Notes on HJM Framework

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1 Formulation

Let $P(t,T)$ denotes the time t price of a zero coupon bond which pays 1 at time T, *i.e.*,

$$P(t,T) = E_t^Q\left[\exp\left(-\int_t^T r(u)du\right)\right], \tag{1}$$

where Q stands for the risk neutral measure, and $r(t)$ is the short rate. Then, under the same measure,

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_P'(t,T)dW(t), \tag{2}$$

where W is a d-dimensional Brownian motion, $\sigma_P(t,T)$ is a d-dimensional stochastic process adapted to the filtration generated by W, and the prime represents matrix transposition. Here, $\sigma_P(t,T)$ must satisfy the consistency condition, $\sigma_P(T,T) = 0$, since $P(T,T) = 1$.

Define the forward discount bond price as

$$P(t,T,T+\tau) = \frac{P(t,T+\tau)}{P(t,T)}, \tag{3}$$

and apply the Ito's lemma, then

$$\frac{dP(t,T,T+\tau)}{P(t,T,T+\tau)} = -\left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]\sigma_P(t,T)dt - \left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]dW(t), \tag{4}$$

under the risk neutral measure. Under the T-forward measure, the forward discount bond price is a martingale, which means

$$\frac{dP(t,T,T+\tau)}{P(t,T,T+\tau)} = -\left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]dW^T(t). \tag{5}$$

Define the forward rate as

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}, \tag{6}$$

then from Eq. (2), the forward rate has the following dynamics,

$$df(t,T) = \sigma_f'(t,T)\sigma_P(t,T)dt + \sigma_f'(t,T)dW(t), \tag{7}$$

where $\sigma_f(t,T) = \partial\sigma_P(t,T)/\partial T$, or equivalently, $\sigma_P(t,T) = \int_t^T \sigma_f(t,u)du$.

2 Markovian Short Rate Dynamics

From Eq. (7), the forward rate is given by

$$f(t,T) = f(0,T) + \int_0^t \sigma_f'(u,T)\left(\int_u^T \sigma_f(u,s)ds\right)du + \int_0^t \sigma_f'(u,T)dW(u). \tag{8}$$

Define the short rate as $r(t) = f(t,t)$, then the short rate dynamics becomes

$$r(t) = f(0,t) + \int_0^t \sigma_f'(u,t)\left(\int_u^t \sigma_f(u,s)ds\right)du + \int_0^t \sigma_f'(u,t)dW(u). \tag{9}$$

Generally, the short rate process is not Markovian. However, after imposing the separation condition $\sigma_f(t,T) = g(t)h(T)$, where $g(t) \equiv g(t;\omega)$ is a $d \times d$ matrix valued function, which can be deterministic or stochastic, and h is a deterministic d-dimensional column vector function, the short rate will be Markovian. Under this condition,

$$r(t) = f(0,t) + h'(t)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du + h'(t)\int_0^t g'(u)dW(u). \tag{10}$$

To cast the short rate dynamics into a convenient form, define

$$H(t) = \mathrm{diag}(h_1(t), ..., h_d(t)) = \begin{pmatrix} h_1(t) & & 0 \\ & \ddots & \\ 0 & & h_d(t) \end{pmatrix}, \tag{11}$$

and

$$\kappa(t) = -\frac{dH(t)}{dt}H^{-1}(t). \tag{12}$$

Here, we have assumed that none of the h_i is zero. Later it will be shown that the diagonal elements of $\kappa(t)$ are the mean reversion parameter for the short rate dynamics. Now, define

$$x(t) = H(t)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du + H(t)\int_0^t g'(u)dW(u), \tag{13}$$

and

$$y(t) = H(t)\left(\int_0^t g'(u)g(u)du\right)H(t), \tag{14}$$

then it can be shown

$$dy(t) = \Big(\sigma_r'(t;\omega)\sigma_r(t;\omega) - \kappa(t)y(t) - y(t)\kappa(t)\Big)dt, \tag{15}$$

with $\sigma_r(t;\omega) = g(t;\omega)H(t)$, which is locally deterministic and can be solved numerically and analytically, if possible. Also,

$$dx(t) = \left[\frac{dH(t)}{dt}\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du\right]dt + \left[H(t)\left(\int_0^t g'(u)g(u)du\right)h(t)dt\right]$$

$$
\begin{aligned}
&\quad +\frac{dH(t)}{dt}\left(\int_0^t g'(u)dW(u)\right)dt + H(t)g'(t)dW(t) \\
&= \Big(y(t)I_{d\times 1} - \kappa(t)x(t)\Big)dt + H(t)g'(t)dW(t) \\
&= \Big(y(t)I_{d\times 1} - \kappa(t)x(t)\Big)dt + \sigma_r'(t;\omega)dW(t),
\end{aligned} \tag{16}
$$

where $I_{d\times 1} = (1, 1, ..., 1)'$. The x and y processes have initial conditions $x(0) = 0$ and $y(0) = 0$, which is evident from their definitions. It can be shown, by Ritchken and Sankarasubramanian, that any interest rate derivative will be completely determined by the two-state Markovian process $(x(t), y(t))$.

Using the above parameterizations, it can be shown, for forward rate,

$$
\begin{aligned}
f(t,T) &= f(0,T) + I_{d\times 1}'H(T)\int_0^t g'(u)g(u)\left(\int_u^T h(s)ds\right)du + I_{d\times 1}'H(T)\int_0^t g'(u)dW(u) \\
&= f(0,T) + I_{d\times 1}'H(T)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du + I_{d\times 1}'H(T)\int_0^t g'(u)dW(u) \\
&\quad + I_{d\times 1}'H(T)\int_0^t g'(u)g(u)\left(\int_t^T h(s)ds\right)du \\
&= f(0,T) + I_{d\times 1}'H(T)H^{-1}(t)\Big(H(t)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du \\
&\qquad\qquad + H(t)\int_0^t g'(u)dW(u)\Big) \\
&\quad + I_{d\times 1}'H(T)H^{-1}(t)H(t)\int_0^t g'(u)g(u)\left(\int_t^T h(s)ds\right)du \\
&= f(0,T) + M'(t,T)\left(x(t) + y(t)\int_t^T M(t,u)du\right),
\end{aligned} \tag{17}
$$

where

$$
M(t,T) = H(T)H^{-1}(t)I_{d\times 1}. \tag{18}
$$

From this, the short rate is given by

$$
r(t) = f(t,t) = f(0,t) + I_{d\times 1}'x(t) = f(0,t) + \sum_{i=1}^{d} x_i(t). \tag{19}
$$

From the above result, we can also derive the bond reconstitution formula. Notice $P(t,T) = \exp\Big(-\int_t^T f(t,u)du\Big)$, we have

$$
\begin{aligned}
P(t,T) &= \exp\Bigg(-\int_t^T f(0,u)du - \left(\int_t^T M'(t,u)du\right)x(t) \\
&\qquad - \int_t^T M'(t,u)y(t)\left(\int_t^u M(t,s)ds\right)du\Bigg).
\end{aligned} \tag{20}
$$

Define

$$G(t,T) = \int_t^T M(t,u)du, \tag{21}$$

and notice that

$$\int_t^T M'(t,u)y(t)\left(\int_t^u M(t,s)ds\right)du = \int_t^T M'(t,u)y(t)\left(\int_u^T M(t,s)ds\right)du, \tag{22}$$

the bond reconstitution formula can be rewritten as

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left(-G'(t,T)x(t) - \frac{1}{2}G'(t,T)y(t)G(t,T)\right). \tag{23}$$

The density relating the T-forward measure and the risk neutral measure is given by

$$\zeta(t) = E_t^Q\left[\frac{dP^T}{dP}\right] = \frac{P(t,T)/P(0,T)}{B(t)/B(0)}. \tag{24}$$

This density is a martingale in the risk neutral measure, and therefore,

$$\frac{d\zeta(t)}{\zeta(t)} = -G'(t,T)\sigma_r'(t)dW(t). \tag{25}$$

The Brownian motion under the T-forward is then given by

$$dW^T(t) = dW(t) + \sigma_r(t)G(t,T)dt. \tag{26}$$

3 Closed Form Results for Gaussian Models

In the above formulation, when $\sigma_f(t,T)$ is a deterministic function, the resulting models will be Gaussian, where the distribution of the forward rate, and the short rate, will be Gaussian. This leads to simplifications to the modeling and pricing of interest rate derivatives, and even closed form expressions for several simple derivatives.

3.1 Zero Coupon Bond Option

Consider a call option with expiry T and strike K on a zero coupon bond with maturity T^*. At expiry, the option pays

$$V(T) = \left(P(T,T^*) - K\right)^+. \tag{27}$$

Using the change of measure trick, the value of the zero coupon bond becomes

$$V(t) = E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)V(T)\right] = P(t,T)E_t^T\left[\left(P(T,T,T^*) - K\right)^+\right], \tag{28}$$

under the T-forward measure. Then, apply the Black-Scholes formula and take into account of Eq. (5),

$$\begin{aligned} V(t) &= P(t,T)\left[P(t,T,T^*)N(d_+) - KN(d_-)\right] \\ &= P(t,T^*)N(d_+) - KP(t,T)N(d_-), \end{aligned} \tag{29}$$

where

$$d_{\pm} = \frac{\log\left(\dfrac{P(t,T,T^*)}{K}\right) \pm \dfrac{v}{2}}{\sqrt{v}} = \frac{\log\left(\dfrac{P(t,T^*)}{KP(t,T)}\right) \pm \dfrac{v}{2}}{\sqrt{v}}, \tag{30}$$

and

$$v = \int_t^T |\sigma_P(s,T^*) - \sigma_P(s,T)|^2 \, ds. \tag{31}$$

3.2 Caplet

Now consider a caplet which resets at time T, and pays at $T+\tau$ the amount

$$V(T+\tau) = \tau \left(L(T,T+\tau) - K\right)^+, \tag{32}$$

where $L(T,T+\tau)$ is the LIBOR rates spanning from T to $T+\tau$, and τ is the year fraction. The value of the caplet at time t is

$$V(t) = E_t^Q \left[\exp\left(-\int_t^{T+\tau} r(s)ds\right) V(T+\tau)\right]. \tag{33}$$

Use the tower rule,

$$\begin{aligned} V(t) &= E_t^Q \left[\exp\left(-\int_t^T r(s)ds\right) E_T^Q \left[\exp\left(-\int_T^{T+\tau} r(s)ds\right) V(T+\tau)\right]\right] \\ &= E_t^Q \left[\exp\left(-\int_t^T r(s)ds\right) P(T,T+\tau)V(T+\tau)\right]. \end{aligned} \tag{34}$$

Notice that $L(T,T+\tau) = \tau^{-1}(1/P(T,T+\tau) - 1)$, the above expression becomes

$$\begin{aligned} V(t) &= E_t^Q \left[\exp\left(-\int_t^T r(s)ds\right) \left(1 - (1+\tau K)\, P(T,T+\tau)\right)^+\right] \\ &= (1+\tau K) E_t^Q \left[\exp\left(-\int_t^T r(s)ds\right) \left(\frac{1}{1+\tau K} - P(T,T+\tau)\right)^+\right]. \end{aligned} \tag{35}$$

Now, the caplet price has been turned into a put option on a zero coupon bond. Using the earlier result, the caplet price is

$$\begin{aligned} V(t) &= (1+\tau K)P(t,T)\left(\frac{1}{1+\tau K} N(-d_-) - P(T,T+\tau)N(-d_+)\right) \\ &= P(t,T)N(-d_-) - (1+\tau K)P(t,T+\tau)N(-d_+), \end{aligned} \tag{36}$$

where

$$d_{\pm} = \frac{\log\left(\dfrac{(1+\tau K)P(t,T+\tau)}{P(t,T)}\right) \pm \dfrac{v}{2}}{\sqrt{v}}, \tag{37}$$

and

$$v = \int_t^T |\sigma_P(s,T+\tau) - \sigma_P(s,T)|^2 \, ds. \tag{38}$$

3.3 Futures

The futures rate is a martingale under the risk neutral measure, *i.e.*, $F(t,T,T+\tau) = E_t^Q[L(T,T,T+\tau)]$. This is equivalent to

$$F(t,T,T+\tau) = \frac{1}{\tau} E_t^Q \left[\frac{1}{P(T,T+\tau)} - 1 \right]. \tag{39}$$

The dynamics of $G(t) = 1/P(t,T,T+\tau) = P(t,T)/P(t,T+\tau)$ under the risk neutral measure is

$$\begin{aligned}
d\left(\frac{1}{P(t,T,T+\tau)}\right) &= -\frac{dP(t,T,T+\tau)}{P^2(t,T,T+\tau)} + \frac{(dP(t,T,T+\tau))^2}{P^3(t,T,T+\tau)} \\
&= \frac{1}{P(t,T,T+\tau)}\Big(\left[\sigma'_P(t,T+\tau) - \sigma'_P(t,T)\right]\sigma_P(t,T)dt \\
&\qquad + \left[\sigma'_P(t,T+\tau) - \sigma'_P(t,T)\right] dW(t)\Big) \\
&\quad + \frac{1}{P(t,T,T+\tau)}\left[\sigma'_P(t,T+\tau) - \sigma'_P(t,T)\right]\left[\sigma_P(t,T+\tau) - \sigma_P(t,T)\right] dt \\
&= \frac{1}{P(t,T,T+\tau)}\Big(\left[\sigma'_P(t,T+\tau) - \sigma'_P(t,T)\right]\sigma_P(t,T+\tau)dt \\
&\qquad + \left[\sigma'_P(t,T+\tau) - \sigma'_P(t,T)\right] dW(t)\Big).
\end{aligned} \tag{40}$$

Therefore,

$$E_t^Q\left[\frac{1}{P(T,T+\tau)}\right] = \frac{1}{P(t,T,T+\tau)} e^{\Omega(t,T)}, \tag{41}$$

where

$$\Omega(t,T) = \int_t^T \left[\sigma'_P(u,T+\tau) - \sigma'_P(u,T)\right]\sigma_P(u,T+\tau)du. \tag{42}$$

So, the futures rate is

$$F(t,T,T+\tau) = \frac{1}{\tau}\left(\frac{1}{P(t,T,T+\tau)} e^{\Omega(t,T)} - 1\right). \tag{43}$$

3.4 European Swaption - Jamshidian Decomposition

The above results do not make any specific assumptions besides of the Gaussianness, and can be applied to general models with deterministic forward rate volatility functions. The trick, Jamshidian decomposition, which will be introduced in the following relies on the monotonicity of the zero coupon bond price on short rate.

Consider a payer swaption expiring at T_0, with the underlying swap paying annualized coupon c at times $T_1 < T_2 < \cdots < T_N$, with $T_1 > T_0$. The swaption payout at T_0 is

$$V(T_0) = \left(1 - P(T_0,T_N) - c\sum_{i=1}^N \tau_i P(T_0,T_i)\right)^+, \tag{44}$$

where $\tau_i = T_i - T_{i-1}$ is the accrual period. Notice the dependence of the zero coupon bond price on x, *i.e.*, $P(T_0, T_N) \equiv P(T_0, T_N; x(T_0))$, through the bond reconstitution formula, Eq. (23). Define a critical value x^* by setting the value of the swaption at time T_0 to exactly zero,

$$P(T_0, T_N; x^*) + c \sum_{i=1}^{N} \tau_i P(T_0, T_i; x^*) = 1. \tag{45}$$

Also define the "strikes" as

$$K_i = P(T_0, T_i; x^*), \tag{46}$$

then

$$K_N + c \sum_{i=1}^{N} \tau_i K_i = 1. \tag{47}$$

Notice that $P(T_0, T_i; x(T_0))$ is monotonically decreasing in $x(T_0)$, which always holds due to the bond reconstitution formula (23), we have

$$\begin{aligned}
V(T_0) &= \left(1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^{N} \tau_i P(T_0, T_i; x(T_0))\right)^+ \\
&= \left(1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^{N} \tau_i P(T_0, T_i; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} \\
&= \left(K_N + c \sum_{i=1}^{N} \tau_i K_i - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^{N} \tau_i P(T_0, T_i; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} \\
&= \left(K_N - P(T_0, T_N; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} + c \sum_{i=1}^{N} \tau_i \left(K_i - P(T_0, T_i; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} \\
&= \left(K_N - P(T_0, T_N; x(T_0))\right)^+ + c \sum_{i=1}^{N} \tau_i \left(K_i - P(T_0, T_i; x(T_0))\right)^+ .
\end{aligned} \tag{48}$$

Then, this can be evaluated with the zero coupon bond option price formula, Eq. (29). This trick can also be applied to coupon bond options.

4 Examples of Classical Gaussian Short Rate Models - One Factor Models

By setting $d = 1$ in the above formulation, we will obtain one-factor models. Since the matrix products will be reduced to scalar products, this will greatly simplify the discussion.

4.1 Ho-Lee Model

The Ho-Lee model can be recovered by taking $h(t) = 1$, and $g(t) = \sigma$ which is constant. It is straightforward to show that $\kappa(t) = 0$, $y(t) = \sigma^2 t$,

$$dx(t) = \sigma^2 t dt + \sigma dW(t), \tag{49}$$

and $r(t) = f(0,t) + x(t)$. Also, the forward rate volatility is $\sigma_f(t,T) = \sigma$, which is constant too, $\sigma_P(t,T) = \sigma(T-t)$, and the bond reconstitution formula is

$$
\begin{aligned}
P(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left(-x(t)(T-t) - \frac{1}{2}\sigma^2(T-t)^2 t\right) \\
&= \frac{P(0,T)}{P(0,t)} \exp\left(-\left(r(t) - f(0,t)\right)(T-t) - \frac{1}{2}\sigma^2(T-t)^2 t\right).
\end{aligned} \tag{50}
$$

4.2 Hull-White Model

Take $h(t) = \exp\left(-\int_0^t \kappa(u)du\right)$, and $g(t) = \sigma(t)\exp\left(\int_0^t \kappa(u)du\right)$, then we have

$$
\frac{dy(t)}{dt} = \sigma^2(t) - 2\kappa(t)y(t), \tag{51}
$$

which has the following analytical solution,

$$
y(t) = \int_0^t \sigma^2(u)\exp\left(-2\int_u^t \kappa(s)ds\right)du, \tag{52}
$$

Then,

$$
dx(t) = \left(y(t) - \kappa(t)x(t)\right)dt + \sigma(t)dW(t), \tag{53}
$$

which can be shown to have the following solution,

$$
x(t) = \int_0^t y(u)\exp\left(-\int_u^t \kappa(s)ds\right)du + \int_0^t \sigma(u)\exp\left(-\int_u^t \kappa(s)ds\right)dW(u). \tag{54}
$$

4.3 Mercurio-Moraleda Model

Similar to the Vasicek model, set

$$
\kappa(t) = \lambda - \frac{\gamma}{1+\gamma t}, \tag{55}
$$

5 Examples of Classical Gaussian Short Rate Models - Multi Factor Models

As an example, let us consider the two factor Hull-White model in terms of the HJM framework. Define

$$
g(t) = \begin{pmatrix} \sigma_{11}(t)\exp\left\{\int_0^t \kappa_1(u)du\right\} & 0 \\ \sigma_{21}(t)\exp\left\{\int_0^t \kappa_1(u)du\right\} & \sigma_{22}(t)\exp\left\{\int_0^t \kappa_2(u)du\right\} \end{pmatrix}, \tag{56}
$$

and

$$
h(t) = \begin{pmatrix} \exp\left\{-\int_0^t \kappa_1(u)du\right\} \\ \exp\left\{-\int_0^t \kappa_2(u)du\right\} \end{pmatrix}. \tag{57}
$$

Then,

$$\sigma_r(t) = g(t)H(t) = \begin{pmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix}, \tag{58}$$

and

$$dx(t) = \Big(y(t)I_{2\times 1} - \kappa(t)x(t)\Big)dt + \sigma_r'(t)dW(t), \tag{59}$$

where $y(t)$ can be found by numerical integration, and $\kappa(t) = \mathrm{diag}(\kappa_1(t), \kappa_2(t))$. The short rate can be written as

$$r(t) = f(0,t) + x_1(t) + x_2(t), \tag{60}$$

Define $\sigma_1(t) = \sqrt{\sigma_{11}^2(t) + \sigma_{21}^2(t)}$, $\sigma_2(t) = \sigma_{22}(t)$, and

$$\rho_x(t) = \frac{\sigma_{21}(t)\sigma_{22}(t)}{\sigma_1(t)\sigma_2(t)}, \tag{61}$$

then the correlation in forward rates can be expressed in these three parameters, together with $\kappa_1(t)$ and $\kappa_2(t)$. For the forward rate, we have

$$\sigma_f(t,T) = g(t)h(T) = \begin{pmatrix} \sigma_{11}(t)\exp\left\{-\int_t^T \kappa_1(u)du\right\} \\ \sigma_{21}(t)\exp\left\{-\int_t^T \kappa_1(u)du\right\} + \sigma_{22}(t)\exp\left\{-\int_t^T \kappa_2(u)du\right\} \end{pmatrix}, \tag{62}$$

and

$$df(t,T) = O(dt) + \sigma_f'(t,T)dW(t). \tag{63}$$

The covariance of the forward rates maturing at T_1 and T_2 is

$$\begin{aligned} &\frac{\langle df(t,T_1), df(t,T_2)\rangle}{dt} \\ =\;& \left(\sigma_{11}^2(t) + \sigma_{21}^2(t)\right)\exp\left\{-\int_t^{T_1}\kappa_1(u)du - \int_t^{T_2}\kappa_1(u)du\right\} \\ &+\sigma_{22}^2(t)\exp\left\{-\int_t^{T_1}\kappa_2(u)du - \int_t^{T_2}\kappa_2(u)du\right\} \\ &+\sigma_{21}(t)\sigma_{22}(t)\Bigg(\exp\left\{-\int_t^{T_1}\kappa_1(u)du - \int_t^{T_2}\kappa_2(u)du\right\} \\ &\qquad + \exp\left\{-\int_t^{T_1}\kappa_2(u)du - \int_t^{T_2}\kappa_1(u)du\right\}\Bigg) \\ =\;& \sigma_1^2(t)\exp\left\{-\int_t^{T_1}\kappa_1(u)du - \int_t^{T_2}\kappa_1(u)du\right\} b(t,T_1,T_2), \end{aligned} \tag{64}$$

where

$$\begin{aligned} b(t,T_1,T_2) =\;& 1 + \rho_x(t)\frac{\sigma_2(t)}{\sigma_1(t)}\Bigg(\exp\left\{-\int_t^{T_1}(\kappa_2(u) - \kappa_1(u))\,du\right\} \\ &\qquad + \exp\left\{-\int_t^{T_2}(\kappa_2(u) - \kappa_1(u))\,du\right\}\Bigg) \\ &+ \left(\frac{\sigma_2(t)}{\sigma_1(t)}\right)^2 \exp\left\{-\int_t^{T_1}(\kappa_2(u) - \kappa_1(u))\,du - \int_t^{T_2}(\kappa_2(u) - \kappa_1(u))\,du\right\}. \end{aligned} \tag{65}$$

Therefore, the correlation between the forward rates is given by

$$\rho(t,T_1,T_2) = \frac{b(t,T_1,T_2)}{\sqrt{b(t,T_1,T_1)b(t,T_2,T_2)}}. \tag{66}$$

It is clear from the above expression that the correlation of the forward rates can be controlled by the five functions: $\rho_x(t)$, $\sigma_1(t)$, $\sigma_2(t)$, $\kappa_1(t)$, and $\kappa_2(t)$. If we have either $\rho_x = 1$, $\kappa_2(t) - \kappa_1(t) = 0$, or $\sigma_2(t)/\sigma_1(t) = 0$, then $\rho(t,T_1,T_2) \equiv 1$, where the model reduces to a one-factor one. Also,

$$\frac{\langle df(t,T), df(t,T)\rangle}{dt} = \sigma_1^2(t)\exp\left\{-2\int_t^T \kappa_1(u)du\right\} b(t,T,T), \tag{67}$$

it can be shown that if ρ_x is negative enough, the term structure of the volatility will develop a humped shape.

6 Quasi-Gaussian Models

We are going to consider general multi-factor quasi-Gaussian models, with the separation condition $\sigma_f(t,T;\omega) = g(t;\omega)h(T)$, where $g(t;\omega)$ is a $d \times d$-matrix valued function, and $h(t)$ is a deterministic, d-dimensional column vector function. We are going to develop efficient approximation for the swap rate, which will also cover the pure Gaussian case.

Consider the following volatility specification which contains both local and stochastic volatilities,

$$\sigma_r(t;\omega) = \sqrt{z(t)}\sigma_x(t,x(t),y(t)), \tag{68}$$

where $z(t)$ follows the CIR process

$$dz(t) = \theta(z_0 - z(t))dt + \eta(t)\sqrt{z(t)}dZ(t), \qquad z(0) = z_0 = 1, \tag{69}$$

and $\langle dZ(t), dW(t)\rangle = 0$.

To proceed with the swap rate dynamics, fix a tenor structure

$$0 < T_0 < T_1 < \cdots < T_N \tag{70}$$

with

$$\tau_n = T_{n+1} - T_n, \tag{71}$$

and the swap rate $S(t)$, with first fix date at T_0 and last payment date at T_N, is given by

$$S(t) = \frac{P(t,T_0) - P(t,T_N)}{A(t)}, \qquad A(t) = \sum_{n=0}^{N-1} \tau_n P(t,T_{n+1}). \tag{72}$$

Notice the functional dependence of the discount bond $P(t,T)$ on the state variables x and y, *i.e.*, $P(t,T) \equiv P(t,T,x(t),y(t))$, the swap rate dynamics is

$$dS(t) = \sqrt{z(t)}(\nabla S)\sigma_x' dW(t) + O(dt), \tag{73}$$

where we are only concerned with the drift term. Since the swap rate is a martingale in the annuity measure, its dynamics in the corresponding annuity measure is given by

$$dS(t) = \sqrt{z(t)}\Big((\nabla S)\sigma_x'\sigma_x(\nabla S)'\Big)^{1/2} dW^A(t). \tag{74}$$

Using the Markovian Projection (MP) method, the above process is equivalent to the following local volatility process in terms of European option pricing,

$$dS(t) = \sqrt{z(t)}\varphi(t, S(t))dW^A(t), \tag{75}$$

where

$$\varphi^2(t,s) = E^A\Big[\, (\nabla S)c_x(\nabla S)' |\, S(t) = s\Big], \tag{76}$$

and $c_x = \sigma_x'\sigma_x$.

We will look for the linear approximation to $\varphi(t,s)$ in s, and will use $E^A(x(t))$ and $E^A(y(t))$ as expansion point. First, define

$$\sigma_x^0(t) = \sigma_x(t, 0, 0), \tag{77}$$

then from Eq. (14), we have

$$y(t) = H(t)\left(\int_0^t H^{-1}(u)\sigma_r'(u)\sigma_r(u)H^{-1}(u)du\right)H(t), \tag{78}$$

and

$$\bar{y}(t) = E^A\left[y(t)\right] \approx H(t)\left(\int_0^t H^{-1}(u)\left(\sigma_x^0(u)\right)'\sigma_x^0(u)H^{-1}(u)du\right)H(t). \tag{79}$$

For the approximation of $x(t)$, replace $y(t)$ with $\bar{y}(t)$ in Eq. (16), which is the Gaussian approximation to $x(t)$,

$$dx_g(t) = \Big(\bar{y}(t)I_{d\times 1} - \kappa(t)x_g(t)\Big)dt + \left(\sigma_x^0(t)\right)' dW(t). \tag{80}$$

The corresponding dynamics for $x_g(t)$ in the T-forward measure, using Eq. (26), is

$$dx_g(t) = \Big(\bar{y}(t)I_{d\times 1} - \left(\sigma_x^0(t)\right)'\sigma_x^0(t)G(t,T) - \kappa(t)x_g(t)\Big)dt + \left(\sigma_x^0(t)\right)' dW^T(t). \tag{81}$$

The solution to the above SDE is

$$x_g(t) = H(t)\left(\int_0^t H^{-1}(u)\left[\Big(\bar{y}(u)I_{d\times 1} - \left(\sigma_x^0(u)\right)'\sigma_x^0(u)G(u,T)\Big)du + \left(\sigma_x^0(t)\right)' dW^T(u)\right]\right), \tag{82}$$

and

$$\begin{aligned}
\bar{x}(t) &= E^A\left[x(t)\right] \approx E^A\left[x_g(t)\right] \\
&= E^Q\left[\frac{dP^A}{dP}x_g(t)\right] = E^Q\left[\frac{A(t)/A(0)}{B(t)}x_g(t)\right] \\
&= \frac{1}{A(0)}\sum_{n=0}^{N-1}\tau_n E^Q\left[\frac{P(t,T_{n+1})}{B(t)}x_g(t)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{A(0)}\sum_{n=0}^{N-1}\tau_n E^{T_{n+1}}\left[\frac{dP}{dP^{T_{n+1}}}\frac{P(t,T_{n+1})}{B(t)}x_g(t)\right] \\
&= \frac{1}{A(0)}\sum_{n=0}^{N-1}\tau_n E^{T_{n+1}}\left[\frac{B(t)}{P(t,T_{n+1})/P(0,T_{n+1})}\frac{P(t,T_{n+1})}{B(t)}x_g(t)\right] \\
&= \sum_{n=0}^{N-1}\frac{\tau_n P(0,T_{n+1})}{A(0)}E^{T_{n+1}}\left[x_g(t)\right] \\
&= \sum_{n=0}^{N-1}\frac{\tau_n P(0,T_{n+1})}{A(0)}H(t)\int_0^t H^{-1}(u)\left(\bar{y}(u)I_{d\times 1}-\left(\sigma_x^0(u)\right)'\sigma_x^0(u)G(u,T)\right)du. \qquad (83)
\end{aligned}
$$

Now, we can proceed to develop further approximation for the swap rate dynamics, around $\bar{x}(t)$ and $\bar{y}(t)$. Expand Eq. (76) to first order in x,

$$
\begin{aligned}
\left((\nabla S)c_x(\nabla S)'\right)(t,x(t),y(t)) &\approx \left((\nabla S)c_x(\nabla S)'\right)(t,\bar{x}(t),\bar{y}(t)) \\
&\quad + \nabla\left((\nabla S)c_x(\nabla S)'\right)\Big|_{(t,\bar{x}(t),\bar{y}(t))}(x(t)-\bar{x}(t)). \qquad (84)
\end{aligned}
$$

Then,

$$
\begin{aligned}
E^A\Big[\,(\nabla S)c_x(\nabla S)'|\,S(t)=s\Big] &\approx \left((\nabla S)c_x(\nabla S)'\right)(t,\bar{x}(t),\bar{y}(t)) \\
&\quad + \nabla\left((\nabla S)c_x(\nabla S)'\right)\Big|_{(t,\bar{x}(t),\bar{y}(t))}E^A\Big[x(t)-\bar{x}(t)|S(t)=s\Big]. \qquad (85)
\end{aligned}
$$

Using a Gaussian approximation for the conditional expectation,

$$
\begin{aligned}
E^A\Big[x(t)-\bar{x}(t)|S(t)=s\Big] &\approx E^A\Big[x_g(t)-\bar{x}_g(t)|S(t)=s\Big] \\
&= \frac{\mathrm{Cov}\left(S_g(t),x_g(t)\right)}{\mathrm{Var}\left(S_g(t)\right)}(s-S(0)) \\
&= \frac{c_x\,(\nabla S)'}{(\nabla S)c_x(\nabla S)'}(s-S(0)). \qquad (86)
\end{aligned}
$$

Notice that

$$
\nabla\Big((\nabla S)c_x(\nabla S)'\Big)c_x\,(\nabla S)' = (\nabla S)d_x(\nabla S)' + 2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)', \qquad (87)
$$

where

$$
d_x = \sum_{l=1}^{d}\left(c_x(\nabla S)'\right)_l\left(\frac{\partial\sigma_x'}{\partial x_l}\sigma_x+\sigma_x'\frac{\partial\sigma_x}{\partial x_l}\right). \qquad (88)
$$

The local volatility function is

$$
\varphi^2(t,s) = (\nabla S)c_x(\nabla S)' + \frac{(\nabla S)d_x(\nabla S)'+2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)'}{(\nabla S)c_x(\nabla S)'}(s-S(0)), \qquad (89)
$$

and linearizing around $s \approx S(0)$,

$$\varphi(t,s) = \Big((\nabla S)c_x(\nabla S)'\Big)^{1/2} + \frac{1}{2\Big((\nabla S)c_x(\nabla S)'\Big)^{1/2}} \times \frac{(\nabla S)d_x(\nabla S)' + 2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)'}{(\nabla S)c_x(\nabla S)'}(s - S(0)). \tag{90}$$

7 The Affine Model

$\nabla\nabla$

A Girsanov Theorem

Consider two measures P and $P(\theta)$ related by a density

$$\zeta^\theta(t) = \mathrm{E}_t^P\left[\frac{dP(\theta)}{dP}\right], \tag{91}$$

where $\zeta^\theta(t)$ is an exponential martingale given by

$$\frac{d\zeta^\theta(t)}{\zeta^\theta(t)} = -\theta'(t)dW(t), \tag{92}$$

where $W(t)$ is a d-dimensional P-Brownian motion. Then,

$$W^\theta(t) = W(t) + \int_0^t \theta(s)ds \tag{93}$$

is a $P(\theta)$-Brownian motion, and

$$dW^\theta(t) = dW(t) + \theta(t)dt. \tag{94}$$