BİL 331/531 Design and Analysis of Algorithms

HOMEWORK 5 (55 Points)

Due Date: March 20, 2018

1 [5 Points] Dijkstra May Not Work If Some Edge Weights Are Negative

Prove that Dijkstra's algorithm may not find the shortest paths if some of the edge weights are negative, by giving a counterexample. So, you should give a directed graph $G = (V, E)$, and a specified node s. You should describe the iterations of your algorithm. You shall show that for at least one node, the path found by the algorithm is not the shortest. *Make sure that your counterexample does not have a negative cost cycle.* (Hint: There are very small graphs that you can use as counterexample.)

2 [10 Points] Shortest Paths vs. MST

[5 Points] Given an undirected and connected graph $G = (V, E)$ with positive edge lengths on the edges, and a vertex $s \in S$, we can find the shortest paths from s to every other vertex of G. Prove that the union of all these $n-1$ paths forms a tree.
[5 Points] Prove that the tree described above may be different than the minimum spanning tree of G.

3 [5 Points] Counting Argument

What is the maximum number of simple cycles an undirected graph with n nodes can have?

4 [5 POINTS] EITHER THIS OR THAT

Let $G = (V, E)$ be a undirected and connected graph such that each edge $e \in E$ is associated with a distinct positive weight. Prove that each edge $e \in E$ is the smallest weighted edge of some cut of G, or it is the largest weighted edge of some cycle in G. (Hint: Understand what the MST algorithms we have given in class are doing.)

5 [10 POINTS] DECIDING WHETHER AN EDGE IS PART OF MST OR NOT IN LINEAR TIME

Suppose you are given an undirected connected graph G, with edge costs that you may assume are all positive and distinct. G has n vertices and m edges. A particular edge e of G is specified. Give an algorithm with running time $O(m+n)$ to decide whether e is contained in the Minimum Spanning Tree of G or not.

6 [10 POINTS] YOUR MILITARY SERVICE

Assume that both males and females have obligatory military service so that none of you is exempt from solving this question :) Assume your rank is lieutenant and you are in charge of conducting the exercise program of the soldiers. The exercise program is as follows: Each soldier must swim 20 laps of a pool, then bike 10 miles, then run 3 miles. **Each soldier must use the pool one at a time**. Several soldiers may be biking at the same time, or several soldiers may be running at the same time, however, at most one soldier is swimming at any time.

For each soldier, assume that you know the *swimming time*, *biking time*, and *running time* specific to her/himself. Your job is to decide on a *schedule* for this exercise program: an order in which to sequence the starts of the soldiers.

Let's say that the exercise program starts at time $t = 0$. *Completion time* of a schedule s is the time at which all soldiers completed their exercises if they are sequenced with respect to s. You should come up with the schedule that minimizes the completion time among all possible schedules.
(Hint: Use the exchange argument in order to prove your greedy algorithm.)

7 [10 POINTS] MINIMIZING MAKESPAN

Assume that you have m jobs and n machines. Each machine i has a *nondecreasing* latency function $l_i : \mathbb{N} \to \mathbb{R}$ that only depends on the number of jobs assigned to machine i. To illustrate, if $l_j(5) = 7$, then machine j needs to work 7 units of time if it is assigned (any) five of the jobs. Assume that $l_i(0) = 0$ for all machines i.

Given a set of m jobs, and n machines, where each machine is associated with a nondecreasing latency function as described above. You are asked to give a $O(m \cdot \lg n)$ algorithm that assigns each job to a machine such that the *makespan*(the maximum amount of time any machine executes) is minimized. Needless to say, but just in case, you need to prove that your algorithm is correct.