Homework 3

Requirements:

1. Digital format (can be typeset or photos, ought to write clearly if written by hand), upload to `https://course.pku.edu.cn/`.
2. Submit by next class
3. A problem is not counted if nobody can work it out
4. Each homework 10 points; 1 point deducted for each week's delay

Problems:

1. Suppose $z \sim p_\theta(z) = \begin{cases} \theta \exp(-\theta z), & z \geq 0, \\ 0, & z < 0. \end{cases}$. Design a reparameterization trick for $L(\theta) = \mathbb{E}_{z \sim p_\theta(z)}[f(z)]$.

2. Which of the following sets are convex?

(a) A slab, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n | \ \alpha \leq \mathbf{a}^T \mathbf{x} \leq \beta\}$.

(b) A wedge, i.e., $\{\mathbf{x} \in \mathbb{R}^n | \ \mathbf{a}_1^T \mathbf{x} \leq b_1, \ \mathbf{a}_2^T \mathbf{x} \leq b_2\}$.

(c) The set of points closer to a given point than a given set, i.e., $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2$ for all $\mathbf{y} \in S\}$ where $S \subseteq \mathbb{R}^n$.

(d) The set of points closer to one set than another, i.e., $\{\mathbf{x} | \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\}$, where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 | \ \mathbf{z} \in S\}.$$

(e) The set $\{\mathbf{x} | \mathbf{x} + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

(f) The set of points whose distance to $\mathbf{a}$ does not exceed a fixed fraction θ of the distance to $\mathbf{b}$, i.e., the set $\{\mathbf{x} | \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta \|\mathbf{x} - \mathbf{b}\|_2\}$ ($\mathbf{a} \neq \mathbf{b}$ and $0 \leq \theta \leq 1$).

3. Find the convex hulls of the following sets:

$$\{\mathbf{x} \in \mathbb{R}^2 | \mathbf{x}_1^2 = \mathbf{x}_2\}, \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{x}_1^2 = \mathbf{x}_2, \mathbf{x}_1 \geq 0\}, \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{x}_1 \mathbf{x}_2 = 1\}.$$

4. Find the convex hull of the set $\{\pm \mathbf{u}\mathbf{u}^T | \|\mathbf{u}\| = 1\}$. Express it in a compact form.

5. Find the dual cone of $\{\mathbf{A}\mathbf{x} | \ \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$.

6. We define the monotone nonnegative cone as

$$K_{m+} = \{\mathbf{x} \in \mathbb{R}^n | \ x_1 \geq x_2 \geq ... \geq x_n \geq 0\}.$$

i.e., all nonnegative vectors with components sorted in nonincreasing order. Find the dual cone K_{m+}^*.

7. Give an expression $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ for the unit ball $\{\mathbf{X} | \|\mathbf{X}\|_2 \leq 1\}$, where S_α is a half-space.