2010 Bulgarian IPhO Team Selection Test – Solutions

Short Exam 1

Problem. An axially symmetric body (e.g. a hoop, a cylinder, or a ball) of mass m, radius r, and moment of inertia I has zero initial velocity and an initial angular velocity ω_0. The body is placed at the base of an incline, as shown on Figure 1. The coefficient of friction between the body and the incline is k. The acceleration due to gravity is g.

(a) For what values of k will the body go up? **(0.5 pt)**

(b) Assume k is such that the body starts moving along the incline.
What is the distance L_1 the body has moved until the slipping stops? **(1.25 pt)**
What is the velocity of the body V_1 at that instant? **(0.5 pt)**

(c) What additional distance L_2 does the body move until it stops? **(1.0 pt)**
We place three bodies at the base of the plane – a hoop, a cylinder, and a ball (with moments of inertia mr^2, $\frac{1}{2}mr^2$ and $\frac{2}{5}mr^2$). The bodies are given identical initial angular velocities. Which body will rise up the most, and which the least? **(0.25 pt)**

(d) What is the velocity V_2 of the body when it returns to the base of the incline? **(1.25 pt)**
Calculate V_2 for a hoop with $k = \sqrt{3}/2$ and $\alpha = 30°$. **(0.25 pt)**

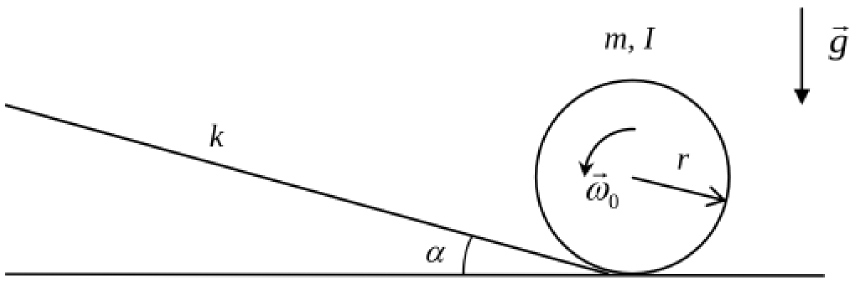
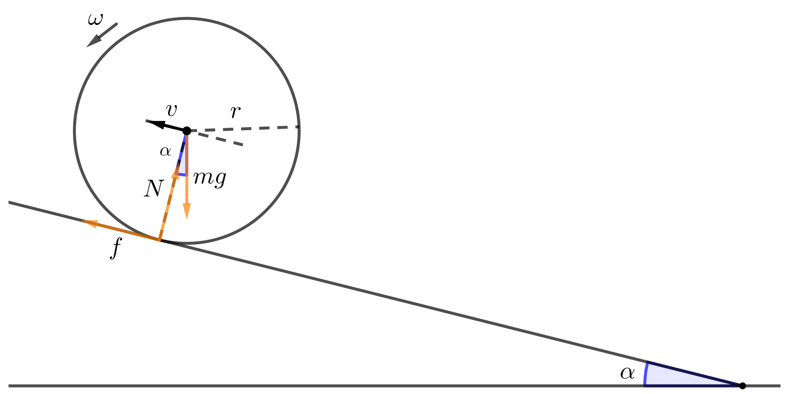


Figure 1

Solution. (a) There are only three forces at play, the weight of the body mg, the normal force $N = mg\cos\alpha$ from the plane, and a frictional force f. We will denote the velocity of the centre of mass by v and the angular velocity by ω. For the body to roll without slipping, we want the velocity of the body relative to the incline to be zero at its contact points with the incline, meaning $v - \omega r = 0$. Initially $v = 0$ and $\omega = \omega_0$, so the body has to slip instead. Then, if it does move, we will have sliding friction, or $f = kN = kmg\cos\alpha$. To move upwards, this force must overcome the component of gravity directed along the incline, $mg\sin\alpha$. We get $kmg\cos\alpha - mg\sin\alpha \geq 0$, or $\boxed{k \geq \tan\alpha.}$



(b) The motion of the centre of mass is set by $kmg\cos\alpha - mg\sin\alpha = ma$. At $t=0$ the velocity is zero, which implies $v = (k\cos\alpha - \sin\alpha)gt$. Working with positive angular velocities when the rotation is counterclockwise, the torque equation with respect to the centre of mass looks like $I\frac{\mathrm{d}\omega}{\mathrm{d}t} = -fr$. Integrating this with the initial condition $\omega(0)=\omega_0$, we find $\omega = \omega_0 - \left(\frac{kmgr\cos\alpha}{I}\right)t$. Now, v and ω will evolve this way until a time t_1 when $v = \omega r$. From then on, the body is locked into rolling without slipping and the friction force will change. To get t_1, we write

$$(k\cos\alpha - \sin\alpha)gt_1 = \left(\omega_0 - \frac{kmgr\cos\alpha}{I}t_1\right)r \quad \Rightarrow \quad t_1 = \frac{\omega_0 r/g}{k\cos\alpha\left(1+\frac{mr^2}{I}\right) - \sin\alpha}.$$

The total distance travelled is

$$L_1 = \frac{at_1^2}{2} = \boxed{\frac{\omega_0^2 r^2}{2g}\cdot\frac{k\cos\alpha - \sin\alpha}{\left(k\cos\alpha\left(1+\frac{mr^2}{I}\right)-\sin\alpha\right)^2}.}$$

The velocity at t_1 is

$$V_1 = at_1 = \boxed{\omega_0 r\cdot\frac{k\cos\alpha - \sin\alpha}{k\cos\alpha\left(1+\frac{mr^2}{I}\right)-\sin\alpha}.}$$

(c) After the slipping stops, we always have $v=\omega r$. We differentiate this to get an additional relation, $a = \frac{\mathrm{d}\omega}{\mathrm{d}t}r$. Together with the previous rotational and translational equations of motion, this is a system of three equations with three variables: a, $\frac{\mathrm{d}\omega}{\mathrm{d}t}$, and f. The solution is

$$a = -\frac{g\sin\alpha}{1+\frac{I}{mr^2}}, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{g\sin\alpha}{r\left(1+\frac{I}{mr^2}\right)}, \qquad f = \left(\frac{mr^2}{I+mr^2}\right)mg\sin\alpha.$$

Then, the additional distance before stopping is

$$L_2 = \frac{V_1^2}{2|a|} = \boxed{\frac{\omega_0^2 r^2}{2g\sin\alpha}\left(1+\frac{I}{mr^2}\right)\left(\frac{k\cos\alpha - \sin\alpha}{k\cos\alpha\left(1+\frac{mr^2}{I}\right)-\sin\alpha}\right)^2.}$$

The sum of L_1 and L_2 is

$$L = \frac{\omega^2 r^2}{2g}\cdot\frac{\left(k\cos\alpha\left(1+\frac{I}{mr^2}\right) - \frac{I}{mr^2}\sin\alpha\right)(k\cos\alpha - \sin\alpha)}{\left(k\cos\alpha\left(1+\frac{mr^2}{I}\right)-\sin\alpha\right)^2\sin\alpha}.$$

Because $k > \tan\alpha$, increasing I would both increase the numerator and decrease the denominator of this expression. A larger I means a greater distance L. Thus, the $\boxed{\text{hoop}}$ rises up the $\boxed{\text{most}}$ and the $\boxed{\text{ball}}$ rises up the $\boxed{\text{least.}}$

(d) When the body rolls downhill, the motion is still restricted to rolling without slipping. Because there is no relative motion with respect to the incline, the friction force does zero work, and energy is conserved. We will take the base of the incline as the zero point for the potential energy. At the start of the descent, the kinetic energy is zero and the potential energy is $mgL\sin\alpha$. At the end, the potential energy is zero and the kinetic energy is $\frac{mV_2^2}{2} + \frac{I(V_2/r)^2}{2}$:

$$mgL\sin\alpha = \frac{mV_2^2}{2}\left(1+\frac{I}{mr^2}\right).$$

We conclude that

$$\boxed{V_2 = \omega_0 r\left(\frac{\left(k\cos\alpha\left(1+\frac{I}{mr^2}\right) - \frac{I}{mr^2}\sin\alpha\right)(k\cos\alpha - \sin\alpha)}{\left(1+\frac{I}{mr^2}\right)\left(k\cos\alpha\left(1+\frac{mr^2}{I}\right)-\sin\alpha\right)^2}\right)^{1/2}.}$$

Substituting the values for k and α, we get $\boxed{V_2 = \omega_0 r/\sqrt{8}.}$

Problem 1. Assume the Earth rotates around the Sun in a circular orbit of radius $r_0 = 1\,\mathrm{au}$ with velocity $v_0 = 30\,\mathrm{km/s}$ and period $T_0 = 1\,\mathrm{yr}$. Halley's comet has an orbital period of $T_1 = 76\,\mathrm{yr}$ and its closest approach to the Sun is at a distance $r_{\min} = 0.59\,\mathrm{au}$.

(a) Find the maximum distance between the comet and the Sun $r_{\max}$. **(1.0 pt)**

(b) Find the minimum and maximum velocities of the comet, $v_{\min}$ and $v_{\max}$. **(2.0 pt)**

Your formulae should only include the data in the problem statement, and your numerical values should be in the same units.

Solution. (a) We can find the semi-major axis a of the comet's elliptical orbit from Kepler's third law,

$$\frac{r_0^3}{T_0^2} = \frac{a^3}{T_1^2} = \frac{GM}{4\pi^2} \quad \Rightarrow \quad a = r_0 \left(\frac{T_1}{T_0}\right)^{2/3}.$$

The minimum and maximum distances from the focus where the Sun is add up to the major axis, $r_{\min} + r_{\max} = 2a$, yielding

$$\boxed{r_{\max} = 2r_0 \left(\frac{T_1}{T_0}\right)^{2/3} - r_{\min} = 35.28\,\mathrm{au}.}$$

We can assume the Sun doesn't move at all (it does, but we need the comet's mass m to account for this, and the resulting corrections are tiny anyway). The total energy of the system is then $\frac{mv^2}{2} - \frac{GMm}{r}$, and it is conserved. We see that smaller distances correspond to larger velocities, and we can write

$$\frac{mv_{\min}^2}{2} - \frac{GMm}{r_{\max}} = \frac{mv_{\max}^2}{2} - \frac{GMm}{r_{\min}}.$$

The gravitational pull from the Sun is a central force, so the angular momentum of the comet about the Sun's position is conserved. In particular,

$$mv_{\min}r_{\max} = mv_{\max}r_{\min}.$$

We solve this set of equations and find

$$v_{\max} = \sqrt{\frac{2GM(r_{\max}/r_{\min})}{r_{\max} + r_{\min}}}, \qquad v_{\min} = \sqrt{\frac{2GM(r_{\min}/r_{\max})}{r_{\max} + r_{\min}}}.$$

Now we need to express the velocities in the original variables. To this end, we use

$$\frac{2GM}{r_{\min} + r_{\max}} = \frac{GM}{a} = \frac{4\pi^2 a^2}{T_1^2} = \left(\frac{2\pi r_0}{T_0}\left(\frac{T_0}{T_1}\right)^{1/3}\right)^2,$$

and we end up with

$$\boxed{v_{\max} = \frac{2\pi r_0}{T_0}\left(\frac{T_0}{T_1}\right)^{1/3}\left(2\left(\frac{r_0}{r_{\min}}\right)\left(\frac{T_1}{T_0}\right)^{2/3} - 1\right)^{1/2} = 54.78\,\mathrm{km/s},}$$

$$\boxed{v_{\min} = \frac{2\pi r_0}{T_0}\left(\frac{T_0}{T_1}\right)^{1/3}\left(2\left(\frac{r_0}{r_{\min}}\right)\left(\frac{T_1}{T_0}\right)^{2/3} - 1\right)^{-1/2} = 0.92\,\mathrm{km/s}.}$$

Problem 2. A metal puck of mass m has an outer radius b and an inner radius a.

(a) Find the moment of inertia of the puck about an axis perpendicular to the puck and passing through its centre of mass. **(1.0 pt)**

(b) The puck is tied on a very thin string and is left to oscillate around its equilibrium position. Find the oscillation period T. The acceleration due to gravity is g. **(2.0 pt)**

Solution. (a) Assume the puck is uniform and denote its thickness by d. Its density is then $\rho = \frac{m}{\pi(b^2-a^2)d}$. The moment of inertia is calculated similarly to that of a cylinder,

$$I = \int_a^b (\rho d 2\pi r \mathrm{d}r) r^2 = \left(\frac{m}{\pi(b^2-a^2)d}\right) 2\pi d \left(\frac{b^4-a^4}{4}\right) = \boxed{\frac{1}{2}m(b^2+a^2).}$$

(b) The problem statement is unclear, but because length of the string isn't given, the only feasible option is that the string is kept horizontal and taut. We will also have to assume that the puck doesn't slide on the string. Now we have a physical pendulum with its pivot at a from its centre of mass. Using the parallel axis theorem, the moment of inertia about the pivot is $I' = I + ma^2$. For small deviations θ from the equilibrium position, the torque equation about the pivot is

$$I'\ddot{\theta} + mga\theta = 0.$$

This corresponds to harmonic oscillations with a period of

$$T = 2\pi\sqrt{\frac{I'}{mga}} = \boxed{2\pi\sqrt{\frac{b^2+3a^2}{2ga}}.}$$

Problem 3. A neutron at rest decays into an electron, an electron antineutrino, and a proton at rest, $n \to p + e^- + \tilde{\nu}_e$. The neutrino is assumed massless.

(a) Find the momentum of the electron p_e and calculate it. **(2.0 pt)**

(b) Find the velocity of the electron v_e and calculate it. **(1.0 pt)**

Solution. (a) As with all other problems in special relativity, we will set $c = 1$ and bring the c's back at the end. The total energy before the decay is m_n, and the total momentum is zero. After the decay, we have a proton at rest, an electron with some momentum $\mathbf{p_e}$, and a neutrino whose momentum has to be $-\mathbf{p_e}$. The neutrino is massless, so its energy is simply p_e. Then, the total energy after the decay is $p_e + \sqrt{p_e^2 + m_e^2} + m_p$. Going through the algebra,

$$\sqrt{p_e^2 + m_e^2} = m_n - m_p - p_e,$$

$$-2(m_n - m_p)p_e + (m_n - m_p)^2 = m_e^2,$$

$$p_e = \frac{(m_n - m_p)^2 - m_e^2}{2(m_n - m_p)} \quad \Leftrightarrow \quad \boxed{\frac{(m_n - m_p)^2 - m_e^2}{2(m_n - m_p)}c = 0.54\,\mathrm{MeV/c} = 2.9 \times 10^{-22}\,\mathrm{kg{\cdot}m/s}.}$$

We work with only two significant digits because that's how the electron mass was given.

(b) For the velocity, we will use $v_e = \frac{p_e}{E_e} = \frac{p_e}{\sqrt{p_e^2+m_e^2}}$. Then

$$v = \frac{(m_n - m_p)^2 - m_e^2}{\sqrt{((m_n - m_p)^2 - m_e^2)^2 + 4m_e^2(m_n - m_p)^2}} = \frac{(m_n - m_p)^2 - m_e^2}{(m_n - m_p)^2 + m_e^2},$$

$$\boxed{v = \frac{(m_n - m_p)^2 - m_e^2}{(m_n - m_p)^2 + m_e^2}c = 0.73c = 2.2 \times 10^8\,\mathrm{m/s}.}$$

Experimental Exam

Problem 1. N-resistor black box.

Equipment:
Sealed paper box with a closed circuit (Figure 2), multimeter, ruler, graph paper. The circuit consists of N identical resistors in series, each of resistance R_0. You can only access the terminals of 12 adjacent resistors.

Figure 2

The aim of this problem is to find the resistance R_0 and the number of resistors in the box N.

(a) Find a formula for the resistance $R(k)$ of the circuit when measuring between terminals which are k resistors away from each other. **(1.0 pt)**

(b) Describe a method for finding the resistance R_0 and the number of resistors N by plotting a series of measurements. **(1.0 pt)**

(c) Take the necessary measurements. Present them in a table and explain how they were obtained. **(3.0 pt)**

(d) State the variables which, when plotted, can easily give you R_0 and N. **(1.0 pt)**

(e) Plot the relevant graph. **(4.0 pt)**

(f) Using the graph, determine R_0. **(1.5 pt)**
Likewise, determine N. **(2.5 pt)**

(g) Estimate your error in finding R_0. **(0.5 pt)**
Estimate your error in finding N. **(0.5 pt)**

Call the examiner in case of any technical difficulties.

Note: If there is evidence that the box has been unsealed or that the circuit has been interrupted, you will be disqualified.