Conjugate Functions - Summaries and Concepts

Shyam Bhagwat , shyambhagwat@utexas.edu

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1 Convex Conjugate

1.1 Conjugate functions

Conjugate function f: $R^n -> R$ conjugate f^*- $R^n -> R$

$$f^*(y) = \max_x y^T x - f(x) \tag{1}$$

where f^* is always convex because its max of affine functions

1.2 Conjugate function Properties

- $f(x) + f^*(y) \geq x^T y$
- $f(u, v) = f1(u) + f2(v) => f^*(w, z) = f1(w) + f2(z)$
- Double Conjugate is always convex $f^{**}z = max_y z^T y - f^*(y) => f^{**}(z) \leq f(z)$
- if f is convex,

  Given $\bar{y}, let\ \bar{x}\ st\ \bar{y} \in \partial f(\bar{x})$
  from convexity $f(x) \geq f(\bar{x}) + y^T(x - \bar{x})$
  $\bar{y}^T.x - f(x) \leq \bar{y}^T.\bar{x} - f(\bar{x})$

  Hence, $f^*(y) = \bar{y}^T \bar{x} - f(\bar{x})$
  (2)

- If f is closed and convex, then
  - $f^{**} = f$
  - $x \in \partial f^*(y) \Leftrightarrow y \in \partial f(x) \Leftrightarrow f(x) + f^*(y) = x^T y$
- If f is not convex, then , f^{**} is the convex envelope of f

if $g(x) \leq f(x) \forall x$
and g is convex then
$g(x) = \leq f^{**}(x)$
(3)

1.3 Conjugate functions Gradient Property

If f is closed and convex, derivatives of f and f^* are inverse of each other

$$y = \nabla f(x). => x = \nabla f^*(y)$$

, also $y \in \partial f(x). => x = \partial f^*(y)$(4)

1.4 Examples

Support function

$$f(x) = supp(x)$$

on the domain

$$\{x : max(x_i| \leq 1\}$$

$$f^{**}(x) = ||x||_1$$

Simple quadratic

$$f(x) = \frac{1}{2} x^T Q x$$

$$f^*(y) = \frac{1}{2} y^T Q^{-1} y$$

Indicator function

$$f(x) = I_C(x)$$
$$f^*(y) = I_C^*(y) = \max_{x \in C} y^T x$$

Norm

$$f(x) = ||x||$$
$$f^*(y) = I_{z:||z||_* \leq 1}(y)$$

1.5 Conjugate functions and Duality

Conjugates also appear frequently in derivation of dual problems, via

$$-f^*(u) = \min_x f(x) - u^T x \tag{5}$$

in minimization of the Lagrangian.

For example, consider

$$\min_x f(x) + g(x) \tag{6}$$

The dual problem is

$$\max_u -f^*(-u) - g^*(-u) \tag{7}$$

We can find the dual of indicator functions and norms with the last equation.

Dual formulation can also help us by "shifting" a linear transformation between one part of the objective and another. Consider

$$\min_x f(x) + g(Ax) \tag{8}$$

By reparameterizing $z = Ax$, we have the dual problem as

$$\max_u -f^*(A^T u) - g^*(-u) \tag{9}$$