

# Notes on HJM Framework

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## 1 Formulation

Let  $P(t, T)$  denotes the time  $t$  price of a zero coupon bond which pays 1 at time  $T$ , *i.e.*,

$$P(t, T) = E_t^Q \left[ \exp \left( - \int_t^T r(u) du \right) \right], \quad (1)$$

where  $Q$  stands for the risk neutral measure, and  $r(t)$  is the short rate. Then, under the same measure [1],

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma_P(t, T)^\top dW(t), \quad (2)$$

where  $W$  is a  $d$ -dimensional Brownian motion,  $\sigma_P(t, T)$  is a  $d$ -dimensional stochastic process adapted to the filtration generated by  $W$ , and  $\top$  represents matrix transposition. Here,  $\sigma_P(t, T)$  must satisfy the consistency condition,  $\sigma_P(T, T) = 0$ , since  $P(T, T) = 1$ . From Eq. (2), we have

$$P(t, T) = \exp \left( - \int_t^T r(u) du + \int_t^T \sigma_P(u, T)^\top dW(u) + \frac{1}{2} \int_t^T \sigma_P(u, T)^\top \sigma_P(u, T) du \right), \quad (3)$$

or, equivalently,

$$\begin{aligned} \frac{B(t)}{B(T)} &= \exp \left( - \int_t^T r(u) du \right) \\ &= P(t, T) \exp \left( - \int_t^T \sigma_P(u, T)^\top dW(u) - \frac{1}{2} \int_t^T \sigma_P(u, T)^\top \sigma_P(u, T) du \right), \end{aligned} \quad (4)$$

where  $B(t) = \exp \left( \int_0^t r(u) du \right)$  is the numeraire of the risk neutral measure.

Define the forward discount bond price as

$$P(t, T, T + \tau) = \frac{P(t, T + \tau)}{P(t, T)}, \quad (5)$$

and apply the Ito's lemma, then

$$\frac{dP(t, T, T + \tau)}{P(t, T, T + \tau)} = - [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] \sigma_P(t, T) dt - [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW(t), \quad (6)$$

under the risk neutral measure. This leads to a convenient form of the bond reconstitution formula,

$$P(T, T + \tau) = \frac{P(t, T + \tau)}{P(t, T)} \exp \left( - \int_t^T [\sigma_P(u, T + \tau)^\top - \sigma_P(u, T)^\top] dW(u) - \frac{1}{2} \int_t^T [\sigma_P(u, T + \tau)^\top \sigma_P(u, T + \tau) - \sigma_P(u, T)^\top \sigma_P(u, T)] du \right). \quad (7)$$

Under the  $T$ -forward measure, the forward discount bond price is a martingale, which means

$$\frac{dP(t, T, T + \tau)}{P(t, T, T + \tau)} = - [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW^T(t). \quad (8)$$

Define the forward rate as

$$f(t, T) = - \frac{\partial \log P(t, T)}{\partial T}, \quad (9)$$

then from Eq. (2), the forward rate has the following dynamics,

$$df(t, T) = \sigma_f(t, T)^\top \sigma_P(t, T) dt + \sigma_f(t, T)^\top dW(t), \quad (10)$$

where  $\sigma_f(t, T) = \partial \sigma_P(t, T) / \partial T$ , or equivalently,  $\sigma_P(t, T) = \int_t^T \sigma_f(t, u) du$ .

## 2 Markovian Short Rate Dynamics

From Eq. (10), the forward rate is given by

$$f(t, T) = f(0, T) + \int_0^t \sigma_f(u, T)^\top \left( \int_u^T \sigma_f(u, s) ds \right) du + \int_0^t \sigma_f(u, T)^\top dW(u). \quad (11)$$

Define the short rate as  $r(t) = f(t, t)$ , then the short rate dynamics becomes

$$r(t) = f(0, t) + \int_0^t \sigma_f(u, t)^\top \left( \int_u^t \sigma_f(u, s) ds \right) du + \int_0^t \sigma_f(u, t)^\top dW(u). \quad (12)$$

Generally, the short rate process is not Markovian. However, after imposing the separation condition  $\sigma_f(t, T) = g(t)h(T)$ , where  $g(t) \equiv g(t; \omega)$  is a  $d \times d$  matrix valued function, which can be deterministic or stochastic, and  $h$  is a deterministic  $d$ -dimensional column vector function, the short rate will be Markovian. Under this condition,

$$r(t) = f(0, t) + h(t)^\top \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du + h(t)^\top \int_0^t g(u)^\top dW(u). \quad (13)$$

To cast the short rate dynamics into a convenient form, define

$$H(t) = \text{diag}(h_1(t), \dots, h_d(t)) = \begin{pmatrix} h_1(t) & & 0 \\ & \ddots & \\ 0 & & h_d(t) \end{pmatrix}, \quad (14)$$

and

$$\kappa(t) = -\frac{dH(t)}{dt}H^{-1}(t). \quad (15)$$

Here, we have assumed that none of the  $h_i$  is zero. Later it will be shown that the diagonal elements of  $\kappa(t)$  are the mean reversion parameter for the short rate dynamics. Now, define

$$x(t) = H(t) \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du + H(t) \int_0^t g(u)^\top dW(u), \quad (16)$$

and

$$y(t) = H(t) \left( \int_0^t g(u)^\top g(u) du \right) H(t), \quad (17)$$

then it can be shown

$$dy(t) = \left( \sigma_r(t; \omega)^\top \sigma_r(t; \omega) - \kappa(t)y(t) - y(t)\kappa(t) \right) dt, \quad (18)$$

with  $\sigma_r(t; \omega) = g(t; \omega)H(t)$ , which is locally deterministic and can be solved numerically and analytically, if possible. Also,

$$\begin{aligned} dx(t) &= \left[ \frac{dH(t)}{dt} \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du \right] dt + \left[ H(t) \left( \int_0^t g(u)^\top g(u) du \right) h(t) dt \right] \\ &\quad + \frac{dH(t)}{dt} \left( \int_0^t g(u)^\top dW(u) \right) dt + H(t)g(t)^\top dW(t) \\ &= \left( y(t)I_{d \times 1} - \kappa(t)x(t) \right) dt + H(t)g(t)^\top dW(t) \\ &= \left( y(t)I_{d \times 1} - \kappa(t)x(t) \right) dt + \sigma_r(t; \omega)^\top dW(t), \end{aligned} \quad (19)$$

where  $I_{d \times 1} = (1, 1, \dots, 1)^\top$ . The  $x$  and  $y$  processes have initial conditions  $x(0) = 0$  and  $y(0) = 0$ , which is evident from their definitions. It can be shown, by Ritchken and Sankarasubramanian, that any interest rate derivative will be completely determined by the two-state Markovian process  $(x(t), y(t))$ .

Using the above parameterizations, it can be shown, for forward rate,

$$\begin{aligned} f(t, T) &= f(0, T) + I_{d \times 1}^\top H(T)^\top \int_0^t g(u)^\top g(u) \left( \int_u^T h(s) ds \right) du + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top dW(u) \\ &= f(0, T) + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top g(u) \left( \int_u^T h(s) ds \right) du + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top dW(u) \\ &\quad + I_{d \times 1}^\top H(T) \int_0^t g(u)^\top g(u) \left( \int_t^T h(s) ds \right) du \\ &= f(0, T) + I_{d \times 1}^\top H(T)H^{-1}(t) \left( H(t) \int_0^t g(u)^\top g(u) \left( \int_u^t h(s) ds \right) du \right. \\ &\quad \left. + H(t) \int_0^t g(u)^\top dW(u) \right) \end{aligned}$$

$$\begin{aligned}
& + I_{d \times 1}^\top H(T) H^{-1}(t) H(t) \int_0^t g(u)^\top g(u) \left( \int_t^T h(s) ds \right) du \\
& = f(0, T) + M(t, T)^\top \left( x(t) + y(t) \int_t^T M(t, u) du \right),
\end{aligned} \tag{20}$$

where

$$M(t, T) = H(T) H^{-1}(t) I_{d \times 1}. \tag{21}$$

From this, the short rate is given by

$$r(t) = f(t, t) = f(0, t) + I_{d \times 1}^\top x(t) = f(0, t) + \sum_{i=1}^d x_i(t). \tag{22}$$

From the above result, we can also derive the bond reconstitution formula. Notice  $P(t, T) = \exp \left( - \int_t^T f(t, u) du \right)$ , we have

$$\begin{aligned}
P(t, T) &= \exp \left( - \int_t^T f(0, u) du - \left( \int_t^T M(t, u)^\top du \right) x(t) \right. \\
&\quad \left. - \int_t^T M(t, u)^\top y(t) \left( \int_t^u M(t, s) ds \right) du \right).
\end{aligned} \tag{23}$$

Define

$$G(t, T) = \int_t^T M(t, u) du, \tag{24}$$

and notice that

$$\int_t^T M(t, u)^\top y(t) \left( \int_t^u M(t, s) ds \right) du = \int_t^T M(t, u)^\top y(t) \left( \int_u^T M(t, s) ds \right) du, \tag{25}$$

the bond reconstitution formula can be rewritten as

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( -G(t, T)^\top x(t) - \frac{1}{2} G(t, T)^\top y(t) G(t, T) \right). \tag{26}$$

The density relating the  $T$ -forward measure and the risk neutral measure is given by

$$\zeta(t) = E_t^Q \left[ \frac{dP^T}{dP} \right] = \frac{P(t, T)/P(0, T)}{B(t)/B(0)}. \tag{27}$$

This density is a martingale in the risk neutral measure, and therefore,

$$\frac{d\zeta(t)}{\zeta(t)} = -G(t, T)^\top \sigma_r(t)^\top dW(t). \tag{28}$$

The Brownian motion under the  $T$ -forward, following the Girsanov's theorem (see Appendix A), is then given by

$$dW^T(t) = dW(t) + \sigma_r(t) G(t, T) dt. \tag{29}$$

### 3 Closed Form Results for Gaussian Models

In the above formulation, when  $\sigma_f(t, T)$  is a deterministic function, the resulting models will be Gaussian, where the distribution of the forward rate, and the short rate, will be Gaussian. This leads to simplifications to the modeling and pricing of interest rate derivatives, and even closed form expressions for several simple derivatives.

#### 3.1 Zero Coupon Bond Option

Consider a call option with expiry  $T$  and strike  $K$  on a zero coupon bond with maturity  $T^*$ . At expiry, the option pays

$$V(T) = (P(T, T^*) - K)^+. \quad (30)$$

Using the change of measure trick, the value of the zero coupon bond becomes

$$V(t) = E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) V(T) \right] = P(t, T) E_t^T \left[ (P(T, T, T^*) - K)^+ \right], \quad (31)$$

under the  $T$ -forward measure. Then, apply the Black-Scholes formula and take into account of Eq. (8),

$$\begin{aligned} V(t) &= P(t, T) [P(t, T, T^*)N(d_+) - KN(d_-)] \\ &= P(t, T^*)N(d_+) - KP(t, T)N(d_-), \end{aligned} \quad (32)$$

where

$$d_{\pm} = \frac{\log \left( \frac{P(t, T, T^*)}{K} \right) \pm \frac{v}{2}}{\sqrt{v}} = \frac{\log \left( \frac{P(t, T^*)}{KP(t, T)} \right) \pm \frac{v}{2}}{\sqrt{v}}, \quad (33)$$

and

$$v = \int_t^T |\sigma_P(s, T^*) - \sigma_P(s, T)|^2 ds. \quad (34)$$

#### 3.2 Caplet

Now consider a caplet which resets at time  $T$ , and pays at  $T + \tau$  the amount

$$V(T + \tau) = \tau (L(T, T + \tau) - K)^+, \quad (35)$$

where  $L(T, T + \tau)$  is the LIBOR rates spanning from  $T$  to  $T + \tau$ , and  $\tau$  is the year fraction. The value of the caplet at time  $t$  is

$$V(t) = E_t^Q \left[ \exp \left( - \int_t^{T+\tau} r(s) ds \right) V(T + \tau) \right]. \quad (36)$$

Use the tower rule,

$$\begin{aligned} V(t) &= E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) E_T^Q \left[ \exp \left( - \int_T^{T+\tau} r(s) ds \right) V(T + \tau) \right] \right] \\ &= E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) P(T, T + \tau) V(T + \tau) \right]. \end{aligned} \quad (37)$$

Notice that  $L(T, T + \tau) = \tau^{-1}(1/P(T, T + \tau) - 1)$ , the above expression becomes

$$\begin{aligned} V(t) &= E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) (1 - (1 + \tau K) P(T, T + \tau))^+ \right] \\ &= (1 + \tau K) E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \left( \frac{1}{1 + \tau K} - P(T, T + \tau) \right)^+ \right]. \end{aligned} \quad (38)$$

Now, the caplet price has been turned into a put option on a zero coupon bond. Using the earlier result, the caplet price is

$$\begin{aligned} V(t) &= (1 + \tau K) P(t, T) \left( \frac{1}{1 + \tau K} N(-d_-) - P(T, T + \tau) N(-d_+) \right) \\ &= P(t, T) N(-d_-) - (1 + \tau K) P(t, T + \tau) N(-d_+), \end{aligned} \quad (39)$$

where

$$d_{\pm} = \frac{\log \left( \frac{(1 + \tau K) P(t, T + \tau)}{P(t, T)} \right) \pm \frac{v}{2}}{\sqrt{v}}, \quad (40)$$

and

$$v = \int_t^T |\sigma_P(s, T + \tau) - \sigma_P(s, T)|^2 ds. \quad (41)$$

### 3.3 Futures

The futures rate is a martingale under the risk neutral measure, *i.e.*,  $F(t, T, T + \tau) = E_t^Q[L(T, T, T + \tau)]$ . This is equivalent to

$$F(t, T, T + \tau) = \frac{1}{\tau} E_t^Q \left[ \frac{1}{P(T, T + \tau)} - 1 \right]. \quad (42)$$

The dynamics of  $G(t) = 1/P(t, T, T + \tau) = P(t, T)/P(t, T + \tau)$  under the risk neutral measure is

$$\begin{aligned} d \left( \frac{1}{P(t, T, T + \tau)} \right) &= - \frac{dP(t, T, T + \tau)}{P^2(t, T, T + \tau)} + \frac{(dP(t, T, T + \tau))^2}{P^3(t, T, T + \tau)} \\ &= \frac{1}{P(t, T, T + \tau)} \left( [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] \sigma_P(t, T) dt \right. \\ &\quad \left. + [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW(t) \right) \\ &\quad + \frac{1}{P(t, T, T + \tau)} [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] [\sigma_P(t, T + \tau) - \sigma_P(t, T)] dt \\ &= \frac{1}{P(t, T, T + \tau)} \left( [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] \sigma_P(t, T + \tau) dt \right. \\ &\quad \left. + [\sigma_P(t, T + \tau)^\top - \sigma_P(t, T)^\top] dW(t) \right). \end{aligned} \quad (43)$$

Therefore,

$$E_t^Q \left[ \frac{1}{P(T, T + \tau)} \right] = \frac{1}{P(t, T, T + \tau)} e^{\Omega(t, T)}, \quad (44)$$

where

$$\Omega(t, T) = \int_t^T [\sigma_P(u, T + \tau)^\top - \sigma_P(u, T)^\top] \sigma_P(u, T + \tau) du. \quad (45)$$

So, the futures rate is

$$\begin{aligned} F(t, T, T + \tau) &= \frac{1}{\tau} \left( \frac{1}{P(t, T, T + \tau)} e^{\Omega(t, T)} - 1 \right) \\ &= L(t, T, T + \tau) + \left( \frac{1}{\tau} + L(t, T, T + \tau) \right) (e^{\Omega(t, T)} - 1), \end{aligned} \quad (46)$$

where the last term is the futures convexity adjustment in Gaussian short rate models.

The same result can also be obtained by direct application of the bond reconstitution formula, Eq. (7). For a futures rate fixing at  $T_0$  for the time period between  $T$  and  $T + \tau$ , we have

$$F(t, T, T + \tau) = E_t^Q [L(T_0, T, T + \tau)] = E_t^Q \left[ \frac{1}{\tau} \left( \frac{P(T_0, T)}{P(T_0, T + \tau)} - 1 \right) \right]. \quad (47)$$

From Eq. (7), we have

$$\begin{aligned} \frac{P(T_0, T)}{P(T_0, T + \tau)} &= \frac{P(t, T)}{P(t, T + \tau)} \exp \left( - \int_t^{T_0} [\sigma_P(u, T)^\top - \sigma_P(u, T + \tau)^\top] dW(u) \right. \\ &\quad \left. - \frac{1}{2} \int_t^{T_0} [\sigma_P(u, T)^\top \sigma_P(u, T) - \sigma_P(u, T + \tau)^\top \sigma_P(u, T + \tau)] du \right). \end{aligned} \quad (48)$$

Notice that

$$\begin{aligned} &E \left[ \exp \left( - \int_t^{T_0} [\sigma_P(u, T)^\top - \sigma_P(u, T + \tau)^\top] dW(u) \right) \right] \\ &= \exp \left( \frac{1}{2} \int_t^{T_0} [\sigma_P(u, T)^\top - \sigma_P(u, T + \tau)^\top] [\sigma_P(u, T) - \sigma_P(u, T + \tau)] du \right), \end{aligned}$$

then we will obtain the same result, except that the spot lag has been explicitly included [4].

### 3.4 Option on futures

### 3.5 European Swaption - Jamshidian Decomposition

The above results do not make any specific assumptions besides of the Gaussianness, and can be applied to general models with deterministic forward rate volatility functions. The trick, Jamshidian decomposition, which will be introduced in the following relies on the monotonicity of the zero coupon bond price on short rate.

Consider a payer swaption expiring at  $T_0$ , with the underlying swap paying annualized coupon  $c$  at times  $T_1 < T_2 < \dots < T_N$ , with  $T_1 > T_0$ . The swaption payout at  $T_0$  is

$$V(T_0) = \left( 1 - P(T_0, T_N) - c \sum_{i=1}^N \tau_i P(T_0, T_i) \right)^+, \quad (49)$$

where  $\tau_i = T_i - T_{i-1}$  is the accrual period. Notice the dependence of the zero coupon bond price on  $x$ , *i.e.*,  $P(T_0, T_N) \equiv P(T_0, T_N; x(T_0))$ , through the bond reconstitution formula, Eq. (26). Define a critical value  $x^*$  by setting the value of the swaption at time  $T_0$  to exactly zero,

$$P(T_0, T_N; x^*) + c \sum_{i=1}^N \tau_i P(T_0, T_i; x^*) = 1. \quad (50)$$

Also define the “strikes” as

$$K_i = P(T_0, T_i; x^*), \quad (51)$$

then

$$K_N + c \sum_{i=1}^N \tau_i K_i = 1. \quad (52)$$

Notice that  $P(T_0, T_i; x(T_0))$  is monotonically decreasing in  $x(T_0)$ , which always holds due to the bond reconstitution formula (26), we have

$$\begin{aligned} V(T_0) &= \left( 1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^N \tau_i P(T_0, T_i; x(T_0)) \right)^+ \\ &= \left( 1 - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^N \tau_i P(T_0, T_i; x(T_0)) \right) \mathbf{1}_{x(T_0) > x^*} \\ &= \left( K_N + c \sum_{i=1}^N \tau_i K_i - P(T_0, T_N; x(T_0)) - c \sum_{i=1}^N \tau_i P(T_0, T_i; x(T_0)) \right) \mathbf{1}_{x(T_0) > x^*} \\ &= (K_N - P(T_0, T_N; x(T_0))) \mathbf{1}_{x(T_0) > x^*} + c \sum_{i=1}^N \tau_i (K_i - P(T_0, T_i; x(T_0))) \mathbf{1}_{x(T_0) > x^*} \\ &= (K_N - P(T_0, T_N; x(T_0)))^+ + c \sum_{i=1}^N \tau_i (K_i - P(T_0, T_i; x(T_0)))^+. \end{aligned} \quad (53)$$

Then, this can be evaluated with the zero coupon bond option price formula, Eq. (32). This trick can also be applied to coupon bond options.

## 4 Examples of Classical Gaussian Short Rate Models - One Factor Models

By setting  $d = 1$  in the above formulation, we will obtain one-factor models. Since the matrix products will be reduced to scalar products, this will greatly simplify the discussion.



## 4.1 Ho-Lee Model

The Ho-Lee model can be recovered by taking  $h(t) = 1$ , and  $g(t) = \sigma$  which is constant. It is straightforward to show that  $\kappa(t) = 0$ ,  $y(t) = \sigma^2 t$ ,

$$dx(t) = \sigma^2 t dt + \sigma dW(t), \quad (54)$$

and  $r(t) = f(0, t) + x(t)$ . Also, the forward rate volatility is  $\sigma_f(t, T) = \sigma$ , which is constant too,  $\sigma_P(t, T) = \sigma(T - t)$ , and the bond reconstitution formula is

$$\begin{aligned} P(t, T) &= \frac{P(0, T)}{P(0, t)} \exp \left( -x(t)(T - t) - \frac{1}{2} \sigma^2 (T - t)^2 t \right) \\ &= \frac{P(0, T)}{P(0, t)} \exp \left( -(r(t) - f(0, t))(T - t) - \frac{1}{2} \sigma^2 (T - t)^2 t \right). \end{aligned} \quad (55)$$

## 4.2 Hull-White Model

Take  $h(t) = \exp \left( -\int_0^t \kappa(u) du \right)$ , and  $g(t) = \sigma(t) \exp \left( \int_0^t \kappa(u) du \right)$ , then we have

$$\frac{dy(t)}{dt} = \sigma^2(t) - 2\kappa(t)y(t), \quad (56)$$

which has the following analytical solution,

$$y(t) = \int_0^t \sigma^2(u) \exp \left( -2 \int_u^t \kappa(s) ds \right) du, \quad (57)$$

Then,

$$dx(t) = (y(t) - \kappa(t)x(t)) dt + \sigma(t) dW(t), \quad (58)$$

which can be shown to have the following solution,

$$x(t) = \int_0^t y(u) \exp \left( -\int_u^t \kappa(s) ds \right) du + \int_0^t \sigma(u) \exp \left( -\int_u^t \kappa(s) ds \right) dW(u). \quad (59)$$

Also,

$$\sigma_f(t, T) = g(t)h(T) = \sigma(t) \exp \left( -\int_t^T \kappa(u) du \right), \quad (60)$$

and

$$\sigma_P(t, T) = \int_t^T \sigma_f(t, u) du = \int_t^T \sigma(t) \exp \left( -\int_t^u \kappa(s) ds \right) du. \quad (61)$$

Moreover,

$$M(t, T) = \frac{h(T)}{h(t)} = \exp \left( -\int_t^T \kappa(u) du \right), \quad (62)$$

and

$$G(t, T) = \int_t^T M(t, u) du = \int_t^T \exp \left( -\int_t^u \kappa(s) ds \right) du. \quad (63)$$

The bond reconstitution formula follows Eq. (26).

The Vasicek model can be recovered by setting  $\kappa$  and  $\sigma$  to constants, and the mathematical details will not be reproduced here.

### 4.3 Mercurio-Moraleda Model

The Mercurio-Moraleda model [2] can be obtained by setting

$$\kappa(t) = \lambda - \frac{\gamma}{1 + \gamma t}, \quad (64)$$

then

$$h(t) = (1 + \gamma t) e^{-\lambda t}, \quad (65)$$

and

$$g(t) = \sigma \frac{e^{\lambda t}}{1 + \gamma t}. \quad (66)$$

The forward rate volatility is given by

$$\sigma_f(t, T) = g(t)h(T) = \sigma \frac{1 + \gamma T}{1 + \gamma t} e^{-\lambda(T-t)}, \quad (67)$$

which satisfies the separability condition, and the short rate dynamics will be Markovian. Moreover, this volatility specification can also generate a humped shape in the volatility term structure, which is often observed in the cap market. However, the above dynamics is not time-stationary. This can be addressed by expanding the forward volatility to the first order in  $\gamma$ , leading to the following expression,

$$\sigma_f(t, T) = \sigma \left[ 1 + \gamma(T - t) \right] e^{-\lambda(T-t)}, \quad (68)$$

sacrificing the Markov property of the short rate dynamics. In the following, we will focus on the volatility structure specified by Eq. (67).

From Eq. (17), we have

$$\begin{aligned} y(t) &= \int_0^t \sigma^2 \exp \left( -2 \int_u^t \kappa(s) ds \right) du \\ &= \sigma^2 \int_0^t \left( \frac{1 + \gamma t}{1 + \gamma u} \right)^2 e^{-2\lambda(t-u)} du. \end{aligned} \quad (69)$$

Notice that

$$\int \frac{e^{\lambda t}}{1 + \gamma t} dt = \frac{e^{-\lambda/\gamma}}{\gamma} \text{Ei} \left( \frac{\lambda}{\gamma} (1 + \gamma t) \right) + \text{const.}, \quad (70)$$

where

$$\text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt \quad (71)$$

is the exponential integral function, it can be shown that

$$\begin{aligned} y(t) &= \frac{\sigma^2(1 + \gamma t)}{\gamma^2} \left[ 2\lambda(1 + \gamma t) \exp \left( \frac{2\lambda}{\gamma} (1 + \gamma t) \right) \text{Ei} \left( \frac{2\lambda}{\gamma} (1 + \gamma t) \right) - \gamma \right] \\ &\quad + \frac{\sigma^2(1 + \gamma t)^2}{\gamma^2} \left[ 2\lambda \exp \left( \frac{2\lambda}{\gamma} (1 + \gamma t) \right) \text{Ei} \left( \frac{2\lambda}{\gamma} \right) - \gamma e^{-2\lambda t} \right]. \end{aligned} \quad (72)$$

Also,

$$M(t, T) = \frac{h(T)}{h(t)} = \frac{1 + \gamma T}{1 + \gamma t} e^{-\lambda(T-t)}, \quad (73)$$

and

$$\begin{aligned} G(t, T) &= \int_t^T M(t, u) du = \frac{e^{\lambda t}}{1 + \gamma t} \int_t^T (1 + \gamma u) e^{-\lambda u} du \\ &= \frac{1}{\lambda^2(1 + \gamma t)} \left[ \gamma + \lambda + \gamma \lambda t - (\gamma + \lambda + \gamma \lambda T) e^{-\lambda(T-t)} \right]. \end{aligned} \quad (74)$$

The bond reconstitution formula follows Eq. (26).

## 5 Examples of Classical Gaussian Short Rate Models - Multi Factor Models

As an example, let us consider the two factor Hull-White model in terms of the HJM framework. Define

$$g(t) = \begin{pmatrix} \sigma_{11}(t) \exp \left\{ \int_0^t \kappa_1(u) du \right\} & 0 \\ \sigma_{21}(t) \exp \left\{ \int_0^t \kappa_1(u) du \right\} & \sigma_{22}(t) \exp \left\{ \int_0^t \kappa_2(u) du \right\} \end{pmatrix}, \quad (75)$$

and

$$h(t) = \begin{pmatrix} \exp \left\{ - \int_0^t \kappa_1(u) du \right\} \\ \exp \left\{ - \int_0^t \kappa_2(u) du \right\} \end{pmatrix}. \quad (76)$$

Then,

$$\sigma_r(t) = g(t)H(t) = \begin{pmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix}, \quad (77)$$

and

$$dx(t) = \left( y(t)I_{2 \times 1} - \kappa(t)x(t) \right) dt + \sigma_r(t)^\top dW(t), \quad (78)$$

where  $y(t)$  can be found by numerical integration, and  $\kappa(t) = \text{diag}(\kappa_1(t), \kappa_2(t))$ . The short rate can be written as

$$r(t) = f(0, t) + x_1(t) + x_2(t), \quad (79)$$

Define  $\sigma_1(t) = \sqrt{\sigma_{11}^2(t) + \sigma_{21}^2(t)}$ ,  $\sigma_2(t) = \sigma_{22}(t)$ , and

$$\rho_x(t) = \frac{\sigma_{21}(t)\sigma_{22}(t)}{\sigma_1(t)\sigma_2(t)}, \quad (80)$$

then the correlation in forward rates can be expressed in these three parameters, together with  $\kappa_1(t)$  and  $\kappa_2(t)$ . For the forward rate, we have

$$\sigma_f(t, T) = g(t)h(T) = \begin{pmatrix} \sigma_{11}(t) \exp \left\{ - \int_t^T \kappa_1(u) du \right\} \\ \sigma_{21}(t) \exp \left\{ - \int_t^T \kappa_1(u) du \right\} + \sigma_{22}(t) \exp \left\{ - \int_t^T \kappa_2(u) du \right\} \end{pmatrix}, \quad (81)$$

and

$$df(t, T) = O(dt) + \sigma_f(t, T)^\top dW(t). \quad (82)$$

The covariance of the forward rates maturing at  $T_1$  and  $T_2$  is

$$\begin{aligned} & \frac{\langle df(t, T_1), df(t, T_2) \rangle}{dt} \\ &= (\sigma_{11}^2(t) + \sigma_{21}^2(t)) \exp \left\{ - \int_t^{T_1} \kappa_1(u) du - \int_t^{T_2} \kappa_1(u) du \right\} \\ & \quad + \sigma_{22}^2(t) \exp \left\{ - \int_t^{T_1} \kappa_2(u) du - \int_t^{T_2} \kappa_2(u) du \right\} \\ & \quad + \sigma_{21}(t) \sigma_{22}(t) \left( \exp \left\{ - \int_t^{T_1} \kappa_1(u) du - \int_t^{T_2} \kappa_2(u) du \right\} \right. \\ & \quad \quad \left. + \exp \left\{ - \int_t^{T_1} \kappa_2(u) du - \int_t^{T_2} \kappa_1(u) du \right\} \right) \\ &= \sigma_1^2(t) \exp \left\{ - \int_t^{T_1} \kappa_1(u) du - \int_t^{T_2} \kappa_1(u) du \right\} b(t, T_1, T_2), \end{aligned} \quad (83)$$

where

$$\begin{aligned} b(t, T_1, T_2) &= 1 + \rho_x(t) \frac{\sigma_2(t)}{\sigma_1(t)} \left( \exp \left\{ - \int_t^{T_1} (\kappa_2(u) - \kappa_1(u)) du \right\} \right. \\ & \quad \left. + \exp \left\{ - \int_t^{T_2} (\kappa_2(u) - \kappa_1(u)) du \right\} \right) \\ & \quad + \left( \frac{\sigma_2(t)}{\sigma_1(t)} \right)^2 \exp \left\{ - \int_t^{T_1} (\kappa_2(u) - \kappa_1(u)) du - \int_t^{T_2} (\kappa_2(u) - \kappa_1(u)) du \right\}. \end{aligned} \quad (84)$$

Therefore, the correlation between the forward rates is given by

$$\rho(t, T_1, T_2) = \frac{b(t, T_1, T_2)}{\sqrt{b(t, T_1, T_1)b(t, T_2, T_2)}}. \quad (85)$$

It is clear from the above expression that the correlation of the forward rates can be controlled by the five functions:  $\rho_x(t)$ ,  $\sigma_1(t)$ ,  $\sigma_2(t)$ ,  $\kappa_1(t)$ , and  $\kappa_2(t)$ . If we have either  $\rho_x = 1$ ,  $\kappa_2(t) - \kappa_1(t) = 0$ , or  $\sigma_2(t)/\sigma_1(t) = 0$ , then  $\rho(t, T_1, T_2) \equiv 1$ , where the model reduces to a one-factor one. Also,

$$\frac{\langle df(t, T), df(t, T) \rangle}{dt} = \sigma_1^2(t) \exp \left\{ -2 \int_t^T \kappa_1(u) du \right\} b(t, T, T), \quad (86)$$

it can be shown that if  $\rho_x$  is negative enough, the term structure of the volatility will develop a humped shape.

## 6 The Quasi-Gaussian Models

We are going to consider general multi-factor quasi-Gaussian models, with the separation condition  $\sigma_f(t, T; \omega) = g(t; \omega)h(T)$ , where  $g(t; \omega)$  is a  $d \times d$ -matrix valued function, and  $h(t)$  is a deterministic,  $d$ -dimensional column vector function. We are going to develop efficient approximation for the swap rate, which will also cover the pure Gaussian case.

### 6.1 Approximate Swap Rate Dynamics

Consider the following volatility specification which contains both local and stochastic volatilities,

$$\sigma_r(t; \omega) = \sqrt{z(t)}\sigma_x(t, x(t), y(t)), \quad (87)$$

where  $z(t)$  follows the CIR process

$$dz(t) = \theta(z_0 - z(t))dt + \eta(t)\sqrt{z(t)}dZ(t), \quad z(0) = z_0 = 1, \quad (88)$$

and  $\langle dZ(t), dW(t) \rangle = 0$ .

To proceed with the swap rate dynamics, fix a tenor structure

$$0 < T_0 < T_1 < \dots < T_N \quad (89)$$

with

$$\tau_n = T_{n+1} - T_n, \quad (90)$$

and the swap rate  $S(t)$ , with first fix date at  $T_0$  and last payment date at  $T_N$ , is given by

$$S(t) = \frac{P(t, T_0) - P(t, T_N)}{A(t)}, \quad A(t) = \sum_{n=0}^{N-1} \tau_n P(t, T_{n+1}). \quad (91)$$

Notice the functional dependence of the discount bond  $P(t, T)$  on the state variables  $x$  and  $y$ , *i.e.*,  $P(t, T) \equiv P(t, T, x(t), y(t))$ , the swap rate dynamics is

$$dS(t) = \sqrt{z(t)}(\nabla S)\sigma_x^\top dW(t) + O(dt), \quad (92)$$

where we are only concerned with the drift term. Since the swap rate is a martingale in the annuity measure, its dynamics in the corresponding annuity measure is given by

$$dS(t) = \sqrt{z(t)}\left((\nabla S)\sigma_x^\top \sigma_x (\nabla S)^\top\right)^{1/2} dW^A(t). \quad (93)$$

Using the Markovian Projection (MP) method, the above process is equivalent to the following local volatility process in terms of European option pricing,

$$dS(t) = \sqrt{z(t)}\varphi(t, S(t))dW^A(t), \quad (94)$$

where

$$\varphi^2(t, s) = E^A\left[(\nabla S)c_x(\nabla S)^\top \mid S(t) = s\right], \quad (95)$$

and  $c_x = \sigma_x^\top \sigma_x$ .

We will look for the linear approximation to  $\varphi(t, s)$  in  $s$ , and will use  $E^A(x(t))$  and  $E^A(y(t))$  as expansion point. First, define

$$\sigma_x^0(t) = \sigma_x(t, 0, 0), \quad (96)$$

then from Eq. (17), we have

$$y(t) = H(t) \left( \int_0^t H^{-1}(u) \sigma_r(u)^\top \sigma_r(u) H^{-1}(u) du \right) H(t), \quad (97)$$

and

$$\bar{y}(t) = E^A[y(t)] \approx H(t) \left( \int_0^t H^{-1}(u) (\sigma_x^0(u))^\top \sigma_x^0(u) H^{-1}(u) du \right) H(t). \quad (98)$$

For the approximation of  $x(t)$ , replace  $y(t)$  with  $\bar{y}(t)$  in Eq. (19), which is the Gaussian approximation to  $x(t)$ ,

$$dx_g(t) = \left( \bar{y}(t) I_{d \times 1} - \kappa(t) x_g(t) \right) dt + (\sigma_x^0(t))^\top dW(t). \quad (99)$$

The corresponding dynamics for  $x_g(t)$  in the  $T$ -forward measure, using Eq. (29), is

$$dx_g(t) = \left( \bar{y}(t) I_{d \times 1} - (\sigma_x^0(t))^\top \sigma_x^0(t) G(t, T) - \kappa(t) x_g(t) \right) dt + (\sigma_x^0(t))^\top dW^T(t). \quad (100)$$

The solution to the above SDE is

$$x_g(t) = H(t) \left( \int_0^t H^{-1}(u) \left[ \left( \bar{y}(u) I_{d \times 1} - (\sigma_x^0(u))^\top \sigma_x^0(u) G(u, T) \right) du + (\sigma_x^0(u))^\top dW^T(u) \right] \right), \quad (101)$$

and

$$\begin{aligned} \bar{x}(t) &= E^A[x(t)] \approx E^A[x_g(t)] \\ &= E^Q \left[ \frac{dP^A}{dP} x_g(t) \right] = E^Q \left[ \frac{A(t)/A(0)}{B(t)} x_g(t) \right] \\ &= \frac{1}{A(0)} \sum_{n=0}^{N-1} \tau_n E^Q \left[ \frac{P(t, T_{n+1})}{B(t)} x_g(t) \right] \\ &= \frac{1}{A(0)} \sum_{n=0}^{N-1} \tau_n E^{T_{n+1}} \left[ \frac{dP}{dP^{T_{n+1}}} \frac{P(t, T_{n+1})}{B(t)} x_g(t) \right] \\ &= \frac{1}{A(0)} \sum_{n=0}^{N-1} \tau_n E^{T_{n+1}} \left[ \frac{B(t)}{P(t, T_{n+1})/P(0, T_{n+1})} \frac{P(t, T_{n+1})}{B(t)} x_g(t) \right] \\ &= \sum_{n=0}^{N-1} \frac{\tau_n P(0, T_{n+1})}{A(0)} E^{T_{n+1}} [x_g(t)] \\ &= \sum_{n=0}^{N-1} \frac{\tau_n P(0, T_{n+1})}{A(0)} H(t) \int_0^t H^{-1}(u) \left( \bar{y}(u) I_{d \times 1} - (\sigma_x^0(u))^\top \sigma_x^0(u) G(u, T) \right) du. \quad (102) \end{aligned}$$

Now, we can proceed to develop further approximation for the swap rate dynamics, around  $\bar{x}(t)$  and  $\bar{y}(t)$ . Let us define  $S_g(t)$  to be the Gaussian approximation to  $S(t)$ , and its dynamics will be represented by

$$dS_g(t) = \nabla S(\sigma_x^0)^\top dW^A(t). \quad (103)$$

Expand Eq. (95) to first order in  $x$ ,

$$\begin{aligned} \left( (\nabla S)c_x(\nabla S)^\top \right) (t, x(t), y(t)) &\approx \left( (\nabla S)c_x(\nabla S)^\top \right) (t, \bar{x}(t), \bar{y}(t)) \\ &+ \nabla \left( (\nabla S)c_x(\nabla S)^\top \right) \Big|_{(t, \bar{x}(t), \bar{y}(t))} (x(t) - \bar{x}(t)). \end{aligned} \quad (104)$$

Then,

$$\begin{aligned} E^A \left[ (\nabla S)c_x(\nabla S)^\top | S(t) = s \right] &\approx \left( (\nabla S)c_x(\nabla S)^\top \right) (t, \bar{x}(t), \bar{y}(t)) \\ &+ \nabla \left( (\nabla S)c_x(\nabla S)^\top \right) \Big|_{(t, \bar{x}(t), \bar{y}(t))} E^A \left[ x(t) - \bar{x}(t) | S(t) = s \right]. \end{aligned} \quad (105)$$

Using a Gaussian approximation for the conditional expectation,

$$\begin{aligned} E^A \left[ x(t) - \bar{x}(t) | S(t) = s \right] &\approx E^A \left[ x_g(t) - \bar{x}_g(t) | S_g(t) = s \right] \\ &= \frac{\text{Cov}(S_g(t), x_g(t))}{\text{Var}(S_g(t))} (s - S(0)) \\ &= \frac{c_x(\nabla S)^\top}{(\nabla S)c_x(\nabla S)^\top} (s - S(0)). \end{aligned} \quad (106)$$

Notice that

$$\nabla \left( (\nabla S)c_x(\nabla S)^\top \right) c_x(\nabla S)^\top = (\nabla S)d_x(\nabla S)^\top + 2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)^\top, \quad (107)$$

where

$$d_x = \sum_{l=1}^d \left( c_x(\nabla S)^\top \right)_l \left( \frac{\partial \sigma_x^\top}{\partial x_l} \sigma_x + \sigma_x^\top \frac{\partial \sigma_x}{\partial x_l} \right). \quad (108)$$

The local volatility function is

$$\varphi^2(t, s) = (\nabla S)c_x(\nabla S)^\top + \frac{(\nabla S)d_x(\nabla S)^\top + 2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)^\top}{(\nabla S)c_x(\nabla S)^\top} (s - S(0)), \quad (109)$$

and linearizing around  $s \approx S(0)$ ,

$$\begin{aligned} \varphi(t, s) &= \left( (\nabla S)c_x(\nabla S)^\top \right)^{1/2} + \frac{1}{2 \left( (\nabla S)c_x(\nabla S)^\top \right)^{1/2}} \\ &\quad \times \frac{(\nabla S)d_x(\nabla S)^\top + 2(\nabla S)c_x(\nabla^2 S)c_x(\nabla S)^\top}{(\nabla S)c_x(\nabla S)^\top} (s - S(0)). \end{aligned} \quad (110)$$

Setting

$$\lambda_S(t) = \frac{\varphi(t, S(0))}{S(0)} = \frac{1}{S(0)} \left( (\nabla S) c_x (\nabla S)^\top \right)^{1/2} (t, \bar{x}(t), \bar{y}(t)), \quad (111)$$

$$b_S(t) = S(0) \frac{\frac{\partial \varphi}{\partial s}(t, S(0))}{\varphi(t, S(0))} = S(0) \frac{(\nabla S) d_x (\nabla S)^\top + 2(\nabla S) c_x (\nabla^2 S) c_x (\nabla S)^\top}{2(\nabla S) c_x (\nabla S)^\top (\nabla S) c_x (\nabla S)^\top} \Big|_{t, \bar{x}(t), \bar{y}(t)}, \quad (112)$$

the swap rate dynamics can be recast into a Heston form,

$$dS(t) = \sqrt{z(t)} \lambda_S(t) \left( b_S(t) S(t) + (1 - b_S(t)) S(0) \right) dU^A(t), \quad (113)$$

and the European swaption price can be obtained through the parameter averaging method.

## 6.2 Approximate Swap Rate Dynamics in Gaussian Short Rate Models

For Gaussian short rate models, the volatility is a deterministic function, and

$$dS(t) = \varphi(t, S(t)) dW^A(t), \quad (114)$$

with  $\varphi(t, S(t))$  identical to Eq. (95). Set  $x(t)$  in the above SDE to  $\bar{x}(t)$  as discussed earlier, the swap rate dynamics can be reduced to a normal process. The European payer swaption price is then given by the Bachelier formula,

$$A(0) \left[ (S(0) - c) \Phi(d) + \sqrt{v} \phi(d) \right], \quad (115)$$

with

$$d = \frac{S(0) - c}{\sqrt{v}}, \quad v = \int_0^{T_0} \left[ (\nabla S) \sigma_x(t)^\top \sigma_x(t) (\nabla S)^\top \right] dt. \quad (116)$$

Similarly, we can also expand the  $\varphi(t, S(t))$  term to the first order in  $x(t)$  and obtain a shifted lognormal process for the swap rate, and the corresponding models parameters are identical to those in Eq. (110).

## 6.3 Volatility Parameterization

The local volatility function  $\sigma_x(t, x, y)$  still needs to be specified. To this end, we consider  $d$  rolling benchmark rates,  $f_i(t) \equiv f_i(t, t + \delta_i)$ ,  $i = 1, \dots, d$ , with  $\delta_1 < \dots < \delta_d$ . We require that the benchmark rates have the following dynamics,

$$df_i(t) = O(dt) + \sqrt{z(t)} \lambda_i^f(t) \left( \alpha_i^f(t) + b_i^f(t) f_i(t) \right) dU_i(t), \quad (117)$$

with  $U(t)$  is a  $d$ -dimensional Brownian motion, with correlation matrix  $X^f(t) = \{\chi_{i,j}(t)\}$ . Define the decomposition of  $X^f$  as

$$X^f(t) = D^f(t)^\top D^f(t), \quad (118)$$



and  $H^f$  by

$$H^f(t) = \begin{pmatrix} h(t + \delta_1)^\top \\ \vdots \\ h(t + \delta_d)^\top \end{pmatrix}, \quad (119)$$

and  $\sigma^f$  by

$$\sigma^f(t, f(t)) = \text{diag} \left( \lambda_1^f(t) \left( \alpha_1^f(t) + b_1^f(t) f_1(t) \right), \dots, \lambda_d^f(t) \left( \alpha_d^f(t) + b_d^f(t) f_d(t) \right) \right), \quad (120)$$

then we can set

$$\sigma_r(t; \omega)^\top = \sqrt{z(t)} \sigma_x(t, x, y)^\top = \sqrt{z(t)} H(t) H^f(t)^{-1} \sigma^f(t, f(t)) D^f(t)^\top, \quad (121)$$

to be consistent with the benchmark rates dynamics. To see this, let  $f(t) = (f_1(t), \dots, f_d(t))^\top$ , and it can be shown from the bond reconstitution formula, Eq. (26),

$$f(t) = f(0) + H^f(t) H(t)^{-1} \left( x(t) + y(t) G(t, T) \right). \quad (122)$$

The dynamics is then given by

$$\begin{aligned} df(t) &= O(dt) + H^f(t) H(t)^{-1} dx(t) \\ &= O(dt) + H^f(t) H(t)^{-1} \sigma_r(t; \omega)^\top dW(t) \\ &= O(dt) + \sqrt{z(t)} \sigma^f(t, f(t)) D^f(t)^\top dW(t), \end{aligned} \quad (123)$$

consistent with Eq. (117).

### 6.3.1 Case 1: One Factor Model

For one factor model, there is only one benchmark rate, and we can simply write  $\sigma_r$  as

$$\sigma_r(t; \omega) \equiv \sigma_r(t, x(t), y(t)) = \sqrt{z(t)} \lambda_r(t) (\alpha_r(t) + b_r(t) x(t)), \quad (124)$$

where  $\alpha_r(t)$  is actually redundant and can be absorbed into  $\lambda_r(t)$ . We will get onto this shortly.

In the one dimensional case, the matrix products appearing in the approximate swap rate dynamics (see Sec. 6.1) will reduce to scalar products, and the result is given by

$$\lambda_S(t) = \lambda_r(t) \frac{1}{S(0)} \frac{\partial S}{\partial x} \left( t, \bar{x}(t), \bar{y}(t) \right) \left( \alpha_r(t) + b_r(t) \bar{x}(t) \right), \quad (125)$$

$$b_S(t) = \frac{S(0)}{\alpha_r(t) + b_r(t) \bar{x}(t)} \frac{\partial S}{\partial x} \left( t, \bar{x}(t), \bar{y}(t) \right) + \frac{S(0) \frac{\partial^2 S}{\partial x^2} \left( t, \bar{x}(t), \bar{y}(t) \right)}{\left( \frac{\partial S}{\partial x} \left( t, \bar{x}(t), \bar{y}(t) \right) \right)^2}, \quad (126)$$

and then the short rate dynamics takes the same form as in the multi-factor case.

From the above result, we can see that the local swap rate skew is related to the local short rate skew, ignoring small terms, by

$$b_S(t) \approx \frac{S(0)}{\alpha_r(t)} \frac{b_r(t)}{\frac{\partial S}{\partial x}(t, \bar{x}(t), \bar{y}(t))}. \quad (127)$$

It will be convenient to have these two skews of the same order of magnitude. Therefore, we can choose  $\alpha_r(t)$  to be the same as the swap rate, and scale  $b_r(t)$  by  $\partial S / \partial x$ .

### 6.3.2 Case 2: Gaussian Model

For the volatility parameterization of Gaussian short rate models, we can simply choose the volatilities of the benchmark rates to be deterministic functions,  $\lambda_i^f(t)$ ,  $i = 1, \dots, d$ .

## 6.4 Calibration

# 7 The Affine Model

## 7.1 One-Factor Model

### 7.1.1 Discount Bond Price

Consider the following short rate dynamics,

$$dr(t) = \kappa(t) (\theta(t) - r(t)) dt + \sigma(t) \sqrt{\alpha(t) + \beta(t)r(t)} dW(t), \quad (128)$$

with some technical conditions which will be omitted here. The details can be found in [1].

Even though we are interested in the discount bond price (1), let us consider the more general problem and find

$$g(t, T; c_1, c_2) = E_t^Q \left[ \exp \left( -c_1 r(T) - c_2 \int_t^T r(u) du \right) \right], \quad (129)$$

which can be viewed as the joint characteristic function for  $r(t)$  and  $\int r(t) dt$ . Then, the bond price can be obtained through

$$P(t, T) = g(t, T; 0, 1). \quad (130)$$

The affine structure of the model enable us to write

$$g(t, T; c_1, c_2) = \exp (A(t, T; c_1, c_2) - B(t, T; c_1, c_2) r(t)), \quad (131)$$

and the its PDE is given by

$$\frac{\partial g}{\partial t} + \kappa(t) (\theta(t) - r) \frac{\partial g}{\partial r} + \frac{1}{2} \sigma^2(t) (\alpha(t) + \beta(t)r) \frac{\partial^2 g}{\partial r^2} = c_2 r g. \quad (132)$$

Matching terms, the above PDE reduces to the following system of Riccati ODEs,

$$\frac{dA}{dt} - \kappa(t)\theta(t)B + \frac{1}{2}\alpha(t)\sigma^2(t)B^2 = 0, \quad (133)$$

$$-\frac{dB}{dt} + \kappa(t)B + \frac{1}{2}\beta(t)\sigma^2(t)B^2 = c_2, \quad (134)$$

subject to the terminal condition  $A(T, T; c_1, c_2) = B(T, T; c_1, c_2) = 0$ . For constant model parameters, this system of ODEs can analytically solved.

### 7.1.2 European Option Pricing

Since caplet are equivalent to options on zero coupon bond, the joint characteristic function from last section can be used for simple option pricing by the Fourier transform method, as briefly discussed in Appendix C. For European swaption, we will seek approximate swap rate martingale dynamics.

To this end, we introduce another variable,  $x(t) = r(t) - f(0, t)$ , whose dynamics is given by

$$dx(t) = dr(t) - \frac{\partial f(0, t)}{\partial t}dt = (\omega(t) - \kappa(t)x(t))dt + \sigma(t)\sqrt{\xi(t) + \beta(t)x(t)}dW(t), \quad (135)$$

where  $x(0) = 0$ ,  $\xi(t) = \alpha(t) + \beta(t)f(0, t)$ , and

$$\omega(t) = \kappa(t)\theta(t) - \frac{\partial f(0, t)}{\partial t} - \kappa(t)f(0, t). \quad (136)$$

Notice that even  $\omega(t)$  contains the derivative of the initial forward rate, which is generally interpolated from market data and is likely not smooth,  $\omega(t)$  is likely to be smooth.  $\omega(t)$  ensures that the affine model will exactly match the initial yield curve, in contrast to  $\theta(t)$ .

The swap rate dynamics is a martingale in the annuity measure, that is, in terms of  $x(t)$ ,

$$dS(t) = \frac{\partial S}{\partial x}\sigma(t)\sqrt{\xi(t) + \beta(t)x(t)}dW^A(t). \quad (137)$$

Similar to the quasi-Gaussian model, we can expand  $S(t)$  around some point to the first order in  $x(t)$ ,

$$S(t) \approx \zeta(t) + \chi(t)x(t). \quad (138)$$

Then, the swap rate dynamics becomes

$$\begin{aligned} dS(t) &\approx \chi(t)\sigma(t)\sqrt{\xi(t) + \beta(t)\left(\frac{S(t) - \zeta(t)}{\chi(t)}\right)}dW^A(t) \\ &= \sigma(t)\sqrt{\alpha_s(t) + \beta_s(t)S(t)}dW^A(t), \end{aligned} \quad (139)$$

with

$$\alpha_s(t) = \xi(t)\chi^2(t) - \beta(t)\zeta(t)\chi(t), \quad \beta_s(t) = \beta(t)\chi(t). \quad (140)$$

Since the value of a European payer swaption is, in the annuity measure,

$$A(0)E^A \left[ \left( S(T_0) - c \right)^+ \right], \quad (141)$$

it can be valued using the Fourier transform method. Alternatively, it can also be valued by the parameter averaging (PA) method, as discussed in Appendix D.

## 8 The Quadratic Gaussian Model

The quadratic Gaussian model can be obtained by having the short rate in a quadratic form,

$$r(t) = z(t)^\top \gamma(t) z(t) + h(t)^\top z(t) + a(t), \quad (142)$$

where  $\gamma(t)$  is a  $d \times d$  symmetric matrix,  $h(t)$  is a  $d$ -dimensional vector, and  $a(t)$  is used to fit the initial yield curve. Here, the  $d$ -dimensional state variable  $z(t)$  satisfies

$$dz(t) = g(t)^\top dW(t), \quad z(0) = 0, \quad (143)$$

where  $W(t)$  is a  $d$ -dimensional Brownian motion.

### 8.1 Relation to the Affine Model

To see the connection to the affine model, consider the one dimensional constant parameter case. In this case, Eq. (142) reduces to

$$r(t) = \gamma z^2(t) + h z(t) + a, \quad (144)$$

which can be solved,

$$z(t) = \frac{\sqrt{h^2 + 4\gamma(r(t) - a)} - h}{2}, \quad (145)$$

by keeping the positive root. Then, it can be shown

$$dr(t) = \frac{1}{\gamma} \sqrt{h^2 + 4\gamma(r(t) - a)} g(t) dW(t), \quad (146)$$

which is clearly affine.

### 8.2 Analytical Results

Due to the affine nature of the quadratic Gaussian model, the discount bond price is similarly given in a quadratic form,

$$P(t, T) = \exp \left( -z(t)^\top \gamma(t, T) z(t) - h(t, T)^\top z(t) - a(t, T) \right). \quad (147)$$

Applying the Feynman-Kac theorem, the discount bond price satisfies the following PDE,

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sum_{ijk} \frac{\partial^2 P}{\partial z_i \partial z_j} g_{ki}(t) g_{kj}(t) = r(t) P. \quad (148)$$

Calculate the derivatives and match the  $z$  terms on both sides of the above equation, we will obtain a system of Riccati ODEs,

$$-\frac{d}{dt} \gamma(t, T) + 2\gamma(t, T) g(t)^\top g(t) \gamma(t, T) = \gamma(t), \quad (149)$$

$$-\frac{d}{dt} h(t, T) + 2\gamma(t, T) g(t)^\top g(t) h(t, T) = h(t), \quad (150)$$

$$-\frac{d}{dt} a(t, T) - \text{Tr} \left( \gamma(t, T) g(t)^\top g(t) \right) + \frac{1}{2} h(t, T)^\top g(t)^\top g(t) h(t, T) = a(t), \quad (151)$$

with terminal conditions  $\gamma(T, T) = 0$ ,  $h(T, T) = 0$ , and  $a(T, T) = 0$ .

## 9 The Libor Market Model (LMM)

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### A Girsanov's Theorem

Consider two measures  $P$  and  $P(\theta)$  related by a density

$$\zeta^\theta(t) = \mathbb{E}_t^P \left[ \frac{dP(\theta)}{dP} \right], \quad (152)$$

where  $\zeta^\theta(t)$  is an exponential martingale given by

$$\frac{d\zeta^\theta(t)}{\zeta^\theta(t)} = -\theta(t)^\top dW(t), \quad (153)$$

where  $W(t)$  is a  $d$ -dimensional  $P$ -Brownian motion. Then,

$$W^\theta(t) = W(t) + \int_0^t \theta(s) ds \quad (154)$$

is a  $P(\theta)$ -Brownian motion, and

$$dW^\theta(t) = dW(t) + \theta(t) dt. \quad (155)$$

### B Futures Convexity Adjustment

#### B.1 Ho-Lee Model

In the Ho-Lee model,  $\sigma_P(t, T) = \sigma(T - t)$ , and

$$\Omega(t, T) = \sigma^2 \tau \int_t^T (T + \tau - u) du = \frac{\sigma^2}{2} (T - t)(T - t + 2\tau). \quad (156)$$

By expanding Eq. (46) to first order in  $\tau$ , the futures rate in the Ho-Lee model is given by

$$F(t, T, T + \tau) = L(t, T, T + \tau) + \frac{\sigma^2}{2} (T - t)(T - t + 2\tau). \quad (157)$$

#### B.2 Vasicek Model

In the Vasicek model, the forward rate volatility function is set to  $\sigma_f(t, T) = \sigma e^{-\kappa(T-t)}$ . Therefore,

$$\sigma_P(t, T) = \frac{\sigma}{\kappa} (1 - e^{-\kappa(T-t)}), \quad (158)$$

and

$$\Omega(t, T) = \frac{\sigma^2}{\kappa^2} \int_t^T (e^{-\kappa(T-u)} - e^{-\kappa(T+\tau-u)}) (1 - e^{-\kappa(T+\tau-u)}) du. \quad (159)$$

This can be shown to give

$$\Omega(t, T) = \frac{\sigma^2}{2\kappa^3} (1 - e^{-\kappa\tau}) (1 - e^{-\kappa(T-t)}) (2 - e^{-\kappa\tau} - e^{-\kappa(T+\tau-t)}). \quad (160)$$

The above result can be exactly reduced to the Ho-Lee model when taking the limit  $\kappa \rightarrow 0$ , while under this limit, the vasicek model also reduces to the Ho-Lee model.

### B.3 Continuously Compounded Forward Rate

Rather than using the simply compounded forward rate, we can also formulate the convexity adjustment in terms of continuously compounded forward rate [3],

$$f(t, T, T + \tau) = -\frac{1}{\tau} \log \left( \frac{P(t, T + \tau)}{P(t, T)} \right) = \frac{1}{\tau} (\log P(t, T) - \log P(t, T + \tau)), \quad (161)$$

and the dynamics can be derived by simple application of Ito's lemma,

$$df(t, T, T + \tau) = \frac{\sigma_P^2(t, T + \tau) - \sigma_P^2(t, T)}{2\tau} dt + \frac{\sigma_P(t, T + \tau) - \sigma_P(t, T)}{\tau} dW(t). \quad (162)$$

Then, the expected change in the forward rate from  $t$  to the expiry of the futures contract  $T$ , is given by the integral over the drift term,

$$\text{ConvAdj}(t, T, T + \tau) = \frac{1}{2\tau} \int_t^T [\sigma_P^2(u, T + \tau) - \sigma_P^2(u, T)] du. \quad (163)$$

For the Ho-Lee model, the convexity adjustment is given by

$$\frac{\sigma^2}{2} (T - t)(T - t + \tau), \quad (164)$$

and the vasicek convexity adjustment is

$$\frac{\sigma^2}{4\kappa^3\tau} (1 - e^{-\kappa\tau}) (1 - e^{-\kappa(T-t)}) (3 - e^{-\kappa\tau} - e^{-\kappa(T-t)} - e^{-\kappa(T-t+\tau)}). \quad (165)$$

Again, in the  $\kappa \rightarrow 0$  limit, this reduces to the Ho-Lee convexity adjustment.

## C Transform Method for Option Pricing

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## D Parameter Averaging Method for Option Pricing

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## References

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