2008 Bulgarian IPhO Team Selection Test

Short Exam 1

Problem. A solid ball of mass m and radius a (moment of inertia $I = \frac{2}{5}ma^2$) starts rolling without slipping from the top of another fixed ball of radius b. Its initial velocity is negligible. The acceleration due to gravity is g.

(a) Find the angle $\theta = \theta_0$ at which the rolling ball will lose contact with the fixed ball. The angle θ is measured between the upward direction and the segment connecting the centres of the balls.

(b) Find the velocity of the centre of mass v of the rolling ball when it detaches.

(c) What coefficient of friction k would make the upper ball start slipping at an angle $\theta = \alpha < \theta_0$?

Solution. (a) Let us find the dependence of the centre of mass velocity v on θ. Our ball rolls without slipping, so the friction force at the point of contact with the other ball does not do any work. Applying conservation of energy,

$$\frac{mv^2}{2} + \frac{I\omega^2}{2} = mg(a+b)(1-\cos\theta),$$

and using $v = \omega a$ (no slipping), we find that $v^2 = \frac{10}{7}g(a+b)(1-\cos\theta)$. Now we can determine the normal force $N(\theta)$. There are three forces on the ball: gravity mg, friction f, and a normal force N. Projecting these along the normal, we find

$$mg\cos\theta - N = ma_n,$$

where $a_n = v^2/(a+b)$ is the normal acceleration of the centre of mass. After substituting for v^2, we get $N = \left(\frac{17}{7}\cos\theta - \frac{10}{7}\right)mg$. The ball loses contact with the surface when $N = 0$. Thus $\cos\theta_0 = \frac{10}{17}$, or $\boxed{\theta_0 \approx 54^\circ.}$

(b) Plugging the value for θ back into the formula for $v(\theta)$, we get $\boxed{v = \sqrt{\frac{10}{17}g(a+b)}.}$

(c) First, we need to find the friction force $f(\theta)$. Projecting forces along the tangent, we obtain

$$mg\sin\theta - f = ma_\tau,$$

where the tangential acceleration a_τ of the centre of mass is related to the angular acceleration of the ball ε by $a_\tau = \varepsilon a$ (no slipping). Taking torques about the centre of mass, we get $fa = I\varepsilon$. Now, solving for f, we find $f(\theta) = \frac{2}{7}mg\sin\theta$. For the ball to start slipping at $\theta = \alpha$, we need $f(\alpha) = kN(\alpha)$. Plugging in our formulae for f and N, we get

$$\boxed{k = \frac{2\sin\alpha}{17\cos\alpha - 10}}.$$

You would be well advised to verify that all the results behave appropriately in special cases.

Theoretical Exam

Problem 1. A satellite of mass m moves in a circular orbit of radius r around a planet of mass M. Because of a drag force of the form $F_{\mathrm{dr}} = Av^n$, the orbital radius decreases at a constant rate

$$\frac{\mathrm{d}r}{\mathrm{d}t} = D \ll \frac{r}{T},$$

where T is the orbital period. The gravitational constant is γ.

(a) Find the number n.

(b) Determine $D = f(\gamma, M, m, A)$.

Solution. Since gravity $F_{\mathrm{g}} = \frac{\gamma m M}{r^2}$ acts as a centripetal force of the form $F_{\mathrm{c}} = \frac{mv^2}{r}$, the orbital velocity must be $v = \sqrt{\frac{\gamma M}{r}}$. The mechanical energy of the satellite is then

$$E = \frac{mv^2}{2} - \frac{\gamma m M}{r} = -\frac{\gamma m M}{2r}.$$

Of course, this is in agreement with the general result for an elliptical orbit of semi-major axis a, which is $E = -\frac{\gamma m M}{2a}$. The power of the drag force is

$$P = \mathbf{F} \cdot \mathbf{v} = -Fv = -Av^{n+1}.$$

Nonconservative forces act to change the mechanical energy, meaning that $P = \frac{\mathrm{d}E}{\mathrm{d}t}$. Then

$$-Av^{n+1} = \frac{\gamma m M}{2r^2}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\gamma m M}{2r^2}D.$$

The equation above must always hold, and $v^{n+1} \propto r^{-(n+1)/2}$, so $\boxed{n = 3.}$ Going back to that equation,

$$-A\gamma^2 M^2 = \frac{\gamma m M}{2}D \qquad \Rightarrow \qquad \boxed{D = \frac{2\gamma M A}{m}}.$$

Problem 2. An incompressible fluid of viscosity η flows along a cylindrical pipe of length L and radius R. The pressures at the two ends of the pipe are p_1 and p_2, respectively. The flow is stationary.

(a) Find the flow velocity $v(r)$ in terms of the distance from the axis of the pipe r.

(b) Find the volumetric flow rate through the pipe Q.

Solution. (a) The problem is symmetric with respect to θ and z, and the velocity v depends only on r. In other words, the velocity gradient is along r only. The equation for the viscous force acting on a piece of fluid is then greatly simplified. Choose a cylindrical piece of fluid that extends from the axis to some distance r. Its area of contact with the outside fluid is $A = 2\pi r L$. The outside fluid then acts on our piece with a force $F_{\mathrm{d}} = \eta(2\pi r L)\frac{\mathrm{d}v}{\mathrm{d}r}$. Our piece is also pushed forwards by a force $F_{\mathrm{p}} = (p_1 - p_2)\pi r^2$ due to the pressure difference at the two ends. The flow is stationary, so the two forces must balance. This yields

$$-\frac{(p_1 - p_2)}{2\eta L} r\mathrm{d}r = \mathrm{d}v.$$

This holds throughout $r \in [0, R]$. Using that $v = 0$ at the pipe's boundary, integrate it as follows:

$$-\frac{(p_1 - p_2)}{2\eta L}\int_r^R r\mathrm{d}r = \int_v^0 \mathrm{d}v,$$

$$\boxed{v(r) = \frac{(p_1 - p_2)}{4L\eta}(R^2 - r^2)}.$$

(b) The total volumetric flow rate can be found by integrating the flow rates for all cylindrical rings between $r = 0$ and $r = R$.

$$Q = \int v\mathrm{d}S = \int_0^R 2\pi r v(r)\mathrm{d}r = \frac{\pi(p_1 - p_2)}{2\eta L}\int_0^R (rR^2 - r^3)\mathrm{d}r = \boxed{\frac{\pi R^4(p_1 - p_2)}{8\eta L}}.$$

This is called Poiseulle's law.

Problem 3. A ball of mass M has velocity v_0. It strikes a ball of mass m ($M > m$) at rest. The collision is elastic. The angle between the velocity vectors of M before and after the collision is α.

(a) Find the maximum value of α.

(b) Find the velocities u_M and u_m of the two balls after the collison in the case where maximum α is realised.

Solution. (a) Consider the collision in the reference frame of the centre of mass (CM). The CM's velocity with respect to the lab frame is

$$\mathbf{v}_{\mathrm{CM}} = \frac{M\mathbf{v_0}}{m + M}.$$

The velocities of the two balls in the CM frame are then

$$\mathbf{v_M} = \mathbf{v_0} - \mathbf{v}_{\mathrm{CM}} = \frac{m\mathbf{v_0}}{m + M} \quad \text{and} \quad \mathbf{v_m} = \mathbf{0} - \mathbf{v_{CM}} = -\frac{M\mathbf{v_0}}{m + M}.$$

Let us examine the momenta of the balls in the CM frame before the collision ($\mathbf{p_M}$, $\mathbf{p_m}$) and after the collision ($\mathbf{p'_M}$, $\mathbf{p'_m}$). The collision is elastic in this frame as well (it is impossible to release heat in some reference frames but not in others). This leads us to the set of equations

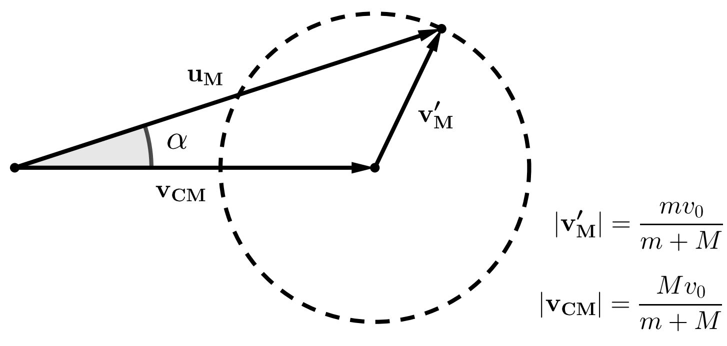
$$\mathbf{p_M} + \mathbf{p_m} = \mathbf{p'_M} + \mathbf{p'_m} = \mathbf{0},$$

$$\frac{|\mathbf{p_M}|^2}{2M} + \frac{|\mathbf{p_m}|^2}{2m} = \frac{|\mathbf{p'_M}|^2}{2M} + \frac{|\mathbf{p_m}|^2}{2m}.$$

We can see that these equations can hold only when $|\mathbf{p_M}| = |\mathbf{p'_M}|$ and $|\mathbf{p_m}| = |\mathbf{p'_m}|$ (with no other constraints). In other words, in the CM frame the velocities after the collision have the same magnitudes as before. The new velocity vector of M in the CM frame $\mathbf{v'_M}$ can point in any direction. So, if we fix one end of this vector, the other end is constrained to a circle.

To get us back to the new velocity vector of M in the lab frame $\mathbf{u_M}$, we need to add $\mathbf{v_{CM}}$:

$$\mathbf{u_M} = \mathbf{v'_M} + \mathbf{v_{CM}}.$$



From the diagram we see that the deviation of $\mathbf{u_M}$ from $\mathbf{v_0}$ is largest when the vector $\mathbf{u_M}$ ends up tangent to the circle of $\mathbf{v'_M}$. In that case,

$$\sin\alpha = \frac{mv_0}{m+M}\bigg/\frac{Mv_0}{m+M} = m/M\ .$$

Then $\boxed{\alpha = \arcsin\left(m/M\right).}$

(b) When α is largest, the velocities are related by $|\mathbf{u_M}|^2 + |\mathbf{v'_M}|^2 = |\mathbf{v_{CM}}|^2$. This gives us

$$\boxed{u_M = \sqrt{\frac{M-m}{M+m}}v_0}\ .$$

To find u_m, we will use $\mathbf{u_m} = \mathbf{v'_m} + \mathbf{v_{CM}}$. The vector $\mathbf{v'_m}$ makes an angle $\frac{\pi}{2} - \alpha$ with the x-axis and has the same magnitude as $\mathbf{v_{CM}}$. Using the law of cosines, we find

$$|\mathbf{u_m}| = \sqrt{2(1+\sin\alpha)}\,|\mathbf{v_{CM}}| \qquad \Rightarrow \qquad \boxed{u_m = \sqrt{\frac{2M}{m+M}}v_0}\ .$$

Experimental Exam

Problem 1. Diode and paperclip circuit.

Equipment:
Circuit consisting of two identical diodes and a paperclip (the diodes are connected in parallel and the paperclip is in series with one of the diodes), rectifier which can supply either constant voltage or constant current, two multimeters, resistor substitution box (current not to exceed 100 mA), wires, screwdriver, graph paper.

Task 1. Finding the resistance of the paperclip R.
In this part of the problem you will measure the I-V curve of the circuit (without using the substitution box) for both positive and negative (i.e. with reversed polarity) voltages.

Note: Do not exceed a current of 2.5 A.

(a) Sketch the circuit that you have assembled.

(b) Write down the ranges that you use for the multimeters.

(c) Describe how R can be calculated from your measurements.

(d) How will you use the rectifier – to supply a constant voltage or a to supply a constant current?

Note: The characteristics of the diodes have a strong dependence on temperature.

(e) **Quickly** measure the I-V curve of the circuit as the voltage/current is raised. After you have reached the maximum voltage/current, wait until the open diode reaches its equilibrium temperature (be careful not to burn yourself on one of the diodes). Then, **quickly** measure the I-V curve of the circuit as the voltage/current is lowered. Repeat this for voltages of the opposite polarity. Present your results in a table.

(f) Write down whether a diode is open when a positive potential is applied on the terminal with the white band, or vice versa.

(g) Decide on the dataset that you will use for determining R. Choose between the values taken when raising the current/voltage and those taken when lowering the current/voltage.

(h) Plot a graph from which you can find R.

(i) Find R from the graph.

(j) Using the graph, find your error ΔR.

Task 2. Finding the reverse-bias saturation current of the diodes I_S.
The current I_S is the maximum current through a closed diode. The I-V curve of a diode can be modelled by the Shockley diode equation,

$$I = I_S \left(e^{\frac{eU}{nkT}} - 1 \right),$$

where e is the charge of the electron, k is the Boltzmann constant, T is the absolute temperature, and n is a number close to 1.

(a) Find an approximation of the formula above which can be used when measuring the forward I-V curve for voltages on the order of a few hundred mV at room temperature.

(b) Apply a voltage of such polarity that the diode with no paperclip attached to it is open. Measure an appropriate part of the I-V curve for currents under 100 mA. Use the resistor substitution box if necessary. Present your results in a table.

(c) Plot your data in appropriate variables.

(d) Using the plot, find I_S and n.

Call the examiner if you suspect that a multimeter's fuse has blown, or in case of any other technical difficulties.