Homework 5

Requirements:

1. Digital format (can be typeset or photos, ought to write clearly if written by hand), upload to `https://course.pku.edu.cn/`.
2. Submit by next class
3. A problem is not counted if nobody can work it out
4. Each homework 10 points; 1 point deducted for each week's delay

Problems:

1. Use the trick of checking the convexity of 1D sections to prove that

   (a) $f(\mathbf{X}) = \mathrm{tr}(\mathbf{X}^{-1})$ is convex on $\mathbf{dom}\, f = \mathbb{S}^n_{++}$.

   (b) $f(\mathbf{X}) = (\det \mathbf{X})^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbb{S}^n_{++}$.

2. For strictly convex and differentiable f, prove the identity:

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{z} \rangle = B_f(\mathbf{z}, \mathbf{x}) + B_f(\mathbf{x}, \mathbf{y}) - B_f(\mathbf{z}, \mathbf{y}).$$

3. Show that $B_f(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$ but not necessarily in $\mathbf{y}$.
4. Let

$$\phi(\mathbf{x}) = \frac{\beta \|\mathcal{A}\|_2^2}{2} \|\mathbf{x} - \mathbf{u}\|^2 - \frac{\beta}{2} \|\mathcal{A}\mathbf{x} - \mathbf{v}\|^2,$$

where $\mathbf{u}$ and $\mathbf{v}$ are constant vectors. Show that $B_\phi(\mathbf{x}, \mathbf{x}')$ actually does not depend on $\mathbf{u}$ and $\mathbf{v}$.

5. Compute the subgradients of $f(\mathbf{x}) = \frac{1}{2} x_1^2 + |x_2|$, $\max(x, x^2)$, and $\|\mathbf{X}\|_{2,1}$ (matrix (p,q)-norm).

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that $\partial f : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a monotone mapping, i.e.,

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq 0, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2.$$

Further, if f is μ-strongly convex, then the above inequality can be strengthened as

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq \mu \|\mathbf{x}_1 - \mathbf{x}_2\|^2, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2.$$

7. Show that $f(\mathbf{x}) = \sum_{i=1}^r \alpha_i x_{[i]}$ is a convex function of $\mathbf{x}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the ith largest component of $\mathbf{x}$.

8. Prove that $f_k(\mathbf{X}) = \sum\limits_{i=1}^{k} \sigma_i(\mathbf{X})$ is a convex function of $\mathbf{X}$ for all $k = 1, \cdots,$ rank$(\mathbf{X})$.