Concrete Mathematics Notes

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Chapter 1

Recursion Problems

1.1 The Tower of Hanoi

Problem setup. There are 3 pegs and some disks stacked in decreasing order.

- Rule. Move 1 disk at a time and never moving a large one to a smaller one
- Objective. Move all disk from one peg to another.

Technique 1. Look at small examples. Transfer with 1, 2 and 3 disks is somehow obvious. General patterns are easier to precieze while extreme cases are well understood.

Technique. Divide and conquer. We first transfer $n-1$ smallest to a different peg (T_{n-1} times in total), then move the largest, then move the smallest on the idea. So there are $2T_{n-1}+1$ times. So we have

$$T_n \leq 2T_{n-1}+1, \text{for } n > 0.$$

We have to move the biggest one, so when moving the biggest one, we have already moved the ones at the bottom. As we have to move it up. So we have

$$T_n \geq T_{n-1}+1, \text{for} n > 0.$$

Recurrence Relation.
Representation: Like

$$\begin{aligned} T_n &= 0 \\ T_n &= 2T_{n-1}+1, \text{for } n > 0 \end{aligned}$$

is called a recursion.

Solution: a closed form for this, for example, $T_n = 2^n - 1$. A closed form involves only addition, subtraction, multiplication and division and exponentiation in explicit ways.

Mathematical Induction. First we prove a statement when it has the smallest value(basis), then we prove it for $n \geq n_0$, assuming it has been proved in $[n_0, n-1]$(induction).

1.2 Lines on a plane

Problem setup. What's the maxim number L_n of regions defines by n lines on the plane?

Observing small cases.

Observation 1: Adding a new line seems to double the region.

Observation 2: adding the 3rd line, it can at most hit at least 3 regions. So the desired generalization is it splits k old regions if and only if it hits all previous regions. So the upper bound is

$$L_n \leq L_{n-1} + n, n > 0$$

This is a arithmetic series, so this the answer is $S_n = n\frac{n+1}{2}$.

Variation problem. With lines on the plane problem: Suppose instead of straight lines, using zigzag lines. What's the max. number?

Observation: If we extend all the lines up, it will be like the lines on the plane problem. Each line will lose n regions, so it will be

$$Z_n = L_{2n} - 2n, n \geq 0.$$

1.3 The Josephus problem

Problem setup. Start with n people $1 \sim n$, eliminate every second remaining person till only 1 survives. We shall determine the survival number.

Using proper notation. Denote $J(n)$ as the final survivor's number. Note then n is odd, we have $J(2n) = 2J_n - 1$. For after one round, the configuration is 2 times and minus 1. So the $J(2n+1) = 2J(n) + 1$.

Simplify. We note that this is closely related to power of 2's. After a few listing it can be shown that $J(2^m + l) = 2l+!$. We can prove it by induction.

It may be also helpful to see radix 2 representations. Suppose n's binary representation is $n = (b_m, b_{m-1}, \cdots, b_1, b_0)$. According to the previous ones, we have that $J(b_m, b_{m-1}, \cdots, b_1, b_0) = (b_{m-1}, \cdots, b_1, b_0, b_m)$. That is moving one bit cyclic shift left!

Guessing in a smarter way. Assume our recurrence had something like

$$\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n) + \beta \\ f(2n+1) &= 2f(n) + \gamma \end{aligned}$$

We can assume the general form like $f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$. After writing a few, we might investigate $A(n) = 2^m, B(n) = 2^m - 1 - l, C(n) = l$.

This method will work perfectly well when the function is linear.

1.4 Exercises

Here are some answers to the exercises. The answer might not be always correct.

W 1. In this problem, the set of the horses doesn't meet well ordering principle, that is no linear order for a set.

W 2. Note that splitting it in the first one and then other ones is not correct, for when it was set like this, it will cause big ones over small ones. This will lead to $T_n = 2 + T_{n-1} + 2 + T_{n-1} + 2$ – this is a wrong answer. The correct answer is using top $n-1$ as a group and one at last one, so hence swapping all Ts and 2s in the previous equation, hence getting $T_n = T_{n-1} + 2 + T_{n-1} + 2 + T_{n-1}$. Using techniques of inductions will get $3^n - 1$.

W 3. From the previous question we have totally $3^n - 1$ cases will be shown. We may find that no two configurations are the same, since each move at least one plate is in the different place. On the other hand, there are $3^n - 1$ cases according to rules of multiplication. So we are done.

W 4. No. Since it has reached the highest part.

W 5. We can derive the formula for circles. That is $T(n) = T(n-1) + 2n$. Calculating for $n = 4$, we have at most 14 areas. So this is impossible.

W 6. We can deduce it firstly to L_n problems. Subtracting surrounding unbounded regions, we have $B_n = L_n - 2n$.

W 7. This argument can only make sure the recurrence is true, without proving the basis one is correct.

H 8. We note that $Q_0 = \alpha$, $Q_1 = \beta$. Calculating for $Q_3 = \frac{1+\beta}{\alpha}$, and $Q_4 = \frac{\alpha+\beta+1}{\alpha\beta}$, and we find that $Q_4 = \frac{1+\alpha}{\beta}$, and

$$\begin{aligned} &Q_5 = \frac{\alpha\beta + \alpha^2 + \alpha}{\alpha + \beta + 1} = \alpha. \\ &Q_6 = \beta. \end{aligned}$$

So we end up in a loop. So the answer is:

$$Q_n = \begin{cases} \alpha & n \bmod 4 \text{ is } 0 \\ \beta & n \bmod 4 \text{ is } 1 \\ \frac{1}{\alpha\beta} + \frac{1}{\alpha} + \frac{1}{\beta} & n \bmod 4 \text{ is } 2 \\ \frac{1+\alpha}{\beta} & n \bmod 4 \text{ is } 3 \end{cases}.$$

HW 9. (a) We show that

$$x_1 x_2 \cdots x_{n-1} x_n \leq \left(\frac{x_1 + \cdots + x_n}{n} \right)^n$$
$$x_1 x_2 \cdots x_{n-1} \frac{x_1 + \cdots + x_{n-1}}{n-1} \leq \left(\frac{x_1 + \cdots + x_{n-1} + \frac{x_1 + \cdots + x_{n-1}}{n-1}}{n} \right)^n$$

Multiplying and making some ideas, we have

$$x_1 x_2 \cdots x_{n-1} \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \leq \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}$$

and $P(n-1)$ is true now.

(b) We can set $x_1 = \frac{y_1 + \cdots + y_n}{n}$, and set $x_2 = \frac{y_{n+1} + \cdots + y_{2n}}{n}$.

(c) We may use the binary representation for each number, hence we can repeat the case over and over again until implying all the possible $P(n)$s.

HW 10.

Chapter 2

Manipulating sums

2.1 Basic Notations

Summation notation. We can use $\sum$ to denote add things together.

- $\sum_a$: sum over all items in a
- $\sum_{i=1}^n f(i)$: sum from $i = 1$ to $i = n$, substituting all is in the right $f(i)$.

Iverson's Bracket. We have the notation $[p]$: p is a proposition, and when p is true, the equation is evaluated 1, otherwise evaluated as 0.

Using Aversion's bracket to simplifying sums. For example we have

$$\sum_{p \text{ prime}, p \leq N} p = \sum_p [p \text{ prime}][p \leq N]p.$$

2.2 Manipulation of sums

Distributive law. $\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$.

Associative law. $\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$.

Commutative law. $\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}$.

This rule is for mainly substitution use. When the condition is not specified, the $p(k)$ should be a permutation of all integers.

Example. We have

$$\sum_{k \in K, k \texttt{ even}} a_k = \sum_{n \in K, n \texttt{ even}} a_n = \sum_{2k \in k}^{a_{2k}}$$

Example 1: Arithmetic's progression. We have $S = \sum_{0 \leq k \leq n} (a + bk)$. By commutative law, replace k by $n - k$, obtaining $S = \sum_{0 \leq n-k \leq n} (a + b(n - k)) = \sum_{0 \leq k \leq n} (a + bn - bk)$. These two can be added up to $2S = \sum_{0 \leq k \leq n} (2a + bn)$. Then this problem would be trivial.

Exclusion and Inclusive Principle with Sums. Here is a important rule for combining different set of indicies.

Suppose K and K' are any set of integers, then

$$\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in k\cap K'} a_k + \sum_{K\in K\cup K'} a_k.$$

In the case of Iverson's bracket, we have

$$[k \in K] + [k \in K'] = [k \in K \cap K'] + [K \in K \cup K']$$

Perturbation method. We first write $S_n = \sum_{0\le k\le n} a_k$, then we rewrite S_{n+1} in 2 ways:

$$\begin{aligned} S_n + a_{n+1} = \sum_{0\le k\le n+1} \qquad\qquad & = a_0 + \sum_{1\le k\le n+1} a_k \\ & = a_0 + \sum_{1\le k+1\le n+1} a_{k+1} \\ & = a_0 + \sum_{0\le k\le n} a_{k+1}. \end{aligned}$$

Now we can work on the last term trying to solving the closed form for it.

Example 2. Geometry progression. We have $S_n = \sum_{0\le k\le n} ax^k$. And the sum is needed. Using *perturbation* method we have

$$S_n + ax^{n+1} = ax^0 + \sum_{0\le k\le n} ax^{k+1}$$

Factoring out an x, we have

$$S_n + ax^{n+1} = a + xS_n$$

$$\sum_{k=0}^{n} ax^k = \frac{a - ax^{n+1}}{1-x}, \qquad \text{for } x \neq 1$$

Example 3. Arithmetic Geometric progression. We have the sequence

$$S_n = \sum_{0\le k\le n} k2^k$$

Using the perturbation technique, we have

$$S_n + (n+1)^{2n+1} = \sum_{0\le k\le n} (k+1)2^{k+1}$$

Rewrite RHS as sums, we have

$$S_n + (n+1)^{2n+1} = \sum_{0\le k\le n} k2^{k+1} + \sum_{0\le k\le n} 2^{k+1}$$

Hence $\sum_{0\le k\le n} k2^k = (n-1)2^{n+1} + 2$.

2.3 Multiple sums

Notations. Stacking multiple sums in a row helps to derivate more complex sums.

Example: $\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} = a_{11} + a_{12} + a_{13} + \cdots + a_{33}$.

Using the simplified Iverson's bracket, getting another way of expressing multiple sums: $\sum_{1\le j,k\le 3} a_j b_k = \sum_{j,k} a_j b_k [1 \le j \le 3][1 \le k \le 3]$.

General distributive law. For distinct lower set index i, j, we have the following distributive law:

$$\sum_{j\in J, k\in K} a_j b_k = \big(\sum_{j\in J} a_j\big) + \big(\sum_{k\in K} b_k\big)$$

Another general form for this is $\sum_{j\in J}\sum_{k\in K(j)} a_{j,k} = \sum_{k\in K'}\sum_{j\in J'(k)} a_{j,k}$. Here, the sets should satisfy $[j \in J][k \in K(j)] = [k \in K'][j \in J'(k)]$.

Example 1. Consecutive integers. We may rewrite $[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k]$.

Example 2. Sum over an matrix. We try to find a simplified idea of the matrix

$$\begin{pmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{pmatrix}$$

we shall find a sum $S_1 = \sum_{1\le j\le k\le n} a_j a_k$.

According to the associativity of the multiplication, we find that $a_j a_k = a_k a_j$. Hence we have $S_\Delta = \sum_{1\le j\le k\le n} a_j a_k = \sum_{1\le k\le j\le n} a_k a_j = \sum_{1\le j\le n} a_j a_k = S_\nabla$. According to the problem above, we have $[1 \le j \le k \le n] + [1 \le k \le j \le n] = [1 \le j, k \le n] + [1 \le j = k \le n]$. We have

$$2S_\nabla = S_\nabla + S_\Delta = \sum_{1\le j,k\le n} a_j a_k + \sum_{1\le j=k\le n} a_j a_k.$$

The first sum is $\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k-1}^{n} a_k\right) = \left(\sum_{k=1}^{n} a_k\right)^2$, and the second sum is that $\sum_{k-1}^{n} a_k^2$.

So we have the sum

$$\sum_{\nabla} = \sum_{1\le j\le k\le n} a_j a_k = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2\right).$$

The idea is like the aligning the data and subtracting the overlapped off. That's a more religious form.

Another double sum. We try to evaluate

$$S = \sum_{1\le j<k\le n} (a_k - a_j)(b_k - b_j).$$

We find that this sum satisfies summation:

$$S = \sum_{1 \le j < k \le n} (a_k - a_j)(b_k - b_j) = \sum_{1 \le j < k \le n} (a_j - a_k)(b_j - b_k).$$

We add S to itself, with equation

$$[1 \le j < k \le n] + [1 \le k < j \le n] = [1 \le j, k \le n] - [1 \le j = k \le n].$$

yields

$$2S = \sum_{1 \le j, k \le n} (a_j - a_k)(b_j - b_k) - \sum_{1 \le j = k \le n} (a_j - a_k)(b_j - b_k)$$

The second part is 0, so we have to evaluate the first sum. The first sum expand it and swapping the summation index we have the identity

$$2 \sum_{1 \le j, k \le n}^{n} a_k b_k + 2 \sum_{1 \le j, k \le n} a_j b_k$$

We have sums over j, k, extracting j out, we have

$$2n \sum_{1 \le j \le n}^{n} a_k b_k + 2 \left(\sum_{1 \le j \le n} a_j \right)^2 \left(\sum_{i \le k \le n} b_k \right)^2$$

Chebyshev's monotonic inequalities . This is a special case for the last case as an example, that is

$$\left(\sum_{i=1}^{n} a_k \right) \left(\sum_{k=1}^{n} b_k \right) \le n \sum_{k=1}^{n} a_k b_k$$

if $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$. and vice versa.

Changing the index as the single sums. In the index is changed in single sums, we have that

$$\sum_{k \in K} a_k = \sum_{p(k) \in k} a_k,$$

if $p(k)$ as the permutation of the integers.

We now generalize k by $f(j)$, where f is an arbitrary function:

$$f : j \to K$$

that takes an integer $j \in J$ into an integer $f(j) \in K$. In this case we have the replace formula:

$$\sum_{j \in J} a_{f(j)} = \sum_{k \in K} a_k \# f^{-}(k)$$

where the $\#f^{-}(k)$ stands for number of sets in the set $f^{-}(k) = \{j|f(j) = k\}$.

Proof. $\sum_{j\in J} a_{f(j)} = \sum_{j\in J, k\in K} a_k[f(j) = k] = \sum_{k\in K} a_k \sum_{j\in J}[f(j) = k]$. This yields the answer.

Example: A fraction sum. We wish to sum

$$S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}.$$

An attempt shows that

$$\begin{aligned} S_n &= \sum_{j=1}^{n}\sum_{k=j}^{n} \frac{1}{k-j} \\ &= \sum_{1\le k\le n}^{1\le k-j<k} && \text{Replacing } j \text{ by } k-i \\ &= \sum_{1\le k\le n}^{0\le k\le j-1} \frac{1}{j} \\ &= \sum_{0\le k\le n} H_k. \end{aligned}$$

The harmonic number must be evaluated to make sure it is evaluated.

2.4 Finite Calculus

The diff operator. By definition, the Difference operator is defined as

$$\Delta f(x) = f(x+1) - f(x).$$

Falling factorial power.
The falling factorial power is defined as (positive int)

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1)$$

The rising factorial power is defined as (positive int)

$$x^{\overline{m}} = x(x+1)\cdots(x+m-1)$$

We have the property of

$$\begin{aligned} \Delta(x^{\underline{m}}) &= x+1^{\underline{m}} - x^{\underline{m}} \\ &= mx(x-1)(x-2)\cdots(x-m+2) \\ &= mx^{\underline{m-1}} \end{aligned}$$

And the inverse operator of Δ is $\sum$. Still we have the fundamental theorem of differential.

$$\sum_a^b g(x)\delta x = f(x)_a^b = f(b) - f(a).$$

Example: calculating a falling sum.

$$\sum_{0\le k<n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

and it's easy to analoog to the integrate formula.

<u>The conversion between ordinary powers and falling powers</u>. We observe the following fact:

$$k^2 = k^{\underline{3}}/3 + k^{\underline{2}}/2$$

and the fact that

$$k^3 = k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}$$

The general method of converting can be found at the chapter Stirling numbers.

The falling/rising sum also satisfies binomial theorem.

$$(a+b)^{\underline{m}} = \sum_{i=1}^n \binom{n}{i} a^{\underline{i}} b^{\underline{m-i}},$$

and

$$(a+b)^{\overline{m}} = \sum_{i=1}^n \binom{n}{i} a^{\overline{i}} b^{\overline{m-i}},$$

<u>Generalization fp and rp to neg int</u>. We observe as the degree decrease the division are made in the context. hence we define

$$x^{\underline{-1}} = \frac{1}{x+1}$$

and so on. In general, $f^{\underline{-m}} = \frac{1}{(x+1)(x+2)\cdots(x+m}$, for $m > 0$.

And we can also define real or even complex valued values here, omitted.

<u>Properties about fp and rp</u>. Law 1. $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$.

Law 2. The $\Delta x^{\underline{m}} = mx^{\underline{m-1}}$, forall m.

Law 3. The summation

$$\sum_a^b x^{\underline{m}}\delta x = \begin{cases} \frac{x^{\underline{m+1}}}{m+1}|_a^b & \text{if } x \neq 1 \\ H_x|_a^b & \text{if} m = -1 \end{cases}$$

This is similar to the derivative formula for $1/x$.

Law 4. $\Delta 2^x = 2^x$. We can see that $f(x+1) - f(x) = f(x)$ hence getting $f(x+1) = 2f(x)$.

Now have a look at the ways of manipulation:

Compared to calculus, there's no such thing as chain rule, but we still have

Rule 1. $\Delta cf = c\Delta f$. This can be proved by extracting the constant around the function f.

Rule 2. $\Delta(f+g) = \Delta f + \Delta g$. This can also be proved via a similar technique.

Rule 3. $\Delta fg = f\Delta g + g\Delta f$. This can be proved with the technique of adding and subtracting the terms:

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)\\ &= u(x)\Delta v(x) + v(x+1) - \Delta u(x)\end{aligned}$$

Making the shifting operator E into this formula will simplify it a lot namely defined as

$$Ef(x) = f(x+1).$$

this will lead to $\Delta(uv) = u\Delta v + Ev\Delta u$

Rule 4. Summation by parts: $\sum u\Delta v = uv - \sum v\Delta u$. This rule is similar to the one by integrating by parts.

Example. Do summation to $x2^x\delta x$.

$$\begin{aligned}\sum x2^x\delta x &= x2^x - \sum 2^{x+1}\delta x\\ &= x2^x - 2^{x+1} + C\end{aligned}$$

We may get the result by attaching limits.

Another example: find $\sum xH_x\delta x$.

$$\begin{aligned}\sum xH_x\delta x &= \frac{x^{\underline{2}}}{2}x^{\underline{-1}}\delta x\\ &= \frac{x^{\underline{2}}}{2}H_x - \frac{1}{2}\sum x^{\underline{1}}\delta x\\ &= \frac{x^{\underline{2}}}{2}H_x - \frac{x^{\underline{2}}}{4} + C.\end{aligned}$$

2.5 Infinite series

We cannot manipulate things with the restriction of infinite sums. Here, associativity and exchanging things may not be true. However, we can observe the following sequence:

Good summation sequences. There's a general sum $\sum_{k\in K} a_k$ as K can be infinite. We assume that all the $a_k \geq 0$, if there's a bounding constant A s.t. $\sum_{k\in F} a_k \leq A$ holds for all finite subsets $F \subset K$, we define then $\sum_{k\in K} a_k$ to be the least such A. If there's no bounding constant, we say the result is ∞.

This implies that $\sum_{k\geq 0} a_k = \lim_{n\to\infty}\sum_{k=0}^{n} a_k$, for example,

$$\sum_{k\geq 0} x^k = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \begin{cases}\frac{1}{1-x}, & 0 \leq x < 1;\\ \infty, & x \geq 1.\end{cases}$$

Simplified definition on infinite sums. Let K be any set, and let a_k be a real-valued term defined for each $k \in K$, any real number x can be written as the difference of its positive and negative parts, that is: $x = x^+ - x^-$. Hence, we can come up with ideas like converges and converges absolutely and so on. More in the Calculus book.

2.6 The repertoire method

Introductory Example. We have the fact that

$$S_n = \sum_{k=0}^{n} a_k$$

that is we have $S_0 = a_0$, $S_n = S_{n-1} + a_n (n > 0)$. Now this recurrence gives all the prefix sum in the series.

Suppose that we have the linear combination form $a_n = \beta + \gamma n$, and this will define a as:

$$\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma r, \qquad n > 0 \end{aligned}$$

The key idea is that if we can convince ourselves that the final solution is like the form of

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

with independent solutions for three different relations as what α, β, γ is, then we can claim what our closed forms are!

Example 1.

Assume that we have that

$$\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma n \quad n > 0 \end{aligned}$$

and has the table that

n	R_n
0	α
1	$\alpha + \beta + \gamma$
2	$\alpha + 2\beta + 3\gamma$
3	$\alpha + 3\beta + 6\gamma$
4	$\alpha + 4\beta + 10\gamma$
5	$\alpha + 5\beta + 15\gamma$

Then we can start guessing what the solution of the function is $A(n) = 1, \forall n$, and that $B(n) = n$, and that $C(n) = n(n+1)/2$. If the function is too complicated, we cant guess that out. Hence we need the repertoire method.

The idea is to take the linear combination of the solutions like follows:

Basic setting 1: Consider that when $\alpha = 1, \beta = \gamma = 0$, the equation will be easy to solve – that's an constant solution.

Basic setting 2: $\alpha = 0, \beta = 1, \gamma = 0$, this yields $R_n = R_{n-1} + 1$. This makes a simple sequence. This can be used reversed the finding progress.

Basic setting 3: $R_n = n^2$, guessing its recurrence relation:

$$\begin{aligned} 1 &= \beta + \gamma \\ 4 &= 1 + \beta + 2\gamma \end{aligned}$$

and this implies $\alpha = 0, \beta = -1, \gamma = 2$. Then by induction we found it matches the pattern of all n^2 s. so setting all these three cases we get

$$\begin{aligned} 1 &= 1A(n) + 0B(n) + 0C(n) \\ n &= 0A(n) + 1B(n) + 0C(n) \\ 1 &= 1A(n) + (-1)B(n) + 2C(n) \end{aligned}$$

Thus, after guessing the constants, we can solve all the $A(n), B(n)$ and $C(n)$. That is

$$\begin{aligned} A(n) &= 1 \\ B(n) &= n \\ C(n) &= \frac{n^2 + n}{2} \end{aligned}$$

The main idea. This method is to find coefficients for a linear combination. We guess that the solutions can be expressed as a sum of coefficients multiplied by functions of n .

The recipe of repertoire method.

- Relax the recurrence by adding an extra function term.
- Substitute known function into the recurrence to derive identities similar to the recurrence.
- Take linear combination of such identities to derive an equation identical to the recurrent.

That is, firstly check $a_n = a_{n-1} + \text{sth}$, generalize sth to $f(n)$. For example, $a_n = a_{n-1}$ will be converted to $a_n = a_{n-1} + f(n)$. We easily retrieve $f(n) = a_n - a_{n-1}$. Next we build a table of indigents in which we can construct $f(n)$. Finally, we determine coefficients of each component so that they satisfy the basis of the basis and recurrence.

Example 1. Find the closed form sum of

$$S_n = \sum_{k=0}^{n} k^3.$$

The sum can be seen as $a_0 = 0$, and $a_n = a_{n-1} + n^3$. Next we build a table of a_n and $a_n - a_{n-1}$.

We find some candidates that may be useful.

Table 2.1: The table built

a_n	$a_n - a_{n-1}$
1	0
n	1
n^2	$2n-1$
n^3	$3n^2-3n+1$
n^4	$4n^3-6n^2+4n-1$

How is the coeffs useful? The simplest way we come up is

$$n^3 = \alpha\left(4n^3 - 6n^2 + 4n - 1\right) \implies \alpha = 1/4.$$

We shall also eliminate n^2 and so on and we have that

$$\alpha(4n^3 - 6n + 4n - 1) + \beta(3n^2 - 3n + 1) + \gamma(2n - 1) = n^3$$

Solve this will be useful.

2.7 Exercises

W 1. This means 0. For the lower bound is larger than upper bound.

W 2. We do this by splitting the cases:

If $x = 0$, that is simply 0. If $x > 0$, it will become x; otherwise, it will become $-x$.

To sum up, this is equivalent to absolute value. Namely $|x|$.

W 3. The first one is

$$a_1 + a_2 + a_3 + a_4 + a_5$$

while the second one is

$$a_{1^2} + a_{\sqrt{2}^2} + a_{\sqrt{3}^2} + a_{\sqrt{4}^2} + a_{\sqrt{5}^2}.$$

W 4. (a) We have $\sum[1 \le i < j < k \le 4] = \sum[1 \le k \le 4][k > j][j > i] = \sum_{k=1}^{4} \sum_{j=k}^{4} \sum_{i=j}^{4} a_{ijk}$.

(b) We have $\sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{j=1}^{4} a_{ijk}$. This is pretty similar to the determination in double integral and triple integral.

W 5. The change of variable is wrong, for i and j represents different meaning. The change of variables must be fresh. That is it's not able to be collision to the existing value.

W 6. The value of that is $n-j+1$, as a counting problem.

W 7. We have the fact that $x^{\overline{m}} = x(x+1)\cdots(x+m-1)$, and that $x^{\overline{m-1}} = x(x+1)\cdots(x+m-2)$. So the answer is

$$(x+m-2)x^{\overline{m-1}}$$

W 8. According to the definition, we have that the answer is 0. For whatever we do the answer must be 0.

W 9. We define that $x^{\overline{-n}}$ according to the division rule that

$$x^{\overline{-1}} = \frac{1}{x-1},$$

and more generally, we have

$$x^{\overline{-m}} = \frac{1}{(x-1)(x-2)\cdots(x-m)}.$$

W 10. If we swap this, getting

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

and have

$$\Delta(vu) = v\Delta u + Eu\Delta v$$

subtracting both, we have

$$\begin{aligned}\Delta(uv) - \Delta(vu) &= u\Delta v + Ev\Delta u - (v\Delta u + Eu\Delta v)\\ &= (u - Eu)\Delta v + (Ev - v)\Delta u\\ &= (u(x) - u(x+1))(v(x+1) - v(x)) + (v(x+1) - v(x))(u(x+1) - u(x))\\ &= 0\end{aligned}$$

and the derivation is symmetric, although.

B 11. We first note this by setting $n = 3$, and we have

$$a_1b_0 - a_0b_0 + a_2b_1 - a_1b_1 + a_3b_2 - a_2b_2,$$

that is

$$\begin{aligned}\sum_{0\le k\le n}(a_{k+1} - a_k)b_k &= -\sum_{0\le k<n} a_{k+1}(b_{k+1} - b_k) - \sum_{0\le k<n} a_{k+1}b_k - \sum_{0\le k<n} a_kb_k\\ &= -\sum_{0\le k<n} a_{k+1}(b_{k+1} - b_k) - \sum_{0\le k<n} b_k(a_{k+1} + a_k)\\ &= -\sum_{0\le k<n} a_{k+1}(b_{k+1} - b_k) + (a_nb_n - a_0b_0).\end{aligned}$$

B 12. A permutation of integer means that $p(k)$ and all integers are bijection. That is: Consider $\mathbb{R} \to \mathbb{R}$, $x \mapsto x + (-1)^k c$, the result is unique. Substituting back, we have that: if k is odd, we have $k = f(x) - c$; if k is even, we have $k = f(x) + c$. We can find that these are identical and is able to make a bijection, hence there's a permutation on every integer.

B 13. We can generate the following list of basic components:
$$\begin{array}{c|l} a_n & a_n - a_{n-1} \\ (-1)^n n^2 & (-1)^n(2n^2 - 2n + 1) \\ (-1)^n & 2(-1)^n \\ (-1)^n n & (-1)^n(2n-1) \end{array}$$

And we have the idea that

$$S_n = \frac{1}{2}(-1)^n n(n+1).$$

B 14. According to geometric series formula, we have $k = \sum_{1\le j\le k} 1$, gives that

$$\sum_{1\le j\le n} (2^{n+1} = 2^j) = n_2^{n+1} - (2^{n+1} - 2).$$

B 15. We have $c_n + s_n = \left(\sum_{k=1}^n k\right)^2 + s_n$. according to the instruction given.

B 16. Multiplying for both sides will yield the answer. That is $x^{\underline{m}}(x-m)^{\underline{n}} = x^{\underline{n}}(x-n)^{\underline{m}}$. And by definition, this two will always equal.

B 17. Without the loss of the generality, we can prove the equation that

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = 1/(x-1)^{\underline{-m}}.$$

By induction, the foundation is that

$$x^{\overline{1}} = (-1)^1(-x)^{\underline{1}} = x = 1/$$

and this is true obviously.

Assume $x = k$ is true, then consider $x = k+1$. We can prove that

$$\begin{aligned} x^{\overline{k+1}} = x^{\overline{k}}(x+k) &= (-1)^k(-x)^{\underline{k}}(x+k) \\ &= (-1)^2 k x^{\underline{k}}(x+k) \\ &= x^{\underline{k}}(x+k) \\ &= (-1)^{k+1}(-x)^{\underline{k+1}} \end{aligned}$$

and $k+1$ is true, and by induction, this theorem is always true for positive integers.

Other cases can be shown to be correct by the property of rising and falling powers.

The second equation shown can be simply show by expanding then all.

The last equation follows immediately by the property of the negative falling numbers.

B 18.