2024 Saint Petersburg Astronomy Olympiad – Practical Round, Grade 11

You're given the apparent trajectory of an object in the sky as observed from the centre of the Earth. The circles' size corresponds to the angular size as seen from Earth. You're also given an equatorial sky chart. Find the elements of the object's orbit around the Sun – inclination, longitude of the ascending node, argument of perihelion, eccentricity, and semi-major axis. Find the semi-minor axis, the perihelion distance, and the perihelion velocity. What else can you say about this object?

Solution:

Firstly, note that the trajectory isn't closed, and instead asymptotes to two endpoints. This implies a hyperbolic orbit. The object must then originate from outside the Solar system. We notice that it nevertheless sweeps through a large angle on the sky within a year, meaning that it approaches the Sun quite closely. As the Earth orbits around the Sun, the object will get projected on different areas of the sky. So, as the object approaches the Sun, its geocentric trajectory will differ significantly from its heliocentric trajectory. Since the parallax vanishes as the object moves further away from the Sun, it will be much more convenient to derive the orbital parameters by looking at the endpoints of the trajectory.

Using the sky chart, we can measure their equatorial coordinates as $\alpha_1 = 18.6^{\mathrm{h}}$, $\delta_1 = +33°$, and $\alpha_2 = 23.9^{\mathrm{h}}$, $\delta_2 = +25°$. Some of the orbital elements are measured with respect to the ecliptic, so we will need to convert to ecliptic coordinates (λ, β). We draw the celestial sphere with the equator and the ecliptic. Then, we form the relevant spherical triangle, and the sine/cosine rules yield

$$\sin\beta = \sin\delta\cos\epsilon - \cos\delta\sin\epsilon\sin\alpha, \qquad \cos\lambda = \frac{\cos\alpha\cos\delta}{\cos\beta}.$$

We find $\lambda_1 = 284°$, $\beta_1 = +56°$, and $\lambda_2 = 9°$, $\beta_2 = +23°$. Alternatively, we could have noted the proximity of the Lyra endpoint to the 18^{h} meridian and the proximity of the Pegasus endpoint to the 0^{h} meridian and the equator. Then, using planar approximations, $\lambda_1 \approx \alpha_1$, $\beta_1 \approx \delta_1 + \epsilon$, $\lambda_2 \approx \delta_2 \sin\epsilon$, $\beta_2 \approx \delta_2\cos\epsilon$, with similar numerical results.

The object's motion in space is constrained to a plane, so the heliocentric trajectory is a great circle. We've now got the coordinates of two points on this circle, which allows us to find its equation. We will first determine the angular distance χ between the endpoints L and P. Applying the cosine rule on triangle πLP,

$$\cos\chi = \sin\beta_1\sin\beta_2 + \cos\beta_1\cos\beta_2\cos\Delta\lambda \approx \sin\beta_1\sin\beta_2 \qquad \Rightarrow \qquad \chi = 68°.$$

Or, if we're feeling brave, planar geometry yields

$$\chi \approx \sqrt{\Delta\beta^2 + (\Delta\lambda\cos\beta_{\mathrm{avg}})^2} = 73°.$$

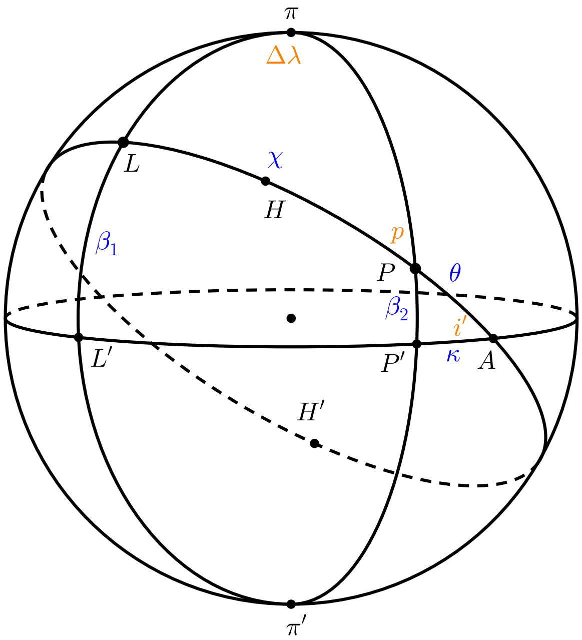
Though here at least it's better to opt for spherical trigonometry.

To proceed, find $p = 31°$ with the sine rule. Switching to the small triangle $PP'A$ and using planar geometry, we immediately recover

$$\kappa \approx \beta_2\tan p = 14°, \gamma \approx \frac{\beta_2}{\cos p} = 27°, \text{ and } i' \approx \frac{\pi}{2} - p = 59°.$$

Alternatively, we can do it the long way – the cosine rule on triangle πPA gives $\gamma = 27°$. A cosine rule and a sine rule in triangle $PP'A$ then give $\kappa = 14°$ and $i' = 61°$, respectively.

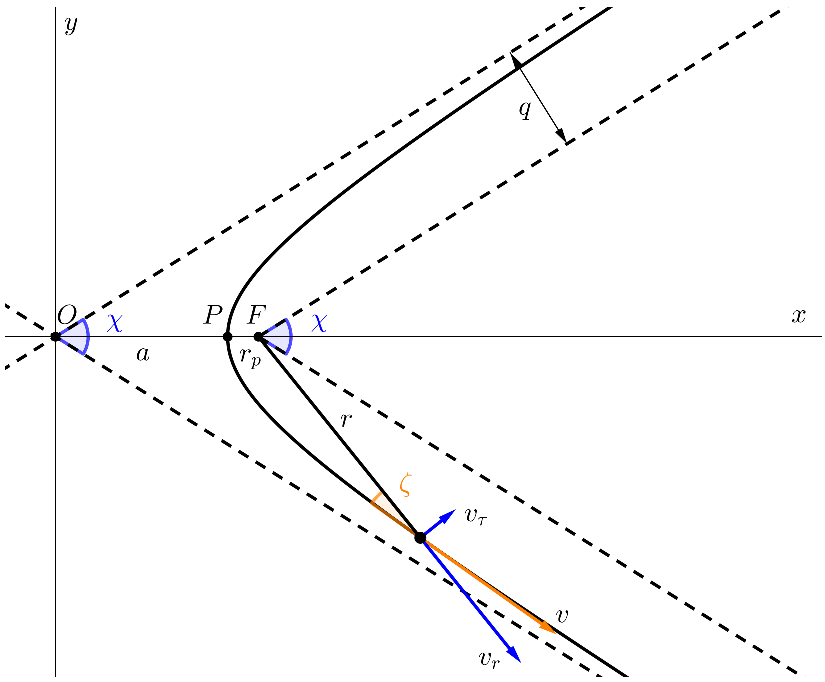
Now we are ready to draw some conclusions. Our object moves on a great circle, taking the long arc from L to P. This means that A corresponds to the ascending node. Its longitude is found as $\Omega = \lambda_2 + \kappa = 24°$. Now, note that the ecliptic longitude of the object as it moves from L to P is decreasing. Therefore, this object's orbital motion is retrograde. Its orbital inclination is given as $i = \pi - i' = 121°$. The argument of perihelion ω is the length of the arc starting from A and following the direction of motion until we reach the perihelion point H'. From symmetry, H' lies exactly opposite the point midway between P and L. Hence, $\omega = \pi + \gamma + \frac{\chi}{2} = 241°$.



To find the remaining orbital elements, consider the geometry of a hyperbolic orbit. We will work within the convention of a positive semi-major axis a. Even if we don't know immediately how to interpret a, b, and e geometrically, one can find this out 'on the go'. All Keplerian orbits are conic sections, so it is intuitive that the usual polar and Cartesian equations of an ellipse extend to hyperbolae with very minor changes,

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1, \qquad r = \frac{a(e^2-1)}{1+e\cos\theta}.$$

Any sign changes can be worked out by drawing a diagram:



If we do write down the correct Cartesian equation, we immediately note that no y are allowed until $x \geq a$, i.e. a is the distance from the centre of the hyperbola to its periapsis. We also see that at large x and y, $\frac{y}{x} \to \frac{b}{a}$, implying slant asymptotes which pass through the centre and have slope $\frac{b}{a}$. The angle between the asymptotes corresponds to the angle χ between the endpoints of the trajectory as seen from the focus, so $\tan\frac{\chi}{2} = \frac{b}{a}$.

Within the context of the polar equation, the asymptotes mean that as $\pi - \theta \to \frac{\chi}{2}$, we get $r \to \infty$. Thus $\cos\frac{\chi}{2} = \frac{1}{e} = \frac{a}{f}$. We may now also interpret the impact parameter q of the hyperbola. On the one hand, $\sin\frac{\chi}{2} = \frac{q}{f}$. On the other,

$$\sin\frac{\chi}{2} = \tan\frac{\chi}{2}\cos\frac{\chi}{2} = \frac{b}{a}\cdot\frac{a}{f} = \frac{b}{f} \qquad \Rightarrow \qquad q \Leftrightarrow b.$$

Returning to our problem, the eccentricity of the object's orbit is

$$e = \frac{1}{\cos\frac{\chi}{2}} = 1.20.$$

To find a, we will consider the motion of the object at one of the endpoints (we pick P, where measurements will be easier). The trajectory at the endpoints is due to the superposition of parallax and proper motion. As the object goes further and further away from the Sun, its velocity will tend to the velocity at infinity v_∞, with the tangential component v_τ tending to zero (see the diagram above). Therefore, the heliocentric radial velocity v_r far from the Sun will suffice as a good estimate for v_∞. To find it, we can track the heliocentric distance of the object with time. We do this by estimating the mean parallax of the object as it recedes from year to year. If we do this for later years, our answer will be more physically sound, but our measurement error will grow rapidly. To compromise, we will consider measurements only for 2021, 2022, 2023, and 2024.

Using the sky chart, we find the distance between Algenib (γ Peg) and Alpheratz (α And) as 14°. We use this as our scale on the closeup of the endpoint P. We measure the 'major axis' of each year's loop and consider it twice that year's mean parallax. As a result,

Year	Parallax [°]	Distance [AU]	Increment [AU]
2021	4.94	11.6	
2022	3.24	17.7	6.1
2023	2.43	23.6	5.9
2024	1.99	28.8	5.2

We estimate $v_\infty = 5.5\,\mathrm{AU/yr}$. Extending our knowledge for an ellipse, the specific orbital energy of the object must be $\varepsilon = \frac{GM_\odot}{2a}$. Since the potential energy is negligible at infinity, we can also write down $\varepsilon = \frac{v_\infty^2}{2}$, concluding that $a = \frac{GM_\odot}{v_\infty^2}$. For the Earth we can write down $r_E = \frac{GM_\odot}{v_E^2}$, where $r_E = 1\,\mathrm{AU}$ and $v_E = 2\pi\,\mathrm{AU/yr}$. Thus

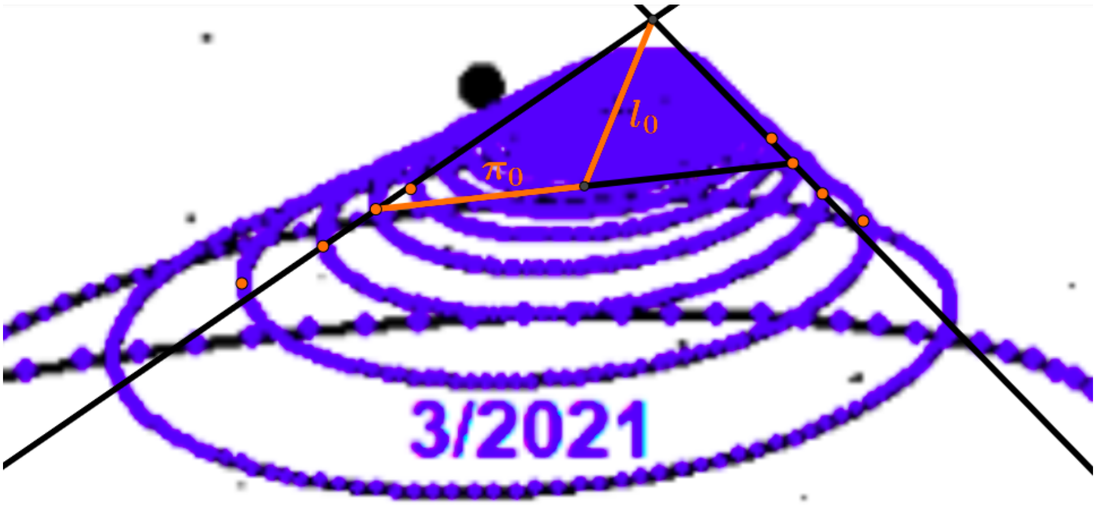
$$a = r_E\left(\frac{v_E}{v_\infty}\right)^2 = 1.30\,\mathrm{AU}.$$

The rest follows from the geometry of a hyperbola. We get $b = a\tan\frac{\chi}{2} = 0.88\,\mathrm{AU}$ for the semi-minor axis and $r_p = a(e-1) = 0.26\,\mathrm{AU}$ for the perihelion distance. As for the perihelion velocity, it is easiest to use the conservation of angular momentum,

$$v_p r_p = v_\infty b \qquad \Rightarrow \qquad v_p = 89\,\mathrm{km/s}.$$

We will also present another approach for finding the physical dimensions of the orbit. We notice that the spiralling motion of the object at large distances has some asymptotic behaviour. In particular, the motion is confined to a triangle, meaning that the angular displacement and the respective change in parallax maintain a constant ratio. To find this ratio, we draw the tangents that bound the spiral. Then, using linearity, and referring to the diagram of a hyperbolic orbit above, we find

$$\frac{\pi_0}{l_0} = \frac{\mathrm{d}\pi}{\mathrm{d}l} = \frac{\mathrm{d}\pi/\mathrm{d}t}{\mathrm{d}l/\mathrm{d}t} = \frac{\mathrm{d}\pi/\mathrm{d}t}{\mu} = \frac{r_E}{r^2}\cdot\frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{1}{\mu} = \frac{r_E}{r^2}\cdot v_r\cdot\frac{r}{v_\tau} = \frac{r_E}{r\tan\zeta} \approx \frac{r_E}{r\sin\zeta} \approx \frac{r_E}{p} = \frac{r_E}{b}.$$



Hence $\frac{r_E}{b} = 1.16$, giving us $b = 0.86\,\mathrm{AU}$. A few other parameters follow immediately, $a = 1.27\,\mathrm{AU}$, $r_p = 0.25\,\mathrm{AU}$. Finally, v_∞ and v_p can be obtained by conserving energy and angular momentum,

$$v_p^2 - 2v_E^2\frac{r_E}{r_p} = v_\infty^2, \qquad v_p r_p = v_\infty b \qquad \Rightarrow \qquad v_\infty = 26\,\mathrm{km/s}, \qquad v_p = 89\,\mathrm{km/s}.$$

Considering the eccentricity and the retrograde motion, we conclude that this object is likely of interstellar origin. We do not have enough information to comment on the composition of the object. For example, our object gets quite close to the Sun, but many sungrazing comets come even closer without evaporating.

Our object is in fact 1I/'Oumuamua, the first interstellar body detected in the Solar system. From the literature, its orbital elements are $\Omega = 25°$, $i = 123°$, $\omega = 242°$, $e = 1.20$, $a = 1.27\,\mathrm{AU}$. Additionally, $b = 0.84\,\mathrm{AU}$, $r_p = 0.25\,\mathrm{AU}$, $v_p = 88\,\mathrm{km/s}$.

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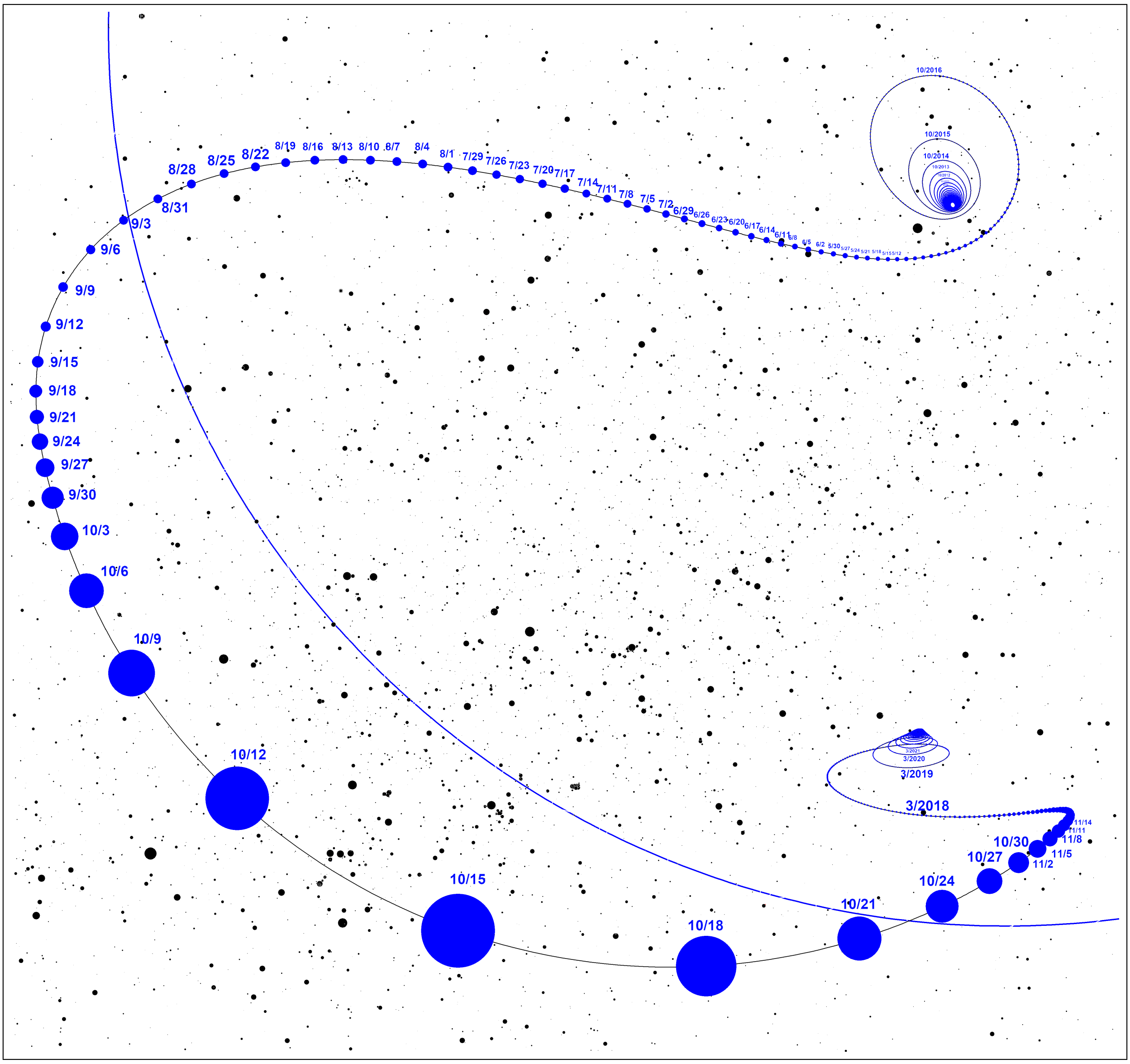
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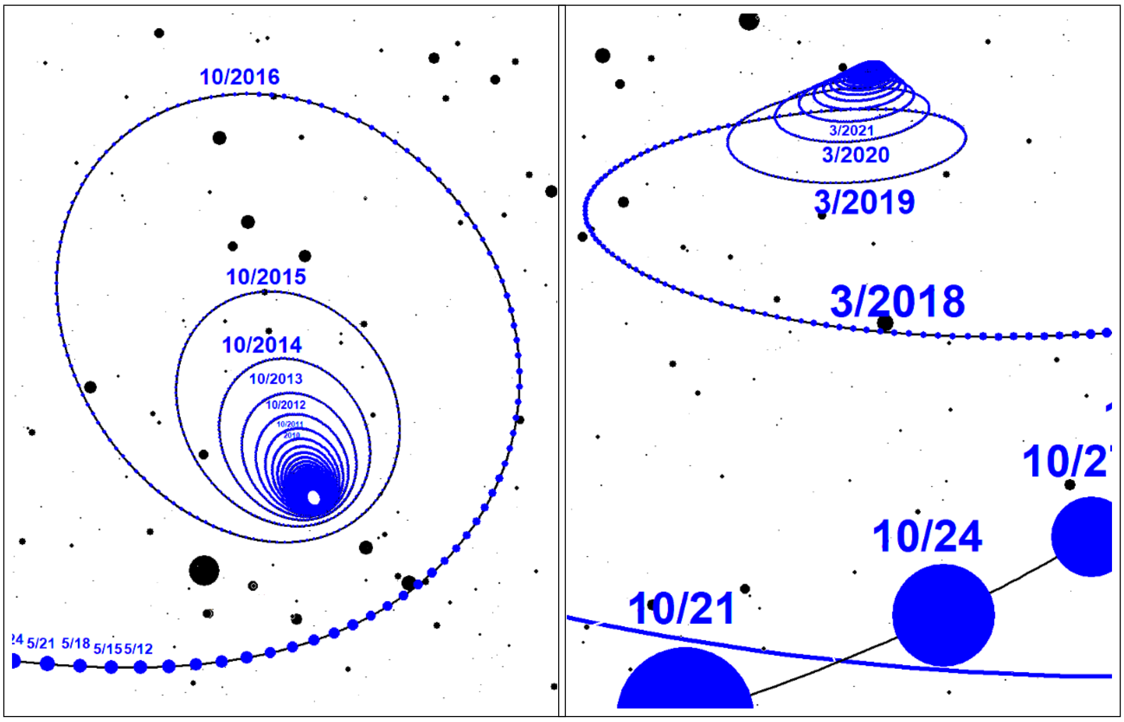
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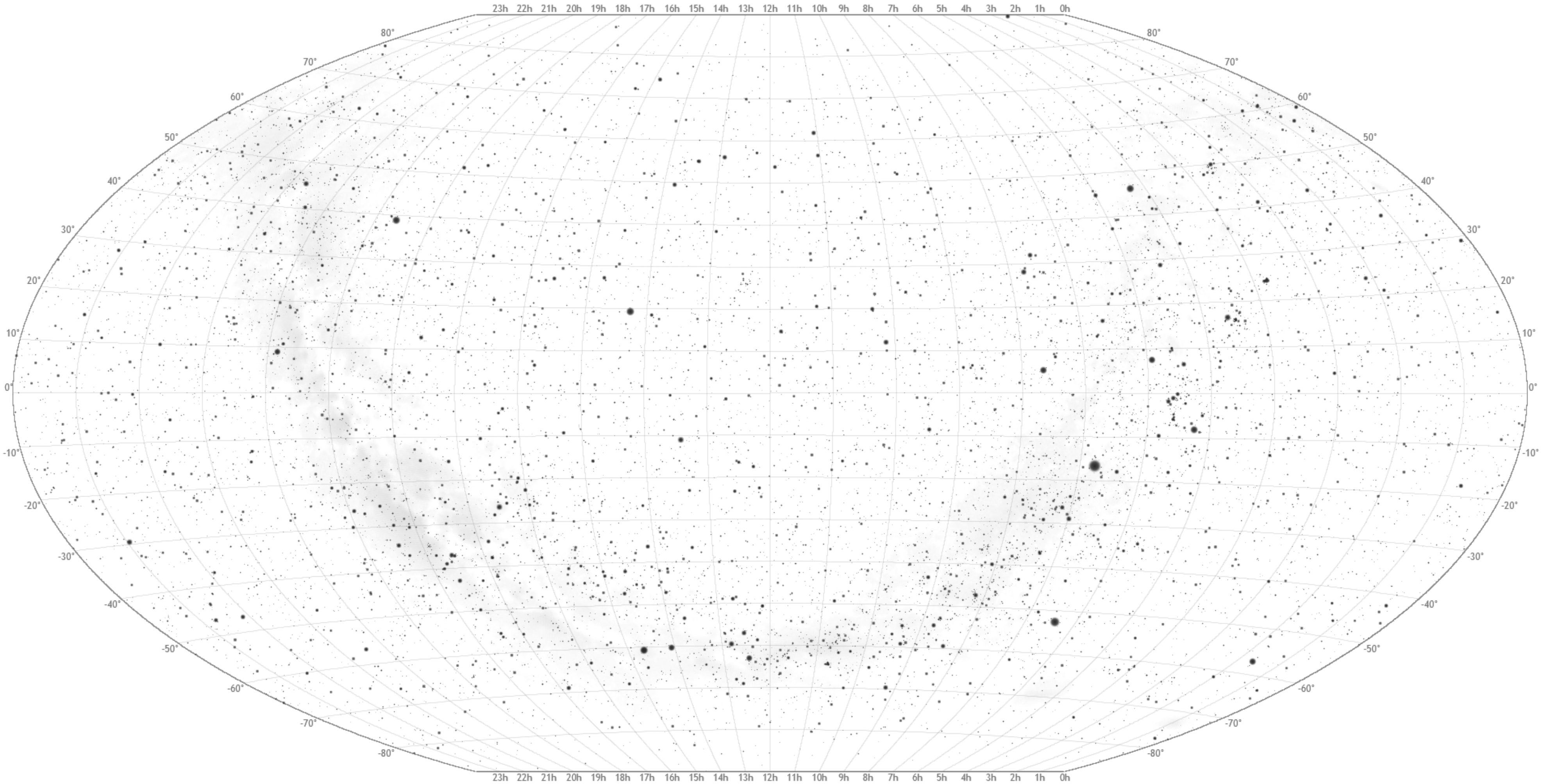
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