Convex Sets and Convex Functions - Summaries (CS395T Optimization)

Shyam Bhagwat , shyambhagwat@utexas.edu

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1 Convex Sets

A set of points, C $\subset Rn$, is **convex** if for any two points $x_1, x_2 \in C$, and any $\alpha \subset [0,1]$ every point on the line segment between x_1 and x_2 is also in C.

$$\forall x_1, x_2 \in C, \forall \alpha \in [0,1], \quad \alpha x_1 + (1-\alpha)x_2 \in C.$$

1.1 Operations Preserving Convexity

Shifting , Scaling,Rotating - preserves convexity $C = \{Ax + b | x \in C\}$

Intersection - Let X_1, X_2 be two convex sets. The intersection, $X \doteq X_1 \cap X_2$, is also convex.

Example 1 (Polyhedron)**.** *A **polyhedron** is an intersection of halfspaces.*

Note - Unions are not convex

1.2 Examples of Convex Sets

1.2.1 Convex hull of n points

Let there be $x1, x2, x3..xn \in C, X = \theta1x1 + \theta2x2 + \theta3x3$....n for any $\theta1$+$\theta2 + \theta3..n = 1$

Note : Sets can also have convex hulls. Let S be a non convex set . Set of a non convex set S is a convex hull of original non covex set S and the additional elements not in the original set S.

1.2.2 Affine Sets

A set is **affine** if for any two points $x_1, x_2 \in X$, every point on the entire line containing x_1 and x_2 is also in X.

$$\forall x_1, x_2 \in X, \forall \alpha \in R, \quad \alpha x_1 + (1 - \alpha)x_2 \in X.$$

1.2.3 Hyperplanes and Halfspaces

A **hyperplane** is a set of the form

$$\{x \ : \ a^T x = b\},$$

for some fixed vector a and fixed scalar b. **Hyperplanes are convex sets.**

A **halfspace** is a set of the form

$$\{x \ : \ a^T x \leq b\}.$$

Geometrically speaking, a halfspace is a hyperplane and all of the points to one side of it.

1.2.4 Conic Combination

Let there be $x1, x2, x3..xn \in C, X = \theta1x1 + \theta2x2 + \theta3x3$....n for any $\theta1$+$\theta2 + \theta3..n >= 0$

1.2.5 Ellipses

Let X be all points in ellipse, center C and matrix M defining how much ellipse strech in direction can be defined as

$(X - C)^T M(X - C) <= 1$ where M is positive semi definite

1.2.6 Polyhedra

Intersection of halfspaces and hyperplanes $x : Ax <= b, Cx = d$

1.2.7 Set of all positive semi definite matrices is also convex

1.2.8 Norm Balls

Euclidian, L1 etc are also convex. TODO - proof

1.2.9 Polynomials with lower bound

Polynomial with lower bound is also convex. $P_(d, l) = \{p(x) : p(x) >= l \forall x \in S\}$

Note: Polynomial whose max on set is lower bound is not convex.

2 Convex Functions

A function $f : Rn \to R$ is **a convex function** if f is a convex set, and for all points $x, y \in f$, and all $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

A function is **concave** if $-f$ is convex.

2.1 Examples of Convex Functions

The following are some examples of convex functions of scalar variables.

- *Linear function.* $f(x) = ax + b$ is convex on $\mathbb{R}^n$.

- *Exponential.* $f(x) = e^{ax}$ is convex on , for any $a \in$.
- *Polynomial Powers.* $f(x) = x^a$ is convex for any $a \geq 1$ or $a \leq 0$ and x¿=0
- *Negative entropy* xlogx is convex
- *Logarithm det(x)* . $f(x) = -\log x$ is convex

- *Norms.* All norms on $\mathbb{R}^n$ are convex (note this follows from the triangle inequality and using it to directly prove $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$.
- *Max function.* $f(x) = \max\{x_1, \ldots, x_n\}$ is convex on $\mathbb{R}^n$.

2.2 Operations that preserve convexity

Usually in optimization, we have functions that are composed with one another and not nice functions that we can directly compute hessians or use the definition of convexity to show functions are convex. We revert to rules that preserve convexity. Below are some key operations that preserve convexity. All the rules below are worth memorizing.

- Nonnegative weighted sum of convex functions: $\sum_{i=1}^{n} a_i f_i \quad a_i \geq 0$ and f_i convex $\forall i$ (extends to infinite sums, integrals)
- Composition with linear function: f convex $\Rightarrow f(Ax+b)$ is convex
- Pointwise maximum of convex functions: if f_i are convex for $i = 1, \ldots, n$ then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex
- Pointwise supremum: if $f(x,y)$ is convex in x, then for $y \in C \quad g(x) = \sup_{y \in C} f(x,y)$ is convex (note C does not need to be convex!) **This rule you will use repeatedly**
- Pointwise Infimum: if $f(x,y)$ is jointly-convex in (x,y) (not the same as separately being convex in x and in y) and C is a convex set then $g(x) = \inf y \in C f(x,y)$ is convex
- Composition: g is convex, h convex and nondecreasing, then $f(x) = h(g(x))$ is convex
- Composition: g is concave, h convex and nonincreasing, then $f(x) = h(g(x))$ is convex
- Composition: g is concave, h is concave and nondecreasing, then $f(x) = h(g(x))$ is concave

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Example : $h(x) = x^2, g(h(x)) -- > f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$. Note for the compositions, we need to be careful about the range of $g(x)$ and domain of $h(x)$ - if the range of $g(x)$ contains values that are not in the domain of $h(x)$ then $h(g(x))$ is undefined.

2.3 First-Order Conditions for Convexity

Suppose f is differentiable. Then f is convex if and only if f is convex, and for every $x, y \in f$,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

This is saying that the linearization of f at x is a *global underestimator* for f.

2.4 Second-Order Conditions for Convexity

Suppose f is twice-differentiable. Then f is convex if and only if f is convex and the Hessian of f, $\nabla^2 f$, is positive semi-definite everywhere: $\forall x \in f$,

$$\nabla^2 f(x) \succeq 0.$$