Notes on CEV

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1 Fokker-Planck Equation

Consider the CEV process,

$$dS_t = \sigma S_t^\beta dW_t, \tag{1}$$

with $\beta < 1$, the corresponding Fokker-Planck equation is given by

$$-\frac{\partial p(t_0, S_0; t, s)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial s^2}\left(s^{2\beta} p(t_0, S_0; t, s)\right) = 0, \tag{2}$$

with the initial condition

$$p(t_0, S_0; t_0, s) = \delta(s - S_0), \tag{3}$$

where $\delta(x)$ is the Dirac delta function. In the following, we will write $p(t, s) \equiv p(t_0, S_0; t, s)$.

Define the Laplace transform of the forward transition density $p(t, s)$ as

$$g(\lambda, s) = \int_0^{+\infty} e^{-\lambda t} p(t, s) dt, \tag{4}$$

the Fokker-Planck equation (2) becomes

$$\frac{\partial^2 g(\lambda, s)}{\partial s^2} + \frac{4\beta}{s}\frac{\partial g(\lambda, s)}{\partial s} + \left(\frac{2\lambda}{\sigma^2 s^{2\beta}} + \frac{2\beta(2\beta - 1)}{s^2}\right) g(\lambda, s) = 0. \tag{5}$$

Following [1], if $u(x)$ satisfies the Bessel equation of order ν,

$$u'' + \frac{1}{x}u' + \left(1 - \frac{\nu^2}{x^2}\right) u = 0, \tag{6}$$

and let $y(x) = x^a u(bx^c)$, then $y(x)$ satisfies

$$y'' + \frac{1 - 2a}{x}y' + \left(b^2 c^2 x^{2c-2} + \frac{a^2 - \nu^2 c^2}{x^2}\right) y = 0. \tag{7}$$

Comparing the coefficients with Eq. (5), we have

$$a = \frac{1}{2}(1 - 4\beta), \qquad b = \frac{2\nu\sqrt{2\lambda}}{\sigma} = \frac{\sqrt{2\lambda}}{\sigma(1-\beta)}, \qquad c = \frac{1}{2\nu} = 1 - \beta, \qquad \nu = \frac{1}{2(1-\beta)}. \tag{8}$$

Then, the general solution to Eq. (5) is given by

$$g(\lambda, s) = s^{(1-4\beta)/2}\left[A(\lambda)J_\nu\left(\frac{2\nu\sqrt{2\lambda}}{\sigma}s^{\frac{1}{2\nu}}\right) + B(\lambda)Y_\nu\left(\frac{2\nu\sqrt{2\lambda}}{\sigma}s^{\frac{1}{2\nu}}\right)\right], \tag{9}$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively. To determine the coefficients $A(\lambda)$ and $B(\lambda)$, and subsequently the forward transition density $p(t, s)$, boundary conditions in the s direction is needed.

2 Absorbing boundary condition

3 Reflection boundary condition

4 European option pricing

5 SABR

6 Free boundary CEV

7 Free boundary SABR

References

[1] See `http://mathworld.wolfram.com/BesselDifferentialEquation.html`.