot\tilde{A}}(x) = a\cdot\mu_{\tilde{A}}(x) \qquad (6.31)$$

Example

$$\tilde{A} = \{(x_1,0.4),(x_2,0.6),(x_3,0.8)\}$$

For $a = 0.3$

$$a\cdot\tilde{A} = \{(x_1,0.12),(x_{2,}0.18),(x_3,0.24)\}$$

since,

$$\mu_{a\cdot\tilde{A}}(x_1) = a\cdot\mu_{\tilde{A}}(x_1)$$
$$= 0.3 \cdot 0.4$$
$$= 0.12$$

$$\mu_{a\cdot\tilde{A}}(x_2) = a\cdot\mu_{\tilde{A}}(x_2)$$
$$= 0.3 \cdot 0.6$$
$$= 0.18$$

$$\mu_{a\cdot\tilde{A}}(x_3) = a\cdot\mu_{\tilde{A}}(x_3)$$
$$= 0.3 \cdot 0.8$$
$$= 0.24$$

Power of a fuzzy set

The α power of a fuzzy set $\tilde{A}$ is a new fuzzy set A^{α} whose membership function is given by

$$\mu_{A^{\alpha}}(x) = (\mu_{\tilde{A}}(x))^{\alpha} \tag{6.32}$$

Raising a fuzzy set to its second power is called *Concentration* (CON) and taking the square root is called *Dilation* (DIL).

Example

$$\tilde{A} = \{(x_1,0.4),(x_2,0.2),(x_3,0.7)\}$$

For $\alpha = 2$

$$\mu_{\tilde{A}^2}(x) = (\mu_{\tilde{A}}(x))^2$$

Hence, $(\tilde{A})^2 = \{(x_1,0.16),(x_2,0.04),(x_3,0.49)\}$

Since
$$\mu_{\tilde{A}^2}(x_1) = (\mu_{\tilde{A}}(x_1))^2 = (0.4)^2 = 0.16$$
$$\mu_{\tilde{A}^2}(x_2) = (\mu_{\tilde{A}}(x_2))^2 = (0.2)^2 = 0.04$$
$$\mu_{\tilde{A}^2}(x_3) = (\mu_{\tilde{A}}(x_3))^2 = (0.7)^2 = 0.49$$

Difference

The difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A}-\tilde{B}$ defined as

$$\boxed{\tilde{A}-\tilde{B} = (\tilde{A}\cap\tilde{B}^c)} \tag{6.33}$$

Example

$$\tilde{A} = \{(x_1,0.2),(x_2,0.5),(x_3,0.6)\};\ \tilde{B} = \{(x_1,0.1),(x_2,0.4),(x_3,0.5)\}$$
$$\tilde{B}^c = \{(x_1,0.9),(x_2,0.6),(x_3,0.5)\}$$
$$\tilde{A}-\tilde{B} = \tilde{A}\cap\tilde{B}^c$$
$$= \{(x_1,0.2)(x_2,0.5)(x_3,0.5)\}$$

Disjunctive sum

The disjunctive sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A}\oplus\tilde{B}$ defined as

$$\boxed{\tilde{A}\oplus\tilde{B} = (\tilde{A}^c\cap\tilde{B})\cup(\tilde{A}\cap\tilde{B}^c)} \tag{6.34}$$

Example

$$\tilde{A} = \{(x_1,0.4)(x_2,0.8)(x_3,0.6)\}$$

$$\tilde{B} = \{(x_1,0.2)(x_2,0.6)(x_3,0.9)\}$$

Now,

$$\tilde{A}^c = \{(x_1,0.6)(x_2,0.2)(x_3,0.4)\}$$

$$\tilde{B}^c = \{(x_1,0.8)(x_2,0.4)(x_3,0.1)\}$$

$$\tilde{A}^c \cap \tilde{B} = \{(x_1,0.2)(x_2,0.2)(x_3,0.4)\}$$

$$\tilde{A} \cap \tilde{B}^c = \{(x_1,0.4)(x_2,0.4)(x_3,0.1)\}$$

$$\tilde{A} \oplus \tilde{B} = \{(x_1,0.4)(x_2,0.4)(x_3,0.4)\}$$

6.3.3 Properties of Fuzzy Sets

Fuzzy sets follow some of the properties satisfied by crisp sets. In fact, crisp sets can be thought of as special instances of fuzzy sets. Any fuzzy set $\tilde{A}$ is a subset of the reference set X. Also, the membership of any element belonging to the null set $\varnothing$ is 0 and the membership of any element belonging to the reference set is 1.

The properties satisfied by fuzzy sets are

Commutativity:

$$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$$

$$\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A} \tag{6.35}$$

Associativity:

$$\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$$

$$\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C} \tag{6.36}$$

Distributivity:

$$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$$

$$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C}) \tag{6.37}$$

Idempotence:

$$\tilde{A} \cup \tilde{A} = \tilde{A}$$

$$\tilde{A} \cap \tilde{A} = \tilde{A} \tag{6.38}$$

Identity:

$$\tilde{A} \cup \varnothing = \tilde{A}$$

$$\tilde{A} \cup X = \tilde{A}$$

$$\tilde{A} \cap \varnothing = \varnothing$$

$$\tilde{A} \cup X = X \tag{6.39}$$

Transitivity: If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $\tilde{A} \subseteq \tilde{C}$ (6.40)

Involution:

$$(\tilde{A}^c)^c = \tilde{A} \tag{6.41}$$

De Morgan's laws:

$$(\tilde{A} \cap \tilde{B})^c = (\tilde{A}^c \cup \tilde{B}^c)$$

$$(\tilde{A} \cup \tilde{B})^c = (\tilde{A}^c \cap \tilde{B}^c) \tag{6.42}$$

Since fuzzy sets can overlap, the laws of excluded middle do not hold good. Thus,

$$\tilde{A} \cup \tilde{A}^c \neq X \tag{6.43}$$

$$\tilde{A} \cap \tilde{A}^c \neq \varnothing \tag{6.44}$$

Example 6.5

The task is to recognize English alphabetical characters (F, E, X, Y, I, T) in an image processing system.

Define two fuzzy sets $\tilde{I}$ and $\tilde{F}$ to represent the identification of characters I and F.

$$\tilde{I} = \{(F, 0.4), (E, 0.3), (X, 0.1), (Y, 0.1), (I, 0.9), (T, 0.8)\}$$

$$\tilde{F} = \{(F, 0.99), (E, 0.8), (X, 0.1), (Y, 0.2), (I, 0.5), (T, 0.5)\}$$

Find the following.

(a) (i) $\tilde{I} \cup \tilde{F}$ (ii) $(\tilde{I} - \tilde{F})$ (iii) $\tilde{F} \cup \tilde{F}^c$

(b) Verify De Morgan's Law, $(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$

Solution

(a) (i)

$$\tilde{I} \cup \tilde{F} = \{(F,0.99), (E,0.8), (X,0.1), (Y,0.2), (I,0.9), (T,0.8)\}$$

(ii)

$$\tilde{I} - \tilde{F} = (\tilde{I} \cap \tilde{F}^c)$$

$$= \{(F,0.01), (E,0.2), (X,0.1), (Y,0.1), (I,0.5), (T,0.5)\}$$

(iii)

$$\tilde{F} \cup \tilde{F}^c = \{(F,0.99), (E,0.8), (X,0.9), (Y,0.8), (I,0.5), (T,0.5)\}$$

(b) De Morgan's Law

$$(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$$

$$\tilde{I} \cup \tilde{F} = \{(F,0.99), (E,0.8), (X,0.1), (Y,0.2), (I,0.9), (T,0.8)\}$$

$$(\tilde{I} \cup \tilde{F})^c = \{(F,0.01), (E,0.2), (X,0.9), (Y,0.8), (I,0.1), (T,0.2)\}$$

$$\tilde{I}^c = \{(F,0.6), (E,0.7), (X,0.9), (Y,0.9), (I,0.1), (T,0.2)\}$$

$$\tilde{F}^c = \{(F,0.01), (E,0.2), (X,0.9), (Y,0.8), (I,0.5), (T,0.5)\}$$

and

$$\tilde{I}^c \cap \tilde{F}^c = \{(F,0.01), (E,0.2), (X,0.9), (Y,0.8), (I,0.1), (T,0.2)\}$$

Hence,

$$(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$$

Example 6.6

Consider the fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined on the interval $X = [0, 5]$ of real numbers, by the membership grade functions

$$\mu_{\tilde{A}}(x) = \frac{x}{x+1}, \quad \mu_{\tilde{B}}(x) = 2^{-x}$$

Determine the mathematical formulae and graphs of the membership grade functions of each of the following sets

(a) A^c, B^c

(b) $A \cup B$

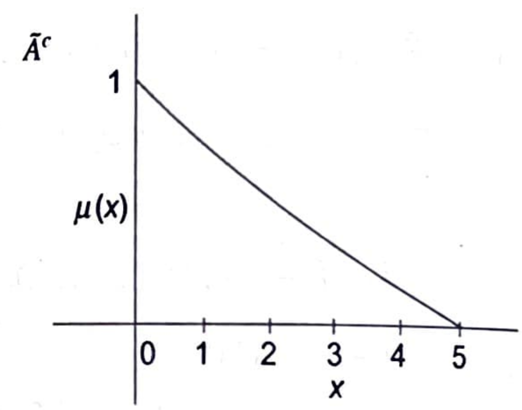
(c) $A \cap B$

(d) $(A \cup B)^c$

Solution

(a) $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) = 1 - \dfrac{x}{x+1}$

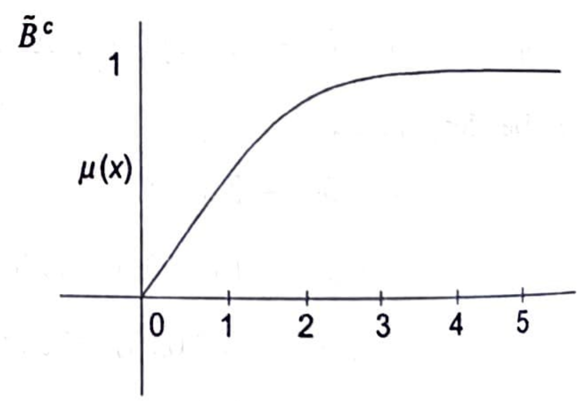
$$= \frac{1}{x+1}$$



$\mu_{\tilde{B}^c}(x) = 1 - \mu_{\tilde{B}}(x)$

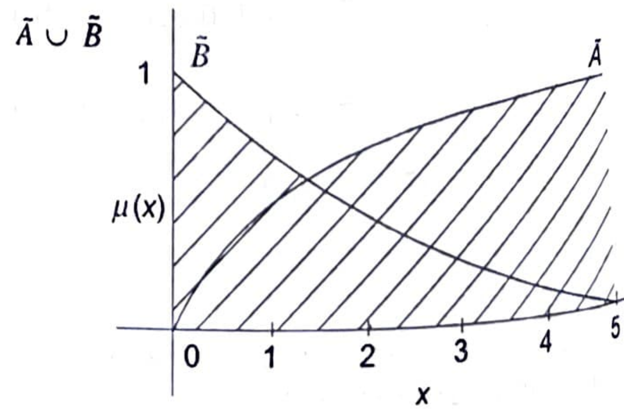
$= 1 - 2^{-x}$

$$= \frac{2^x - 1}{2^x}$$



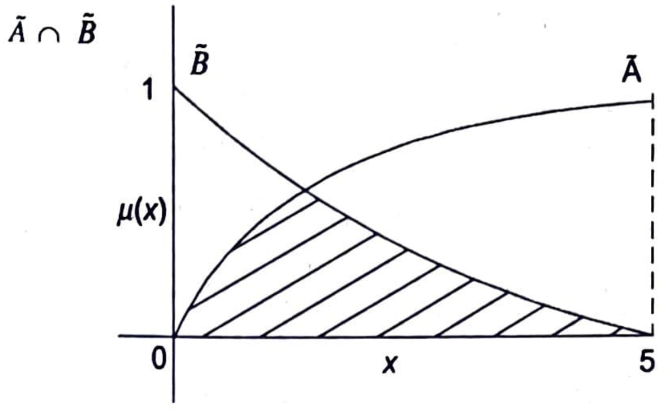
(b) $\mu_{\tilde{A} \cup \tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

$$= \max\left(\frac{x}{x+1}, 2^{-x}\right)$$



(c) $\mu_{\tilde{A}\cap\tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

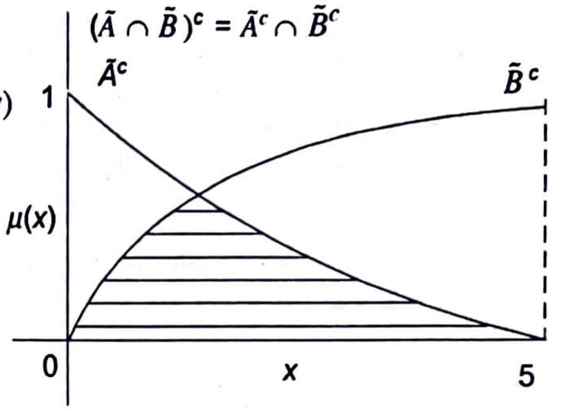
$$= \min\left(\frac{x}{x+1}, 2^{-x}\right)$$



(d) $\mu_{(\tilde{A}\cup\tilde{B})^c}(x) = \mu_{\tilde{A}^c\cap\tilde{B}^c}(x)$ (Θ De Morgan's law)

$$= \min(\mu_{\tilde{A}^c}(x),\ \mu_{\tilde{B}^c}(x))$$

$$= \min\left(\frac{1}{x+1}, \frac{2^x-1}{2^x}\right)$$



6.4 CRISP RELATIONS

In this section, we review crisp relations as a prelude to fuzzy relations. The concept of relations between sets is built on the Cartesian product operator of sets.

6.4.1 Cartesian Product

The *Cartesian product* of two sets A and B denoted by $A \times B$ is the set of all ordered pairs such that the first element in the pair belongs to A and the second element belongs to B.

i.e. $$A \times B = \{(a,b)/a \in A,\ b \in B\}$$

If $A \neq B$ and A and B are non-empty then $A \times B \neq B \times A$.

The Cartesian product could be extended to n number of sets

$$\mathop{\times}_{i=1}^{n} A_i = \{(a_1, a_2, a_3, \ldots, a_n)/a_i \in A_i \text{ for every } i = 1, 2, \ldots, n\} \tag{6.45}$$

Observe that $$\left|\mathop{\times}_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i| \tag{6.46}$$

Example

Given $A_1 = \{a, b\}, A_2 = \{1, 2\}, A_3 = \{\alpha\}$,

$$A_1 \times A_2 = \{(a, 1), (b, 1), (a, 2), (b, 2)\}, \; |A_1 \times A_2| = 4, \quad \text{and} \quad |A_1| = |A_2| = 2$$

Here, $$|A_1 \times A_2| = |A_1| \cdot |A_2|$$

Also, $$A_1 \times A_2 \times A_3 = \{(a, 1, \alpha), (a, 2, \alpha), (b, 1, \alpha), (b, 2, \alpha)\}$$

$$|A_1 \times A_2 \times A_3| = 4 = |A_1| \cdot |A_2| \cdot |A_3|$$

6.4.2 Other Crisp Relations

An *n-ary* relation denoted as $R(X_1, X_2, \ldots, X_n)$ among crisp sets $X_1, X_2, \ldots, X_n$ is a subset of the Cartesian product $\underset{i=1}{\overset{n}{\times}} X_i$ and is indicative of an association or relation among the tuple elements.

For $n = 2$, the relation $R(X_1, X_2)$ is termed as a *binary* relation; for $n = 3$, the relation is termed *ternary*; for $n = 4$, *quarternary*; for $n = 5$, *quinary* and so on.

If the universe of discourse or sets are finite, the *n-ary* relation can be expressed as an *n*-dimensional *relation matrix*. Thus, for a binary relation $R(X, Y)$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$, the relation matrix R is a two dimensional matrix where X represents the rows, Y represents the columns and $R(i, j) = 1$ if $(x_i, y_j) \in R$ and $R(i, j) = 0$ if $(x_i, y_j) \notin R$.

Example

Given $X = \{1, 2, 3, 4\}$,

$$X \times X = \begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4) \\ (3,1)(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(4,4) \end{Bmatrix}$$

Let the relation R be defined as

$$R = \{(x, y) / y = x + 1, \; x, y \in X\}$$

$$R = \{(1, 2)(2, 3)(3, 4)\}$$

The relation matrix R is given by

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.4.3 Operations on Relations

Given two relations R and S defined on $X \times Y$ and represented by relation matrices, the following operations are supported by R and S

Union: $R \cup S$

$$R \cup S(x, y) = \max (R(x, y), S(x, y)) \tag{6.47}$$

Intersection: $R \cap S$

$$R \cap S(x, y) = \min (R(x, y), S(x, y)) \tag{6.48}$$

Complement: $\bar{R}$

$$\bar{R}(x, y) = 1 - R(x, y) \tag{6.49}$$

Composition of relations: $R \circ S$

Given R to be a relation on X, Y and S to be a relation on Y, Z then $R \circ S$ is a composition of relation on X, Z defined as

$$R \circ S = \{(x, z)/(x, z) \in X \times Z,\ \exists\ y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\} \tag{6.50}$$

A common form of the composition relation is the *max-min composition.*

Max-min composition:

Given the relation matrices of the relation R and S, the max-min composition is defined as

For $$T = R \circ S$$

$$T(x, z) = \max_{y \in Y} (\min(R(x, y), S(y, z))) \tag{6.51}$$

min-max composin

Example

Let R, S be defined on the sets $\{1, 3, 5\} \times \{1, 3, 5\}$ 4

Let $$R: \{(x, y) \mid y = x + 2\},\quad S: \{(x, y) \mid x < y\}$$

$$R = \{(1, 3)(3, 5)\},\quad S = \{(1, 3)(1, 5)\ (3, 5)\}$$

The relation matrices are

$$R: \begin{array}{c|ccc} & 1 & 3 & 5 \\ \hline 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 \end{array} \qquad 4 \qquad S: \begin{array}{c|ccc} & 1 & 3 & 5 \\ \hline 1 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 \end{array}$$

Using max-min composition

$$R \circ S = \begin{array}{c|ccc} & 1 & 3 & 5 \\ \hline 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{array}$$

since

$$R \circ S\,(1,\,1) = \max\{\min\,(0,\,0),\,\min(1,\,0),\,\min(0,\,0)\}$$
$$= \max\,(0,\,0,\,0) = 0.$$
$$R \circ S\,(1,\,3) = \max\{0,\,0,\,0\} = 0$$
$$R \circ S\,(1,\,5) = \max\{0,\,1,\,0\} = 1.$$

Similarly,

$$R \circ S\,(3,\,1) = 0.$$
$$R \circ S\,(3,\,3) = R \circ S\,(3,\,5) = R \circ S\,(5,\,1) = R \circ S\,(5,\,3) = R \circ S\,(5,\,5) = 0$$

$R \circ S$ from the relation matrix is $\{(1,\,5)\}$.

Also,

$$S \circ R = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.5 FUZZY RELATIONS

Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets $X_1, X_2,\ldots, X_n$ where the n-tuples $(x_1,\, x_2,\ldots,\, x_n)$ may have varying degrees of membership within the relation. The membership values indicate the strength of the relation between the tuples.

Example

Let R be the fuzzy relation between two sets X_1 and X_2 where X_1 is the set of diseases and X_2 is the set of symptoms.

X_1 = {typhoid, viral fever, common cold}

X_2 = {running nose, high temperature, shivering}

The fuzzy relation R may be defined as

	Running nose	*High temperature*	*Shivering*
Typhoid	0.1	0.9	0.8
Viral fever	0.2	0.9	0.7
Common cold	0.9	0.4	0.6

6.5.1 Fuzzy Cartesian Product

Let $\tilde{A}$ be a fuzzy set defined on the universe X and $\tilde{B}$ be a fuzzy set defined on the universe Y, the Cartesian product between the fuzzy sets $\tilde{A}$ and $\tilde{B}$ indicated as $\tilde{A} \times \tilde{B}$ and resulting in a fuzzy relation $\tilde{R}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} \subset X \times Y \tag{6.52}$$

where $\tilde{R}$ has its membership function given by

$$\begin{aligned} \mu_{\tilde{R}}(x, y) &= \mu_{\tilde{A}\times\tilde{B}}(x, y) \\ &= \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \end{aligned} \tag{6.53}$$

← cartesian product

Example

Let $\tilde{A} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.4)\}$ and $\tilde{B} = \{(y_1, 0.5), (y_2, 0.6)\}$ be two fuzzy sets defined on the universes of discourse $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ respectively. Then the fuzzy relation $\tilde{R}$ resulting out of the fuzzy Cartesian product $\tilde{A} \times \tilde{B}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

since,

$$\begin{aligned} \tilde{R}(x_1, y_1) &= \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(y_1)) = \min(0.2, 0.5) = 0.2 \\ \tilde{R}(x_1, y_2) &= \min(0.2, 0.6) = 0.2 \\ \tilde{R}(x_2, y_1) &= \min(0.7, 0.5) = 0.5 \\ \tilde{R}(x_2, y_2) &= \min(0.7, 0.6) = 0.6 \\ \tilde{R}(x_3, y_1) &= \min(0.4, 0.5) = 0.4 \\ \tilde{R}(x_3, y_2) &= \min(0.4, 0.6) = 0.4 \end{aligned}$$

6.5.2 Operations on Fuzzy Relations

Let $\tilde{R}$ and $\tilde{S}$ be fuzzy relations on $X \times Y$.

Union

$$\mu_{\tilde{R}\cup\tilde{S}}(x, y) = \max(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y)) \tag{6.54}$$

Intersection

$$\mu_{\tilde{R}\cap\tilde{S}}(x, y) = \min(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y)) \tag{6.55}$$

Complement

$$\mu_{\tilde{R}^c}(x, y) = 1 - \mu_{\tilde{R}}(x, y) \tag{6.56}$$

Composition of relations

The definition is similar to that of crisp relation. Suppose $\tilde{R}$ is a fuzzy relation defined on $X \times Y$, and $\tilde{S}$ is a fuzzy relation defined on $Y \times Z$, then $\tilde{R} \circ \tilde{S}$ is a fuzzy relation on $X \times Z$. The fuzzy max-min composition is defined as

$$\mu_{\tilde{R}\circ\tilde{S}}(x,z) = \max_{y \in Y} \left(\min(\mu_{\tilde{R}}(x,y),\ \mu_{\tilde{S}}(y,z))\right) \tag{6.57}$$

Example

$X = \{x_1, x_2, x_3\}$ $Y = \{y_1, y_2\}$ $Z = \{z_1, z_2, z_3\}$

Let $\tilde{R}$ be a fuzzy relation

$$\begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} \cancel{0.5} & \cancel{0.1} \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \quad 3\times 2 \tag{6.58}$$

Let $\tilde{S}$ be a fuzzy relation

$$\begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array} \quad 2\times 3$$

Then $R \circ S$, by max-min composition yields,

$$R \circ S = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$$\begin{aligned} \mu_{\tilde{R}\circ\tilde{S}}(x_1,z_1) &= \max(\min(0.5, 0.6), \min(0.1, 0.5)) \\ &= \max(0.5, 0.1) \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} \mu_{\tilde{R}\circ\tilde{S}}(x_1,z_2) &= \max(\min(0.5, 0.4), \min(0.1, 0.8)) \\ &= \max(0.4, 0.1) \\ &= 0.4 \end{aligned}$$

Similarly,

$$\begin{aligned} \mu_{\tilde{R}\circ\tilde{S}}(x_1,z_3) &= \max(0.5, 0.1) = 0.5 \\ \mu_{\tilde{R}\circ\tilde{S}}(x_2,z_1) &= \max(0.2, 0.5) = 0.5 \\ \mu_{\tilde{R}\circ\tilde{S}}(x_2,z_2) &= \max(0.2, 0.8) = 0.8 \\ \mu_{\tilde{R}\circ\tilde{S}}(x_2,z_3) &= \max(0.2, 0.9) = 0.9 \\ \mu_{\tilde{R}\circ\tilde{S}}(x_3,z_1) &= \max(0.6, 0.5) = 0.6 \\ \mu_{\tilde{R}\circ\tilde{S}}(x_3,z_2) &= \max(0.4, 0.6) = 0.6 \\ \mu_{\tilde{R}\circ\tilde{S}}(x_3,z_3) &= \max(0.7, 0.6) = 0.7 \end{aligned}$$

Example 6.7

Consider a set $P = \{P_1, P_2, P_3, P_4\}$ of four varieties of paddy plants, set $D = \{D_1, D_2, D_3, D_4\}$ of the various diseases affecting the plants and $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let $\tilde{R}$ be a relation on $P \times D$ and $\tilde{S}$ be a relation on $D \times S$

For,
$$\tilde{R} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.1 & 0.2 \end{bmatrix} \end{array} \qquad \tilde{S} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1 & 1 & 0.4 & 0.6 \\ 0 & 0 & 0.5 & 0.9 \\ 0.9 & 1 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of the plants with the different symptoms of the diseases using max-min composition.

Solution

To obtain the association of the plants with the symptoms, $R \circ S$ which is a relation on the sets P and S is to be computed.

Using max-min composition,

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

SUMMARY

- Fuzzy set theory is an effective tool to tackle the problem of uncertainty.
- In crisp logic, an event can take on only two values, either a 1 or 0 depending on whether its occurrence is true or false respectively. However, in fuzzy logic, the event may take a range of values between 0 and 1.
- Crisp sets are fundamental to the study of fuzzy sets. The basic concepts include universal set, membership, cardinality of a set, family of sets, Venn diagrams, null set, singleton set, power set, subset, and super set. The basic operations on crisp sets are union, intersection, complement, and difference. A set of properties are satisfied by crisp sets. Also, the concept of partition and covering result in the two important rules, namely rule of addition and principle of inclusion and exclusion.
- Fuzzy sets support a flexible sense of membership and is defined to be the pair $(x, \mu_{\tilde{A}}(x))$ where $\mu_{\tilde{A}}(x)$ could be discrete or could be described by a continuous function. The membership functions could be triangular, trapezoidal, curved or its variations.

- The basic fuzzy operations used often are,

$$\tilde{A} \cup \tilde{B} = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

$$\tilde{A} \cap \tilde{B} = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

$$\tilde{A}^c = 1 - \mu_{\tilde{A}}(x)$$

- Fuzzy sets, similar to crisp sets satisfy properties such as commutativity, associativity, distributivity, De Morgan's laws and so on.
- Crisp relations on sets are subsets of the Cartesian product of the given sets. A crisp relation associates the tuples by means of a relation. A Cartesian relation could be represented by a relation matrix.
- Fuzzy relations also associate tuples but to a varying degree of membership. Some of the fuzzy relation operations are,

$$R \cup S\,(x, y) = \max\,(R(x, y), S(x, y))$$

$$R \cap S\,(x, y) = \min\,(R(x, y), S(x, y))$$

$$R^c(x, y) = 1 - R(x, y)$$

$$R \circ S\,(x, y) = \max_{y \in Y}(\min(R(x,y), S(y,z))) \quad \text{(using the max-min composition)}$$

PROGRAMMING ASSIGNMENT

P6.1 (a) Design and implement a fuzzy library FUZZYLIB.H comprising the basic fuzzy set operations such as union, intersection, complement etc.

(b) Also provide routines to implement fuzzy relations and their operations, namely union, intersection, complement, and max-min composition.

Note: Make use of relation matrix representation for the relations.

(c) Define an appropriate fuzzy problem and apply FUZZYLIB.H to solve the problem.

SUGGESTED FURTHER READING

Fuzzy Logic with Engineering Applications (Ross, 1997) is a lucid treatise on fuzzy logic. *Introduction to the Theory of Fuzzy Subsets,* Vol. 1, (Kaufmann, 1975), *Fuzzy Sets and Systems: Theory and Applications* (Dubois and Prade, 1980), *Fuzzy Set Theory and its Applications* (Zimmerman, 1987) and *Fuzzy Mathematical Techniques with Applications* (Kandel, 1986) are some of the early literature in this field. *Fuzzy Sets and Fuzzy Logic* (Klir and Yuan Bo, 1997) provides good material on fuzzy systems and its applications.

REFERENCE

Zadeh, Lotfi A. (1965), *Fuzzy Sets, Inf. Control*, Vol. 8, pp. 338–353.

Chapter 7

Fuzzy Systems

Logic is the science of reasoning. Symbolic or *mathematical* logic has turned out to be a powerful computational paradigm. Not only does symbolic logic help in the description of events in the real world but has also turned out to be an effective tool for inferring or deducing information from a given set of facts.

Just as mathematical sets have been classified into crisp sets and fuzzy sets (Refer Chapter 6), logic can also be broadly viewed as *crisp logic* and *fuzzy logic*. Just as crisp sets survive on a 2-state membership (0/1) and fuzzy sets on a multistate membership [0–1], crisp logic is built on a 2-state truth value (True/False) and fuzzy logic on a multistate truth value (True/False/very True/partly False and so on.)

We now briefly discuss crisp logic as a prelude to fuzzy logic.

7.1 CRISP LOGIC

Consider the statements "Water boils at 90°C" and "Sky is blue". An agreement or disagreement with these statements is indicated by a "True" or "False" value accorded to the statements. While the first statement takes on a value *false*, the second takes on a value ***true***.

Thus, a statement which is either 'True' or 'False' but not both is called a *proposition*. A proposition is indicated by upper case letters such as P, Q, R and so on.

Example: P: Water boils at 90°C.
Q: Sky is blue.

are propositions.

A simple proposition is also known as an *atom*. Propositions alone are insufficient to represent phenomena in the real world. ~~In order to~~ represent complex information, one has to build a sequence of propositions linked using *connectives* or *operators*. Propositional logic recognizes five major operators as shown in Table 7.1.

Table 7.1 Propositional logic connectives

Symbol	Connective	Usage	Description
$\wedge$	and	$P \wedge Q$	P and Q are true.
$\vee$	or	$P \vee Q$	Either P or Q is true.
$\neg$ or $\sim$	not	$\sim P$ or $\neg P$	P is not true.
$\Rightarrow$	implication	$P \Rightarrow Q$	P implies Q is true.
$=$	equality	$P = Q$	P and Q are equal (in truth values) is true.

Observe that ∧, ∨, ⇒, and = are 'binary' operators requiring two propositions while ~ is a 'unary' operator requiring a single proposition. ∧ and ∨ operations are referred to as *conjunction* and *disjunction* respectively. In the case of ⇒ operator, the proposition occurring before the '⇒' symbol is called as the *antecedent* and the one occurring after is called as the *consequent*.

The semantics or meaning of the logical connectives are explained using a *truth table*. A truth table comprises rows known as *interpretations*, each of which evaluates the logical formula for the given set of truth values. Table 7.2 illustrates the truth table for the five connectives.

Table 7.2 Truth table for the connectives ∧, ∨, ~, ⇒ , =

P	Q	$P \wedge Q$	$P \vee Q$	$\sim P$	$P \Rightarrow Q$	$P = Q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	F	F	F	T	T	T
F	T	F	T	T	T	F

T : True, F : False

A logical formula comprising n propositions will have 2^n interpretations in its truth table. A formula which has all its interpretations recording true is known as a *tautology* and the one which records false for all its interpretations is known as *contradiction*.

Example 7.1

Obtain a truth table for the formula $(P \vee Q) \Rightarrow (\sim P)$. Is it a tautology?

Solution

The truth table for the given formula is

P	Q	$P \vee Q$	$\sim P$	$P \vee Q \Rightarrow \sim P$
T	F	T	F	F
F	T	T	T	T
T	T	T	F	F
F	F	F	T	T

⇒ T default right & same
∨ T present
∧ T×T = T or false

No, it is not a tautology since all interpretations do not record 'True' in its last column.

Example 7.2

Is $((P \Rightarrow Q) \wedge (Q \Rightarrow P)) = (P = Q)$ a tautology?

Solution

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	A: $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	B: $P = Q$	$A = B$
T	F	F	T	F	F	T
F	T	T	F	F	F	T
T	T	T	T	T	T	T
F	F	T	T	T	T	T

Yes, the given formula is a tautology.

Example 7.3

Show that $(P \Rightarrow Q) = (\sim P \vee Q)$

Solution

The truth table for the given formula is

P	Q	$A: P \Rightarrow Q$	$\sim P$	$B: \sim P \vee Q$	$A = B$
T	T	T	F	T	T
T	F	F	F	F	T
F	F	T	T	T	T
T	T	T	F	T	T

Since the last column yields 'True' for all interpretations, it is a tautology.

The logical formula presented in Example 7.3 is of practical importance since $(P \Rightarrow Q)$ is shown to be equivalent to $(\sim P \vee Q)$, a formula devoid of '$\Rightarrow$' connective. This equivalence can therefore be utilised to eliminate '$\Rightarrow$' in logical formulae.

It is useful to view the '$\Rightarrow$' operator from a set oriented perspective. If X is the universe of discourse and A, B are sets defined in X, then propositions P and Q could be defined based on an element $x \in X$ belonging to A or B. That is,

$$P: x \in A$$

$$Q: x \in B \tag{7.1}$$

Here, P, Q are true if $x \in A$ and $x \in B$ respectively, and $\sim P$, $\sim Q$ are true if $x \notin A$ and $x \notin B$ respectively. In such a background, $P \Rightarrow Q$ which is equivalent to $(\sim P \vee Q)$ could be interpreted as

$$(P \Rightarrow Q) : x \notin A \quad \text{or} \quad x \in B \tag{7.2}$$

However, if the '$\Rightarrow$' connective deals with two different universes of discourse, that is, $A \subset X$ and $B \subset Y$ where X and Y are two universes of discourse then the '$\Rightarrow$' connective is represented by the relation R such that

$$R = (A \times B) \cup (\bar{A} \times Y) \tag{7.3}$$

In such a case, $P \Rightarrow Q$ is linguistically referred to as IF A THEN B. The compound proposition $(P \Rightarrow Q) \vee (\sim P \Rightarrow S)$ linguistically referred to as IF A THEN B ELSE C is equivalent to

$$\text{IF } A \text{ THEN } B \; (P \Rightarrow Q)$$

$$\text{IF } \sim A \text{ THEN } C \; (\sim P \Rightarrow S) \tag{7.4}$$

where P, Q, and S are defined by sets A, B, C, $A \subset X$, and B, $C \subset Y$.

7.1.1 Laws of Propositional Logic

Crisp sets as discussed in Section 6.2.2. exhibit properties which help in their simplification.

Similarly, propositional logic also supports the following laws which can be effectively used for their simplification. Given P, Q, R to be the propositions,

(i) *Commutativity*

$$(P \vee Q) = (Q \vee P)$$
$$(P \wedge Q) = (Q \wedge P) \tag{7.5}$$

(ii) *Associativity*

$$(P \vee Q) \vee R = P \vee (Q \vee R)$$
$$(P \wedge Q) \wedge R = P \wedge (Q \wedge R) \tag{7.6}$$

(iii) *Distributivity*

$$(P \vee Q) \wedge R = (P \wedge R) \vee (Q \wedge R)$$
$$(P \wedge Q) \vee R = (P \vee R) \wedge (Q \vee R) \tag{7.7}$$

(iv) *Identity*

$$P \vee \text{false} = P$$
$$P \wedge \text{True} = P$$
$$P \wedge \text{False} = \text{False}$$
$$P \vee \text{True} = \text{True} \tag{7.8}$$

(v) *Negation*

$$P \wedge {\sim}P = \text{False}$$
$$P \vee {\sim}P = \text{True} \tag{7.9}$$

(vi) *Idempotence*

$$P \vee P = P$$
$$P \wedge P = P \tag{7.10}$$

(vii) *Absorption*

$$P \wedge (P \vee Q) = P$$
$$P \vee (P \wedge Q) = P \tag{7.11}$$

(viii) *De Morgan's laws*

$${\sim}(P \vee Q) = ({\sim}P \wedge {\sim}Q)$$
$${\sim}(P \wedge Q) = ({\sim}P \vee {\sim}Q) \tag{7.12}$$

(ix) *Involution*

$${\sim}({\sim}P) = P \tag{7.13}$$

Each of these laws can be tested to be a tautology using truth tables.

Example 7.4

Verify De Morgan's laws.

(a) ${\sim}(P \vee Q) = ({\sim}P \wedge {\sim}Q)$

(b) ${\sim}(P \wedge Q) = ({\sim}P \vee {\sim}Q)$

Solution

(a)

P	Q	$P \vee Q$	A: $\sim(P \vee Q)$	$\sim P$	$\sim Q$	B: $\sim P \wedge \sim Q$	$A = B$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Therefore, $\sim(P \vee Q) = (\sim P \wedge \sim Q)$

(b)

P	Q	$P \wedge Q$	A: $\sim(P \wedge Q)$	$\sim P$	$\sim Q$	B: $\sim P \vee \sim Q$	$A = B$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	T	T	T
F	F	F	T	T	F	T	T

Therefore $\sim(P \wedge Q) = (\sim P \vee \sim Q)$

Example 7.5

Simplify $(\sim(P \wedge Q) \Rightarrow R) \wedge P \wedge Q$

Solution

Consider $\quad (\sim(P \wedge Q) \Rightarrow R) \wedge P \wedge Q$

$= (\sim \sim(P \wedge Q) \vee R) \wedge P \wedge Q$

(by eliminating '$\Rightarrow$' using $(P \Rightarrow Q) = (\sim P \vee Q)$)

$= ((P \wedge Q) \vee R) \wedge P \wedge Q$ (by the law of involution)

$= (P \wedge Q)$ (by the law of absorption)

7.1.2 Inference in Propositional Logic

Inference is a technique by which, given a set of *facts* or *postulates* or *axioms* or *premises* F_1, $F_2, \ldots, F_n$, a *goal* G is to be derived. For example, from the facts "Where there is smoke there is fire", and "There is smoke in the hill", the statement "Then the hill is on fire" can be easily deduced.

In propositional logic, three rules are widely used for inferring facts, namely

(i) *Modus Ponens*
(ii) *Modus Tollens*, and
(iii) *Chain rule*

Modus ponens (mod pons)

Given $P \Rightarrow Q$ and P to be true, Q is true.

$$\begin{array}{c} P \Rightarrow Q \\ P \\ \hline Q \end{array} \tag{7.14}$$

Here, the formulae above the line are the *premises* and the one below is the *goal* which can be inferred from the premises.

Modus tollens

Given $P \Rightarrow Q$ and $\sim Q$ to be true, $\sim P$ is true.

$$\begin{array}{c} P \Rightarrow Q \\ \sim Q \\ \hline \sim P \end{array} \tag{7.15}$$

Chain rule

Given $P \Rightarrow Q$ and $Q \Rightarrow R$ to be true, $P \Rightarrow R$ is true.

$$\begin{array}{c} P \Rightarrow Q \\ Q \Rightarrow R \\ \hline P \Rightarrow R \end{array} \tag{7.16}$$

a = b
b = c
a; c / c = a

Note that the chain rule is a representation of the *transitivity* relation with respect to the '$\Rightarrow$' connective.

Example 7.6

Given

(i) $C \vee D$
(ii) $\sim H \Rightarrow (A \wedge \sim B)$
(iii) $(C \vee D) \Rightarrow \sim H$
(iv) $(A \wedge \sim B) \Rightarrow (R \vee S)$

Can $(R \vee S)$ be inferred from the above?

Solution

From (i) and (iii) using the rule of Modus Ponens , $\sim H$ can be inferred.

$$\begin{array}{lc} \text{(i)} & C \vee D \\ \text{(iii)} & (C \vee D) \Rightarrow \sim H \\ \hline & \sim H \quad \text{(v)} \end{array}$$

From (ii) and (iv) using the chain rule, $\sim H \Rightarrow (R \vee S)$ can be inferred.

$$\begin{array}{lc} \text{(ii)} & \sim H \Rightarrow (A \wedge \sim B) \\ \text{(iv)} & (A \wedge \sim B) \Rightarrow (R \vee S) \\ \hline & \sim H \Rightarrow (R \vee S) \quad \text{(vi)} \end{array}$$

From (v) and (vi) using the rule of Modus Ponens $(R \vee S)$ can be inferred.

$$\begin{array}{lc} \text{(vi)} & \sim H \Rightarrow (R \vee S) \\ \text{(v)} & \sim H \\ \hline & R \vee S \end{array}$$

Hence, the result.

7.2 PREDICATE LOGIC

In propositional logic, events are symbolised as propositions which acquire either 'True/False' values. However, there are situations in the real world where propositional logic falls short of its expectation. For example, consider the following statements:

P : All men are mortal.

Q : Socrates is a man.

From the given statements it is possible to infer that Socrates is mortal. However, from the propositions P, Q which symbolise these statements nothing can be made out. The reason being, propositional logic lacks the ability to symbolise *quantification*. Thus, in this example, the quantifier "All" which represents the entire class of men encompasses Socrates as well, who is declared to be a man, in proposition Q. Therefore, by virtue of the first proposition P, Socrates who is a man also becomes a mortal, giving rise to the deduction Socrates is mortal. However, the deduction is not directly perceivable owing to the shortcomings in propositional logic. Therefore, propositional logic needs to be augmented with more tools to enhance its logical abilities.

Predicate logic comprises the following apart from the connectives and propositions recognized by propositional logic.

(i) Constants
(ii) Variables
(ii) Predicates
(iv) Quantifiers
(v) Functions

Constants represent objects that do not change values.

Example Pencil, Ram, Shaft, 100°C.

Variables are symbols which represent values acquired by the objects as qualified by the quantifier with which they are associated with.

Example *x, y, z.*

Predicates are representative of associations between objects that are constants or variables and acquire truth values 'True' or 'False'. A *predicate* carries a name representing the association followed by its arguments representing the objects it is to associate.

Example

likes (Ram, tea) (Ram likes tea)

plays (Sita, x) (Sita plays anything)

Here, likes and plays are predicate names and Ram, tea and Sita, x are the associated objects. Also, the predicates acquire truth values. If Ram disliked tea, likes (Ram, tea) acquires the value *false* and if Sita played any sport, plays (Sita, x) would acquire the value *true* provided x is suitably qualified by a quantifier.

Quantifiers are symbols which indicate the two types of quantification, namely, *All* ($\forall$) and *Some* ($\exists$). '$\forall$' is termed *universal quantifier* and '$\exists$' is termed *existential quantifier.*

Example Let,

man (x)	:	x is a man.
mortal (x)	:	x is mortal.
mushroom (x)	:	x is a mushroom.
poisonous (x)	:	x is poisonous.

Then, the statements

All men are mortal.
Some mushrooms are poisonous.

are represented as

$$\forall x \text{ (man } (x) \Rightarrow \text{mortal } (x))$$

$$\exists x \text{ (mushroom } (x) \wedge \text{poisonous } (x))$$

Here, a useful rule to follow is that a universal quantifier goes with implication and an existential quantifier with conjunction. Also, it is possible for logical formula to be quantified by multiple quantifiers.

Example Every ship has a captain.

$\forall x \, \exists y$ (ship (x) $\Rightarrow$ captain (x, y))

where, ship (x) : x is a ship

captain (x, y) : y is the captain of x.

Functions are similar to predicates in form and in their representation of association between objects but unlike predicates which acquire truth values alone, functions acquire values other than truth values. Thus, functions only serve as object descriptors.

Example

plus (2, 3) (2 plus 3 which is 5)

mother (Krishna) (Krishna's mother)

Observe that plus () and mother () indirectly describe "5" and "Krishna's mother" respectively.

Example 7.7

Write predicate logic statements for

(i) Ram likes all kinds of food.

(ii) Sita likes anything which Ram likes.

(iii) Raj likes those which Sita and Ram both like.

(iv) Ali likes some of which Ram likes.

Solution

Let food (x) : x is food.

likes (x, y) : x likes y

Then the above statements are translated as

(i) $\forall x$ food $(x) \Rightarrow$ likes (Ram, x))

(ii) $\forall x$ (likes (Ram, x) $\Rightarrow$ likes (Sita, x))

(iii) $\forall x$ (likes (Sita, x) $\wedge$ likes (Ram, x)) $\Rightarrow$ likes (Raj, x))

(iv) $\exists x$ (likes (Ram, x) $\wedge$ likes (Ali, x))

The application of the rule of universal quantifier and rule of existential quantifier can be observed in the translations given above.

7.2.1 Interpretations of Predicate Logic Formula

For a formula in propositional logic, depending on the truth values acquired by the propositions, the truth table interprets the formula. But in the case of predicate logic, depending on the truth values acquired by the predicates, the nature of the quantifiers, and the values taken by the constants and functions over a domain D, the formula is interpreted.

Example

Interpret the formulae

(i) $\forall x\, p(x)$

(ii) $\exists x\, p(x)$

where the domain $D = \{1, 2\}$ and

$p(1)$	$p(2)$
True	False

Solution

(i) $\forall x\ p(x)$ is true only if $p(x)$ is true for all values of x in the domain D, otherwise it is false. Here, for $x = 1$ and $x = 2$, the two possible values for x chosen from D, namely $p(1)$ = true and $p(2)$ = false respectively, yields (i) to be false since $p(x)$ is not true for $x = 2$. Hence, $\forall x\ p(x)$ is false.

(ii) $\exists x\ p(x)$ is true only if there is atleast one value of x for which $p(x)$ is true. Here, for $x = 1$, $p(x)$ is true resulting in (ii) to be true. Hence, $\exists x\ p(x)$ is true.

Example 7.8

Interpret $\forall x \exists y\ P(x, y)$ for $D = \{1, 2\}$ and

$P(1, 1)$	$P(1, 2)$	$P(2, 1)$	$P(2, 2)$
True	False	False	True

Solution

For $x = 1$, there exists a y, ($y = 1$) for which $P(x, y)$, i.e. ($P(1,1)$) is true.

For $x = 2$, there exists a y, ($y = 2$) for which $P(x, y)$ ($P(2, 2)$) is true.

Thus, for all values of x there exists a y for which $P(x, y)$ is true.

Hence, $\forall x \exists y\ P(x, y)$ is true.

7.2.2 Inference in Predicate Logic

The rules of inference such as Modus Ponens, Modus Tollens and Chain rule, and the laws of propositional logic are applicable for inferring predicate logic but not before the quantifiers have been appropriately eliminated (refer Chang & Lee, 1973).

Example

Given (i) All men are mortal.
(ii) Confucius is a man.

Prove: Confucius is mortal.

Translating the above into predicate logic statements

(i) $\forall x$ (man $(x) \Rightarrow$ mortal (x))
(ii) man (Confucius)
(iii) mortal (Confucius)

Since (i) is a tautology qualified by the universal quantifier for x = Confucius, the statement is true, i.e.

man (Confucius) $\Rightarrow$ mortal (Confucius)
$\Rightarrow$ ~man (Confucius) $\vee$ mortal (Confucius)

But from (ii), man (Confucius) is true.
Hence (iv) simplifies to

False $\vee$ mortal (Confucius)
= mortal (Confucius)

Hence, Confucius is mortal has been proved.

Example 7.9

Given (i) Every soldier is strong-willed.
(ii) All who are strong-willed and sincere will succeed in their career.
(iii) Indira is a soldier.
(iv) Indira is sincere.

Prove: Will Indira succeed in her career?

Solution

Let soldier (x) : x is a soldier.
strong-willed (x) : x is a strong-willed.
sincere (x) : x is sincere.
succeed_career (x) : x succeeds in career.

Now (i) to (iv) are translated as

$$\forall x \text{ (soldier } (x) \Rightarrow \text{strong-willed } (x)) \quad \text{(i)}$$
$$\forall x \text{ ((strong-willed } (x) \wedge \text{sincere } (x)) \Rightarrow \text{succeed\_career } (x)) \quad \text{(ii)}$$
$$\text{soldier (Indira)} \quad \text{(iii)}$$
$$\text{sincere (Indira)} \quad \text{(iv)}$$

To show whether Indira will succeed in her career, we need to show

$$\text{succeed\_career(Indira) is true.} \quad \text{(v)}$$

Since (i) and (ii) are quantified by $\forall$, they should be true for x = Indira.
Substituting x = Indira in (i) results in (soldier (Indira) $\Rightarrow$ strong-willed (Indira),

i.e. $$\sim\text{soldier (Indira)} \vee \text{strong-willed (Indira)} \quad \text{(vi)}$$

Since from (iii) soldier (Indira) is true, (vi) simplifies to

$$\text{strong-willed (Indira)} \quad \text{(vii)}$$

Substituting x = Indira in (ii),

$$\text{(strong-willed (Indira)} \wedge \text{sincere (Indira))} \Rightarrow \text{succeed\_career (Indira)}$$

i.e. $$\sim\text{(strong-willed (Indira)} \wedge \text{sincere (Indira))} \vee \text{succeed\_career (Indira)}$$
$$(\Theta\ P \Rightarrow Q = \sim P \vee Q))$$

i.e. $$\sim\text{(strong-willed (Indira)} \vee \sim\text{sincere (Indira))} \vee \text{succeed\_career (Indira)}$$
(De Morgan's law) (viii)

From (vii), strong-willed (Indira) is true and from (iv) sincere (Indira) is true. Substituting these in (viii),

$$\text{False} \vee \text{False} \vee \text{succeed\_career (Indira)}$$

i.e. $$\text{succeed\_career (Indira)} \quad \text{(using law of identity)}$$

Hence, Indira will succeed in her career is true.

7.3 FUZZY LOGIC

In crisp logic, the truth values acquired by propositions or predicates are 2-valued, namely *True*, *False* which may be treated numerically equivalent to (0, 1). However, in fuzzy logic, truth values are multivalued such as *absolutely true, partly true, absolutely false, very true,* and so on and are numerically equivalent to (0–1).

Fuzzy propositions

A *fuzzy proposition* is a statement which acquires a fuzzy truth value. Thus, given $\tilde{P}$ to be a fuzzy proposition, $T(\tilde{P})$ represents the truth value (0–1) attached to $\tilde{P}$. In its simplest form, fuzzy propositions are associated with fuzzy sets. The fuzzy membership value associated with the fuzzy set $\tilde{A}$ for $\tilde{P}$ is treated as the fuzzy truth value $T(\tilde{P})$.

i.e. $T(\tilde{P}) = \mu_{\tilde{A}}(x)$ where $0 \le \mu_{\tilde{A}}(x) \le 1$ (7.17)

Example

$\tilde{P}$: Ram is honest.

$T(\tilde{P}) = 0.8$, if $\tilde{P}$ is partly true.

$T(\tilde{P}) = 1$, if $\tilde{P}$ is absolutely true.

Fuzzy connectives

Fuzzy logic similar to crisp logic supports the following connectives:

(i) *Negation* : $-$

(ii) *Disjunction* : $\vee$

(iii) *Conjunction* : $\wedge$

(iv) *Implication* : $\Rightarrow$

Table 7.3 illustrates the definition of the connectives. Here $\tilde{P}$, $\tilde{Q}$ are fuzzy propositions and $T(\tilde{P})$, $T(\tilde{Q})$, are their truth values.

Table 7.3 Fuzzy connectives

Symbol	Connective	Usage	Definition
$-$	Negation	$\bar{\tilde{P}}$	$1 - T(\tilde{P})$
$\vee$	Disjunction	$\tilde{P} \vee \tilde{Q}$	$\max(T(\tilde{P}), T(\tilde{Q}))$
$\wedge$	Conjunction	$\tilde{P} \wedge \tilde{Q}$	$\min(T(\tilde{P}), T(\tilde{Q}))$
$\Rightarrow$	Implication	$\tilde{P} \Rightarrow \tilde{Q}$	$\bar{\tilde{P}} \vee \tilde{Q} = \max(1 - T(\tilde{P}), T(\tilde{Q}))$

$\tilde{P}$ and $\tilde{Q}$ related by the '$\Rightarrow$' operator are known as antecedent and consequent respectively. Also, just as in crisp logic, here too, '$\Rightarrow$' represents the IF-THEN statement as

IF x is $\tilde{A}$ THEN y is $\tilde{B}$, and is equivalent to

$$\boxed{\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times Y)} \tag{7.18}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x,y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), 1 - \mu_{\tilde{A}}(x)) \tag{7.19}$$

Also, for the compound implication IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$ the relation R is equivalent to

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{C}) \tag{7.20}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x,y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(y))) \tag{7.21}$$

Example

$\tilde{P}$: Mary is efficient, $T(\tilde{P}) = 0.8$

$\tilde{Q}$: Ram is efficient, $T(\tilde{Q}) = 0.65$

(i) $\bar{\tilde{P}}$: Mary is not efficient.

$$T(\bar{\tilde{P}}) = 1 - T(\tilde{P}) = 1 - 0.8 = 0.2$$

(ii) $\tilde{P} \wedge \tilde{Q}$: Mary is efficient and so is Ram.

$$\begin{aligned} T(\tilde{P} \wedge \tilde{Q}) &= \min(T(\tilde{P}), T(\tilde{Q})) \\ &= \min(0.8\ ,\ 0.65) \\ &= 0.65 \end{aligned}$$

(iii) $T(\tilde{P} \vee \tilde{Q})$: Either Mary or Ram is efficient.

$$\begin{aligned} T(\tilde{P} \vee \tilde{Q}) &= \max(T(\tilde{P}), T(\tilde{Q})) \\ &= \max(0.8, 0.65) \\ &= 0.8 \end{aligned}$$

(iv) $\tilde{P} \Rightarrow \tilde{Q}$: If Mary is efficient then so is Ram.

$$\begin{aligned} T(\tilde{P} \Rightarrow \tilde{Q}) &= \max(1 - T(\tilde{P}), T(\tilde{Q})) \\ &= \max(0.2, 0.65) \\ &= 0.65 \end{aligned}$$

Example 7.10

Let $X = \{a, b, c, d\}$ $Y = \{1, 2, 3, 4\}$

0.2 .0.4 0.

and $\tilde{A} = \{(a, 0)(b, 0.8)(c, 0.6)(d, 1)\}$

$\tilde{B} = \{(1, 0.2)(2, 1)(3, 0.8)(4, 0)\}$

$\tilde{C} = \{(1, 0)(2, 0.4)(3, 1)(4, 0.8)\}$

1-A

(a,1) (b,0.2) (c,0.4) (d,0)

Determine the implication relations

(i) IF x is $\tilde{A}$ THEN y is $\tilde{B}$.

(ii) IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

Solution

To determine (i) compute

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times Y) \quad \text{where}$$

$$\mu_{\tilde{R}}(x, y) = \max\left(\min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\right), 1 - \mu_{\tilde{A}}(x)\right)$$

$$\tilde{A} \times \tilde{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{A}} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here, Y the universe of discourse could be viewed as $\{(1, 1)\ (2, 1)\ (3, 1)\ (4, 1)\}$ a fuzzy set all of whose elements x have $\mu(x) = 1$.

Therefore,

$$\tilde{R} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 0.1 & 0.8 & 0 \end{bmatrix} \end{array}$$

MAX

which represents IF x is $\tilde{A}$ THEN y is $\tilde{B}$.

To determine (ii) compute

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{C}) \text{ where}$$

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(y)))$$

$$\tilde{A} \times \tilde{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{A}} \times \tilde{C} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore,

$$\tilde{R} = \max((\tilde{A} \times \tilde{B}), (\bar{\tilde{A}} \times \tilde{C})) \text{ gives}$$

$$\tilde{R} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

The above $\tilde{R}$ represents IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

7.3.1 Fuzzy Quantifiers

Just as in crisp logic where predicates are quantified by quantifiers, fuzzy logic propositions are also quantified by *fuzzy quantifiers*. There are two classes of fuzzy quantifiers such as

(i) Absolute quantifiers and

(ii) Relative quantifiers

While absolute quantifiers are defined over $\mathfrak{R}$, relative quantifiers are defined over [0–1].

Example

Absolute quantifier	Relative quantifier
round about 250	almost
much greater than 6	about
some where around 20	most

7.3.2 Fuzzy Inference

Fuzzy inference also referred to as *approximate reasoning* refers to computational procedures used for evaluating linguistic descriptions. The two important inferring procedures are

(i) Generalized Modus Ponens (GMP)

(ii) Generalized Modus Tollens (GMT)

GMP is formally stated as

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \quad x \text{ is } \tilde{A}' \\ \hline & \quad y \text{ is } \tilde{B}' \end{array} \tag{7.22}$$

Here, $\tilde{A}$, $\tilde{B}$, $\tilde{A}'$ and $\tilde{B}'$ are fuzzy terms. Every fuzzy linguistic statement above the line is analytically known and what is below is analytically unknown.

To compute the membership function of $\tilde{B}'$, the max-min composition of fuzzy set A' with $\tilde{R}$ (x, y) which is the known implication relation (IF-THEN relation) is used. That is,

$$\tilde{B}' = \tilde{A}' \circ \tilde{R}(x, y) \tag{7.23}$$

In terms of membership function,

$$\mu_{\tilde{B}'}(y) = \max\left(\min\left(\mu_{\tilde{A}'}(x), \mu_{\tilde{R}}(x, y)\right)\right) \tag{7.24}$$

where $\mu_{\tilde{A}'}(x)$ is the membership function of $\tilde{A}'$, $\mu_{\tilde{R}}(x, y)$ is the membership function of the implication relation and $\mu_{\tilde{B}'}(y)$ is the membership function of $\tilde{B}'$

On the other hand, GMT has the form

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \quad y \text{ is } \tilde{B}' \\ \hline & \quad x \text{ is } \tilde{A}' \end{array}$$

The membership of $\tilde{A}'$ is computed on similar lines as

$$\tilde{A}' = \tilde{B}' \circ \tilde{R}(x, y)$$

In terms of membership function,

$$\mu_{\tilde{A}'}(x) = \max\left(\min\left(\mu_{\tilde{B}'}(y), \mu_{\tilde{R}}(x, y)\right)\right) \tag{7.25}$$

Example

Apply the fuzzy Modus Ponens rule to deduce Rotation is quite slow given

(i) If the temperature is high then the rotation is slow.

(ii) The temperature is very high.

Let $\tilde{H}$ (High), $\tilde{VH}$ (Very High), $\tilde{S}$ (Slow) and $\tilde{QS}$ (Quite Slow) indicate the associated fuzzy sets as follows:

For $X = \{30, 40, 50, 60, 70, 80, 90, 100\}$, the set of temperatures and $Y = \{10, 20, 30, 40, 50, 60\}$, the set of rotations per minute,

$$\tilde{H} = \{(70, 1)\ (80, 1)\ (90, 0.3)\}$$

$$\tilde{VH} = \{(90, 0.9)\ (100, 1)\}$$

$$\tilde{QS} = \{(10, 1)\ (20, 0.8)\}$$

$$\tilde{S} = \{(30, 0.8)\ (40, 1)\ (50, 0.6)\}$$

To derive $\tilde{R}(x, y)$ representing the implication relation (i), we need to compute

$$\tilde{R}(x, y) = \max(\tilde{H} \times \tilde{S}, \bar{\tilde{H}} \times Y)$$

$$\tilde{H} \times \tilde{S} = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{H}} \times Y = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\tilde{R}(x,y) = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

To deduce Rotation is quite slow we make use of the composition rule

$$\tilde{QS} = V\tilde{H} \circ \tilde{R}(x,y)$$

$$= [0\ 0\ 0\ 0\ 0\ 0\ 0.9\ 1] \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= [1\ 1\ 1\ 1\ 1\ 1]$$

7.4 FUZZY RULE BASED SYSTEM

Fuzzy linguistic descriptions are formal representations of systems made through fuzzy IF-THEN rules. They encode knowledge about a system in statements of the form—
IF (a set of conditions) are satisfied THEN (a set of consequents) can be inferred.
Fuzzy IF-THEN rules are coded in the form—

$$\text{IF } (x_1 \text{ is } \tilde{A}_1, x_2 \text{ is } \tilde{A}_2, \ldots, x_n \text{ is } \tilde{A}_n) \text{ THEN } (y_1 \text{ is } \tilde{B}_1, y_2 \text{ is } \tilde{B}_2, \ldots, y_n \text{ is } \tilde{B}_n).$$

where linguistic variables x_i, y_j take the values of fuzzy sets A_i and B_j respectively.

Example

If there is heavy rain and strong winds
then there must be severe flood warning.

Here, heavy, strong, and severe are fuzzy sets qualifying the variables rain, wind, and flood warning respectively.

A collection of rules referring to a particular system is known as a *fuzzy rule base*. If the conclusion C to be drawn from a rule base R is the conjunction of all the individual consequents C_i of each rule, then

$$C = C_1 \cap C_2 \cap \ldots \cap C_n \tag{7.26}$$

where

$$\mu_C(y) = \min\,(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y \tag{7.27}$$

where Y is the universe of discourse.

On the other hand, if the conclusion C to be drawn from a rule base R is the disjunction of the individual consequents of each rule, then

$$C = C_1 \cup C_2 \cup C_3 \ldots \cup C_n \tag{7.28}$$

where

$$\mu_C(y) = \max\,(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y \tag{7.29}$$

7.5 DEFUZZIFICATION

In many situations, for a system whose output is fuzzy, it is easier to take a crisp decision if the output is represented as a single scalar quantity. This conversion of a fuzzy set to single crisp value is called *defuzzification* and is the reverse process of *fuzzification*.

Several methods are available in the literature (Hellendoorn and Thomas, 1993) of which we illustrate a few of the widely used methods, namely *centroid method, centre of sums,* and *mean of maxima.*

Centroid method / CG / centroid of area

Also known as the *centre of gravity* or the *centre of area* method, it obtains the centre of area (x^*) occupied by the fuzzy set. It is given by the expression

$$x^* = \frac{\int \mu(x)\, x\, d\,x}{\int \mu(x)\, d\,x} \tag{7.30}$$

for a continuous membership function, and

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)} \tag{7.31}$$

for a discrete membership function.

Here, n represents the number of elements in the sample, x_i's are the elements, and $\mu(x_i)$ is its membership function.

Centre of sums (COS) method

In the centroid method, the overlapping area is counted once whereas in *centre of sums*, the overlapping area is counted twice. COS builds the resultant membership function by taking the algebraic sum of outputs from each of the contributing fuzzy sets $\tilde{A}_1$, $\tilde{A}_2$..., etc. The defuzzified value x^* is given by

$$x^* = \frac{\sum_{i=1}^{N} x_i \cdot \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)} \tag{7.32}$$

Here n is the number of fuzzy sets and N the number of fuzzy variables. COS is actually the most commonly used defuzzification method. It can be implemented easily and leads to rather fast inference cycles.

Mean of maxima (MOM) defuzzification

One simple way of defuzzifying the output is to take the crisp value with the highest degree of membership. In cases with more than one element having the maximum value, the mean value of the maxima is taken. The equation of the defuzzified value x^* is given by

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|} \tag{7.33}$$

where $M = \{x_i \mid \mu(x_i)$ is equal to the *height* of fuzzy set$\}$
$|M|$ is the cardinality of the set M. In the continuous case, M could be defined as

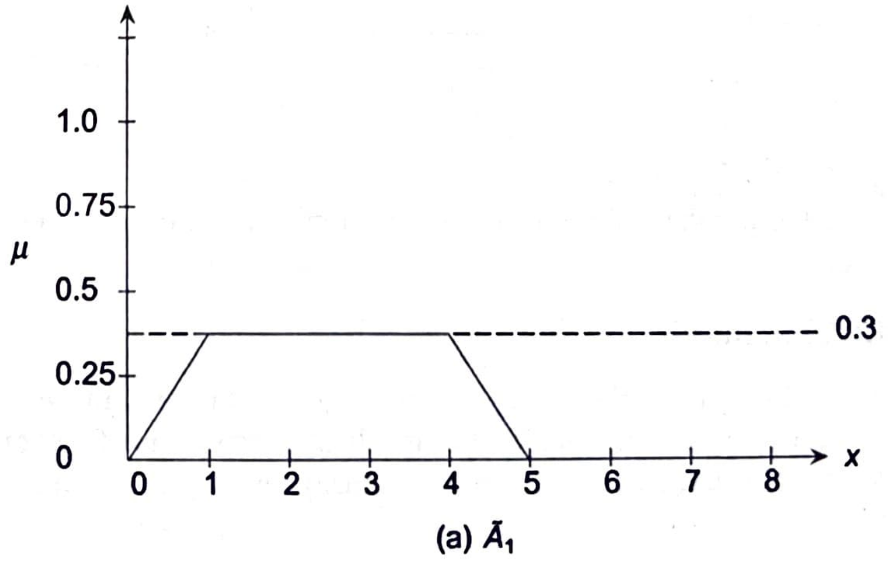
$$M = \{x \in [-c, c] \mid \mu(x) \text{ is equal to the height of the fuzzy set}\} \tag{7.34}$$

In such a case, the *mean of maxima* is the arithmetic average of mean values of all intervals contained in M including zero length intervals.

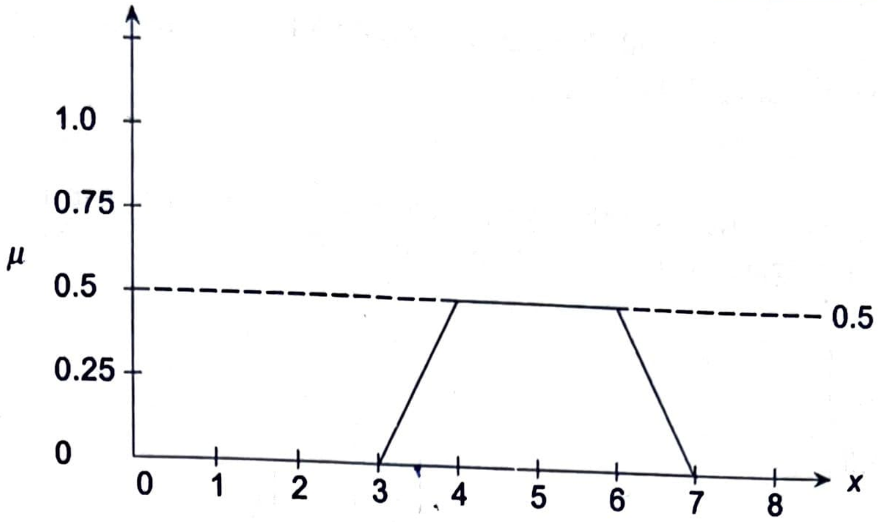
The *height* of a fuzzy set A, i.e. $h(A)$ is the largest membership grade obtained by any element in that set.

Example

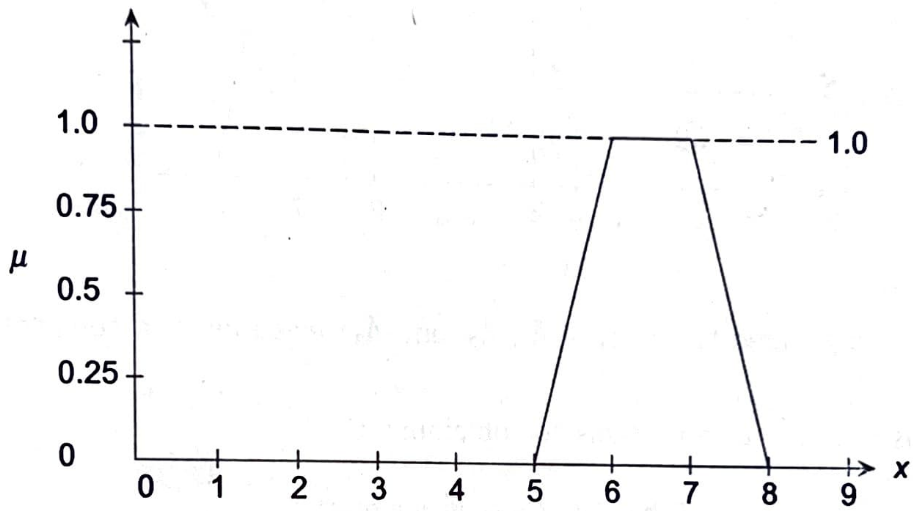
$\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ are three fuzzy sets as shown in Fig. 7.1(a), (b), and (c). Figure 7.2 illustrates the aggregate of the fuzzy sets.



(a) $\tilde{A}_1$



(b) $\tilde{A}_2$



(c) $\tilde{A}_3$

Fig. 7.1 Fuzzy sets $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$.

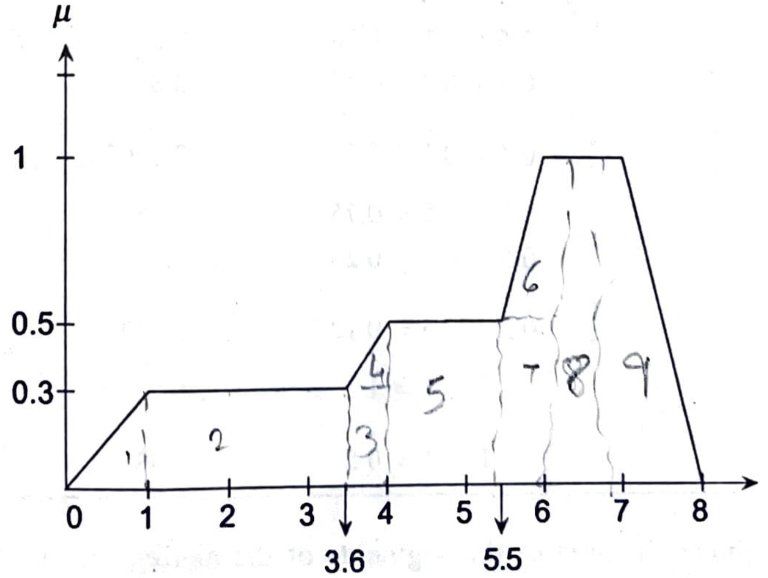


Fig. 7.2 Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.

The defuzzification using (i) centroid method, (ii) centre of sums method, and (iii) mean of maxima method is illustrated as follows.

Centroid method

To compute x^*, the centroid, we view the aggregated fuzzy sets as shown in Figs. 7.2 and 7.3. Note that in Fig. 7.3 the aggregated output has been divided into areas for better understanding.

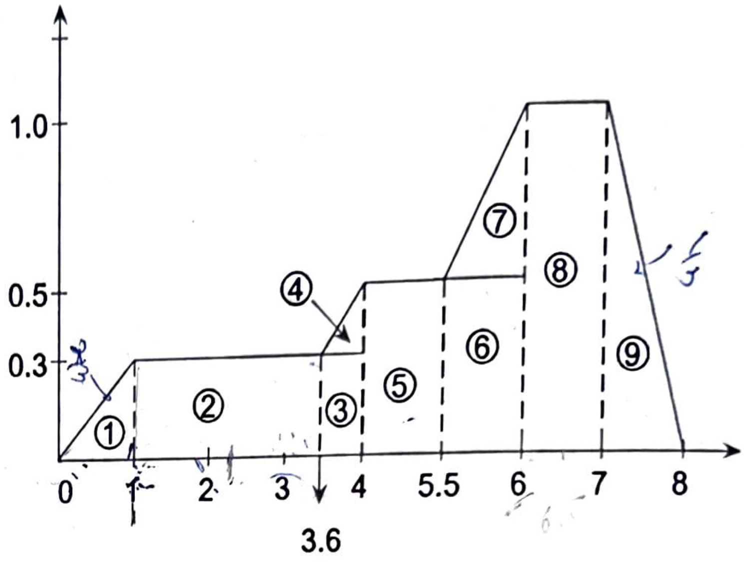


Fig. 7.3 Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ viewed as area segments.

Table 7.4 illustrates the computations for obtaining x^*.

Table 7.4 Computation of x^*

Area segment no.	Area (A)	$\bar{x}$	$A\bar{x}$
1	$\frac{1}{2} \times 0.3 \times 1 = 0.15$	0.67	0.1005
2	$2.6 \times 0.3 = 0.78$	2.3	1.794
3	$0.3 \times 0.4 = 0.12$	3.8	0.456
4	$\frac{1}{2} \times 0.4 \times 0.2 = 0.04$	3.8667	0.1546
5	$1.5 \times 0.5 = 0.75$	4.75	3.5625
6	$0.5 \times 0.5 = 0.25$	5.75	1.4375
7	$\frac{1}{2} \times 0.5 \times 0.5 = 0.125$	5.833	0.729
8	$1 \times 1 = 1$	6.5	6.5
9	$\frac{1}{2} \times 1 \times 1 = 0.5$	7.33	3.665

In Table 7.4, Area (A) shows the area of the segments of the aggregated fuzzy set and $\bar{x}$ shows the corresponding centroid. Now,

$$x^* = \frac{\sum A\bar{x}}{\sum A}$$

i.e.
$$x^* = 18.353/3.715$$
$$= 4.9$$

Centre of sums method

Here, unlike centroid method the overlapping area is counted not once but twice. Making use of the aggregated fuzzy set shown in Fig.7.2, the centre of sums, x^* is given by

$$x^* = \frac{\frac{1}{2}\times 0.3\times(3+5)\times 2.5+\frac{1}{2}\times 0.5\times(4+2)\times 5+\frac{1}{2}\times 1\times(3+1)\times 6.5}{\frac{1}{2}\times 0.3\times(3+5)+\frac{1}{2}\times 0.5\times(4+2)+\frac{1}{2}\times 1\times(3+1)}$$
$$= 5$$

Here, the areas covered by the fuzzy sets $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ (Refer Figs. 7.1(a), (b), and (c)) are given by

$$\frac{1}{2}\times 0.3\times(3+5),\quad \frac{1}{2}\times 0.5\times(4+2),\ \text{and}\ \frac{1}{2}\times 1\times(3+1)\ \text{respectively.}$$

Mean of maxima method

Since the aggregated fuzzy set shown in Fig. 7.2 is a continuous set, x^* the mean of maxima is computed as $x^* = 6.5$.

Here, $M = \{X \in [6, 7] \mid \mu(x) = 1\}$ and the height of the aggregated fuzzy set is 1.

Figure 7.4 shows the defuzzified outputs using the above three methods.

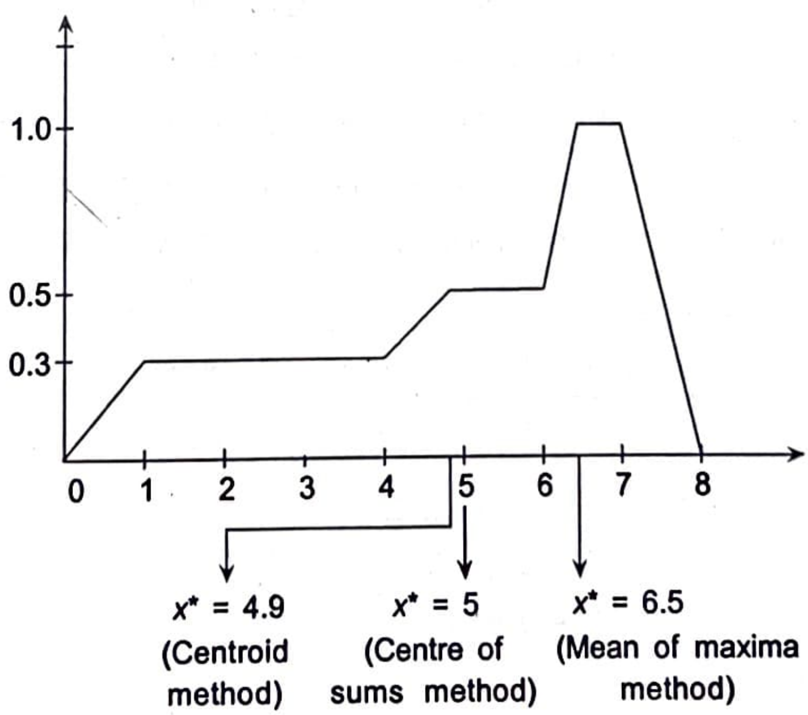


Fig. 7.4 Defuzzified outputs of the aggregate of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.

7.6 APPLICATIONS

In this section we illustrate two examples of Fuzzy systems, namely

(i) *Greg Viot's* (Greg Viot, 1993) *Fuzzy Cruise Control System*

(ii) *Yamakawa's* (Yamakawa, 1993) *Air Conditioner Controller*

7.6.1 Greg Viot's Fuzzy Cruise Controller

This controller is used to maintain a vehicle at a desired speed. The system consists of two fuzzy inputs, namely speed difference and acceleration, and one fuzzy output, namely throttle control as illustrated in Fig. 7.5.

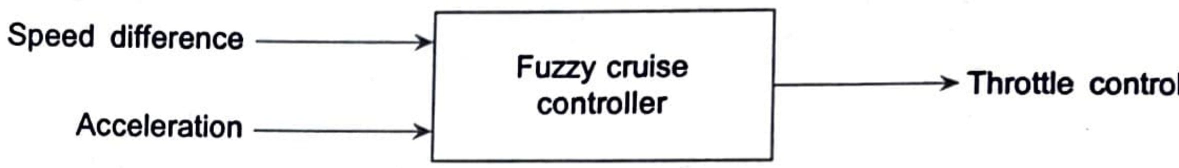


Fig. 7.5 Fuzzy cruise controller.

Fuzzy rule base

A sample fuzzy rule base *R* governing the cruise control is as given in Table 7.5.

Table 7.5 Sample cruise control rule base

Rule 1	If (speed difference is NL) and (acceleration is ZE) then (throttle control is PL).
Rule 2	If (speed difference is ZE) and (acceleration is NL) then (throttle control is PL).
Rule 3	If (speed difference is NM) and (acceleration is ZE) then (throttle control is PM).
Rule 4	If (speed difference is NS) and (acceleration is PS) then (throttle control is PS).
Rule 5	If (speed difference is PS) and (acceleration is NS) then (throttle control is NS).
Rule 6	If (speed difference is PL) and (acceleration is ZE) then (throttle control is NL).
Rule 7	If (speed difference is ZE) and (acceleration is NS) then (throttle control is PS).
Rule 8	If (speed difference is ZE) and (acceleration is NM) then (throttle control is PM).

Key

NL - Negative Large

ZE - Zero

PL - Positive Large

NM - Negative Medium

PM - Positive Medium

NS - Negative Small

PS - Positive Small

Fuzzy sets

The fuzzy sets which characterize the inputs and output are as given in Fig. 7.6.

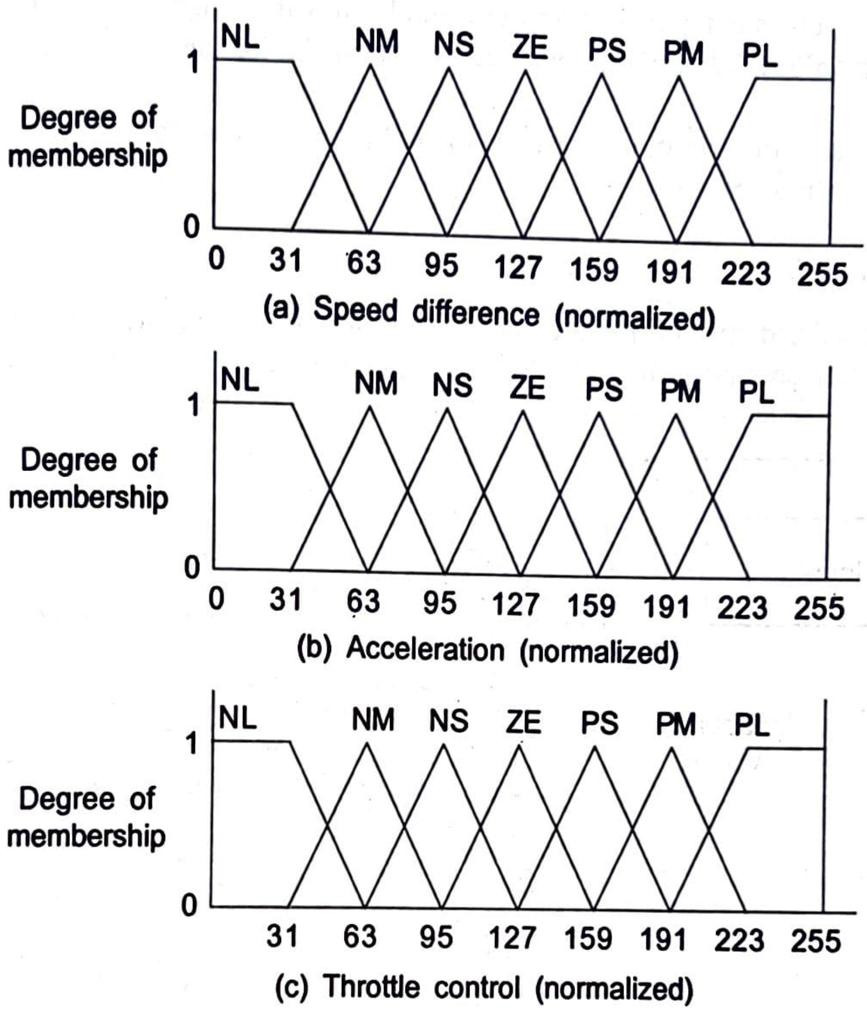
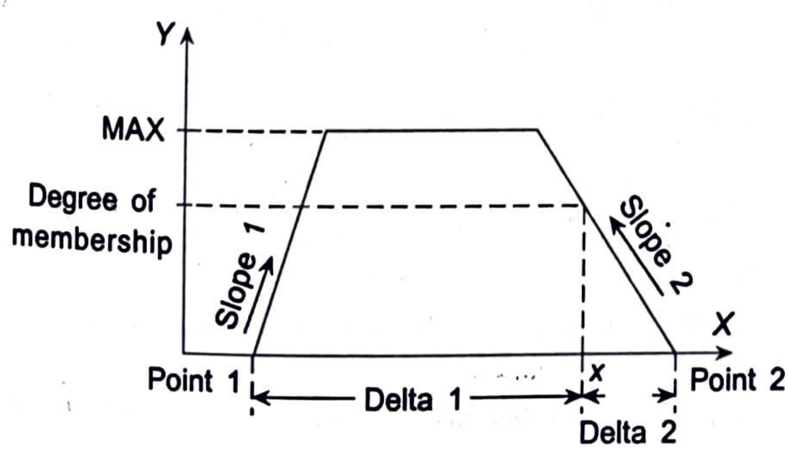


Fig. 7.6 Fuzzy sets characterising fuzzy cruise control.

Fuzzification of inputs

For the *fuzzification* of inputs, that is, to compute the membership for the antecedents, the formula illustrated in Fig. 7.7 is used.



Degree of membership

- Compute Delta 1 = x – point 1
  Delta 2 = Point 2 – x
- If (Delta 1 ≤ 0) or (Delta 2 ≤ 0)
  then Degree of membership = 0
  else Degree of membership

$$= \min \begin{pmatrix} \text{Delta 1} * \text{Slope 1} \\ \text{Delta 2} * \text{Slope 2} \\ \text{Max} \end{pmatrix}$$

Fig. 7.7 Computation of fuzzy membership value.

Here, x which is the system input has its membership function values computed for all fuzzy sets. For example, the system input speed difference deals with 7 fuzzy sets, namely NL, NM, NS, ZE, PS, PM, and PL. For a measured value of the speed difference x', the membership function of x' in each of the seven sets is computed using the formula shown in Fig. 7.7. Let $\mu_1', \mu_2', \ldots, \mu_7'$ be the seven membership values. Then, all these values are recorded for the input x' in an appropriate data structure.

Similarly, for each of the other system inputs (acceleration in this case), the fuzzy membership function values are recorded.

Example

Let the measured normalized speed difference be 100 and the normalized acceleration be 70, then the fuzzified inputs after computation of the fuzzy membership values are shown in Fig. 7.8.

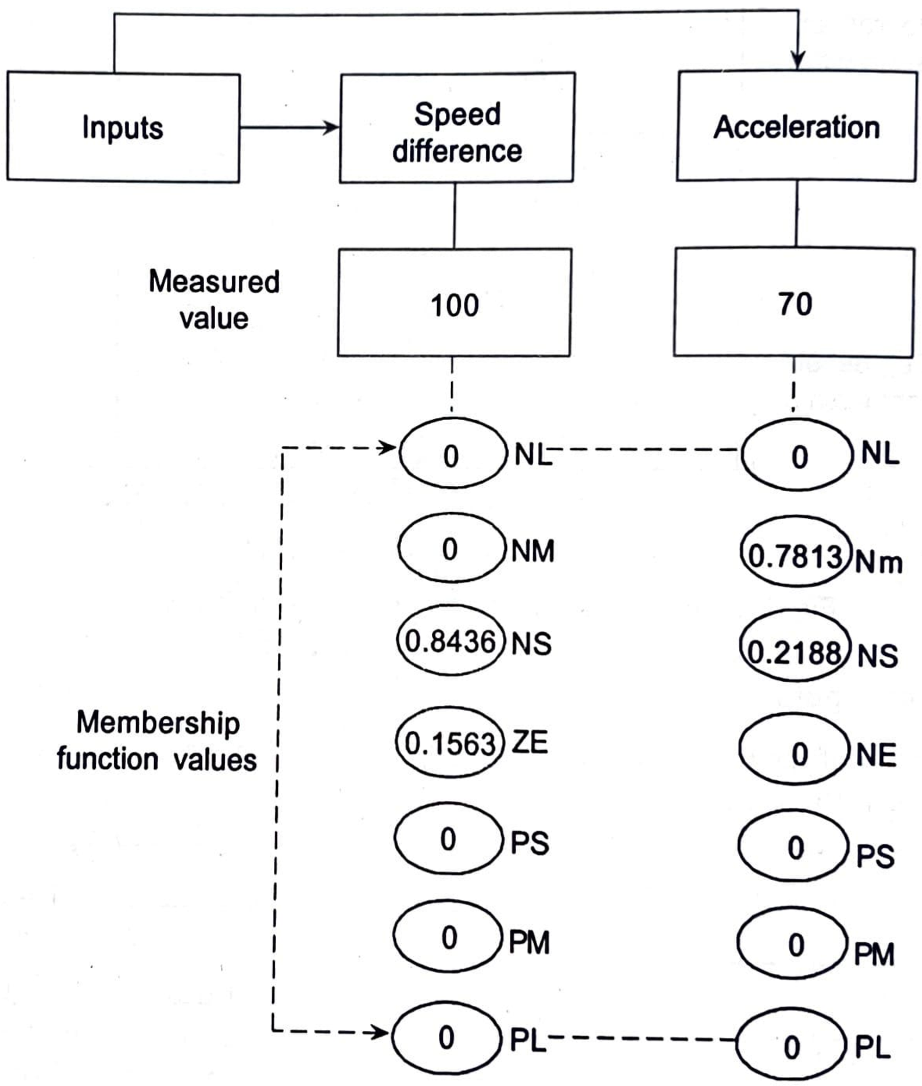


Fig. 7.8 Fuzzy membership values for speed difference = 100 and acceleration = 70.

The computations of the fuzzy membership values for the given inputs have been shown in Fig. 7.9.

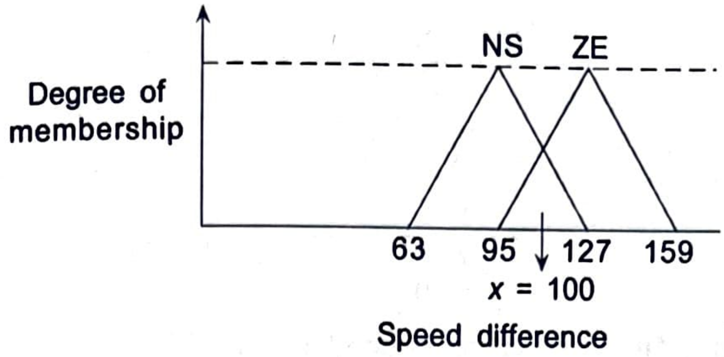


Fig. 7.9 Fuzzification of speed difference = 100.

For speed difference (x = 100), the qualifying fuzzy sets are as shown in Fig. 7.9. Fuzzy membership function of x for NS where

$$\text{Delta } 1 = 100 - 63 = 37$$

$$\text{Delta } 2 = 127 - 100 = 27$$

$$\text{Slope } 1 = 1/32 = 0.03125$$

$$\text{Slope } 2 = 1/32 = 0.03125$$

Degree of membership function

$$\mu_{NS}(x) = \min \begin{pmatrix} 37 \times 0.03125 \\ 27 \times 0.03125 \\ 1 \end{pmatrix}$$

$$= 0.8438$$

Fuzzy membership function of x for ZE where

$$\text{Delta } 1 = 100 - 95 = 5$$

$$\text{Delta } 2 = 159 - 100 = 59$$

$$\text{Slope } 1 = \frac{1}{32} = 0.03125$$

$$\text{Slope } 2 = 0.03125$$

Degree of membership function

$$\mu_{ZE}(x) = \min \begin{pmatrix} 5 \times 0.03125 \\ 59 \times 0.03125 \\ 1 \end{pmatrix} = 0.1563$$

The membership function of x with the remaining fuzzy sets, namely NL, NM, PS, PM, PL is zero. Similarly for acceleration (x = 70), the qualifying fuzzy sets are as shown in Fig. 7.10.

The fuzzy membership function of x = 70 for NM is $\mu_{NM}(x)$ = 0.7813 and for NS is $\mu_{NS}(x) = 0.2188$.

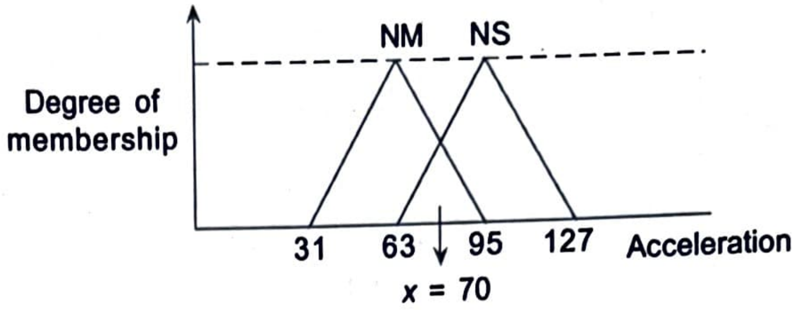


Fig. 7.10 Fuzzification of acceleration = 70.

Rule strength computation

The *rule strengths* are obtained by computing the minimum of the membership functions of the antecedents.

Example

For the sample rule base *R* given in Table 7.5, the rule strengths using the fuzzy membership values illustrated in Fig. 7.8 are

Rule 1: min (0, 0) = 0

Rule 2: min (0.1563, 0) = 0

Rule 3: min (0, 0) = 0

Rule 4: min (0.8438, 0) = 0

Rule 5: min (0, 0.2188) = 0

Rule 6: min (0,0) = 0

Rule 7: min (0.1563, 0.2188) = 0.1563

Rule 8: min (0.1563, 0.7813) = 0.1563

Fuzzy output

The *fuzzy output* of the system is the 'fuzzy OR' of all the fuzzy outputs of the rules with non-zero rule strengths. In the event of more than one rule qualifying for the same fuzzy output, the stronger among them is chosen.

Example

In the given rule base *R*, the competing fuzzy outputs are those of Rules 7 and 8 with strengths of 0.1563 each.

However, the fuzzy outputs computed here do not aid a clear-cut decision on the throttle control. Hence, the need for defuzzification arises.

Defuzzification

The centre of gravity method is applied to defuzzify the output. Initially, the centroids are computed for each of the competing output membership functions. Then, the new output membership areas are determined by shortening the height of the membership value on the Y axis as dictated by the rule strength value. Finally, the Centre of Gravity (CG) is computed using the weighted average of the X-axis centroid points with the newly computed output areas, the latter serving as weights.

Example

Figure 7.11 illustrates the computation of CG for the two competing outputs of rules 7 and 8 with strength of 0.1563 each.

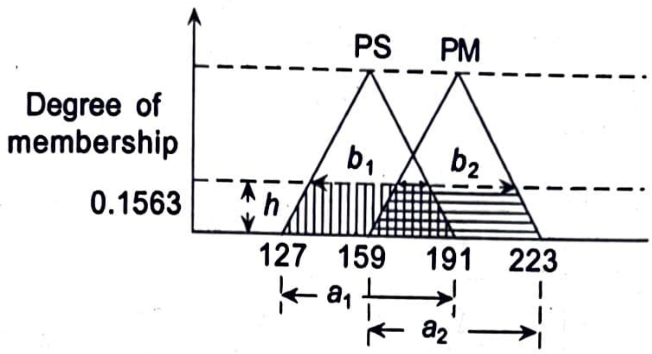


Fig. 7.11 Computation of CG for fuzzy cruise control system.

For the fuzzy set PS,

X-axis centroid point $= 159$

Rule strength applied to determine output area $= 0.1563$

Shaded area $= \frac{1}{2} h \cdot (a_1 + b_1)$

$= \frac{1}{2}(0.1563)(64 + 63.82)$

$= 9.99$

For the fuzzy set PM,

X-axis centroid point $= 191$

Rule strength applied to determine output area $= 0.1563$

Shaded area $= \frac{1}{2} h \cdot (a_1 + b_1)$

$= \frac{1}{2}(0.1563)(64 + 63.82)$

$= 9.99$

Therefore,

Weighted average, $(CG) = \dfrac{9.99 \times 159 + 9.99 \times 191}{19.98} = 175$

In crisp terms, the throttle control (normalized) is to be set as 175.

7.6.2 Air Conditioner Controller

The system as illustrated in Fig. 7.12 comprises a dial to control the flow of warm/hot or cool/cold air and a thermometer to measure the room temperature (T°C). When the dial is turned positive, warm/hot air is supplied from the air conditioner and if it is turned negative, cool/cold air is supplied. If set to zero, no air is supplied.

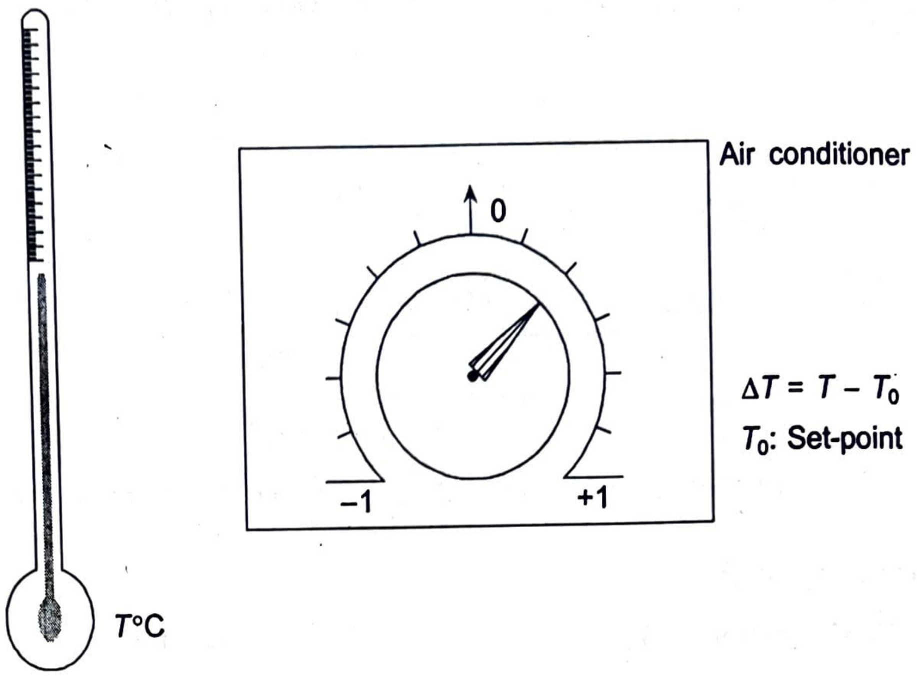


Fig. 7.12 Air conditioner control system.

A person now notices the difference in temperature (ΔT °C) between the room temperature (T °C) as measured by the thermometer and the desired temperature (T_0°C) at which the room is desired to be kept (set-point). The problem now is to determine to what extent the dial should be turned so that the appropriate supply of air (hot/warm/cool/cold) will nullify the change in temperature.

For the above problem the rule base is as shown in Table 7.6.

Table 7.6 Fuzzy rule base for the air conditioner control

S.no.		Fuzzy rule (Descriptive)		Fuzzy rule (Notational)
1	*If*	the room temperature is approximately equal to the set point T_0°C, ΔT is approximately Zero (ZE) *and* the temperature is rapidly changing higher, i.e. $\frac{d\Delta T}{dt}$ is positively large (PL)	*If*	ΔT is ZE *and* $\frac{d\Delta T}{dt}$ is PL
	then	blow cold air rapidly, i.e. turn the dial negative large (NL).	*then*	dial should be NL

(*Cont.*)

Table 7.6 Fuzzy rule base for the air conditioner control *(Cont)*

S.no.		Fuzzy rule (Descriptive)		Fuzzy rule (Notational)
2	*If*	the room temperature is high *and* there is no change in temperature, i.e. ΔT is positive large (PL) and $\frac{d\Delta T}{dt}$ is approximately zero (ZE)	*If*	ΔT is PL *and* $\frac{d\Delta T}{dt}$ is ZE *then* dial should be NM.
	then	blow cold air at an intermediate level, i.e. turn the dial negative medium (NM).		
3	*If*	the room temperature is a little bit higher than the set-point *and* the temperature is gradually decreasing, i.e. ΔT is positively small (PS) and $\frac{d\Delta T}{dt}$ is negatively small (NS)	*If*	ΔT is PS *and* $\frac{d\Delta T}{dt}$ is NS *then* dial should be ZE.
	then	there is no need to blow hot or cold air, i.e. turn the dial to approximately zero (ZE).		

The fuzzy sets for the system inputs, namely ΔT and $\frac{d\Delta T}{dt}$, and the system output, namely turn of the dial are as shown in Fig. 7.13.

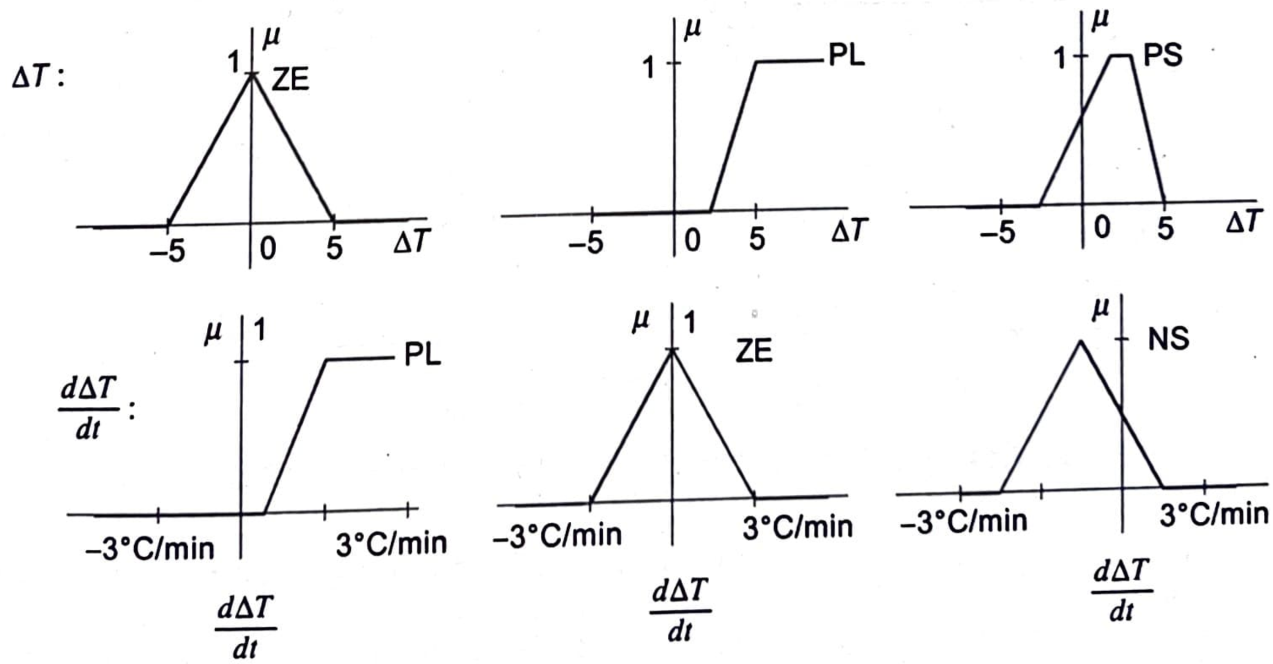


Fig. 7.13 Cont.

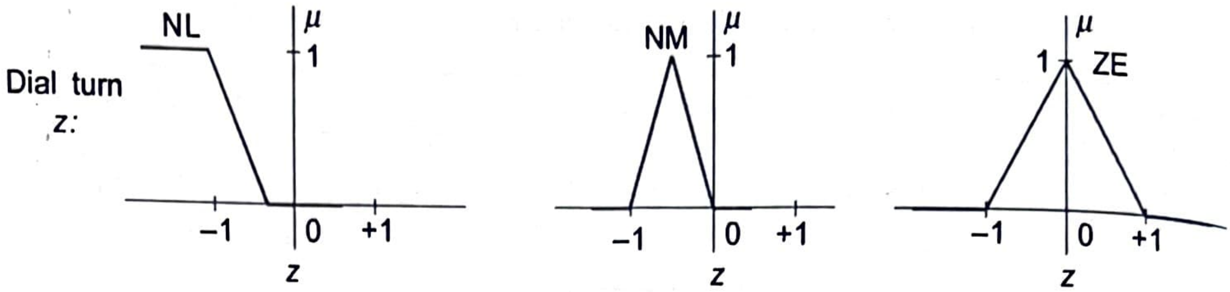


Fig. 7.13 Fuzzy sets for the air conditioner control system.

Consider the system inputs, $\Delta T = 2.5°C$ and $\dfrac{d\Delta T}{dt} = -1°C/min$. Here the fuzzification of system inputs has been directly done by noting the membership value corresponding to the system inputs as shown in Fig. 7.14.

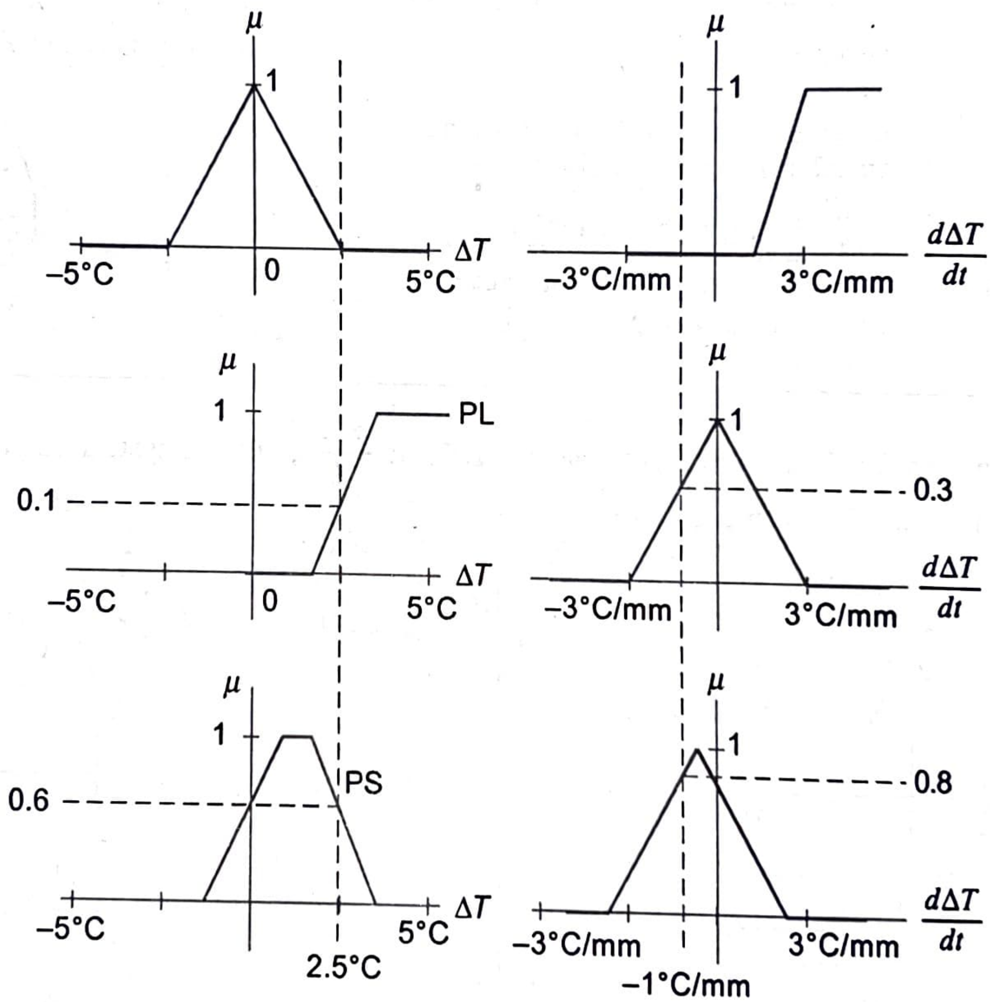


Fig. 7.14 Fuzzification of inputs $\Delta T = 2.5°C$, $d\Delta T/dt = -1°C/min$.

The rule strengths of rules 1, 2, 3 choosing the minimum of the fuzzy membership value of the antecedents are 0, 0.1 and 0.6 respectively. The fuzzy output is as shown in Fig 7.15.

The defuzzification of the fuzzy output yields $Z = -0.2$ for $\Delta T = 2.5°C$ and $y = -1°C/min$. Hence, the dial needs to be turned in the negative direction, i.e. −0.2 to achieve the desired temperature effect in the room.

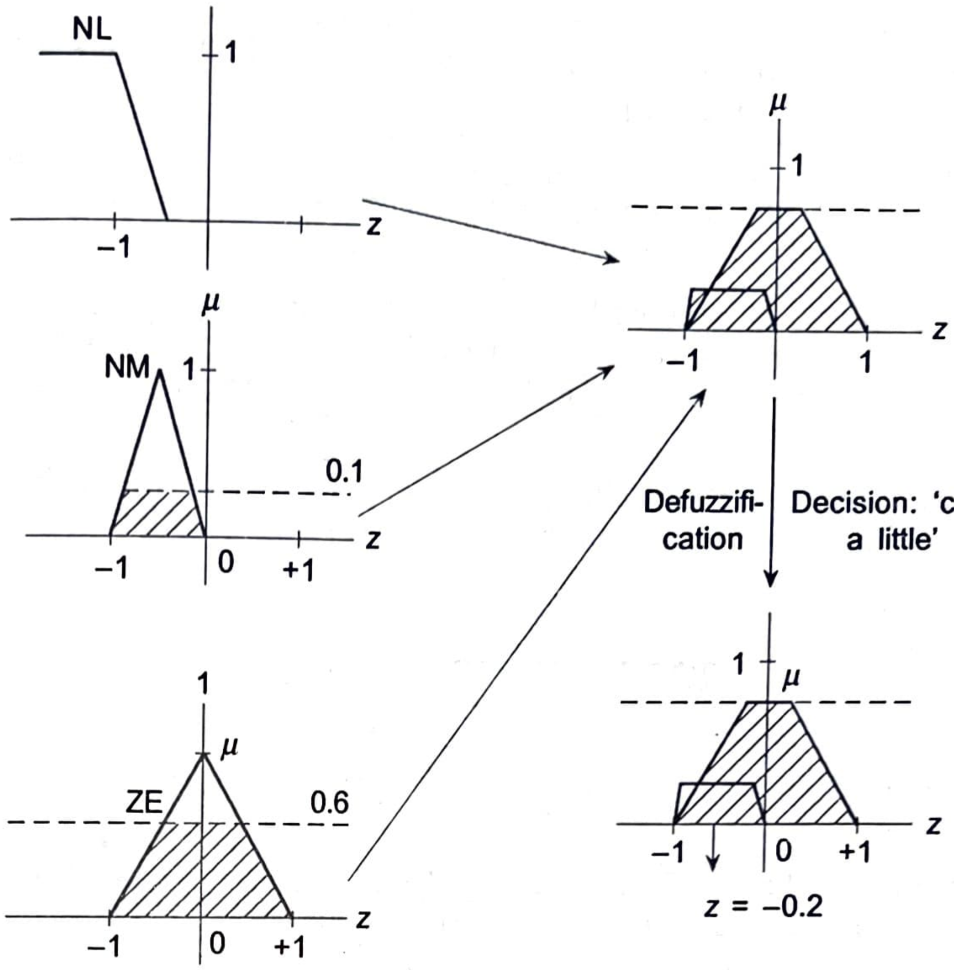


Fig. 7.15 Defuzzification of fuzzy outputs for z (turn of the dial).

SUMMARY

- *Crisp logic* is classified into *propositional logic* and *predicate logic*.
- *Propositions* are statements which are either true or false but not both.
- Propositional logic supports the five major *connectives* $\wedge$, $\vee$, ~, $\Rightarrow$, $=$.

  Truth tables describe the semantics of these connectives.
- The laws of propositional logic help in the simplification of formulae.
- *Modus Ponens* ($P \Rightarrow Q$ and P, infers Q), *Modus Tollens* ($P \Rightarrow Q$ and $\sim Q$, infers $\sim P$), and *Chain rule* ($P \Rightarrow Q$ and $Q \Rightarrow R$ infers $P \Rightarrow R$) are useful rules of inference in propositional logic.
- Propositional logic is handicapped owing to its inability to quantify. Hence, the need for *predicate logic arises*. Besides propositions and connectives, predicate logic supports *predicates, functions, variables, constants and quantifiers* ($\forall, \exists$). The interpretation of predicate logic formula is done over a domain D. The three rules of inference of propositional logic are applicable here as well.

- Fuzzy logic on the other hand accords multivalued truth values such as *absolutely true*, *partly true*, *partly false* etc. to fuzzy propositions. While crisp logic is two valued, fuzzy logic is multivalued [0–1].
- Fuzzy logic also supports fuzzy quantifiers classified as relative and absolute quantifiers and the Fuzzy rules of inference *Generalized Modus Ponens* (GMP) and *Generalized Modus Tollens* (GMT).
- A set of fuzzy *if-then* rules known as a *fuzzy rule base* describes a *fuzzy rule based system*. However, for effective decision making, defuzzification techniques such as center of gravity method are employed which render the fuzzy outputs of a system in crisp terms.
- Fuzzy systems have been illustrated using two examples, namely Greg Viot's fuzzy cruise control system and Yamakawa's air conditioner control system.

PROGRAMMING ASSIGNMENT

P7.1 Solve the Air conditioner controller problem (Sec. 7.6.2) using MATLAB®'s fuzzy logic tool box.

(a) Make use of the FIS (Fuzzy Inference System) editor to frame the rule base and infer from it. Employ the centroid method of defuzzification.

(b) Download Robert Babuska's fuzzy logic tool box. (http://lcewww.et.tudelft.nl/~babuska/) and implement the same problem.

SUGGESTED FURTHER READING

Fuzzy logic concepts are discussed in *A First Course in Fuzzy Logic* (Nguyen and Walker, 1999). The design and properties of fuzzy systems and fuzzy control systems could be found in *A Course in Fuzzy Systems and Control* (Wang, 1997). Several fuzzy system case studies have been discussed in *The Fuzzy Systems Handbook* (Earl Cox, 1998). The book is also supplemented by a CD-ROM containing Windows 95 fuzzy logic library with code to generate 32 bit DLLs for Visual BASIC and Visual C++. The applications of fuzzy systems for neural networks, knowledge engineering and chaos are discussed in *Foundations of Neural Networks, Fuzzy Systems and Knowledge Engineering* (Kasabov, 1996).

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