**NBPhO 2010 – Corrected Solution to Problem 2**

**(i)** The vessels radiate some net power each, and we wish to study what happens in the cavity that separates them. The inner vessel radiates power $P_1 = \varepsilon\sigma T_1^4 4\pi R_1^2$ outwards. This radiation can then bounce between the vessels in all sorts of ways, but only two things can happen to it in the end – either it ends up absorbed at the outer vessel 2 ($P_{1\to2}$) or reabsorbed at the inner vessel 1 ($P_{1\to1}$), such that $P_1 = P_{1\to2} + P_{1\to1}$. Note that when we track the radiation, we only care for reflections and absorptions. At no point are we allowed to consider reemission, because $P_1$ comprises by definition *all* the emission that comes from vessel 1.

As for the emission of the outer vessel, we only track the part radiated inwards, which is $P_2 = \varepsilon\sigma T_2^4 4\pi R_2^2$. It similarly separates into two parts, $P_2 = P_{2\to1} + P_{2\to2}$. Now, note that the presence of $P_1$ results in a heat flux $P_{1\to2}$ outwards, while the presence of $P_2$ gives $P_{2\to1}$ inwards. Accounting for signs, the net heat flux in the cavity is $P' = P_{2\to1} - P_{1\to2}$ inwards. This is the quantity we want to calculate.

Let's start with $P_{2\to1}$. Each little segment of the outer sphere radiates in all directions. We'll denote the fraction of its emission that strikes the inner sphere by $\kappa$ (and bother with finding the exact value of this purely geometrical factor $\kappa$ later). Out of the radiation that hits vessel 1, a part $P_2\kappa\epsilon$ is absorbed and a part $P_2\kappa(1-\varepsilon)$ is reflected, after which it's due to strike 2. This is because of Kirchhoff's law, whereby the absorptivity equals the emissivity.

Notice also that the power $P_2(1-\kappa)$ which missed 1 is due to strike 2 as well. We can treat it on equal footing with $P_2\kappa(1-\varepsilon)$, given that the same things will happen to both beams afterwards (reflection and absorbtion by 2, etc.). Their sum equals $P_2(1-\kappa\varepsilon)$, and the part of this radiation which reflects off 2 is $P_{\mathrm{I}} = P_2(1-\kappa\varepsilon)(1-\varepsilon)$. We'll now look at the fate of this radiation. A piece $P_{\mathrm{I}}\kappa\varepsilon$ gets absorbed at 1, while the pieces $P_{\mathrm{I}}(1-\kappa)$ and $P_{\mathrm{I}}\kappa(1-\varepsilon)$ avoid absorption and are now headed towards 2. After reflection there, we're left with $P_{\mathrm{II}} = P_{\mathrm{I}}(1-\kappa\varepsilon)(1-\varepsilon)$. And then we repeat. We see that the total absorbed power is

$$P_{2\to1} = P_2\kappa\varepsilon + P_{\mathrm{I}}\kappa\varepsilon + P_{\mathrm{II}}\kappa\varepsilon + \cdots,$$

$$P_{2\to1} = P_2\kappa\varepsilon\left(1 + (1-\kappa\varepsilon)(1-\varepsilon) + (1-\kappa\varepsilon)^2(1-\varepsilon)^2 + \cdots\right),$$

$$P_{2\to1} = \frac{P_2\kappa\varepsilon}{1-(1-\kappa\varepsilon)(1-\varepsilon)} = \frac{P_2\kappa}{1+\kappa(1-\varepsilon)}.$$

This whole procedure can be repeated for $P_{1\to2}$, which will give us $P_{1\to2} = \frac{P_1}{1+\kappa(1-\varepsilon)}$. Our answer for $P'$ is then

$$P' = \frac{P_2\kappa - P_1}{1+\kappa(1-\varepsilon)}.$$

It remains to find $\kappa$. We'll do this in two ways. The first approach is to make use of the laws of thermodynamics. Temperature is a measure of the direction of heat flow. In a hypothetical situation where the two spheres have equal temperature $T$ and are left to themselves, ignoring everything else save for their interaction, we would expect no net heat exchange between them. Of course, in a such a situation $\kappa$ stays the same, so we can write

$$P' = \frac{\varepsilon\sigma T^4 4\pi(R_2^2\kappa - R_1^2)}{1+\kappa(1-\varepsilon)} = 0,$$

which gives us $\kappa = R_1^2/R_2^2$. The second way to find $\kappa$ is by brute strength. Consider a little planar element of area $\Delta A$ on the inside surface of the outer sphere. Think of it as a window through which photons can leave into the outside world. If a photon is to leave through the window at an angle $\theta$ to its normal, it must really deal with a projected area $\Delta A\cos\theta$ rather than $\Delta A$. Hence, the intensity of radiation at angle $\theta$ is modulated by a factor $\cos\theta$. With this in mind, let's now see what fraction of the radiation emitted from the element $\Delta A$ strikes the inner sphere, i.e. fits between angles $\theta = 0$ and $\theta = \theta_0$, where $\sin\theta_0 = R_1/R_2$. Scanning across all solid angles,

$$\kappa = \frac{\int_0^{\theta_0} 2\pi \sin\theta \cos\theta \, \mathrm{d}\theta}{\int_0^{\pi/2} 2\pi \sin\theta \cos\theta \, \mathrm{d}\theta} = \frac{\int_0^{2\theta_0} \sin 2\theta \, \mathrm{d}(2\theta)}{\int_0^{\pi} \sin 2\theta \, \mathrm{d}(2\theta)} = \frac{1 - \cos 2\theta_0}{2} = \sin^2\theta_0 = \frac{R_1^2}{R_2^2}.$$

All that is left is to substitute $\kappa$ into our formula for $P'$, giving us

$$P' = \frac{\varepsilon\sigma(T_2^4 - T_1^4) 4\pi R_1^2}{1 + (1-\varepsilon)(R_1/R_2)^2} = \boxed{1.78\,\mathrm{W}.}$$

**(ii)** While we're undergoing a phase transition, the temperature of the inner sphere does not change. The heat flux $P'$ can therefore be taken as constant. The required time $\tau$ can now be found from

$$P'\tau = \frac{\lambda}{\mu}\left(\frac{4}{3}\pi R_1^3 \rho\right),$$

where we divided $\lambda$ by the molar mass $\mu = 28\,\mathrm{g/mol}$ to find the specific latent heat of vaporisation, which we then multiplied by the total mass of the nitrogen. The value of the molar mass came from the fact that nitrogen (mass number 14) forms diatomic molecules. We obtain $\tau = \boxed{36\,\mathrm{h}.}$

Stefan Ivanov

Last updated: October 2, 2025