Wrench (2023 Bulgarian Team Selection Test). A homogeneous asteroid is shaped like a cube of side $a = 1\,\text{km}$. A narrow shaft is dug through the cube along one of its space diagonals. An astronaut of mass $M = 100\,\text{kg}$ is located at the centre of the cube. A wrench of mass $m = 0.4\,\text{kg}$ is dropped from rest at the entrance of the shaft (i.e. at one of the corners of the cube). The wrench goes down the shaft and eventually undergoes a perfectly inelastic collision with the astronaut. What is the maximum subsequent displacement s of the astronaut from the centre of the cube? Answer in metres, accurate to one decimal place. You can assume that the maximum displacement is much less than the side of the cube. You can use that the escape velocity at the corner of a homogeneous cube of side a and density ρ is given by $v = ka\sqrt{G\rho}$, where G is the gravitational constant and $k = 1.543$.

Solution: Denote the potential energy of the wrench at the corner of the cube by V. From the definition of escape velocity v', we have

$$\frac{mv'^2}{2} + V = 0.$$

First, we will need to find the velocity v_0 of the wrench right before striking the astronaut. To this end, we need to express the potential energy V' at the centre of the cube in terms of V. Let's break up the cube into 8 little cubes of side $a/2$. Since the potential energies are scalar quantities, we have $V' = V''$, where V'' is the potential energy at the corner of a cube of side $a/2$. Additionally, we know that the potential energy at the surface of an object of mass μ and characteristic size a is of the form $\sim \frac{Gm\mu}{a}$. If the density of the object is ρ, this reduces to $\sim G\mu\rho a^2$. It follows that $V'' = \frac{1}{4}V$, and then $V' = 2V$. We can now apply conservation of energy:

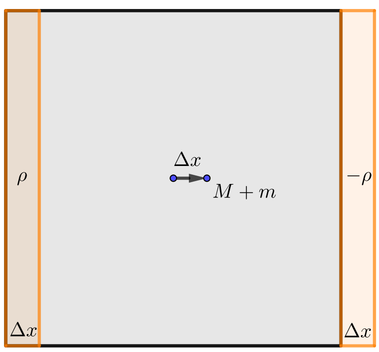
$$0 + V = \frac{mv_0^2}{2} + 2V.$$

Combining this with the previous energy conservation statement, it turns out that $v_0 = v'$.

The collision is perfectly inelastic, so the velocity of the astronaut after the collision u_0 can be found from

$$mv_0 = (M+m)u_0.$$

Let's study the motion that ensues. We will use a coordinate system with the origin at the centre of the cube and axes x, y, z directed parallel to the edges. We want to find the gravitational force on the astronaut along x due to a small displacement from the origin Δx. The mass which attracts the astronaut in that case is a superposition of a cube of side a with the astronaut at its centre, and two thin square plates of side a and thickness Δx, one with density ρ and the other with density $-\rho$.



Now we will derive an expression for the force due to the plate ρ. Consider a small element $\mathrm{d}S$ at an angle θ from the x-axis as observed by the astronaut. The force along x due to this element is given by

$$F_x = -\frac{G(M+m)\rho\Delta x\mathrm{d}S}{r^2}\cos\theta.$$

We note that $\frac{\Delta S}{r^2}\cos\theta$ is actually the solid angle $\mathrm{d}\Omega$ subtended by the small element. Repeating this for each area element of the plate, we see that the total force due to the plate is $-G(M+m)\rho\Delta x\Omega$, where Ω is the total solid angle subtended by the plate as seen by the astronaut. This corresponds to one sixth of all space, so $\Omega = \frac{2}{3}\pi$. The other plate (of density $-\rho$) gives rise to the same force, and the total force along x due to the plates is then $-\frac{4}{3}\pi G(M+m)\rho\Delta x$.

The same arguments can be repeated for displacements along y and z. Adding vectorially, we conclude that the total restoring force for a small displacement $\Delta\mathbf{r}$ from the centre of the cube is

$$\mathbf{F} = -\frac{4}{3}\pi G(M+m)\rho\Delta\mathbf{r}.$$

This corresponds to harmonic oscillations about the centre of the cube, with a 'spring constant' of $K \equiv \frac{4}{3}\pi G(M+m)\rho$. The potential energy of this motion is then given by $\frac{1}{2}K(\Delta r)^2$. To obtain the maximum displacement, we need to apply conservation of energy again:

$$\frac{1}{2}(M+m)u_0^2 + 0 = 0 + \frac{1}{2}Ks^2.$$

After expanding all the variables, G and ρ will cancel, and we're left with

$$s = \sqrt{\frac{3k^2}{4\pi}\frac{m}{M+m}}a = 3.0\,\mathrm{m}.$$