Short Exam 1

Problem. Spider web. Consider a hexagonal spider web. It has six radial threads, each of relaxed length $l_0 = 45\,\mathrm{cm}$, radius $r = 0.01\,\mathrm{mm}$, and Young modulus $E = 2\times10^8\,\mathrm{Pa}$. The threads are given tension $F_0 = 6\,\mathrm{mN}$ after being attached to some nearby surface. Four concentric hexagonal threads are woven around the radial threads equidistantly (Figure 1). These are initially relaxed. Each of the threads breaks when its strain $\varepsilon = \Delta l/l$ reaches $\varepsilon_{\max} = 0.2$. The spider web can be considered massless.

(a) A fly sticks to the centre of the web with velocity $v = 2\,\mathrm{m/s}$ perpendicular to the web. What is the maximum mass $M_{\max}$ of the fly so that the web does not break? **(1.5 pt)**

(b) A fly of mass $m = 0.1\,\mathrm{g}$ sticks to the centre of the web. Find the period T of its small oscillations perpendicular to the web. **(1.5 pt)**

(c) Consider the same web as before, but without tension in any of the threads. The web is carefully taken off its supporting surfaces. It is then loaded at its outermost vertices using six radial forces of equal magnitude. At what magnitude $F_{\max}$ does the web break? Where will it break? **(2.0 pt)**

Hint: An elastic material is loaded in some direction using force F. The cross section of the material perpendicular to that direction is S. The relaxed length of the material in that direction is l and its extension is Δl. Hooke's law states that

$$\frac{F}{S} = E\frac{\Delta l}{l},$$

where E is the Young modulus of the material.

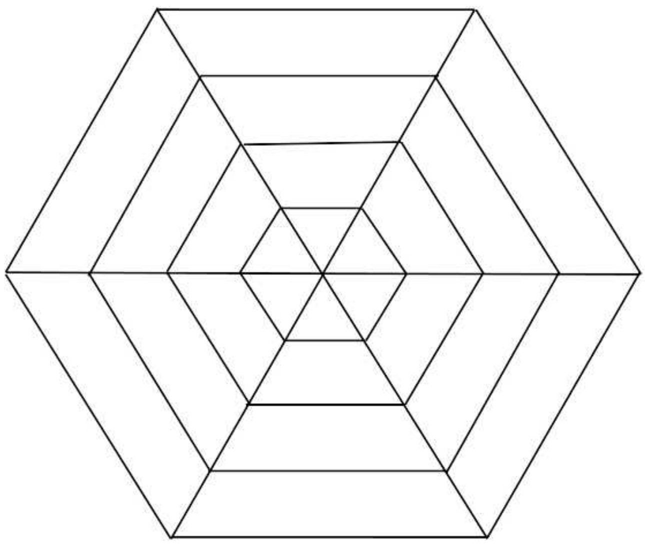

Figure 1

Solution. (a) We'll always assume that the strain on the web is relatively small, which will help us avoid all sorts of complications (for example, stretching a cord will generally reduce its cross-section, but we can just set $r = \mathrm{const}$). The elastic energy stored in a thread of relaxed length l_0 is

$$U = \int_{l_0}^{l} F\,\mathrm{d}l = \int_0^{\varepsilon} E\pi r^2\varepsilon(l_0\,d\varepsilon) = \left(\frac{E\varepsilon^2}{2}\right)\pi r^2 l_0.$$

Initially the strain in each radial thread is $\varepsilon_0 = \frac{F_0}{\pi r^2 E} = 0.095$. When the fly comes in, it will stretch the radial threads until its velocity turns to zero. While this happens, the lateral threads do not stretch – their motion is purely translational. In the limiting case, the radial threads are on the verge of breaking right when the fly is at rest. We can then apply energy conservation:

$$\frac{M_{\max} v^2}{2} + 6\left(\frac{E\varepsilon_0^2}{2}\right)\pi r^2 l_0 = 6\left(\frac{E\varepsilon_{\max}^2}{2}\right)\pi r^2 l_0,$$

$$\boxed{M_{\max} = \frac{6\pi r^2 l_0 E}{v^2}\left(\varepsilon_{\max}^2 - \left(\frac{F_0}{\pi r^2 E}\right)^2\right) = 1.3\,\mathrm{g}.}$$

(b) This is easier to solve with forces. At equilibrium, the radial threads are already taut with force F_0. A small displacement x of the fly from the equilibrium position doesn't change the tension much. The main effect is instead that the tension acquires a component against the motion of the fly, equal to $F_0(\frac{x}{l}) = F_0(\frac{x}{l_0(1+\varepsilon_0)})$. This acts as a restoring force with an elastic constant $k = \frac{F_0}{l_0(1+\varepsilon_0)}$. The oscillation period is thus

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ml_0(1+\varepsilon_0)}{F_0}},$$

$$\boxed{T = 2\pi\sqrt{\frac{ml_0}{6F_0}\left(1 + \frac{F_0}{\pi r^2 E}\right)} = 0.23\,\mathrm{s}.}$$

(c) Since the web is loaded symmetrically, all radial threads must stretch identically and the 60° angles at the centre of the web will be preserved. This implies that all equilateral triangles will stay equilateral. This places a constraint on how the lateral threads stretch.

The four segments of each radial thread will all stretch in a different manner. Counting inwards, we'll label their tensions with $F_{1,2,3,4}$ and their strains with $\varepsilon_{1,2,3,4}$. We do the same with the lateral pieces, labelling the tensions as $T_{1,2,3,4}$ and the strains as $\varepsilon'_{1,2,3,4}$. Our goal is to find expressions for all the strains, see which of those is largest, express it in terms of F, and then set it to $\varepsilon_{\max} = 0.2$ to find $F = F_{\max}$. Let's first write down the "legs = base" constraints on the strains:

$$\frac{l_0}{4}(1+\varepsilon_4) = \frac{l_0}{4}(1+\varepsilon'_4), \qquad \frac{l_0}{4}(1+\varepsilon_4) + \frac{l_0}{4}(1+\varepsilon_3) = \left(2\cdot\frac{l_0}{4}\right)(1+\varepsilon'_3), \qquad \cdots$$

These simplify to

$$\varepsilon'_4 = \varepsilon_4, \quad \varepsilon'_3 = \frac{\varepsilon_3+\varepsilon_4}{2}, \quad \varepsilon'_2 = \frac{\varepsilon_2+\varepsilon_3+\varepsilon_4}{3}, \quad \varepsilon'_1 = \frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{4}.$$

Now we'll do a force balance on each of the vertices that lie on a given radial thread. For the outermost vertex, we have

$$F = 2T_1\cos 60^\circ + F_1 = T_1 + F_1.$$

Similarly,

$$F_1 = T_2 + F_2, \qquad F_2 = T_3 + F_3, \qquad F_3 = T_4 + F_4.$$

For any force F, the strain can be found from $F = (E\pi r^2)\varepsilon$, and the conversion factor $E\pi r^2$ is the same for all threads. Thus

$$\frac{F}{E\pi r^2} = \varepsilon'_1 + \varepsilon_1, \qquad \varepsilon_1 = \varepsilon'_2 + \varepsilon_2, \qquad \varepsilon_2 = \varepsilon'_3 + \varepsilon_3, \qquad \varepsilon_3 = \varepsilon'_4 + \varepsilon_4.$$

At this point we notice that the largest strain is ε_1, so the web will break at the outermost quarter of the radial threads. Let us express everything in terms of ε_4':

$$\varepsilon_4' = \varepsilon_4, \quad \varepsilon_3 = 2\varepsilon_4, \quad \varepsilon_3' = \frac{3}{2}\varepsilon_4, \quad \varepsilon_2 = \frac{7}{2}\varepsilon_4, \quad \varepsilon_2' = \frac{13}{6}\varepsilon_4, \quad \varepsilon_1 = \frac{17}{3}\varepsilon_4, \quad \varepsilon_1' = \frac{73}{24}\varepsilon_4.$$

We've obtained that $\varepsilon_1' = \frac{73}{136}\varepsilon_1$, and so $\frac{F}{E\pi r^2} = \frac{209}{136}\varepsilon_1$. Finally,

$$\boxed{F_{\max} = \frac{209}{136}E\pi r^2\varepsilon_{\max} = 19\,\mathrm{mN}.}$$

Incidentally, note that we just solved a set of 16 equations with 16 variables!

Short Exam 2

Problem. Discharge. Even though air is an insulator, it has a finite resistivity that depends on pressure, temperature, and humidity. This is why charged bodies in air gradually lose their charge through currents directed towards other conductors, including the ground. In this problem we study dry air under standard temperature and pressure. Its resistivity is $\rho = 1.0 \times 10^{14}\,\Omega\mathrm{m}$.

A metal ball with charge $q = 1.0\,\mathrm{nC}$ is suspended in air at a height of $h = 0.10\,\mathrm{m}$ above the ground. The ground can be considered as an infinite conducting plane.

(a) Find the maximum current density $j_{\max}$ (in $\mathrm{A/m^2}$) flowing through the ground.

(b) Find the total current I that flows through the ground.

(c) Find the time $T_{1/2}$ taken for the ball to lose half of its initial charge.

The dielectric constant of air is $\varepsilon = 1$.

Solution. (a) We'll use coordinates such that the ground is at $z = 0$ and the metall ball is at $(x, y, z) = (0, 0, h)$. The current density at a point with electric field $\mathbf{E}$ is be given by $\mathbf{j} = \mathbf{E}/\rho$, So, our task boils down to finding $\mathbf{E}$ everywhere in the air ($z > 0$), and after that determining the maximum electric field right above the ground (essentially at $z = 0$). Since the ground is an infinite conducting plane, induced charges will pop up on the surface, and they will certainly contribute to the field $\mathbf{E}$ above. We know that the superposition of the fields of all the charges should give us a situation where the potential is zero at $z = 0$. This is because the solution for $\mathbf{E}$ should comply with the fact that at $z = 0$ there's a conductor, and those are equipotential (setting this constant potential to zero is then a matter of convenience).

There's a theorem which says that there is only one way this can happen in terms of charge configurations. Luckily, it's not that hard to spot it. If the induced charges at $z = 0$ arrange themselves in such a way so that their total field replicates the field of an imaginary lone charge $-q$ at $(0, 0, -h)$, then the job's done, because everywhere on the plane we'll have $V = (-kq/r) + (kq/r) = 0$. By uniqueness, that *is* what they do in reality.

We conclude that at $z = 0$ and a distance a from the axis $(x, y) = (0, 0)$ the electric field points precisely downwards and is equal to

$$E(a) = 2 \cdot \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{a^2 + h^2} \cdot \frac{h}{\sqrt{a^2 + h^2}}.$$

The maximum value is at $a = 0$, so

$$\boxed{j_{\max} = \frac{q}{2\pi\varepsilon_0\rho h^2} = 1.8 \times 10^{-11}\,\mathrm{A/m^2}.}$$

(b) We need to integrate along the whole plane, such that $I = \int \mathbf{j}\cdot \mathrm{d}\mathbf{S}$. Making use of symmetry,

$$I = \frac{1}{\rho}\int_0^\infty \frac{q}{2\pi\varepsilon_0}\frac{h}{(a^2+h^2)^{3/2}}\, 2\pi a\, \mathrm{d}a,$$

$$\boxed{I = \frac{q}{\rho\varepsilon_0} = 1.1\times 10^{-12}\,\mathrm{A}.}$$

(c) We just found that

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{q}{\rho\varepsilon_0}.$$

After integrating, we get

$$\boxed{T_{1/2} = \rho\varepsilon_0 \ln 2 = 610\,\mathrm{s}.}$$

Short Exam 3

Problem. Tea in a vacuum flask. A liter of tea at temperature 90 °C is poured into a vacuum flask (a vessel with an imperfect vaccum between its inner and outer walls). The outer surface of the flask is $S = 600\,\mathrm{cm}^2$. The pressure in the volume between the walls is $P_0 \approx 5\times 10^{-6}\,\mathrm{atm}$ at room temperature 20 °C. The walls have an emissivity of $\varepsilon = 0.1$ compared to a black body at the same temperature. The specific heat capacity of water is $c = 4.2\times 10^3\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. Ignore heat loss through the cap of the flask.

(a) Estimate the total rate of heat loss for the tea due to thermal radiation and thermal conduction between the walls of the flask.

(b) Estimate the time taken for the temperature of the tea to decrease from 90 °C to 70 °C.

Solution. (a) This is an expanded version of a problem from the 1978 USSR Olympiad. Let's first consider the radiative heat transfer. We aren't given any information about the gap between the inner and the outer walls, so we'll just assume that it's very narrow, such that both surfaces have an area S. The temperature of the inner surface is $T_1 = 363\,\mathrm{K}$, and the net power it emits outwards is $P_1 = \varepsilon\sigma T_1^4 S$. Likewise, for the outer surface $T_2 = 293\,\mathrm{K}$ and a net power $P_2 = \varepsilon\sigma T_2^4 S$ is emitted inwards. Since the outer surface is always in contact with the environment, its temperature will stay fixed at T_2.

We want to find the overall heat flow away from the inner surface. This is what draws energy away from the tea, bringing its temperature down. Alas, finding the heat flow is a bit more complicated than just writing down $P_1 - P_2$.

Let us follow what happens to the photons radiated from the inner surface. After reaching the outer surface, a fraction ε of them will get absorbed there, because by Kirchhoff's law the absorptivity equals the emissivity. The remaining part $1-\varepsilon$ is all reflected diffusely back towards the inner surface (because we assumed a narrow gap, there's essentially no chance for these photons to miss the inner surface). Now, a part ε of this absorbs onto the inner surface, while a part ε is sent back to the outer surface. In the end, all photons end up absorbed at one of the two surfaces, even though they might have bounced back and forth many times. Note that we aren't discussing reemission of photons at any point, because working with the net power P_1 means that we've already included all the photons that'd get emitted within unit time – and absorption and reflection are separate physical processes from emission.

We see that due to the presence of P_1, the outer surface absorbs power

$$P_{1\to 2} = P_1\varepsilon + P_1(1-\varepsilon)^2\varepsilon + P_1(1-\varepsilon)^4\varepsilon + \cdots = \frac{P_1\varepsilon}{1-(1-\varepsilon)^2} = \frac{P_1}{2-\varepsilon},$$

while the inner surface absorbs power

$$P_{1\to1} = P_1(1-\varepsilon)\varepsilon + P_1(1-\varepsilon)^3\varepsilon + P_1(1-\varepsilon)^5\varepsilon + \cdots = P_1\frac{1-\varepsilon}{2-\varepsilon}.$$

These sum to P_1, as they should. Likewise, for P_2 we will find $P_{2\to1} = \frac{P_2}{2-\varepsilon}$ and $P_{2\to2} = P_2\frac{1-\varepsilon}{2-\varepsilon}$. Since only $P_{1\to2}$ and $P_{2\to1}$ carry energy away from the surfaces, we find that the net outwards heat flow is

$$P_{\text{rad}} = P_{1\to2} - P_{2\to1} = \sigma(T_1^4 - T_2^4)S\frac{\varepsilon}{2-\varepsilon} = 1.79\,\text{W}.$$

Now we turn to the conductive heat transfer. However, a value for the thermal conductivity of air κ is conspicuously missing. Perhaps we're acquainted with the formula for thermal conductivity of a gas

$$\kappa = \frac{1}{3}nC_V\langle v\rangle\lambda,$$

where n is the number density, C_V is the heat capacity per particle, $\langle v\rangle$ is the arithmetic mean speed of the particles, and λ is the mean free path. But we have no way of estimating the mean free path! Indeed, the situation here calls for something different. The formula above applies when heat is exchanged between layers of gas molecules that interact by collisions. But in our case we're working with quite a rarefied gas (pressure $P_0 = 0.5\,\text{Pa}$), so the molecules will travel between the inner and the outer surface without any collisions. Let's build up a very crude model of our own for the heat exchange.

We model the surfaces as parallel plates separated by distance L. The molecules (all of mass m) will bounce back and forth between the two plates. Whenever a molecule strikes the inner surface, it acquires a velocity which corresponds to temperature T_1, thereby taking out some energy from the surface. This is the RMS speed $v_1 = \sqrt{\frac{3k_BT_1}{m}}$. After integrating across a hemisphere, we see that the projection of v_1 along the direction normal to the plates is on average $v_1/2 = \sqrt{\frac{3k_BT_1}{4m}}$. The molecule soon strikes the outer surface and acquires a velocity $v_2 = \sqrt{\frac{3k_BT_2}{m}}$, depositing some heat on the surface. We find that for each time interval

$$t = \frac{L}{v_1/2} + \frac{L}{v_2/2},$$

a single molecule will transfer energy $\frac{3}{2}k_B(T_1 - T_2)$ to the outward surface. The total number of molecules between the two surfaces can be found from $P_0(SL) = Nk_BT_2$, which gives us the net heat flow

$$P_{\text{cond}} = \frac{3}{2}\left(\frac{P_0SL}{k_BT_2}\right)k_B(T_1 - T_2)\frac{1}{2L}\sqrt{\frac{3k_B}{m}}\frac{\sqrt{T_1T_2}}{\sqrt{T_1}+\sqrt{T_2}},$$

$$P_{\text{cond}} = \frac{3\sqrt{3}}{4}\sqrt{\frac{R}{\mu}}P_0S\sqrt{\frac{T_1}{T_2}}\left(\sqrt{T_1}-\sqrt{T_2}\right) = 1.42\,\text{W}.$$

Here we are expected to know that $R = 8.314\,\text{J}\cdot\text{K}^{-1}\cdot\text{mol}^{-1}$ and $\mu = 29\,\text{g/mol}$. We now add the two contributions to the heat transfer to find

$$P = P_{\text{rad}} + P_{\text{cond}} = 3.21\,\text{W} \approx \boxed{3\,\text{W}.}$$

(b) When the tea cools down to $70\,^\circ\text{C}$, the rate of heat loss will change to

$$P = P_{\text{rad}} + P_{\text{cond}} = 1.16\,\text{W} + 1.00\,\text{W} = 2.16\,\text{W}.$$

This is a signicant change. However, solving a differential equation this difficult is out of the question. Instead, we will channel our inner caveman. Let's approximate that the rate of heat loss is constant and equal to the average of the values at the endpoints,

$$P = \frac{3.21\,\mathrm{W} + 2.16\,\mathrm{W}}{2} = 2.69\,\mathrm{W}.$$

Then we can find the time τ from $P\tau = cm\Delta T$, where $m = 1\,\mathrm{kg}$ (a liter of water) and $\Delta T = 20\,^\circ\mathrm{C}$. We get a time of $\boxed{9\,\mathrm{h}.}$

Theoretical Exam

The problems from EuPhO 2021 (held online) were used in place of the usual theoretical exam.

Experimental Exam

The problems from EuPhO 2021 (held online) were used in place of the usual experimental exam.