Notes on Quadratic Gaussian Model

Dec. 32, 2999

1 The Quadratic Gaussian Model

The quadratic Gaussian model can be obtained by having the short rate in a quadratic form,

$$r(t) = z(t)^\top \gamma(t) z(t) + h(t)^\top z(t) + a(t), \tag{1}$$

where $\gamma(t)$ is a $d \times d$ symmetric matrix, $h(t)$ is a d-dimensional vector, and $a(t)$ is used to fit the initial yield curve. Here, the d-dimensional state variable $z(t)$ satisfies

$$dz(t) = g(t)^\top dW(t), \qquad z(0) = 0, \tag{2}$$

where $W(t)$ is a d-dimensional Brownian motion.

1.1 Relation to the Affine Model

To see the connection to the affine model, consider the one dimensional constant parameter case. In this case, Eq. (1) reduces to

$$r(t) = \gamma z^2(t) + h z(t) + a, \tag{3}$$

which can be solved,

$$z(t) = \frac{\sqrt{h^2 + 4\gamma\big(r(t) - a\big)} - h}{2}, \tag{4}$$

by keeping the positive root. Then, it can be shown

$$dr(t) = \frac{1}{\gamma}\sqrt{h^2 + 4\gamma\big(r(t) - a\big)}\, g(t) dW(t), \tag{5}$$

which is clearly affine.

1.2 Analytical Results

Due to the affine nature of the quadratic Gaussian model, the discount bond price is similarly given in a quadratic form,

$$P(t,T) = \exp\Big(- z(t)^\top \gamma(t,T) z(t) - h(t,T)^\top z(t) - a(t,T) \Big). \tag{6}$$

Applying the Feynman-Kac theorem, the discount bond price satisfies the following PDE,

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sum_{ijk} \frac{\partial^2 P}{\partial z_i \partial z_j} g_{ki}(t) g_{kj}(t) = r(t) P. \tag{7}$$

Calculate the derivatives and match the z terms on both sides of the above equation, we will obtain a system of Riccati ODEs,

$$-\frac{d}{dt}\gamma(t,T) + 2\gamma(t,T) g(t)^\top g(t) \gamma(t,T) = \gamma(t), \tag{8}$$

$$-\frac{d}{dt} h(t,T) + 2\gamma(t,T) g(t)^\top g(t) h(t,T) = h(t), \tag{9}$$

$$-\frac{d}{dt} a(t,T) - \mathrm{Tr}\Big(\gamma(t,T) g(t)^\top g(t)\Big) + \frac{1}{2} h(t,T)^\top g(t)^\top g(t) h(t,T) = a(t), \tag{10}$$

with terminal conditions $\gamma(T,T) = 0$, $h(T,T) = 0$, and $a(T,T) = 0$.

References

[1] Leif Andersen and Vladimir Piterbarg, *Interest Rate Modeling*, Atlantic Financial Press, 2010.