Notes on SABR Model

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The SABR model is characterized by the following stochastic process,

$$\begin{aligned} dS_t &= \sigma_t S_t^\beta dW_t, \\ d\sigma_t &= \gamma \sigma_t dZ_t, \\ \mathrm{E}[dW_t dZ_t] &= \rho dt, \end{aligned} \tag{1}$$

where the volatility of the forward process is now stochastic, γ is the volatility of volatility (volvol), and ρ is the correlation. The initial volatility of σ_t is assumed to be σ_0. We also assume that the absorbing boundary condition is imposed for the forward process to preserve the martingale property. In the following, we will present several approximate results.

1 Projection to SABR model with zero correlation

When the correlation between the underlying and the volatility vanishes, the SABR model admits a (semi) closed form solution for the option prices. We can start from this and build an approximation for general correlations.

1.1 Zero correlation

As in the last section, define

$$q_S = \frac{S^{1-\beta}}{1-\beta}, \quad q_K = \frac{K^{1-\beta}}{1-\beta}, \quad \bar{q} = q_S q_K, \quad b = \frac{q_S^2 + q_K^2}{2 q_S q_K}, \tag{2}$$

the call option price is given by

$$\begin{aligned} C(T,S,K) &= (S-K)^+ \\ &+ \frac{\sqrt{SK}}{\pi}\left(\int_0^\pi \frac{\sin\phi \sin(\nu\phi)}{b-\cos\phi} \frac{G(\gamma^2 T, s(\phi))}{\cosh(s(\phi))} d\phi \right. \\ &\qquad \left. + \sin(\nu\pi) \int_0^{+\infty} \frac{e^{-\nu\psi}\sinh\psi}{b+\cosh\psi} \frac{G(\gamma^2 T, s(\psi))}{\cosh(s(\psi))} d\psi \right), \end{aligned} \tag{3}$$

where

$$\begin{aligned}\sinh\left(s(\phi)\right) &= \frac{\gamma}{\sigma_0}\sqrt{2\bar{q}(b-\cos\phi)},\\ \sinh\left(s(\psi)\right) &= \frac{\gamma}{\sigma_0}\sqrt{2\bar{q}(b+\cosh\psi)}.\end{aligned} \tag{4}$$

Here,

$$G(t,s) = \frac{2e^{-t/8}}{t\sqrt{\pi t}}\int_s^{+\infty} ue^{-u^2/2t}\sqrt{\cosh u - \cosh s}du, \tag{5}$$

is an integration kernel, which has a highly accurate series expansion approximation,

$$G(t,s) = \sqrt{\frac{\sinh s}{s}}e^{-\frac{s^2}{2t}-\frac{t}{8}}\left(R(t,s)+\delta R(t,s)\right), \tag{6}$$

where

$$R(t,s) = 1+\frac{3tg(s)}{8s^2} - \frac{5t^2(-8s^2+3g^2(s)+24g(s))}{128s^4} + \frac{35t^3(-40s^2+3g^3(s)+24g^2(s)+120g(s))}{1024s^6}, \tag{7}$$

with $g(s) = s\coth(s)-1$, and

$$\delta R(t,s) = e^{\frac{t}{8}} - \frac{3072+384t+24t^2+t^3}{3072}. \tag{8}$$

In the limit of $s\to 0$, $G(t,s)$ has the following MacLaurin expansion,

$$G(t,s) = e^{-\frac{s^2}{2t}}\left(1+\left(\frac{1}{12}-\frac{2688+80t^2+21t^3}{322560}e^{-\frac{t}{8}}\right)s^2\right). \tag{9}$$

1.2 Non-zero correlation

For non-zero correlation model, we need to map it to a zero correlation model. Denote the mapped parameters with a tilde, then the mapped β and γ can be chosen as

$$\tilde{\beta} = \beta, \quad \tilde{\gamma}^2 = \gamma^2 - \frac{3}{2}\Big(\gamma^2\rho^2 + \sigma_0\gamma\rho(1-\beta)S^{\beta-1}\Big). \tag{10}$$

Then, the mapped initial volatility can be expressed as an expansion in maturity

$$\tilde{\sigma}_0 = \tilde{\sigma}_0^{(0)} + T\tilde{\sigma}_0^{(1)}. \tag{11}$$

The zeroth order term is given by

$$\tilde{\sigma}_0^{(0)} = \frac{2\Phi\delta\tilde{q}\tilde{\gamma}}{\Phi^2-1}, \tag{12}$$

where

$$\Phi = \left(\frac{\sigma_{\min}+\rho\sigma_0+\gamma\delta q}{(1+\rho)\sigma_0}\right)^{\frac{\tilde{\gamma}}{\gamma}}, \tag{13}$$

and

$$\delta q = \frac{K^{1-\beta} - S^{1-\beta}}{1-\beta}, \quad \delta\tilde{q} = \frac{K^{1-\tilde{\beta}} - S^{1-\tilde{\beta}}}{1-\tilde{\beta}} = \delta q, \tag{14}$$

since $\tilde{\beta} = \beta$. Also,

$$\sigma_{\min}^2 = \gamma^2\delta q^2 + 2\rho\gamma\delta q\sigma_0 + \sigma_0^2. \tag{15}$$

The first order term is given by

$$\frac{\tilde{\sigma}_0^{(1)}}{\tilde{\sigma}_0^{(0)}} = \tilde{\gamma}^2\sqrt{1+\tilde{R}^2}\frac{\frac{1}{2}\log\left(\frac{\sigma_0\sigma_{\min}}{\tilde{\sigma}_0^{(0)}\tilde{\sigma}_{\min}}\right) - \mathcal{B}_{\min}}{\tilde{R}\log\left(\tilde{R}+\sqrt{\tilde{R}^2+1}\right)}, \tag{16}$$

where

$$\tilde{R} = \frac{\delta q\tilde{\gamma}}{\tilde{\sigma}_0^{(0)}}, \quad \tilde{\sigma}_{\min}^2 = \tilde{\gamma}^2\delta q^2 + \left(\tilde{\sigma}_0^{(0)}\right)^2, \tag{17}$$

and $\mathcal{B}_{\min}$ is given by

$$\mathcal{B}_{\min} = -\frac{1}{2}\frac{\beta}{1-\beta}\frac{\rho}{\sqrt{1-\rho^2}}\left(\pi - \varphi_0 - \arccos\rho - I\right). \tag{18}$$

Here,

$$\varphi_0 = \arccos\left(-\frac{\gamma\delta q + \rho\sigma_0}{\sigma_{\min}}\right), \quad L = \frac{\sigma_{\min}}{\gamma q_K\sqrt{1-\rho^2}}, \tag{19}$$

and

$$I = \frac{2}{\sqrt{1-L^2}}\left(\arctan\left(\frac{u_0+L}{\sqrt{1-L^2}}\right) - \arctan\left(\frac{L}{\sqrt{1-L^2}}\right)\right) \qquad L<1 \tag{20}$$

$$I = \frac{1}{\sqrt{L^2-1}}\log\left(\frac{1+u_0\left(L+\sqrt{L^2-1}\right)}{1+u_0\left(L-\sqrt{L^2-1}\right)}\right) \qquad L>1 \tag{21}$$

with the limit $I(L=1) = 2u_0/(1+u_0)$, where

$$u_0 = \frac{\delta q\gamma\rho + \sigma_0 - \sigma_{\min}}{\delta q\gamma\sqrt{1-\rho^2}}. \tag{22}$$

Finally, the call option price of the non-zero correlation model can be obtained by replace the parameters in Eq. (3) with the mapped parameters.

1.2.1 Limiting cases

In the following limiting cases, the mapped γ parameter will be simplified.

i) At-the-money strike. For $K = S$, we have

$$\tilde{\sigma}_0^{(0)} = \sigma_0, \tag{23}$$

and

$$\frac{\tilde{\sigma}_0^{(1)}}{\tilde{\sigma}_0^{(0)}} = \frac{1}{12}\left(1 - \frac{\tilde{\gamma}^2}{\gamma^2} - \frac{3}{2}\rho^2\right)\gamma^2 + \frac{1}{4}\beta\rho\gamma\sigma_0 S^{\beta-1}. \tag{24}$$

ii) Negative $\tilde{\gamma}^2$. If the mapping of $\tilde{\gamma}$ leads to a negative $\tilde{\gamma}^2$, we can set $\tilde{\gamma} = 0$, and the mapped $\tilde{\sigma}_0$ will take the following limiting form,

$$\tilde{\sigma}_0^{(0)} = \frac{\gamma\delta\tilde{q}}{\log\left(\frac{\sigma_{\min} + \rho\sigma_0 + \gamma\delta q}{(1+\rho)\sigma_0}\right)}, \tag{25}$$

and

$$\frac{\tilde{\sigma}_0^{(1)}}{\tilde{\sigma}_0^{(0)}} = \frac{\gamma^2}{\log^2\left(\frac{\sigma_{\min} + \rho\sigma_0 + \gamma\delta q}{(1+\rho)\sigma_0}\right)}\left(\frac{1}{2}\log\left(\frac{\sigma_0\sigma_{\min}}{\tilde{\sigma}_0^{(0)}\tilde{\sigma}_{\min}}\right) - \mathcal{B}_{\min}\right). \tag{26}$$

Since $\tilde{\gamma} = 0$, the mapped model will degenerate to the CEV model, and the call option price can be obtained by replacing the parameters in Eq. (**??**) with the mapped parameters.

2 Free boundary SABR

Similar to the free boundary CEV process, the forward process of the SABR model can also be modified to allow negative forward in the following way,

$$dS_t = \sigma\left|S_t\right|^{\beta} dW_t, \tag{27}$$

with $0 < \beta < 1/2$, which becomes the free boundary SABR model. When the correlation is zero, $\rho = 0$, the call option value has the similar form to the SABR model,

$$\begin{aligned} C(T,S,K) &= (S-K)^+ \\ &+ \frac{\sqrt{|SK|}}{\pi}\Bigg(\mathbf{1}_{K\geq 0}\int_0^{\pi}\frac{\sin\phi\sin(\nu\phi)}{b-\cos\phi}\frac{G(\gamma^2 T, s(\phi))}{\cosh\left(s(\phi)\right)}d\phi \\ &\qquad + \sin(\nu\pi)\int_0^{+\infty}\frac{d\psi}{b+\cosh\psi}\frac{G(\gamma^2 T, s(\psi))}{\cosh\left(s(\psi)\right)} \\ &\qquad\qquad \times\Big(\mathbf{1}_{K\geq 0}\cosh(\nu\psi) + \mathbf{1}_{K<0}\sinh(\nu\psi)\Big)\sinh\psi\Bigg). \end{aligned} \tag{28}$$

For non-zero correlation, the parameters can be mapped to those of a zero correlation model as in the SABR case. The only difference is that

$$\delta q = \frac{|k|^{1-\beta} - |S|^{1-\beta}}{1-\beta}, \quad \delta\tilde{q} = \frac{|k|^{1-\tilde{\beta}} - |S|^{1-\tilde{\beta}}}{1-\tilde{\beta}} = \delta q, \tag{29}$$

where

$$k = \max(K, 0.1S) \tag{30}$$

and

$$L = \frac{\sigma_{\min}(1-\beta)}{\gamma k^{1-\beta}\sqrt{1-\rho^2}}, \tag{31}$$

If the mapped $\tilde{\gamma}^2$ is negative, we can similarly set $\tilde{\gamma}$ to 0, and the free boundary SABR model will degenerate to the corresponding free boundary CEV model.

References

[1] See `http://www.lesniewski.us/papers/working/NotesOnCEV.pdf`.

[2] See `http://mathworld.wolfram.com/BesselDifferentialEquation.html`.

[3] See `http://dlmf.nist.gov/10.22.E67`.

[4] See `http://dlmf.nist.gov/10.32.E4`.