Notes on Replication

Dec. 32, 2999

1 Replication of a European Payoff

For a vanilla European call option with spot price S_t, strike K, expiration T, its value at t can be determined by

$$C(K) = e^{-r\tau} \int_0^{+\infty} (S_T - K)^+ p(t, S_t; T, S_T) dS_T = e^{-r\tau} \int_K^{+\infty} (S_T - K) p(t, S_t; T, S_T) dS_T, \quad (1)$$

where r is the risk free rate, $p(t, S_t; T, S_T)$ is the transition density for the asset price starting from S_t at t and ending with S_T at T, and $\tau = T - t$. In the Black-Scholes world, the underlying process is

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t, \quad (2)$$

where q is the dividend rate. The corresponding transition density is given by

$$p(t, S_t; T, S_T) = \frac{1}{S_T \sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{\left(\log\left(\frac{S_T}{S_t}\right) - \left(r - q - \frac{\sigma^2}{2}\right)\tau\right)^2}{2\sigma^2\tau}\right). \quad (3)$$

On the other hand, if we have the call option prices for a continuum of strikes, the transition density can be recovered by differentiating the call option price twice,

$$\frac{\partial^2 C(K)}{\partial K^2} = e^{-r\tau} p(t, S_t; T, K). \quad (4)$$

We can obtain the same result from put option prices, *i.e.*,

$$\frac{\partial^2 P(K)}{\partial K^2} = e^{-r\tau} p(t, S_t; T, K). \quad (5)$$

Given the transition density, the price of a European contract with arbitrary payoff $f(S_T)$ at maturity is given by

$$V = e^{-r\tau} \int_0^{+\infty} f(K) p(t, S_t; T, K) dK. \quad (6)$$

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Using Eqs. (4) and (5), Eq. (6) can be represented in terms of out-of-the-money (OTM) option prices,

$$V = \int_0^F f(K) \frac{\partial^2 P(K)}{\partial K^2} dK + \int_F^{+\infty} f(K) \frac{\partial^2 C(K)}{\partial K^2} dK, \tag{7}$$

where F is an arbitrary positive number. Integrating the above equation by parts twice, we have

$$V = e^{-r\tau} f(F) + f'(F) \left[e^{-q\tau} S_t - e^{-r\tau} F \right] + \int_0^F f''(K) P(K) dK + \int_F^{+\infty} f''(K) C(K) dK. \tag{8}$$

Here, we have used the call-put parity,

$$C(K) - P(K) = e^{-q\tau} S_t - e^{-r\tau} K. \tag{9}$$

Therefore, if F is chosen to be the forward price $F_T = S_t e^{(r-q)\tau}$, the linear term will drop out,

$$V = e^{-r\tau} f(F_T) + \int_0^{F_T} f''(K) P(K) dK + \int_{F_T}^{+\infty} f''(K) C(K) dK. \tag{10}$$

This means that we can replicate a European contract with arbitry payoff with plain vanilla options.

Another useful representation of the static replication is given by

$$\begin{aligned} f(S_T) &= f(F) + f'(F) \left[(S_T - F)^+ - (F - S_T)^+ \right] \\ &+ \int_0^F f''(K) (F - S_T)^+ dK + \int_F^{\infty} f''(K) (S_T - F)^+ dK. \end{aligned} \tag{11}$$

This can be viewed as the forward version of Eq. (8).

1.1   An Exmaple: Libor-in-arrears

For a Libor rate with a start date T and an end date M, the forward Libor rate is given by

$$L(t) \equiv L(t, T, M) = \frac{1}{\tau} \left(\frac{P(t, T)}{P(t, M)} - 1 \right), \tag{12}$$

where $P(t, T)$ is the time t price of a zero-coupon bond maturing at T, and $\tau = M - T$. A standard Libor payment will be on the end date of the related Libor rate, and the discounted cash flow is

$$\begin{aligned} E^Q \left[\frac{B(0)}{B(M)} L(T) \right] &= E^M \left[\frac{P(0, M)}{P(M, M)} L(T) \right] \\ &= P(0, M) E^M [L(T)] = P(0, M) L(0). \end{aligned} \tag{13}$$

Here, Q and M stand for the risk neutral and M-forward measures, respectively. We have also used the fact that $L(t)$ is a martingale in the M-forward measure.

This relation does not hold any more for a Libor-in-arrears (LIA) payment, which pays immediately after the Libor rate is fixed, *i.e.*, the discounted cash flow is

$$V_{LIA}(0) = E^Q \left[\frac{B(0)}{B(T)} L(T) \right],$$ (14)

where the payment is on the start date T, rather than the end date M as in Eq. (13). If we apply the same change-of-numeraire technique as before, and use M-forward measure for convenience, we will have

$$
\begin{aligned}
V_{LIA}(0) &= E^M \left[\frac{P(0, M)}{P(T, M)} L(T) \right] = P(0, M) E^M \left[(1 + \tau L(T)) L(T) \right] \\
&= P(0, M) \left(L(0) + \tau E^M \left[L(T)^2 \right] \right).
\end{aligned}
$$ (15)

The $E^M \left[L(T)^2 \right]$ term can be replicated with call and put options on the Libor rate, known as caplets and floorlets, which leads to

$$E^M \left[L(T)^2 \right] = L(0)^2 + 2 \int_{-\infty}^{L(0)} p(0, L(0); T, K) dK + 2 \int_{L(0)}^{+\infty} c(0, L(0); T, K) dK,$$ (16)

where $c(0, L(0); T, K)$ and $p(0, L(0); T, K)$ are the undiscounted caplet and floorlet prices with maturity T and strike K.

2 Static Replication of Barrier Options

Consider a down-and-in barrier option with barrier level $H < S_t$ with terminal payoff $f(S_T)$, the option price is given by

$$V_{DI} = e^{-r\tau} \int_0^{+\infty} f(K) p(t, S_t; T, K; H) dK,$$ (17)

where $p(t, S_t; T, K; H)$ is the transition density from S_t at t to K at T, while the underlying process has hit barrier H at least once in this time frame. The option price can be decomposed into two parts,

$$V_{DI} = e^{-r\tau} \int_0^H f(K) p(t, S_t; T, K) dK + e^{-r\tau} \int_H^{+\infty} f(K) p(t, S_t; T, K; H) dK.$$ (18)

The first term is identical to its European counterpart, since the terminal underlying price is below the barrier level, and the underlying process must have cross the barrier level at least once. Therefore, the transition density reduces to a plain vanilla one. For the second term, assume the density for the first passage time to the barrier H starting from S_t at t is given by $\phi(t, S_t; \delta, H)$, then

$$p(t, S_t; T, K; H) = \int_t^T \phi(t, S_t; \delta, H) p(\delta, H; T, K) d\delta.$$ (19)

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The second term of Eq. (18) becomes

$$e^{-r\tau}\int_H^{+\infty} f(K)p(t, S_t; T, K; H)dK$$

$$= e^{-r\tau}\int_t^T d\delta\phi(t, S_t; \delta, H)\int_H^{+\infty} dK f(K)p(\delta, H; T, K)$$

$$= e^{-r\tau}\int_t^T d\delta\phi(t, S_t; \delta, H)\int_0^H dK \left(\frac{H}{K}\right)^2 f\left(\frac{H^2}{K}\right)p\left(\delta, H; T, \frac{H^2}{K}\right). \tag{20}$$

From Eq. (3), we have

$$\left(\frac{H}{K}\right)^2 p\left(\delta, H; T, \frac{H^2}{K}\right)$$

$$= \left(\frac{H}{K}\right)^2 \frac{1}{\frac{H^2}{K}\sqrt{2\pi\sigma^2(T-\delta)}}\exp\left(-\frac{\left(\log\left(\frac{H}{K}\right) - \left(r - q - \frac{\sigma^2}{2}\right)(T-\delta)\right)^2}{2\sigma^2(T-\delta)}\right)$$

$$= \left(\frac{K}{H}\right)^k \frac{1}{K\sqrt{2\pi\sigma^2(T-\delta)}}\exp\left(-\frac{\left(\log\left(\frac{K}{H}\right) - \left(r - q - \frac{\sigma^2}{2}\right)(T-\delta)\right)^2}{2\sigma^2(T-\delta)}\right)$$

$$= \left(\frac{K}{H}\right)^k p\left(\delta, H; T, K\right), \tag{21}$$

and the second term of Eq. (18) becomes

$$e^{-r\tau}\int_t^T d\delta\phi(t, S_t; \delta, H)\int_0^H dK f\left(\frac{H^2}{K}\right)\left(\frac{K}{H}\right)^k p\left(\delta, H; T, K\right)$$

$$= e^{-r\tau}\int_0^H \left(\frac{K}{H}\right)^k f\left(\frac{H^2}{K}\right)p\left(t, S_t; T, K\right)dK, \tag{22}$$

where $k = 1 - 2(r - q)/\sigma^2$. Now, the down-and-in option price becomes

$$V_{DI} = e^{-r\tau}\int_0^\infty \hat{f}(K)p\left(t, S_t; T, K\right)dK, \tag{23}$$

which means that the down-and-in option with final payoff $f(S)$ at maturity can be statically replicated with a European contract with terminal payoff [1]

$$\hat{f}(S) = \left[f(S) + \left(\frac{S}{H}\right)^k f\left(\frac{H^2}{S}\right)\right]\mathcal{I}_{S<H} \tag{24}$$

at maturity. It can be shown that this terminal payoff will lead to exactly the same expression for the down-and-in call option price.

3 Variance swap

A variance swap is a forward contract on future realized variance. At maturity, the floating leg pays

$$A \times \frac{N}{n} \sum_{i=1}^{n} \log^2 \left(\frac{S_i}{S_{i-1}} \right), \tag{25}$$

where A is the notional amount of the variance swap, N is an annualization factor and is usually set to 252, n is the number of monitor dates, and S is the stock price at market close. The fixed leg payment is so determined that the variance swap has zero value at inception. In this section, we will discuss the valuation and replication of the variance swap, for both continuously and discretely monitored cases. Let us start with a very general result.

3.1 Profit and loss of a delta-hedged position

Consider the price of a European payoff $\phi(S)$ maturing at T, $V_{BS}(t, S_t; \sigma_{BS})$, where a constant volatility is assumed throughout the life time of the contract. For the discounted price, $f(t) = e^{-rt} V_{BS}(t, S_t; \sigma_{BS})$, we have

$$f(T) = f(0) + \int_0^T df(t) = f(0) + \int_0^T e^{-rt} \left[\left(-rV_{BS} + \frac{\partial V_{BS}}{\partial t} \right) dt + \frac{\partial V_{BS}}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 V_{BS}}{\partial S^2} (dS_t)^2 \right]. \tag{26}$$

Assume that the underlying follows a geometric Brownian motion,

$$dS_t = (r - q) S_t dt + \sigma_t S_t dW_t, \tag{27}$$

then

$$\begin{aligned}
f(T) &= f(0) + \int_0^T e^{-rt} \left(-rV_{BS} + \frac{\partial V_{BS}}{\partial t} + (r - q) S_t \frac{\partial V_{BS}}{\partial S} + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S^2} \right) dt \\
&\quad + \int_0^T e^{-rt} \sigma_t S_t \frac{\partial V_{BS}}{\partial S} dW_t.
\end{aligned} \tag{28}$$

Notice that $V(t, S_t; \sigma_{BS})$ satisfies the Black-Scholes PDE,

$$\frac{\partial V_{BS}}{\partial t} + (r - q) S \frac{\partial V_{BS}}{\partial S} + \frac{1}{2} \sigma_{BS}^2 S^2 \frac{\partial^2 V_{BS}}{\partial S^2} = rV_{BS}, \tag{29}$$

and finally, we have

$$f(T) = f(0) + \int_0^T e^{-rt} \frac{S_t^2}{2} \frac{\partial^2 V_{BS}}{\partial S^2} \left(\sigma_t^2 - \sigma_{BS}^2 \right) dt + \int_0^T e^{-rt} \sigma_t S_t \frac{\partial V_{BS}}{\partial S} dW_t. \tag{30}$$

The expected value of the hedged position is then given by

$$E\left[f(T) \right] = f(0) + E\left[\int_0^T e^{-rt} \frac{S_t^2}{2} \frac{\partial^2 V_{BS}}{\partial S^2} \left(\sigma_t^2 - \sigma_{BS}^2 \right) dt \right]. \tag{31}$$

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From this, it is clear that, a delta-hedged option is sensitive to the difference between realized and implied variances. However, if one wants to trade the realized variance, the delta-hedged option is not a good choice, since the dollar gamma is peaked at the strike price. It can be shown, for a portfolio of options weighted by their inversed strike squared, $1/K^2$, the portfolio will be insensitive to the stock price move.

As a side result, notice that, in an arbitrage free world, we have

$$E\left[f(T)\right] = f(0),\tag{32}$$

which leads to

$$\sigma_{BS}^2 = \frac{E\left[\int_0^T e^{-rt}\sigma_t^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S^2}dt\right]}{E\left[\int_0^T e^{-rt}S_t^2 \frac{\partial^2 V_{BS}}{\partial S^2}dt\right]}.\tag{33}$$

This result implies that, the implied variance is the average of realized variance, weighted by dollar gamma. It can lead to interesting approximation for the implied volatility, for example, the so called "most likely path" method. However, in practice, the approximation is rather crude.

3.2   Replication of a continuously monitored variance swap

In the limit of continuous monitoring, the floating leg payment of a variance swap becomes the integrated instantaneous variance, $\int_0^T \sigma_u^2 du$. We can use the result from previous section to determine the replication of a continuously monitored variance swap. Suppose we risk manage our position with zero implied volatility, $i.e.$, $\sigma_{BS} = 0$ in Eq. (30). Then, Eq. (30) can be written as

$$\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{\partial^2 V_{BS}}{\partial S^2}\sigma_t^2 dt = f(T) - f(0) - \int_0^T e^{-rt}\sigma_t S_t \frac{\partial V_{BS}}{\partial S}dW_t.\tag{34}$$

If we assume the underlying follows the stochastic process

$$dS_t = (r - q)S_t dt + \sigma_t S_t dW_t,\tag{35}$$

the last term becomes

$$\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{\partial^2 V_{BS}}{\partial S^2}\sigma_t^2 dt = f(T) - f(0) - \int_0^T e^{-rt}\frac{\partial V_{BS}}{\partial S}\left(dS_t - (r - q)S_t dt\right).\tag{36}$$

Now, choose

$$\phi(S) = 2\left[\log\left(\frac{F}{S}\right) + \frac{S}{F} - 1\right],\tag{37}$$

then its price at time t with spot S_t is

$$V_{BS}(t, S_t; \sigma_{BS} = 0) = 2e^{-r(T-t)}\left[\log\left(\frac{F}{S_t}\right) - (r - q)(T - t) + \frac{S_t e^{(r-q)(T-t)}}{F} - 1\right].\tag{38}$$

The first and second order derivatives are

$$\frac{\partial V_{BS}}{\partial S} = 2e^{-r(T-t)} \left[\frac{e^{(r-q)(T-t)}}{F} - \frac{1}{S_t} \right], \tag{39}$$

and

$$\frac{\partial^2 V_{BS}}{\partial S^2} = \frac{2e^{-r(T-t)}}{S_t^2}, \tag{40}$$

repectively. With these derivatives, Eq. (36) becomes

$$
\begin{aligned}
\int_0^T \sigma_t^2 dt &= 2 \left[\log\left(\frac{F}{S_T}\right) + \frac{S_T}{F} - 1 \right] \\
&\quad -2 \left[\log\left(\frac{F}{S_0}\right) - (r-q)T + \frac{S_0 e^{(r-q)T}}{F} - 1 \right] \\
&\quad -2 \int_0^T \left[\frac{e^{(r-q)(T-t)}}{F} - \frac{1}{S_t} \right] (dS_t - (r-q)S_t dt).
\end{aligned} \tag{41}
$$

Expand the last line, we have

$$
\begin{aligned}
&\int_0^T \left[\frac{e^{(r-q)(T-t)}}{F} - \frac{1}{S_t} \right] (dS_t - (r-q)S_t dt) \\
&= \int_0^T \left[\frac{e^{(r-q)(T-t)}}{F} dS_t - \frac{dS_t}{S_t} - (r-q)S_t \frac{e^{(r-q)(T-t)}}{F} dt + (r-q)dt \right] \\
&= \frac{S_T}{F} - \frac{S_0 e^{(r-q)T}}{F} + \int_0^T \frac{dS_t}{S_t} + (r-q)T,
\end{aligned} \tag{42}
$$

where we have performed integration by parts for the third term on the second line in the above equations. Finally, we have

$$
\begin{aligned}
\int_0^T \sigma_t^2 dt &= 2 \left[\log\left(\frac{F}{S_T}\right) + \frac{S_T}{F} - 1 \right] - 2 \left[\log\left(\frac{F}{S_0}\right) + \frac{S_0}{F} - 1 \right] \\
&\quad -2 \int_0^T \left[\frac{1}{F} - \frac{1}{S_t} \right] dS_t \\
&= 2 \left[\log\left(\frac{S_0}{S_T}\right) + \int_0^T \frac{dS_t}{S_t} \right].
\end{aligned} \tag{43}
$$

This means that a continuously monitored variance swap can be replicated by a static position of a log contract, and a dynamically traded underlying position.

Now, write $\log(S_0/S_T)$ as $\log(S_0/S_*) + \log(S_*/S_T)$, and notice that the fair strike of the

variance swap is the risk neutral expectation of time averaged realized variance, we have

$$
\begin{aligned}
K_{var} &= E\left[\frac{1}{T}\int_0^T \sigma_t^2 dt\right] \\
&= \frac{2}{T}\left[(r-q)T - \frac{S_0}{S_*}e^{(r-q)T} + 1 - \log\left(\frac{S_*}{S_0}\right)\right. \\
&\qquad \left. +e^{rT}\int_0^{S_*} P(K)\frac{dK}{K^2} + e^{rT}\int_{S_*}^{+\infty} C(K)\frac{dK}{K^2}\right].
\end{aligned}
\tag{44}
$$

3.3   VIX

In Eq. (44), if we set the separator S_* with the forward price of the underlying, $i.e.$,

$$
S_* = F_T = S_0 e^{(r-q)T},
\tag{45}
$$

the fair strike K_{var} will become

$$
\begin{aligned}
K_{var} &= \frac{2e^{rT}}{T}\left[\int_0^{F_T} P(K)\frac{dK}{K^2} + \int_{F_T}^{+\infty} C(K)\frac{dK}{K^2}\right] \\
&= \frac{2e^{rT}}{T}\left[\int_0^{K_0} P(K)\frac{dK}{K^2} + \int_{K_0}^{+\infty} C(K)\frac{dK}{K^2} + \int_{K_0}^{F_T}\left(P(K)-C(K)\right)\frac{dK}{K^2}\right] \\
&= \frac{2e^{rT}}{T}\left[\int_0^{K_0} P(K)\frac{dK}{K^2} + \int_{K_0}^{+\infty} C(K)\frac{dK}{K^2}\right] + \frac{2}{T}\int_{K_0}^{F_T}\left(K-F_T\right)\frac{dK}{K^2} \\
&= \frac{2e^{rT}}{T}\left[\int_0^{K_0} P(K)\frac{dK}{K^2} + \int_{K_0}^{+\infty} C(K)\frac{dK}{K^2}\right] - \frac{1}{T}\left(\frac{F_T}{K_0}-1\right)^2.
\end{aligned}
\tag{46}
$$

After discretization, the fair strike is

$$
K_{var} = \frac{2e^{rT}}{T}\sum_i \frac{\Delta K_i}{K_i^2}Q(K_i) - \frac{1}{T}\left(\frac{F_T}{K_0}-1\right)^2.
\tag{47}
$$

Here, F_T is the forward index value derived from option prices. Specifically, from a strip of option prices, we determine the forward index value by identifying the strike price K_c at which the absolute difference between call and put option prices are the smallest. Here, the call and put option prices are taken to be the average of the corresponding bid and ask values. Then, the forward can be determined by

$$
F_T = K_c + e^{rT}\left(C(K_c) - P(K_c)\right),
\tag{48}
$$

where r is the prevailing risk-free rate for option expiry T, which is usually taken to be the corresponding US treasury yield. K_0 will be set to the strike level immediately below the forward value, and ΔK_i is half the difference between the two adjacent strikes below and above K_i, $i.e.$,

$\Delta K_i = (K_{i+1} - K_{i-1})/2$. For the highest and lowest strike to be used in the option strip, ΔK_i is calculated from only one side. Now, for K_i greater than K_0, $Q(K_i)$ will be set to the out-of-money call price with strike K_i. For K_i less than K_0, the out-of-money put price is used. For K_0, it is the average of the call and put option prices.

This is the basic component for the VIX index. For more details, see the VIX White Paper. [4]

3.4   Discretely monitored variance swap

Now, let us consider the floating leg payment of a discretely monitored variance swap [2]

$$V = \frac{N}{n} \sum_{i=1}^{n} \log^2 \left(\frac{S_i}{S_{i-1}} \right). \tag{49}$$

Define the simple return as

$$R_i = \frac{S_i - S_{i-1}}{S_i}, \tag{50}$$

the log return can be Taylor expanded to third order in terms of the simple return,

$$\log \left(\frac{S_i}{S_{i-1}} \right) = R_i - \frac{1}{2} R_i^2 + \frac{1}{3} R_i^3. \tag{51}$$

Squaring both sides, we have

$$\log^2 \left(\frac{S_i}{S_{i-1}} \right) = R_i^2 - R_i^3. \tag{52}$$

Solving for R_i^2 and plug back into Eq. (51),

$$\log \left(\frac{S_i}{S_{i-1}} \right) = R_i - \frac{1}{2} \log^2 \left(\frac{S_i}{S_{i-1}} \right) - \frac{1}{6} R_i^3, \tag{53}$$

which finally leads to

$$\log^2 \left(\frac{S_i}{S_{i-1}} \right) = 2R_i - 2 \log \left(\frac{S_i}{S_{i-1}} \right) - \frac{1}{3} R_i^3. \tag{54}$$

Therefore, the floating leg payment becomes

$$\sum_{i=1}^{n} \log^2 \left(\frac{S_i}{S_{i-1}} \right) = \sum_{i=1}^{n} \frac{2}{S_{i-1}} (S_i - S_{i-1}) - 2 \log \left(\frac{S_n}{S_0} \right) - \frac{1}{3} \sum_{i=1}^{n} R_i^3. \tag{55}$$

Using Eq. (11), $\log S_n$ can be expanded around S_0,

$$\log S_n = \log S_0 + \frac{1}{S_0} (S_n - S_0) - \int_0^{S_0} (K - S_n)^+ \frac{dK}{K^2} - \int_{S_0}^{\infty} (S_n - K)^+ \frac{dK}{K^2}, \tag{56}$$

we can obtain the desired decomposition,

$$
\sum_{i=1}^{n} \log^2 \left(\frac{S_i}{S_{i-1}} \right) = 2 \sum_{i=1}^{n} \left(\frac{1}{S_{i-1}} - \frac{1}{S_0} \right) (S_i - S_{i-1})
$$

$$
+ 2 \left[\int_0^{S_0} (K - S_n)^+ \frac{dK}{K^2} + \int_{S_0}^{\infty} (S_n - K)^+ \frac{dK}{K^2} \right] - \frac{1}{3} \sum_{i=1}^{n} R_i^3 \qquad (57)
$$

If the third order term sums to a negative number, the variance swap will be mispriced when only the first two terms are taken into account for replication.

If we use the simple return in the variance swap floating leg payment, instead of the log return, we can obtain a similar decomposition,

$$
\sum_{i=1}^{n} \left(\frac{S_i}{S_{i-1}} \right)^2 = 2 \sum_{i=1}^{n} \left(\frac{1}{S_{i-1}} - \frac{1}{S_0} \right) (S_i - S_{i-1})
$$

$$
+ 2 \left[\int_0^{S_0} (K - S_n)^+ \frac{dK}{K^2} + \int_{S_0}^{\infty} (S_n - K)^+ \frac{dK}{K^2} \right] + \frac{2}{3} \sum_{i=1}^{n} R_i^3, \qquad (58)
$$

which differs only by the third order term. We can see that, if the cubed simple return sums to a negative number and the variance swap is priced and risk managed with static positions in options, then the variance swap with log return is preferred from a buyer's perspective.

4 Weighted variance swap

The method developed in the previous section can be applied to other types of variance swap. We will show a few examples in this section.

4.1 Corridor variance swap

The floating leg payment of the corridor variance swap is

$$
A \times \frac{N}{n} \sum_{i=1}^{n} \mathcal{I}_{L \leq S_i \leq H} \log^2 \left(\frac{S_i}{S_{i-1}} \right), \qquad (59)
$$

i.e., the contribution to the realized variance is only taken into account if the stock price falls in the corridor $[L, H]$.

Take [3]

$$
\phi(S) = 2 \left[\log \left(\frac{F}{S'} \right) + S \left(\frac{1}{F} - \frac{1}{S'} \right) \right], \qquad (60)
$$

where

$$
S' = \max \left[L, \min (S, H) \right] \qquad (61)
$$

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is floored at L and capped at H, *i.e.*, the value of S' is restricted in the corridor $[L, H]$. Following the same argument as for the variance swap, we have

$$\int_0^T \sigma_t^2 \mathcal{I}_{L \leq S \leq H} dt = 2\left[\log\left(\frac{F}{S_T'}\right) + \frac{S_T'}{F} - 1\right] - 2\left[\log\left(\frac{F}{S_0}\right) + \frac{S_0}{F} - 1\right]$$
$$- 2\int_0^T \left[\frac{1}{F} - \frac{1}{S_t'}\right] dS_t. \tag{62}$$

4.2 Gamma swap

The floating leg payment of the gamma swap is

$$A \times \frac{N}{n} \sum_{i=1}^n \frac{S_i}{S_0} \log^2\left(\frac{S_i}{S_{i-1}}\right). \tag{63}$$

Take

$$\phi(S) = \frac{2}{S_0}\left[S\log\left(\frac{S}{F}\right) - S + F\right], \tag{64}$$

then

$$\int_0^T \frac{S}{S_0}\sigma_t^2 dt = \frac{2}{S_0}\left[S_T\log\left(\frac{S_T}{F}\right) - S_T + F\right] - \frac{2}{S_0}\left[S_0\log\left(\frac{S_0}{F}\right) - S_0 + F\right]$$
$$- \frac{2}{S_0}\int_0^T \log\left(\frac{S_t}{F}\right) dS_t. \tag{65}$$

5 Volatility swap

A volatility swap is a forward contract on future realized volatility. The volatility swap payoff lies closely to the variance swap, and the floating leg payment is

$$A\sqrt{\times\frac{N}{n}\sum_{i=1}^n \log^2\left(\frac{S_i}{S_{i-1}}\right)}, \tag{66}$$

i.e., the square root of Eq. (25). Due to the square root, volatility swap is more complicated than variance swap in valuation and hedging. The valuation of a volatility swap generally requires a stochastic volatility model, and depends on the correlation between the underlying and the stochastic volatility processes. However, as shown by Carr and Lee [5], there exists model-free replication strategies for the volatility swaps, insensitive to the correlation in the first oder.

5.1 Valuation under stoachstic volatility

For a European payoff $F(S, \omega)$, where ω represents other stochastic factors that affect the payoff, define

$$F^{BS}(S, \sigma, \omega) = \int_0^{+\infty} F(Sy, \omega)\frac{1}{y\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y + \sigma^2/2)^2}{2\sigma^2}\right) dy. \tag{67}$$

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This gives the Black-Scholes result if there is no ω dependence.

Consider the following stochastic process for an underlying,

$$dS_t = \sqrt{1-\rho^2}\sigma_t S_t dW_{1t} + \rho\sigma_t S_t dW_{2t}, \tag{68}$$

where the volatility process σ_t is independent on the Brownian motion W_1, and the underlying is correlated with the volatility with correlation ρ. Let $X_t \equiv \log(S_t/S_0)$, then

$$dX_t = -\frac{1-\rho^2}{2}\sigma_t^2 dt + \sqrt{1-\rho^2}\sigma_t dW_{1t} - \frac{\rho^2}{2}\sigma_t^2 dt + \rho\sigma_t S_t dW_{2t}. \tag{69}$$

Conditioning on W_1, X_t is normally distributed,

$$X_T \sim N\left(X_t + \log M_{t,T}(\rho) - \frac{1-\rho^2}{2}\bar{\sigma}_{t,T}^2 dt, \sqrt{1-\rho^2}\bar{\sigma}_{t,T}\right), \tag{70}$$

where

$$M_{t,T}(\rho) = \exp\left(-\frac{\rho^2}{2}\int_t^T \sigma_u^2 du + \rho\int_t^T \sigma_u dW_u\right), \tag{71}$$

and

$$\bar{\sigma}_{t,T}^2 = \int_t^T \sigma_u^2 du. \tag{72}$$

Now, apply repeated expectation, we have

$$E_t[F(S_T)] = E_t\left[F^{BS}\left(S_t M_{t,T}(\rho), \bar{\sigma}_{t,T}\sqrt{1-\rho^2}\right)\right]. \tag{73}$$

Clearly, this result depends on the correlation ρ. If we can find a payoff $F(S)$ which can render a correlation-insensitive result, or at least correlation-insensitive locally around $\rho = 0$, then we can have (nearly) model-free valuation for the volatility swap. This can be achieved by expanding the above result around $\rho = 0$,

$$\begin{aligned}
E_t[F(S_T)] &= E_t\left[F^{BS}\left(S_t M_{t,T}(\rho), \bar{\sigma}_{t,T}\sqrt{1-\rho^2}\right)\right] \\
&= E_t\left[F^{BS}(S_t, \bar{\sigma}_{t,T})\right] + \rho S_t E_t\left[\frac{\partial F^{BS}(S_t, \bar{\sigma}_{t,T})}{\partial S}\int_t^T \sigma_u dW_u\right]. \tag{74}
\end{aligned}$$

If $\partial F^{BS}/\partial S$ does not depend on $\bar{\sigma}_{t,T}$, then the expectation in the last line will be zero, and the above result will be correlation-insensitive, with an error of order $O(\rho^2)$.

5.2 Characteristic function of realized variance

Define

$$X_t \equiv \log\frac{S_t}{S_0}, \tag{75}$$

and

$$\langle X\rangle_t \equiv \int_0^t \sigma_u^2 du. \tag{76}$$

Assuming the underlying process Eq. (68), these are related by

$$\langle X \rangle_t = -2X_t + \int_0^t \frac{2}{S_u} dS_u. \tag{77}$$

Consider the realized variance from t to T,

$$X_T - X_t = -\frac{1}{2} \left(\langle X \rangle_T - \langle X \rangle_t \right) + \int_t^T \sigma_u dW_u, \tag{78}$$

which means that $X_T - X_t$ is normally distributed, *i.e.*,

$$X_T - X_t \sim N \left(-\frac{1}{2} \left(\langle X \rangle_T - \langle X \rangle_t \right), \langle X \rangle_T - \langle X \rangle_t \right). \tag{79}$$

Therefore,

$$E_t \left[\exp \left\{ p \left(X_T - X_t \right) \right\} \right] = E_t \left[\exp \left\{ \frac{1}{2} \left(p^2 - p \right) \left(\langle X \rangle_T - \langle X \rangle_t \right) \right\} \right]. \tag{80}$$

Denote $\lambda = (p^2 - p)/2$, we can solve p for λ, and get

$$p = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}. \tag{81}$$

So, the characteristic function for the realized variance is

$$E_t \left[\exp \left\{ \lambda \left(\langle X \rangle_T - \langle X \rangle_t \right) \right\} \right] = E_t \left[\exp \left\{ p_\pm \left(X_T - X_t \right) \right\} \right]. \tag{82}$$

5.3   Correlation-insensitive replication of a volatility swap

Now, let us consider the replication and valuation of the volatility swap. Use the following identity,

$$\sqrt{q} = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \frac{1 - 2^{-zq}}{z^{3/2}} dz, \tag{83}$$

the floating leg payment of the volatility swap can be expressed as

$$E_t \left[\sqrt{\langle X \rangle_T - \langle X \rangle_t + q} \right] = \frac{1}{2\sqrt{\pi}} E_t \left[\int_0^{+\infty} \left(1 - \exp \left\{ -z \left(\langle X \rangle_T - \langle X \rangle_t + q \right) \right\} \right) \right] \frac{dz}{z^{3/2}}. \tag{84}$$

Here, we have considered a very general scenario, so that we can handle both a newly issued volatility swap and a seasoned one.

The integration and the expectation in the above equation can be exchanged, then

$$\begin{aligned}
E_t \left[\sqrt{\langle X \rangle_T - \langle X \rangle_t + q} \right] &= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \sum_\pm \theta_\pm \left(1 - e^{-zq} E_t \left[\exp \left\{ p_\pm \left(X_T - X_t \right) \right\} \right] \right) \frac{dz}{z^{3/2}} \\
&= \frac{1}{2\sqrt{\pi}} E_t \left[\int_0^{+\infty} \sum_\pm \theta_\pm \left(1 - e^{-zq} \exp \left\{ p_\pm \left(X_T - X_t \right) \right\} \right) \frac{dz}{z^{3/2}} \right].
\end{aligned} \tag{85}$$

Here,

$$p_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2z}, \tag{86}$$

and $\theta_+ + \theta_- = 1$. We can use the freedom in choosing $\theta_\pm$ to achieve the correlation immunity. With zero correlation, $\rho = 0$, S_T is lognormally distributed. The expectation in the above equation can be transform into an integration with respect to a lognormal density with parameters $(-v/2, v)$. Then, the derivative in Eq. (74) becomes

$$\begin{aligned}
\frac{\partial F^{BS}(S_t)}{\partial s} &= \frac{1}{2\sqrt{\pi}} \left.\frac{\partial}{\partial s}\right|_{s=S_t} \int_0^{+\infty} \left[\int_0^{+\infty} \sum_\pm \theta_\pm \left(1 - e^{-zq} (sy/S_t)^{p_\pm} \right) \frac{dz}{z^{3/2}} \right] \phi_v(y) dy \\
&= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \int_0^{+\infty} \frac{-e^{-zq} \left(\theta_+ p_+ y^{p_+} + \theta_- p_- y^{p_-} \right)}{S_t z^{3/2}} \phi_v(y) dz dy.
\end{aligned} \tag{87}$$

Since

$$\int_0^{+\infty} y^{p_+} \phi_v(y) dy = \int_0^{+\infty} y^{p_-} \phi_v(y) dy, \tag{88}$$

the correlation immunity can be achieved by

$$\theta_+ p_+ + \theta_- p_- = 0, \tag{89}$$

which leads to

$$\theta_\pm = \frac{1}{2} \mp \frac{1}{2\sqrt{1-8z}}. \tag{90}$$

Finally, the correlation-immune price of a volatility swap is given by

$$E_t\left[\sqrt{\langle X \rangle_T} \right] = E_t\left[G_{SVS}\left(S_T, S_t, \langle X \rangle_t \right) \right], \tag{91}$$

where

$$G_{SVS}\left(S_T, S_t, q \right) = \frac{1}{2\sqrt{\pi}} \int_t^{+\infty} \left(\theta_+ \left[1 - e^{-zq} \left(\frac{S_T}{S_t} \right)^{p_+} \right] + \theta_- \left[1 - e^{-zq} \left(\frac{S_T}{S_t} \right)^{p_-} \right] \right) \frac{dz}{z^{3/2}}. \tag{92}$$

At inception, the above expression can be simplified,

$$E_0\left[\sqrt{\langle X \rangle_T} \right] = E_0\left[\psi(S_T) \right], \tag{93}$$

where $\psi(S) = \phi(\log(S/S_0))$, and

$$\phi(x) = \sqrt{\frac{\pi}{2}} e^{x/2} \left| x I_0\left(\frac{x}{2} \right) - x I_1\left(\frac{x}{2} \right) \right|. \tag{94}$$

Here, I_ν is the modified Bessel function of order ν.

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