Notes on Feller Condition

Dec. 32, 2999

1 Fokker-Planck Equation

Consider the following paraboic partial differential equation (PDE) [1]

$$\frac{\partial u(t,x)}{\partial t} = -\frac{\partial}{\partial x}\left((bx+c)u(t,x)\right) + \frac{\partial^2}{\partial x^2}\left(axu(t,x)\right), \tag{1}$$

with $x > 0$. This can be viewed as the corresponding Fokker-Planck equation for the Cox-Ingersoll-Ross (CIR) process

$$dx_t = \kappa(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \tag{2}$$

with $a = \sigma^2/2$, $b = -\kappa$, and $c = \kappa\theta$.

We want to find the fundamental solution of the PDE (1), *i.e.*, the initial condition is given by

$$u(0,x) = \delta(x-y), \tag{3}$$

where $\delta(x)$ is the Dirac delta function. To this end, introduce the Laplace transform of $u(t,x)$ as

$$v(t,\lambda) = \int_0^{+\infty} e^{-\lambda x} u(t,x), \tag{4}$$

for $\lambda > 0$. Consider the Laplace transform of the right hand side of Eq. (1),

$$\begin{aligned}
& \int_0^{+\infty} e^{-\lambda x}\left[(axu(t,x))_{xx} - ((bx+c)u(t,x))_x\right]dx \\
= \quad & e^{-\lambda x}\left[(axu(t,x))_x - ((bx+c)u(t,x))\right]\Big|_0^{+\infty} \\
& \qquad \lambda\int_0^{+\infty} e^{-\lambda x}\left[(axu(t,x))_x - ((bx+c)u(t,x))\right]dx \\
= \quad & f(t) + \lambda(b-\lambda a)v_s - c\lambda v,
\end{aligned} \tag{5}$$

where

$$f(t) = \lim_{x\to 0}\left[(bx+c)u - (axu)_x\right], \tag{6}$$

is the flux at $x = 0$ and cannot be arbitrarily specified.

After the Laplace transform, Eq. (1) becomes

$$v_t + \lambda(\lambda a - b)v_s = f(t) - c\lambda v. \tag{7}$$

This first order PDE can be solved by the method of characteristics.

References

[1] W. Feller, *Two Singular Diffusion Problems*, Annals of Mathematics **54**, 173 (1951).