Notes on Characteristic Functions

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1 Characteristic Function

A stochastic process X_t is called a *Lévy process* if it has the following properties:

1. $X_0 = 0$ almost surealy;
2. *Independence of increments.* For any $0 \leq t_1 < t_2 < \cdots < t_n < +\infty$, $X_{t_2} - X_{t_1}$, $X_{t_3} - t_2$, $\cdots$, $X_{t_n} - X_{t_{n-1}}$ are independent;
3. *Stationary increments.* For any $s < t$, $X_t - X_s$ is equal in distribution to X_{t-s};
4. *Continuity in probability.* For any $\epsilon > 0$ and $t \geq 0$, $\lim_{h\to 0} P(|X_{t+h} - X_t > \epsilon|) = 0$.

For the Lévy process, the characteristic function is given by

$$\varphi(u) = \mathrm{E}\left[e^{iuX_t}\right] = \exp\left\{t\left(iua - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}n\{0\}}\left(e^{iux} - 1 - iux\mathcal{I}_{|x|<1}\right)\Pi(dx)\right)\right\}. \tag{1}$$

In the following, we will give several examples of Lévy process and their corresponding characteristic functions.

1.1 Black-Scholes Model

For the Black-Scholes model,

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t, \tag{2}$$

we can define

$$X_t = \log\left(\frac{S_t}{S_0 e^{(r-q)t}}\right), \tag{3}$$

it can be shown that

$$dX_t = -\frac{1}{2}\sigma^2 dt + \sigma dW_t. \tag{4}$$

Therefore, $X_t \sim N(-\sigma^2 t/2, \sigma^2 t)$, where $N(\mu, \sigma^2)$ is a Gaussian distribution with mean and variance as μ and σ^2, respectively. The characteristic function can be found by direct integration

$$\varphi(u) = \frac{1}{2\pi\sigma^2 t}\int \exp\left(iux - \frac{\left(x + \frac{1}{2}\sigma^2 t\right)^2}{2\sigma^2 t}\right) = \exp\left(-\frac{\sigma^2 t}{2}u(u+i)\right). \tag{5}$$

Remark 1. *In many applications, the characteristic function is alternatively defined as* $\mathrm{E}\left[e^{iu\log S_t}\right]$, *which is related to (5) by a factor,* $\exp\left[iu(\log S_0 + (r-q)t)\right]$. *We will use these two definitions interchangeably, and which definition is used is clear from the context.*

1.2 Heston Model

The state variable dynamics of the Heston model is given by

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dW_t, && (6) \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dB_t, && (7) \end{aligned}$$

with $\langle dW_t, dB_t \rangle = \rho dt$. Consider the bivariate characteristic function,

$$\varphi(u_1, u_2) = \mathrm{E}\left[\exp\left(iu_1 x_T + iu_2 v_T\right) \middle| x_t, v_t\right], \tag{8}$$

where $u_t = \log S_t$. Define $\tau = T - t$, we are seeking a solution in the form of

$$\varphi(u_1, u_2) = \exp\left(A(\tau, u_1, u_2) + B(\tau, u_1, u_2)x_t + C(\tau, u_1, u_2)v_t\right). \tag{9}$$

According to the Feynman-Kac theorem, the bivariate characteristic function satisfies the following PDE,

$$-\frac{\partial \varphi}{\partial \tau} + \left(r - q - \frac{1}{2}v\right)\frac{\partial \varphi}{\partial x} + \kappa(\theta - v)\frac{\partial \varphi}{\partial x} + \frac{1}{2}v\frac{\partial^2 \varphi}{\partial x^2} + \rho\sigma v\frac{\partial^2 \varphi}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \varphi}{\partial v^2} = 0. \tag{10}$$

2 Fourier Inversion Formula

Consider the problem of finding the in-the-money (ITM) probability of the underlying asset at maturity, $P(\log S_T > \log K)$, which is given by

$$P(\log S_T > k) = \int_k^{+\infty} f(x)dx, \tag{11}$$

with $k = \log K$. Here, $f(x)$ is the probability density function of $X_T = \log S_T$, and can be recovered from the characteristic function (**??**) by the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\varphi(u)du. \tag{12}$$

Therefore,

$$\begin{aligned} P(\log S_T > k) &= \frac{1}{2\pi}\int_k^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-iux}\varphi(u)du\right)dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(u)\left(\int_k^{+\infty} e^{-iux}dx\right)du \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(u)\frac{e^{-iuk}}{iu}du - \frac{1}{2\pi}\lim_{R\to+\infty}\int_{-\infty}^{+\infty}\varphi(u)\frac{e^{-iuR}}{iu}du. && (13) \end{aligned}$$

To evaluate the second term, we again use the definition of the characteristic function,

$$\begin{aligned}
\frac{1}{2\pi}\lim_{R\to+\infty}\int_{-\infty}^{+\infty}\varphi(u)\frac{e^{-iuR}}{iu}du &= \frac{1}{2\pi}\lim_{R\to+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}e^{iux}f(x)dx\right)\frac{e^{-iuR}}{iu}du \\
&= \frac{1}{2\pi}\lim_{R\to+\infty}\int_{-\infty}^{+\infty}f(x)\left(\int_{-\infty}^{+\infty}\frac{e^{iu(x-R)}}{iu}du\right)dx \\
&= \frac{1}{2\pi}\lim_{R\to+\infty}\int_{-\infty}^{+\infty}f(x)\cdot\pi\mathrm{sgn}(x-R)dx \\
&= \frac{1}{2}\lim_{R\to+\infty}(1-2F(R)) = -\frac{1}{2}. \qquad (14)
\end{aligned}$$

Here, we have used the fact that

$$\int_{-\infty}^{+\infty}\mathrm{sgn}(x-y)f(x)dx = \int_{y}^{+\infty}f(x)dx - \int_{-\infty}^{y}f(x)dx = 1-2F(x), \qquad (15)$$

where $F(x)$ is the cumulative distribution function of X_T.

Finally, the ITM probability is given by

$$P(\log S_T > k) = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(u)\frac{e^{-iuk}}{iu}du. \qquad (16)$$

3 Vanilla option pricing with characteristic functions

In the following, we are going to explore several formulations of vanilla option prices in terms of characteristic functions.

3.1 Heston [1]

Consider the call option price

$$C(K) = e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[(S_T-K)^+\right], \qquad (17)$$

under the risk neutral measure. We are seeking a representation similar to the Black-Scholes formula. To this end, we can write the call option prices as

$$C(K) = e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[(S_T-K)\mathcal{I}_{S_T>K}\right], \qquad (18)$$

where $\mathcal{I}$ is the indicator function. Now, we have

$$C(K) = e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[S_T\mathcal{I}_{S_T>K}\right] - Ke^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}\left[\mathcal{I}_{S_T>K}\right] = S_tP_1 - Ke^{-r(T-t)}P_2, \qquad (19)$$

where

$$P_1 = \mathrm{E}^{\mathbb{Q}}\left[\frac{S_T/S_t}{B_T/B_t}\mathcal{I}_{S_T>K}\right], \qquad P_2 = \mathrm{E}^{\mathbb{Q}}\left[\mathcal{I}_{S_T>K}\right]. \qquad (20)$$

It is obvious that P_2 is the risk neutral probability for the underlying asset maturing in-the-money. Also, P_1 can be represented as the ITM probability in another measure. Notice that

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S} = \frac{B_T/B_t}{S_T/S_t} = \frac{\mathrm{E}^{\mathbb{Q}}[e^{X_T}]}{e^{X_T}}, \tag{21}$$

where we have used the fact that

$$\mathrm{E}^{\mathbb{Q}}[S_T] = \mathrm{E}^{\mathbb{Q}}[e^{X_T}] = S_t\frac{B_T}{B_t}, \tag{22}$$

then

$$P_1 = \mathrm{E}^{\mathbb{Q}^S}\left[\frac{S_T/S_t}{B_T/B_t}\mathcal{I}_{S_T>K}\frac{dQ}{dQ^S}\right] = \mathrm{E}^{\mathbb{Q}^S}\left[\mathcal{I}_{S_T>K}\right]. \tag{23}$$

From Eq. (16), the two probabilities P_1 and P_2 can be written as

$$P_j = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi_j(u)\frac{e^{-iuk}}{iu}du, \qquad j = 1, 2. \tag{24}$$

It seems that two characteristic functions corresponding to the two different measures are required in the valuation of the option price. However, these two characteristic functions are related due to the measure change. To see this, notice that

$$\varphi_1(u) = \int_{-\infty}^{+\infty} e^{iux} f^S(x)dx, \tag{25}$$

where $f^S(x)$ is the probability density function of X_T under the $\mathbb{Q}^S$-measure. It is related to $f(x)$, the probability density function of X_T under the $\mathbb{Q}$-measure, through the measure change,

$$f^S(x) = f(x)\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = f(x)\frac{e^{X_T}}{\mathrm{E}^{\mathbb{Q}}[e^{X_T}]}. \tag{26}$$

Now,

$$\varphi_1(u) = \frac{1}{\mathrm{E}^{\mathbb{Q}}[e^{X_T}]}\int_{-\infty}^{+\infty} e^{ix(u-i)}f(x)dx = \frac{\varphi(u-i)}{\varphi(-i)}, \tag{27}$$

since

$$\mathrm{E}^{\mathbb{Q}}[e^{X_T}] = \int_{-\infty}^{+\infty} e^{x}f(x)dx = \varphi(-i), \tag{28}$$

where $\varphi(u)$ is the characteristic function as defined in (**??**), and coincides with $\varphi_2(u)$. Therefore,

$$P_1 = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-iu\log K}\varphi(u-i)}{iu\varphi(-i)}du, \tag{29}$$

$$P_2 = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-iu\log K}\varphi(u)}{iu}du. \tag{30}$$

$$\tag{31}$$

3.2 Carr and Madan [2]

The call option price can be represented as

$$C(k) = e^{-r(T-t)} \int_k^{+\infty} (e^x - e^k) f(x) dx, \tag{32}$$

where $k = \log K$, $x = \log S_T$, and $f(x)$ is the probability density function for the distribution of x at maturity. We want to find the Fourier transform of the above call option price, but it is not integrable. To remedy this, we introduce a damping factor and modify the call option price accodingly,

$$c(k) = e^{\alpha k} C(k), \tag{33}$$

and the corresponding Fourier transform is given by

$$\hat{c}(v) = \int_{-\infty}^{+\infty} e^{ivk} c(k) dk. \tag{34}$$

Rearrange the integration order, we have

$$\begin{aligned}
\hat{c}(v) &= e^{-r(T-t)} \int_{-\infty}^{+\infty} e^{ivk} \left(\int_k^{+\infty} e^{\alpha k} (e^x - e^k) f(x) dx \right) dk \\
&= e^{-r(T-t)} \int_{-\infty}^{+\infty} f(x) \left(\int_{-\infty}^{x} \left(e^{(\alpha+iv)k} e^x - e^{(\alpha+iv+1)k} \right) dk \right) dx \\
&= e^{-r(T-t)} \int_{-\infty}^{+\infty} f(x) \left[\frac{e^{(\alpha+iv)k} e^x}{\alpha + iv} - \frac{e^{(\alpha+iv+1)k}}{\alpha + iv + 1} \right] \Bigg|_{-\infty}^{x} dx \\
&= e^{-r(T-t)} \int_{-\infty}^{+\infty} f(x) \left[\frac{e^{(\alpha+iv+1)x}}{\alpha + iv} - \frac{e^{(\alpha+iv+1)x}}{\alpha + iv + 1} \right] dx \\
&= e^{-r(T-t)} \int_{-\infty}^{+\infty} f(x) \frac{e^{(\alpha+iv+1)x}}{(\alpha + iv)(\alpha + iv + 1)} dx \\
&= \frac{e^{-r(T-t)} \varphi(v - (1+\alpha)i)}{(\alpha + iv)(\alpha + iv + 1)},
\end{aligned} \tag{35}$$

where $\alpha > 0$ is necessary to ensure convergence at $k = -\infty$. Once the Fourier transform of the modified call option price is known, we can use the inverse Fourier transform to recover the call option price,

$$C(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \hat{c}(v) dv. \tag{36}$$

3.3 Lewis [3]

Denote the payoff the contingent claim at maturity as $g(x)$, then the value of the derivative is given by

$$V = \int_{-\infty}^{+\infty} f(x) g(x) dx. \tag{37}$$

References

[1] S. Heston, *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, Review of Financial Studies **6**, 327 (1993).

[2] P. Carr and D. Madan, *Option valuation using the fast Fourier transform*, The Journal of Computational Finance **2**, 61 (1999).

[3] A. Lewis, *A Simple Option Formula for General Jump-Diffusion and Other Exponential Levy Processes*, `https://ssrn.com/abstract=282110`.