Notes on HJM Framework

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This mainly derives from Andersen and Piterbarg's *Interest Rate Modeling*, and Brigo and Mercurio's *Interest Rate Models - Theory and Practice*.

1 Formulation

Let $P(t,T)$ denotes the time t price of a zero coupon bond which pays 1 at time T, *i.e.*,

$$P(t,T) = E_t^Q \left[\exp\left(-\int_t^T r(u)du \right) \right], \tag{1}$$

where Q stands for the risk neutral measure, and $r(t)$ is the short rate. Then, under the same measure,

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_P'(t,T)dW(t), \tag{2}$$

where W is a d-dimensional Brownian motion, $\sigma_P(t,T)$ is a d-dimensional stochastic process adapted to the filtration generated by W, and the prime represents matrix transposition. Here, $\sigma_P(t,T)$ must satisfy the consistency condition, $\sigma_P(T,T) = 0$, since $P(T,T) = 1$.

Define the forward discount bond price as

$$P(t,T,T+\tau) = \frac{P(t,T+\tau)}{P(t,T)}, \tag{3}$$

and apply the Ito's lemma, then

$$\frac{dP(t,T,T+\tau)}{P(t,T,T+\tau)} = -\left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]\sigma_P(t,T)dt - \left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]dW(t), \tag{4}$$

under the risk neutral measure. Under the T-forward measure, the forward discount bond price is a martingale, which means

$$\frac{dP(t,T,T+\tau)}{P(t,T,T+\tau)} = -\left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]dW^T(t). \tag{5}$$

Define the forward rate as

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}, \tag{6}$$

then from Eq. (2), the forward rate has the following dynamics,

$$df(t,T) = \sigma'_f(t,T)\sigma_P(t,T)dt + \sigma'_f(t,T)dW(t), \tag{7}$$

where $\sigma_f(t,T) = \partial\sigma_P(t,T)/\partial T$, or equivalently, $\sigma_P(t,T) = \int_t^T \sigma_f(t,u)du$.

2 Markovian Short Rate Dynamics

From Eq. (7), the forward rate is given by

$$f(t,T) = f(0,T) + \int_0^t \sigma'_f(u,T)\left(\int_u^T \sigma_f(u,s)ds\right)du + \int_0^t \sigma'_f(u,T)dW(u). \tag{8}$$

Define the short rate as $r(t) = f(t,t)$, then the short rate dynamics becomes

$$r(t) = f(0,t) + \int_0^t \sigma'_f(u,t)\left(\int_u^t \sigma_f(u,s)ds\right)du + \int_0^t \sigma'_f(u,t)dW(u). \tag{9}$$

Generally, the short rate process is not Markovian. However, after imposing the separation condition $\sigma_f(t,T) = g(t)h(T)$, where $g(t) \equiv g(t;\omega)$ is a $d \times d$ matrix valued function, which can be deterministic or stochastic, and h is a deterministic d-dimensional vector, the short rate will be Markovian. Under this condition,

$$r(t) = f(0,t) + h'(t)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du + h'(t)\int_0^t g'(u)dW(u). \tag{10}$$

To cast the short rate dynamics into a convenient form, define

$$H(t) = \mathrm{diag}(h_1(t), ..., h_d(t)) = \begin{pmatrix} h_1(t) & & 0 \\ & \ddots & \\ 0 & & h_d(t) \end{pmatrix}, \tag{11}$$

and

$$\kappa(t) = -\frac{dH(t)}{dt}H^{-1}(t). \tag{12}$$

Here, we have assumed that none of the h_i is zero. Later it will be shown that the diagonal elements of $\kappa(t)$ are the mean reversion parameter for the short rate dynamics. Now, define

$$x(t) = H(t)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du + H(t)\int_0^t g'(u)dW(u), \tag{13}$$

and

$$y(t) = H(t)\left(\int_0^t g'(u)g(u)du\right)H(t), \tag{14}$$

then it can be shown

$$\frac{dy}{dt} = H(t)g'(t)g(t)H(t) - \kappa(t)y(t) - y(t)\kappa(t), \tag{15}$$

which is deterministic and can be solved numerically and analytically, if possible. Also,

$$\begin{aligned} dx(t) &= \left[\frac{dH(t)}{dt}\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du\right]dt + \left[H(t)\left(\int_0^t g'(u)g(u)du\right)h(t)dt\right] \\ &\quad +\frac{dH(t)}{dt}\left(\int_0^t g'(u)dW(u)\right)dt + H(t)g'(t)dW(t) \\ &= (y(t)I_{d\times 1} - \kappa(t)x(t))dt + H(t)g'(t)dW(t) \\ &= (y(t)I_{d\times 1} - \kappa(t)x(t))dt + \sigma_x'(t)dW(t), \end{aligned} \tag{16}$$

where $I_{d\times 1} = (1, 1, ..., 1)'$, and $\sigma_x(t) = g(t)H(t)$. The x and y processes have initial conditions $x(0) = 0$ and $y(0) = 0$, which is evident from their definitions. It can be shown, by Ritchken and Sankarasubramanian, that any interest rate derivative will be completely determined by the two-state Markovian process $(x(t), y(t))$.

Using the above parameterizations, it can be shown, for forward rate,

$$\begin{aligned} f(t,T) &= f(0,T) + I_{d\times 1}'H(T)\int_0^t g'(u)g(u)\left(\int_u^T h(s)ds\right)du + I_{d\times 1}'H(T)\int_0^t g'(u)dW(u) \\ &= f(0,T) + I_{d\times 1}'H(T)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du + I_{d\times 1}'H(T)\int_0^t g'(u)dW(u) \\ &\quad +I_{d\times 1}'H(T)\int_0^t g'(u)g(u)\left(\int_t^{\prime} h(s)ds\right)du \\ &= f(0,T) + I_{d\times 1}'H(T)H^{-1}(t)\Big(H(t)\int_0^t g'(u)g(u)\left(\int_u^t h(s)ds\right)du \\ &\qquad\qquad +H(t)\int_0^t g'(u)dW(u)\Big) \\ &\quad +I_{d\times 1}'H(T)H^{-1}(t)H(t)\int_0^t g'(u)g(u)\left(\int_t^T h(s)ds\right)du \\ &= f(0,T) + M'(t,T)\left(x(t) + y(t)\int_t^T M(t,u)du\right), \end{aligned} \tag{17}$$

where

$$M(t,T) = H(T)H^{-1}(t)I_{d\times 1}. \tag{18}$$

From this, the short rate is given by

$$r(t) = f(t,t) = f(0,t) + I_{d\times 1}'x(t) = f(0,t) + \sum_{i=1}^d x_i(t). \tag{19}$$

From the above result, we can also derive the bond reconstitution formula. Notice $P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$, we have

$$\begin{aligned} P(t,T) &= \exp\Bigg(-\int_t^T f(0,u)du - \left(\int_t^T M'(t,u)du\right)x(t) \\ &\qquad -\int_t^T M'(t,u)y(t)\left(\int_t^u M(t,s)ds\right)du\Bigg). \end{aligned} \tag{20}$$

Define

$$G(t,T) = \int_t^T M(t,u)du, \tag{21}$$

and notice that

$$\int_t^T M'(t,u)y(t)\left(\int_t^u M(t,s)ds\right)du = \int_t^T M'(t,u)y(t)\left(\int_u^T M(t,s)ds\right)du, \tag{22}$$

the bond reconstitution formula can be rewritten as

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left(-G'(t,T)x(t) - \frac{1}{2}G'(t,T)y(t)G(t,T)\right). \tag{23}$$

3 Closed Form Results for Gaussian Models

In the above formulation, when $\sigma_f(t,T)$ is a deterministic function, the resulting models will be Gaussian, where the distribution of the forward rate, and the short rate, will be Gaussian. This leads to simplifications to the modeling and pricing of interest rate derivatives, and even closed form expressions for several simple derivatives.

3.1 Zero Coupon Bond Option

Consider a call option with expiry T and strike K on a zero coupon bond with maturity T^*. At expiry, the option pays

$$V(T) = \left(P(T,T^*) - K\right)^+. \tag{24}$$

Using the change of measure trick, the value of the zero coupon bond becomes

$$V(t) = E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)V(T)\right] = P(t,T)E_t^T\left[\left(P(T,T,T^*) - K\right)^+\right], \tag{25}$$

under the T-forward measure. Then, apply the Black-Scholes formula and take into account of Eq. (5),

$$\begin{aligned} V(t) &= P(t,T)\left[P(t,T,T^*)N(d_+) - KN(d_-)\right] \\ &= P(t,T^*)N(d_+) - KP(t,T)N(d_-), \end{aligned} \tag{26}$$

where

$$d_\pm = \frac{\log\left(\dfrac{P(t,T,T^*)}{K}\right) \pm \dfrac{v}{2}}{\sqrt{v}} = \frac{\log\left(\dfrac{P(t,T^*)}{KP(t,T)}\right) \pm \dfrac{v}{2}}{\sqrt{v}}, \tag{27}$$

and

$$v = \int_t^T \left|\sigma_P(s,T^*) - \sigma_P(s,T)\right|^2 ds. \tag{28}$$

3.2 Caplet

Now consider a caplet which resets at time T, and pays at $T+\tau$ the amount

$$V(T+\tau) = \tau\left(L(T,T+\tau) - K\right)^+, \tag{29}$$

where $L(T,T+\tau)$ is the LIBOR rates spanning from T to $T+\tau$, and τ is the year fraction. The value of the caplet at time t is

$$V(t) = E_t^Q\left[\exp\left(-\int_t^{T+\tau} r(s)ds\right)V(T+\tau)\right]. \tag{30}$$

Use the tower rule,

$$\begin{aligned} V(t) &= E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)E_T^Q\left[\exp\left(-\int_T^{T+\tau} r(s)ds\right)V(T+\tau)\right]\right] \\ &= E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)P(T,T+\tau)V(T+\tau)\right]. \end{aligned} \tag{31}$$

Notice that $L(T,T+\tau) = \tau^{-1}(1/P(T,T+\tau)-1)$, the above expression becomes

$$\begin{aligned} V(t) &= E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\left(1-(1+\tau K)P(T,T+\tau)\right)^+\right] \\ &= (1+\tau K)E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\left(\frac{1}{1+\tau K} - P(T,T+\tau)\right)^+\right]. \end{aligned} \tag{32}$$

Now, the caplet price has been turned into a put option on a zero coupon bond. Using the earlier result, the caplet price is

$$\begin{aligned} V(t) &= (1+\tau K)P(t,T)\left(\frac{1}{1+\tau K}N(-d_-) - P(T,T+\tau)N(-d_+)\right) \\ &= P(t,T)N(-d_-) - (1+\tau K)P(t,T+\tau)N(-d_+), \end{aligned} \tag{33}$$

where

$$d_\pm = \frac{\log\left(\dfrac{(1+\tau K)P(t,T+\tau)}{P(t,T)}\right) \pm \dfrac{v}{2}}{\sqrt{v}}, \tag{34}$$

and

$$v = \int_t^T \left|\sigma_P(s,T+\tau) - \sigma_P(s,T)\right|^2 ds. \tag{35}$$

3.3 Futures

The futures rate is a martingale under the risk neutral measure, *i.e.*, $F(t,T,T+\tau) = E_t^Q[L(T,T,T+\tau)]$. This is equivalent to

$$F(t,T,T+\tau) = \frac{1}{\tau}E_t^Q\left[\frac{1}{P(T,T+\tau)} - 1\right]. \tag{36}$$

The dynamics of $G(t) = 1/P(t,T,T+\tau) = P(t,T)/P(t,T+\tau)$ under the risk neutral measure is

$$
\begin{aligned}
d\left(\frac{1}{P(t,T,T+\tau)}\right) &= -\frac{dP(t,T,T+\tau)}{P^2(t,T,T+\tau)} + \frac{(dP(t,T,T+\tau))^2}{P^3(t,T,T+\tau)} \\
&= \frac{1}{P(t,T,T+\tau)}\Big(\left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]\sigma_P(t,T)dt \\
&\qquad + \left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]dW(t)\Big) \\
&\quad + \frac{1}{P(t,T,T+\tau)}\left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]\left[\sigma_P(t,T+\tau) - \sigma_P(t,T)\right]dt \\
&= \frac{1}{P(t,T,T+\tau)}\Big(\left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]\sigma_P(t,T+\tau)dt \\
&\qquad + \left[\sigma_P'(t,T+\tau) - \sigma_P'(t,T)\right]dW(t)\Big). \qquad (37)
\end{aligned}
$$

Therefore,

$$
E_t^Q\left[\frac{1}{P(T,T+\tau)}\right] = \frac{1}{P(t,T,T+\tau)}e^{\Omega(t,T)}, \qquad (38)
$$

where

$$
\Omega(t,T) = \int_t^T \left[\sigma_P'(u,T+\tau) - \sigma_P'(u,T)\right]\sigma_P(u,T+\tau)du. \qquad (39)
$$

So, the futures rate is

$$
F(t,T,T+\tau) = \frac{1}{\tau}\left(\frac{1}{P(t,T,T+\tau)}e^{\Omega(t,T)} - 1\right). \qquad (40)
$$

3.4 European Swaption - Jamshidian Decomposition

The above results do not make any specific assumptions besides of the Gaussianness, and can be applied to general models with deterministic forward rate volatility functions. The trick, Jamshidian decomposition, which will be introduced in the following relies on the monotonicity of the zero coupon bond price on short rate.

Consider a payer swaption expiring at T_0, with the underlying swap paying annualized coupon c at times $T_1 < T_2 < \cdots < T_N$, with $T_1 > T_0$. The swaption payout at T_0 is

$$
V(T_0) = \left(1 - P(T_0,T_N) - c\sum_{i=1}^N \tau_i P(T_0,T_i)\right)^+, \qquad (41)
$$

where $\tau_i = T_i - T_{i-1}$ is the accrual period. Notice the dependence of the zero coupon bond price on $x(T_0)$ through short rate $r(T_0)$, *i.e.*, $P(T_0,T_N) \equiv P(T_0,T_N;x(T_0))$. Define a critical value x^* by setting the value of the swaption at time T_0 to be exactly zero,

$$
P(T_0,T_N;x^*) + c\sum_{i=1}^N \tau_i P(T_0,T_i;x^*) = 1. \qquad (42)
$$

Also defint the "strikes" as

$$K_i = P(T_0, T_i; x^*), \tag{43}$$

then

$$K_N + c \sum_{i=1}^{N} \tau_i K_i = 1. \tag{44}$$

Notice that if $P(T_0, T_i; x(T_0))$ is monotonically decreasing in $x(T_0)$, which will hold for Markovian Gaussian short rate models from the bond reconstitution formula (23), then we will have

$$\begin{aligned}
V(T_0) &= \left(1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^{N} \tau_i P(T_0, T_i; x(T_0))\right)^+ \\
&= \left(1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^{N} \tau_i P(T_0, T_i; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} \\
&= \left(K_N + c \sum_{i=1}^{N} \tau_i K_i - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^{N} \tau_i P(T_0, T_i; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} \\
&= \left(K_N - P(T_0, T_N; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} + c \sum_{i=1}^{N} \tau_i \left(K_i - P(T_0, T_i; x(T_0))\right) \mathbf{1}_{x(T_0) > x^*} \\
&= \left(K_N - P(T_0, T_N; x(T_0))\right)^+ + c \sum_{i=1}^{N} \tau_i \left(K_i - P(T_0, T_i; x(T_0))\right)^+ . \qquad (45)
\end{aligned}$$

Then, this can be evaluated with the zero coupon bond option price formula, Eq. (26). This trick can also be applied to coupon bond options.

4 Examples of Classical Gaussian Short Rate Models - One Factor Models

By setting $d = 1$ in the above formulation, we will obtain one-factor models. Since the matrix products will be reduced to scalar products, this will greatly simplify the discussion.

4.1 Ho-Lee Model

The Ho-Lee model can be recovered by taking $h(t) = 1$, and $g(t) = \sigma$ which is constant. It is straightforward to show that $\kappa(t) = 0$, $y(t) = \sigma^2 t$,

$$dx(t) = \sigma^2 t dt + \sigma dW(t), \tag{46}$$

and $r(t) = f(0, t) + x(t)$. Also, the forward rate volatility is $\sigma_f(t, T) = \sigma$, which is constant too, $\sigma_P(t, T) = \sigma(T - t)$, and the bond reconstitution formula is

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp\left(-x(t)(T - t) - \frac{1}{2}\sigma^2 (T - t)^2 t\right)$$

$$= \frac{P(0,T)}{P(0,t)} \exp\left(-\left(r(t) - f(0,t)\right)(T-t) - \frac{1}{2}\sigma^2 (T-t)^2 t\right). \tag{47}$$

4.2 Hull-White Model

Take $h(t) = \exp\left(-\int_0^t \kappa(u)du\right)$, and $g(t) = \sigma(t)\exp\left(\int_0^t \kappa(u)du\right)$, then we have

$$\frac{dy(t)}{dt} = \sigma^2(t) - 2\kappa(t)y(t), \tag{48}$$

which has the following analytical solution,

$$y(t) = \int_0^t \sigma^2(u) \exp\left(-2\int_u^t \kappa(s)ds\right) du, \tag{49}$$

Then,

$$dx(t) = \left(y(t) - \kappa(t)x(t)\right)dt + \sigma(t)dW(t), \tag{50}$$

which can be shown to have the following solution,

$$x(t) = \int_0^t y(u) \exp\left(-\int_u^t \kappa(s)ds\right) du + \int_0^t \sigma(u) \exp\left(-\int_u^t \kappa(s)ds\right) dW(u). \tag{51}$$

4.3 Mercurio-Moraleda Model

Similar to the Vasicek model, set

$$\kappa(t) = \lambda - \frac{\gamma}{1+\gamma t}, \tag{52}$$

5 Examples of Classical Gaussian Short Rate Models - Multi Factor Models

5.1 Deficiencies of One-Factor Models

5.2 Hull-White Two-Factor Model

6 Quasi-Gaussian Models

7 Quadratic Gaussian Models