2019 Bulgarian IPhO Team Selection Test – Solutions

Short Exam 1

Problem. Linear crystal. A large number of balls (atoms) of mass m are positioned on a line at a distance a from each other. Adjacent balls are connected by identical springs of constant k. The balls are indexed with numbers $n = 0, 1, 2, \dots$, where $n = 0$ corresponds to the ball at the left end of the chain (see Figure 1). The leftmost ball oscillates longitudinally with an amplitude A ($A \ll a$), as given by

$$x_0 = A \sin(\omega t),$$

where x_0 is the displacement from its equilibrium position. As a result, a longitudinal wave of wavelength λ propagates to the right. In what follows, you may use the magnitude of the wavevector $q = \frac{2\pi}{\lambda}$ instead of the wavelength λ.

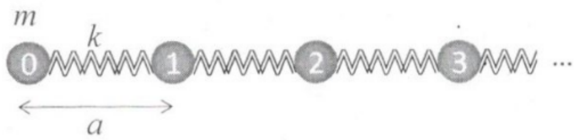


Figure 1

(a) Find an expression for the displacement x_n of the n-th ball ($n > 0$) as a function of time.

(b) By considering the motion of the n-th ball ($n > 0$), find the dependence $\omega(\lambda)$ (or $\omega(q)$) of the wave's angular frequency on the wavelength λ (or the wavevector q).

(c) Acoustic waves have a wavelength much larger than the atomic distance, i.e. $\lambda \gg a$. The speed of sound in a crystal is essentially independent of the sound's frequency. Find an expression for the speed of the acoustic waves in the chain in terms of k, m, and a.

(d) The power of an acoustic wave is defined as the mean energy carried by the wave per unit time. Find an expression for the power of an acoustic wave P in terms of its angular frequency ω, its amplitude A, as well as the parameters k, m, and a.

Hint: You may use that

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Solution. (a) We are told that the motion of the balls conforms to a wavelike solution. We will proceed to guess an equation for this wave, and later we will find its exact parameters in order for it to actually work. We expect the wave to be harmonic (sinusoidal), because that is how the first ball moves. Mathematically, any wave propagating to the right is represented by a function where the time dependence of the oscillating variable $x(t)$ comes grouped together with the position r, or $x = f(vt - r)$, where after a quick think you can see that v may be interpreted as the velocity of propagation. In the case of harmonic waves, the exact form of the function is

$$x(t) = A \sin(\omega t - qr),$$

where A is the amplitude, i.e. the maximum value of x. Note that $v = \omega/q$, but we write $(\omega t - qr)$ rather than $q(vt - r)$ simply because the quantities ω and q have neat interpretations.

If T is the period of the wave and λ is its wavelength, then $\omega = 2\pi/T$ and $q = 2\pi/\lambda$. This is evident from the definitions of a period or a wavelength. One period is the time after which the pattern of the wave repeats at a fixed position r. So, $x(t)$ should be the same whether you sit at t or $t+T$. This implies $\omega T = 2\pi$. Similarly, one wavelength is the distance after which the pattern repeats if you freeze time. Hence, $x(r)$ is the same at r and $r+\lambda$, which yields $k\lambda = 2\pi$.

But let's return to our problem. We intentionally chose the form $A\sin(\omega t - qr)$ rather than something like $A\sin(qr-\omega t)$ or $A\cos(\omega t - qr)$ because we wanted to match $x_0 = A\sin(\omega t)$ at $r=0$. For ball n we have $r = na$, so

$$\boxed{x_n = A\sin(\omega t - qna).}$$

However, we still need to find the dependence $\omega(q)$, which will then tell us how the wave velocity depends on the wavevector q.

(b) Every ball has a spring to the right, stretched by $x_{n+1} - x_n$, and a spring to the left, stretched by $x_n - x_{n-1}$. In total,

$$m\ddot{x}_n = k(x_{n+1} - x_n) - k(x_n - x_{n-1}),$$

Now we plug in our guess for x_n. It will satisfy the force equation only for a specific form of ω:

$$-\left(\frac{m\omega^2}{k}\right)\sin(\omega t - qna) = \sin(\omega t - qna + qa) + \sin(\omega t - qna - qa) - 2\sin(\omega t - qna),$$

$$-\left(\frac{m\omega^2}{k}\right)\sin(\omega t - qna) = 2\sin(\omega t - qna)\cos(qa) - 2\sin(\omega t - qna),$$

$$\frac{m\omega^2}{k} = 4\left(\frac{1-\cos(qa)}{2}\right) \quad\Rightarrow\quad \boxed{\omega(q) = 2\sqrt{\frac{k}{m}}\sin\left(\frac{qa}{2}\right).}$$

We can also write this in terms of the wavelength:

$$\boxed{\omega(\lambda) = 2\sqrt{\frac{k}{m}}\sin\left(\frac{\pi a}{\lambda}\right).}$$

(c) At the limit $\lambda \gg a$ we can use $\sin x \approx x$, and then

$$\omega(q) = \left(a\sqrt{\frac{k}{m}}\right) q.$$

Therefore the acoustic waves have speed

$$\boxed{v = a\sqrt{\frac{k}{m}}.}$$

Indeed, this doesn't depend on ω or q.

(d) We'll solve this in two ways. The first approach is to find the energy stored in the system within a length a. We need to consider one ball and one spring. The ball has an average kinetic energy

$$\langle E_{\text{kin}}\rangle = \frac{m\langle v_n^2\rangle}{2} = \frac{m(\omega A)^2}{2}\langle\cos^2(\omega t - qna)\rangle = \frac{m\omega^2 A^2}{4},$$

while the spring has an average potential energy

$$\begin{aligned}\langle E_{\text{pot}}\rangle &= \frac{k}{2}\langle (x_{n+1}-x_n)^2\rangle = \frac{kA^2}{2}\langle [\sin(\omega t - qna - qa) - \sin(\omega t - qna)]^2\rangle \\ &= \frac{kA^2}{2}\left\langle \left[2\sin\left(\frac{qa}{2}\right)\cos\left(\omega t - qna - \frac{qa}{2}\right)\right]^2 \right\rangle = \frac{kA^2q^2a^2}{4} = \frac{m\omega^2A^2}{4}.\end{aligned}$$

The two terms are equal (as is usually the case with waves), and they add up to a total of $E = \frac{m\omega^2A^2}{2}$. Now note that the pulse we send along the system covers a length a in time $t = \frac{a}{v}$. The power is then

$$P = \frac{E}{t} = \boxed{\frac{1}{2}\omega^2A^2\sqrt{km}.}$$

The second approach is more tricky. The point here is that the leftmost ball cannot oscillate like this unless it is supported by some external force F. This force is associated with a power input Fv_0, and this nowhere to go except towards the energy of the propagating wave. Thus $P = \langle Fv_0\rangle$. To find F, we write

$$m\ddot{x}_0 = F + k(x_1 - x_0).$$

After a bit of algebra, we get

$$F = -m\omega^2A\sin(\omega t) + 2kA\sin\left(\frac{qa}{2}\right)\cos\left(\omega t - \frac{qa}{2}\right).$$

We need to multiply this with

$$v_0 = \omega A\cos(\omega t)$$

and then take the time average. The first term in the expression for F promptly goes away, while the second needs extra massaging:

$$P = \left\langle 2k\omega A^2\sin\left(\frac{qa}{2}\right)\cdot\frac{1}{2}\left(\cos\left(2\omega t - \frac{qa}{2}\right) + \cos\left(\frac{qa}{2}\right)\right)\right\rangle.$$

The term $\frac{qa}{2}$ is small, so our expression simplifies to

$$P = \frac{1}{2}k\omega A^2qa = \boxed{\frac{1}{2}\omega^2A^2\sqrt{km}.}$$

Short Exam 2

Problem. Conducting sphere. The centre of a neutral conducting sphere is collinear with two point charges q and $-q$. The charges and the sphere are in vacuum. The distance between the sphere and the charges is l, the radius of the sphere is r, and the distance between the charges is d, such that $l \gg r$ and $l \gg d$. Find a formula for the force F on the sphere due to the charges.

Solution. This image charge problem is exactly the same as Example 4 in Kevin Zhou's Handout E2, so I'll be quite brief in sketching a solution. We'll look for the total force on the charges due to the sphere, which is equivalent by Newton's third law. The induced charges of the sphere must arrange themselves in such a way so as to ensure a constant potential on the sphere. To compensate for a single charge q that is l away from the centre of the sphere, the induced charges must form a net field like that of a point charge $-qr/l$ located at a distance r^2/l from the centre of the sphere along the axis of the outside charges. You can verify that this gives you zero potential everywhere on the sphere. But we're not quite there yet. A surface integral of the electric field right above the sphere has to equal zero because the conductor is neutral. And one equivalent image charge corresponding to the induced charges is not enough to satisfy this condition.

But if we also imagine a charge $+qr/l$ at the centre of the sphere, this issue is resolved, and the potential on the sphere is just shifted by a constant, which is allowed. So, the response of the sphere to the pair of outside charges can be represented by four image charges: two for $+q$ at distance l and two for $-q$ at distance $l+d$ (without loss of generality). At the centre of the sphere there is net charge

$$\frac{qr}{l} - \frac{qr}{l+d} = \frac{qrd}{l^2} = \frac{pr}{l^2},$$

where p is the dipole moment of the outside pair. Apart from that, we have $-qr/l$ at r^2/l and

$$+\frac{qr}{l+d} \approx +\frac{qr}{l}\left(1 - \frac{d}{l}\right) \quad \text{at} \quad \frac{r^2}{l+d} \approx \frac{r^2}{l}\left(1 - \frac{d}{l}\right).$$

Let's split up this charge into $+qr/l$ and $-qdr/l^2$. The first piece is a distance r^2d/l^2 from the charge $-qr/l$. Together they form a dipole of magnitude pr^3/l^3 which is collinear with the outside dipole. As for the remaining piece $-pr/l^2$, it will form a dipole together with the net central charge. Its magnitude is also pr^3/l^3, and it is also collinear with the outside dipole.

We've thus found that the charges outside (a dipole p) will experience the field of a net induced dipole with magnitude $p' = 2pr^3/l^3$. The two dipoles are aligned, so we know to expect an attractive force. On the z-axis pointing along the dipoles, the field of the induced dipole is $E = -2kp'/z^3$. The outside dipole then experiences a force

$$F = p\,\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{6kpp'}{z^4} = \frac{12kp^2r^3}{l^3z^4} = \boxed{\frac{3q^2d^2r^3}{\pi\varepsilon_0 l^7}.}$$

Here we introduced z as a temporary variable that is independent of l. We needed this because the formula for the force from a dipole comes only from considering the variation of its field in space. We're not really moving anything around, so the magnitude of the induced dipole should stay constant.

Short Exam 3

Problem. Surface gravity waves. A monochromatic surface gravity wave propagates along a channel of depth H and width $l \gg H$. The wavelength of the wave greatly exceeds the width of the channel. For such waves the relation between the angular frequency of the wave ω and the wavenumber $k = \frac{2\pi}{\lambda}$ is $\omega = uk$, where u is the wave velocity. Here u is independent of k.

(a) Propose a form for the dependence of the wavenumber k on depth H for such waves.

(b) Consider the reflection and the transmission of such a wave at a point where the channel suddenly changes depth from H_1 to $H_2 = 4H_1$. Compare the amplitudes of the reflected wave B and the transmitted wave C with that of the incident wave A.

(c) Find the reflection coefficient $R = \left(\frac{B}{A}\right)^2$ and the transmission coefficient $T = \frac{k_2}{k_1}\left(\frac{C}{A}\right)^2$. Verify that $R + T = 1$.

Solution. (a) Let's take a peek at the next subpart. Any monochromatic wave can be decomposed into sinusoidal building blocks, so it suffices to check what will happen to a sinusoidal wave incident on a boundary at $x = 0$. Let the incident wave be described by

$y_I = A\sin(\omega t - kx)$. After striking the boundary, it splits up into a reflected part $y_R = B\sin(\omega_1 t + k_1 x)$ and a transmitted part $y_T = C\sin(\omega_2 t - k_2 x)$. Here the plus sign corresponds to reverse propagation. Now, at $x = 0$ we need to have $y_I = y_R + y_T$ at all times. This can happen only when $\omega = \omega_1 = \omega_2$. So, it seems that ω must stay constant.

Consider the equation $\omega = uk$ from the problem statement. Here, u can depend only on some length scale L and the acceleration due to gravity g (this is because gravity acts as a restoring force for water waves; see also the pointer in the title of the problem). Using dimensional analysis, we find that $u \sim \sqrt{gL}$. Now we need to think about what L might be. It turns out that it's just the depth H. All other lengths are way larger, so they cannot influence the small-scale behaviour of the wave. In contrast, for deep water ($H \gg \lambda$) we'd have $u \sim \sqrt{g\lambda}$. The quantities u and k yield a constant when multiplied, and $u \propto \sqrt{H}$. We conclude that

$$\boxed{k \propto 1/\sqrt{H}.}$$

(b) We now know that $k_1 = k$ and $k_2 = k\sqrt{H_1/H_2}$. But we still need B and C. Setting $y_I = y_R + y_T$ at $x = 0$ gives us $A = B + C$. Since the transition at the boundary shouldn't have any kinks, the derivatives at $x = 0$ must also be continuous:

$$-kA = kB - k\sqrt{\frac{H_1}{H_2}}C.$$

Solving the set of equations gives us

$$\boxed{\frac{B}{A} = \frac{\sqrt{H_1} - \sqrt{H_2}}{\sqrt{H_1} + \sqrt{H_2}} = -\frac{1}{3},} \qquad \boxed{\frac{C}{A} = \frac{2\sqrt{H_2}}{\sqrt{H_1} + \sqrt{H_2}} = \frac{4}{3}.}$$

(c) We compute

$$\boxed{R = \left(\frac{\sqrt{H_1} - \sqrt{H_2}}{\sqrt{H_1} + \sqrt{H_2}}\right)^2 = \frac{1}{9},} \qquad \boxed{T = \frac{4\sqrt{H_1 H_2}}{(\sqrt{H_1} + \sqrt{H_2})^2} = \frac{8}{9}.}$$

These expressions evidently add up to 1.

Theoretical Exam

Problem 1. Unwinding a string. A string of length l is wound around a uniform cylinder of mass m and radius R, where $l \gg R$. The cylinder is initially at rest on a horizontal plane along which it can only roll without slipping. The free end of the string A is at the top of the cylinder. A constant horizonal force F is applied at the free end of the string, and the cylinder starts rolling (Figure 2).

(a) Find the acceleration of the centre of the cylinder a.

(b) Find the velocity of the centre of the cylinder v when the string is fully unwound.

Solution. (a) There are two relevant forces, the pull F and a friction force f, which we'll assume points backwards. Take the system consisting of the cylinder and the string that remains wound. For this system, we can write $F - f = ma$. Looking at the torques about the centre of mass, one finds $(F + f)R = (1/2)mR^2\varepsilon$. There's rolling without slipping, so $a = \varepsilon R$. We solve the equations and find $\boxed{a = 4F/3m.}$

(b) When the string is fully unwound, it covers a length l on the ground. So we're really asking what the centre of mass velocity of the cylinder is after it has covered a distance l. Since $v = a\tau$ and $l = a\tau^2/2$, we get $v = \sqrt{2al}$, or $\boxed{v = \sqrt{8Fl/3m}.}$

Problem 2. Spiral motion. A point mass m moves under a central force. It begins its motion at a distance r_0 from the centre of force. For some initial velocity of magnitude v_0 the particle moves along a spiral trajectory where the velocity vector maintains a constant angle θ with the radius vector (Figure 3).

(a) Find the equation of the trajectory in polar coordinates, i.e. find the dependence $r(\varphi)$ of the distance to the centre r on the angle of rotation of the radius vector φ. The equation may include r_0 and θ.

(b) Find an expression (up to a constant) for the potential energy W of the particle in terms of the distance to the centre r.

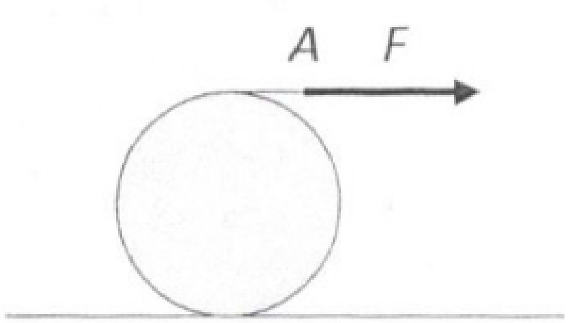


Figure 2

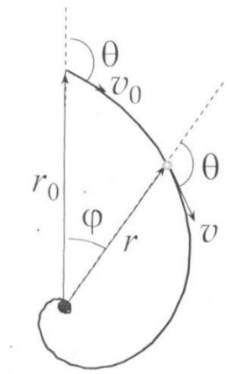


Figure 3

Solution. (a) This is a maths problem. We know that the tangential displacement and the radial displacement always maintain the same ratio, such that

$$\tan(\pi - \theta) = -r\frac{\mathrm{d}\varphi}{\mathrm{d}r}.$$

After integrating, we find

$$\boxed{r = r_0 e^{(\varphi/\tan\theta)}.}$$

Note that when $\tan\theta$ is negative, the distance to the centre will tend to zero. Otherwise, the distance will tend to infinity. The trajectory is a logarithmic spiral.

(b) The trick here is that the force is central, so we can conserve angular momentum. Since θ is constant, the ratio of the tangential velocities at any two points is the same as the ratio of the total velocities. And so $vr = v_0 r_0$. Applying energy conservation would give us

$$\frac{mv^2}{2} + W(r) = \frac{mv_0^2}{2},$$

where we chose the potential energy to equal zero at r_0. Thus

$$\boxed{W(r) = \frac{mv_0^2}{2}\left(1 - \left(\frac{r_0}{r}\right)^2\right).}$$

Problem 3. Hodograph. Let us draw the velocity vector $\mathbf{V}$ of a point mass on a diagram with axes V_x and V_y which correspond to the components of the velocity. If the velocity of the point mass varies, the end of the vector $\mathbf{V}$ will trace a curve called a hodograph (Figure 4). The hodograph can be interpreted as the trajectory of the vector $\mathbf{V}$ in velocity space.

(a) Draw the hodograph for a body launched at an angle α to the horizon with an initial velocity V_0. Mark the points which correspond to the launch and to the landing. Mark the point that corresponds to maximum height.

(b) Consider a simple string pendulum of length l with no friction at the pivot. The pendulum is initially at an angle of 90° to the vertical (Figure 5). The pendulum is then let go from rest. Draw the hodograph for the velocity of the bob *qualitatively*. Mark the points which correspond to maximum angular displacement. Mark the points which correspond to the pendulum's equilibrium position. Calculate the coordinates of the extrema of the hodograph. Annotate your diagram with your results.

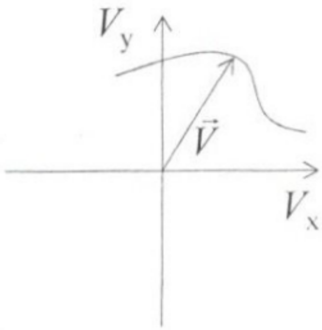


Figure 4

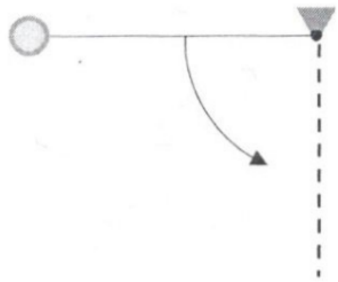


Figure 5

Solution. (a) The horizontal velocity stays fixed at $V_x = V_0 \cos\alpha$, while the vertical velocity starts off as $V_y = V_0 \sin\alpha$, turns to zero at the moment of maximum height, and goes down to $V_y = V_0 \sin\alpha$ at touchdown. The hodograph is then a straight line, see below.

(b) Let the bob start its motion at an angular displacement $\alpha = -90°$. The velocity of the bob can be found from energy conservation:

$$\frac{mV^2}{2} = mgl\cos\alpha \quad \Rightarrow \quad V \equiv \sqrt{2gl\cos\alpha} = V_{\max}\sqrt{\cos\alpha}.$$

Its components are

$$V_x = V_{\max}\sqrt{\cos\alpha}\cos\alpha, \qquad V_y = V_{\max}\sqrt{\cos\alpha}\sin\alpha.$$

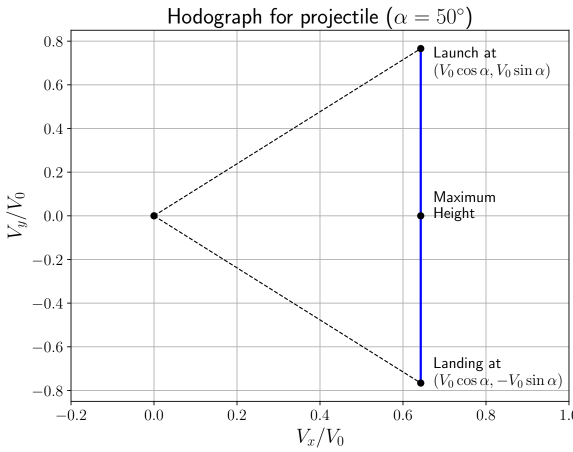
We want an expression for the dependence of V_y on V_x. First, we see that $\cos\alpha = \frac{V_x^2+V_y^2}{V_{\max}^2}$. After substituting this in the equation for V_x, we reach

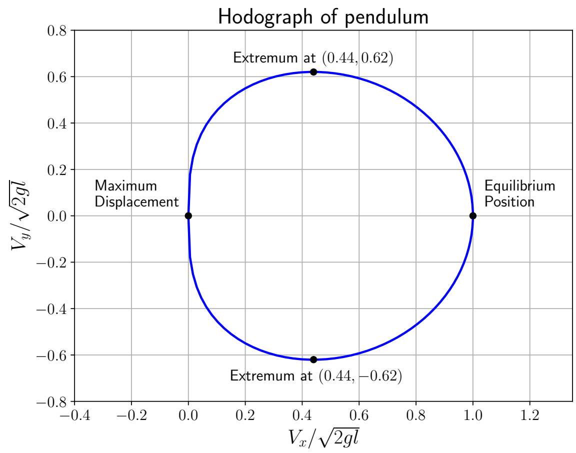
$$\frac{(V_x^2+V_y^2)^3}{V_x^2} = V_{\max}^4.$$

We differentiate this to find

$$\frac{\mathrm{d}V_y}{\mathrm{d}V_x} = \frac{(2/3)V_{\max}^{4/3}V_x^{-1/3} - 2V_x}{2V_y}.$$

From this expression we see that there's a vertical asymptote at the points of maximum displacement $(V_x, V_y) = (0,0)$. To find the extrema, we set the derivative to zero, which yields $V_x = 3^{-3/4}V_{\max} \approx \boxed{0.44V_{\max}}$. The respective values for the y-velocity are $V_y = \pm(2^{1/2}\,3^{-3/4})V_{\max}$, or $V_y \approx \boxed{\pm 0.62V_{\max}}$. The hodograph is shown below.





Problem 4. The Wien constant. For a black body of temperature T, the spectral radiance in frequency $r(\nu, T)$ (the energy emitted in a unit frequency interval per unit emitter area per target solid angle per unit time, $\frac{\mathrm{W}}{\mathrm{Hz}\cdot\mathrm{m}^2\cdot\mathrm{sr}}$) is given by

$$r(\nu, T) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1}.$$

(a) Derive a formula for the spectral radiance in wavelength $r'(\lambda, T)$ (the energy emitted in a unit wavelength interval per unit emitter area per target solid angle per unit time, $\frac{\mathrm{W}}{\mathrm{m}^3\cdot\mathrm{sr}}$).

(b) Express the Wien constant b (from Wien's displacement law $\lambda_{\max} T = b$) in terms of the Planck constant h, the Boltzmann constant k, the speed of light c, and some unknown number A.

(c) Calculate the number A, rounded to five significant figures. Calculate the Wien constant, rounded to five significant figures.

Solution. (a) We are given the formula for $r(\nu, T) = \frac{\mathrm{d}E}{\mathrm{d}\nu\,\mathrm{d}S\,\mathrm{d}\Omega\,\mathrm{d}t}$, and we want a formula for $r'(\lambda, T) = \frac{\mathrm{d}E}{\mathrm{d}\lambda\,\mathrm{d}S\,\mathrm{d}\Omega\,\mathrm{d}t}$. We see that

$$r' = r\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| \quad \overset{c=\lambda\nu}{\Longrightarrow} \quad r' = r\frac{c}{\lambda^2}.$$

All that remains is to write r in terms of λ rather than ν:

$$\boxed{r'(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}.}$$

(b) Wien's law relates the wavelength of maximum spectral radiance to the temperature by $\lambda_{\max} = b/T$. This $\lambda_{\max}$ corresponds to an extremum of r'. After differentiating and setting the derivative to zero, we find

$$-5 + \left(\frac{hc}{\lambda_{\max} kT}\right) \frac{e^{hc/\lambda_{\max} kT}}{e^{hc/\lambda_{\max} kT} - 1} = 0.$$

We can use this equation to find the dimensionless combination $hc/\lambda_{\max} kT = A$. This matches Wien's law, with $\boxed{b = hc/kA.}$

(c) To find A, we need to solve $(5 - A)e^A = 5$. After lots of trial and error, the solutions are $A = 0$ (which is useless) and $\boxed{A = 4.9651}$. For better accuracy, we'll use 4.96511 when calculating b. Our result is $\boxed{b = 2.8978 \times 10^{-3}\,\mathrm{m}\cdot\mathrm{K}.}$

Problem 5. Prism. The cross section of a prism with refractive index n is an isosceles trapezium. The angle between its larger base and its legs is θ, its height is h, and the length of the larger base is s. Consider a ray of light (lying in the plane of the cross-section) which is incident on one of the legs of the prism, at an angle α with respect to its normal. The ray is reflected by one of the bases and leaves the prism at the other leg. The prism is surrounded by air of refractive index unity.

(a) Derive a formula for the deviation angle φ of the ray due to the prism (that is, the angle between the incident ray and the outgoing ray).

(b) Assume that the incident ray is parallel to the bases. The ray strikes the prism at a distance x above the larger base. There exist ratios s/h for which the outgoing ray is also parallel to the bases for all $x \in (0, h)$.Find a formula for the smallest such ratio.

(c) Calculate the ratio s/h (as a decimal) for $\theta = 60.0°$ and $n = 1.500$.

(d) Consider a prism with this ratio. You are looking at the Cyrillic letter Б through the legs. Draw what you will see.

Solution. (a) Assuming that the ray indeed reflects at the base of the prism, it will exit the prism at the same angle α with the respect to the normal of the other leg. We'll use the notation on the diagram. Extend the incoming and the outgoing rays and let them meet a point Q. We're interested in the quadrilateral $MNPQ$, specifically in the angle external to $\angle Q$, because that's what corresponds to the deviation φ. We have $\angle M = \angle P = \alpha - \beta$ and $\angle N = 2(\theta + \beta)$, so $\boxed{\varphi = 2(\alpha + \theta) - \pi.}$

(b) Horizontal rays approach at $\alpha = \pi/2 - \theta$, so in principle $\varphi = 0$. However, it's not guaranteed that the incoming rays will hit the lower base in the first place – they might just land at the other leg. The worst offender is the ray incident at D, so the lowest possible ratio $r \equiv s/h$ makes it so that even that ray gets to land on the lower base, at point B. The angle of refraction β' can be found from $n \sin \beta' = \cos \theta$. From the diagram, we see that in the limiting case the deflection of the incident ray $\pi/2 - \theta - \beta'$ corresponds to one of the angles of $\triangle DHB$. We can then write

$$\tan\left(\frac{\pi}{2} - \theta - \beta'\right) = \frac{DH}{HB} = \frac{DH}{AB - AH} = \frac{h}{s - (h/\tan\theta)} = \frac{1}{r - (1/\tan\theta)}.$$

Then we have

$$r - \frac{1}{\tan\theta} = \tan(\theta + \beta') = \frac{\tan\theta + \tan\beta'}{1 - \tan\theta\tan\beta'},$$

$$r = \frac{1 + \tan^2\theta}{\tan\theta(1 - \tan\theta\tan\beta')} = \frac{1}{\sin\theta\cos\theta(1 - \tan\theta\tan\beta')}.$$

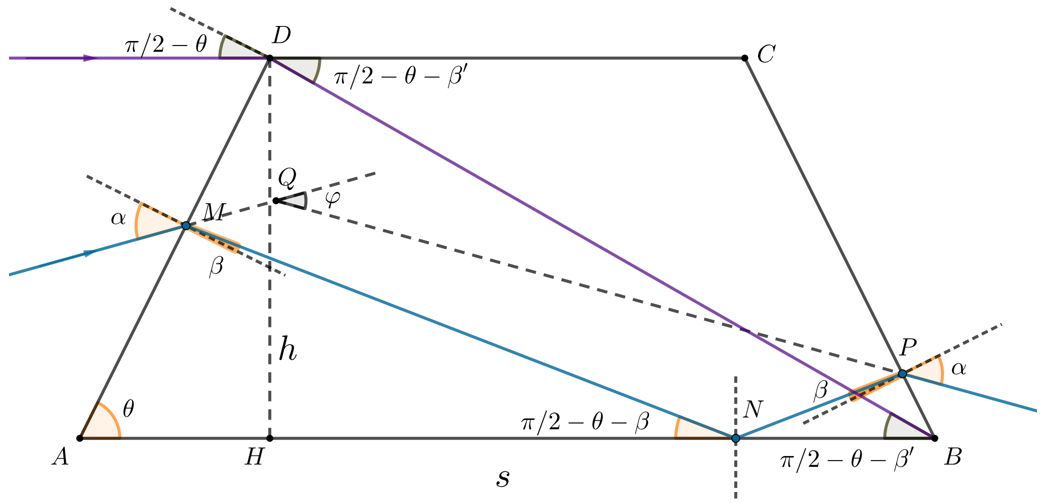
It turns out that

$$\boxed{r = \frac{\sqrt{n^2 - \cos^2\theta}}{\sin\theta\cos\theta\left(\sqrt{n^2 - \cos^2\theta} - \sin\theta\right)}.}$$

We also need to be sure that a ray which hits the lower base doesn't ever get to hit the upper base. Actually, we're in the clear here, because any ray reflected from the lower base will make an angle $\pi/2 - \theta - \beta'$ with AB. In the worst case scenario, that reflected ray emanates from point A. But we had just arranged things so that such a ray would hit the other leg at point C, or lower!

(c) The answer is $\boxed{s/h = 5.958.}$

(d) Rays which are incident at lower x end up leaving the prism at higher x. So, the letter will look like it's been reflected about a horizontal line: $\boxed{\text{P}}$



Problem 6. Magnet. Assume the simplest possible model for the Earth's magnetic field: the source is located at the centre of the Earth, and its size is negligible compared to the radius of the Earth.

(a) Find the ratio of the magnetic fields at the magnetic poles and the magnetic equator.

(b) Find the angle which the magnetic field in Sofia (latitude 42.7°) makes with the horizon.

Solution. (a) We're basically told to assume a dipole field, so it's enough to know the formula for the magnetic field a dipole,

$$\mathbf{B}(r,\theta) = \frac{\mu_0 p}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right),$$

where the angle θ is measured from the direction of the dipole moment. The formula is easier to make sense of when you split the dipole moment $\mathbf{p}$ into $\mathbf{p}_\parallel + \mathbf{p}_\perp$, where $\mathbf{p}_\parallel$ points towards the observer and $\mathbf{p}_\perp$ is perpendicular to them. This means that there's a contribution $p_\parallel = p\cos\theta$ which creates a field $\frac{\mu_0 p_\parallel}{2\pi r^3}$ purely along $\hat{\mathbf{r}}$ and a contribution $p_\perp = p\sin\theta$ which gives rise to a field $\frac{\mu_0 p_\perp}{4\pi r^3}$ purely along $\hat{\boldsymbol{\theta}}$. In our problem, the poles are at $\theta = \{0,\pi\}$ and the equator is at $\theta = \pi/2$. Immediately we see that the ratio of the magnetic fields equals $\boxed{2.}$

(b) We'll have to assume that the north magnetic pole coincides with the north geographic pole. But be careful! The dipole will point from the north geographic pole to the south geographic pole, so the magnetic field in Sofia will point into the ground. The angle α at which it does this is determined by $\tan\alpha = |B_r/B_\theta|$ (draw a picture here if you're not feeling convinced). We find $\tan\alpha\tan\theta = 2$, or $\boxed{\alpha = 65.2°.}$

Problem 7. Photocathode. Compare the maximum velocities $v_{1,\max}$ and $v_{2,\max}$ of the photoelectrons for silver (work function $A = 4.7\,\text{eV}$) in the following cases:

1. The photocathode is irradiated with ultraviolet light of wavelength $\lambda_1 = 155\,\text{nm}$.
2. The photocathode is irradiated with gamma rays of wavelength $\lambda_2 = 2.47\times10^{-3}\,\text{nm}$.

Solution. In the two cases, the energy of the photons $\varepsilon = hc/\lambda$ is respectively $8.0\,\text{eV}$ and $0.50\,\text{MeV}$. We see that the first case is classical, while the second case is relativistic, with the work function completely swamped by the photon energy. For the UV rays, we have

$$\varepsilon_1 - A = \frac{mv_{1,\max}^2}{2} \quad\Rightarrow\quad v_{1,\max} = 1.1\times10^6\,\text{m/s}.$$

This is indeed much less than c. As for the gamma rays, we can safely neglect A when incrementing the electron rest energy by the photon energy ε_2:

$$mc^2 + \varepsilon_2 = \frac{mc^2}{\sqrt{1 - v_{2,\max}^2/c^2}} \quad \Rightarrow \quad v_{2,\max} = c\frac{\sqrt{\varepsilon_2(2mc^2 + \varepsilon_2)}}{mc^2 + \varepsilon_2} = 0.86c = 2.59 \times 10^8\,\text{m/s}.$$

The ratio of the velocities is $v_{2,\max}/v_{1,\max} = \boxed{230.}$

Problem 8. Hydrogen. A hydrogen atom at rest emits a photon in a transition from the first excited state to the ground state. What is the difference (in percent) between the energy of the emitted photon and the energy of the transition?

Solution. The energy of the transition ΔE is defined as the magnitude of the change in the rest energy associated with the atom. We will write the initial rest energy as mc^2 and the final rest energy as $m'c^2$. Then ΔE is just $(m - m')c^2$.

The emission of a photon obeys both energy conservation and momentum conservation, which means that the atom will acquire some kinetic energy. So, the change of the atom's total energy (which is the energy of the photon) is a bit more subtle. The conservation laws are

$$E_{\text{ph}} + \frac{m'c^2}{\sqrt{1 - v^2/c^2}} = mc^2, \qquad \frac{E_{\text{ph}}}{c} = \frac{mv}{\sqrt{1 - v^2/c^2}}.$$

At this point we'll note that the ionisation energy of hydrogen is $E_{\text{ion}} = 13.6\,\text{eV}$, meaning that the energy of the transition is

$$\Delta E = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) E_{\text{ion}} = 10.2\,\text{eV}.$$

This is tiny compared to the rest energy of the proton $mc^2 = 938\,\text{MeV}$, so we're comfortably outside the relativistic regime, and we're free to use $v/c \ll 1$ and $m' \approx m$. Let's get rid of the c factors:

$$E_{\text{ph}} + \frac{m'}{\sqrt{1 - v^2}} = m, \qquad E_{\text{ph}} = \frac{mv}{\sqrt{1 - v^2}}.$$

We'll simplify, retaining factors of v only up to order v^2:

$$E_{\text{ph}} + m'\left(1 + \frac{v^2}{2}\right) = m, \qquad E_{\text{ph}} = mv.$$

This brings us to

$$\frac{E_{\text{ph}} - \Delta E}{E_{\text{ph}}} = -E_{\text{ph}}\left(\frac{m'}{2m^2}\right).$$

After some more approximations, we bring back the c's and we get

$$q \equiv \frac{E_{\text{ph}} - \Delta E}{\Delta E} = -\frac{\Delta E}{2mc^2} = \boxed{-10^{-6}\,\%.}$$

This process is covered in more detail in USAPhO 1997 A4.

Problem 9. Accelerator. A particle accelerator works with particles of rest mass m_0. For what kinetic energies should the accelerator be designed if it is to be used for probing structures of size l? Make numerical estimates for electrons and protons, with $l = 10^{-15}\,\text{m}$ (the length scale of atomic nuclei).

Solution. This is Problem 2.26 from MIPT, Volume 3. We need to impose the condition that the de Broglie wavelength of the incoming particles is less than l. One relevant example is electron microscopes. We prefer them to optical microscopes precisely because the lower wavelength of electrons compared to optical photons allows for better resolution. Now, working with $c = 1$,

$$\frac{h}{p} = l \quad \Rightarrow \quad \frac{h}{\sqrt{E^2 - m^2}} = l \quad \Rightarrow \quad E = m\sqrt{1 + \left(\frac{h}{ml}\right)}.$$

After restoring the c's and subtracting mc^2, we see that the kinetic energies should exceed

$$\boxed{T = mc^2\left(\sqrt{1 + \left(\frac{h}{mcl}\right)^2} - 1\right).}$$

For electrons, we calculate $\boxed{T_e = 1200\,\text{MeV}}$, while for protons we find $\boxed{T_p = 620\,\text{MeV}}$.

Problem 10. Helium.

(a) A vessel is filled with helium at temperature $T = 300\,\text{K}$. A hole of size $S = 1\,\text{mm}^2$ is cut in the vessel. What should the pressure in the vessel be so that the gas flows out from the hole at a rate of $\omega = 1\,\text{g/h}$? Assume that the vessel is in vacuum and that the pressure inside does not change significantly.

(b) A disc of radius $r = 1\,\text{cm}$ moves along its axis in a medium filled with helium at temperature $T = 300\,\text{K}$ and pressure $p = 1\,\text{kPa}$. Its velocity is $u = 10\,\text{m/s}$. Find the drag force acting on the disc.

Solution. (a) This is a repeat of Problem 3 from the 2001 Bulgarian National Round, but I will translate the original solution for our non-Bulgarian readers. We observe that the value of ω is very small, so the gas must be rarefied, meaning it cannot be treated as a continuous fluid. The molecules will escape from the hole without colliding with each other very much, or stating it more formally, the mean free path of the molecules greatly exceeds the size of the hole. So, we need to work with kinetic theory.

Let's assume that all particles move with the RMS speed $v = \sqrt{3RT/\mu}$ in one of six directions $(\pm x, \pm y, \pm z)$, such that one of those is directed right into the hole. This implies that $1/6$ of all molecules will approach the hole frontally with velocity v. The particles which leave through the hole within time $\mathrm{d}t$ are all contained in a cylinder with volume $(v\,\mathrm{d}t)S$. The number density of the particles n can be found from the pressure p through $n = p/k_BT$. And so, the mass loss within $\mathrm{d}t$ is

$$\mathrm{d}m = \frac{1}{6}\left(\frac{\mu}{N_A}\right)\left(\frac{p}{k_BT}\right) vS\,\mathrm{d}t.$$

We know that $\omega = \frac{\mathrm{d}m}{\mathrm{d}t} = 1\,\text{g/h}$. The pressure is then

$$\boxed{p = \frac{2\omega}{S}\sqrt{\frac{3RT}{\mu}} = 760\,\text{Pa}.}$$

(b) We'll make use of the fact that the disc is much slower than the gas molecules. As with the previous subpart, we'll let all molecules move with $v = \sqrt{3RT/\mu}$ along six directions, and we'll assume that a sixth of the molecules can collide with the disc at the front, and a sixth can collide at the back.

Working in the rest frame of the disc, when a molecule collides elastically at the front, it approaches with a velocity $v+u$ and bounces off with velocity $v+u$ in the opposite direction. Going back to the lab frame, these molecules will change their speed from v to $v+2u$. This requires a change in the kinetic energy

$$\Delta\varepsilon = \frac{1}{2}m\left((v+2u)^2 - v^2\right) = 2mu^2.$$

This increment has to come from the kinetic energy of the disc, so in a time interval $\mathrm{d}t$ the collisions at the front of the disc cause a loss of

$$\mathrm{d}E_{\text{front}} = \frac{1}{6}\left(\frac{p}{k_B T}\right) v(\pi r^2)\Delta\varepsilon\,\mathrm{d}t.$$

You can do a similar calculation for the collisions at the back of the disk, and it'd give you the same result. This makes for a total rate of energy loss

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{3}\left(\frac{p}{k_B T}\right) v(\pi r^2)\Delta\varepsilon.$$

This can be interpreted as the result of the negative work done by a drag force, $\frac{\mathrm{d}E}{\mathrm{d}t} = Fu$. After some rearranging, we obtain

$$\boxed{F = 2pu\sqrt{\frac{\mu}{3RT}}\pi r^2 = 4.6\,\mathrm{mN}.}$$

Experimental Exam

Problem 1. Separation of magnets.

Equipment:

1. Two identical neodymium magnets of size 15 mm × 10 mm × 5 mm and mass 5.6 g. They are magnetised parallel to their shortest edge.
2. Two identical plastic rails with an L-shaped cross section. The length of the rails is $L = 1.000$ m and their thickness is $d = 3.0$ mm.
3. Spring scale with a range of 5 N and an adjustable zero. The scale can measure both tension and compression forces.
4. Plastic tape measure, accurate to 1 mm.
5. Stopwatch.
6. Ball of plasticine.
7. Two sheets of graph paper (you will not be given extra sheets).
8. Blank paper (you can ask for extra sheets).

Figure 6 shows two magnets in the shape of rectangular cuboids with edges a, b, and c. They are magnetised in the direction of edge c, i.e. their poles are on the parallel faces ab. If the magnets are placed with their unlike poles together, for small distances d between the poles the attractive force between the magnets is given by

$$F(d) = F_0 e^{-kd}, \tag{1}$$

where k is a constant which depends on the size of the magnets and F_0 is the so called breakaway force. This is the minimum force necessary to separate the magnets when their poles touch. The principal aim of this problem is to find the breakaway force for two neodymium magnets without a tool that can measure this force directly.

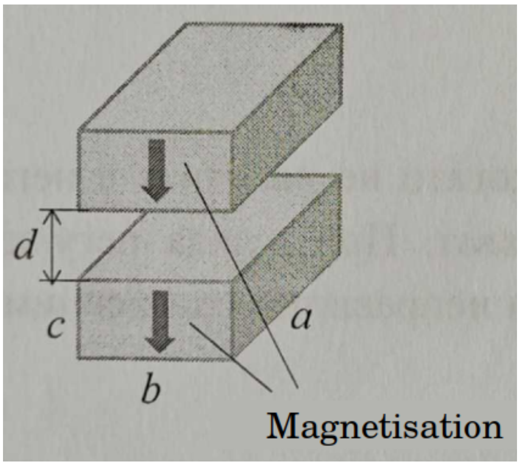


Figure 6

Tasks:

Design an experiment and determine:

(a) The coefficient of friction μ between the magnets and the plastic rail.

(b) The breakaway force F_0 between the two magnets.

Depending on your methods, you may not need some of the equipment. You can find the two parameters in any order. Your mark will depend on the explanation of your methods, the tabular and the graphical presentation of your data, your final values, and their error estimates.

Note: Do not load the spring scale when not working with it. Do not apply forces beyond the range of the spring scale. You may need to adjust the zero of the spring scale. If you damage the spring scale through your fault, you will not be given a spare.

Problem 2. AC circuits.

Equipment:
Unknown resistance R, unknown capacitance C, unknown inductance L (these are inside a box, each between neighouring terminals, respectively 1-2, 2-3, or 3-4, see Figure 7), multimeter (with instructions), alternating voltage source (with instructions), wires, ruler, graph paper.

Record all measurements in tables. Write down your results in the answer sheet.

Figure 7

Task 1. Determining the positions of the components.
Using only the multimeter, find out the positions of the resistor, the capacitor, and the inductor. Record your answers on the answer sheet. **(1.0 pt)**

Task 2. RC circuit measurements.
Set the alternating voltage source (the generator) to sine wave mode. Set the amplitude to maximum using the 'AMPL' potentiometer. Use the 'OUTPUT' terminal. The display will show the frequency $\nu = \frac{1}{T}$ of the output voltage. Use the multimeter's needle probles for faster measurements.

Notes:

1. The multimeter's output voltage is not constant. Its amplitude may vary depending on the load.
2. The multimeter measures alternating voltages accurately only in the range (40 Hz – 400 Hz). However, you can also use it for measuring much higher frequencies. Assume that the measured voltage U_{meas} is related to the true value U by $U_{\text{meas}} = k(\nu)U$, where $k(\nu)$ is some slowly decreasing function of the frequency ν.
3. In addition to the alternating voltage, there may be a constant voltage at the 'OUTPUT' terminal. Adjust the 'DC OFFSET' knob so as to minimise it. You can measure the output with the multimeter in DC voltage mode.

(a) Measure the resistance R. **(0.5 pt)**

(b) Assemble a circuit with R and C in series. Measure the dependence of the voltages U_R (across the resistor) and U_Z (across the RC circuit) on frequency. Work in the range (80 Hz – 800 Hz). Always measure U_R and U_Z in pairs at the same frequencies. Record your results in a table. **(2.0 pt)**

(c) Using appropriate variables, plot a linearised graph from which C can be calculated. The impedance of the RC circuit is $Z = \sqrt{R^2 + \frac{1}{(\omega C)^2}}$, where $\omega = 2\pi\nu$ is the angular frequency. **(2.0 pt)**

(d) Using the graph, find the capacitance C. **(1.0 pt)**

Task 3. RLC circuit measurements.

(a) Assemble a circuit where C and L are in parallel, and this pair is in series with the resistance R. The impedance of such an RLC circuit is $Z = \sqrt{R^2 + \left(\frac{\omega L}{1-\omega^2 LC}\right)^2}$, with $\omega = 2\pi\nu$. This circuit exhibits resonance properties. Study the range (2 kHz – 20 kHz) and determine the resonant frequency ν_{res} with an accuracy of 100 Hz. **(1.5 pt)**

(b) Using the data obtained so far, find the value of L. **(1.0 pt)**

(c) Measure the dependence of the voltages U_R (across the resistor) and U_Z (across the RLC circuit) on frequency. Work in an appropriate range around ν_{res}. Always measure U_R and U_Z in pairs at the same frequencies. Record your results in a table. **(2.0 pt)**

(d) Using appropriate variables, plot a linearised graph. **(2.0 pt)**

(e) Using the graph, recalculate the resonant frequency ν_{res}, the capacitance C, and the inductance L. **(2.0 pt)**

Call the examiner in case of any technical difficulties.