Homework 10

Requirements:

1. Digital format (can be typeset or photos, ought to write clearly if written by hand), upload to `https://course.pku.edu.cn/`.
2. Submit by next class
3. A problem is not counted if nobody can work it out
4. Each homework 10 points; 1 point deducted for each week's delay

Problems:

1. With $f(\mathbf{x}) := x_1^2 + x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$ consider

$$(P) \begin{cases} \min\limits_{\mathbf{x}} f(\mathbf{x}) \\ -x_2 \leq 0 \\ x_1^3 - x_2 \leq 0 \\ x_1^3(x_2 - x_1^3) \leq 0 \end{cases}.$$

a) Determine the linearizing cone, the tangent cone and the feasible direction cones at the (strict global) minimal point $\mathbf{x}_0 := (0,0)^T$.

b) Find all its KKT points. Do they all correspond to local minima?

c) Check whether SCQ holds.

d) Check whether GCQ and ACQ hold at its KKT points.

e) Find its dual function, with the domain specified.

f) Write down its dual problem.

2. Find the point $\mathbf{x} \in \mathbb{R}^2$ that lies closest to the point $\mathbf{p} := (2,3)^T$ under the constraints $g_1(\mathbf{x}) := x_1 + x_2 \leq 0$ and $g_2(\mathbf{x}) := x_1^2 - 4 \leq 0$.

a) Verify that the problem fulfills SCQ.

b) Determine the KKT points by differentiating between three cases: none is active, exactly the first one is active, exactly the second one is active.

c) Find its dual function, with the domain specified.

d) Write down its dual problem.

3. Given a support vector machine:

$$\min_{\mathbf{w},\beta} \frac{1}{2}\|\mathbf{w}\|^2,$$
$$s.t.\ y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + \beta) \geq 1, (i = 1, \cdots, m).$$

a) Check whether the problem fulfills SCQ. What does SCQ mean in this scenario?

b) Find its dual function, with the domain specified.

c) Write down its dual problem.

4. Write down the dual problem of the Regularized Empirical Risk Minimization problem:

$$\min_{\mathbf{x},\mathbf{y}} F(\mathbf{x}) \equiv \frac{1}{n}\sum_{i=1}^{n} \phi_i(y_i) + \frac{\mu}{2}\|\mathbf{x}\|^2,$$
$$s.t.\ y_i = \mathbf{a}_i^T \mathbf{x}.$$

5. Express the dual problem of

$$\min_{\mathbf{x}} \ \mathbf{c}^T \mathbf{x}$$
$$s.t.\ f(\mathbf{x}) \leq 0,$$

with $\mathbf{c} \neq \mathbf{0}$, in terms of the conjugate f^*. Explain why the problem you get is convex. We do not assume that f is convex.