优化第6

1. (a) $\forall x_1, x_2 \in \text{dom} f$ $\theta \in [0,1]$ $x = \theta x_1 + (1-\theta) x_2$.

$\sum_{i=1}^m e^{a_i^T x + b_i} \le \theta \sum_{i=1}^m e^{a_i^T x_1 + b_i} + (1-\theta) \sum_{i=1}^m e^{a_i^T x_2 + b_i} \le 1$ $\therefore x \in \text{dom} f$ $\therefore \text{dom} f$ 凸.

↑ e^x 凸且↗, $a_i^T x + b_i$ 凸.

设 $g(x) = \log \sum e^{a_i^T x + b_i}$ 则 $g(x)$ 是凸函数 ($\because$ 因 $\log \sum e^{y_i}$ 凸且↗, $a_i^T x + b_i$ 凸)

又 $x \in \text{dom} f$ $\therefore g(x) < 0$ $\Rightarrow -\log(-g(x))$ 凸 也即 $f(x)$ 凸.

(b) $g(x,u) = \frac{x^T x}{u}$ 凸 $\Rightarrow \text{epi} g = \{(x,u,t) \mid t \ge g(x)\} = \{(x,u,t) \mid ut \ge x^T x\}$ = dom f 凸.

$h(z_1, z_2) = -\sqrt{z_1 z_2}$ $z_1 = u$ $z_2 = g(x,u)$.

则 $s(x,u) = h(z_1(x,u), z_2(x,u)) = -\sqrt{u(u - \frac{x^T x}{u})} = -\sqrt{u - x^T x}$

z_1, z_2 凹. h 凸且↓ $\Rightarrow s(x,u)$ 是凸函数

$ts(\frac{x}{t}, \frac{u}{t}) = -t\sqrt{\frac{u}{t} - \frac{x^T x}{t^2}} = -\sqrt{ut - x^T x} = f(x,u,t)$ 由透视知 f 凸.

(c) 同(b)知 dom f 凸

$g_1(u,v,x) = uv$ 在 dom f 上凹. ($\nabla^2 g$ 半负定 $\nabla^2 g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$).

$g_2(u,v,x) = -x^T x$ 在 dom f 上凹.

$\therefore -uv + x^T x$ 在 dom f 上凸. 且 $-uv + x^T x < 0$.

$\therefore f(x) = -\log(-(-uv + x^T x))$ 在 dom f 上凸.

(d) $\|x\|_p$ 是凸函数 $\therefore \text{epi} \|x\|_p = \{(x,t) \mid t \ge \|x\|_p\}$ = dom f 是凸函数.

~~设 $g(x,y) = -x^{\frac{1}{p}} y^{1-\frac{1}{p}}$ 则 g 是凸函数且单调不增.~~

~~$z_1 = t$ 凹~~ $z_1(x,t) = t$ 凹 ~~$h(s) = -\|s\|_p$~~

$z_2(x,t) = t - \frac{\|x\|_p^p}{t^{p-1}} = t h(\frac{x}{t}) = t - \frac{\|x\|_p^p}{t^{p-1}}$

设 $g(x,y) = -x^{\frac{1}{p}} y^{1-\frac{1}{p}}$ g 凸且不增

$z_1(x,t) = t$ 凹

$z_2(x,t) = t - \frac{\|x\|_p^p}{t^{p-1}}$ 凹

则 $g(z_1(x,t), z_2(x,t))$

$= -(t - \frac{\|x\|_p^p}{t^{p-1}})^{\frac{1}{p}} \cdot t^{1-\frac{1}{p}}$

$= -((t - \frac{\|x\|_p^p}{t^{p-1}}) \cdot t^{p-1})^{\frac{1}{p}} = f(x,t)$.

$\therefore f(x,t)$ 凸.

(2) 同(d)知 dom f 凸.

~~$g(x,y) = g(xy) = -\ln(xy)$ 凸~~

$g(x,y) = -\ln xy$ 凸且不增 ($\nabla^2 g = \begin{bmatrix} \frac{1}{x^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix} > 0$)

$z_1(x,t)$, $z_2(x,t)$ 同(d) 则 $g(z_1(x,t), z_2(x,t)) = f(x,t)$ ∴ f 凸.

凹.

2.

$$g_i(w) = -y_i \ln h(x_i) - (1-y_i)\ln(1-h(x_i))$$

$$= -y_i \ln\left(\frac{1}{1+e^{-w^T x_i}}\right) - (1-y_i)\ln\left(\frac{e^{-w^T x_i}}{1+e^{-w^T x_i}}\right)$$

$$= y_i \ln(1+e^{-w^T x_i}) + (1-y_i)(w^T x_i + \ln(1+e^{-w^T x_i}))$$

$$= (1-y_i) w^T x_i + \ln(1+e^{-w^T x_i})$$

~~$f(x) = \ln(1+e^{-x})$~~ $f(x) = \ln(1+e^{-x})$ $f'(x) = \frac{-e^{-x}}{1+e^{-x}}$ $f''(x) = \frac{e^{-x}(1+e^{-x}) + e^{-x}e^{-x}}{(1+e^{-x})^2} > 0$

∴ $f(x)$ 凸且不增 又 $w^T x_i$ 关于 w 凹 ∴ $\ln(1+e^{-w^T x_i})$ 关于 w 凸.

$\Rightarrow g_i(w)$ 关于 w 凸 则 $H(w) = \sum_{i=1}^{N} g_i(w)$ 关于 w 凸.

3. (a) 若 $\exists i$ $y_i < 0$ 取 $x_i = -t$, $x_j = 0$ $j \neq i$ 则 $f^*(y) = -ty_i \to +\infty$ 当 $t \to \infty + \infty$

若 $\exists i$ $y_i > 1$ 取 $x_i = t$. $x_j = 0$ $j \neq i$ 则 $f^*(y) = ty_i - t \to +\infty$ 当 $t \to +\infty$.

当 $0 \le y \le 1$ ($0 \le y_i \le 1$ $\forall i$)

① $1^T y \neq r$ 取 $x = t1$ ~~$\langle y, x \rangle - f(x) = t1^T y - rt = t(1^T y - r) \to +\infty$~~

($t \to -\infty$ 或 $t \to +\infty$).

② $1^T y = r$ ~~$f^*(y) = \langle y, x \rangle - f(x) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]}$~~

~~y_i 为 $x_{[i]}$ 对应 $\sum x_i y_i = \sum x_i y_i$~~

不妨 $x_1 \ge x_2 \ge \cdots \ge x_r \ge \cdots \ge x_n$

$$\langle y, x \rangle - f(x) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_i = \sum_{i=1}^{r} x_i(y_i - 1) + \sum_{i=r+1}^{n} x_i y_i$$

$$\le x_r\left(\sum_{i=1}^{r}(y_i - 1) + \sum_{i=r+1}^{n} y_i\right) = 0.$$

($y_i \le 1$, $x_j \le x_r$, $\forall j > r$)

$x = 0$ 时 $\langle y, x \rangle - f(x) = 0$ ∴ $f^*(y) = 0$

综上 $f^*(y) = \begin{cases} 0 & 0 \le y_i \le 1,\ \sum y_i = 1 \\ +\infty & \text{else } y \end{cases}$ $\text{dom } f^* = \{y \mid 0 \le y_i \le 1,\ \sum_{i=1}^{n} y_i = 1\}$.

(b)

$y \notin [a_1, a_m]$ 时 $\langle y, x\rangle - f(x) = xy - \max\limits_{1\le i\le n} a_i x + b_i$

$= \max\limits_{1\le i\le n}(xy - (a_i x + b_i))$

∵ $y \notin [a_1, a_m]$ ∴ $xy - (a_i x + b_i)$ 均无界. $\Rightarrow f^*(y) = +\infty$.

$y \in [a_i, a_{i+1}]$ 时 ~~$xy - a_j x - b$~~ ~~$x \ge 0$ 时~~

~~$xy - (a_j x + b_j) = (y - a_j)x + b \le (y - a_{i+1})x + b_i$~~

根据几何意义 x取 $a_i x + b_i$ 和 $a_{i+1}x + b_{i+1}$ 相交的点时 $\langle y, x\rangle - f(x)$ 最大.

∴ $f^*(y) = \langle y, \frac{b_{i+1} - b_i}{-a_{i+1} + a_i}\rangle - f(\frac{b_{i+1} - b_i}{a_i - a_{i+1}}) = (y - a_i)\left(\frac{b_{i+1} - b_i}{a_i - a_{i+1}}\right) - b_i$

$\mathrm{dom} f^* = [a_1, a_m]$

(c) $p > 1$ $\langle y, x\rangle - f(x) = xy - x^p := g(x)$

$g'(x) = y - p x^{p-1}$ 若 $y \le 0$ 则 $g'(x) \le 0$ $\Rightarrow \sup\limits_x g(x) = g(0) = 0$.

若 $y > 0$ 则 $g'(x) = 0 \Rightarrow x_0 = (y/p)^{\frac{1}{p-1}}$

$\sup\limits_x g(x) = g(x_0) = x_0 y - \frac{x_0 y}{p} = (1 - \frac{1}{p})(\frac{y}{p})^{\frac{1}{p-1}} \cdot y$

$= (p-1)(\frac{y}{p})^{\frac{p}{p-1}}$

∴ $f^*(y) = \begin{cases} 0 & y \le 0 \\ (p-1)(\frac{y}{p})^{\frac{p}{p-1}} & y > 0 \end{cases}$ $\mathrm{dom} f^* = R$

$p < 0$ $g'(x) = y - p x^{p-1}$ 同理有 $y \ge 0$ 时 $\sup\limits_x g(x) = g(0) = +\infty$

$y < 0$ 时 $g'(x) = 0 \Rightarrow x_0 = (\frac{y}{p})^{\frac{1}{p-1}}$

$\sup g(x) = g(x_0) = (p-1)(\frac{y}{p})^{\frac{p}{p-1}}$

∴ $f^*(y) = (p-1)(\frac{y}{p})^{\frac{p}{p-1}}$ $y < 0$ $\mathrm{dom} f^* = -R_{++}$

(d) $\langle y, x\rangle - f(x) = \sum x_i y_i + \sqrt[n]{x_1 \cdots x_n}$

若∃ $y_i \ge 0$ 取 $x_i = t$, $x_j = 0$ $j \ne i$ $\langle y, x\rangle - f(x) = t x_i \to +\infty$ $t \to +\infty$

若∃ $y_i = 0$ 取 $x_i = t$ $x_j = 1$ $j \ne i$ $\langle y, x\rangle - f(x) = \sum x_i y_i + \sqrt[n]{t} \to +\infty$ $t \to \infty$.

∴ $y_i < 0$ $-\sum x_i y_i = \sum(-y_i x_i) \ge n(\prod(-y_i x_i))^{\frac{1}{n}} = n(\prod(-y_i))^{\frac{1}{n}}(\prod x_i)^{\frac{1}{n}}$

∴ $\langle y, x\rangle - f(x) \le (\prod x_i)^{\frac{1}{n}}(-n(\prod(-y_i))^{\frac{1}{n}} + 1)$. ①

若 $(\prod(-y_i))^{\frac{1}{n}} < \frac{1}{n}$ 取 $x_i = \frac{t}{y_i}$ 则 等号成立. $\langle y, x\rangle - f(x) = t(n(\prod(-y_i))^{\frac{1}{n}} - 1) \to +\infty$ $t \to \infty$

∴ $\prod(-y_i)^{\frac{1}{n}} \ge \frac{1}{n}$ 此时 由① 知 $\langle y, x\rangle - f(x) \le 0$.

当 $x \to 0^+$ 时 $\langle x, y\rangle - f(x) \to 0$ ∴ $f^*(y) = 0$.

综上 $\mathrm{dom} f^* = \{y \mid y_i < 0, \prod(-y_i)^{\frac{1}{n}} \ge \frac{1}{n}\}$ $f^*(y) = \begin{cases} 0 & y \in \mathrm{dom} f^* \\ \infty & \text{else} \end{cases}$

(2)

$h(x,t) = y^Tx + tu + \log(t^2 - x^Tx)$

对 x, t 求导并等于 0，可解得 $\begin{cases} x_0 = \dfrac{2y_0}{u^2 - y^Ty} \\ t_0 = -\dfrac{2u}{u^2 - y^Ty} \end{cases}$

$\|x_0\|_2 < t_0 \Rightarrow \|y_0\| < -u$.

若 $\|y\|_2 \geq -u$　若 $u > 0$　$h(x,t) \to +\infty$　$t \to +\infty$, $x^Tx = t\,\log$.

若 $u \leq 0$　取 $x_i = \dfrac{y_i \lambda}{\|y\|_2}$　$t^2 = x^Tx + 1$　$x^Tx = \lambda^2$.

则 $h(x,t) = \dfrac{\|y\|_2^2 \lambda}{\|y\|_2} + tu + \log 1 = \lambda\|y_2\| + \sqrt{x^Tx+1}\, u$

$= \lambda\|y_2\| + \sqrt{\lambda^2+1}\, u$.

$= \lambda(\|y_2\| - (-u)) + (\sqrt{\lambda^2+1} - \lambda) u$.

$= \lambda(\|y_2\| - (-u)) + \dfrac{u}{\sqrt{\lambda^2+1} + \lambda}$

当 $\lambda \to +\infty$ 时 $h(x,t) \to +\infty$　（取 $t = \|x\|_2 + 1$ 可处理 $\|y\|_2 = -u$ 时的情况）.

$\therefore \mathrm{dom} f^* = \{(y,u) \mid (y,u) \in R^n \times R,\ \|y\|_2 < -u.\}$

$f^*(y,u) = \begin{cases} h(x_0,t_0) = -2 + \ln 4 - \ln(y^Ty - u^2) & y,u \in \mathrm{dom} f^* \\ \infty & \text{else.} \end{cases}$

4. (a)　设 $f^*(y) = \sup_x \langle y,x\rangle - f(x)$.

$f^*(y-c) = \sup_x \langle y-c, x\rangle - f(x) = \sup_x \langle y,x\rangle - c^Tx - f(x) = g^*(y) + d$.

$\Rightarrow g^*(y) = f^*(y-c) - d$　　$\mathrm{dom}\, g^* = \{y \mid c + y \in \mathrm{dom} f^*\}$.

(b)　$g(x,t) = t f(\frac{x}{t})$　$t > 0$

$g^*(y,s) = \sup_{x,t,\frac{x}{t} \in \mathrm{dom} f} y^Tx + ts - t f(\frac{x}{t}) = \sup_{x,t,\frac{x}{t} \in \mathrm{dom} f} t\left(y^T\frac{x}{t} - f(\frac{x}{t})\right) + ts$

设 $z = \frac{x}{t} \in \mathrm{dom} f$.

$= \sup_{t>0,\ z \in \mathrm{dom} f} t(y^Tz - f(z)) + ts$

$= \sup_{t>0} t f^*(y) + ts$

$\mathrm{dom}\, g^* = \{(y,s) \mid s \leq -f^*(y)$

~~① $f^*(y) + s = +\infty \Rightarrow f^*(y) = +\infty$　$g^*(y,s) = +\infty$~~

① $f^*(y) + s > 0$　$t \to \infty$ 时 $t(f^*(y) + s) \to +\infty$　$g^*(y,s) = +\infty$

② $f^*(y) + s \leq 0$　$t \to 0$ 有 $t(f^*(y) + s) \to 0$　$g^*(y,s) = 0$.

$\Rightarrow g^*(y,s) = \begin{cases} 0 & y,s \in \mathrm{dom}\, g \\ +\infty & \text{else} \end{cases}$

(c) $g^*(y) = \sup_x (y^Tx - \inf_z f(x,z)) = \sup_x (y^Tx + \sup_z (-f(x,z))) = \sup_x \sup_z (y^Tx - f(x,z))$

$= \sup_{x,z} (y^Tx + 0\cdot z - f(x,z)) = f^*(y,0)$ $\quad dom\, g^* = \{y \mid (y,0) \in dom\, f^*\}$

$f(x,z) = \begin{cases} h(z) & Az+b=x \\ \infty & \text{else} \end{cases}$

$f^*(y,s) = \sup_{\substack{x,z \\ Az+b=x}} (y^Tx + s^Tz - h(z))$

$= \sup_z (y^T(Az+b) + s^Tz - h(z))$

$= y^Tb + \sup_z (y^TAz + s^Tz - h(z))$

$= y^Tb + h^*(A^Ty + s)$

$\Rightarrow g^*(y) = f^*(y,0) = y^Tb + h^*(A^Ty)$ $\quad dom\, g^* = \{y \mid (y,0) \in dom\, f^*\}$

$= \{y \mid A^Ty \in dom\, h^*\}$