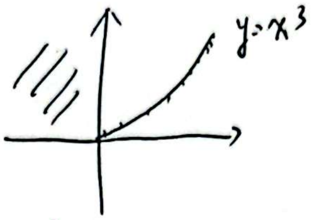
优化方法 10



1. a) $C_{fd}(x_0)=\{d \mid d_2\ge 0,\ d_1\le 0\}$.

$C_t(x_0)=\{d \mid d_2\ge 0, d_1\le 0$ 或 $d=(d_1,0), d_1\ge 0\}$

$C_l(x_0)=\{d \mid d_2\ge 0\}$

$g_1'(x_0)=(0,-1)$

$g_2'(x_0)=(3x_1^2,-1)|_0=(0,-1)$

$g_3'(x_0)=(3x_2x_1^2-6x_1^5,\ x_1^3)|_0=(0,0)$

b) $\nabla_x L(x,\lambda)=(2x_1+3\lambda_2x_1^2+\lambda_3(3x_2x_1^2-6x_1^5),\ -\lambda_1-\lambda_2+\lambda_3x_1^3+2x_2)=0$

$\lambda_1x_2=0$

$\lambda_2(x_1^3-x_2)=0$ $\qquad \lambda_i\ge 0$

$\lambda_3(x_1^3(x_2-x_1^3))=0$.

解之得 $x^*=(0,0)^T$，是全局最小

c) g_1 线性 ⇒ 凸

$\nabla^2 g(x)=\begin{bmatrix}6x_1 & 0\\ 0 & 0\end{bmatrix}$ 非凸. ∴ SCQ 不成立.

d) 由a) ACQ 不成立 GCQ 成立 $(C_t(x_0))^*=C_l(x_0)^*=\{d \mid d_1=0, d_2\le 0\}$.

e) $g(\lambda)=\inf\limits_x \left(x_1^2+x_2^2-\lambda_1x_2+\lambda_2(x_1^3-x_2)+\lambda_3(x_1^3(x_2-x_1^3))\right)$

~~关于x_2二次~~ $\cancel{x_2=\frac{\lambda_1+\lambda_2-\lambda_3x_1^3}{2}}$

$\lambda_3\ne 0$时 x_1最高6次且系数为负 $\Rightarrow g(\lambda)=-\infty$

同理 $\lambda_2\ne 0$时 $g(\lambda)=-\infty$.

$\lambda_2=\lambda_3=0$时

$g(\lambda)=\inf\limits_x(x_1^2+x_2^2-\lambda_1x_2)=-\frac{\lambda_1^2}{4}$

$\therefore g(\lambda)=\begin{cases}-\frac{\lambda_1^2}{4} & \lambda_2=\lambda_3=0,\ \lambda_1\ge 0\\ -\infty & \text{else.}\end{cases}$ $\qquad \mathrm{dom}\, g=\{\lambda \mid \lambda_1\ge 0, \lambda_2=\lambda_3=0\}$.

f) $\max\limits_\lambda g(\lambda)=-\frac{\lambda_1^2}{4}$ s.t. $\lambda_1\ge 0,\ \lambda_2=\lambda_3=0$.

2. a) $g_1(x)$ 仿射 ∴只考 $\nabla g_2(x) = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ 即可

$\tilde{x} = (1,-2)$ $\quad g_2(\tilde{x}) = -3 < 0$ ∴ SCQ 成立

b) $f(x) = (x_1-2)^2 + (x_2-3)^2$

解 ① 均不活跃 $\lambda_1 = \lambda_2 = 0$ $\quad \nabla f(x) = (2(x_1-2), 2(x_2-3)) = 0 \Rightarrow x_1 = 2, x_2 = 3$ $\quad x = (2,3)^T \notin F$ 不可行

② $g_1(x)$ 活跃 $\lambda_2 = 0$ $\quad x_1 + x_2 = 0$ $\quad \nabla L(x,\lambda) = (2(x_1-2)+\lambda_1, 2(x_2-3)+\lambda_1) = 0$

$\Rightarrow x = (-\frac{1}{2}, \frac{1}{2})^T$ $\quad \lambda_1 = 5$ 可行

③ $g_2(x)$ 活跃 $\lambda_1 = 0$ $\quad x_1^2 - 4 = 0$ $\quad \nabla L(x,\lambda) = (2(x_1-2) + 2x_1\lambda_2, 2(x_2-2)) = 0$

$\Rightarrow x_2 = 2$ $\cancel{x_1 = -2 \quad \lambda_2 =}$ λ_1 无解.

c) $g(\lambda) = \inf\limits_{x} (x_1-2)^2 + (x_2-3)^2 + \lambda_1(x_1+x_2) + \lambda_2(x_1^2-4)$

$$= -\frac{(\lambda_1-4)^2}{4(1+\lambda_2)} - \frac{(\lambda_1-6)^2}{4} + 9 + 4 - 4\lambda_2$$ (分别对 x_1, x_2 均为二次函数)

$$= -\frac{(\lambda_1-4)^2}{4(1+\lambda_2)} - \frac{(\lambda_1-6)^2}{4} + 13 - 4\lambda_2 \qquad \lambda_1, \lambda_2 \in \mathbb{R}_+^2 = \operatorname{dom} g$$

d) $\max g(\lambda) = \max -\frac{(\lambda_1-4)^2}{4(1+\lambda_2)} - \frac{(\lambda_1-6)^2}{4} + 13 - 4\lambda_2 \qquad \lambda_1, \lambda_2 \geq 0.$

3. a) 均为线性约束 故满足 SCQ 强对偶成立. 数据线性可分

b) $g(\lambda) = \inf\limits_{w,\beta} \frac{1}{2}\|w\|^2 + \sum \lambda_i [1 - y_i(\langle w, x_i\rangle + \beta)]$ [illegible]

对 β, w 求导得 $\sum \lambda_i y_i = 0$ $\quad \sum \lambda_i y_i x_i = w$

代回有 $g(\lambda) = \|\sum \lambda_i y_i x_i\|^2 \cdot \frac{1}{2} + \sum \lambda_i - \|\sum \lambda_i y_i x_i\|^2$

$$= \sum \lambda_i - \frac{1}{2}\|\sum \lambda_i y_i x_i\|^2$$

$\operatorname{dom} g = \{\lambda \mid \sum \lambda_i y_i = 0, \lambda_i \geq 0\}.$

c) $\max \sum \lambda_i - \frac{1}{2}\|\sum \lambda_i y_i x_i\|_2^2 \quad$ s.t. $\lambda_i \geq 0, \sum \lambda_i y_i = 0$

4. $L(x,y,\lambda) = F(x) + \sum \lambda_i (y_i - a_i^T x)$

$$g(\lambda) = \inf_{x,y} \frac{1}{n}\sum \phi_i(y_i) + \frac{\mu}{2}\|x\|_2^2 + \sum \lambda_i (y_i - a_i^T x)$$

$$= \inf_{y} \frac{1}{n}\sum \phi_i(y_i) + \sum \lambda_i y_i + \inf_{x} \frac{\mu}{2}\|x\|_2^2 - \sum \lambda_i a_i^T x$$

$$= \sum -\frac{1}{n}\phi_i^*(-n\lambda_i) + \left(-\frac{1}{2\mu}\|\sum \lambda_i a_i\|_2^2\right)$$

∴对偶问题 $\max_{\lambda} -\frac{1}{n}\sum \phi_i^*(-n\lambda_i) - \frac{1}{2\mu}\|\sum \lambda_i a_i\|_2^2$ $\lambda_i \in R$. 且 $\phi_i^*(-n\lambda_i) < +\infty$

$(-n\lambda_i \in \mathrm{dom}\,\phi_i^*)$

5. $g(\lambda) = \inf_{x} c^T x + \lambda f(x)$

$$= -\sup_{x} -c^T x - \lambda f(x)$$

$$= -\lambda \sup_{x} \left[-\frac{c^T}{\lambda} x - f(x)\right]$$

$$= -\lambda f^*\left(-\frac{c}{\lambda}\right) \quad \lambda > 0 \quad \mathrm{dom}\,g = \{\lambda \mid \lambda > 0\} \quad (\lambda = 0 \text{时} \to +\infty)$$

对偶 $\max_{\lambda} -\lambda f^*\left(-\frac{c}{\lambda}\right)$ s.t. $\lambda > 0$ $\Leftrightarrow \min_{\lambda} \lambda f^*\left(-\frac{c}{\lambda}\right)$ $\lambda > 0$

$F = \{\lambda \mid \lambda > 0\}$ 是凸的 f^*是凸的 ∴ $\lambda f^*\left(-\frac{c}{\lambda}\right)$ 是凸的 (透视)

∴是CP.