Notes on Piterbarg's Formulation of Heston Model

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1 Shifted Lognormal Model

Consider the following shifted lognormal process for underlying S,

$$dS_t = \lambda\Big(bS_t + (1-b)\,L\Big)dW_t, \tag{1}$$

the price of a call option written on S is given by

$$C(t, S_t, T, K) = \frac{1}{b}\Big[\Big(bS_t + (1-b)\,L\Big)N\left(d_+\right) - \Big(bK + (1-b)\,L\Big)N\left(d_-\right)\Big], \tag{2}$$

where

$$d_\pm = \frac{1}{|\lambda b|\sqrt{T-t}}\left(\log\left(\frac{bS_t + (1-b)\,L}{bK + (1-b)\,L}\right) \pm \frac{1}{2}\lambda^2 b^2 (T-t)\right), \tag{3}$$

S_t is the spot price of the underlying at time t, and the strike and the expiry of the call option are K and T, respectively. The parameters λ and b are responsible for the overall level and the slope of the implied volatility smile, respectively.

2 Stochastic Volatility Model

We consider a stochastic volatility model in the following form,

$$dS_t = \lambda\Big(bS_t + (1-b)\,L\Big)\sqrt{z_t}dW_t, \tag{4}$$

$$dz_t = \theta(z_0 - z_t)dt + \eta\sqrt{z_t}dZ_t, \tag{5}$$

with $\langle dW_t, dZ_t\rangle = \rho dt$. If $b = 1$, this model will reduce to the Heston model. All the parameters have similar meaning as their counterparts of the Heston model. Since we have already explicitly included the volatility λ of the underlying process, we can set $z_0 = 1$. The volatlity of variance η is responsible for the convexity of the implied volatility smile.

The volatility follows the Cox-Ingersoll-Ross (CIR) process. Its properties have been discussed in [1], and will not be repeated here.

Define

$$X_t = \frac{bS_t + (1-b)\,L}{bS_0 + (1-b)\,L}, \tag{6}$$

we have

$$\frac{dX_t}{X_t} = \lambda b \sqrt{z_t} dW_t, \tag{7}$$

which has the following solution,

$$X_t = \exp\left(-\frac{1}{2}\lambda^2 b^2 \int_0^t z_u du + \lambda b \int_0^t \sqrt{z_u} dW_u\right). \tag{8}$$

References

[1] *Notes on Feller Condition*, personal notes.