

Notes on Feller Condition

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1 Fokker-Planck Equation

Consider the following parabolic partial differential equation (PDE) [1]

$$\frac{\partial u(t, x)}{\partial t} = -\frac{\partial}{\partial x} ((bx + c)u(t, x)) + \frac{\partial^2}{\partial x^2} (axu(t, x)), \quad (1)$$

with $x > 0$. This can be viewed as the corresponding Fokker-Planck equation for the Cox-Ingersoll-Ross (CIR) process

$$dx_t = \kappa(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad (2)$$

with $a = \sigma^2/2$, $b = -\kappa$, and $c = \kappa\theta$.

We want to find the solution of the PDE (1), with certain initial condition. In particular, we are interested in the fundamental solution of the PDE (1), *i.e.*, the initial condition is given by

$$u(0, x) = \delta(x - y), \quad (3)$$

where $\delta(x)$ is the Dirac delta function. To this end, introduce the Laplace transform of $u(t, x)$ as

$$v(t, \lambda) = \int_0^{+\infty} e^{-\lambda x} u(t, x) dx, \quad (4)$$

for $\lambda > 0$. We denote the Laplace transform of the initial condition as

$$\pi(\lambda) = \int_0^{+\infty} e^{-\lambda x} u(0, x) dx, \quad (5)$$

and for the fundamental solution, $\pi(\lambda) = e^{-\lambda y}$. Consider the Laplace transform of the right hand side of Eq. (1),

$$\begin{aligned} & \int_0^{+\infty} e^{-\lambda x} \left[(axu(t, x))_{xx} - ((bx + c)u(t, x))_x \right] dx \\ = & e^{-\lambda x} \left[(axu(t, x))_x - ((bx + c)u(t, x)) \right] \Big|_0^{+\infty} \\ & + \lambda \int_0^{+\infty} e^{-\lambda x} \left[(axu(t, x))_x - ((bx + c)u(t, x)) \right] dx \\ = & f(t) + \lambda(b - \lambda a)v_\lambda - c\lambda v, \end{aligned} \quad (6)$$

where

$$f(t) = \lim_{x \rightarrow 0} [(bx + c)u - (axu)_x], \quad (7)$$

is the flux at $x = 0$ and cannot be arbitrarily specified.

After the Laplace transform, Eq. (1) becomes

$$v_t + \lambda(\lambda a - b)v_\lambda = f(t) - c\lambda v. \quad (8)$$

This first order PDE can be solved by the method of characteristics.

References

- [1] W. Feller, *Two Singular Diffusion Problems*, Annals of Mathematics **54**, 173 (1951).