Duality- LP- Summaries and Concepts

Shyam Bhagwat , shyambhagwat@utexas.edu

February 24, 2022

1 Duality

Duality is a subject where we study the relationships between the optimization problems in context of LP. For every LP there is an associated alternative LP that has the same optimal value can guarantee optimality of a particular point . It provides structural insights into problems provides new algorithms to solve LPs.

Making the Dual for below

$$\min_{x>=0} C'x \ st \ Ax = b$$

Step 1: Introduce penalty variables

$$\min_{x>=0} \max_{P} C'x + p'(b - Ax)$$

Step2. Interchange max and min

$$\max_{P} \min_{x>=0} C'x + p'(b - Ax)$$

Step3. Rewrite , eliminating the primal variables to get the dual

$$\max_{P} p'b \ st \ c' - p'A >= 0$$

1.1 Weak Duality

The principle of **weak duality** is as follows: for every set , set , and function $f : \times \to$, the following inequality holds:

$$p^* \doteq \min_{x \in X} \max_{p \in P} C'x \geq \max_{p \in P} \min_{x \in X} P'b \tag{1}$$

The difference $p^* - d^*$ is called the **duality gap**.

Weak duality *always holds*, and can come in handy sometimes, even though we'll generally prefer strong duality.

1.2 Strong Duality

We say that strong duality holds for a min-max problem if primal and dual are both feasible, and if primal optimum x and dual optimum p

$$p^* \doteq \min_{x \in X} \max_{p \in P} C'x \geq \max_{p \in P} \min_{x \in X} P'b \tag{2}$$

Strong duality does not generally hold. If you want to claim that it does, you have to justify it in some sense. We will talk about the conditions for strong duality to hold later in the notes.

1.3 Farkas' lemma

For any vector c and ai, $i \in I$ either of below should hold but not both

$$\exists p_i \geq 0 \ st \ c = \sum a_i, p_i \tag{3}$$

$$\exists d' a_j \geq 0 \ but \ d'c \leq 0 \tag{4}$$

1.4 Strong Duality Proof

Using Farkas lemma - TODO

1.5 Complementary Slackness

Using Farkas lemma - TODO

1.6 Theorem of Alternatives

Using Farkas lemma - TODO