Magnetic Charges

Stefan Ivanov

This note assumes you're familiar with all the standard formulae for dipoles (field of a dipole, potential energy of a dipole in an external field, force on a dipole in an external field). You can find these in **E4** from Kevin Zhou's handouts, or in Griffiths' Introduction to Electrodynamics, Chapter 5.

Consider the magnetic field that corresponds to a dipole moment of magnitude p_{m}, expressed in spherical polar coordinates:

$$\mathbf{B} = \frac{\mu_0 p_{\mathrm{m}}}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right), \tag{1}$$

where the angle θ is measured from the direction of the dipole moment. Here "dipole" really only means that the field scales as $1/r^3$. What gives rise to such fields is a different matter entirely. Of course, we know from experiment that the physical origin of dipole fields is current loops. In particular, the field due to a planar loop of area A carrying current I has the form from (1) at large distances r, and in that case $p_{\mathrm{m}} = IA$.

However, consider how an outside observer perceives such a dipole field as long as they don't approach the source too closely. By moving around, they can figure out the magnitude of p_{m}, but they have zero information about the origin of this p_{m}. They can't know if it's due to a current loop or something else. And they shouldn't really care either. The torques and forces due to the field in (1) depend only on the dipole moment p_{m}, not the machinery which gives rise to it.

We will see that working with current loops is rather unwieldy, so it would be very convenient to find another configuration that produces the dipole field in (1). For this purpose, let us introduce an object called a magnetic charge. A magnetic charge q_{m} produces a radial magnetic field

$$\mathbf{B} = \frac{\mu_0 q_{\mathrm{m}}}{4\pi r^2}\hat{\mathbf{r}}. \tag{2}$$

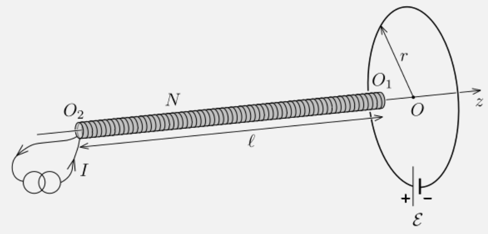
In addition, if placed in an external field $\mathbf{B}_0$, this charge would feel a force $\mathbf{F} = q_{\mathrm{m}}\mathbf{B}_0$. This is the same as Coulomb's law in electrostatics, but with μ_0 in place of $1/\varepsilon_0$. Of course, magnetic charges have not been observed in real life, but their existence would not be at odds with the laws of electrodynamics. So we're not doing anything illegal.

Take a pair of magnetic charges $\pm q_{\mathrm{m}}$ at a distance d from each other. We want to find the total field they produce at large distances ($r \gg d$). Since the field from a single magnetic charge is analogous to that of an electric charge, the total field will be analogous to that of an electric dipole. And if you know the formula for that, you can directly conclude that in our case

$$\mathbf{B} = \frac{\mu_0 (q_{\mathrm{m}} d)}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right), \tag{3}$$

where θ is measured starting from the ray which originates at $-q_{\mathrm{m}}$ and passes through $+q_{\mathrm{m}}$. Now compare this with (1) and you can see that for an outside observer in a dipole field, the moment p_{m} might as well be due to a pair of magnetic charges, as long as their q_{m} and d comply with $q_{\mathrm{m}} d = p_{\mathrm{m}}$. This perspective can often simplify the problem at hand. Let's have a look at an example.

Problem 1. Solenoid and loop (EuPhO 2020, 1a). A closed circular loop of radius r consists of an ideal battery of electromotive force $\mathcal{E}$ and a wire of resistance R. A long thin air-core solenoid is aligned with the axis of the loop (z-axis). Its length is $l \gg r$, its cross-sectional area is A ($\sqrt{A} \ll r$), and the number of turns is N. The solenoid is powered by a constant current I provided by an ideal current source. The directions of the currents in the solenoid and in the loop are the same (clockwise in the figure). Find the force F_1 acting on the solenoid when its head O_1 is positioned at the loop's centre O. What is the force F_2 acting on the solenoid when its tail O_2 is located at the centre of the loop?

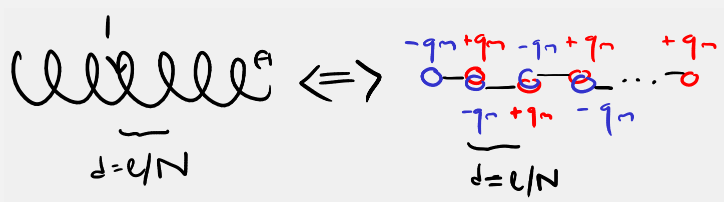


Solution. The magnetic flux through the loop is constant, so there is no induced EMF. The current through the loop is $I' = \mathcal{E}/R$. This current in turn gives rise to a magnetic field

$$\mathbf{B} = \frac{\mu_0 I' r^2}{2(r^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

along the axis of the loop. The force F_1 is due to the interaction of this field with the currents in the solenoid. The standard way to find F_1 is as follows. Assuming the solenoid is tightly wound, we can treat it as a stack of N rings, each with an area A and current I. Since the solenoid is thin ($\sqrt{A} \ll r$), each ring behaves as a magnetic dipole with moment $p_\mathrm{m} = IA$ in a field $B(z)$ directed along $\hat{\mathbf{z}}$. The force on each dipole is then given by $p_\mathrm{m} \frac{\mathrm{d}B}{\mathrm{d}z}$. The total force F_1 corresponds to the sum of these forces. We will need to integrate along the length of the solenoid. This is doable, if a bit clunky.

Alternatively, here's how to approach this with magnetic charges. The force on each dipole in the solenoid depends only on the dipole moment, so it's fair game to imagine our dipoles as pairs of magnetic charges $\pm q_\mathrm{m}$ at a distance d as long $q_\mathrm{m} d = IA$. Each ring takes up a fraction $1/N$ of the total length of the solenoid, so we can ascribe a length l/N to each. We'll choose $d = l/N$ for our magnetic charge pairs as well (implying $q_\mathrm{m} = NIA/l$). We could have picked any d, but this choice in particular allows for a neat trick. The stack of rings is equivalent to a chain of magnetic charges. When the distance between the charge pairs is equal to the distance between the rings, the magnetic charges will overlap: the positive charge of a given pair is exactly where the negative charge of the next pair is. In those regions there is effectively no charge at all. The only places with no cancellation along the solenoid are the two ends.



Thus the whole solenoid behaves exactly like two magnetic charges $\pm q_\mathrm{m}$ at each end, at least in terms of the fields it produces and how it's affected by external fields. The total force on the solenoid is then simply the sum of the force on $-q_\mathrm{m}$ at the left end and the force on $+q_\mathrm{m}$ at the right end:

$$\mathbf{F}_1 = +q_\mathrm{m} \mathbf{B}|_{z=0} - q_\mathrm{m} \mathbf{B}|_{z=-l} = \frac{NIA}{l} \left(\frac{\mu_0 I'}{2r} - \frac{\mu_0 I' r^2}{2(r^2 + l^2)^{3/2}} \right) \hat{\mathbf{z}} \approx \frac{\mu_0 \mathcal{E} NIA}{2rRl} \hat{\mathbf{z}}.$$

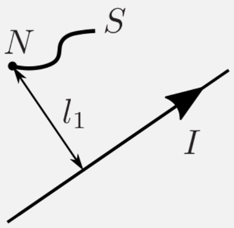
In the case of F_2, the negative charge $-q_\mathrm{m}$ is at the centre of the loop. Hence $\mathbf{F}_2 = -\mathbf{F}_1$.

If you're familiar with magnetic charges, this part of the problem takes about 5 minutes. All in all, not too bad for 5.5 points on EuPhO! Indeed, if you know your physics, you will get gold on EuPhO, period. In contrast, your mark on IPhO depends just as much on your exam technique.

For extra practice, try your hand at IPhO 2022 1.A5 and IPhO 2012 1.C3. Keep in mind that in the first problem you're just looking for the field due to a chain of dipoles, and in the second one you're looking for the total force on a chain of dipoles due to another chain of dipoles.

Let's explore another example. This is one of the most beloved Russian problems, and everyone I show this to finds it astonishing. But who knows, you might find it trivial now!

Problem 2. Magnetic cord (Russia 2019). A thin homogeneous flexible inextensible cord of length l is made from a ferromagnetic material such that the magnetic moment of each small piece is directed along the cord. One end of the cord is held at a distance l_1 ($l_1 > l$) from an infinite straight wire carrying current I. The cord eventually reaches equilibrium. Find the distance l_0 between the ends of the cord in equilibrium, as well as the distance x between the free end of the cord and the wire. Neglect gravity and the magnetic field of the cord itself.



Solution 1. The following solution doesn't involve any tricks, but it requires some experience with curvilinear coordinates. We will use cylindrical polar coordinates with the z-axis directed along the wire. The magnetic field of the wire is $\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}$. The cord has some uniform dipole moment per unit length, which we will denote by β. The magnetic moment of a piece $\mathrm{d}\mathbf{l}$ of the cord will then be $\beta \mathrm{d}\mathbf{l}$. The potential energy of this piece in a local field $\mathbf{B}$ is then

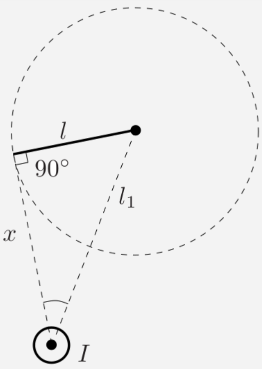
$$\mathrm{d}U = -\beta \mathrm{d}\mathbf{l} \cdot \mathbf{B} = -\beta\left((\mathrm{d}r)\hat{\mathbf{r}} + (r\mathrm{d}\phi)\hat{\boldsymbol{\phi}} + (\mathrm{d}z)\hat{\mathbf{z}}\right) \cdot \left(\frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}\right) = -\frac{\mu_0 \beta I}{2\pi}(\mathrm{d}\phi),$$

which is proportional to the angle $\mathrm{d}\phi$ that this piece subtends in a plane perpendicular to the wire[a]. The total potenial energy is then

$$U = -\frac{\mu_0 \beta I \Delta\phi}{2\pi},$$

where $\Delta\phi$ is the total subtended angle between the two ends of the cord (again, in a projection perpendicular to the wire). In equilibrium the potential energy of the cord is minimised. So the equilibrium configuration is such that $\Delta\phi$ is maximised.

This is a geometry problem. We seek to maximise $\Delta\phi$ in a plane perpendicular to the wire. Displacing the cord *along* the wire contributes nothing to this, so it would just be a waste of length. The problem can then be reduced to 2D. The free end of the cord can access all points in space up to l away from the fixed end, which corresponds to a circle of radius l centered at the fixed end. Think about the segment connecting the wire and the free end of the cord. We want to pull it as far away as possible from the segment connecting the wire and the fixed end. We can keep pulling it away until this segment becomes tangent to the circle of radius l. Remember, the free end is constrained to this circle, so you cannot improve on this. We then conclude that the cord is taut, $l_0 = l$. The distance between the wire and the free end is $x = \sqrt{l_1^2 - l^2}$.



Solution 2. In essence, we want to understand how a string of magnetic dipoles will align itself under an azimuthal external magnetic field. Each small piece of the cord can be treated

as a pair of magnetic charges. Let us choose the distance between the charges to be the actual length of the piece. The cord behaves as a superposition of magnetic charges, but these charges will cancel everywhere inside the cord. There are only two charges $\pm q_{\mathrm{m}}$ left at the two ends of the cord. The interaction between them can be ignored according to the problem statement. The charge at the fixed end of the cord is, well, fixed, so we don't need to account for it in what follows.

Let's now think about what happens to the charge q_{m} at the free end of the cord[b]. The only force acting on it is a magnetic force $q_{\mathrm{m}}\mathbf{B}$ from the wire. But the charge cannot be in equilibrium under a single force. That is, unless the cord is taut. In that case there would also be a tension force which can indeed balance the magnetic force. Therefore $l_0 = l$.

The tension will be directed along the cord, and the magnetic force on q_{m} is directed along the local $\mathbf{B}$. These need to be antiparallel. The magnetic field does not have a component along the wire, and the same should be true for the tension, so this is a 2D problem. Finally, note that the magnetic force on q_{m} is perpendicular to the line connecting the wire and q_{m}. Because the two forces are antiparallel, the tension force at q_{m} should also be perpendicular to this line. The cord will then align itself as in the figure above, from which we find $x = \sqrt{l_1^2 - l^2}$. We didn't have to write down a single equation!

[a]There is actually a deeper reason why the distances r cancelled here. The field $\mathbf{B}$ is in fact precisely set up so that this happens. For we know that $\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l}$ is a constant ($= \mu_0 I$), and there is no preferred azimuth ϕ in the case of a straight wire, so we could have concluded that our integral should only depend on the total subtended angle $\Delta\phi$ without knowing the exact form of $\mathbf{B}$.

[b]I opt for positive q_{m} in keeping with the figure in the previous solution.

Last updated: 22nd December 2024
Send comments and corrections to sivanov.mail@proton.me