Notes on Feller Condition

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1 Fokker-Planck Equation

Consider the following parabolic partial differential equation (PDE) [1]

$$\frac{\partial u(t,x)}{\partial t} = -\frac{\partial}{\partial x}\left((bx+c)u(t,x)\right) + \frac{\partial^2}{\partial x^2}\left(axu(t,x)\right), \tag{1}$$

with $x > 0$. This can be viewed as the corresponding Fokker-Planck equation for the Cox-Ingersoll-Ross (CIR) process

$$dx_t = \kappa(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \tag{2}$$

with $a = \sigma^2/2$, $b = -\kappa$, and $c = \kappa\theta$.

We want to find the solution of the PDE (1), with certain initial condition. In particular, we are interested in the fundamental solution of the PDE (1), *i.e.*, the initial condition is given by

$$u(0,x) = \delta(x-y), \tag{3}$$

where $\delta(x)$ is the Dirac delta function. To this end, introduce the Laplace transform of $u(t,x)$ as

$$v(t,\lambda) = \int_0^{+\infty} e^{-\lambda x} u(t,x)dx, \tag{4}$$

for $\lambda > 0$. We denote the Laplace transform of the initial condition as

$$\pi(\lambda) = \int_0^{+\infty} e^{-\lambda x} u(0,x)dx, \tag{5}$$

and for the fundamental solution, $\pi(\lambda) = e^{-\lambda y}$. Consider the Laplace transform of the right hand side of Eq. (1),

$$\begin{aligned}
& \int_0^{+\infty} e^{-\lambda x}\Big[(axu(t,x))_{xx} - ((bx+c)u(t,x))_x\Big]dx \\
= \quad & e^{-\lambda x}\Big[(axu(t,x))_x - ((bx+c)u(t,x))\Big]\Big|_0^{+\infty} \\
& \qquad +\lambda \int_0^{+\infty} e^{-\lambda x}\Big[(axu(t,x))_x - ((bx+c)u(t,x))\Big]dx \\
= \quad & f(t) + \lambda(b - \lambda a)v_\lambda - c\lambda v,
\end{aligned} \tag{6}$$

where

$$f(t) = \lim_{x\to 0} \left[(bx+c)u - (axu)_x\right], \tag{7}$$

is the flux at $x=0$ and cannot be arbitrarily specified.

After the Laplace transform, Eq. (1) becomes

$$v_t + \lambda(\lambda a - b)v_\lambda = f(t) - c\lambda v. \tag{8}$$

This first order PDE can be solved by the method of characteristics. The characteristics can be determined by integrating

$$dt = \frac{d\lambda}{\lambda(a\lambda - b)}, \tag{9}$$

which leads to

$$e^{-bt}\frac{a\lambda - b}{\lambda} = C_1, \tag{10}$$

with $C_1 = a - b/\lambda$. Then, Eq. (8) can be integrated,

$$v(t,\lambda) = \left|C_1 - ae^{-bt}\right|^{c/a}\left\{C_2 + \int_0^t \frac{f(\tau)}{\left|C_1 - ae^{-bt}\right|^{c/a}}\right\}. \tag{11}$$

Together with the initial condition, the general solution becomes

$$v(t,\lambda) = \left(\frac{b}{\lambda a(e^{bt}-1)+b}\right)^{c/a}\pi\left(\frac{\lambda b e^{bt}}{\lambda a(e^{bt}-1)+b}\right) + \int_0^t f(\tau)\left(\frac{b}{\lambda a(e^{b(t-\tau)}-1)+b}\right)^{c/a} d\tau. \tag{12}$$

References

[1] W. Feller, *Two Singular Diffusion Problems*, Annals of Mathematics **54**, 173 (1951).