Homework 7

1. Digital format (can be typeset or photos, ought to write clearly if written by hand), upload to `https://course.pku.edu.cn/`.
2. Submit by next class
3. A problem is not counted if nobody can work it out
4. Each homework 10 points; 1 point deducted for each week's delay

Problems:

1. Find the proximal mappings of the following functions.

(a) $\|\mathbf{x}\|_2$, by direct computation and using Moreau decomposition.

(b) $-\sum_{i=1}^{n} \log x_i$.

(c) $g(x) = \begin{cases} \lambda x^3, & x \geq 0, \\ \infty, & x < 0. \end{cases}$

2. Let $d(\mathbf{x})$ be the distance from $\mathbf{x}$ to a closed convex set $\mathcal{C}$: $d(\mathbf{x}) = \inf_{\mathbf{y}\in\mathcal{C}} \|\mathbf{x}-\mathbf{y}\|_2$. Find the proximal mappings of $d(\mathbf{x})$ and $\frac{1}{2}d^2(\mathbf{x})$.

3. Prove that:

(a) If $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{a}^T\mathbf{x}$, then $Prox_c f(\mathbf{x}) = Prox_c g(\mathbf{x} - c\mathbf{a})$.

(b) If $f(\mathbf{x}) = g(\mathbf{x}) + \frac{1}{2\mu}\|\mathbf{x}-\mathbf{a}\|^2$, then $Prox_c f(\mathbf{x}) = Prox_\lambda g(\lambda(c^{-1}\mathbf{x} + \mu^{-1}\mathbf{a}))$, where $\lambda^{-1} = \mu^{-1} + c^{-1}$.

4. Let $\mathbf{x}_k = Prox_\eta g(\mathbf{y}_k - \eta\nabla f(\mathbf{y}_k))$. Prove that

$$g(\mathbf{x}_k) \leq g(\mathbf{y}_k) - \langle \nabla f(\mathbf{y}_k), \mathbf{x}_k - \mathbf{y}_k\rangle - \frac{1}{2\eta}\|\mathbf{x}_k - \mathbf{y}_k\|^2.$$

5. Use Moreau decomposition to prove that $\mathbf{x} = P_L(\mathbf{x}) + P_{L^\perp}(\mathbf{x})$, where L is a subspace and $L^\perp$ is its orthogonal complement.

6. Prove that if f is μ-strongly convex and L-smooth, then

$$\mu\|\mathbf{x}_1 - \mathbf{x}_2\|^2 \leq \langle \nabla f(\mathbf{x}_1) - \nabla f(\mathbf{x}_2), \mathbf{x}_1 - \mathbf{x}_2\rangle \leq L\|\mathbf{x}_1 - \mathbf{x}_2\|^2, \quad \forall \mathbf{x}_1, \mathbf{x}_2.$$

In particular,

$$\mu\|\mathbf{x}_1 - \mathbf{x}_2\| \leq \|\nabla f(\mathbf{x}_1) - \nabla f(\mathbf{x}_2)\|.$$