NBPhO 2007 – Corrected Solution to Problem 3

(1) We'll set the voltage on the trajectory to zero (except on the segment BC, where the voltage is $-U(t)$ as mentioned in the problem statement). Close to the source, the electrons are quickly accelerated because the voltage rises from $-U_0$ to 0. In that case, we can use conservation of energy, $\frac{mv^2}{2} - eU_0 = \text{const}$, where we've introduced the mass of the electron as m. The velocity of the electrons comes out as $v = \sqrt{2eU_0/m}$. Then, the time required to cross a distance $a+b$ will be

$$\boxed{t_0 = (a+b)\sqrt{\frac{m}{2eU_0}}.}$$

(2) At point B the potential changes by $-U$. Following the same approach, the velocity of the electrons on BC should be

$$v_a = \sqrt{2e(U_0 - U)/m}.$$

The total travel time is $t = (a/v_a) + (b/v)$, hence, using the binomial approximation,

$$t = b\sqrt{\frac{m}{2eU_0}} + a\sqrt{\frac{m}{2e(U_0 - U)}} \approx b\sqrt{\frac{m}{2eU_0}} + a\sqrt{\frac{m}{2eU_0}}\left(1 + \frac{U}{2U_0}\right),$$

$$t = (a+b)\sqrt{\frac{m}{2eU_0}} + \frac{aU}{2U_0}\sqrt{\frac{m}{2eU_0}} = \boxed{t_0 + \frac{aU}{2U_0}\sqrt{\frac{m}{2eU_0}}.}$$

(3) Here we should keep in mind that the jump in the electron velocity at B and C only depends on what the local voltage is at that instant. Now let's consider two electrons, the first one arriving at B at time 0, and the second one arriving a time Δt later. The voltage $U(t)$ should vary such that these two electrons end up at D simultaneously for any value of Δt.

The first electron reaches the target at time

$$t_1 = \frac{a}{v_a} + \frac{b}{v_b} = a\sqrt{\frac{m}{2e(U_0 - U(0))}} + b\sqrt{\frac{m}{2e(U_0 - U(0) + U(\tau'))}},$$

where τ' is the time required to cross BC (and this is different for each electron). Given that $|U(t)| \ll U_0$, we can work within the approximation $U(\tau') = U(\tau)$, where τ corresponds to the unmodulated velocity v, so that $\tau = a/v = a\sqrt{m/2eU_0}$. We can then simplify the expression for t_1 as follows:

$$t_1 \approx a\sqrt{\frac{m}{2eU_0}}\left(1 + \frac{U(0)}{2U_0}\right) + b\sqrt{\frac{m}{2eU_0}}\left(1 + \frac{U(0)}{2U_0} - \frac{U(\tau)}{2U_0}\right).$$

Similarly, after accounting for the time delay Δt, we find that the second electron reaches point D at time

$$t_2 \approx \Delta t + a\sqrt{\frac{m}{2eU_0}}\left(1 + \frac{U(\Delta t)}{2U_0}\right) + b\sqrt{\frac{m}{2eU_0}}\left(1 + \frac{U(\Delta t)}{2U_0} - \frac{U(\Delta t + \tau)}{2U_0}\right).$$

Setting $t_1 = t_2$, we obtain

$$\frac{1}{2U_0}\sqrt{\frac{m}{2eU_0}}\Big((a+b)\big(U(0) - U(\Delta t)\big) - b\big(U(\tau) - U(\Delta t + \tau)\big)\Big) = \Delta t.$$

We need to find a function that satisfies this functional equation for all Δt. We're physicists, so we'll poke around in the hope of finding at least one function $U(t)$ which works. Proving that this function is the unique solution is left to the mathematicians. First consider the special case $\Delta t \to \mathrm{d}t$. This gives us a constraint on the derivative of $U(t)$:

$$-\frac{1}{2U_0}\sqrt{\frac{m}{2eU_0}}\left((a+b)\frac{\mathrm{d}U}{\mathrm{d}t}\bigg|_0 - b\,\frac{\mathrm{d}U}{\mathrm{d}t}\bigg|_\tau\right) = 1.$$

We're told that $a \ll b$, so we'll neglect a for now. It seems that the derivative can't just be a constant, so we'll try the next simplest thing, a linear function of the form $\mathrm{d}U/\mathrm{d}t = At + B$. This fits the bill when $A = 4eU_0^2/mab$. We integrate and find

$$U(t) = \left(\frac{2eU_0^2}{mab}\right)t^2 + Bt + C.$$

Next, we need to substitute this in the general functional equation and confirm that it still works. And indeed, it does, as long as $B = 0$. Still, the focusing won't be perfect across large time intervals Δt, because on the left hand side there will remain a term

$$-\frac{\Delta t^2}{b}\sqrt{\frac{eU_0}{2m}}$$

which doesn't cancel out. That aside, we've almost arrived at an answer. We can set the voltage at $t = 0$ to zero without loss of generality, and then

$$\boxed{U(t) = \frac{2eU_0^2}{mab}t^2.}$$

(4) Now we'll use t to denote the time of arrival of the electrons at B. We know that the waveform $U(t)$ consists of a series of identical parabolic arcs spanning 0 to T, then T to $2T$, then $2T$ to $3T$, and so on. Each arc corresponds to a batch of focused electrons that should sense a change $-U(t)$ at point B and $U(t+\tau)$ at point C. However, it may occur that $t+\tau$ already corresponds to the next parabolic arc in the waveform, in which case these elecrons don't arrive at D together with the rest. This happens for the electrons arriving at B between $T-\tau$ and T, $2T-\tau$ and $2T$, etc. Hence, the fraction that we lose is

$$q = \frac{\tau}{T} = \boxed{\frac{a}{T}\sqrt{\frac{m}{2eU_0}}.}$$