Mathematics Notes

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Chapter 1

Number Systems

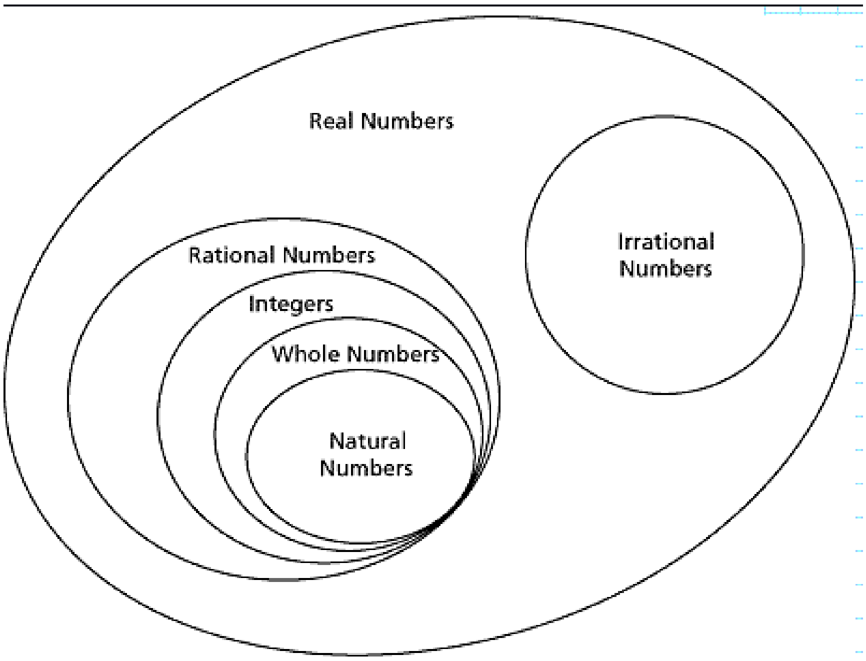
1.1 Introduction

- **N, Natural Numbers**: 1,2,3...
- **W, Whole Numbers**: 0,1,2,3,...
- **Z, Integers**: ...-3,-2,-1,0,1,2,3,...
- **Q, Rational Numbers**
- **P, Irrational Numbers**
- **R (Real Numbers)**: Collection of all rational and irrational numbers, i.e. a real number is either rational or irrational.

1.2 Real Number Line

Number Line On the number line, distances from a fixed point are marked in equal units positively in one direction and negatively in the other . The point from which the distances are marked is called the **origin**. We use the number line to represent the numbers by marking points on a line at equal distances. If one unit distance represents the number 1, then 3 units distance represents the number 3, 0 being at the origin. The point in the positive direction at a distance r from the origin represents the number r . The point in the negative direction at a distance r from the origin represents the number $-r$.

Real Number Line Every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number - this is why the number line is called the Real Number Line.



1.3 Rational Numbers

Definition: A number 'r' is called a rational number if it can be written in the form p/q, where:

- p and q are integers
- $q \neq 0$
- p,q have no common factors other than 1 (i.e. p and q are co-prime).

Important properties of Rational Numbers:

- An integer, for example 25, can be written as 25/1, therefore rational numbers include natural numbers, whole numbers and integers.
- Rational numbers do **NOT** have a unique representation in the form p/q. For example, $1/2 = 2/4 = 10/20$ and so on. These are **equivalent rational numbers**. When we say p/q is a rational number or represent it on the number line, we assume that p and q are co-prime. On the number line, among the infinitely many fractions equivalent to 1/2, we will choose 1/2 to represent them all.
- There are infinitely many rational numbers between any two given rational numbers.
- 0 is a rational number
- Decimal expansion of a rational number is either terminating (example $1/4 = 0.25$) or non-terminating recurring (example $1/3 = 0.333.....$)
- Number of recurring entries is less than the divisor. For example, in 1/7, there are six entries 326451 in the repeating string of remainders and 7 is the divisor. If the remainder's repeat, then we get a repeating block of digits in the quotient (142857 in case of 1/7).
- If x is a rational number (of the form p/q, where p and q are co-prime), whose decimal expansion terminates, then prime factorization of q is of the form 2^m5^n, where n and m are non-negative numbers. Vice versa is also true, i.e. if prime factorization of q sis of the form 2^m5^n, then x has a decimal expansion which terminates. If the prime factorization of q is not of the form 2^m5^n then the decimal expansion is non-terminating and recurring.

1.4 Irrational Numbers

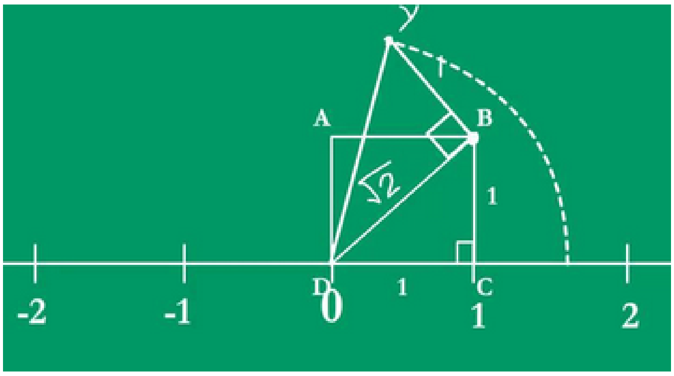
- A number s is called irrational, if it **cannot** be written in the form p/q, where p and q are integers and $q \neq 0$
- The decimal expansion of an irrational number is non-terminating and non-recurring.

1.5 Examples

Example: Express $0.333..$ as a rational number in the p/q form.

Solution:
$x = 0.333.....$
$10x = 3.333.... = 3 + 0.333... = 3 + x => 9x = 3 => x = 1/3$

Example: Locate $\sqrt{3}$ on the number line



Solution

Draw triangle BCD, where $BC = CD = 1 => BD = \sqrt{2}$
Draw BY perpendicular to DB , therefore $DY = \sqrt{3}$
With center D and radius DY draw an arc intersecting the number line to locate $\sqrt{3}$
Using this approach, you can locate the square root of other integers.

1.6 Square Root of Real Numbers

If a is a natural number, then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$. In general, if $a > 0$ be a real number and n is a positive integer, then $\sqrt[n]{a} = b$ if $b^n = a$ and $b > 0$. We follow the below convention:

$\sqrt{x}$ represents the **positive square root** of x and is called the **principle square root** of x.

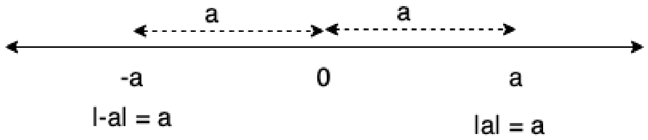
For $y = \sqrt[n]{x}$:

x	n	Result	Example
$x < 0$	*odd*	There exists one negative n^{th} root of x	-2 is the principle cube root of -8
$x < 0$	*even*	n^{th} root of x is not a real number	$\sqrt{-16}$ is not Real

We must emphasize that $\sqrt{x^2} = \mid x \mid \neq x$. For example, $\sqrt{-3^2} = \sqrt{9} = 3$ which is the absolute value of $\mid -3 \mid$.

1.7 Absolute Value

We now introduce the geometric concept of distance in the real number system. Let a be the coordinate of a point on the number line - the number of units between this point and the origin 0 is called the absolute value of a and is denoted by $\mid a \mid$. Thus the absolute value of a number **gives only its distance from the origin and not its direction.**



In general, if a is any real number, then $\mid a \mid = \mid -a \mid$, therefore the absolute value of a number is never negative.
Concept of absolute value can be used to measure the distance between two points - if x_1 and x_2 are co-ordinates for two points A and B on the real number line, then the distance between A and B is $\mid x_1 - x_2 \mid$

Important Propositions related to Absolute Value:

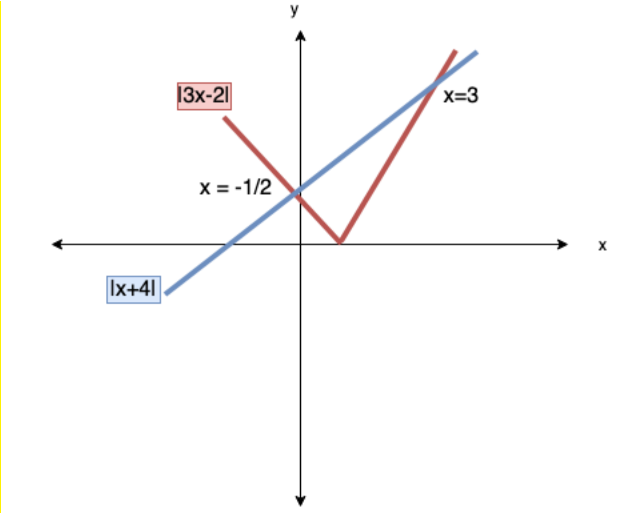
- $\mid a \mid = \mid -a \mid$
- $\mid a.b \mid = \mid a \mid . \mid b \mid$
- $ab \leq \mid ab \mid$
- $\mid a+b \mid \leq \mid a \mid + \mid b \mid$
- $\mid a \mid - \mid b \mid \leq \mid a - b \mid$
- $\mid a/b \mid = \mid a \mid / \mid b \mid$
- $\mid\mid x \mid - \mid y \mid\mid \leq \mid x + y \mid \leq \mid x + \mid y \mid$
- $\mid x+y \mid = \mid x \mid + \mid y \mid$ when x and y have the same sign, i.e. $x.y \geq 0$
- Equations in absolute value must be equal to each other or be negative to each other. For example, if $\mid 3x+1 \mid = \mid x-2 \mid$. This implies that $3x+1 = x-2$ or $3x+1 = -(x-2)$ and the solutions are $x = -3/2, 1/4$

Example: Solve $\mid 3x-2 \mid \geq \mid x+4 \mid$
Solution Lets first identify the critical points (where the value equates to zero) for each side: in this case its $2/3, -4$

- $\mid 3x-2 \mid = 3x-2 \forall x > 2/3$ and $\mid 3x-2 \mid = -(3x-2) \forall x < 2/3$
- $\mid x+4 \mid = (x+4) \forall x > -4$ and $\mid x+4 \mid = -(x+4) \forall x < -4)$

For $x < 2/3$, our equation becomes $x+4 = -(3x-2)$, therefore $x = -1/2$
For $x > -4$, our equation becomes $x+4 = 3x-2$, therefore $x = 3$

Solution is the values for which graph of $\mid 3x-2 \mid$ is above the graph of $\mid x+4 \mid$



1.8 Intervals

Open Interval: If a and b are real numbers and $a < b$, then the open interval from a to b is written (a, b) , is the collection of all real numbers greater than a and less than b, i.e. $(a, b) = x|a < x < b$. Geometrically, its represented on the real number line as shown below - the open circles represent the fact that a and b are **not included** in the interval.

Closed Interval: If a and b are real numbers and $a < b$, then the closed interval from a to b is written $[a, b]$, is the collection of all real numbers greater than equal to a and less than equal to b, i.e. $[a, b] = x|a \leq x \leq b$. Geometrically, its represented on the real number line as shown below - the closed circles represent the fact that a and b are also **included** in the interval.

1.9 Euclid's Division Lemma

Given positive integers a and b, there exist unique integers q and r satisfying $a = bq + r$ where $0 \leq r < b$

1.9.1 Highest Common Factor

HCF of two positive integers a and b is the largest positive integer d that divides both a and b. We can use Euclid's division lemma to find the HCF of two positive integers. For example, to find the HCF of 455 and 42:
1. Start with highest number, $455 = 42 \times 10 + 35$
2. Now consider the divisor and remainder: $42 = (35 \times 1) + 7$
3. Repeat the above step with divisor and remainder, $35 = (7 \times 5) + 0$ 4. Since the remainder is zero, the divisor at this stage, i..e 7 will be the HCF

$HCF(a, b) \times LCM(a, b) = a \times b$

1.10 Fundamental Theorem of Arithmetic

Every composite (i.e. non-prime) number can be factored as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

Example: $32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$

- $HFC(a, b)$ is the product of smallest power of each "common" prime factor. Example, $6 = 2^1 \times 3^1$ and $20 = 2^2 \times 5^1$, so the HCF $= 2^1$ which is the only common prime factor.
- $LCM(a, b)$ is the product of "greatest" power of "each" prime factor involved in the numbers. So in the above example, $LCM(6, 20)$ will be $2^2 \times 3^1 \times 5^1 = 60$

Chapter 2

Polynomials

2.1 Introduction

Consider $-x^3 + 4x^2 + 7x - 2$

- **Terms**: This polynomial has four terms, $-x^3, 4x^3, 7x, -2$
- **Coefficient**: Coefficients of the four terms are $-1, 4, 7, -2$
- **Exponents should be whole numbers.** For exampple, $1 + 1/x$ is not a polynomial, as the coefficient of the second term is -1 which is nota whole number. Another example is $\sqrt{x} + 3$.
- **Constant Polynomial**: Example, 2 is also a polynomial
- **Zero Polynomial**: Constant 0 is called the zero polynomial
- **Monomial**: Polynomial with only one term
- **Binomial**: Polynomial with two terms
- **Degree of a polynomial**: Highest power of the variable in a polynomial. For example the degree of $-x^3 + 4x^2 + 7x + 2$ is 3
- Degree of a non-zero constant polynomial is zero, degree of a zero polynomial is not defined
- **Linear Polynomial**: Degree is 1. For example, $p(x) = 4x + 5$ is a linear polynomial and is of the form $ax + b$, can contain a maximum of two terms.
- **Quadratic Polynomial**: Degree is 2, is of the form $ax^2 + bx + c$ and will contain a maximum of three terms. In general, a polynomial of degree n will contain a maximum of $n + 1$ terms.
- **Zero of a Polynomial**: Consider $p(x) = x - 1$. When we equate $p(x)$ to 0, we get $x = 1$, and 1is the root (or zero) of the polynomial.
- Non-zero constant polynomial has no zero
- Every real number is a zero of the zero polynomial
- A linear polynomial has one and only one zero

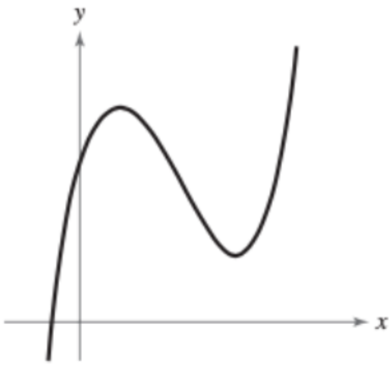
In general, a polynomial on one variable x of degree n is an expression of the form:

$$\boxed{p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0}$$

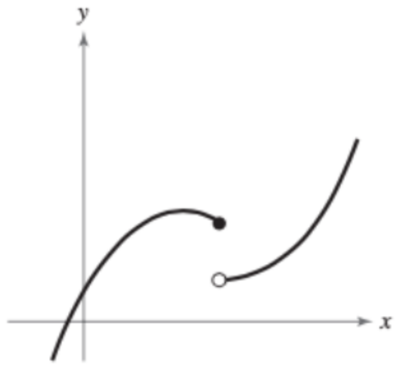
where $a_0, a_1, ...$ are constants and $a_n \neq 0$.

2.1.1 Graphs of Polynomial Functions

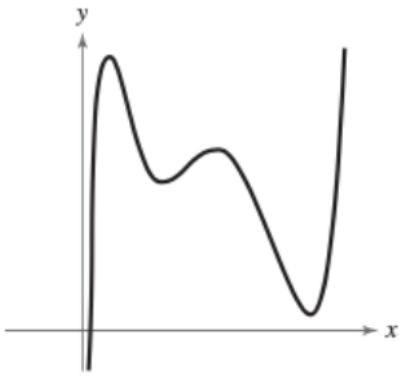
- The graph of a polynomial function is continuous. Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps. Informally, you can say that a function is continuous when its graph can be drawn with a pencil without lifting the pencil from the paper.
- Another feature of the graph of a polynomial function is that it has only smooth, rounded turns. It cannot have a sharp turn such as the one shown below.



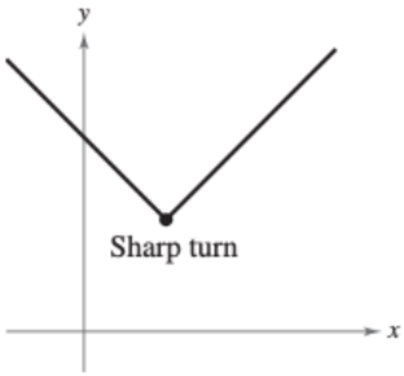
(a) Polynomial functions have continuous graphs.



(b) Functions with graphs that are not continuous are not polynomial functions.



(a) Polynomial functions have graphs with smooth, rounded turns.



(b) Functions with graphs that have sharp turns are not polynomial functions.

2.2 Remainder Theorem

If $p(x)$and $g(x)$ are two polynomials such that degree of $p(x)$ > degree of $g(x)$ and $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that:

$$p(x) = g(x)q(x) + r(x)$$

where $r(x) = 0$ or degree of $r(x)$ < degree of $g(x)$. Essentially, what this means is $p(x)$ divided by $g(x)$ gives us $q(x)$ as the quotient and $r(x)$ as the remainder.

Remainder Theorem: Let $p(x)$ be any polynomial of degree ≥ 1 and let a be any real number. If $p(x)$ is divided by $x - a$, then the remainder is $p(a)$

2.3 Factor Theorem

Factor Theorem:If p(x) is a polynomial of degree $n \geq 1$ and a is any real number, then:

- $x - a$ is a factor of $p(x)$ if $p(a) = 0$, and
- $p(a) = 0$ if $x - a$ is a factor of $p(x)$

2.4 Useful Identities

- $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$
- $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$
- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
- $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$
- $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$
- $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

2.5 Synthetic Division

Synthetic division is used to divide one polynomial, $P(x)$ by a polynomial of the form $x - k$, where the coefficient of x is 1. It involves the below steps:

1. Arrange $P(x)$ in descending powers of x. If a term is missing, write a 0 for its coefficient.
2. Place k, the additive inverse of $-k$ in the divisor.
3. Bring down the leading coefficient of largest power of x to the 3rd row.
4. Multiply the leading coefficient we bought down by k, place the product in the next position under the 2nd coefficient of the dividend and add the two numbers
5. Multiply this sum by k, place the product under the 3rd coefficient and add.
6. Continue this process until all coefficients in the dividend are used.

Example: Divide $P(x) = 2x^4 - 3x^3 - 4x + 8$ by $x - 3$

Solution: : Using Synthetic Division, we can see from the below figure that the quotient is $2x^3 + 3x^2 + 9x + 23$ and remainder is 77

3	2	-3	0	-4	8
		6	9	27	69
	2	3	9	23	77 = remainder

2.6 Zeros of a Polynomial

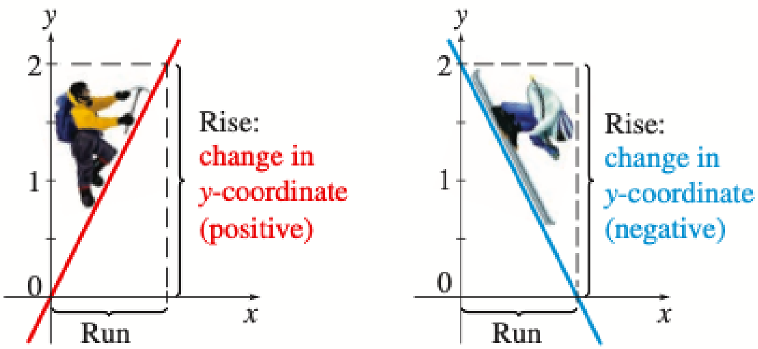
The zeros of a polynomial are the solutions to the equation $p(x) = 0$, where p(x) represents the polynomial. If we graph this polynomial as $y = p(x)$, then you can see that these are the values of x where $y = 0$. In other words, they are the x-intercepts of the graph. **In general, given a polynomial $p(x)$ of degree n, will have a maximum of n zeroes**

Chapter 3

Graphs of Polynomial Functions

3.1 Linear Polynomial - Straight Line

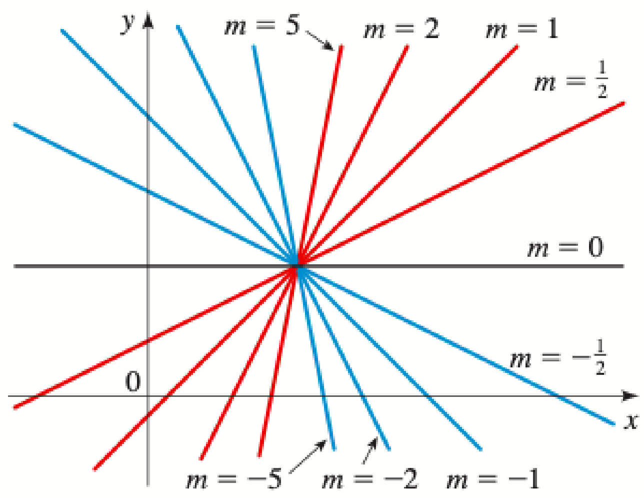
A linear polynomial is of the form $p(x) = mx + c$, and its graph is a straight line. Here m is the slope or "steepness" of this straight line, i.e. how quickly it rises (or falls) as we move along the $x - axis$ and the y-intercept is $(0, c)$. The slope of a line is the ratio of $rise/run$:



The slope m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = rise/run = (y_2 - y_1)/(x_2 - x_1)$$

- A horizontal line has slope zero.
- The slope of a vertical line is not defined
- If two lines are parallel, they have the same slope
- If two lines are perpendicular to each other, the product of their slopes is -1
- Graph of $y = ax+b$ is a straight line which intersects the x-axis at exactly one point, $(-b/a, 0)$. This means that a linear polynomial has exactly one zero, namely the point where the graph intersects the x-axis.
- Figure below shows several lines with their slopes - lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right.

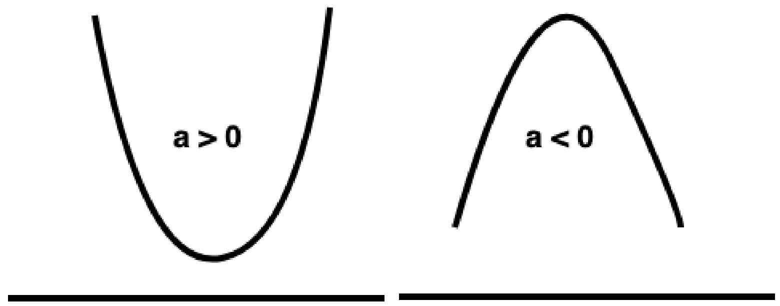


3.2 Quadratic Polynomial - Parabola

A quadratic polynomial can be written in the form $p(x) = ax^2 + bx + c$ where $a \neq 0$ and a, b, c are constants.
The **standard form** of a quadratic function is $p(x) = a(x - h)^2 + k, a \neq 0$
The graph of a quadratic function is called a parabola

3.2.1 Leading Coefficient

- In either form: If leading coefficient, $a > 0$ then the parabola opens up. If leading coefficient, $a < 0$ then the parabola opens down.
- The leading coefficient indicates how "fat" or how "skinny" the parabola will be:
  - If $\mid a \mid > 1$ the parabola will be "skinny" because it grows more quickly
  - If $\mid a \mid < 1$, the parabola will be "fat" because it grows more slowly.



3.2.2 Vertex

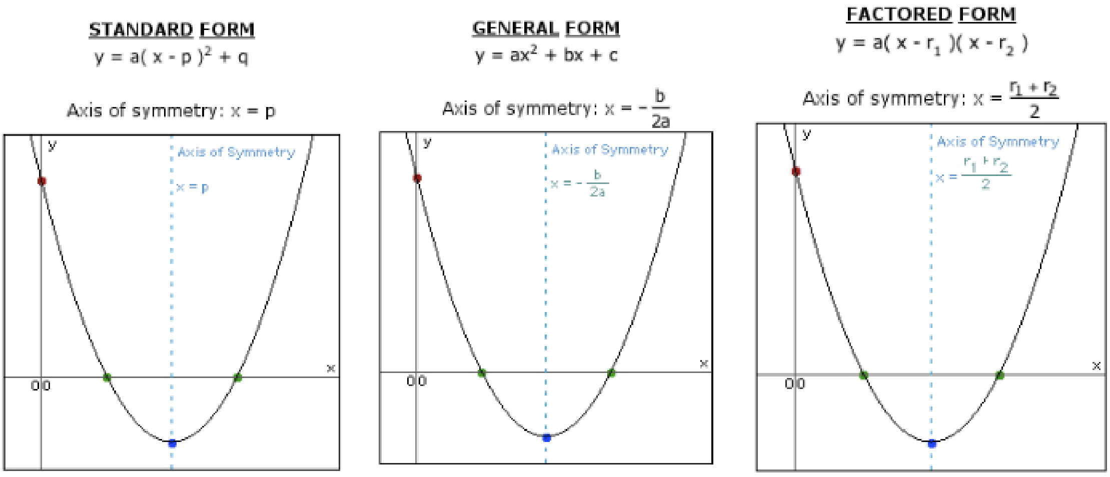
The vertex is the lowest or highest point (depending on direction) on the graph of a quadratic function.

- In the non-standard form, i.e. $p(x) = ax^2 + bx + c$, the vertex is at the coordinate $-b/2a, p(-b/2a)$.
- In the standard form, $p(x) = a(x - h)^2 + k, a \neq 0$ the vertex is at the coordinate (h, k).
- Let's assume we need to find the vertex of $p(x) = -(x - 5)(x - 9)$ which is in the factored form:
  1. Determine the zeros, in this case it is $-2, 4$
  2. Determine the x-coordinate of the vertex by averaging the zeros, $(-2 + 4)/2 = 1$
  3. Determine the y-coordinate of the vertex by substituting the x-coordinate of vertex and solving for y, $y = (1 + 2)(1 - 4) = -9$
  4. Vertex is (1,-9)

3.2.3 Axis of Symmetry

Each parabola is symmetric about a vertical line called the axis of symmetry. This vertical line goes through the vertex.

- In the non-standard form, i.e. $p(x) = ax^2 + bx + c$, the line of symmetry is represented by the vertical line $x = -b/2a$
- In the standard form, $p(x) = a(x - h)^2 + k, a \neq 0$ the line of symmetry is the vertical line $x = k$
- In the factored form, $p(x) = a(x-r_1)(x-r_2)$, the line of symmetry is the vertical line $x = (r_1+r_2)/2$

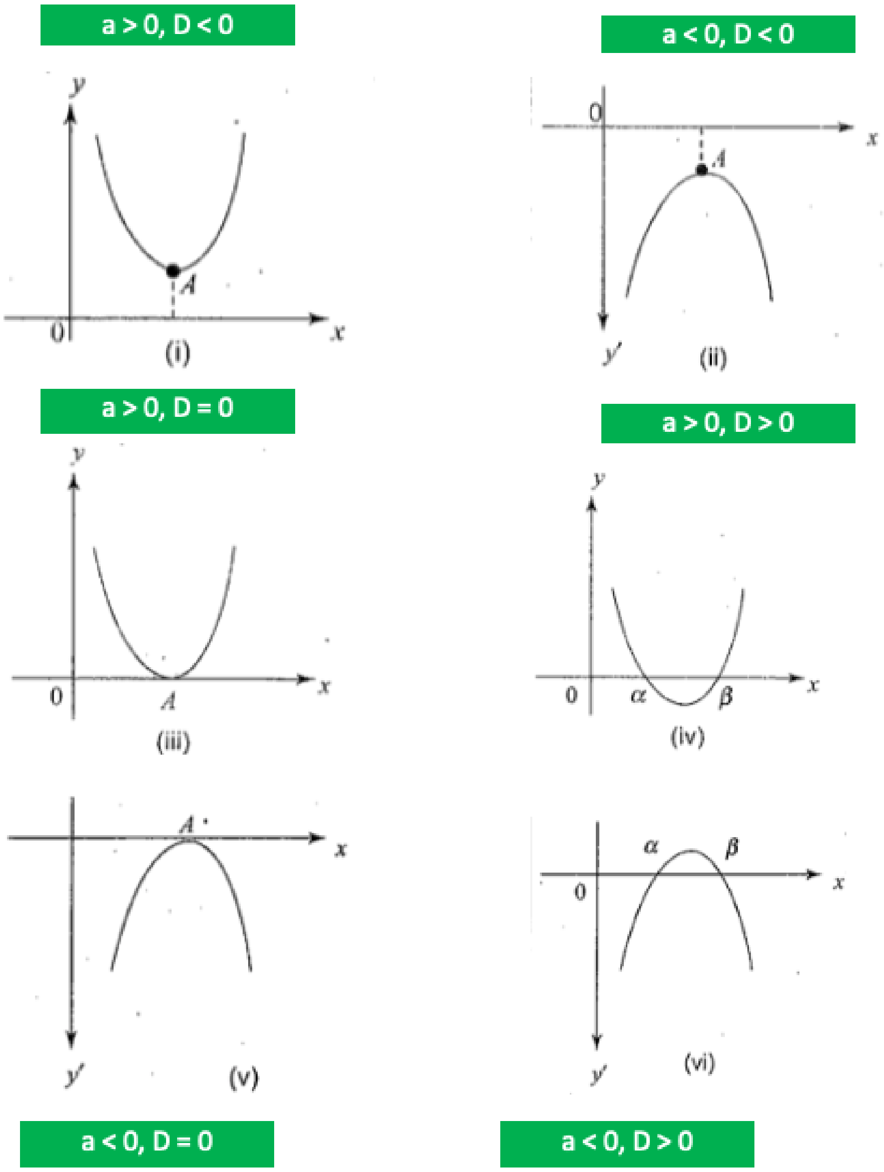


3.2.4 Discriminant

In the polynomial $p(x) = ax^2 + bx + c$, the $D = b^2 - 4ac$ is the discriminant.

a	D	p(x)
$a > 0$	$D < 0$	$p(x) > 0 \forall x$
$a < 0$	$D < 0$	$p(x) > 0 \forall x$
$a > 0$	$D = 0$	$p(x) > 0 \forall x$ except at vertex.
$a > 0$	$D > 0$	$p(x)$ has two real roots α, β, where $\alpha < \beta$. $p(x) > 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$. $p(x) < 0 \forall x \in (\alpha, \beta)$
$a < 0$	$D = 0$	$p(x) < 0 \forall x$ except at the vertex
$a < 0$	$D > 0$	$p(x)$ has two real roots α, β, where $\alpha < \beta$. $p(x) < 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$. $p(x) > 0 \forall x \in (\alpha, \beta)$

Please refer to the figure in the next page for the graphs associated with these conditions.

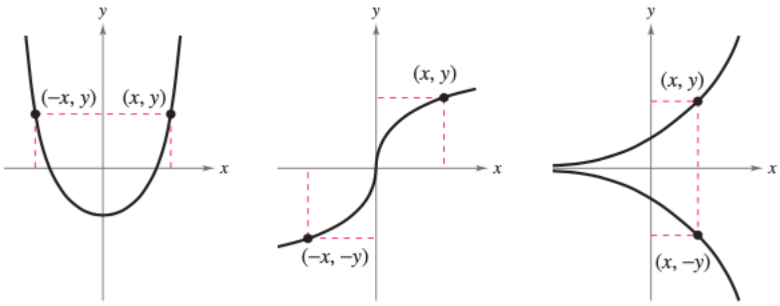


3.3 Graphs of Other Polynomial Functions

We will consider the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, where $a_n \neq 0$ and the leading term is $a_n x^n$

3.3.1 Symmetry

- If the function is an even function, that is, $f(-x) = f(x)$, its graph is symmetric with respect to the y-axis.
- If the function is an odd function, that is, $f(-x) = -f(x)$, its graph is symmetric with respect to the origin.



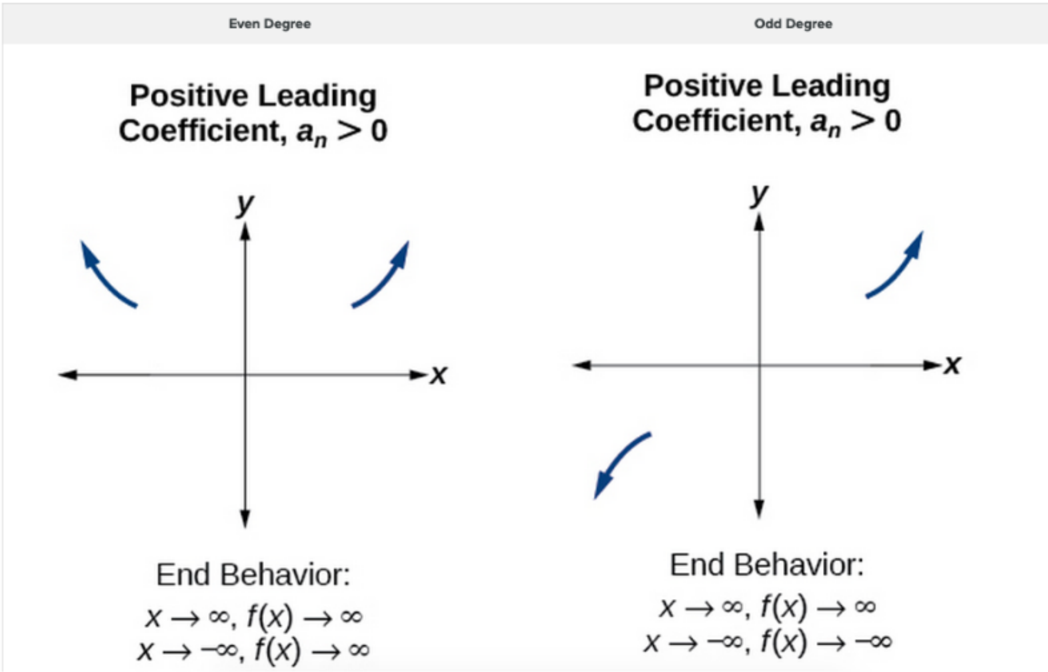
Symmetric to y-axis
Even function

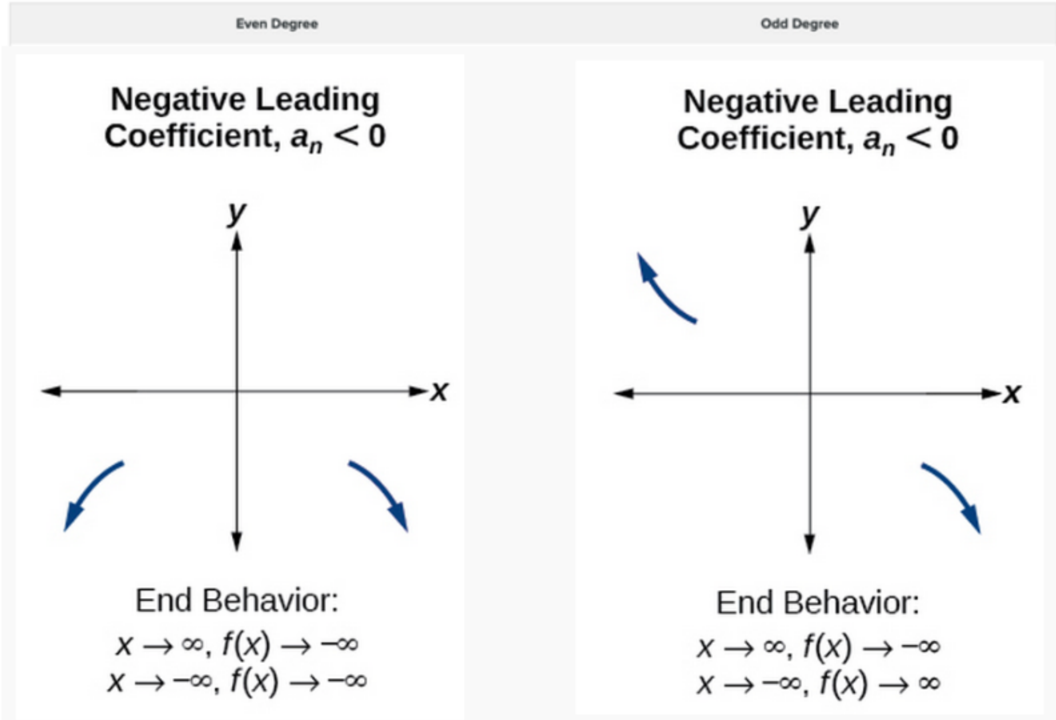
Symmetric to origin
Odd function

Symmetric to x-axis
Not a function

3.3.2 End Behavior - Leading Coefficient and Degree

The graph of a polynomial function will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large or very small inputs, the **leading coefficient dominates the size of the output**. We call this behavior the **end behavior** of a function.

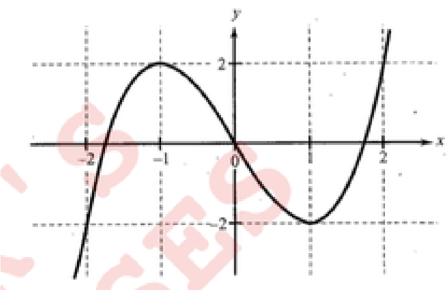




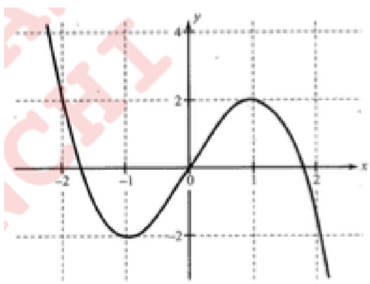
n	a_n	Graph
Odd	Positive	Graph falls to the left and rises to the right
Odd	Negative	Graph rises to the left and falls to the right
Even	Positive	Graph rises to the left and to the right
Even	Negative	Graph falls to the left and to the right

Refer to the below figure for examples:-

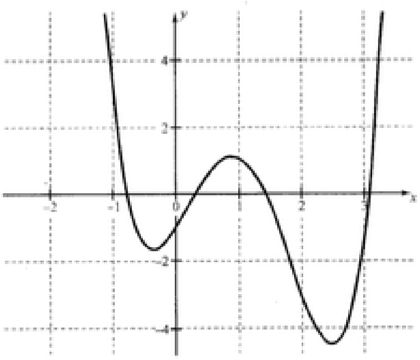
n = odd, a_n +ve



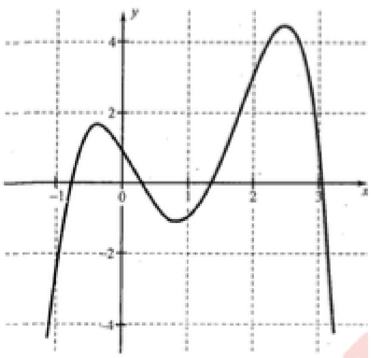
n = odd, a_n -ve



n = even, a_n +ve



n = even, a_n -ve



3.3.3 Zeroes and their multiplicities

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x=h$ is a zero of multiplicity p.

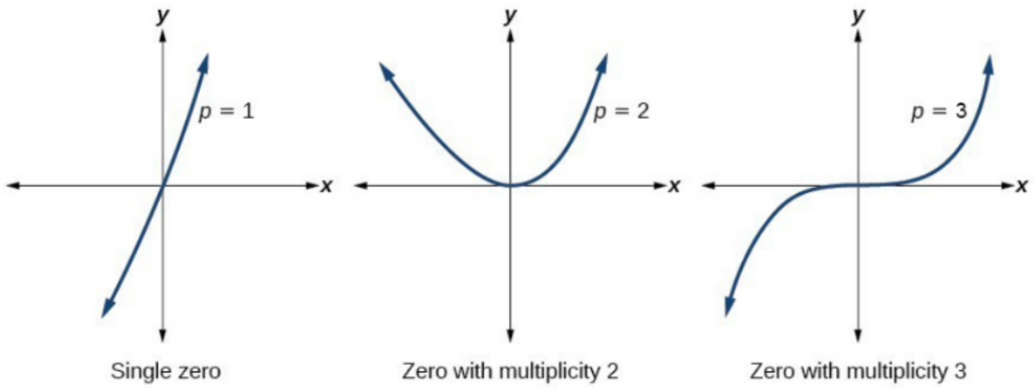
For example, consider the polynomial $p(x)=3(x+5)^3(x+2)^4(x-1)^2)(x-5)$. The multiplicity of each zero is the number of times that its corresponding factor appears:

- $x=-5$ has an odd multiplicty of 3
- $x=-2$ has an even multiplicity of 4
- $x=1$ has an even multiplicity of 2
- $x=5$ has an odd multiplicity of 1

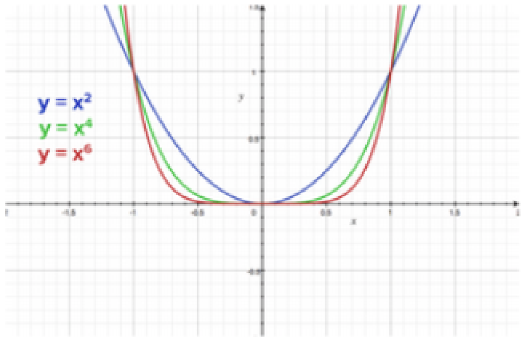
The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

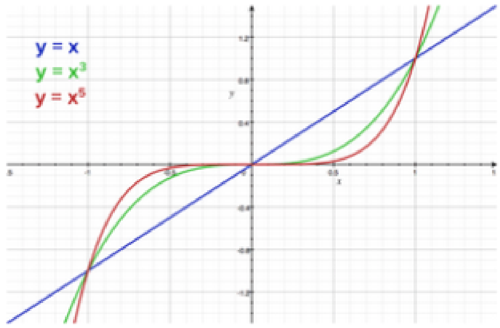
The point of multiplicities with respect to graphing is that any factors that occur an even number of times are squares, so they don't change sign. Squares are always positive. This means that the x-intercept corresponding to an even-multiplicity zero can't cross the x-axis, because the zero can't cause the graph to change sign from positive (above the x-axis) to negative (below the x-axis), or vice versa. An even-multiplicity zero makes the graph just barely touch the x-axis, and then turns it back around the way it came. This means that the sign of $p(x)$ does not change on either side of zero with even multiplicity.

- If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero
- If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity. For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the x-axis, but for each increasing even power the graph will appear flatter as it approaches and leaves the x-axis
- If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity. For higher odd powers, such as 5, 7, and 9, the graph will still cross through the x-axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.
- The sum of the multiplicities is the degree of the polynomial function



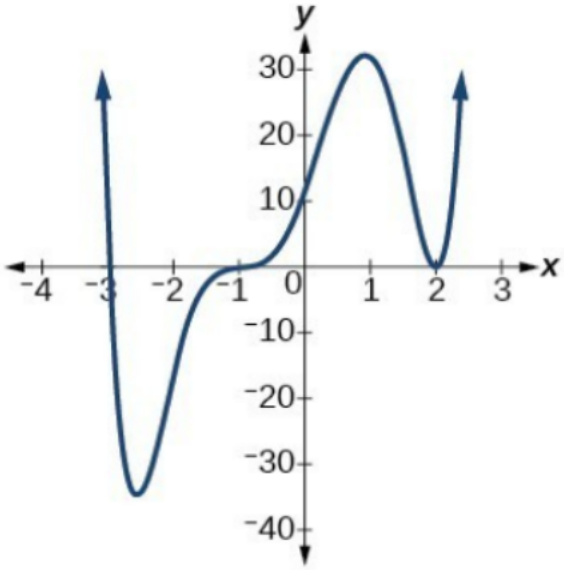
Polynomial zeroes with even and odd multiplicities will always behave in this way. From the graphs below, we can see that for higher even or odd powers, the graph still either bounces or crosses the x-axis, but it becomes flatter as it approaches and leaves the x-axis.





Suppose for example we graph the function $f(x) = (x + 3)(x - 2)^2(x + 1)^3$

- At $x = -3$, we have a single zero (i.e. multiplicity is 1) and the factor is linear, so the behavior near the intercept is like that of a line; it passes directly through the intercept
- At $x = 2$, multiplicity is 2 - graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.
- At $x = -1$, multiplicity is 3. The graph passes through the axis at the intercept but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic with the same S-shape near the intercept



3.3.4 Turning Points

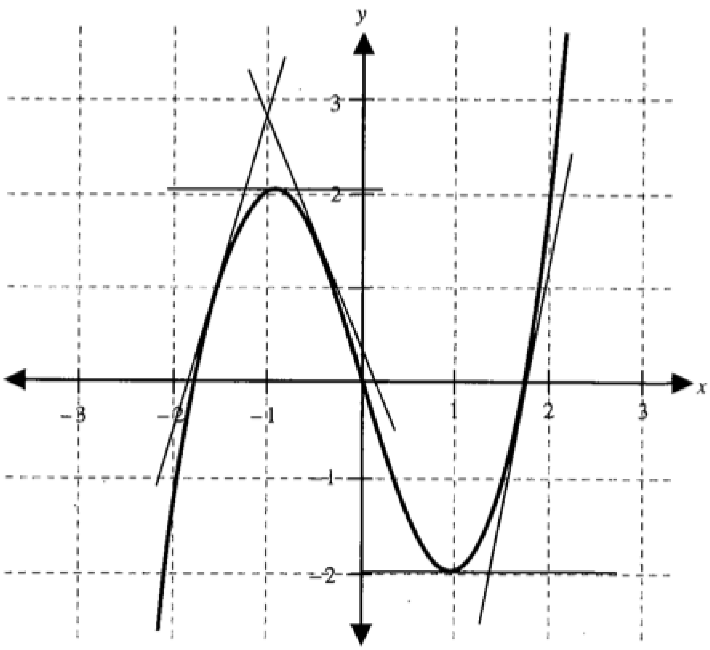
From the previous subsection we can see that the graph of a polynomial consists of a smooth line with a series of hills and valleys called turning points.

- The maximum number of turning points is one less than the degree of the polynomial
- The point where the graph has a turning point, the derivative of the function/polynomial (i.e tangent/slope at that point will be parallel to x-axis) becomes zero, which provides the point of local minima or maxima.

Consider the below graph for example

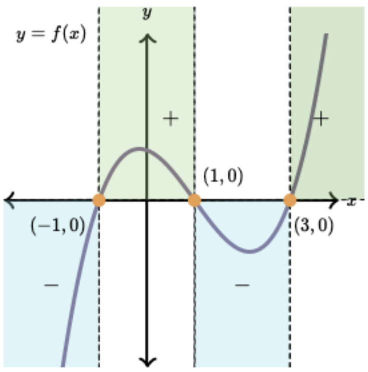
- Tangent to the curve at point for which $x < -1$ and $x > 1$ makes an acute angle with the positive direction of x-axis, hence derivative is positive for these points.
- For $-1 < x < 1$, tangent to the curve makes an obtuse angle with the positive direction of x-axis, hence derivative is negative at these points.

- At $x = 1$ and $x = -1$, the tangent is parallel to x-axis, where derivative is zero.

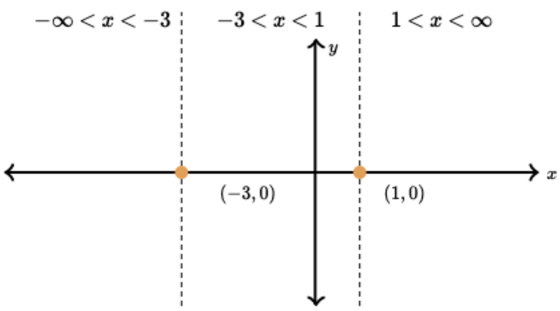


3.3.5 Intermediate Value Theorem

The sign of a polynomial between any two **consecutive zeros** is either always positive or always negative. This is because polynomial functions are continuous functions (no breaks in the graph), which means that the only way to change signs is to cross the x-axis. But if this happened, the given zeros would not be consecutive! It is not necessary, however, for a polynomial function to change signs between zeros.

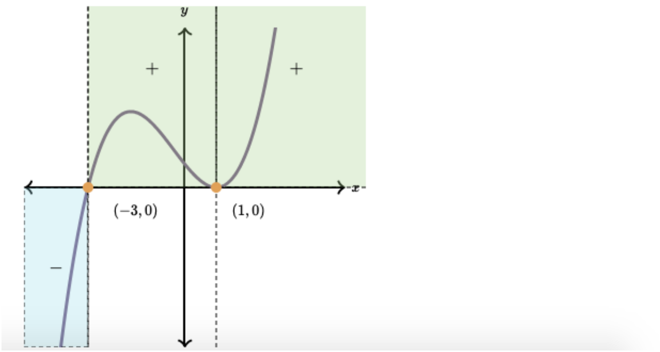


Let's consider the polynomial $p(x) = (x+3)(x-1)^2$. The zeros are $-3, 1$. This creates three intervals over which the sign of $p(x)$ is constant:

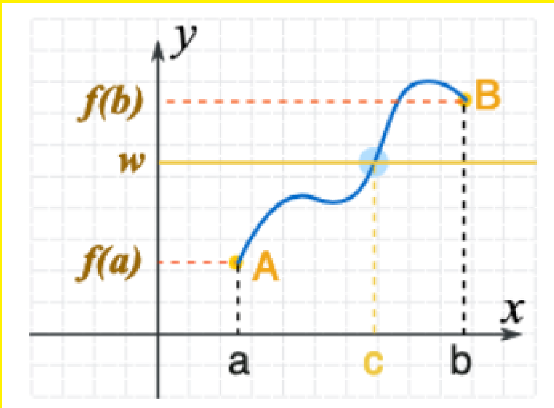


Interval	The value of a specific $f(x)$ within the interval	Sign of f on interval	Connection to graph of f
$-\infty < x < -3$	$f(-4) < 0$	negative	Below the x-axis
$-3 < x < 1$	$f(0) > 0$	positive	Above the x-axis
$1 < x < \infty$	$f(2) > 0$	positive	Above the x-axis

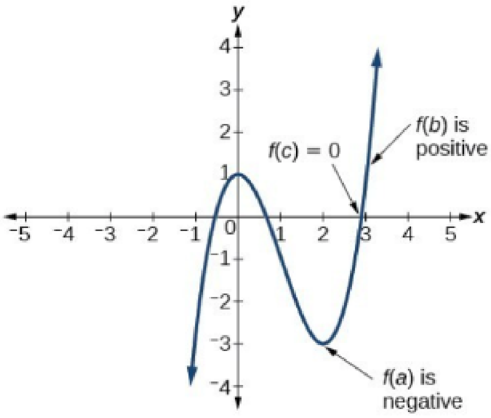
This is consistent with the graph of $y = f(x)$.



Intermediate Value Theorem If $y = f(x)$ is a continuous curve on the interval $[a, b]$, and w is a number between $f(a)$ and $f(b)$, then there must be at least one value, c, within $[a, b]$ such that $f(c) = w$.

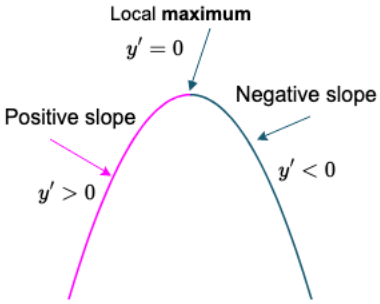


We can use the special case of Intermediate Value Theorem for graphing polynomials - **If there are two points are on opposite sides of the x-axis, we can confirm that there is a zero between them**. If a point on the graph of a continuous function $f(x)$ at $x = a$ lies above the x-axis and another point at $x = b$ lies below the x-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. **In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis**. The figure below shows that there is a zero between a and b. In general, if f(a) and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.



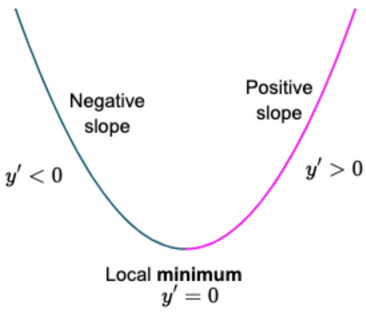
3.3.6 Concavity - Curve Sketching using Differentiation

A local **maximum** occurs when the first derivative = 0 and it changes sign from positive to negative (as we go left to right):



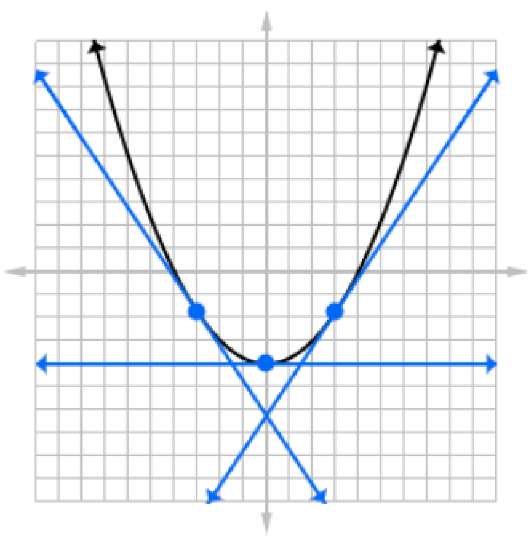
Curve showing portion with positive slope in pink, and negative slope in green.

A local **minimum** occurs when first derivative = 0 and it changes sign from negative to positive

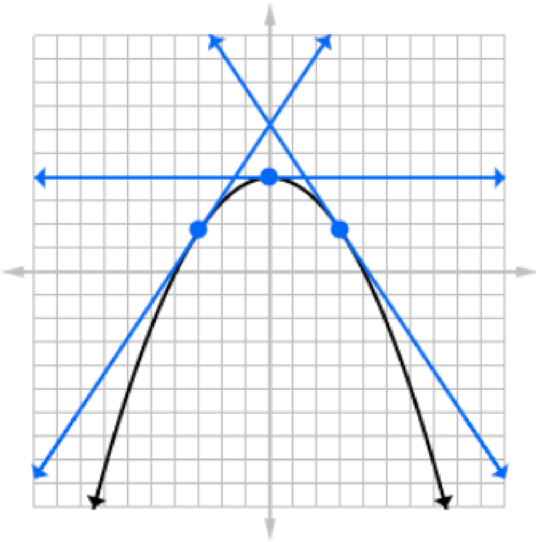


Curve showing positive slope in pink, and negative slope in green.

- We say that a function $y = f(x)$ is concave up (CU) on a given interval if the graph of the function always lies above its tangent lines on that interval. In other words, if you draw a tangent line at any given point, then the graph seems to curve upwards, away from the line.
- Conversely, a function is concave down (CD) on a given interval if the graph of the function always lies below its tangent lines on that interval. That is the graph seems to curve downwards, away from its tangent line at any given point.

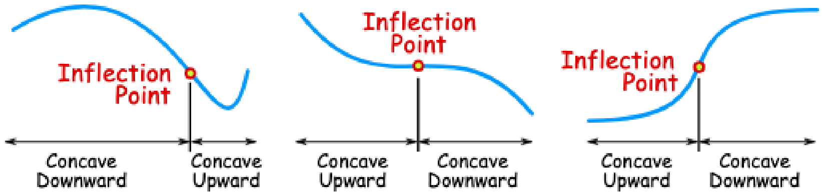


This graph is concave up. At each point, the graph curves upward, away from its tangent line.



This graph is concave down. At each point, the graph curves downward, away from its tangent line.

An **Inflection Point** is a point in a graph at which the concavity changes, i.e. where a curve changes from Concave upward to Concave downward (or vice versa):



Inflection points may be difficult to spot on the graph itself, so we must rely on calculus to find them:

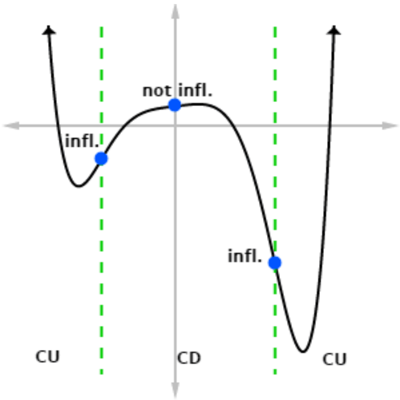
- When $f^{''}(x) > 0$ on an interval, the function is **concave upward** on that interval.
- When $f^{''}(x) < 0$ on an interval, the function is **concave downward** on that interval
- When $f^{''}(x)$ changes sign (from positive to negative, or from negative to positive) at a point $x = c$, then there is an inflection point located at $x = c$ on the graph. In particular, the point $(c, f(c))$ is an inflection point for the function f.

Not all solutions of $f^{''}(x) = 0$ are going to be inflection points. We have to find out how the concavity changes from interval to interval. Once you determine the roots of $f^{''}(x) = 0$, find out the sign (negative or positive) of $f^{''}(x)$ in those intervals.

Consider $f(x) = x^6/30 - x^5/20 - x^4 + 3x + 20$
We see that $f^{''}(x) = x^4 - x^3 - 12x^2 = x^2(x-4)(x+3)$ There are three solutions, $x = 0, 4, and - 3$. These are not necessarily all going to be inflection points, though! We have to find out how the concavity changes from interval to interval first.

Interval	Sample	f"	Concavity
(-∞, -3)	-4	128 (> 0)	CU
(-3, 0)	-1	-10 (< 0)	CD
(0, 4)	1	-12 (< 0)	CD
(4, ∞)	5	200 (> 0)	CU

Notice that there is no change in concavity at x = 0, and the only inflection points are at x = -3 and 4:



Graph of $y = x^6/30 - x^5/20 - x^4 + 3x + 20$, showing intervals of concavity and inflection points. The green vertical lines are not part of the graph, but show where concavity changes.

Chapter 4

Quadratic Equations

4.1 Introduction

When a quadratic polynomial, $p(x) = ax^2 + bx + c$ is equated to zero, we get a quadratic equation. Therefore, a quadratic equation is of the form $ax^2 + bx + c = 0$ where a, b, c are Real Numbers and $a \neq 0$. $b^2 - 4ac$ is called the Discriminant(D):

- If $D \geq 0$ then the roots are real, $(-b \pm \sqrt{D})/2a$
- If $D = 0$, then we have two equal roots, $-b/2a$
- If $D < 0$, then there are no real roots and the graph **does not** cut the x-axis.
- If $D > 0$, then there are two **distinct** real roots and the graph will cut the x-axis at two places.

When coefficients a, b, c are rational or integers:

- $D > 0$ and its a perfect square: Roots are rational.
- $D > 0$ and an imperfect square: Roots are irrational and exist as **conjugate pairs**.

Note 1: If α, β are the roots of a quadratic equation, then the equation can be written in the following ways:

- $(x - \alpha)(x - \beta) = 0$ (by factor theorem)
- $x^2 - (\alpha + \beta)x + \alpha\beta$: This is useful when we know the sum and product of roots.

Note 2: If α, β are the roots of $ax^2 + bx + c = 0$, then:

- Sum of Roots, $\alpha + \beta = (-b/2a)$
- Product of Roots: $\alpha\beta = c/a$

4.2 Common Roots

Let's consider the following quadratic equations:

$$a_1x^2 + b_1x + c_1 = 0$$

$$a_2x^2 + b_2x + c_2 = 0$$

4.2.1 Condition for one common root

If α is the common root for the above equations, then:

$$a_1\alpha^2 + b_1\alpha + c_1 = 0$$

$$a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Solving these two equations by cross-multiplication, we get the condition for one common root

$$\alpha^2/(b_1c_2 - b_2c_1) = \alpha/(c_1a_2 - c_2a_1) = 1/(a_1b_2 - a_2b_1)$$

4.2.2 Condition for both roots to be common

Let α, β be the roots common to the above equation, which means that both equations are identical - the condition for this is:

$$a_1/a_2 = b_1/b_2 = c_1/c_2$$

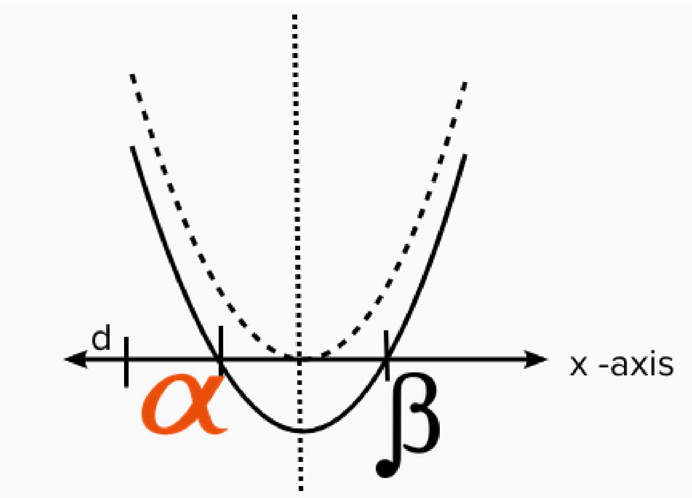
4.3 Location of Roots

For a given quadratic equation $ax^2 + bx + c = 0$, where $a > 0$, the Discriminant(D) $= b^2 - 4ac$, we can have the following conditions:

4.3.1 Both roots are greater than a number d

All the conditions below must be true for both roots α, β to be greater than a number d:

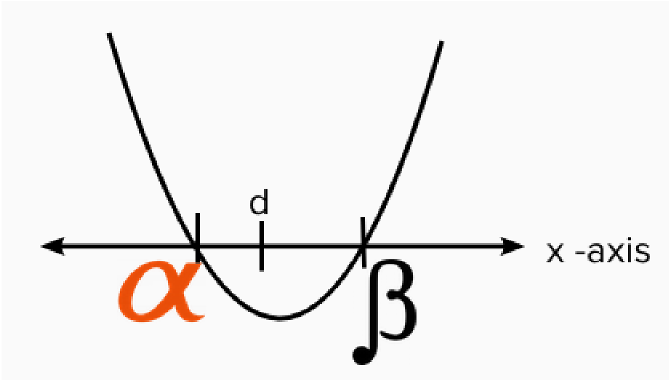
- $D > 0$
- Line of Symmetry, $-b/2a > d$
- Value of $f(x) = ax^2 + bx + c$ at d will be greater than 0



4.3.2 Both roots lie on opposite sides of a number d

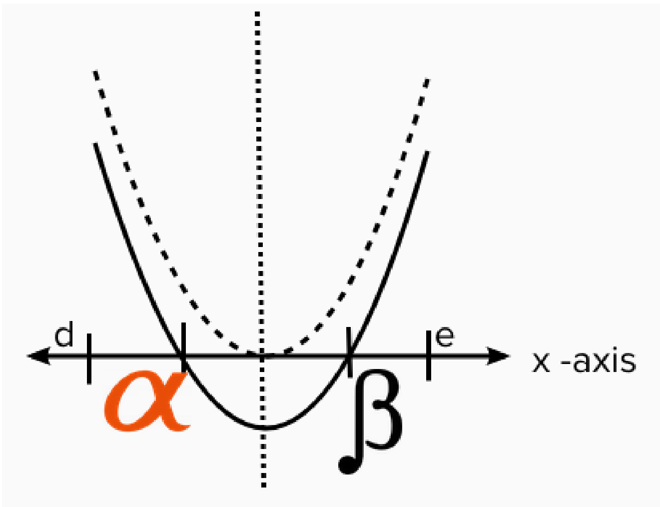
In this case we cannot comment on the line of symmetry, which can lie on either side of d. Below conditions must be true:

- $D > 0$
- Value of $f(x) = ax^2 + bx + c$ at d is less than 0.



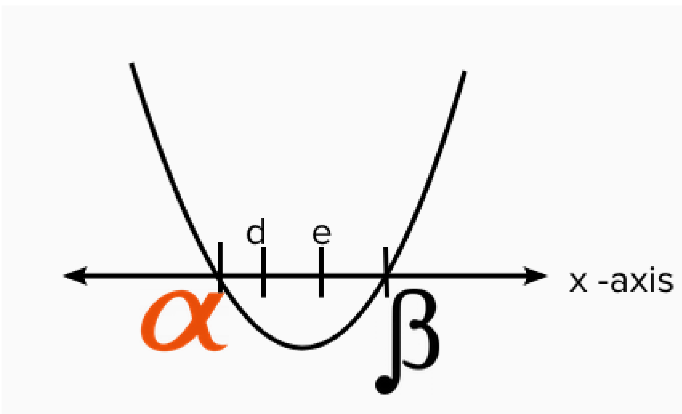
4.3.3 Both roots lie between (d, e)

- $D > 0$
- Line of Symmetry, $d < -b/2a < e$
- Value of $f(x) = ax^2 + bx + c$ at d and e is greater than zero.



4.3.4 (d, e) lie between the roots

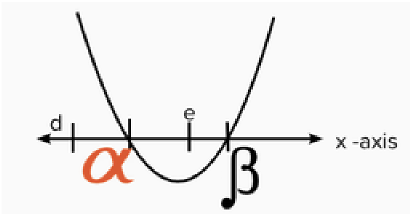
- Value of $f(x) = ax^2 + bx + c$ at / d and e is less than zero.

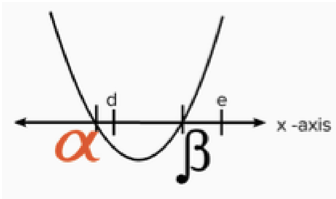


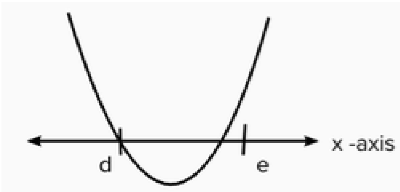
4.3.5 Exactly one root lies in interval (d, e)

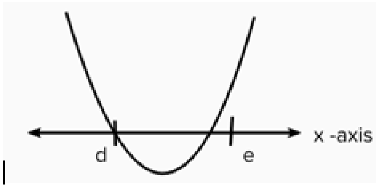
- $f(d)f(e) < 0$, where $f(d), f(e)$ are values of $f(x) = ax^2 + bx + c$ at d, e

- $f(d)f(e) = 0$ - In this case, verify for extraneous roots that are not applicable









Chapter 5

Logarithms

5.1 Introduction

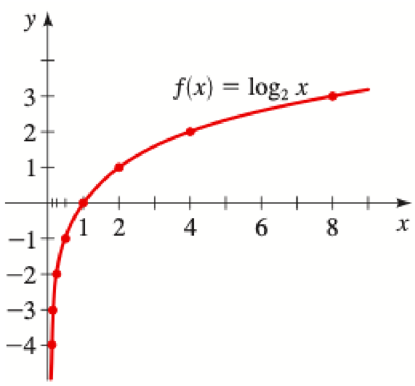
Exponent Form: $6^2 = 36$
Logarithmic Form: $log_6 36 = 2$

In general, $log_a N = x$, where:

- Base $a > 0$
- Base, $a \neq 1$
- $N > 0$ and $N \in R^+$, i.e. N is a positive real number

5.2 Identities

- $log_N N = 1$
- In $log_a N$, if $a \times N = 1$, (i.e. if a and N are reciprocal to each other) then it means that $logaN = -1$
- $log_a 1 = 0$
- $log_a N + log_a M = log_a NM$
- $log_a N - log_a M = log_a(N/M)$
- $log_a N^\beta = \beta log_a N$
- $x^{log_x N} = N$
- $\ln x = log_e x$, Natural Logarithm (Base e). The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. The approximate value of e is 2.718,and it's an irrational number.
- $\ln x = y \leftrightarrow e^x = y$
- $log_{10} x$, Common Log (Base 10)



Here is the graph of $log_2 x$:

Chapter 6

Exponential Functions

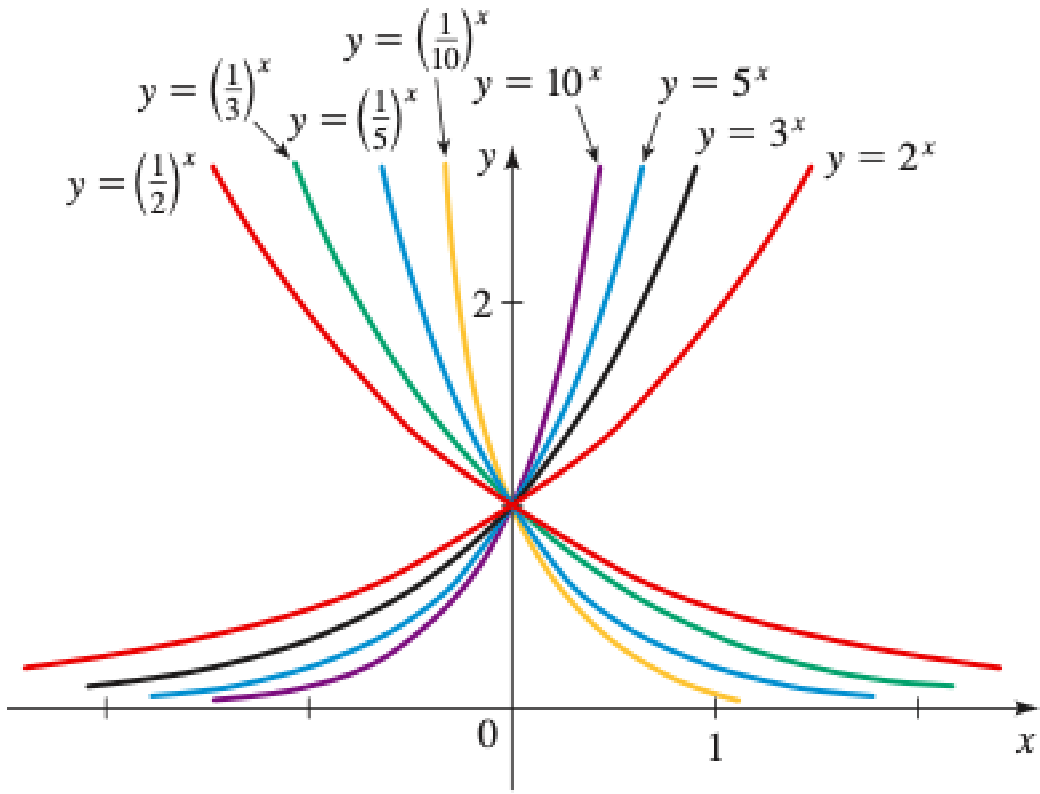
6.1 Introduction

An exponential function with base a is defined for all real numbers x by $f(x) = a^x$ where $a > 0$ and $a \neq 1$

6.2 Graphs of Exponential Functions

Figure below shows the graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a. All of these graphs pass through the point $(0, 1)$ because $a^0 = 1$ for $a \neq 1$.

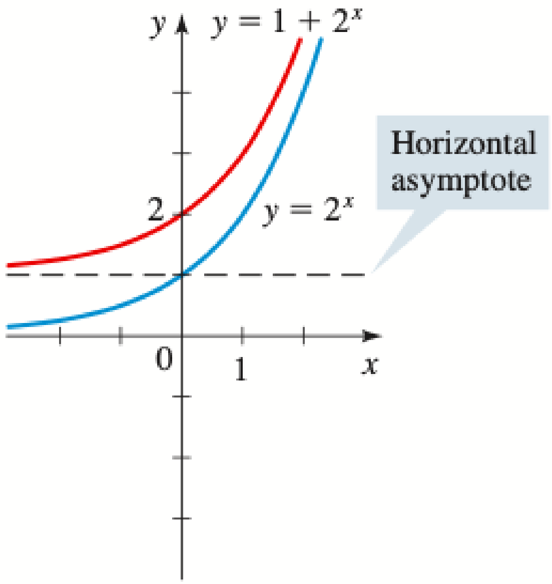
- If $0 < a < 1$, the exponential function **decreases** rapidly.
- If $a > 1$, the exponential function **increases** rapidly.



6.3 Transformation of Exponential Functions

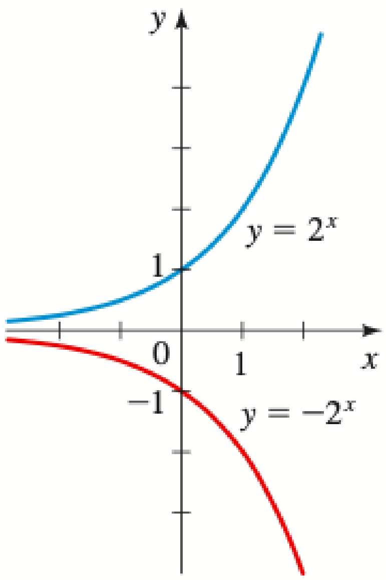
6.3.1 $f(x) = 1 + a^x$

To obtain the graph of $1 + a^x$, we start with the graph of $f(x) = a^x$ and shift it upward 1 unit.



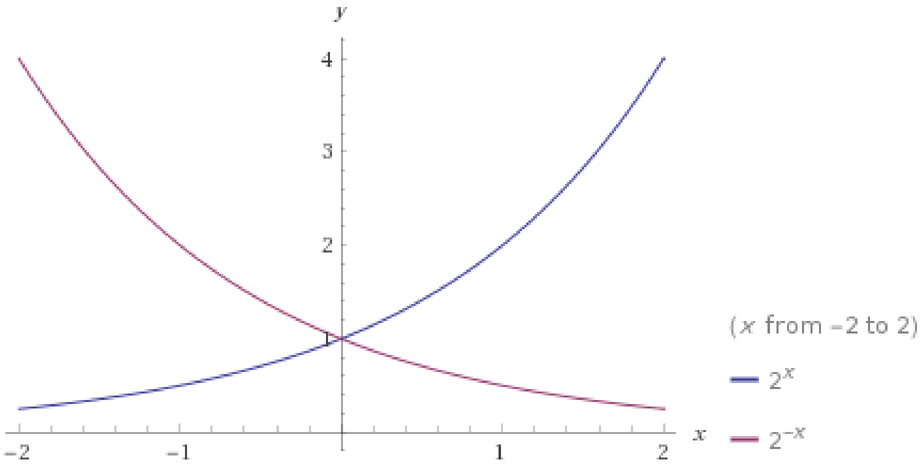
6.3.2 $f(x) = -a^x$

We start with the graph of $f(x) = a^x$, but here we reflect in the x-axis



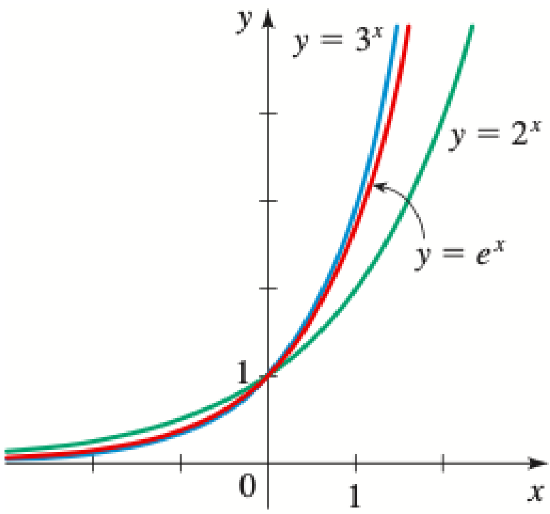
6.3.3 $f(x) = a^{-x}$

In this case, the graph is the mirror image of $f(x) = a^x$



6.4 The Natural Exponential Function

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. The approximate value is $e = 2.71828$. The number e is the base for natural exponential function, which in certain applications is easier to work with than log_{10}.
The **natural exponential function** is the function $f(x) = e^x$ with base e. Since $2 < e < 3$, the graph of e^x lies between the graphs of 2^x and 3^x:

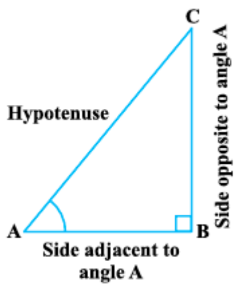


Chapter 7

Trigonometry

7.1 Trigonometric Ratios

Consider the right angle triangle ABC as shown below:
$\sin A = (Side\ opposite\ to\ angle\ A)/Hypotnuse$
$\cos A = (Side\ adjacent\ to\ angle\ A)/Hypotnuse$
$\tan A = (Side\ opposite\ to\ angle\ A)/(Side\ adjacent\ to\ angle\ A)$
$\csc A = 1/(\sin A)$
$\sec A = 1/(\cos A)$
$\cot A = 1/(\tan A)$

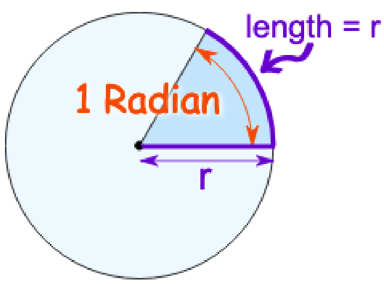


7.2 Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\sec^2\theta - \tan^2\theta = 1$
- $\csc^2\theta - \cot^2\theta = 1$
- $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$
- $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$
- $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$
- $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

7.3 Measurement of Angles - Radian

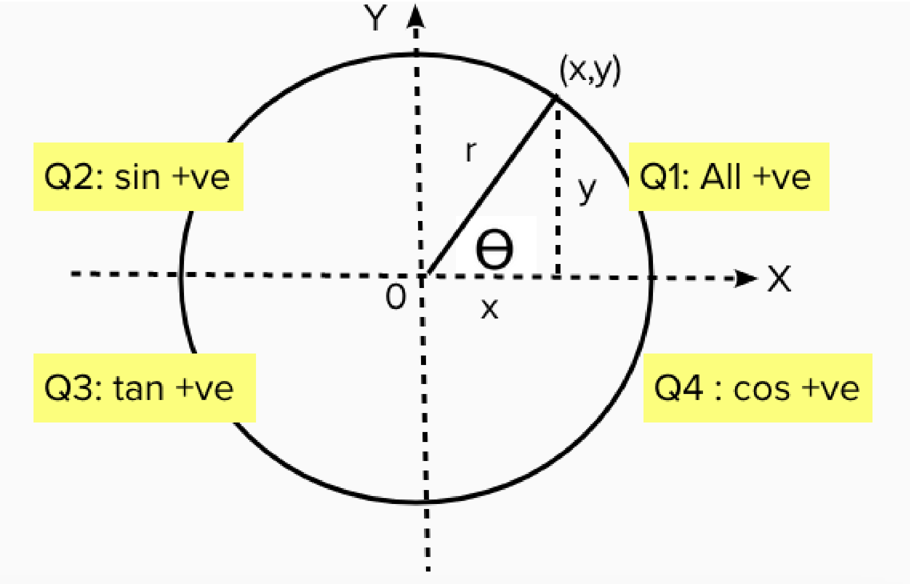
1 Radian, 1^c is the angle created by an arc of length r in a circle of radius r.
$\pi^c = 180\,\text{deg}$



From the below figure, we can see that:

- $\sin\theta = (Y - Coordinate)/Radius$
- $\cos\theta = (X - Coordinate)/Radius$
- $\tan\theta = (Y - Coordinate)/(X - Coordinate)$

We can then determine if $\sin\theta, \cos\theta, \tan\theta$ will be positive or negative in each quadrant.



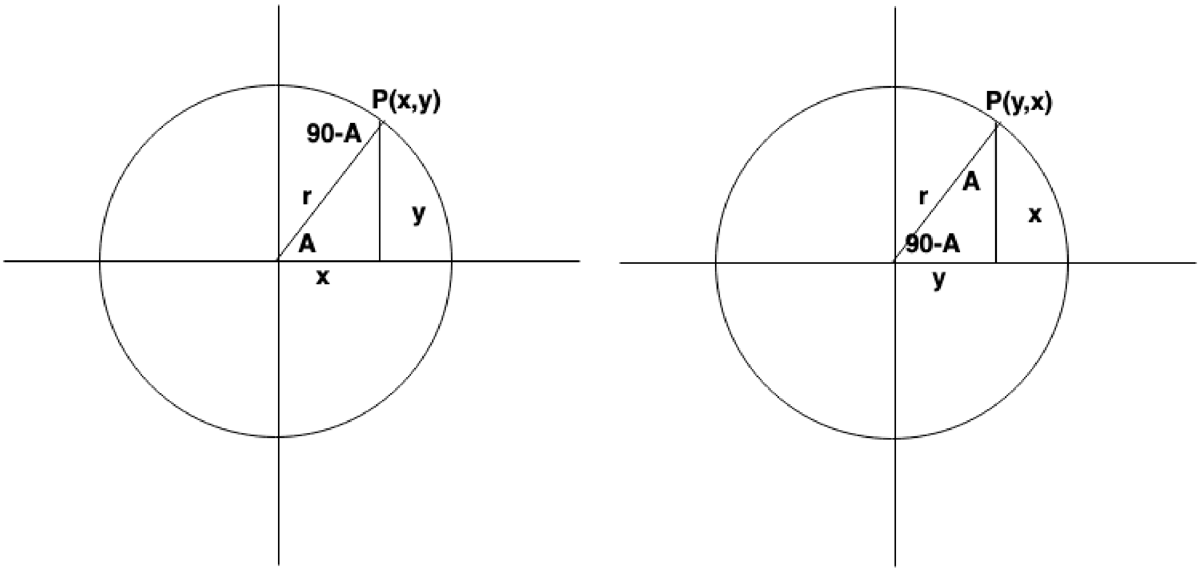
7.4 Values of trigonometric functions for $0^{\circ}, 30^{\circ}, 45^{\circ}, 60^{\circ}, 90^{\circ}$

α	0°	30°	45°	60°	90°
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–

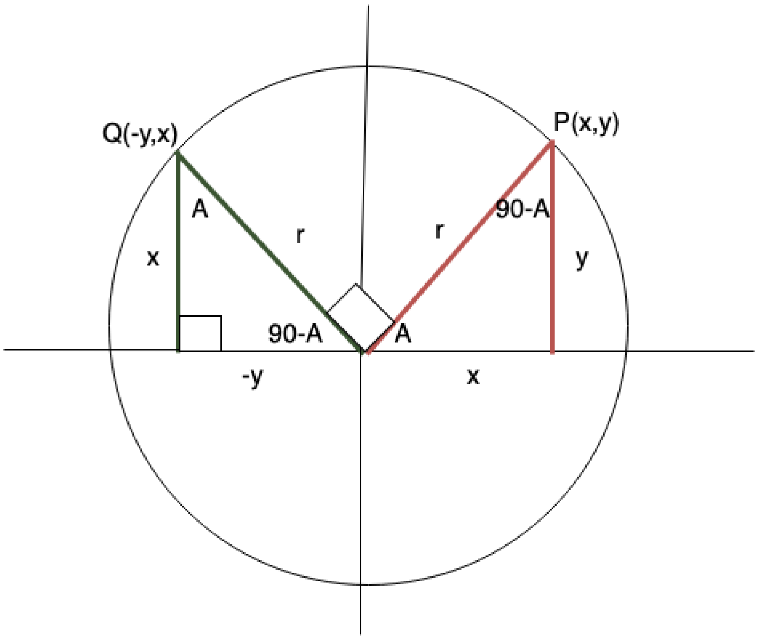
7.5 Complementary Angles

Consider the below figure - the two triangles are congruent (Angle-Side-Angle) and therefore the side opposite to the same angles will be equal.
From the second figure, we can see that $\sin(90 - A) = x/r$. However, we can also see that $(x/r) = \cos A$ from the first figure. Therefore, $\sin(90 - A) = \cos A$

- $\sin(90 - A) = \cos A$
- $\cos(90 - A) = \sin A$



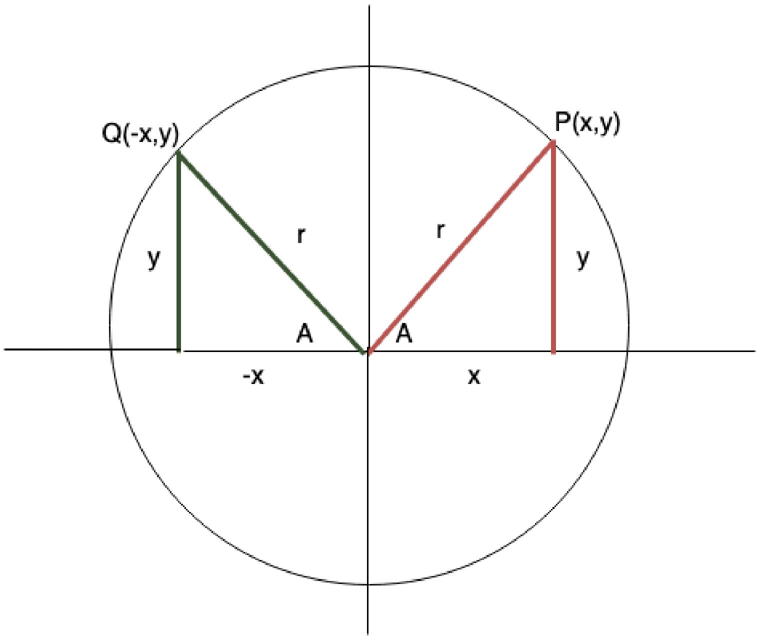
Now, lets see how we can find values of $90 + A$



We apply the same concept of congruent triangle above, and we can see that $\sin(90 + A) = (Y - Coordinate\, of\, Q)/Radius = x/r$. However, $x/r = \cos A$, therefore $\sin(90 + A) = cosA$.
Similarly, $cos(90 + A) = (X - Coordinate\, of\, Q)/Radius = (-y/r) = -\sin A$.

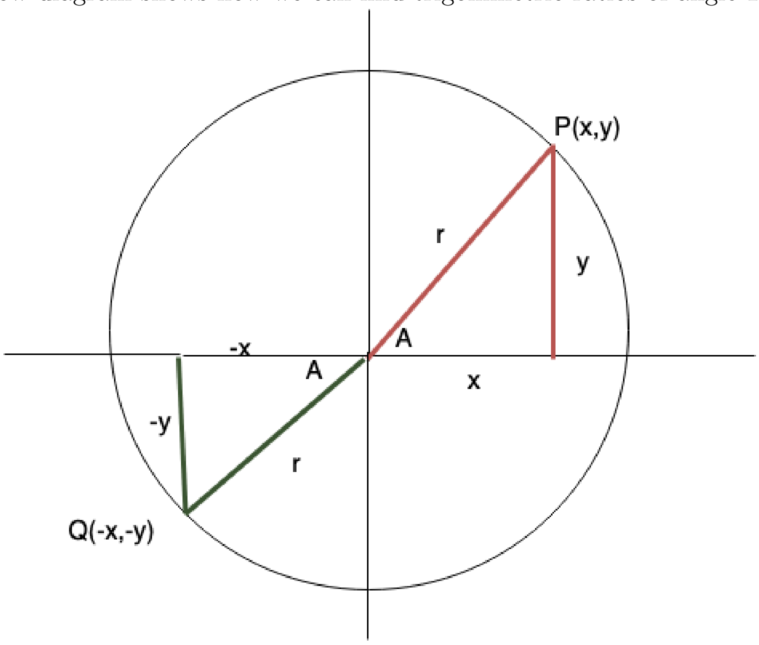
- $\sin(90 + A) = \cos A$
- $\cos(90 + A) = -\sin A$

Below diagram shows how we can find trigonimetric ratios of angle $180 - A$



- $\sin(180 - A) = \sin A$
- $\cos(180 - A) = -\cos A$

Below diagram shows how we can find trigonimetric ratios of angle $180 + A$



- $\sin(180 + A) = -\sin A$
- $\cos(180 + A) = -\cos A$