

# Notes on HJM Framework

Dec. 32, 2999

## 1 Formulation

Let  $P(t, T)$  denotes the time  $t$  price of a zero coupon bond which pays 1 at time  $T$ , *i.e.*,

$$P(t, T) = E_t^Q \left[ \exp \left( - \int_t^T r(u) du \right) \right], \quad (1)$$

where  $Q$  stands for the risk neutral measure, and  $r(t)$  is the short rate. Then, under the same measure [1],

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma_P(t, T)^\top dW(t), \quad (2)$$

where  $W$  is a  $d$ -dimensional Brownian motion,  $\sigma_P(t, T)$  is a  $d$ -dimensional stochastic process adapted to the filtration generated by  $W$ , and  $\top$  represents matrix transposition. Here,  $\sigma_P(t, T)$  must satisfy the consistency condition,  $\sigma_P(T, T) = 0$ , since  $P(T, T) = 1$ .

Define the forward discount bond price as

$$P(t, T, T + \tau) = \frac{P(t, T + \tau)}{P(t, T)}, \quad (3)$$

and apply the Ito's lemma, then

$$\frac{dP(t, T, T + \tau)}{P(t, T, T + \tau)} = - [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] \sigma_P(t, T) dt - [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW(t), \quad (4)$$

under the risk neutral measure. Under the  $T$ -forward measure, the forward discount bond price is a martingale, which means

$$\frac{dP(t, T, T + \tau)}{P(t, T, T + \tau)} = - [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW^T(t). \quad (5)$$

Define the forward rate as

$$f(t, T) = - \frac{\partial \log P(t, T)}{\partial T}, \quad (6)$$

then from Eq. (2), the forward rate has the following dynamics,

$$df(t, T) = \sigma_f(t, T)^\top \sigma_P(t, T) dt + \sigma_f(t, T)^\top dW(t), \quad (7)$$

where  $\sigma_f(t, T) = \partial \sigma_P(t, T) / \partial T$ , or equivalently,  $\sigma_P(t, T) = \int_t^T \sigma_f(t, u) du$ .

## 2 Markovian Short Rate Dynamics

From Eq. (7), the forward rate is given by

$$f(t, T) = f(0, T) + \int_0^t \sigma_f(u, T)^\top \left( \int_u^T \sigma_f(u, s) ds \right) du + \int_0^t \sigma_f(u, T)^\top dW(u). \quad (8)$$

Define the short rate as  $r(t) = f(t, t)$ , then the short rate dynamics becomes

$$r(t) = f(0, t) + \int_0^t \sigma_f(u, t)^\top \left( \int_u^t \sigma_f(u, s) ds \right) du + \int_0^t \sigma_f(u, t)^\top dW(u). \quad (9)$$

Generally, the short rate process is not Markovian. However, after imposing the separation condition  $\sigma_f(t, T) = g(t)h(T)$ , where  $g(t) \equiv g(t; \omega)$  is a  $d \times d$  matrix valued function, which can be deterministic or stochastic, and  $h$  is a deterministic  $d$ -dimensional column vector function, the short rate will be Markovian. Under this condition,

$$r(t) = f(0, t) + h(t)^\top \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du + h(t)^\top \int_0^t g(u)^\top dW(u). \quad (10)$$

To cast the short rate dynamics into a convenient form, define

$$H(t) = \text{diag}(h_1(t), \dots, h_d(t)) = \begin{pmatrix} h_1(t) & & 0 \\ & \ddots & \\ 0 & & h_d(t) \end{pmatrix}, \quad (11)$$

and

$$\kappa(t) = -\frac{dH(t)}{dt} H^{-1}(t). \quad (12)$$

Here, we have assumed that none of the  $h_i$  is zero. Later it will be shown that the diagonal elements of  $\kappa(t)$  are the mean reversion parameter for the short rate dynamics. Now, define

$$x(t) = H(t) \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du + H(t) \int_0^t g(u)^\top dW(u), \quad (13)$$

and

$$y(t) = H(t) \left( \int_0^t g(u)^\top g(u) du \right) H(t), \quad (14)$$

then it can be shown

$$dy(t) = \left( \sigma_r(t; \omega)^\top \sigma_r(t; \omega) - \kappa(t)y(t) - y(t)\kappa(t) \right) dt, \quad (15)$$

with  $\sigma_r(t; \omega) = g(t; \omega)H(t)$ , which is locally deterministic and can be solved numerically and analytically, if possible. Also,

$$dx(t) = \left[ \frac{dH(t)}{dt} \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du \right] dt + \left[ H(t) \left( \int_0^t g(u)^\top g(u) du \right) h(t) dt \right]$$

$$\begin{aligned}
& + \frac{dH(t)}{dt} \left( \int_0^t g(u)^\top dW(u) \right) dt + H(t)g(t)^\top dW(t) \\
& = \left( y(t)I_{d \times 1} - \kappa(t)x(t) \right) dt + H(t)g(t)^\top dW(t) \\
& = \left( y(t)I_{d \times 1} - \kappa(t)x(t) \right) dt + \sigma_r(t; \omega)^\top dW(t),
\end{aligned} \tag{16}$$

where  $I_{d \times 1} = (1, 1, \dots, 1)^\top$ . The  $x$  and  $y$  processes have initial conditions  $x(0) = 0$  and  $y(0) = 0$ , which is evident from their definitions. It can be shown, by Ritchken and Sankarasubramanian, that any interest rate derivative will be completely determined by the two-state Markovian process  $(x(t), y(t))$ .

Using the above parameterizations, it can be shown, for forward rate,

$$\begin{aligned}
f(t, T) &= f(0, T) + I_{d \times 1} H(T)^\top \int_0^t g(u)^\top g(u) \left( \int_u^T h(s) ds \right) du + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top dW(u) \\
&= f(0, T) + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top dW(u) \\
&\quad + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top g(u) \left( \int_t^T h(s) ds \right) du \\
&= f(0, T) + I_{d \times 1}^\top H(T) H^{-1}(t) \left( H(t) \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du \right. \\
&\quad \left. + H(t) \int_0^t g(u)^\top dW(u) \right) \\
&\quad + I_{d \times 1}^\top H(T) H^{-1}(t) H(t) \int_0^t g(u)^\top g(u) \left( \int_t^T h(s) ds \right) du \\
&= f(0, T) + M(t, T)^\top \left( x(t) + y(t) \int_t^T M(t, u) du \right),
\end{aligned} \tag{17}$$

where

$$M(t, T) = H(T)H^{-1}(t)I_{d \times 1}. \tag{18}$$

From this, the short rate is given by

$$r(t) = f(t, t) = f(0, t) + I_{d \times 1}^\top x(t) = f(0, t) + \sum_{i=1}^d x_i(t). \tag{19}$$

From the above result, we can also derive the bond reconstitution formula. Notice  $P(t, T) = \exp \left( - \int_t^T f(t, u) du \right)$ , we have

$$\begin{aligned}
P(t, T) &= \exp \left( - \int_t^T f(0, u) du - \left( \int_t^T M(t, u)^\top du \right) x(t) \right. \\
&\quad \left. - \int_t^T M(t, u)^\top y(t) \left( \int_t^u M(t, s) ds \right) du \right).
\end{aligned} \tag{20}$$

Define

$$G(t, T) = \int_t^T M(t, u) du, \quad (21)$$

and notice that

$$\int_t^T M(t, u)^\top y(t) \left( \int_t^u M(t, s) ds \right) du = \int_t^T M(t, u)^\top y(t) \left( \int_u^T M(t, s) ds \right) du, \quad (22)$$

the bond reconstitution formula can be rewritten as

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( -G(t, T)^\top x(t) - \frac{1}{2} G(t, T)^\top y(t) G(t, T) \right). \quad (23)$$

The density relating the  $T$ -forward measure and the risk neutral measure is given by

$$\zeta(t) = E_t^Q \left[ \frac{dP^T}{dP} \right] = \frac{P(t, T)/P(0, T)}{B(t)/B(0)}, \quad (24)$$

where  $B(t) = \exp \left( \int_0^t r(u) du \right)$  is the numeraire of the risk neutral measure. This density is a martingale in the risk neutral measure, and therefore,

$$\frac{d\zeta(t)}{\zeta(t)} = -G(t, T)^\top \sigma_r(t)^\top dW(t). \quad (25)$$

The Brownian motion under the  $T$ -forward, following the Girsanov's theorem (see Appendix A), is then given by

$$dW^T(t) = dW(t) + \sigma_r(t) G(t, T) dt. \quad (26)$$

### 3 Closed Form Results for Gaussian Models

In the above formulation, when  $\sigma_f(t, T)$  is a deterministic function, the resulting models will be Gaussian, where the distribution of the forward rate, and the short rate, will be Gaussian. This leads to simplifications to the modeling and pricing of interest rate derivatives, and even closed form expressions for several simple derivatives.

#### 3.1 Zero Coupon Bond Option

Consider a call option with expiry  $T$  and strike  $K$  on a zero coupon bond with maturity  $T^*$ . At expiry, the option pays

$$V(T) = (P(T, T^*) - K)^+. \quad (27)$$

Using the change of measure trick, the value of the zero coupon bond becomes

$$V(t) = E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) V(T) \right] = P(t, T) E_t^T \left[ (P(T, T, T^*) - K)^+ \right], \quad (28)$$

under the  $T$ -forward measure. Then, apply the Black-Scholes formula and take into account of Eq. (5),

$$\begin{aligned} V(t) &= P(t, T) [P(t, T, T^*)N(d_+) - KN(d_-)] \\ &= P(t, T^*)N(d_+) - KP(t, T)N(d_-), \end{aligned} \quad (29)$$

where

$$d_{\pm} = \frac{\log\left(\frac{P(t, T, T^*)}{K}\right) \pm \frac{v}{2}}{\sqrt{v}} = \frac{\log\left(\frac{P(t, T^*)}{KP(t, T)}\right) \pm \frac{v}{2}}{\sqrt{v}}, \quad (30)$$

and

$$v = \int_t^T |\sigma_P(s, T^*) - \sigma_P(s, T)|^2 ds. \quad (31)$$

### 3.2 Caplet

Now consider a caplet which resets at time  $T$ , and pays at  $T + \tau$  the amount

$$V(T + \tau) = \tau (L(T, T + \tau) - K)^+, \quad (32)$$

where  $L(T, T + \tau)$  is the LIBOR rates spanning from  $T$  to  $T + \tau$ , and  $\tau$  is the year fraction. The value of the caplet at time  $t$  is

$$V(t) = E_t^Q \left[ \exp \left( - \int_t^{T+\tau} r(s) ds \right) V(T + \tau) \right]. \quad (33)$$

Use the tower rule,

$$\begin{aligned} V(t) &= E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) E_T^Q \left[ \exp \left( - \int_T^{T+\tau} r(s) ds \right) V(T + \tau) \right] \right] \\ &= E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) P(T, T + \tau) V(T + \tau) \right]. \end{aligned} \quad (34)$$

Notice that  $L(T, T + \tau) = \tau^{-1}(1/P(T, T + \tau) - 1)$ , the above expression becomes

$$\begin{aligned} V(t) &= E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) (1 - (1 + \tau K) P(T, T + \tau))^+ \right] \\ &= (1 + \tau K) E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \left( \frac{1}{1 + \tau K} - P(T, T + \tau) \right)^+ \right]. \end{aligned} \quad (35)$$

Now, the caplet price has been turned into a put option on a zero coupon bond. Using the earlier result, the caplet price is

$$\begin{aligned} V(t) &= (1 + \tau K) P(t, T) \left( \frac{1}{1 + \tau K} N(-d_-) - P(T, T + \tau) N(-d_+) \right) \\ &= P(t, T) N(-d_-) - (1 + \tau K) P(t, T + \tau) N(-d_+), \end{aligned} \quad (36)$$

where

$$d_{\pm} = \frac{\log \left( \frac{(1 + \tau K)P(t, T + \tau)}{P(t, T)} \right) \pm \frac{v}{2}}{\sqrt{v}}, \quad (37)$$

and

$$v = \int_t^T |\sigma_P(s, T + \tau) - \sigma_P(s, T)|^2 ds. \quad (38)$$

### 3.3 Futures

The futures rate is a martingale under the risk neutral measure, *i.e.*,  $F(t, T, T + \tau) = E_t^Q[L(T, T, T + \tau)]$ . This is equivalent to

$$F(t, T, T + \tau) = \frac{1}{\tau} E_t^Q \left[ \frac{1}{P(T, T + \tau)} - 1 \right]. \quad (39)$$

The dynamics of  $G(t) = 1/P(t, T, T + \tau) = P(t, T)/P(t, T + \tau)$  under the risk neutral measure is

$$\begin{aligned} d \left( \frac{1}{P(t, T, T + \tau)} \right) &= -\frac{dP(t, T, T + \tau)}{P^2(t, T, T + \tau)} + \frac{(dP(t, T, T + \tau))^2}{P^3(t, T, T + \tau)} \\ &= \frac{1}{P(t, T, T + \tau)} \left( [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] \sigma_P(t, T) dt \right. \\ &\quad \left. + [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW(t) \right) \\ &\quad + \frac{1}{P(t, T, T + \tau)} [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] [\sigma_P(t, T + \tau) - \sigma_P(t, T)] dt \\ &= \frac{1}{P(t, T, T + \tau)} \left( [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] \sigma_P(t, T + \tau) dt \right. \\ &\quad \left. + [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW(t) \right). \end{aligned} \quad (40)$$

Therefore,

$$E_t^Q \left[ \frac{1}{P(T, T + \tau)} \right] = \frac{1}{P(t, T, T + \tau)} e^{\Omega(t, T)}, \quad (41)$$

where

$$\Omega(t, T) = \int_t^T [\sigma_P(u, T + \tau)^\top - \sigma_P(u, T)^\top] \sigma_P(u, T + \tau) du. \quad (42)$$

So, the futures rate is

$$F(t, T, T + \tau) = \frac{1}{\tau} \left( \frac{1}{P(t, T, T + \tau)} e^{\Omega(t, T)} - 1 \right). \quad (43)$$

### 3.4 European Swaption - Jamshidian Decomposition

The above results do not make any specific assumptions besides of the Gaussianness, and can be applied to general models with deterministic forward rate volatility functions. The trick, Jamshidian decomposition, which will be introduced in the following relies on the monotonicity of the zero coupon bond price on short rate.

Consider a payer swaption expiring at  $T_0$ , with the underlying swap paying annualized coupon  $c$  at times  $T_1 < T_2 < \dots < T_N$ , with  $T_1 > T_0$ . The swaption payout at  $T_0$  is

$$V(T_0) = \left( 1 - P(T_0, T_N) - c \sum_{i=1}^N \tau_i P(T_0, T_i) \right)^+, \quad (44)$$

where  $\tau_i = T_i - T_{i-1}$  is the accrual period. Notice the dependence of the zero coupon bond price on  $x$ , *i.e.*,  $P(T_0, T_N) \equiv P(T_0, T_N; x(T_0))$ , through the bond reconstitution formula, Eq. (23). Define a critical value  $x^*$  by setting the value of the swaption at time  $T_0$  to exactly zero,

$$P(T_0, T_N; x^*) + c \sum_{i=1}^N \tau_i P(T_0, T_i; x^*) = 1. \quad (45)$$

Also define the “strikes” as

$$K_i = P(T_0, T_i; x^*), \quad (46)$$

then

$$K_N + c \sum_{i=1}^N \tau_i K_i = 1. \quad (47)$$

Notice that  $P(T_0, T_i; x(T_0))$  is monotonically decreasing in  $x(T_0)$ , which always holds due to the bond reconstitution formula (23), we have

$$\begin{aligned} V(T_0) &= \left( 1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^N \tau_i P(T_0, T_i; x(T_0)) \right)^+ \\ &= \left( 1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^N \tau_i P(T_0, T_i; x(T_0)) \right) \mathbf{1}_{x(T_0) > x^*} \\ &= \left( K_N + c \sum_{i=1}^N \tau_i K_i - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^N \tau_i P(T_0, T_i; x(T_0)) \right) \mathbf{1}_{x(T_0) > x^*} \\ &= (K_N - P(T_0, T_N; x(T_0))) \mathbf{1}_{x(T_0) > x^*} + c \sum_{i=1}^N \tau_i (K_i - P(T_0, T_i; x(T_0))) \mathbf{1}_{x(T_0) > x^*} \\ &= (K_N - P(T_0, T_N; x(T_0)))^+ + c \sum_{i=1}^N \tau_i (K_i - P(T_0, T_i; x(T_0)))^+. \end{aligned} \quad (48)$$

Then, this can be evaluated with the zero coupon bond option price formula, Eq. (29). This trick can also be applied to coupon bond options.

## 4 Examples of Classical Gaussian Short Rate Models - One Factor Models

By setting  $d = 1$  in the above formulation, we will obtain one-factor models. Since the matrix products will be reduced to scalar products, this will greatly simplify the discussion.

### 4.1 Ho-Lee Model

The Ho-Lee model can be recovered by taking  $h(t) = 1$ , and  $g(t) = \sigma$  which is constant. It is straightforward to show that  $\kappa(t) = 0$ ,  $y(t) = \sigma^2 t$ ,

$$dx(t) = \sigma^2 t dt + \sigma dW(t), \quad (49)$$

and  $r(t) = f(0, t) + x(t)$ . Also, the forward rate volatility is  $\sigma_f(t, T) = \sigma$ , which is constant too,  $\sigma_P(t, T) = \sigma(T - t)$ , and the bond reconstitution formula is

$$\begin{aligned} P(t, T) &= \frac{P(0, T)}{P(0, t)} \exp \left( -x(t)(T - t) - \frac{1}{2} \sigma^2 (T - t)^2 t \right) \\ &= \frac{P(0, T)}{P(0, t)} \exp \left( -(r(t) - f(0, t))(T - t) - \frac{1}{2} \sigma^2 (T - t)^2 t \right). \end{aligned} \quad (50)$$

### 4.2 Hull-White Model

Take  $h(t) = \exp \left( - \int_0^t \kappa(u) du \right)$ , and  $g(t) = \sigma(t) \exp \left( \int_0^t \kappa(u) du \right)$ , then we have

$$\frac{dy(t)}{dt} = \sigma^2(t) - 2\kappa(t)y(t), \quad (51)$$

which has the following analytical solution,

$$y(t) = \int_0^t \sigma^2(u) \exp \left( -2 \int_u^t \kappa(s) ds \right) du, \quad (52)$$

Then,

$$dx(t) = (y(t) - \kappa(t)x(t)) dt + \sigma(t) dW(t), \quad (53)$$

which can be shown to have the following solution,

$$x(t) = \int_0^t y(u) \exp \left( - \int_u^t \kappa(s) ds \right) du + \int_0^t \sigma(u) \exp \left( - \int_u^t \kappa(s) ds \right) dW(u). \quad (54)$$

Also,

$$\sigma_f(t, T) = g(t)h(T) = \sigma(t) \exp \left( - \int_t^T \kappa(u) du \right), \quad (55)$$

and

$$\sigma_P(t, T) = \int_t^T \sigma_f(t, u) du = \int_t^T \sigma(t) \exp \left( - \int_t^u \kappa(s) ds \right) du. \quad (56)$$



Moreover,

$$M(t, T) = \frac{h(T)}{h(t)} = \exp \left( - \int_t^T \kappa(u) du \right), \quad (57)$$

and

$$G(t, T) = \int_t^T M(t, u) du = \int_t^T \exp \left( - \int_t^u \kappa(s) ds \right) du. \quad (58)$$

The bond reconstitution formula follows Eq. (23).

The Vasicek model can be recovered by setting  $\kappa$  and  $\sigma$  to constants, and the mathematical details will not be reproduced here.

### 4.3 Mercurio-Moraleda Model

The Mercurio-Moraleda model [2] can be obtained by setting

$$\kappa(t) = \lambda - \frac{\gamma}{1 + \gamma t}, \quad (59)$$

then

$$h(t) = (1 + \gamma t) e^{-\lambda t}, \quad (60)$$

and

$$g(t) = \sigma \frac{e^{\lambda t}}{1 + \gamma t}. \quad (61)$$

The forward rate volatility is given by

$$\sigma_f(t, T) = g(t)h(T) = \sigma \frac{1 + \gamma T}{1 + \gamma t} e^{-\lambda(T-t)}, \quad (62)$$

which satisfies the separability condition, and the short rate dynamics will be Markovian. Moreover, this volatility specification can also generate a humped shape in the volatility term structure, which is often observed in the cap market. However, the above dynamics is not time-stationary. This can be addressed by expanding the forward volatility to the first order in  $\gamma$ , leading to the following expression,

$$\sigma_f(t, T) = \sigma \left[ 1 + \gamma(T - t) \right] e^{-\lambda(T-t)}, \quad (63)$$

sacrificing the Markov property of the short rate dynamics. In the following, we will focus on the volatility structure specified by Eq. (62).

From Eq. (14), we have

$$\begin{aligned} y(t) &= \int_0^t \sigma^2 \exp \left( -2 \int_u^t \kappa(s) ds \right) du \\ &= \sigma^2 \int_0^t \left( \frac{1 + \gamma t}{1 + \gamma u} \right)^2 e^{-2\lambda(t-u)} du. \end{aligned} \quad (64)$$

Notice that

$$\int \frac{e^{\lambda t}}{1 + \gamma t} dt = \frac{e^{-\lambda/\gamma}}{\gamma} \text{Ei} \left( \frac{\lambda}{\gamma} (1 + \gamma t) \right) + \text{const.}, \quad (65)$$

where

$$\text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt \quad (66)$$

is the exponential integral function, it can be shown that

$$\begin{aligned} y(t) = & \frac{\sigma^2(1+\gamma t)}{\gamma^2} \left[ 2\lambda(1+\gamma t) \exp\left(\frac{2\lambda}{\gamma}(1+\gamma t)\right) \text{Ei}\left(\frac{2\lambda}{\gamma}(1+\gamma t)\right) - \gamma \right] \\ & + \frac{\sigma^2(1+\gamma t)^2}{\gamma^2} \left[ 2\lambda \exp\left(\frac{2\lambda}{\gamma}(1+\gamma t)\right) \text{Ei}\left(\frac{2\lambda}{\gamma}\right) - \gamma e^{-2\lambda t} \right]. \end{aligned} \quad (67)$$

Also,

$$M(t, T) = \frac{h(T)}{h(t)} = \frac{1+\gamma T}{1+\gamma t} e^{-\lambda(T-t)}, \quad (68)$$

and

$$\begin{aligned} G(t, T) &= \int_t^T M(t, u) du = \frac{e^{\lambda t}}{1+\gamma t} \int_t^T (1+\gamma u) e^{-\lambda u} du \\ &= \frac{1}{\lambda^2(1+\gamma t)} \left[ \gamma + \lambda + \gamma \lambda t - (\gamma + \lambda + \gamma \lambda T) e^{-\lambda(T-t)} \right]. \end{aligned} \quad (69)$$

The bond reconstitution formula follows Eq. (23).

## 5 Examples of Classical Gaussian Short Rate Models - Multi Factor Models

As an example, let us consider the two factor Hull-White model in terms of the HJM framework. Define

$$g(t) = \begin{pmatrix} \sigma_{11}(t) \exp\left\{\int_0^t \kappa_1(u) du\right\} & 0 \\ \sigma_{21}(t) \exp\left\{\int_0^t \kappa_1(u) du\right\} & \sigma_{22}(t) \exp\left\{\int_0^t \kappa_2(u) du\right\} \end{pmatrix}, \quad (70)$$

and

$$h(t) = \begin{pmatrix} \exp\left\{-\int_0^t \kappa_1(u) du\right\} \\ \exp\left\{-\int_0^t \kappa_2(u) du\right\} \end{pmatrix}. \quad (71)$$

Then,

$$\sigma_r(t) = g(t)H(t) = \begin{pmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix}, \quad (72)$$

and

$$dx(t) = \left( y(t)I_{2 \times 1} - \kappa(t)x(t) \right) dt + \sigma_r^\top(t) dW(t), \quad (73)$$

where  $y(t)$  can be found by numerical integration, and  $\kappa(t) = \text{diag}(\kappa_1(t), \kappa_2(t))$ . The short rate can be written as

$$r(t) = f(0, t) + x_1(t) + x_2(t), \quad (74)$$

Define  $\sigma_1(t) = \sqrt{\sigma_{11}^2(t) + \sigma_{21}^2(t)}$ ,  $\sigma_2(t) = \sigma_{22}(t)$ , and

$$\rho_x(t) = \frac{\sigma_{21}(t)\sigma_{22}(t)}{\sigma_1(t)\sigma_2(t)}, \quad (75)$$

then the correlation in forward rates can be expressed in these three parameters, together with  $\kappa_1(t)$  and  $\kappa_2(t)$ . For the forward rate, we have

$$\sigma_f(t, T) = g(t)h(T) = \begin{pmatrix} \sigma_{11}(t) \exp \left\{ - \int_t^T \kappa_1(u) du \right\} \\ \sigma_{21}(t) \exp \left\{ - \int_t^T \kappa_1(u) du \right\} + \sigma_{22}(t) \exp \left\{ - \int_t^T \kappa_2(u) du \right\} \end{pmatrix}, \quad (76)$$

and

$$df(t, T) = O(dt) + \sigma_f^\top(t, T)dW(t). \quad (77)$$

The covariance of the forward rates maturing at  $T_1$  and  $T_2$  is

$$\begin{aligned} & \frac{\langle df(t, T_1), df(t, T_2) \rangle}{dt} \\ &= (\sigma_{11}^2(t) + \sigma_{21}^2(t)) \exp \left\{ - \int_t^{T_1} \kappa_1(u) du - \int_t^{T_2} \kappa_1(u) du \right\} \\ & \quad + \sigma_{22}^2(t) \exp \left\{ - \int_t^{T_1} \kappa_2(u) du - \int_t^{T_2} \kappa_2(u) du \right\} \\ & \quad + \sigma_{21}(t)\sigma_{22}(t) \left( \exp \left\{ - \int_t^{T_1} \kappa_1(u) du - \int_t^{T_2} \kappa_2(u) du \right\} \right. \\ & \quad \quad \left. + \exp \left\{ - \int_t^{T_1} \kappa_2(u) du - \int_t^{T_2} \kappa_1(u) du \right\} \right) \\ &= \sigma_1^2(t) \exp \left\{ - \int_t^{T_1} \kappa_1(u) du - \int_t^{T_2} \kappa_1(u) du \right\} b(t, T_1, T_2), \end{aligned} \quad (78)$$

where

$$\begin{aligned} b(t, T_1, T_2) &= 1 + \rho_x(t) \frac{\sigma_2(t)}{\sigma_1(t)} \left( \exp \left\{ - \int_t^{T_1} (\kappa_2(u) - \kappa_1(u)) du \right\} \right. \\ & \quad \left. + \exp \left\{ - \int_t^{T_2} (\kappa_2(u) - \kappa_1(u)) du \right\} \right) \\ & \quad + \left( \frac{\sigma_2(t)}{\sigma_1(t)} \right)^2 \exp \left\{ - \int_t^{T_1} (\kappa_2(u) - \kappa_1(u)) du - \int_t^{T_2} (\kappa_2(u) - \kappa_1(u)) du \right\}. \end{aligned} \quad (79)$$

Therefore, the correlation between the forward rates is given by

$$\rho(t, T_1, T_2) = \frac{b(t, T_1, T_2)}{\sqrt{b(t, T_1, T_1)b(t, T_2, T_2)}}. \quad (80)$$

It is clear from the above expression that the correlation of the forward rates can be controlled by the five functions:  $\rho_x(t)$ ,  $\sigma_1(t)$ ,  $\sigma_2(t)$ ,  $\kappa_1(t)$ , and  $\kappa_2(t)$ . If we have either  $\rho_x = 1$ ,  $\kappa_2(t) - \kappa_1(t) = 0$ , or  $\sigma_2(t)/\sigma_1(t) = 0$ , then  $\rho(t, T_1, T_2) \equiv 1$ , where the model reduces to a one-factor one. Also,

$$\frac{\langle df(t, T), df(t, T) \rangle}{dt} = \sigma_1^2(t) \exp \left\{ -2 \int_t^T \kappa_1(u) du \right\} b(t, T, T), \quad (81)$$

it can be shown that if  $\rho_x$  is negative enough, the term structure of the volatility will develop a humped shape.

## 6 The Quasi-Gaussian Models

We are going to consider general multi-factor quasi-Gaussian models, with the separation condition  $\sigma_f(t, T; \omega) = g(t; \omega)h(T)$ , where  $g(t; \omega)$  is a  $d \times d$ -matrix valued function, and  $h(t)$  is a deterministic,  $d$ -dimensional column vector function. We are going to develop efficient approximation for the swap rate, which will also cover the pure Gaussian case.

### 6.1 Approximate swap rate dynamics

Consider the following volatility specification which contains both local and stochastic volatilities,

$$\sigma_r(t; \omega) = \sqrt{z(t)} \sigma_x(t, x(t), y(t)), \quad (82)$$

where  $z(t)$  follows the CIR process

$$dz(t) = \theta(z_0 - z(t))dt + \eta(t)\sqrt{z(t)}dZ(t), \quad z(0) = z_0 = 1, \quad (83)$$

and  $\langle dZ(t), dW(t) \rangle = 0$ .

To proceed with the swap rate dynamics, fix a tenor structure

$$0 < T_0 < T_1 < \dots < T_N \quad (84)$$

with

$$\tau_n = T_{n+1} - T_n, \quad (85)$$

and the swap rate  $S(t)$ , with first fix date at  $T_0$  and last payment date at  $T_N$ , is given by

$$S(t) = \frac{P(t, T_0) - P(t, T_N)}{A(t)}, \quad A(t) = \sum_{n=0}^{N-1} \tau_n P(t, T_{n+1}). \quad (86)$$

Notice the functional dependence of the discount bond  $P(t, T)$  on the state variables  $x$  and  $y$ , *i.e.*,  $P(t, T) \equiv P(t, T, x(t), y(t))$ , the swap rate dynamics is

$$dS(t) = \sqrt{z(t)}(\nabla S)\sigma_x^\top dW(t) + O(dt), \quad (87)$$

where we are only concerned with the drift term. Since the swap rate is a martingale in the annuity measure, its dynamics in the corresponding annuity measure is given by

$$dS(t) = \sqrt{z(t)} \left( (\nabla S) \sigma_x^\top \sigma_x (\nabla S)^\top \right)^{1/2} dW^A(t). \quad (88)$$

Using the Markovian Projection (MP) method, the above process is equivalent to the following local volatility process in terms of European option pricing,

$$dS(t) = \sqrt{z(t)} \varphi(t, S(t)) dW^A(t), \quad (89)$$

where

$$\varphi^2(t, s) = E^A \left[ (\nabla S) c_x (\nabla S)^\top \mid S(t) = s \right], \quad (90)$$

and  $c_x = \sigma_x^\top \sigma_x$ .

We will look for the linear approximation to  $\varphi(t, s)$  in  $s$ , and will use  $E^A(x(t))$  and  $E^A(y(t))$  as expansion point. First, define

$$\sigma_x^0(t) = \sigma_x(t, 0, 0), \quad (91)$$

then from Eq. (14), we have

$$y(t) = H(t) \left( \int_0^t H^{-1}(u) \sigma_r(u)^\top \sigma_r(u) H^{-1}(u) du \right) H(t), \quad (92)$$

and

$$\bar{y}(t) = E^A[y(t)] \approx H(t) \left( \int_0^t H^{-1}(u) (\sigma_x^0(u))^\top \sigma_x^0(u) H^{-1}(u) du \right) H(t). \quad (93)$$

For the approximation of  $x(t)$ , replace  $y(t)$  with  $\bar{y}(t)$  in Eq. (16), which is the Gaussian approximation to  $x(t)$ ,

$$dx_g(t) = \left( \bar{y}(t) I_{d \times 1} - \kappa(t) x_g(t) \right) dt + (\sigma_x^0(t))^\top dW(t). \quad (94)$$

The corresponding dynamics for  $x_g(t)$  in the  $T$ -forward measure, using Eq. (26), is

$$dx_g(t) = \left( \bar{y}(t) I_{d \times 1} - (\sigma_x^0(t))^\top \sigma_x^0(t) G(t, T) - \kappa(t) x_g(t) \right) dt + (\sigma_x^0(t))^\top dW^T(t). \quad (95)$$

The solution to the above SDE is

$$x_g(t) = H(t) \left( \int_0^t H^{-1}(u) \left[ \left( \bar{y}(u) I_{d \times 1} - (\sigma_x^0(u))^\top \sigma_x^0(u) G(u, T) \right) du + (\sigma_x^0(u))^\top dW^T(u) \right] \right), \quad (96)$$

and

$$\begin{aligned} \bar{x}(t) &= E^A[x(t)] \approx E^A[x_g(t)] \\ &= E^Q \left[ \frac{dP^A}{dP} x_g(t) \right] = E^Q \left[ \frac{A(t)/A(0)}{B(t)} x_g(t) \right] \\ &= \frac{1}{A(0)} \sum_{n=0}^{N-1} \tau_n E^Q \left[ \frac{P(t, T_{n+1})}{B(t)} x_g(t) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{A(0)} \sum_{n=0}^{N-1} \tau_n E^{T_{n+1}} \left[ \frac{dP}{dP^{T_{n+1}}} \frac{P(t, T_{n+1})}{B(t)} x_g(t) \right] \\
&= \frac{1}{A(0)} \sum_{n=0}^{N-1} \tau_n E^{T_{n+1}} \left[ \frac{B(t)}{P(t, T_{n+1})/P(0, T_{n+1})} \frac{P(t, T_{n+1})}{B(t)} x_g(t) \right] \\
&= \sum_{n=0}^{N-1} \frac{\tau_n P(0, T_{n+1})}{A(0)} E^{T_{n+1}} [x_g(t)] \\
&= \sum_{n=0}^{N-1} \frac{\tau_n P(0, T_{n+1})}{A(0)} H(t) \int_0^t H^{-1}(u) \left( \bar{y}(u) I_{d \times 1} - (\sigma_x^0(u))^\top \sigma_x^0(u) G(u, T) \right) du. \quad (97)
\end{aligned}$$

Now, we can proceed to develop further approximation for the swap rate dynamics, around  $\bar{x}(t)$  and  $\bar{y}(t)$ . Let us define  $S_g(t)$  to be the Gaussian approximation to  $S(t)$ , and its dynamics will be represented by

$$dS_g(t) = \nabla S(\sigma_x^0)^\top dW^A(t). \quad (98)$$

Expand Eq. (90) to first order in  $x$ ,

$$\begin{aligned}
&\left( (\nabla S)_{c_x} (\nabla S)^\top \right) (t, x(t), y(t)) \approx \left( (\nabla S)_{c_x} (\nabla S)^\top \right) (t, \bar{x}(t), \bar{y}(t)) \\
&\quad + \nabla \left( (\nabla S)_{c_x} (\nabla S)^\top \right) \Big|_{(t, \bar{x}(t), \bar{y}(t))} (x(t) - \bar{x}(t)). \quad (99)
\end{aligned}$$

Then,

$$\begin{aligned}
E^A \left[ (\nabla S)_{c_x} (\nabla S)^\top | S(t) = s \right] &\approx \left( (\nabla S)_{c_x} (\nabla S)^\top \right) (t, \bar{x}(t), \bar{y}(t)) \\
&\quad + \nabla \left( (\nabla S)_{c_x} (\nabla S)^\top \right) \Big|_{(t, \bar{x}(t), \bar{y}(t))} E^A [x(t) - \bar{x}(t) | S(t) = s]. \quad (100)
\end{aligned}$$

Using a Gaussian approximation for the conditional expectation,

$$\begin{aligned}
E^A [x(t) - \bar{x}(t) | S(t) = s] &\approx E^A [x_g(t) - \bar{x}_g(t) | S_g(t) = s] \\
&= \frac{\text{Cov}(S_g(t), x_g(t))}{\text{Var}(S_g(t))} (s - S(0)) \\
&= \frac{c_x (\nabla S)^\top}{(\nabla S)_{c_x} (\nabla S)^\top} (s - S(0)). \quad (101)
\end{aligned}$$

Notice that

$$\nabla \left( (\nabla S)_{c_x} (\nabla S)^\top \right) c_x (\nabla S)^\top = (\nabla S) d_x (\nabla S)^\top + 2(\nabla S)_{c_x} (\nabla^2 S)_{c_x} (\nabla S)^\top, \quad (102)$$

where

$$d_x = \sum_{l=1}^d \left( c_x (\nabla S)^\top \right)_l \left( \frac{\partial \sigma_x^\top}{\partial x_l} \sigma_x + \sigma_x^\top \frac{\partial \sigma_x}{\partial x_l} \right). \quad (103)$$

The local volatility function is

$$\varphi^2(t, s) = (\nabla S)c_x(\nabla S)^\top + \frac{(\nabla S)d_x(\nabla S)^\top + 2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)^\top}{(\nabla S)c_x(\nabla S)^\top} (s - S(0)), \quad (104)$$

and linearizing around  $s \approx S(0)$ ,

$$\begin{aligned} \varphi(t, s) &= \left( (\nabla S)c_x(\nabla S)^\top \right)^{1/2} + \frac{1}{2 \left( (\nabla S)c_x(\nabla S)^\top \right)^{1/2}} \\ &\quad \times \frac{(\nabla S)d_x(\nabla S)^\top + 2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)^\top}{(\nabla S)c_x(\nabla S)^\top} (s - S(0)). \end{aligned} \quad (105)$$

## 6.2 Volatility parameterization

As discussed in the last section, the local volatility function  $\sigma_x(t, x, y)$  still needs to be specified. To this end, we consider  $d$  rolling benchmark rates,  $f_i(t) \equiv f_i(t, t + \delta_i)$ ,  $i = 1, \dots, d$ , with  $\delta_1 < \dots < \delta_d$ . We require that the benchmark rates have the following dynamics,

$$df_i(t) = O(dt) + \sqrt{z(t)}\lambda_i^f(t) \left( \alpha_i^f(t) + b_i^f(t)f_i(t) \right) dU_i(t), \quad (106)$$

with  $U(t)$  is a  $d$ -dimensional Brownian motion, with correlation matrix  $X^f(t) = \{\chi_{i,j}(t)\}$ . Define the decomposition of  $X^f$  as

$$X^f(t) = D^f(t)^\top D^f(t), \quad (107)$$

and  $H^f$  by

$$H^f(t) = \begin{pmatrix} h(t + \delta_1)^\top \\ \vdots \\ h(t + \delta_d)^\top \end{pmatrix}, \quad (108)$$

and  $\sigma^f$  by

$$\sigma^f(t, f(t)) = \text{diag} \left( \lambda_1^f(t) \left( \alpha_1^f(t) + b_1^f(t)f_1(t) \right), \dots, \lambda_d^f(t) \left( \alpha_d^f(t) + b_d^f(t)f_d(t) \right) \right), \quad (109)$$

then we can set

$$\sigma_r(t; \omega)^\top = \sqrt{z(t)}\sigma_x(t, x, y)^\top = \sqrt{z(t)}H(t)H^f(t)^{-1}\sigma^f(t, f(t))D^f(t)^\top, \quad (110)$$

to be consistent with the benchmark rates dynamics. To see this, let  $f(t) = (f_1(t), \dots, f_d(t))^\top$ , and it can be shown from the bond reconstitution formula, Eq. (23),

$$f(t) = f(0) + H^f(t)H(t)^{-1} \left( x(t) + y(t)G(t, T) \right). \quad (111)$$

The dynamics is then given by

$$\begin{aligned} df(t) &= O(dt) + H^f(t)H(t)^{-1}dx(t) \\ &= O(dt) + H(t)H^f(t)^{-1}\sigma_r(t; \omega)^\top dW(t) \\ &= O(dt) + \sqrt{z(t)}\sigma^f(t, f(t))D^f(t)^\top dW(t), \end{aligned} \quad (112)$$

consistent with Eq. (106).

## 6.3 Approximate swap rate dynamics in Gaussian short rate models

## 7 The Affine Model

▽▽

## 8 The Quadratic Gaussian Model

### 8.1 One-Factor Model

Consider the following short rate dynamics,

$$dr(t) = \kappa(t) (\theta(t) - r(t)) dt + \sigma(t) \sqrt{\alpha + \beta r(t)} dW(t), \quad (113)$$

with some technical conditions which will be omitted here. The details can be found in [1].

## 9 The Libor Market Model (LMM)

▽▽

## A Girsanov's Theorem

Consider two measures  $P$  and  $P(\theta)$  related by a density

$$\zeta^\theta(t) = \mathbb{E}_t^P \left[ \frac{dP(\theta)}{dP} \right], \quad (114)$$

where  $\zeta^\theta(t)$  is an exponential martingale given by

$$\frac{d\zeta^\theta(t)}{\zeta^\theta(t)} = -\theta(t)^\top dW(t), \quad (115)$$

where  $W(t)$  is a  $d$ -dimensional  $P$ -Brownian motion. Then,

$$W^\theta(t) = W(t) + \int_0^t \theta(s) ds \quad (116)$$

is a  $P(\theta)$ -Brownian motion, and

$$dW^\theta(t) = dW(t) + \theta(t) dt. \quad (117)$$

## References

- [1] Leif Andersen and Vladimir Piterbarg, *Interest Rate Modeling*, Atlantic Financial Press, 2010.
- [2] Damiano Brigo and Fabio Mercurio, *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*, Springer, 2006.