Notes on Static Hedging

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1 Replication of European Payoff

For a European call option with spot price S_t, strike K, expiration T, its value at t can be determined by

$$C(K) = e^{-r\tau} \int_0^{+\infty} (S_T - K)^+ p(t, S_t; T, S_T) dS_T = e^{-r\tau} \int_K^{+\infty} (S_T - K) p(t, S_t; T, S_T) dS_T, \quad (1)$$

where r is the risk free rate, $p(t, S_t; T, S_T)$ is the transition density for the asset price starting from S_t at t and ending with S_T at T, and $\tau = T - t$. In the Black-Scholes world, the underlying process is

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t, \quad (2)$$

where q is the dividend rate. The corresponding transition density is given by

$$p(t, S_t; T, S_T) = \frac{1}{S_T \sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{\left(\log\left(\frac{S_T}{S_t}\right) - \left(r - q - \frac{\sigma^2}{2}\right)\tau \right)^2}{2\sigma^2\tau} \right). \quad (3)$$

On the other hand, if we have the call option prices for a continuum of strikes, the transition density can be recovered by differentiating the call option price twice,

$$\frac{\partial^2 C(K)}{\partial K^2} = e^{-r\tau} p(t, S_t; T, K). \quad (4)$$

We can obtain the same result from put option prices, *i.e.*,

$$\frac{\partial^2 P(K)}{\partial K^2} = e^{-r\tau} p(t, S_t; T, K). \quad (5)$$

Given the transition density, the price of a European option with arbitrary payoff $f(S_T)$ at maturity is given by

$$V = e^{-r\tau} \int_0^{+\infty} f(K) p(t, S_t; T, K) dK. \quad (6)$$

This option price can also be represented in terms of out-of-the-money (OTM) option prices,

$$V = \int_0^F f(K)\frac{\partial^2 P(K)}{\partial K^2}dK + \int_F^{+\infty} f(K)\frac{\partial^2 C(K)}{\partial K^2}dK, \tag{7}$$

where F is an arbitrary positive number. Integrating the above equation by parts twice, we have

$$V = e^{-r\tau}f(F) + f'(F)\big[e^{-q\tau}S_t - e^{-r\tau}F\big] + \int_0^F f''(K)P(K)dK + \int_F^{+\infty} f''(K)C(K)dK. \tag{8}$$

Here, we have used the call-put parity,

$$C(K) - P(K) = e^{-q\tau}S_t - e^{-r\tau}K. \tag{9}$$

Therefore, if F is chosen to be the forward price $F_T = S_t e^{(r-q)\tau}$, the linear term will drop out,

$$V = e^{-r\tau}f(F_T) + \int_0^{F_T} f''(K)P(K)dK + \int_{F_T}^{+\infty} f''(K)C(K)dK. \tag{10}$$

This means that we can replicate a European option with arbitry payoff with plain vanilla options. The above intergrals in strike have to be discretized in reality, which will introduce some replication errors.

2 Static Hedging of Barrier Options

Consider a down-and-in barrier option with barrier level $H < S_t$ with terminal payoff $f(S_T)$, the option price is given by

$$V_{DI} = e^{-r\tau}\int_0^{+\infty} f(K)p(t, S_t; T, K; H)dK, \tag{11}$$

where $p(t, S_t; T, K; H)$ is the transition density from S_t at t to K at T, while the underlying process has hit barrier H at least once in this time frame. The option price can be decomposed into two parts,

$$V_{DI} = e^{-r\tau}\int_0^H f(K)p(t, S_t; T, K)dK + e^{-r\tau}\int_H^{+\infty} f(K)p(t, S_t; T, K; H)dK. \tag{12}$$

The first term is identical to its European counterpart, since the terminal underlying price is below the barrier level, and the underlying process must have cross the barrier level at least once. Therefore, the transition density reduces to a plain vanilla one. For the second term, assume the density for the first passage time to the barrier H starting from S_t at t is given by $\phi(t, S_t; \delta, H)$, then

$$p(t, S_t; T, K; H) = \int_t^T \phi(t, S_t; \delta, H)p(\delta, H; T, K)d\delta. \tag{13}$$

The second term of Eq. (12) becomes

$$
\begin{aligned}
& e^{-r\tau} \int_H^{+\infty} f(K) p(t, S_t; T, K; H) dK \\
= \quad & e^{-r\tau} \int_t^T d\delta \phi(t, S_t; \delta, H) \int_H^{+\infty} dK f(K) p(\delta, H; T, K) \\
= \quad & e^{-r\tau} \int_t^T d\delta \phi(t, S_t; \delta, H) \int_0^H dK \left(\frac{H}{K}\right)^2 f\left(\frac{H^2}{K}\right) p\left(\delta, H; T, \frac{H^2}{K}\right). \qquad (14)
\end{aligned}
$$

From Eq. (3), we have

$$
\begin{aligned}
& \left(\frac{H}{K}\right)^2 p\left(\delta, H; T, \frac{H^2}{K}\right) \\
= \quad & \left(\frac{H}{K}\right)^2 \frac{1}{\frac{H^2}{K}\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{\left(\log\left(\frac{H}{K}\right) - \left(r - q - \frac{\sigma^2}{2}\right)(T-\delta)\right)^2}{2\sigma^2(T-\delta)}\right) \\
= \quad & \left(\frac{K}{H}\right)^k \frac{1}{K\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{\left(\log\left(\frac{K}{H}\right) - \left(r - q - \frac{\sigma^2}{2}\right)(T-\delta)\right)^2}{2\sigma^2(T-\delta)}\right) \\
= \quad & \left(\frac{K}{H}\right)^k p\left(\delta, H; T, K\right), \qquad (15)
\end{aligned}
$$

and the second term of Eq. (12) becomes

$$
\begin{aligned}
& e^{-r\tau} \int_t^T d\delta \phi(t, S_t; \delta, H) \int_0^H dK f\left(\frac{H^2}{K}\right) \left(\frac{K}{H}\right)^k p\left(\delta, H; T, K\right) \\
= \quad & e^{-r\tau} \int_0^H \left(\frac{K}{H}\right)^k f\left(\frac{H^2}{K}\right) p\left(t, S_t; T, K\right) dK, \qquad (16)
\end{aligned}
$$

where $k = 1 - 2(r-q)/\sigma^2$. Now, the down-and-in option price becomes

$$
V_{DI} = e^{-r\tau} \int_0^\infty \hat{f}(K) p\left(t, S_t; T, K\right) dK, \qquad (17)
$$

which means that the down-and-in option with final payoff $f(S_T)$ at maturity can be statically replicated with an European option with final payoff [1]

$$
\hat{f}(K) = \left[f(K) + \left(\frac{K}{H}\right)^k f\left(\frac{H^2}{K}\right)\right] \mathcal{I}_{K<H} \qquad (18)
$$

at maturity. We can use the exact result from the previous section to further replicate this European option with plain vanilla ones.

For down-and-out options, the replication strategy can be obtained from the difference between a vanilla option and a down-and-in option with the same barrier as the down-and-out option. The is due to the fact that a vanilla option can be decomposed into a down-and-out and a down-and-in options with same features.

3 Variance swap

3.1 Profit and loss of a delta-hedged option

Consider the price of an option, $V_{BS}(t, S_t; \sigma_{BS})$, where a constant volatility is assumed throughout the life time of the option. For the discounted option price, $f(t) = e^{-rt}V_{BS}(t, S_t; \sigma_{BS})$, we have

$$f(T) = f(0) + \int_0^T df(t) = f(0) + \int_0^T e^{-rt}\left[\left(-rV_{BS} + \frac{\partial V_{BS}}{\partial t}\right)dt + \frac{\partial V_{BS}}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V_{BS}}{\partial S^2}(dS_t)^2\right]. \tag{19}$$

Assume that the underlying follows a geometric Brownian motion,

$$dS_t = (r-q)\,S_t dt + \sigma_t S_t dW_t, \tag{20}$$

then

$$\begin{aligned} f(T) &= f(0) + \int_0^T e^{-rt}\left(-rV_{BS} + \frac{\partial V_{BS}}{\partial t} + (r-q)\,S_t\frac{\partial V_{BS}}{\partial S} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S^2}\right)dt \\ &\quad + \int_0^T e^{-rt}\sigma_t S_t \frac{\partial V_{BS}}{\partial S}dW_t. \end{aligned} \tag{21}$$

Notice that the option price $V(t, S_t; \sigma_{BS})$ satisfies the Black-Scholes PDE,

$$\frac{\partial V_{BS}}{\partial t} + (r-q)\,S\frac{\partial V_{BS}}{\partial S} + \frac{1}{2}\sigma_{BS}^2 S^2 \frac{\partial^2 V_{BS}}{\partial S^2} = rV_{BS}, \tag{22}$$

and finally, we have

$$f(T) = f(0) + \int_0^T e^{-rt}\frac{S_t^2}{2}\frac{\partial^2 V_{BS}}{\partial S^2}\left(\sigma_t^2 - \sigma_{BS}^2\right)dt + \int_0^T e^{-rt}\sigma_t S_t \frac{\partial V_{BS}}{\partial S}dW_t. \tag{23}$$

The expected value of the hedged option is given by

$$E\left[f(T)\right] = f(0) + E\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{\partial^2 V_{BS}}{\partial S^2}\left(\sigma_t^2 - \sigma_{BS}^2\right)dt\right]. \tag{24}$$

From this, it can be seen that, a delta-hedged option is sensitive to the difference between realized and implied variances. However, if one want to trade the realized variance, the delta-hedged option is not a good choice, since the dollar gamma is peaked at the strike price. It can be shown, for a portfolio of options weighted by their inversed strike squared, $1/K^2$, the portfolio will be insensitive to the stock price move.

As a side result, notice that, in an arbitrage free world, we have

$$E\left[f(T)\right] = f(0), \tag{25}$$

which leads to

$$\sigma_{BS}^2 = \frac{E\left[\int_0^T e^{-rt}\sigma_t^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S^2} dt\right]}{E\left[\int_0^T e^{-rt} S_t^2 \frac{\partial^2 V_{BS}}{\partial S^2} dt\right]}. \tag{26}$$

This result implies that, the implied variance is the average of realized variance, weighted by dollar gamma. It can lead to interesting approximation for the implied volatility, for example, the so called "most likely path" method. However, in practice, the approximation is rather crude.

3.2 Static replication of variance swap

Results from last subsection relies on the Black-Scholes assumption. Here, we will consider a more general problem of static replication of the realized variance in a model-free way. Consider any continuous stochastic process for the underlying,

$$\frac{dS_t}{S_t} = r dt + \sigma_t dW_t, \tag{27}$$

applying the Ito's lemma, we have

$$d \log S_t = \left(r - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t, \tag{28}$$

where the result holds for either deterministic or stochastic volatility σ_t. For the annualized realized variance, we have

$$E\left[\frac{1}{T}\int_0^T \sigma_t^2 dt\right] = 2r - \frac{2}{T} E\left[\log\left(\frac{S_T}{S_0}\right)\right]. \tag{29}$$

Therefore, the realized variance can be replicated by a log contract. The log contract can be replicated by a portfolio of vanilla options, as shown in the first section.

3.3 Discretely observed variance swap

References

[1] Peter Carr and Andrew Chou, *Breaking Barriers*, Risk **10**(9), 139 (1997).