Semidefinite Optimization - Summaries and Concepts

Shyam Bhagwat , shyambhagwat@utexas.edu

February 27, 2022

1 Semi Definite Optimization

1.1 Eigen values and Eigenvectors

Square matrix M has eigen vector v and eigen value λ if

$$M\dot{v} = \lambda\dot{v}$$

We can compute eigen values by computing the roots of :

$$det(\lambda I - M)$$

1.2 Symmetric Matrices

M is symmetric if $\mathrm{M}_{i,j} = M_{j,i} \forall i, j : M = M^T$.

Key Properties - Symmetric matrices have real eigen values. Eigen vectors of Symmetric matricies are perpendicular to each other. (Orthogonal)

1.3 Eigendecomposition

For real, symmetric matrices $A \in n$, A has an eigendecomposition

$$A = V\Lambda V^T = \sum_{i=1}^{n} \lambda_i v_i v_i^T,$$

where V is the matrix whose columns are the orthonormal eigenvectors v_i. To recap on matrix sizes:

- $V = [v_1, ..., v_n] \in n \times n$
- $\Lambda = \lambda_1, ..., \lambda_n \in n \times n$.

1.4 Positive Semi Definite Matricies

M is positive semi definite if the eigen values is non negative

$$\lambda_i \geq 0$$

Key Def: M is PSD, if below quadratic is non negative :

$$x^T M x \geq 0 \ \forall x \in R^n$$

Properties: For any matrix A, AA^T & $A^T A$ is PSD
Proof:

$$x^T A^T A X = (AX)^T (AX) = ||AX||^2 (Nonnegative)$$

1.5 Schur Complement

Let $X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ be a symmetric matrix with the $X \succ 0$. Then

$$A \succ 0 \ \text{ iff } C - BA^{-1}B^T \succ 0.$$

$$C \succ 0 \ \text{ iff } A - BC^{-1}B^T \succ 0.$$

$$If A \succ 0 \ then \ X \succ 0 \text{ iff } C - BA^{-1}B^T \succ 0.$$

1.6 Semi-definite programming

Two common formats of SDP

Type 1

$$\begin{aligned} p^* = \quad & \min \langle C, X \rangle \\ & \text{s.t.} \qquad \mathcal{A}X = b \quad \text{(linear)} \\ & \qquad\qquad X \succeq 0 \end{aligned}$$

$$\mathcal{A}(\alpha X_1 + \beta X_2) = \alpha \mathcal{A}(X_1) + \beta \mathcal{A}(X_2), \quad \forall X_i, \text{ and scalars } \alpha, \beta$$

Type 2

$$\begin{aligned} p^* = \quad & \min \sum C_i, X_i \\ & \text{s.t.} \qquad \mathcal{A}_\rangle X_i \succeq b \quad \text{(linear)} \end{aligned}$$

1.7 SDP Duality

for the primal

$$\begin{aligned} p^* = \quad & \max \quad CX \\ & \text{s.t.} \quad b \succeq \mathcal{A}X \\ & \qquad\quad X \succeq 0 \end{aligned}$$

dual is

$$\begin{aligned} d^* = \quad & \min \quad b^T y \\ & \text{s.t.} \quad \mathcal{A}^* y = C \end{aligned}$$

1.8 SDP Quadratic Program

TODO

1.9 SDP Polynomials

TODO