IPhO 1992 Theory Problem 1 (Solution)

The answers for all parts are almost the same as the official solutions, but I have condensed the solutions and provided a more general solution for (c).

a) Write the gravitational acceleration at B as the acceleration at P added with a tidal acceleration.

$$\boldsymbol{g_B} = \boldsymbol{g_P} + \boldsymbol{g_{tidal}}, \quad \text{where } g_{\text{tidal}} \ll g_P \tag{1}$$

Expanding the gravitational acceleration to first order,

$$\boldsymbol{g_P} = \frac{K}{R^2}, \quad \boldsymbol{g_{tidal}} = \frac{K}{R^3}(2x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}) \tag{2}$$

where the unit vector $\hat{\boldsymbol{x}}$ is parallel to $\boldsymbol{R}$, and $\hat{\boldsymbol{y}}$ points in the direction of the displacement perpendicular to $\boldsymbol{R}$.

In the nonrotating comoving frame of P, there is a fictitious acceleration opposite to $\boldsymbol{g_P}$. Subtracting $\boldsymbol{g_P}$ from the gravitational acceleration at B, what remains is the tidal acceleration. Hence, in this frame, there are only 2 forces on B: tension and tidal force.

$$(\boldsymbol{T} + m\boldsymbol{g_{tidal}}) \cdot \hat{\boldsymbol{r}} = -m\omega^2 r \tag{3}$$

Defining $\Omega \equiv \sqrt{K/R^3}$, we see that at the parallel and antiparallel positions,

$$T_{\parallel} = mr(\omega^2 + 2\Omega^2) = 30\,706\,\text{N} \tag{4}$$

and at the perpendicular positions,

$$T_{\perp} = mr(\omega^2 - \Omega^2) = 30\,339\,\text{N} \tag{5}$$

Note: A rotating reference frame should not be used in this question, as it will introduce a Coriolis force due to the motion of the bodies B. Check the original solutions for more discussion on this.

b) It is important to take into account the fact that the frequency of the pulling is dependent on the direction of ω, as the period pulling and pushing is half of the synodic period of the satellite. As mentioned in the official solutions, this was only realised after the competition had ended, which explains why the original problem statement seems to ignore it. It is still taken into account in the official solutions though.

The power is

$$P = \frac{2}{t}\left(T_{\parallel} - T_{\perp}\right)0.01r = \frac{0.02}{t}\left(T_{\parallel} - T_{\perp}\right)r, \quad t = \frac{2\pi}{\omega \pm \Omega} \tag{6}$$

where the positive case is antiparallel and negative case is parallel.

$$\begin{aligned} P &= \frac{0.03}{\pi}(\omega \pm \Omega)mr^2\Omega^2 \\ &= 1\,909\,\text{W (parallel) or } \ 2\,168\,\text{W (antiparallel)} \end{aligned} \tag{7}$$

c) Angular momentum is conserved as gravitational force always acts radially. Energy increases at the rate found in the previous subpart.

Defining ω to be positive when it is parallel to Ω,

$$\begin{aligned} L &= 4m\Omega R^2 + I\omega \\ &= 4m\sqrt{KR} + 4mr^2\omega \\ &= 4m\left(\sqrt{KR} + r^2\omega\right) \end{aligned} \tag{8}$$

Since L is conserved, as R increases, ω decreases.

Energy:

$$\begin{aligned} E &= -\frac{K(4m)}{2R} + \frac{1}{2}I\omega^2 \\ &= -2m\frac{K}{R} + 2mr^2\omega^2 \\ &= 2m\left(-\frac{K}{R} + r^2\omega^2\right) \end{aligned} \tag{9}$$

For R to increase with time, at the initial R,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}t} > 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}E}{\mathrm{d}R} > 0 \tag{10}$$

Expressing ω as a function of R,

$$\frac{\mathrm{d}E}{\mathrm{d}R} \propto \frac{K}{R^2} + 2r^2\omega\frac{\mathrm{d}\omega}{\mathrm{d}R} \tag{11}$$

Using Equation 8,

$$\frac{\mathrm{d}\omega}{\mathrm{d}R} = -\frac{1}{2r^2}\sqrt{\frac{K}{R}} \tag{12}$$

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}R} &\propto \sqrt{\frac{K}{R}}(\Omega - \omega) \\ &\propto \Omega - \omega \end{aligned} \tag{13}$$

So R increases if $\omega < \Omega$. As R increases, ω decreases.

Now, returning to the definition of ω in the question, in the antiparallel case, R has to increase. An increase in R causes an increase in ω since it is now defined in the opposite direction. In the parallel case, we just use the inequality found in Equation 13.

I have shortened the table because some rows are really not necessary to obtain the answer. Their answer for the table will also be different as they are solving for the specific quantities given in the problem, where $\omega > \Omega$. Therefore, they have treated $\omega < \Omega$ as impossible. However, I think it is more insightful to show the actual inequalities for a general ω, where it is possible for $\omega < \Omega$.

In the table, a dash (-) indicates that it is not possible.

<table>
<tr><th>Quantity</th><th>increases if...</th><th>decreases if...</th><th>is unchanged if...</th></tr>
<tr><td>Radius of orbit R</td><td>Parallel: $\omega < \Omega$
Antiparallel: any ω</td><td>Parallel: $\omega > \Omega$
Antiparallel: -</td><td>Parallel: $\omega = \Omega$
Antiparallel: -</td></tr>
<tr><td>Magnitude of angular velocity ω</td><td>Parallel: $\omega > \Omega$
Antiparallel: any ω</td><td>Parallel: $\omega < \Omega$
Antiparallel: -</td><td>Parallel: $\omega = \Omega$
Antiparallel: -</td></tr>
<tr><td colspan="4">Could the satellite reach a higher orbit as a result of work done by the machine?

<u>Yes</u> / No</td></tr>
<tr><td colspan="4">Could the satellite reach an arbitrarily high orbit? Why?

Answer: Yes. In the antiparallel case, ω and R both increase, so energy increases with radius at any orbital radius.</td></tr>
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