Notes on Feller Condition

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1 Fokker-Planck Equation

Consider the following parabolic partial differential equation (PDE) [1]

$$\frac{\partial u(t,x)}{\partial t} = -\frac{\partial}{\partial x}\left((bx+c)u(t,x)\right) + \frac{\partial^2}{\partial x^2}\left(axu(t,x)\right), \tag{1}$$

with $x > 0$. This can be viewed as the corresponding Fokker-Planck equation for the Cox-Ingersoll-Ross (CIR) process

$$dx_t = \kappa(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \tag{2}$$

with $a = \sigma^2/2$, $b = -\kappa$, and $c = \kappa\theta$.

We want to find the solution of the PDE (1), with certain initial condition. In particular, we are interested in the fundamental solution of the PDE (1), *i.e.*, the initial condition is given by

$$u(0,x) = \delta(x-\xi), \tag{3}$$

where $\delta(x)$ is the Dirac delta function. To this end, introduce the Laplace transform of $u(t,x)$ as

$$v(t,\lambda) = \int_0^{+\infty} e^{-\lambda x}u(t,x)dx, \tag{4}$$

for $\lambda > 0$. We denote the Laplace transform of the initial condition as

$$\pi(\lambda) = \int_0^{+\infty} e^{-\lambda x}u(0,x)dx, \tag{5}$$

and for the fundamental solution, $\pi(\lambda) = e^{-\lambda\xi}$. Consider the Laplace transform of the right hand side of Eq. (1),

$$\begin{aligned}
&\int_0^{+\infty} e^{-\lambda x}\Big[(axu(t,x))_{xx} - ((bx+c)u(t,x))_x\Big]dx \\
= \quad & e^{-\lambda x}\Big[(axu(t,x))_x - ((bx+c)u(t,x))\Big]\Big|_0^{+\infty} \\
& \qquad +\lambda\int_0^{+\infty} e^{-\lambda x}\Big[(axu(t,x))_x - ((bx+c)u(t,x))\Big]dx \\
= \quad & f(t) + \lambda(b-\lambda a)v_\lambda - c\lambda v,
\end{aligned} \tag{6}$$

where

$$f(t) = \lim_{x\to 0} \left[(bx+c)u - (axu)_x\right], \tag{7}$$

is the flux at $x = 0$ and cannot be arbitrarily specified.

After the Laplace transform, Eq. (1) becomes

$$v_t + \lambda(\lambda a - b)v_\lambda = f(t) - c\lambda v. \tag{8}$$

This first order PDE can be solved by the method of characteristics. The characteristics can be determined by integrating

$$dt = \frac{d\lambda}{\lambda(a\lambda - b)}, \tag{9}$$

which leads to the following characteristic

$$e^{-bt}\frac{a\lambda - b}{\lambda} = C_1, \tag{10}$$

or equivalently,

$$\lambda(t) = -\frac{be^{-bt}}{C_1 - ae^{-bt}}. \tag{11}$$

Then, Eq. (8) becomes an ordinary differential equation,

$$\frac{dv}{dt} - \frac{bce^{-bt}}{C_1 - ae^{-bt}}v = f(t). \tag{12}$$

It can be integrated,

$$v(t,\lambda) = \left|C_1 - ae^{-bt}\right|^{c/a}\left\{C_2 + \int_0^t \frac{f(\tau)}{|C_1 - ae^{-b\tau}|^{c/a}}d\tau\right\}. \tag{13}$$

To determine the unknown constants, assume $C_2 = A(C_1)$, where $A(y)$ is an arbitrary function. For the initial condition $v(0,\lambda) = \pi(\lambda)$, we have

$$\pi(\lambda) = \left|\frac{b}{\lambda}\right|^{c/a} A\left(a - \frac{b}{\lambda}\right), \tag{14}$$

which gives

$$A(y) = |a-y|^{-c/a}\,\pi\left(\frac{b}{a-y}\right). \tag{15}$$

Now, we have

$$v(t,\lambda) = \left|C_1 - ae^{-bt}\right|^{c/a}\left\{|a - C_1|^{-c/a}\,\pi\left(\frac{b}{a-C_1}\right) + \int_0^t \frac{f(\tau)}{|C_1 - ae^{-b\tau}|^{c/a}}d\tau\right\}. \tag{16}$$

Using Eq. (10), the general solution becomes

$$v(t,\lambda) = \left(\frac{b}{\lambda a(e^{bt}-1)+b}\right)^{c/a}\pi\left(\frac{\lambda b e^{bt}}{\lambda a(e^{bt}-1)+b}\right) + \int_0^t f(\tau)\left(\frac{b}{\lambda a(e^{b(t-\tau)}-1)+b}\right)^{c/a}d\tau. \tag{17}$$

2 Fundamental solutions

In the following, we will consider several parameter regimes and the corresponding solutions to the Fokker-Planck equation (1) with initial condition (3).

2.1 $c \leq 0$

We are looking for integrable solutions near $x = 0$, so that u is a probability distribution. To this end, the integrability of u near $x = 0$ is equivalent to $v(t, \lambda) \to 0$ as $\lambda \to +\infty$. Therefore, we must require that $f(t)$ satisfies the following equation,

$$\pi\left(\frac{b}{a(1-e^{-bt})}\right) + \int_0^t f(\tau)\left(\frac{e^{bt}-1}{e^{b(t-\tau)}-1}\right)^{c/a} d\tau = 0. \tag{18}$$

Then, the Fokker-Planck equation (1) with arbitrary initial condition $u(0, x)$ has only one solution corresponding to $f(t)$ in Eq. (18).

For the fundamental solution, we have $\pi(\lambda) = e^{-\lambda\xi}$, and Eq. (18) becomes

$$\int_0^t f(\tau)\left(\frac{e^{bt}-1}{e^{b(t-\tau)}-1}\right)^{c/a} d\tau = -\exp\left(-\frac{b\xi}{a(1-e^{-bt})}\right). \tag{19}$$

Define

$$\frac{1}{z} = 1 - e^{-bt}, \qquad \frac{1}{\zeta} = 1 - e^{-b\tau}, \tag{20}$$

Eq. (19) becomes

$$\int_z^{+\infty} f(\tau)\left(\frac{\zeta}{\zeta - z}\right)^{c/a} \frac{d\zeta}{b\zeta(\zeta-1)} = -\exp\left(-\frac{b\xi z}{a}\right). \tag{21}$$

Let

$$g(\zeta) = f(\tau)\frac{\zeta^{c/a}}{b\zeta(\zeta-1)}, \tag{22}$$

and let $g(\zeta)$ have the following functional form

$$g(\zeta) = Be^{-A\zeta}, \tag{23}$$

then the right hand side of Eq. (21) becomes

$$B\int_z^{+\infty} e^{-A\zeta}(\zeta - z)^{-c/a} d\zeta = \frac{B\Gamma\left(1-\frac{c}{a}\right)}{A^{1-c/a}} e^{-Az}. \tag{24}$$

Matching terms, we can see that

$$A = \frac{b\xi}{a}, \qquad B = -\frac{1}{\Gamma\left(1-\frac{c}{a}\right)}\left(\frac{b\xi}{a}\right)^{1-c/a}. \tag{25}$$

Therefore,

$$f(t) = -\frac{b}{\Gamma\left(1-\dfrac{c}{a}\right)} \frac{e^{-bt}}{1-e^{-bt}} \left(\frac{b\xi}{a(1-e^{-bt})}\right)^{1-c/a} \exp\left(-\frac{b\xi}{a(1-e^{-bt})}\right). \tag{26}$$

With some manipulation, the fundamental solution can be written in the following convenient form,

$$v(t,\lambda) = \left(\frac{b}{\lambda a(e^{bt}-1)+b}\right)^{c/a} \exp\left(-\frac{\lambda b\xi e^{bt}}{\lambda a(e^{bt}-1)+b}\right) \Gamma\left(1-\frac{c}{a}; \frac{b\xi e^{bt}}{a(e^{bt}-1)} \frac{b}{\lambda a(e^{bt}-1)+b}\right), \tag{27}$$

where

$$\Gamma(n;z) = \frac{1}{\Gamma(n)} \int_0^z e^{-x} x^{n-1} dx \tag{28}$$

is the incomplete Gamma function. From this, we can see that

$$\int_0^{+\infty} u(t,x)dx = \lim_{\lambda\to 0} v(t,\lambda) = \Gamma\left(1-\frac{c}{a}; \frac{\lambda b\xi e^{bt}}{a(e^{bt}-1)}\right), \tag{29}$$

which is strictly less than 1. This means that the $x=0$ boundary serves as an absorbing boundary, and once it is hit, the solution will stay there.

We can also find the explicit form of the fundamental solution via inverse Laplace transform. Again, let

$$A = \frac{b\xi e^{bt}}{a(e^{bt}-1)}, \qquad \frac{1}{z} = \frac{b}{\lambda a(e^{bt}-1)+b}, \tag{30}$$

the incomplete Gamma function in Eq. (27) can be expressed as

$$\begin{aligned} \frac{1}{\Gamma\left(1-\dfrac{c}{a}\right)} \int_0^{A/z} e^{-x} x^{n-1} dx &= \frac{1}{\Gamma\left(1-\dfrac{c}{a}\right)} \left(\frac{A}{z}\right)^{1-c/a} \int_0^1 e^{-Au/z} u^{-c/a} du \\ &= \frac{1}{\Gamma\left(1-\dfrac{c}{a}\right)} \left(\frac{A}{z}\right)^{1-c/a} e^{-A/z} \int_0^1 e^{Av/z} (1-v)^{-c/a} dv. \end{aligned} \tag{31}$$

Then,

$$v(t,\lambda) = \frac{e^{-A} A^{1-c/a}}{\Gamma\left(1-\dfrac{c}{a}\right)} \int_0^1 (1-v)^{-c/a} \frac{e^{Av/z}}{z} dv. \tag{32}$$

The inverse Laplace transform can be applied to $e^{Av/z}/z$, which yields

$$\begin{aligned} \int e^{\lambda x} \frac{e^{Av/z}}{z} d\lambda &= \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{b}{a(e^{bt}-1)}\right) \int \exp\left(\frac{bxz}{a(e^{bt}-1)}\right) \frac{e^{Av/z}}{z} dz \\ &= \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{bx}{a(e^{bt}-1)}\right) I_0\left(2\left(\frac{Avbx}{a(e^{bt}-1)}\right)^{1/2}\right) \\ &= \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{bx}{a(e^{bt}-1)}\right) I_0\left(\frac{2b}{a(e^{bt}-1)} \left(e^{bt}\xi v x\right)^{1/2}\right), \end{aligned} \tag{33}$$

where we have used the fact that the inverse Laplace transform of $e^{Av/z}/z$ is $I_0(2\sqrt{Avx})$, where $I_\nu(x)$ is the modified Bessel function of order ν. Finally, we have

$$\begin{aligned} u(t,x) &= \frac{1}{\Gamma\left(1-\frac{c}{a}\right)} \frac{b}{a(e^{bt}-1)} \left(\frac{b\xi}{a(1-e^{-bt})}\right)^{1-c/a} \exp\left(-\frac{b(x+\xi e^{bt})}{a(e^{bt}-1)}\right) \\ &\quad \times \int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi v x\right)^{1/2}\right) dv. \end{aligned} \tag{34}$$

Using the series expansion of the modified Bessel function,

$$I_\nu(x) = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r+1+\nu)} \left(\frac{x}{2}\right)^{2r+\nu}, \tag{35}$$

the integral in (34) can be written as

$$\begin{aligned} &\int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi v x\right)^{1/2}\right) dv \\ &= \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r+1)} \left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right)^{2r} \int_0^1 (1-v)^{-c/a} v^r dv \\ &= \Gamma\left(1-\frac{c}{a}\right) \sum_{r=0}^{\infty} \frac{1}{r!\Gamma\left(r-\frac{c}{a}+2\right)} \left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right)^{2r} \\ &= \Gamma\left(1-\frac{c}{a}\right) \left(\frac{b}{a(1-e^{-bt})}\right)^{-1+c/a} \left(e^{-bt}\xi x\right)^{(c-a)/2a} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right). \end{aligned} \tag{36}$$

Then,

$$u(t,x) = \frac{b}{a(e^{bt}-1)} \exp\left(-\frac{b(x+\xi e^{bt})}{a(e^{bt}-1)}\right) \left(e^{-bt}\frac{x}{\xi}\right)^{(c-a)/2a} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right). \tag{37}$$

2.2 $0 < c \le a$

For $0 < c \le a$, there are infinitely many solutions. We may impose the absorbing boundary condition as in the last section, and obtain the same fundamental solution.

We can also impose the reflecting boundary condition on Eq. (17), $f(t) \equiv 0$, which will render a positive and norm preserving solution. To find the explicit form of the fundamental solution, we can apply the inverse Laplace transform on Eq. (17), with $f(t) \equiv 0$. Use the same substitution as in Eq. (30), we have

$$v(t,\lambda) = z^{-c/a} \exp\left(-b\xi + \frac{A}{z}\right). \tag{38}$$

The same manipulation of the inverse Laplace transform as in last section leads to

$$\begin{aligned}\int e^{\lambda x}\frac{e^{A/z}}{z^{c/a}}d\lambda &= \frac{b}{a(e^{bt}-1)}\int \exp\left(\frac{bxz}{a(e^{bt}-1)}\right)\frac{e^{A/z}}{z^{c/a}}dz \\ &= \frac{b}{a(e^{bt}-1)}\left(\frac{bx}{Aa(e^{bt}-1)}\right)^{(c-a)/2a} I_{-1+c/a}\left(2\left(\frac{Abx}{a(e^{bt}-1)}\right)^{1/2}\right) \\ &= \frac{b}{a(e^{bt}-1)}\left(e^{-bt}\frac{x}{\xi}\right)^{(c-a)/2a} I_{-1+c/a}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right), \qquad (39)\end{aligned}$$

where we have use the fact that the inverse Laplace transform of $e^{A/z}/z^{1+\nu}$ is $(x/A)^{\nu/2}I_\nu(2\sqrt{Ax})$, and the fundamental solution is then given by

$$u(t,x) = \frac{be^{-b\xi}}{a(e^{bt}-1)}\left(e^{-bt}\frac{x}{\xi}\right)^{(c-a)/2a} I_{-1+c/a}\left(\frac{2b}{a(1-e^{-bt})}\left(e^{-bt}\xi x\right)^{1/2}\right). \qquad (40)$$

2.3 $c > a$

3 CEV model

4 Simulation of the CIR process

References

[1] W. Feller, *Two Singular Diffusion Problems*, Annals of Mathematics **54**, 173 (1951).