Notes on Feller Condition

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1 Fokker-Planck Equation

Consider the following parabolic partial differential equation (PDE) [1]

$$\frac{\partial u(t,x)}{\partial t} = -\frac{\partial}{\partial x}\left((bx+c)u(t,x)\right) + \frac{\partial^2}{\partial x^2}\left(axu(t,x)\right), \tag{1}$$

with $x > 0$. This can be viewed as the corresponding Fokker-Planck equation for the Cox-Ingersoll-Ross (CIR) process

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t, \tag{2}$$

with $a = \sigma^2/2$, $b = -\kappa$, and $c = \kappa\theta$.

We want to find the solution of the PDE (1), with certain initial condition,

$$u(t,x) = \phi(x). \tag{3}$$

In addition, we want to the solution is positive, $u(t,x) \geq 0$, and norm preserving or decreasing, that is

$$\int_0^{+\infty} u(t,x)dx \leq \int_0^{+\infty} \phi(x)dx. \tag{4}$$

In particular, we are interested in the fundamental solution of the PDE (1), *i.e.*, the initial condition is given by

$$u(0,x) = \delta(x - \xi), \tag{5}$$

where $\delta(x)$ is the Dirac delta function. To this end, introduce the Laplace transform of $u(t,x)$ as

$$v(t,\lambda) = \int_0^{+\infty} e^{-\lambda x}u(t,x)dx, \tag{6}$$

for $\lambda > 0$. We denote the Laplace transform of the initial condition as

$$\pi(\lambda) = \int_0^{+\infty} e^{-\lambda x}\phi(x)dx, \tag{7}$$

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and for the fundamental solution, $\pi(\lambda) = e^{-\lambda\xi}$. Consider the Laplace transform of the right hand side of Eq. (1),

$$\int_0^{+\infty} e^{-\lambda x} \left[\big(axu(t,x)\big)_{xx} - \big((bx+c)u(t,x)\big)_x \right] dx$$

$$= e^{-\lambda x} \left[\big(axu(t,x)\big)_x - \big((bx+c)u(t,x)\big) \right] \Big|_0^{+\infty}$$

$$+ \lambda \int_0^{+\infty} e^{-\lambda x} \left[\big(axu(t,x)\big)_x - \big((bx+c)u(t,x)\big) \right] dx$$

$$= f(t) + \lambda(b - \lambda a)v_\lambda - c\lambda v, \tag{8}$$

where

$$f(t) = \lim_{x \to 0} \left[(bx+c)u - (axu)_x \right], \tag{9}$$

is the flux at $x = 0$ and generally cannot be arbitrarily specified.

After the Laplace transform, Eq. (1) becomes

$$v_t + \lambda(\lambda a - b)v_\lambda = f(t) - c\lambda v. \tag{10}$$

This first order PDE can be solved by the method of characteristics. The characteristics can be determined by integrating

$$dt = \frac{d\lambda}{\lambda(a\lambda - b)}, \tag{11}$$

which leads to the following characteristic

$$e^{-bt} \frac{a\lambda - b}{\lambda} = C_1, \tag{12}$$

or equivalently,

$$\lambda(t) = -\frac{be^{-bt}}{C_1 - ae^{-bt}}. \tag{13}$$

Then, Eq. (10) becomes an ordinary differential equation,

$$\frac{dv}{dt} - \frac{bce^{-bt}}{C_1 - ae^{-bt}} v = f(t). \tag{14}$$

It can be integrated,

$$v(t, \lambda) = \left| C_1 - ae^{-bt} \right|^{c/a} \left\{ C_2 + \int_0^t \frac{f(\tau)}{|C_1 - ae^{-b\tau}|^{c/a}} d\tau \right\}. \tag{15}$$

To determine the unknown constants, assume $C_2 = A(C_1)$, where $A(y)$ is an arbitrary function. For the initial condition $v(0, \lambda) = \pi(\lambda)$, we have

$$\pi(\lambda) = \left| \frac{b}{\lambda} \right|^{c/a} A\left(a - \frac{b}{\lambda} \right), \tag{16}$$

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which gives

$$A(y) = |a - y|^{-c/a} \, \pi \left(\frac{b}{a - y} \right).$$ (17)

Now, we have

$$v(t, \lambda) = |C_1 - ae^{-bt}|^{c/a} \left\{ |a - C_1|^{-c/a} \, \pi \left(\frac{b}{a - C_1} \right) + \int_0^t \frac{f(\tau)}{|C_1 - ae^{-b\tau}|^{c/a}} d\tau \right\}.$$ (18)

Using Eq. (12), the general solution becomes

$$v(t, \lambda) = \left(\frac{b}{\lambda a(e^{bt} - 1) + b} \right)^{c/a} \pi \left(\frac{\lambda b e^{bt}}{\lambda a(e^{bt} - 1) + b} \right) + \int_0^t f(\tau) \left(\frac{b}{\lambda a(e^{b(t-\tau)} - 1) + b} \right)^{c/a} d\tau.$$ (19)

2 Some exact results

In this section, we are going to gather some exact results regarding the general solution (19) without proof. The implication of these results will be discussed.

Lemma 2.1. If $c \leq 0$, then $v(t, \lambda)$ in Eq. (19) is the Laplace transform of a solution of $u(t, x)$ only if $f(t)$ satisfies

$$\pi \left(\frac{b}{a(1 - e^{-bt})} \right) + \int_0^t f(\tau) \left(\frac{e^{bt} - 1}{e^{b(t-\tau)} - 1} \right)^{c/a} d\tau = 0.$$ (20)

This is due to the fact that $u(t, x)$ is integrable near $x = 0$, which is equivalent to $v(t, \lambda) \to 0$ as $\lambda \to +\infty$. Then, there is only one solution for the original Fokker-Planck equation (1) with arbitrary initial condition (3), with the boundary condition prescribed by (20).

Lemma 2.2. If $c > 0$ and $f(t)$ is continuous for $t \geq 0$, then $v(t, \lambda)$ in Eq. (19) is the Laplace transform of a solution of $u(t, x)$ with initial valules $\phi(x)$. The solution is positive preserving at least whenever $f(t) \geq 0$.

Lemma 2.3. If $c \geq 0$ and $\phi(x)$ is non-negative, then Eq. (19) with $f(t) \equiv 0$ defines a non-negative solution $u(t, x)$ with initial values $\phi(x)$. For this solution,

$$\lim_{x \to 0} u(t, x) = \begin{cases} \infty & \text{if} \quad 0 < c < a, \\ 0 & \text{if} \quad c > a, \\ \frac{b}{a(e^{bt} - 1)} \pi \left(\frac{be^{bt}}{a(e^{bt} - 1)} \right) & \text{if} \quad c = a. \end{cases}$$

Lemma 2.4. If $c > a$ and $f(t) \geq 0$ but $f(t)$ does not vanish identically, then solution $u(t, x)$ is norm increasing. If $f(t_0) < 0$, the $u(t_0, x) < 0$.

From the above results, we can draw the following conclusions:

1. If $c \leq 0$, there is one solution for each initial condition $\phi(x)$. The solution is positive and norm decreasing, and is absorbing at the $x = 0$ boundary;

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2. If $0 < c < a$, there is one positive and norm preserving solution, with vanishing flux at the $x = 0$ boundary, *i.e.*, the reflecting boundary condition. There are infinitely many positive and norm decreasing solutions, with only one solution which is finite at the $x = 0$ boundary, *i.e.*, absorbing boundary condition;

3. If $c \geq a$, the exists one positive and norm preserving solution, with vanishing value and flux at the $x = 0$ boundary. No boundary condition can be specified at the $x = 0$ boundary, and the boundary is unattainable.

3   Fundamental solutions

In the following, we will consider the absorbing and reflecting boundary conditions and find the explicit form of the fundamental solutions to the Fokker-Planck equation (1) with initial condition (5).

3.1   Absorbing boundary condition

If $c \leq a$, we can impose the absorbing boundary condition, (20). For the fundamental solution, we have $\pi(\lambda) = e^{-\lambda\xi}$, and Eq. (20) becomes

$$\int_0^t f(\tau) \left(\frac{e^{bt} - 1}{e^{b(t-\tau)} - 1} \right)^{c/a} d\tau = -\exp\left(-\frac{b\xi}{a(1 - e^{-bt})} \right). \tag{21}$$

Define

$$\frac{1}{z} = 1 - e^{-bt}, \qquad \frac{1}{\zeta} = 1 - e^{-b\tau}, \tag{22}$$

Eq. (21) becomes

$$\int_z^{+\infty} f(\tau) \left(\frac{\zeta}{\zeta - z} \right)^{c/a} \frac{d\zeta}{b\zeta(\zeta - 1)} = -\exp\left(-\frac{b\xi z}{a} \right). \tag{23}$$

Let

$$g(\zeta) = f(\tau) \frac{\zeta^{c/a}}{b\zeta(\zeta - 1)}, \tag{24}$$

and let $g(\zeta)$ have the following functional form

$$g(\zeta) = Be^{-A\zeta}, \tag{25}$$

then the right hand side of Eq. (23) becomes

$$B \int_z^{+\infty} e^{-A\zeta} (\zeta - z)^{-c/a} d\zeta = \frac{B\Gamma\left(1 - \dfrac{c}{a} \right)}{A^{1-c/a}} e^{-Az}. \tag{26}$$

Matching terms, we can see that

$$A = \frac{b\xi}{a}, \qquad B = -\frac{1}{\Gamma\left(1 - \dfrac{c}{a} \right)} \left(\frac{b\xi}{a} \right)^{1-c/a}. \tag{27}$$

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Therefore,

$$f(t) = -\frac{b}{\Gamma\left(1 - \frac{c}{a}\right)} \frac{e^{-bt}}{1 - e^{-bt}} \left(\frac{b\xi}{a(1 - e^{-bt})}\right)^{1-c/a} \exp\left(-\frac{b\xi}{a(1 - e^{-bt})}\right). \tag{28}$$

With some manipulation, the fundamental solution can be written in the following convenient form,

$$v(t, \lambda) = \left(\frac{b}{\lambda a(e^{bt} - 1) + b}\right)^{c/a} \exp\left(-\frac{\lambda b\xi e^{bt}}{\lambda a(e^{bt} - 1) + b}\right) \Gamma\left(1 - \frac{c}{a}; \frac{b\xi e^{bt}}{a(e^{bt} - 1)} \frac{b}{\lambda a(e^{bt} - 1) + b}\right), \tag{29}$$

where

$$\Gamma(n; z) = \frac{1}{\Gamma(n)} \int_0^z e^{-x} x^{n-1} dx \tag{30}$$

is the incomplete Gamma function. From this, we can see that

$$\int_0^{+\infty} u(t, x) dx = \lim_{\lambda \to 0} v(t, \lambda) = \Gamma\left(1 - \frac{c}{a}; \frac{\lambda b\xi e^{bt}}{a(e^{bt} - 1)}\right), \tag{31}$$

which is strictly less than 1. This means that the $x = 0$ boundary serves as an absorbing boundary, and once it is hit, the solution will stay there.

We can also find the explicit form of the fundamental solution via inverse Laplace transform. Again, let

$$A = \frac{b\xi e^{bt}}{a(e^{bt} - 1)}, \qquad \frac{1}{z} = \frac{b}{\lambda a(e^{bt} - 1) + b}, \tag{32}$$

the incomplete Gamma function in Eq. (29) can be expressed as

$$\frac{1}{\Gamma\left(1 - \frac{c}{a}\right)} \int_0^{A/z} e^{-x} x^{n-1} dx = \frac{1}{\Gamma\left(1 - \frac{c}{a}\right)} \left(\frac{A}{z}\right)^{1-c/a} \int_0^1 e^{-Au/z} u^{-c/a} du$$

$$= \frac{1}{\Gamma\left(1 - \frac{c}{a}\right)} \left(\frac{A}{z}\right)^{1-c/a} e^{-A/z} \int_0^1 e^{Av/z} (1 - v)^{-c/a} dv. \tag{33}$$

Then,

$$v(t, \lambda) = \frac{e^{-A} A^{1-c/a}}{\Gamma\left(1 - \frac{c}{a}\right)} \int_0^1 (1 - v)^{-c/a} \frac{e^{Av/z}}{z} dv. \tag{34}$$

The inverse Laplace transform can be applied to $e^{Av/z}/z$, which yields

$$\int e^{\lambda x} \frac{e^{Av/z}}{z} d\lambda = \frac{b}{a(e^{bt} - 1)} \exp\left(-\frac{bx}{a(e^{bt} - 1)}\right) \int \exp\left(\frac{bxz}{a(e^{bt} - 1)}\right) \frac{e^{Av/z}}{z} dz$$

$$= \frac{b}{a(e^{bt} - 1)} \exp\left(-\frac{bx}{a(e^{bt} - 1)}\right) I_0\left(2\left(\frac{Avbx}{a(e^{bt} - 1)}\right)^{1/2}\right)$$

$$= \frac{b}{a(e^{bt} - 1)} \exp\left(-\frac{bx}{a(e^{bt} - 1)}\right) I_0\left(\frac{2b}{a(e^{bt} - 1)} \left(e^{bt} \xi v x\right)^{1/2}\right), \tag{35}$$

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where we have used the fact that the inverse Laplace transform of $e^{Av/z}/z$ is $I_0(2\sqrt{Avx})$, where $I_\nu(x)$ is the modified Bessel function of order ν. Finally, we have

$$
\begin{aligned}
u(t,x) \quad = \quad & \frac{1}{\Gamma\left(1-\frac{c}{a}\right)} \frac{b}{a(e^{bt}-1)} \left(\frac{b\xi}{a(1-e^{-bt})}\right)^{1-c/a} \exp\left(-\frac{b(x+\xi e^{bt})}{a(e^{bt}-1)}\right) \\
& \times \int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi v x\right)^{1/2}\right) dv.
\end{aligned}
\tag{36}
$$

Using the series expansion of the modified Bessel function,

$$
I_\nu(x) = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r+1+\nu)}\left(\frac{x}{2}\right)^{2r+\nu},
\tag{37}
$$

the integral in (36) can be written as

$$
\begin{aligned}
& \int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi v x\right)^{1/2}\right) dv \\
= \quad & \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r+1)}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right)^{2r} \int_0^1 (1-v)^{-c/a} v^r dv \\
= \quad & \Gamma\left(1-\frac{c}{a}\right)\sum_{r=0}^{\infty} \frac{1}{r!\Gamma\left(r-\frac{c}{a}+2\right)}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right)^{2r} \\
= \quad & \Gamma\left(1-\frac{c}{a}\right)\left(\frac{b}{a(1-e^{-bt})}\right)^{-1+c/a}\left(e^{-bt}\xi x\right)^{\frac{c-a}{2a}} I_{1-\frac{c}{a}}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right).
\end{aligned}
\tag{38}
$$

Then,

$$
u(t,x) = \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{b(x+\xi e^{bt})}{a(e^{bt}-1)}\right)\left(e^{-bt}\frac{x}{\xi}\right)^{\frac{c-a}{2a}} I_{1-\frac{c}{a}}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right).
\tag{39}
$$

3.2   Reflecting boundary condition

If $c > 0$, we can impose the reflecting boundary condition on Eq. (19), $f(t) \equiv 0$, which will render a positive and norm preserving solution. To find the explicit form of the fundamental solution, we can apply the inverse Laplace transform on Eq. (19), with $f(t) \equiv 0$. Use the same substitution as in Eq. (32), we have

$$
v(t,\lambda) = e^{-A} z^{-c/a} \exp\left(\frac{A}{z}\right).
\tag{40}
$$

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The same manipulation of the inverse Laplace transform as in last section leads to

$$
\int e^{\lambda x} \frac{e^{A/z}}{z^{c/a}} d\lambda
$$

$$
= \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{bx}{a(e^{bt}-1)}\right) \int \exp\left(\frac{bxz}{a(e^{bt}-1)}\right) \frac{e^{A/z}}{z^{c/a}} dz
$$

$$
= \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{bx}{a(e^{bt}-1)}\right) \left(\frac{bx}{Aa(e^{bt}-1)}\right)^{\frac{c-a}{2a}} I_{-1+\frac{c}{a}}\left(2\left(\frac{Abx}{a(e^{bt}-1)}\right)^{1/2}\right)
$$

$$
= \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{bx}{a(e^{bt}-1)}\right) \left(e^{-bt}\frac{x}{\xi}\right)^{\frac{c-a}{2a}} I_{-1+\frac{c}{a}}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right), \quad (41)
$$

where we have use the fact that the inverse Laplace transform of $e^{A/z}/z^{1+\nu}$ is $(x/A)^{\nu/2} I_\nu(2\sqrt{Ax})$, and the fundamental solution is then given by

$$
u(t,x) = \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{b(x+\xi e^{bt})}{a(e^{bt}-1)}\right) \left(e^{-bt}\frac{x}{\xi}\right)^{\frac{c-a}{2a}} I_{-1+\frac{c}{a}}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right). \quad (42)
$$

4   CEV model

For the underlying asset price, the CEV model assumes the following stochastic process,

$$
dS_t = \sigma S_t^\beta dW_t, \quad (43)
$$

where $\sigma > 0$ is the volatility, and $\beta > 0$. Define [2]

$$
X_t = \frac{S_t^{2(1-\beta)}}{\sigma^2(1-\beta)^2}, \quad (44)
$$

and apply the Ito's lemma, we have

$$
dX_t = \delta dt + 2\sqrt{X_t} dW_t, \quad (45)
$$

where

$$
\delta = \frac{1-2\beta}{1-\beta}. \quad (46)
$$

Thus, the CEV model has been reduced to the problem that we have studied in the past few sections, with $a = 2$, $b = 0$, and $c = \delta$.

The absorbing boundary condition can be imposed when $c \le a$, or $\delta \le 2$, which is equivalent to $0 < \beta \le 1$. The transition density for the transformed state variable X is given by

$$
p_A(X_T, T; X_0) = \frac{1}{2T} \exp\left(-\frac{X_T + X_0}{2T}\right) \left(\frac{X_T}{X_0}\right)^{-\nu/2} I_\nu\left(\frac{\sqrt{X_T X_0}}{T}\right), \quad (47)
$$

where

$$
\nu = \frac{1}{2(1-\beta)}. \quad (48)
$$

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For the underlying, the transition density [3] can be obtained from (47),

$$p_A(S_T, T; S_0) = \frac{\left(S_0 S_T^{1-4\beta}\right)^{1/2}}{(1-\beta)\sigma^2 T} \exp\left(-\frac{S_0^{2(1-\beta)} + S_T^{2(1-\beta)}}{2(1-\beta)^2\sigma^2 T}\right) I_\nu\left(\frac{(S_0 S_T)^{1-\beta}}{(1-\beta)^2\sigma^2 T}\right). \qquad (49)$$

The reflecting boundary condition can be imposed when $c > 0$, which is equivalent to $0 < \beta \leq \frac{1}{2}$ or $\beta > 1$. The transition density for the transformed state variable X is given by

$$p_R(X_T, T; X_0) = \frac{1}{2T} \exp\left(-\frac{X_T + X_0}{2T}\right)\left(\frac{X_T}{X_0}\right)^{-\nu/2} I_{-\nu}\left(\frac{\sqrt{X_T X_0}}{T}\right). \qquad (50)$$

Similarly, transition density for the underlying is

$$p_R(S_T, T; S_0) = \frac{\left(S_0 S_T^{1-4\beta}\right)^{1/2}}{(1-\beta)\sigma^2 T} \exp\left(-\frac{S_0^{2(1-\beta)} + S_T^{2(1-\beta)}}{2(1-\beta)^2\sigma^2 T}\right) I_{-\nu}\left(\frac{(S_0 S_T)^{1-\beta}}{(1-\beta)^2\sigma^2 T}\right). \qquad (51)$$

5   Simulation of the Heston model

In this section, we will consider the Monte Carlo simulation of the Heston model,

$$d\log X_t = -\frac{v_t}{2}dt + \sqrt{v_t}dZ_t, \qquad (52)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t, \qquad (53)$$

with $\langle dZ_t, dW_t\rangle = \rho dt$. Here, the parameters have the usual meaning. Alternatively, the forward process can be written as

$$d\log X_t = -\frac{v_t}{2}dt + \rho\sqrt{v_t}dW_t + \sqrt{1-\rho^2}\sqrt{v_t}dB_t, \qquad (54)$$

where $\langle dB_t, dW_t\rangle = 0$.

For the variance process v_t, we normally impose a reflecting boundary condition. Note that, when the *Feller condition* is satisfied,

$$2\kappa\theta > \sigma^2, \qquad (55)$$

the lower boundary is unattainable. Nevertheless, when the process is discretized in Monte Carlo simulation, it is still possible to reach the lower boundary. From earlier results, the transition density for the variance is given by

$$p\left(v_T|\,v_t\right) = \frac{1}{2}\frac{n(t,T)}{e^{-\kappa(T-t)}} \exp\left(-\frac{1}{2}\left(\frac{n(t,T)}{e^{-\kappa(T-t)}}v_T + n(t,T)v_t\right)\right)\left(\frac{\frac{n(t,T)}{e^{-\kappa(T-t)}}v_T}{n(t,T)v_t}\right)^{d/4-1/2}$$

$$\times I_{d/2-1}\left(\left(\frac{n(t,T)}{e^{-\kappa(T-t)}}v_T \times n(t,T)v_t\right)^{1/2}\right), \qquad (56)$$

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where

$$d = \frac{4\kappa\theta}{\sigma^2}, \qquad n(t,T) = \frac{4\kappa e^{-\kappa(T-t)}}{\sigma^2\left(1 - e^{-\kappa(T-t)}\right)}. \tag{57}$$

Then, the cumulative distribution of v_T conditioning on v_t is given by

$$P\left(v_T \le z \mid v_t\right) = \Upsilon\left(\frac{n(t,T)z}{e^{-\kappa(T-t)}}; d, n(t,T)v_t\right), \tag{58}$$

where $\Upsilon(z; \nu, \lambda)$ is the cumulative distribution function of the non-central chi-squared distribution with ν degrees of freedom and noncentrality parameter λ,

$$\Upsilon\left(z; \nu, \lambda\right) = \frac{1}{2} \int_0^z e^{-(x+\lambda)/2} \left(\frac{x}{\lambda}\right)^{\nu/4-1/2} I_{\nu/2-1}\left(\sqrt{\lambda x}\right) dx. \tag{59}$$

In the following, we will discuss several efficient simulation methods specific for the Heston model.

5.1 Exact simulation of the variance process

From the distribution of v_T conditioning on v_t, (58), we can sample v_T exactly. However, the exact distribution (58) requires the integration of modified Bessel function. This poses a numerical problem, since we need to perform the integration to invert the distribution function.

Alternatively, there are other methods to sample from a non-central chi-squared distribution. Notice that, when $d > 1$, a non-central chi-squared random variable with d degrees of freedom and noncentrality parameter λ can be obtained from a chi-squared random variable by the following relation,

$$\chi_d'^2(\lambda) \stackrel{d}{=} \left(Z + \sqrt{\lambda}\right)^2 + \chi_{d-1}^2, \tag{60}$$

where Z is a standard normal variable, $\stackrel{d}{=}$ denotes equality in distribution. Therefore, by sampling from a standard normal distribution and a chi-squared distribution, we can obtain the desired non-central chi-squared distribution.

The above method only works when $d > 1$. Another sampling method relies on the series expansion of the modified Bessel function. Using Eq. (37), the cumulative distribution function for non-central chi-squared distribution can be written as

$$\Upsilon\left(z; \nu, \lambda\right) = \sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{j!} \frac{1}{2^{\nu/2+j}\Gamma(\nu/2+j)} \int_0^z x^{\nu/2+j-1}e^{-x/2}dx. \tag{61}$$

It can be seen that the integration part of the above expression is a chi-squared distribution with $\nu + 2j$ degrees of freedom, and j follows a Poisson distribution with parameter $\lambda/2$. Therefore, we can sample a non-central chi-squared distribution $\chi_d'^2(\lambda)$ as χ_{d+2N}^2, with $N \sim \text{Poisson}(\lambda/2)$.

5.2 Exact simulation of the Heston model [4]

For the simulation of the forward process, we have

$$\log X_T = \log X_t - \frac{1}{2}\int_t^T v_s ds + \rho \int_t^T \sqrt{v_s}dW_s + \sqrt{1-\rho^2}\int_t^T \sqrt{v_s}dB_s. \tag{62}$$

From the variance process, we have

$$\int_t^T \sqrt{v_s}dW_s = \frac{1}{\sigma}\left(v_T - v_t - \kappa\theta(T-t) + \kappa\int_t^T v_s ds\right). \tag{63}$$

Therefore,

$$\log X_T = \log X_t + \frac{\rho}{\sigma}(v_T - v_t - \kappa\theta(T-t)) + \left[\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right]\int_t^T v_s ds + \sqrt{1-\rho^2}\int_t^T \sqrt{v_s}dB_s. \tag{64}$$

To sample $\log X_T$ conditioning on $\log X_t$ and v_t, we need to sample the following three components:

1. v_T. As discussed in the previous section, v_T can be sampled from a non-central chi-squared distribution exactly.
2. $\int_t^T v_s ds$. We need to sample $\int_t^T v_s ds$ conditioning on v_t and v_T. To this end, we can use the characteristic function,

$$
\begin{aligned}
\varphi(u) &= \mathrm{E}\left[\exp\left(iu\int_t^T v_s ds\right)\middle| v_t, v_T\right] \\
&= \frac{\gamma(u)e^{-(\gamma(u)-\kappa)(T-t)/2}(1-e^{-\kappa(T-t)})}{\kappa(1-e^{-\gamma(u)(T-t)})} \\
&\quad \times \exp\left\{\frac{v_t+v_T}{\sigma^2}\left[\frac{\kappa(1+e^{-\kappa(T-t)})}{1-e^{-\kappa(T-t)}} - \frac{\gamma(u)(1+e^{-\gamma(u)(T-t)})}{1-e^{-\gamma(u)(T-t)}}\right]\right\} \\
&\quad \times \frac{I_{d/2-1}\left[\sqrt{v_t v_T}\dfrac{4\gamma(u)e^{-\gamma(u)(T-t)/2}}{\sigma^2(1-e^{-\gamma(u)(T-t)})}\right]}{I_{d/2-1}\left[\sqrt{v_t v_T}\dfrac{4\kappa e^{-\kappa(T-t)/2}}{\sigma^2(1-e^{-\kappa(T-t)})}\right]},
\end{aligned}
\tag{65}
$$

where $\gamma(u) = \sqrt{\kappa^2 - 2iu\sigma^2}$. Then, for $\int_t^T v_s ds$, its distribution can be inverted from the above characteristic function, and it can be sampled from this distribution.
3. $\int_t^T \sqrt{v_s}dB_s$. Since v_t is independent of B_t, for each realization of v_t, $\int_t^T \sqrt{v_s}dB_s$ is normally distributed with mean 0, and variance $\int_t^T v_s ds$, which has been sampled from the previous step.

Finally, $\log X_T$ is normally distributed, conditioning on $\log X_t$ and one realization of the variance path v_t,

$$\log X_T \sim N\left(\log X_t + \frac{\rho}{\sigma}(v_T - v_t - \kappa\theta(T-t)) + \left[\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right]\int_t^T v_s ds, \quad (1-\rho^2)\int_t^T v_s ds\right). \tag{66}$$

The simulation for the forward and the variance is exact, in the price of numerical complexity.

5.3 Quadratic exponential scheme for the variance process [5]

The variance process can be efficiently simulated with approximation of the non-central chi-squared distribution, depending on the non-centraliy parameter. The approximated distribution can be determined by matching the moments of the conditional mean and variance of the varaince process. To this end, notice that the solution of Eq. (2) is

$$v_T = v_t e^{-\kappa(T-t)} + \theta \left(1 - e^{-\kappa(T-t)}\right) + \sigma \int_t^T e^{-\kappa(T-s)} \sqrt{v_s} dW_s. \tag{67}$$

Therefore,

$$\mathrm{E}\left[v_T\vert v_t\right] = v_t e^{-\kappa(T-t)} + \theta \left(1 - e^{-\kappa(T-t)}\right), \tag{68}$$

and, using Ito's isometry,

$$
\begin{aligned}
\mathrm{Var}\left[v_T\vert v_t\right] &= \sigma^2 \int_t^T e^{-2\kappa(T-s)} \mathrm{E}\left[v_s\vert v_t\right] ds \\
&= \sigma^2 \int_t^T e^{-2\kappa(T-s)} \left(v_t e^{-\kappa(s-t)} + \theta \left(1 - e^{-\kappa(s-t)}\right)\right) ds \\
&= \frac{\sigma^2 \theta}{2\kappa} \left(1 - e^{-\kappa(T-t)}\right)^2 + \frac{\sigma^2 v_t e^{-\kappa(T-t)}}{\kappa} \left(1 - e^{-\kappa(T-t)}\right).
\end{aligned}
\tag{69}
$$

For large non-centrality parameter, the non-central chi-squared random variable can be approximated by a power function applied to the standard normal variable,

$$v_T = a\left(b + Z_v\right)^2. \tag{70}$$

Here, a quadratic form is used to ensure positivity of the random variable. It can be easily shown, that the mean and variance of this approximation are given by

$$\mathrm{E}[v_T] = a(1 + b^2), \tag{71}$$

and

$$\mathrm{Var}[v_T] = 2a^2(1 + 2b^2). \tag{72}$$

Now, denote the conditional mean and variance of the real non-central chi-squared random variable as

$$m = \mathrm{E}\left[v_T\vert v_t\right], \qquad s^2 = \mathrm{Var}\left[v_T\vert v_t\right], \tag{73}$$

then it can be shown that

$$b = \sqrt{\frac{2}{\psi} - 1 + \sqrt{\frac{2}{\psi}\left(\frac{2}{\psi} - 1\right)}}, \qquad a = \frac{m}{1 + b^2}, \tag{74}$$

where $\psi = s^2/m^2$. To ensure the existence of solutions for a and b, the condition $\psi \leq 2$ must be satisfied.

For small non-centrality parameter, the non-central chi-squared density can be approximated by

$$P\left(v_T \in [x, x+dx]\right) = \left(p\delta(0) + (1-p)\beta e^{-\beta x}\right) dx, \tag{75}$$

where $\delta(x)$ is the Dirac δ-function. The inverse cumulative distribution function for this approximate density is given by

$$\Psi^{-1}(u) = \begin{cases} 0 & 0 \le u < p, \\ \dfrac{1}{\beta} \log\left(\dfrac{1-p}{1-u}\right) & p \le u \le 1. \end{cases} \tag{76}$$

Similarly, the mean and variance of the above approximate distribution are given by

$$\mathrm{E}[v_T] = \frac{1-p}{\beta}, \tag{77}$$

and

$$\mathrm{Var}[v_T] = \frac{1-p^2}{\beta^2}. \tag{78}$$

Mathcing with the real distribution, we have

$$p = \frac{\psi - 1}{\psi + 1}, \qquad \beta = \frac{1-p}{m}, \tag{79}$$

where $\psi \ge 1$ must be satisfied.

Therefore, we have the restriction that $1 \le \psi \le 2$, and we can choose a critical value in the range of $[0, 2]$, for example, $\psi_c = 1.5$, to switch between the two approximations for simulation. The simulation procedure for the variance process is summarised as follows:

1. Given v_t, calculate m, s^2, and $\psi = s^2/m^2$;
2. Draw a random number U_v from the uniform distribution on $[0, 1]$;
3. If $\psi > \psi_c$, compute p and β from Eq. (79). Define $v_T = \Psi^{-1}(U_v)$ as in Eq. (76);
4. If $\psi \le \psi_c$, compute a and b from Eq. (74), and $Z_v = \Phi^{-1}(U_v)$, where $\Phi(x)$ is the cumulative distribution function for the standard normal distrubtion. Define $v_T = a\left(b + Z_v\right)^2$ as in Eq. (70);

5.4 Simulation of the forward price process

For the simulation of the forward price process, we can use the equivalent log forward process, Eq. (64). Since we have obtained v_T from the quadratic exponential scheme, we only need to determine the two integrals in Eq. (64). Define $\tau = T - t$, approximate the integrals as

$$\int_t^T v_s ds \simeq \left(\gamma_1 v_t + \gamma_2 v_T\right) \tau, \tag{80}$$

and

$$\int_t^T \sqrt{v_s} dB_s \simeq \sqrt{\gamma_1 v_t + \gamma_2 v_T} \sqrt{\tau} Z_v, \tag{81}$$

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where Z_v is a standard normal random variable. Using these approximate results, Eq. (64) becomes

$$\log X_T = \log X_t + K_0 + K_1 v_t + K_2 v_T + \sqrt{K_3 v_t + K_4 v_T} Z_v, \tag{82}$$

where

$$K_0 = -\frac{\kappa \rho \theta}{\sigma}\tau, \qquad K_1 = \left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\gamma_1\tau - \frac{\rho}{\sigma}, \qquad K_2 = \left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\gamma_2\tau + \frac{\rho}{\sigma},$$

$$K_3 = (1-\rho^2)\gamma_1\tau, \qquad K_4 = (1-\rho^2)\gamma_2\tau. \tag{83}$$

Here, γ_1 and γ_2 can be set to arbitrary numbers as in various numerical integration schemes.

By exponentiating Eq. (82), the forward price is given by

$$X_T = X_t \exp\left(K_0 + K_1 v_t + K_2 v_T + \sqrt{K_3 v_t + K_4 v_T} Z_V\right). \tag{84}$$

Clearly, this discretized forward process is not a martingale, even though the deviation is minor. To fix this issue, we can replace K_0 with

$$K_0^* = -\log M - \left(K_1 + \frac{1}{2}K_3\right)v_t. \tag{85}$$

Denote $A = K_2 + K_4/2$, the martingale correction term M can be expressed in terms of A, depending on the approximate scheme used for sampling v_T. If $\psi \leq \psi_c$, set

$$M = \frac{\exp\left(\dfrac{Ab^2 a}{1 - 2Aa}\right)}{\sqrt{1 - 2Aa}}, \tag{86}$$

as long as $A < (2a)^{-1}$. If $\psi > \psi_c$, we have

$$M = p + \frac{\beta(1-p)}{\beta - A}, \tag{87}$$

for $A < \beta$.

References

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[4] M. Broadie and Ö. Kaya, *Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes*, Operations Research **54** (2), 217 (2006).

[5] L. Andersen, *Efficient Simulation of the Heston Stochastic Volatility Model*, `https://ssrn.com/abstract=946405`.