A Probably More Detailed Explanation of the Shor's Algorithm

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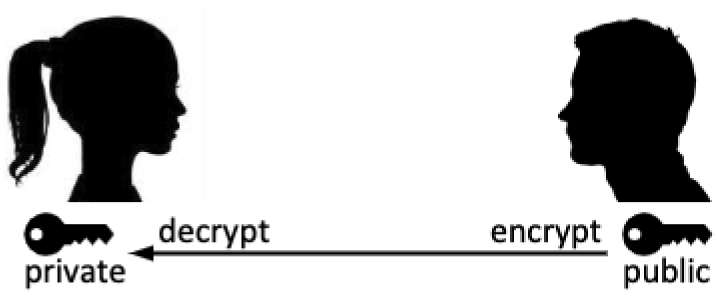
So based on the questions asked during the lab, I feel that probably you need some more detailed explanation of the Shor's algorithm. Here's a video of Shor himself explaining the algorithm: https://www.youtube.com/watch?v=hOlOY7NyMfs.

Public-key Cryptography



- In public-key cryptography Alice publishes a public key, which Bob uses to encode a message. Alice then uses her private key to decrypt the message.
- This relies on the asymmetry of the cryptography: it is easy to encrypt the message using the public key, but hard to decrypt it without knowledge of the private key.
- This in turn relies on the existence of one-way functions.

One-Way Functions

- One way function are functions that are easy to perform "forward", but hard to invert.
- For example, the RSA public-key cryptosystem uses factoring as a one-way function: multiplying two prime number p and q to give a composite number N is easy, but the inverse, factoring a composite number N into factors p and q is mathematically difficult.
- In fact, the best known classical algorithm for factoring, the *number field sieve*, requires $\exp(\Theta(n^{1/3}\log^{2/3} n))$ operations, where $n = \lceil \log_2 N \rceil$, i.e., the number of bits in N.
- For cryptographic applications it is crucial that the mathematical definition of one-way functions is that they must be hard to invert on average, not just in the worst case.

Factoring Quantum Mechanically

In 1994 Peter Shor invented a quantum algorithm which can factor numbers in polynomial time. This remains one of the (or probably the) most important and impressive potential application of quantum computing. Shor's algorithm built on previous query-complexity algorithms, and is founded on two key insights:

- The Quantum Fourier Algorithm can be used to solve the mathematical problem of order (and period) finding.
- Factoring can be reduced to order / period finding.

Period Finding

Let's look at the periodic function:

$$f(x) = a^x \bmod N$$

where a and N are positive integers, a is less than N, and they have no common factors. The period, or order (r), is the smallest (non-zero) integer such that:

$$a^r \bmod N = 1$$

Shor's solution was to use quantum phase estimation on the unitary operator:

$$U|y\rangle \equiv |ay \bmod N\rangle$$

To see how this is helpful, let's work out what an eigenstate of U might look like. If we started in the state $|1\rangle$, we can see that each successive application of U will multiply the state of our register by $a \pmod N$, and after r applications we will arrive at the state $|1\rangle$ again. For example with $a = 3$ and $N = 35$:

$$\begin{aligned} U|1\rangle &= |3\rangle \\ U^2|1\rangle &= |9\rangle \\ U^3|1\rangle &= |27\rangle \\ &\vdots \\ U^{(r-1)}|1\rangle &= |12\rangle \\ U^r|1\rangle &= |1\rangle \end{aligned}$$

So a superposition of the states in this cycle ($|u_0\rangle$) would be an eigenstate of U:

$$|u_0\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |a^k \bmod N\rangle$$

Example with $a = 3$ and $N = 35$

$$\begin{aligned}|u_0\rangle &= \tfrac{1}{\sqrt{12}}(|1\rangle + |3\rangle + |9\rangle \dots + |4\rangle + |12\rangle) \\ U|u_0\rangle &= \tfrac{1}{\sqrt{12}}(U|1\rangle + U|3\rangle + U|9\rangle \dots + U|4\rangle + U|12\rangle) \\ &= \tfrac{1}{\sqrt{12}}(|3\rangle + |9\rangle + |27\rangle \dots + |12\rangle + |1\rangle) \\ &= |u_0\rangle\end{aligned}$$

This eigenstate has an eigenvalue of 1, which isn't very interesting. A more interesting eigenstate could be one in which the phase is different for each of these computational basis states. Specifically, let's look at the case in which the phase of the k^{th} state is proportional to k:

$$\begin{aligned}|u_1\rangle &= \tfrac{1}{\sqrt{r}}\sum_{k=0}^{r-1} e^{-\tfrac{2\pi i k}{r}} |a^k \bmod N\rangle \\ U|u_1\rangle &= e^{\tfrac{2\pi i}{r}} |u_1\rangle\end{aligned}$$

Example with $a = 3$ and $N = 35$

$$\begin{aligned}|u_1\rangle &= \tfrac{1}{\sqrt{12}}(|1\rangle + e^{-\tfrac{2\pi i}{12}}|3\rangle + e^{-\tfrac{4\pi i}{12}}|9\rangle \dots + e^{-\tfrac{20\pi i}{12}}|4\rangle + e^{-\tfrac{22\pi i}{12}}|12\rangle) \\ U|u_1\rangle &= \tfrac{1}{\sqrt{12}}(|3\rangle + e^{-\tfrac{2\pi i}{12}}|9\rangle + e^{-\tfrac{4\pi i}{12}}|27\rangle \dots + e^{-\tfrac{20\pi i}{12}}|12\rangle + e^{-\tfrac{22\pi i}{12}}|1\rangle) \\ U|u_1\rangle &= e^{\tfrac{2\pi i}{12}} \cdot \tfrac{1}{\sqrt{12}}(e^{\tfrac{-2\pi i}{12}}|3\rangle + e^{-\tfrac{4\pi i}{12}}|9\rangle + e^{-\tfrac{6\pi i}{12}}|27\rangle \dots + e^{-\tfrac{22\pi i}{12}}|12\rangle + e^{-\tfrac{24\pi i}{12}}|1\rangle) \\ U|u_1\rangle &= e^{\tfrac{2\pi i}{12}}|u_1\rangle\end{aligned}$$

(We can see $r = 12$ appears in the denominator of the phase.)

This is a particularly interesting eigenvalue as it contains r. In fact, r has to be included to make sure the phase differences between the r computational basis states are equal. This is not the only eigenstate with this behaviour; to generalise this further, we can multiply an integer, s, to this phase difference, which will show up in our eigenvalue:

$$\begin{aligned}|u_s\rangle &= \tfrac{1}{\sqrt{r}}\sum_{k=0}^{r-1} e^{-\tfrac{2\pi i s k}{r}} |a^k \bmod N\rangle \\ U|u_s\rangle &= e^{\tfrac{2\pi i s}{r}} |u_s\rangle\end{aligned}$$

Example with $a = 3$ and $N = 35$

$$|u_s\rangle = \tfrac{1}{\sqrt{12}}(|1\rangle + e^{-\frac{2\pi i s}{12}}|3\rangle + e^{-\frac{4\pi i s}{12}}|9\rangle \cdots + e^{-\frac{20\pi i s}{12}}|4\rangle + e^{-\frac{22\pi i s}{12}}|12\rangle)$$

$$U|u_s\rangle = \tfrac{1}{\sqrt{12}}(|3\rangle + e^{-\frac{2\pi i s}{12}}|9\rangle + e^{-\frac{4\pi i s}{12}}|27\rangle \cdots + e^{-\frac{20\pi i s}{12}}|12\rangle + e^{-\frac{22\pi i s}{12}}|1\rangle)$$

$$U|u_s\rangle = e^{\frac{2\pi i s}{12}} \cdot \tfrac{1}{\sqrt{12}}(e^{-\frac{2\pi i s}{12}}|3\rangle + e^{-\frac{4\pi i s}{12}}|9\rangle + e^{-\frac{6\pi i s}{12}}|27\rangle \cdots + e^{-\frac{22\pi i s}{12}}|12\rangle + e^{-\frac{24\pi i s}{12}}|1\rangle)$$

$$U|u_s\rangle = e^{\frac{2\pi i s}{12}}|u_s\rangle$$

We now have a unique eigenstate for each integer value of s where $0 \leq s \leq r-1$. Very conveniently, if we sum up all these eigenstates, the different phases cancel out all computational basis states except $|1\rangle$:

$$\frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}|u_s\rangle = |1\rangle$$

Example with $a = 7$ and $N = 15$

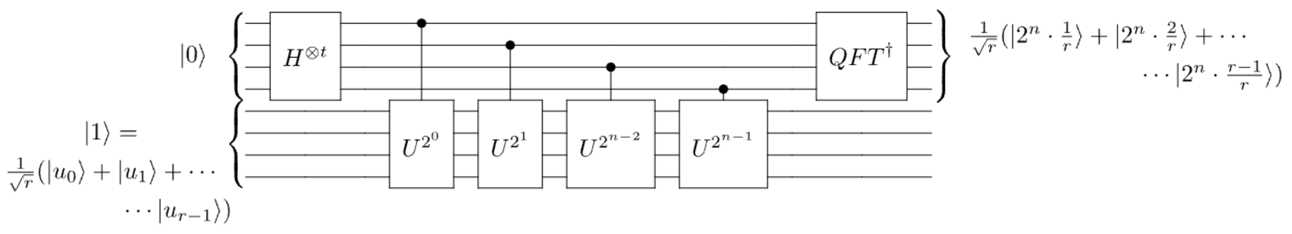
For this, we will look at a smaller example where $a = 7$ and $N = 15$. In this case $r = 4$:

$$\begin{aligned}
\tfrac{1}{2}(\quad |u_0\rangle &= \tfrac{1}{2}(|1\rangle \qquad\quad + |7\rangle \qquad\quad + |4\rangle \qquad\quad + |13\rangle)\dots \\
+|u_1\rangle &= \tfrac{1}{2}(|1\rangle + e^{-\frac{2\pi i}{4}}|7\rangle + e^{-\frac{4\pi i}{4}}|4\rangle + e^{-\frac{6\pi i}{4}}|13\rangle)\dots \\
+|u_2\rangle &= \tfrac{1}{2}(|1\rangle + e^{-\frac{4\pi i}{4}}|7\rangle + e^{-\frac{8\pi i}{4}}|4\rangle + e^{-\frac{12\pi i}{4}}|13\rangle)\dots \\
+|u_3\rangle &= \tfrac{1}{2}(|1\rangle + e^{-\frac{6\pi i}{4}}|7\rangle + e^{-\frac{12\pi i}{4}}|4\rangle + e^{-\frac{18\pi i}{4}}|13\rangle) \quad) = |1\rangle
\end{aligned}$$

Since the computational basis state $|1\rangle$ is a superposition of these eigenstates, which means if we do QPE on U using the state $|1\rangle$, we will measure a phase:

$$\phi = \frac{s}{r}$$

Where s is a random integer between 0 and $r-1$. We finally use the continued fractions algorithm on ϕ to find r. The circuit diagram looks like this (note that this diagram uses Qiskit's qubit ordering convention):



$|0\rangle$
$H^{\otimes t}$
$QFT^\dagger$
$\frac{1}{\sqrt{r}}(|2^n \cdot \frac{1}{r}\rangle + |2^n \cdot \frac{2}{r}\rangle + \cdots \cdots |2^n \cdot \frac{r-1}{r}\rangle)$
$|1\rangle = \frac{1}{\sqrt{r}}(|u_0\rangle + |u_1\rangle + \cdots \cdots |u_{r-1}\rangle)$
U^{2^0}
U^{2^1}
$U^{2^{n-2}}$
$U^{2^{n-1}}$

Factoring from Period Finding

Not all factoring problems are difficult; we can spot an even number instantly and know that one of its factors is 2. In fact, there are specific criteria for choosing numbers that are difficult to factor, but the basic idea is to choose the product of two large prime numbers.

A general factoring algorithm will first check to see if there is a shortcut to factoring the integer (is the number even? Is the number of the form $N = a^b$?), before using Shor's period finding for the worst-case scenario. Since we aim to focus on the quantum part of the algorithm, we will jump straight to the case in which N is the product of two primes.

Example: Factoring $N = 15$

1. The first step is to choose a random number, a, between 1 and $N-1$. We could choose 11, which isn't already a non-trivial factor of N (if it is, then divide N by this a and start again).
2. Next, we do Shor's order finding algorithm for $a = 11$ and $N = 15$. Remember that the phase we measure will be s/r where

   $$a^r \bmod N = 1$$

   and s is a random integer between 0 and $r-1$. If r is not even, we cannot go further and must try again with a different value for a.
3. Now we have r, we might be able to use this to find a factor of N. Since:

   $$a^r \bmod N = 1$$

   then:

   $$(a^r - 1) \bmod N = 0$$

   which means N must divide $a^r - 1$. And if r is also even, then we can write

   $$a^r - 1 = (a^{r/2} - 1)(a^{r/2} + 1).$$

   There is then a high probability that the greatest common divisor of N and either $a^{r/2} - 1$, or $a^{r/2} + 1$ is a proper factor of N. If, during the last step, we have $r = 2$ found from s/r with the continuous fraction algorithm. Then we have $a^{r/2} - 1 = 11^{2/2} - 1 = 10$ and $a^{r/2} + 1 = 11^{2/2} + 1 = 12$. The (non-trivial) greatest common divisor between 10 and 15 is 5, and the (non-trivial) greatest common divisor between 12 and 15 is 3. We can then verify that $3 \times 5 = 15$, indicating that we have successfully factored 15 into 5 and 3.

What is Order Finding?

Some Basic Number Theory

To explain the order-finding problem and how it can be solved using phase estimation, it will be very helpful to explain a couple of basic concepts in number theory and introduce some handy notation along the way.

To begin, for any given positive integer N, we'll define a set

$$\mathbb{Z}_N = \{0, 1, \ldots, N-1\}.$$

For instance, $\mathbb{Z}_1 = \{0\}$, $\mathbb{Z}_2 = \{0, 1\}$, $\mathbb{Z}_3 = \{0, 1, 2\}$, and so on.

These are sets of numbers, but we can think of them as more than sets. In particular, we can think about *arithmetic operations* on $\mathbb{Z}_N$ such as addition and multiplication — and if we agree to always take our answers modulo N, we'll always stay within this set when we perform these operations. (The two specific operations of addition and multiplication, both taken modulo N, turn $\mathbb{Z}_N$ into a *ring*, which is a fundamentally important type of object in algebra.)

For example, 3 and 5 are elements of $\mathbb{Z}_7$, and if we multiply them together we get $3 \cdot 5 = 15$, which leaves a remainder of 1 when divided by 7.

Sometimes we express this as follows.

$$3 \cdot 5 \equiv 1 \pmod 7$$

But we can also simply write $3 \cdot 5 = 1$, provided that it's been made clear that we're working in $\mathbb{Z}_7$, just to keep our notation as simple and clear as possible.

As an example, here are the addition and multiplication tables for $\mathbb{Z}_6$.

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	4	5	0	1	2	3
4	5	0	1	2	3	4
5	0	1	2	3	4	5

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Among the N elements of $\mathbb{Z}_N$, the elements $a \in \mathbb{Z}_N$ that satisfy $\gcd(a, N) = 1$ are special. Frequently the set containing these elements is denoted with a star like this:

$$\mathbb{Z}_N^* = \{a \in \mathbb{Z}_N : \gcd(a, N) = 1\}.$$

If we focus our attention on the operation of multiplication, the set $\mathbb{Z}_N^*$ forms a *group* — and specifically an *abelian group* — which is another important type of object in algebra. It's a basic fact about these sets (and indeed about finite groups in general), that if we pick any element $a \in \mathbb{Z}_N^*$ and repeatedly multiply a to itself, we'll always eventually get the number 1
.

For a first example, let's take $N = 6$. We have that $5 \in \mathbb{Z}_6^*$ because $\gcd(5,6) = 1$, and if we multiply 5 to itself we get 1 (as the table above confirms).

$$5^2 = 1 \quad (\text{working within } \mathbb{Z}_6)$$

As a second example, let's take $N = 21$. If we go through the numbers from 0 to 20, these are the ones that have GCD equal to 1 with 21:

$$\mathbb{Z}_{21}^* = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}.$$

For each of these elements, it is possible to raise that number to a positive integer power to get 1.

Here are the smallest powers for which this works:

$$\begin{aligned} 1^1 &= 1 \\ 2^6 &= 1 \\ 4^3 &= 1 \\ 5^6 &= 1 \\ 8^2 &= 1 \\ 10^6 &= 1 \\ 11^6 &= 1 \\ 13^2 &= 1 \\ 16^3 &= 1 \\ 17^6 &= 1 \\ 19^6 &= 1 \\ 20^2 &= 1 \end{aligned}$$

Naturally we're working within $\mathbb{Z}_{21}$ for all of these equations, which we haven't bothered to write — we take it to be implicit to avoid cluttering things up. We'll continue to do that throughout the rest of the lesson.

You can check each of these equations above, as well as the fact that these are the smallest positive integer powers for which the equations work, using the following code cell (changing the numbers as needed).

PYTHON

```
N = 21
a = 17
max_power = 12
print("k \t a^k \n")
for k in range(1,max_power+1):
    print("%2d \t %2d" %(k, a**k % N)) # The % operation computes the remainder m
```

Notice that after we get back to 1, the cycle repeats — which makes sense because multiplying 1 by a brings us back to a, which is where we started.

Although it isn't essential for the sake of the lesson, we can also check that we never get back to 1 when $\gcd(a, N) \neq 1$ — so we're relying on the fact that $a \in \mathbb{Z}_N^*$ for this to work.

Problem Statement

Now we can state the order-finding problem.

Order finding

- *Input:* positive integers N and a satisfying $\gcd(N, a) = 1$
- *Output:* the smallest positive integer r such that $a^r \equiv 1 \pmod{N}$

Alternatively, in terms of the notation we just introduced above, we're given $a \in \mathbb{Z}_N^*$, and we're looking for the smallest positive integer r such that $a^r = 1$. This number r is called the *order* of a modulo N.

Multiplication by an Element in $\mathbb{Z}_N^*$

To connect the order-finding problem to phase estimation, let's think about the operation defined on a system whose classical states correspond to $\mathbb{Z}_N$, where we multiply by a fixed element $a \in \mathbb{Z}_N^*$.

$$M_a|x\rangle = |ax\rangle \qquad (\text{for each } x \in \mathbb{Z}_N)$$

To be clear, we're doing the multiplication in $\mathbb{Z}_N$, so it's implicit that we're taking the remainder modulo N inside of the ket on the right-hand side of the equation. For example, if $N = 15$ and $a = 2$, we have

$$\begin{aligned}
M_2|0\rangle &= |0\rangle \\
M_2|1\rangle &= |2\rangle \\
M_2|2\rangle &= |4\rangle \\
M_2|3\rangle &= |6\rangle \\
M_2|4\rangle &= |8\rangle \\
M_2|5\rangle &= |10\rangle \\
M_2|6\rangle &= |12\rangle \\
M_2|7\rangle &= |14\rangle \\
M_2|8\rangle &= |1\rangle \\
M_2|9\rangle &= |3\rangle \\
M_2|10\rangle &= |5\rangle \\
M_2|11\rangle &= |7\rangle \\
M_2|12\rangle &= |9\rangle \\
M_2|13\rangle &= |11\rangle \\
M_2|14\rangle &= |13\rangle
\end{aligned}$$

So long as $\gcd(a, N) = 1$, this is a unitary operation. It shuffles the elements of the standard basis $\{|0\rangle, \ldots, |N-1\rangle\}$, so as a matrix it's a *permutation matrix*. It's evident from its definition that this operation is deterministic, and a simple way to see that it's invertible is to think about the order r of a modulo N, and to recognise that the inverse of M_a is M_a^{r-1}.

$$M_a^{r-1} M_a = M_a^r = M_{a^r} = M_1 = I$$

There's another way to think about the inverse that doesn't require any knowledge of r — which, after all, is what we're trying to compute. For every element $a \in \mathbb{Z}_N^*$ there's always a unique element $b \in \mathbb{Z}_N^*$ that satisfies $ab = 1$. We denote this element b by a^{-1}, and it can be computed efficiently. (An extension of Euclid's GCD algorithm does it at cost quadratic in $\lg(N)$.) And thus

$$M_{a^{-1}} M_a = M_{a^{-1}a} = M_1 = I.$$

So, the operation M_a is both deterministic and invertible. That implies that it's described by a permutation matrix, and it's therefore unitary.

Eigenvectors and Eigenvalues of Multiplication Operations

Now let's think about the eigenvectors and eigenvalues of the operation M_a, assuming that $a \in \mathbb{Z}_N^*$.As was just argued, this assumption tells us that M_a is unitary.

There are N eigenvalues of M_a, possibly including the same eigenvalue repeated multiple times, and in general there's some freedom in selecting corresponding eigenvectors — but we won't need to worry about all of the possibilities. Let's start simple and identify just one eigenvector of M_a.

$$|\psi_0\rangle = \frac{|1\rangle + |a\rangle + \cdots + |a^{r-1}\rangle}{\sqrt{r}}$$

The number r is the order of a modulo N — here and throughout the remainder of the note. The eigenvalue associated with this eigenvector is 1 because it isn't changed when we multiply by a.

$$M_a|\psi_0\rangle = \frac{|a\rangle + \cdots + |a^{r-1}\rangle + |a^r\rangle}{\sqrt{r}} = \frac{|a\rangle + \cdots + |a^{r-1}\rangle + |1\rangle}{\sqrt{r}} = |\psi_0\rangle$$

This happens because $a^r = 1$, so each standard basis state $|a^k\rangle$ gets shifted to $|a^{k+1}\rangle$ for $k \leq r-1$, and $|a^{r-1}\rangle$ gets shifted back to $|1\rangle$. Informally speaking, it's like we're slowly stirring $|\psi_0\rangle$, but it's already completely stirred so nothing changes.

Here's another example of an eigenvector of M_a. This one happens to be more interesting in the context of order finding and phase estimation.

$$|\psi_1\rangle = \frac{|1\rangle + \omega_r^{-1}|a\rangle + \cdots + \omega_r^{r-1}|a^{r-1}\rangle}{\sqrt{r}}$$

Alternatively, we can write this vector using a summation as follows:

$$|\psi_1\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \omega_r^{-k} |a^k\rangle$$

Here we see the complex number $\omega_r = e^{2\pi i/r}$ showing up naturally, due to the underlying structure of multiplication by a modulo N. This time the corresponding eigenvalue is ω_r. To see this, we can first compute like this:

$$M_a|\psi_1\rangle = \sum_{k=0}^{r-1} \omega_r^{-k} M_a|a^k\rangle = \sum_{k=0}^{r-1} \omega_r^{-k} |a^{k+1}\rangle = \sum_{k=1}^{r} \omega_r^{-(k-1)} |a^k\rangle = \omega_r \sum_{k=1}^{r} \omega_r^{-k} |a^k\rangle.$$

Then, because $\omega_r^{-r} = 1 = \omega_r^0$ and $|a^r\rangle = |1\rangle = |a^0\rangle$, we see that

$$\sum_{k=1}^{r} \omega_r^{-k} |a^k\rangle = \sum_{k=0}^{r-1} \omega_r^{-k} |a^k\rangle = |\psi_1\rangle,$$

so $M_a|\psi_1\rangle = \omega_r|\psi_1\rangle$.

Using the same reasoning, we can identify additional eigenvector/eigenvalue pairs for M_a.

Indeed, for any choice of $j \in \{0, \ldots, r-1\}$ we have that

$$|\psi_j\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \omega_r^{-jk} |a^k\rangle$$

is an eigenvector of M_a whose corresponding eigenvalue is ω_r^j.

$$M_a|\psi_j\rangle = \omega_r^j |\psi_j\rangle$$

There are other eigenvectors, such as $|0\rangle$, which has eigenvalue 1, but we'll only be concerned with the eigenvectors $|\psi_0\rangle, \ldots, |\psi_{r-1}\rangle$ that we've just identified.

Applying Phase Estimation

To solve the order-finding problem for a given choice of $a \in \mathbb{Z}_N^*$, we can apply the phase-estimation procedure to the operation M_a.

To do this, we need to implement not only M_a efficiently with a quantum circuit, but also M_a^2, M_a^4, M_a^8, and so on, going as far as needed to obtain a precise enough estimate from the phase estimation procedure. Here we'll explain how this can be done, and we'll figure out exactly how much precision is needed a bit later.

Let's start with the operation M_a by itself. Naturally, because we're working with the quantum circuit model, we'll use binary notation to encode the numbers between 0 and $N-1$. The largest number we need to encode is $N-1$, so the number of bits we need is

$$n = \lfloor \log_2(N-1) \rfloor + 1.$$

For example, if $N = 21$ we have $n = 5$. Here's what the encoding of elements of $\mathbb{Z}_{20}$ as binary strings of length 5 looks like.

$$\begin{gathered} 0 \mapsto 00000 \\ 1 \mapsto 00001 \\ \vdots \\ 20 \mapsto 10100 \end{gathered}$$

And now, here's a precise definition of how M_a is defined as an n-qubit operation.

$$M_a|x\rangle = \begin{cases} |ax \pmod N\rangle & 0 \leq x < N \\ |x\rangle & N \leq x < 2^n \end{cases}$$

The point is that although we only care about how M_a works for $|0\rangle, \ldots, |N-1\rangle$, we do have to specify how it works for the remaining $2^n - N$ standard basis states — and we need to do this in a way that still gives us a unitary operation. Defining M_a so that it does nothing to the remaining standard basis states accomplishes this.

Using the algorithms for integer multiplication and division discussed in the previous lesson, together with the methodology for reversible, garbage-free implementations of them, we can build a quantum circuit that performs M_a, for any choice of $a \in \mathbb{Z}_N^*$, at cost $O(n^2)$. Here's one way that this can be done.

1. We build a circuit for performing the operation

$$|x\rangle|y\rangle \mapsto |x\rangle|y \oplus f_a(x)\rangle$$

where

$$f_a(x) = \begin{cases} ax \pmod N & 0 \leq x < N \\ x & N \leq x < 2^n \end{cases}$$

using the method described in https://learn.qiskit.org/course/algorithms/algorithmic-foundations. This gives us a circuit of size $O(n^2)$.
2. We swap the two n-qubit systems using n swap gates to swap the qubits individually.
3. Along similar lines to the first step, we can build a circuit for the operation

$$|x\rangle|y\rangle \mapsto |x\rangle\big|y \oplus f_{a^{-1}}(x)\big\rangle$$

where a^{-1} is the inverse of a in $\mathbb{Z}_N^*$.

By initializing the bottom n qubits and composing the three steps, we obtain this transformation:

$$|x\rangle|0^n\rangle \overset{\text{step 1}}{\mapsto} |x\rangle|f_a(x)\rangle \overset{\text{step 2}}{\mapsto} |f_a(x)\rangle|x\rangle \overset{\text{step 3}}{\mapsto} |f_a(x)\rangle\big|x \oplus f_{a^{-1}}(f_a(x))\big\rangle = |f_a(x)\rangle|0^n\rangle$$

The total cost of the circuit we obtain is $O(n^2)$.

To perform M_a^2, M_a^4, M_a^8, and so on, we can use exactly the same method, except that we replace a with a^2, a^4, a^8, and so on, as elements of $\mathbb{Z}_N^*$. That is, for any power k we choose, we can create a circuit for M_a^k not by iterating k times the circuit for M_a, but instead by computing $b = a^k \in \mathbb{Z}_N^*$ and then using the circuit for M_b.

The computation of powers $a^k \in \mathbb{Z}_N$ is the *modular exponentiation* problem mentioned in the previous lesson. This computation can be done *classically*, using the algorithm for modular exponentiation (often called the *power algorithm* in computational number theory). This time we don't need to implement this algorithm reversibly with a quantum circuit, we just need to do it classically.

And we're fortunate that this is possible. We're effectively offloading the problem of iterating M_a a huge number of times, which can be exponential in the number m we choose in phase estimation, to an efficient classical computation. In terms of the quantum circuit we're running, the cost of M_a iterated k times is simply the cost of M_b, for $b = a^k$ — and so the cost is $O(n^2)$.

For an arbitrary choice of a quantum circuit in the phase estimation problem this won't be possible, resulting in a cost for phase estimation that's *exponential* in the number m we use in phase estimation. By the power of computational number theory, the cost in the case at hand is *linear* in m.

A Convenient Eigenvector/Eigenvalue Pair

To understand how we can solve the order-finding problem using phase estimation, let's start by supposing that we run the phase estimation procedure on the operation M_a using the eigenvector $|\psi_1\rangle$. Getting our hands on this eigenvector isn't easy, as it turns out, so this won't be the end of the story — but it's helpful to start here.

The eigenvalue of M_a corresponding to the eigenvector $|\psi_1\rangle$ is

$$\omega_r = e^{2\pi i \frac{1}{r}}.$$

That is, $\omega_r = e^{2\pi i\theta}$ for $\theta = 1/r$. So, if we run the phase estimation procedure on M_a using the eigenvector $|\psi_1\rangle$, we'll get an approximation to $1/r$. By computing the reciprocal we'll be able to learn r — provided that our approximation is good enough.

To be more precise, when we run the phase-estimation procedure using m control qubits, what we obtain is a number $y \in \{0, \ldots, 2^m - 1\}$, and we take $y/2^m$ as a guess for θ, which is $1/r$ in the case at hand. To figure out what r is from this approximation, the natural thing to do is to compute the reciprocal of our approximation and round to the nearest integer.

$$\left\lfloor \frac{2^m}{y} + \frac{1}{2} \right\rfloor$$

For example, let's suppose $r = 6$ and we perform phase estimation on M_a with the eigenvector $\vert\psi_1\rangle$ using $m = 5$ control bits. The best 5-bit approximation to $1/r = 1/6$ is $5/32$, and we have a pretty good chance (about 68% in this case) to obtain the outcome $y = 5$ from phase estimation. We have

$$\frac{2^m}{y} = \frac{32}{5} = 6.4$$

and rounding to the nearest integer gives 6, which is the correct answer.

On the other hand, if we don't use enough precision we may not get the right answer. For instance, if we take $m = 4$ control qubits in phase estimation, we might obtain the best 4-bit approximation to $1/r = 1/6$, which is $3/16$. Taking the reciprocal yields

$$\frac{2^m}{y} = \frac{16}{3} = 5.333\cdots$$

and rounding to the nearest integer gives an incorrect answer of 5.

How much precision do we need to get the right answer?

We know that the order r is an integer, and intuitively speaking what we need is enough precision to distinguish $1/r$ from nearby possibilities, including $1/(r+1)$ and $1/(r-1)$. The closest number to $1/r$ that we need to be concerned with is $1/(r+1)$, and the distance between these two numbers is

$$\frac{1}{r} - \frac{1}{r+1} = \frac{1}{r(r+1)}.$$

So, if we want to make sure that we don't mistake $1/r$ for $1/(r+1)$, it suffices to use enough precision to guarantee that a best approximation $y/2^m$ to $1/r$ is closer to $1/r$ than it is to $1/(r+1)$. If we use enough precision so that

$$\left\vert \frac{y}{2^m} - \frac{1}{r} \right\vert < \frac{1}{2r(r+1)},$$

so that the error is less than half of the distance between $1/r$ and $1/(r+1)$, then $y/2^m$ will be closer to $1/r$ than to any other possibility, including $1/(r+1)$ and $1/(r-1)$.

We can double-check this as follows. Suppose that

$$\frac{y}{2^m} = \frac{1}{r} + \varepsilon$$

for ε satisfying

$$\vert\varepsilon\vert < \frac{1}{2r(r+1)}.$$

When we take the reciprocal we obtain

$$\frac{2^m}{y} = \frac{1}{\frac{1}{r} + \varepsilon} = \frac{r}{1 + \varepsilon r} = r - \frac{\varepsilon r^2}{1 + \varepsilon r}.$$

By maximizing in the numerator and minimizing in the denominator, we can bound how far away we are from r as follows.

$$\left| \frac{\varepsilon r^2}{1 + \varepsilon r} \right| \leq \frac{\frac{r^2}{2r(r+1)}}{1 - \frac{r}{2r(r+1)}} = \frac{r}{2r + 1} < \frac{1}{2}$$

We're less than $1/2$ away from r, so as expected we'll get r when we round.

Unfortunately, because we don't yet know what r is, we can't use it to tell us how much accuracy we need. What we can do instead is to use the fact that r must be smaller than N to ensure that we use enough precision. In particular, if we use enough accuracy to guarantee that the best approximation $y/2^m$ to $1/r$ satisfies

$$\left| \frac{y}{2^m} - \frac{1}{r} \right| \leq \frac{1}{2N^2},$$

then we'll have enough precision to correctly determine r when we take the reciprocal. Taking $m = 2\lg(N) + 1$ ensures that we have a high chance to obtain an estimation with this precision using the method described previously. (Taking $m = 2\lg(N)$ is good enough if we're comfortable with a lower-bound of 40% on the probability of success.)

Other eigenvector/eigenvalue pairs

As we just saw, if we had the eigenvector $|\psi_1\rangle$ of M_a, we would be able to learn r through phase estimation, so long as we use enough control qubits to get sufficient precision to do this. Unfortunately it's not easy to get our hands on the eigenvector $|\psi_1\rangle$, so we need to figure out how to proceed.

Let's suppose we proceed just like we did above, except with the eigenvector $|\psi_k\rangle$ in place of $|\psi_1\rangle$, for any choice of $k \in \{0, \ldots, r-1\}$ that we choose to think about. The result we get from the phase estimation procedure will be an approximation

$$\frac{y}{2^m} \approx \frac{k}{r}.$$

Working under the assumption that we don't know either k or r, this might or might not allow us to identify r. For example, if $k = 0$ we'll get an approximation $y/2^m$ to 0, which unfortunately tells us nothing. This, however, is an unusual case; for other values of k, we'll at least be able to learn something about r.

We can use an algorithm known as the *continued fraction algorithm* to turn our approximation $y/2^m$ into nearby fractions — including k/r if the approximation is good enough. We won't explain the continued fraction algorithm here. Instead, here's a statement of a known fact about this algorithm.

Fact:

Given an integer $N \geq 2$ and a real number $\alpha \in (0,1)$, there is at most choice of integers $u, v \in \{0, \ldots, N-1\}$ with $v \neq 0$ and $\gcd(u,v) = 1$ satisfying $|\alpha - u/v| < \frac{1}{2N^2}$. Given α and N, the *continued fraction algorithm* finds u and v (or reports that they don't exist).

This algorithm can be implemented as a Boolean circuit having size $O((\log(N))^3)$.

If we have a very close approximation $y/2^m$ to r/k, and we run the continued fraction algorithm for N and $\alpha = y/2^m$, we'll get u and v, as they're described in the fact. A careful reading of the fact allows us to conclude that

$$\frac{u}{v} = \frac{k}{r}.$$

So we don't necessarily learn k and r, we only learn k/r in lowest terms.

For example, and as we've already noticed, we're not going to learn anything from $k = 0$. But that's the only value of k where that happens. When k is nonzero, it might have common factors with r — but the number v we obtain from the continued fraction algorithm must divide r.

It's far from obvious, but it is a known fact that if we have the ability to learn u and v for $u/v = k/r$ for $k \in \{0, \ldots, r-1\}$ chosen *uniformly at random*, then we're very likely to recover r after just a few samples. In particular, if our guess for r is the *least common multiple* of all the values for v that we observe, we'll be right with high probability. Some values of k aren't good because they share common factors with r, and those common factors are hidden to us when we learn u and v. But *random* choices of k aren't likely to hide factors of r for long, and the probability that we don't guess r correctly drops exponentially in the number of samples.

Proceeding without an eigenvector

So far we haven't addressed the issue of how we get our hands on an eigenvector $|\psi_k\rangle$ of M_a to run the phase estimation procedure on. As it turns out, we don't need to create them. What we will do instead is to run the phase estimation procedure on the state $|1\rangle$, by which we mean the n-bit binary encoding of the number 1, in place of an eigenvector $|\psi\rangle$ of M_a.

So far, we've talked about running the phase estimation procedure on a particular eigenvector, but nothing prevents us from running the procedure on an input state that isn't an eigenvector of M_a, and that's what we're doing here with the state $|1\rangle$. (This isn't an eigenvector of M_a for $a \in \mathbb{Z}_N^*$ unless $a = 1$, which is a choice for a that we'll avoid.)

The following equation helps to explain why we choose the state $|1\rangle$ in place of an eigenvector.

$$|1\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |\psi_k\rangle$$

This can be verified by taking the inner product of the right-hand side with standard basis states and using formulas mentioned previously.

In greater detail, let's imagine that we run the phase estimation procedure with the state $|1\rangle$ in place of one of the eigenvectors $|\psi_k\rangle$. After the quantum Fourier transform is performed, this leaves us with the state

$$\frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |\psi_k\rangle |\gamma_k\rangle,$$

where

$$|\gamma_k\rangle = \frac{1}{2^m} \sum_{y=0}^{2^m-1} \sum_{x=0}^{2^m-1} e^{2\pi i x(k/r - y/2^m)} |y\rangle$$

represents the state of the top m qubits after the inverse of the quantum Fourier transform is performed. When the top m qubits are measured, we therefore obtain an approximation $y/2^m$ to the value k/r where $k \in \{0, \ldots, r-1\}$ is chosen uniformly at random.

As we've already discussed, this allows us to learn r with a high degree of confidence after several independent runs, which was our goal.

Total cost

The cost to implement each controlled-unitary M_a^k is $O(n^2)$. There are m controlled-unitary operations, so the total cost for the controlled-unitary operations is $O(n^3)$. In addition, we have m Hadamard gates (which contribute $O(n)$ to the cost), and the quantum Fourier transform contributes $O(n^2)$ to the cost. Thus, the cost of the controlled-unitary operations dominates the cost of the entire procedure — which is therefore $O(n^3)$.

In addition to the quantum circuit itself, there are a few classical computations that need to be performed along the way. This includes computing the powers a^k in $\mathbb{Z}_N$ for $k = 2, 4, 8, \ldots, 2^{m-1}$, which are needed to create the controlled-unitary gates, as well as the continued fraction algorithm that converts approximations of θ into fractions. In both cases, these computations can be performed by Boolean circuits having cost $O(n^3)$.

As is typical, all of these bounds can be improved using asymptotically fast algorithms; these bounds assume we're using standard algorithms for basic arithmetic operations.

Factoring by order-finding

The very last thing we need to discuss is how solving the order-finding problem helps us to factor. This part is completely classical — it has nothing specifically to do with quantum computing.

Here's the basic idea. We want to factorize the number N, and we can do this *recursively*. Specifically, we can focus on the task of *splitting* N, which means finding any two integers $b, c \geq 2$ for which $N = bc$. This isn't possible if N is a prime number, but we can efficiently test to see if N is prime using a primality testing algorithm first, and if N isn't prime we'll try to split it. Once we split N, we can simply recurse on b and c until all of our factors are prime and we obtain the prime factorization of N.

Splitting even integers is easy: we just output 2 and $N/2$.

It's also easy to split perfect powers, meaning numbers of the form $N = s^j$ for integers $s, j \geq 2$, just by approximating the roots $N^{1/2}$, $N^{1/3}$, $N^{1/4}$, etc., and checking nearby integers as suspects for s. We don't need to go further than $\log(N)$ steps into this sequence, because at that point the root drops below 2 and won't reveal additional candidates.

It's good that we can do both of these things because order-finding won't help us for even numbers or for *prime* powers, where the number s happens to be prime.

If N is odd and not a prime power, order-finding allows us to split N.

Input an odd, composite integer N that is not a prime power. Iterate the following steps:

1. Randomly choose $a \in \{2, \ldots, N-1\}$.
2. Compute $d = \gcd(a, N)$.
3. If $d > 1$ then output $b = d$ and $c = N/d$ and stop. Otherwise continue to the next step knowing that $a \in \mathbb{Z}_N^*$.
4. Let r be the order of a modulo N. (Here's where we need order-finding.)
5. If r is even:
    1. Compute $x = a^{r/2} - 1$ (modulo N)
    2. Compute $d = \gcd(x, N)$.
    3. If $d > 1$ then output $b = d$ and $c = N/d$ and stop.
6. If this point is reached, the iteration has failed to find a factor of N.

An iteration of this algorithm may fail to find a factor of N. Specifically, this happens in two situations:

- The order of a modulo N is odd.
- The order of a modulo N is even and $\gcd\left(a^{r/2} - 1, N\right) = 1$.
  Using basic number theory it can be proved that, for a random choice of a, with

probability at least $1/2$ neither of these events happens. In fact, the probability is at most $2^{-(m-1)}$ for m being the number of distinct prime factors of N. This is why the assumption that N is not a prime power is important. The assumption that N is odd is also required for this to be true, which is why the (easy) case that N is even has to be handled separately.

So, if we repeat the process t times, randomly choosing a each time, we'll succeed in splitting N with probability at least $1 - 2^{-t}$.

We won't go through this analysis in detail, but here's the basic idea. If we have a choice of a for which the order r of a modulo N is even, then it makes sense to consider the numbers

$$a^{r/2} - 1 \pmod N \quad \text{and} \quad a^{r/2} + 1 \pmod N.$$

Using the formula $Z^2 - 1 = (Z+1)(Z-1)$, we conclude that

$$\bigl(a^{r/2} - 1\bigr)\bigl(a^{r/2} + 1\bigr) = a^r - 1.$$

We know that $a^r \pmod N = 1$ by the definition of the order, which is another way of saying that N evenly divides $a^r - 1$. So, N evenly divides the number

$$\bigl(a^{r/2} - 1\bigr)\bigl(a^{r/2} + 1\bigr),$$

which means that every prime factor of N must divide either $a^{r/2} - 1$ or $a^{r/2} + 1$ (or both). For a randomly selected a, we're likely to have prime factors of N dividing both terms, which allows us to split N by computing the GCD.