Homework 4

Requirements:

1. Digital format (can be typeset or photos, ought to write clearly if written by hand), upload to `https://course.pku.edu.cn/`.
2. Submit by next class
3. A problem is not counted if nobody can work it out
4. Each homework 10 points; 1 point deducted for each week's delay

Problems:

1. Suppose $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, with $\mathbf{b} \in \mathrm{Range}(\mathbf{A})$. Show that $\exists \mathbf{x}$ satisfying

$$\mathbf{x} > \mathbf{0},\ \mathbf{A}\mathbf{x} = \mathbf{b}$$

iff there exists no $\boldsymbol{\lambda}$ with

$$\mathbf{A}^T\boldsymbol{\lambda} \geq \mathbf{0},\ \mathbf{A}^T\boldsymbol{\lambda} \neq 0,\ \mathbf{b}^T\boldsymbol{\lambda} \leq 0.$$

2. Let $C = \{\mathbf{x} \in \mathbb{R}^n |\ \|\mathbf{x}\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbb{R}^n$, and let $\mathbf{x} \in \partial C$. Identify the supporting hyperplanes of C at $\mathbf{x}$ explicitly.

3. Judge which of the following functions are convex or concave, and find their moduli if they are strongly convex or concave.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$.

(b) $f(x_1, x_2) = x_1x_2$ on $\mathbb{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1x_2)$ on $\mathbb{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 < \alpha < 1$ on $\mathbb{R}^2_{++}$.

4. Show that $f(t) = (t-1)\ln\frac{t}{t+1}$ is strictly convex on $t > 0$.

5. Prove that the negative entropy is 1-strongly convex.

6. Prove that if f is a convex and differentiable function, then for all $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, and $a_1, a_2, a_3 > 0$ such that $a_1 + a_2 = a_3$, we have

$$\langle \nabla f(\mathbf{x}_3), a_1\mathbf{x}_1 + a_2\mathbf{x}_2 - a_3\mathbf{x}_3\rangle \leq a_1 f(\mathbf{x}_1) + a_2 f(\mathbf{x}_2) - a_3 f(\mathbf{x}_3).$$