Homework 2

Requirements:

1. Digital format (can be typeset or photos, ought to write clearly if written by hand), upload to `https://course.pku.edu.cn/`.
2. Submit by next class
3. A problem is not counted if nobody can work it out
4. Each homework 10 points; 1 point deducted for each week's delay

Problems:

1. Compute the condition number of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 9 \end{bmatrix}.$$

2. Suppose $\mathbf{X} \in \mathbb{R}^{3\times 3}$, $\mathcal{A}(\mathbf{X}) = X_{11} + X_{12} - X_{31} + 2X_{33}$, find $\mathcal{A}^*$.

3. Suppose $\mathbf{X} \in \mathbb{R}^{2\times 3}$, $\mathcal{A}(\mathbf{X}) = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{13} & \mathbf{X}_{21} \\ \mathbf{X}_{22} & \mathbf{X}_{23} \end{pmatrix}$, find $\mathcal{A}^*$.

4. Judge the properties of the following sets (openess, closeness, boundedness, compactness) and give their interiors, closures, and boundaries:

a. $\mathcal{C}_1 = \emptyset$.

b. $\mathcal{C}_2 = \mathbb{R}^n$.

c. $\mathcal{C}_3 = \{x|0 \le x < 1\} \cup \{x|2 \le x \le 3\} \cup \{x|4 < x \le 5\}$.

d. $\mathcal{C}_4 = \{(x, y)^T|x \ge 0, y > 0\}$.

e. $\mathcal{C}_5 = \{k|k \in \mathbb{Z}\}$.

f. $\mathcal{C}_6 = \{k^{-1}|k \in \mathbb{Z}, k \ne 0\}$.

g. $\mathcal{C}_7 = \{(1/k, \sin k)^T|k \in \mathbb{Z}, k \ne 0\}$.

5. For each of the following sequences, determine the rate of convergence and the rate constant.

a. $x_k = 1 + 5 \times 10^{-2k}$, for $k = 1, 2, \cdots$.

b. $x_k = 2^{-2^k}$.

c. $x_k = 3^{-k^2}$.

d. $x_k = 1 - 2^{-2^k}$ for k odd, and $x_k = 1 + 2^{-k}$ for k even.

6. Compute the gradient and Hessian of the following functions (write in vector or matrix form, rather than entrywise), give details ($\mathbf{x}$ is a vector and $\mathbf{X}$ is a matrix):

a. $f(\mathbf{x}) = \|\mathbf{x}\|_p$, $p \geq 2$.

b. $f(\mathbf{x}) = (\mathbf{a}^T\mathbf{x})(\mathbf{b}^T\mathbf{x})$.

c. $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.

d. $f(\mathbf{x}) = \mathbf{u}^T g(\mathbf{R}\mathbf{x})$, where $g(\mathbf{y}) = (g_0(y_1), \cdots, g_0(y_n))^T$.

e. $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\mathbf{x}^T - \mathbf{I}\|_F^2$.

f. $f(\mathbf{X}) = \|\mathbf{X}^T\mathbf{A}\mathbf{X}\|_F^2$ and $\mathbf{A}$ is a symmetric matrix.

g. $f(\mathbf{X}) = \mathrm{tr}(\mathbf{A}\mathbf{X}\mathbf{B})$.