

## Short Exam 2

**Problem.** A point charge  $q$  of mass  $m$  is initially at rest. The charge is placed in a homogeneous electric field  $E$  and a homogeneous magnetic field  $B$  at right angles to  $E$ . It turns out that the charge's trajectory is a periodic curve, i.e. the trajectory will consist of many identical segments. Note that the charge's motion can be represented as a superposition of two simpler sorts of motion.

- (a) Find the time  $T$  in which the charge covers one segment of its trajectory. In other words, find the time interval between two consecutive instants at which the charge is at rest.
- (b) Find the spatial periodicity of the curve  $L$ . In other words, find the distance between two neighbouring positions where the charge is at rest.
- (c) Find the maximum velocity of the charge  $v_{\max}$ .
- (d) Find the ‘width’ of the trajectory  $H$ .
- (e) What are the two simpler sorts of motion that give rise to the resultant motion of the charge? What are their parameters?
- (f) Find the radius of curvature  $r_{\text{curv}}$  of the trajectory at the points of maximum velocity  $v_{\max}$ .

**Solution.** (a) We will assume that the electric field  $\mathbf{E}$  is directed along  $\hat{\mathbf{x}}$  and that the magnetic field  $\mathbf{B}$  is directed along  $\hat{\mathbf{z}}$ . The total force on the charge is  $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$ . This force is perpendicular to  $B$ , so the motion is constrained to the  $xy$ -plane. We want to find equations for the velocity components  $v_x$  and  $v_y$ . Writing the fields as  $\mathbf{E} = E\hat{\mathbf{x}}$  and  $\mathbf{B} = B\hat{\mathbf{z}}$ , we find

$$ma_x = qv_y B + qE, \quad (1)$$

$$ma_y = -qv_x B. \quad (2)$$

We differentiate (1) and plug in (2), resulting in

$$\ddot{v}_x = -\left(\frac{qB}{m}\right)^2 v_x.$$

Setting  $\frac{qB}{m} \equiv \omega_c$  and using the initial condition  $v_x(0) = 0$ , we get the solution

$$v_x = v_{0x} \sin(\omega_c t).$$

To obtain  $v_{0x}$ , we substitute our solution into (1) and apply  $v_y(0) = 0$ . Then  $v_{0x} = \frac{E}{B}$ . We end up with the following equations for the velocity:

$$\begin{aligned} v_x &= \frac{E}{B} \sin(\omega_c t), \\ v_y &= \frac{E}{B} (\cos(\omega_c t) - 1). \end{aligned}$$

The charge is at rest  $t = 0$ , and it's at rest again when  $t$  is such that both  $\sin(\omega_c t)$  and  $\cos(\omega_c t)$  take the same values. This happens at

$$T = \frac{2\pi}{\omega_c} = \boxed{\frac{2\pi m}{qB}}.$$

(b) After integrating the equations for the velocity and applying the initial conditions  $x(0) = 0$  and  $y(0)$  to fix the integration constants, we get

$$x = \frac{E}{B\omega_c} (1 - \cos(\omega_c t)),$$

$$y = \frac{E}{B\omega_c} \sin(\omega_c t) - \frac{E}{B} t.$$

Using  $t = T$ , we find that  $x$  has stayed the same as at  $t = 0$ , while  $y = -\frac{E}{B}T$ . The distance  $L$  we seek is the absolute value of this shift, so

$$L = \frac{2\pi m E}{qB^2}.$$

(c) The square of the velocity at time  $t$  is

$$v^2 = v_x^2 + v_y^2 = 2 \left( \frac{E}{B} \right)^2 (1 - \cos(\omega_c t)).$$

This is maximised when  $\cos(\omega_c t) = -1$ , meaning  $\omega_c t = \pi + 2k\pi$  for integer  $k$ . We conclude

$$v_{\max} = \frac{2E}{B}.$$

(d) We see that  $y$  will drift to  $-\infty$ , but  $x$  is always confined between 0 and  $\frac{2E}{B\omega_c}$ . The trajectory's width is therefore

$$H = \frac{2mE}{qB^2}.$$

(e) Along  $y$  there's uniform motion with velocity  $\mathbf{v}_d = -\frac{E}{B}\hat{\mathbf{y}}$ . Additionally, there's rotation with an angular frequency  $\omega_c = \frac{qB}{m}$  along a circle of radius  $\frac{E}{B\omega_c} = \frac{mE}{qB^2}$ . The centre of rotation is a point with coordinates  $(\frac{mE}{qB^2}, -\frac{E}{B}t)$ .

(f) Referring to the equations for the velocity, we note that at the instant of maximum velocity we have  $v_x = 0$  and  $v_y = -\frac{2E}{B}$ . The equations of motion then imply  $F_x = -qE$ ,  $F_y = 0$ . The radius of curvature can now be found from  $|F_x| = \frac{mv_{\max}^2}{r_{\text{curv}}}$ . Thus

$$r_{\text{curv}} = \frac{4mE}{qB^2}.$$

## Theoretical Exam

### Problem 4.

- (a) Find the electric field  $E$  at a point  $A$  on the axis of an ideal electric dipole  $p$ . The distance between point  $A$  and the dipole is  $r$ .
- (b) Find the force  $F$  on another dipole of the same magnitude and orientation, located at point  $A$ .
- (c) Find the energy of this system of dipoles  $W$ .
- (d) Find the torque  $M$  acting on the dipole at point  $A$  if we rotate it at an angle  $\theta$  with respect to its initial orientation.

**Solution.** (a) Let's treat the dipole as a charge  $-q$  at  $z = 0$  and a charge  $+q$  at  $z = d \ll r$  (indeed, that's what a dipole actually is). Point  $A$  is at  $z = \pm r$ . Let's initially work in the case where  $A$  is at  $z = +r$ . The total field at  $A$  points along the  $z$ -axis and can be found as follows:

$$\mathbf{E} = \left( \frac{kq}{(r-d)^2} - \frac{kq}{r^2} \right) \hat{\mathbf{z}} = \left( \frac{kq(r^2 - (r-d)^2)}{r^2(r-d)^2} \right) \hat{\mathbf{z}} \approx \frac{2kqd}{r^3} \hat{\mathbf{z}} \Leftrightarrow \boxed{\frac{2kp}{r^3} \hat{\mathbf{z}}}.$$

If  $A$  were at  $z = -r$ , we would get exactly the same answer in terms of direction and magnitude. If we swapped the charges of the dipole, the direction of  $\mathbf{E}$  at  $z = -r$  would reverse. But now the field points radially outwards, because the configuration is now the same as in the first case. So the field is actually directed radially inwards, i.e. parallel to  $+\hat{\mathbf{z}}$ .

(b) Again, let the charges of the dipole be  $-q$  at  $z = r$  and  $+q$  at  $z = r + d$ , such that  $p = qd$ . The total force is then

$$\mathbf{F} = -q\mathbf{E}(r) + q\mathbf{E}(r+d) = q\hat{\mathbf{z}} \left( \frac{2kp}{(r+d)^3} - \frac{2kp}{r^3} \right) \approx q\hat{\mathbf{z}} \frac{2kp}{r^6} (r^3 - (r^3 + 3r^2d)) = \boxed{-\frac{6kp^2}{r^4} \hat{\mathbf{z}}}.$$

If the second dipole were at  $z = -r$ , the force has the same magnitude, but it's directed along  $+\hat{\mathbf{z}}$  this time, meaning the force is still attractive. The sign change is because now it's the positive charge that is closer to the first dipole. To check your work, recall that like dipoles attract and unlike dipoles repel.

(c) We will derive a general formula for the energy of the system, valid even when the dipoles are not collinear. We'll need this for the final part of the problem. Let the field of the first dipole be  $\mathbf{E}(\mathbf{r})$ . The potential energy of the system is the work required to bring another dipole from infinity to its final position, where  $+q$  is at some  $\mathbf{r}_+$  and  $-q$  is at some  $\mathbf{r}_-$ . This energy does not include the interaction between the charges in each pair, so there is no problem with treating the second dipole as two independent point charges. Thus

$$W = - \left( \int_{\infty}^{\mathbf{r}_+} \mathbf{F}_+ \cdot d\mathbf{r} + \int_{\infty}^{\mathbf{r}_-} \mathbf{F}_- \cdot d\mathbf{r} \right) = -q \left( \int_{\infty}^{\mathbf{r}_+} \mathbf{E} \cdot d\mathbf{r} + \int_{\infty}^{\mathbf{r}_-} -\mathbf{E} \cdot d\mathbf{r} \right),$$

Now we'll use the fact that the integral of  $\mathbf{E}$  along a closed loop is zero (provided that there are no time-dependent magnetic fields, but we're in the clear). In particular,

$$\int_{\infty}^{\mathbf{r}_+} \mathbf{E} \cdot d\mathbf{r} + \int_{\mathbf{r}_+}^{\mathbf{r}_-} \mathbf{E} \cdot d\mathbf{r} + \int_{\mathbf{r}_-}^{\infty} \mathbf{E} \cdot d\mathbf{r} = 0 \Rightarrow W = q \int_{\mathbf{r}_-}^{\mathbf{r}_+} \mathbf{E} \cdot d\mathbf{r}.$$

And since the dipole is tiny, you can just write this as  $W = -q\mathbf{E} \cdot \mathbf{d}$ , where  $\mathbf{d}$  is the vector connecting  $-q$  to  $+q$ . But then  $W = -\mathbf{p} \cdot \mathbf{E}$ . In our particular problem the angle between  $\mathbf{p}$  and  $\mathbf{E}$  is zero, so

$$\boxed{W = -\frac{2kp^2}{r^3}}.$$

(d) In general, if the angle between  $\mathbf{p}$  and  $\mathbf{E}$  is  $\theta$ , the energy of the configuration is  $W = -pE \cos \theta$ . The favoured state is  $\theta = 0$ , and any deviation from this incurs an energy cost. It follows that there's an associated torque  $M$  you need to overcome in order to rotate the dipole through  $d\theta$ . This can be written as  $dW = M d\theta$ . Hence

$$pE \sin \theta d\theta = M d\theta \Rightarrow M = pE \sin \theta = \boxed{\frac{2kp^2}{r^3} \sin \theta}.$$

Even if you are already familiar with the formulae

$$\mathbf{E} = \frac{kp}{r^3} \left( 2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}} \right), \quad \mathbf{F} = \mathbf{p} \cdot \nabla \mathbf{E}, \quad W = -\mathbf{p} \cdot \mathbf{E}, \quad \mathbf{M} = \mathbf{p} \times \mathbf{E},$$

you're not supposed to just plug them in for this sort of problem. You will have to provide some working of your own for full marks.

**Problem 5.** Find the equivalent resistance of the circuit on Figure 1. Each segment represents a resistance  $R$ . Draw the equivalent circuits that you have used to reach the answer.

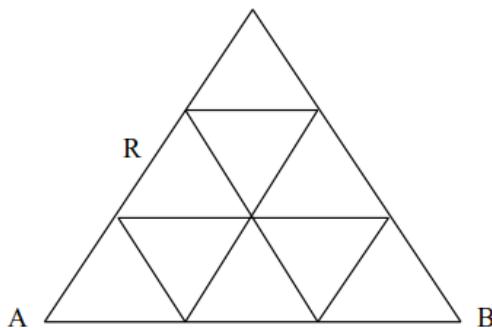
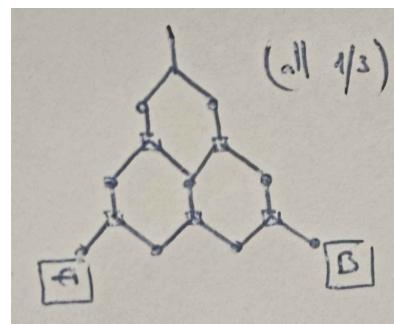
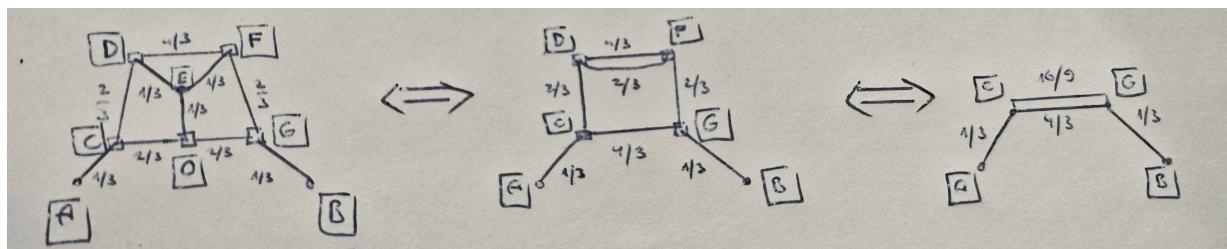


Figure 1

**Solution.** Similarly to how we set  $c = 1$  in relativity, let's set  $R = 1$  and restore the  $R$  at the end. First we apply a Y- $\Delta$  transformation on all triangles that point up. We are left with a simpler circuit:



The resistance at the top doesn't lead anywhere, so we can take it out. After that we can simplify the series connections. Now examine points  $E$  and  $O$ . If the potential at  $A$  is  $\varphi$  and the potential at  $B$  is zero, the potentials at both  $E$  and  $O$  have to be  $\varphi/2$ . This is because they lie on a symmetry axis that divides the circuit between  $A$  and  $B$  into two identical parts. More rigorously, think about how the currents along the path  $ACDE$  are the same as those along the path  $EFGB$ . The resistances are also the same, so the voltage drop between  $A$  and  $E$  should be equal to that between  $E$  and  $B$ . The total voltage across the circuit is  $\varphi$ , so each voltage drop is  $\varphi/2$ .

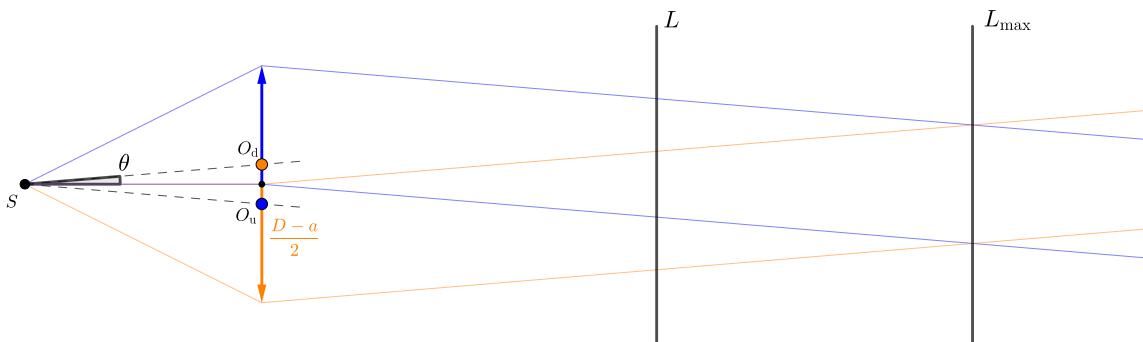


The potentials at  $E$  and  $O$  are equal, meaning that no current can flow between them through the resistance  $EO$ . As a result, removing  $EO$  from the circuit then wouldn't change anything. Following that, we will only have to deal with series and parallel connections. The equivalent resistance turns out to be  $\frac{10}{7}$ , or, after we put the  $R$  back,  $\boxed{\frac{10}{7}R}$ .

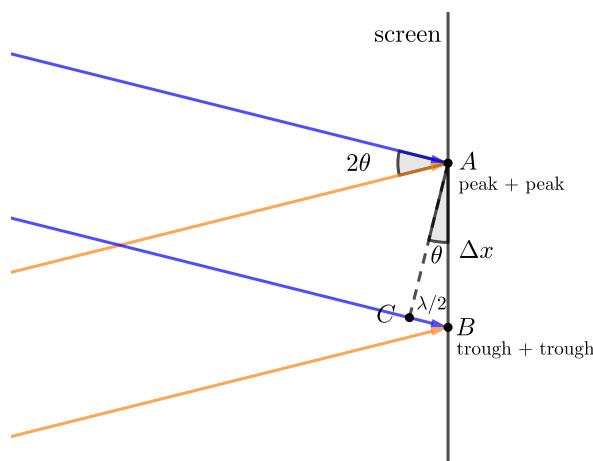
**Problem 6.** A convex lens of diameter  $D = 10\text{ cm}$  and focal length  $f = 1\text{ m}$  is cut into two identical halves, removing a glass layer of thickness  $a = 0.5\text{ mm}$ . The two halves are then stuck back together. A point source of monochromatic light of wavelength  $\lambda = 500\text{ nm}$  is placed in the focal plane of the system. A screen is placed perpendicularly to the optical axis at a distance of  $L = 10\text{ m}$  on the other side of the lens.

- Find the distance  $\Delta x$  between the maxima of the resulting interference pattern.
- Find the number of maxima on the screen  $N$ .
- Approximately at what size of the point source  $\delta$  does the interference pattern vanish?
- Find the distance  $L_{\max}$  at which the screen should be placed so that the number of maxima is largest. What is this number  $N_{\max}$ ?

**Solution.** (a) Each of two halves acts exactly like the lens it came from, in that any ray refracted through that half doesn't know the other half is now missing. After we stick the two halves back together, it's as if we have two identical lenses with centres at  $y = -a/2$  and  $y = +a/2$ . The source is now off-axis for both of them, so the rays passing through the two halves will diverge in different directions. These directions are given by the two lines that connect the source to the centres. The angle these make with the optical axis is  $\theta = \frac{a}{2f}$ . The parallel beams from the two halves of the lens will overlap, resulting in interference:



Let us find the distance between the maxima on the screen, given that the angle between the beams is  $2\theta$ . Consider some maximum at point  $A$ . At that point on the screen, the two interfering waves have a phase difference of  $2\pi k$ ,  $k \in \mathbb{Z}$ . As we move downwards along the screen (towards the next maximum  $B$ ), the phases of the waves when they touch the screen will vary. The rays coming from the top will strike the screen with a greater phase, while those coming from the bottom will strike it with a lesser phase. Eventually, at the next maximum  $B$ , the phase of the rays coming from the top will have increased by  $\pi$ , while the phase of those from the bottom will have gone down by  $\pi$ , for a total extra phase difference of  $2\pi$ .



We then deduce that the segment  $BC$  has to correspond to a phase difference of  $\pi$ , meaning that its length is  $\lambda/2$ . Since  $\theta$  is small, we can write  $\theta = \frac{\lambda/2}{\Delta x}$ . This yields

$$\Delta x = \frac{\lambda f}{a} = 1 \text{ mm.}$$

Note that it doesn't matter how far the screen is.

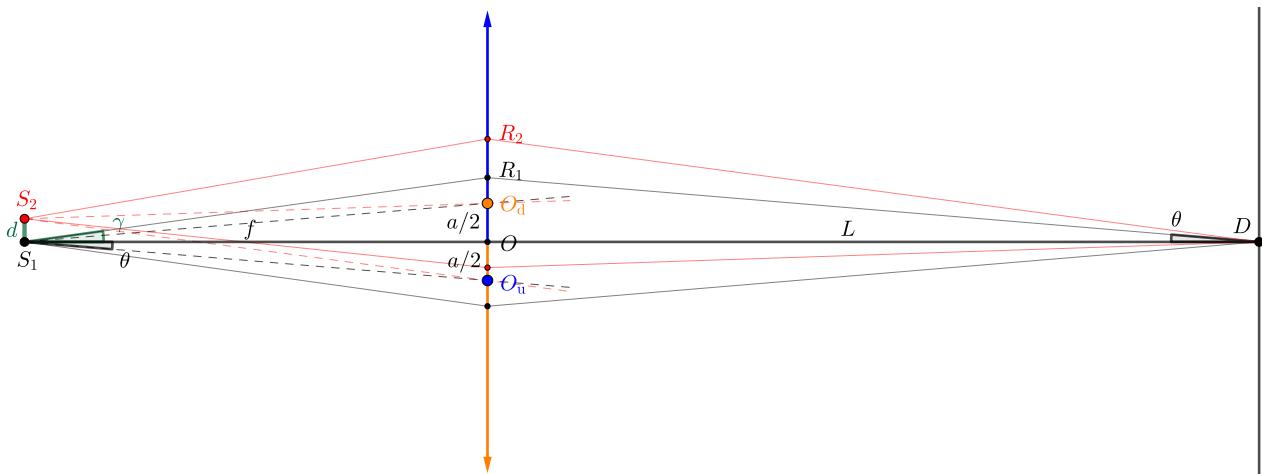
(b) We first need to find the dimensions of the zone where the rays can interfere. This zone is a parallelogram with orthogonal diagonals, meaning it is a rhombus with an angle of  $2\theta$ . Also, we see that its maximum width is  $\frac{D-a}{2}$ , which is the size of each of the halves. This width is reached at some distance  $L_{\max}$  from the lens which satisfies  $(2\theta)L_{\max} = \frac{D-a}{2}$ . Then  $L_{\max} \approx \frac{(D-a)f}{2a} = 99.5 \text{ m}$ . In this part of the problem, the distance to the screen is much less,  $L = 10 \text{ m}$ . The width of the interference zone at that distance is

$$w = (2\theta)L = \frac{aL}{f} = 5 \text{ mm.}$$

This isn't that much compared to  $\Delta x$ , so we need to check explicitly how all the maxima fit within this zone. By symmetry, there will be a maximum at the centre of the screen where the optical axis is. Now we arrange maxima in steps of  $\Delta x$  around the central maximum. We have  $\frac{w}{2} = 2.5 \text{ mm}$  of available space on each side for that. We can only go up to  $2\Delta x = 2 \text{ mm}$  on each side. This is a total of  $1 + 2 \cdot 2 = 5$  maxima. The general form of the answer is

$$N = 1 + 2 \left\lfloor \frac{w/2}{\Delta x} \right\rfloor = 1 + 2 \left\lfloor \left( \frac{a}{f} \right)^2 \frac{L}{2\lambda} \right\rfloor = 5.$$

(c) Assume the source is a sphere of diameter  $\delta$ . Each individual surface element of the sphere can be treated as a separate point source with its own interference pattern. The surface elements have different positions on the  $y$ -axis, which means their interference patterns are offset from each other ( $x$ -axis displacements won't matter to first order). In order to find the offset, we will consider two surface elements  $S_1$  and  $S_2$ , one at  $y = 0$  and the other at  $y = d \leq \delta$ . Let us see what happens at some fixed point on the screen  $D$ . To simplify the calculations, we will work with a point at  $x = L$  and  $y = 0$ . Each source emits a pair of rays that end up at  $D$  after passing through the top and bottom half of the lens, respectively. We now need to construct these rays on a diagram. The sources are in the focal plane, meaning that all refracted rays travel in the same direction as the lines connecting the source to the centre of each half. We draw these lines ( $S_1O_d$ ,  $S_1O_u$ ,  $S_2O_d$ ,  $S_2O_u$ ), and work back from  $D$  to construct rays parallel to them.

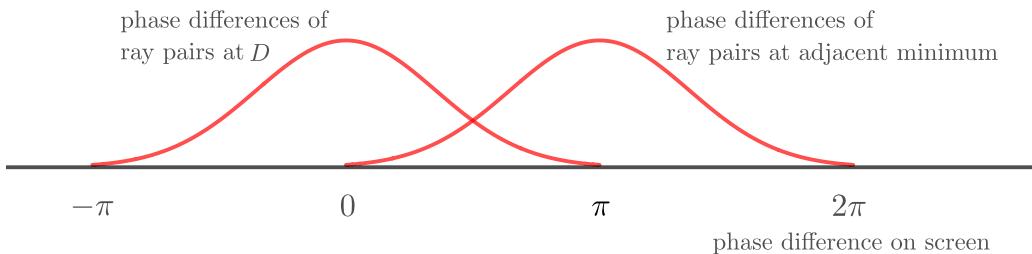


Due to symmetry, the rays coming from  $S_1$  terminate at  $D$  in phase. However, the two rays from  $S_2$  will carry some phase difference. To find it, we first compare the phase difference of rays  $S_1R_1D$  and  $S_2R_2D$ . Here you need to consider the path difference both in the air and within the lens. This is hard to do by hand, but there's a trick. The reversed rays  $DR_1S_1$  and  $DR_2S_2$  both seem to emanate from point  $D$ , and they will eventually converge at the image of  $D$  from the top half (say,  $D'$ , not shown in the figure). From Fermat's principle of equal travel time, we know that these rays will arrive at  $D'$  with no phase difference. Therefore, the phase difference of our two rays is equal to that of the two segments  $D'S_2$  and  $D'S_1$ . But this is a standard configuration with a path difference of  $d \sin \gamma$ , or a phase difference of  $\frac{2\pi}{\lambda} d \sin \gamma$ . Repeating this procedure for the second ray from  $S_2$ , we find the same phase difference with respect to the ray from  $S_1$  passing through the bottom half, albeit with the opposite sign. We conclude that the two rays from  $S_2$  interfere with a phase difference of  $\frac{4\pi}{\lambda} d \sin \gamma \approx \frac{4\pi}{\lambda} d \gamma$ .

To obtain  $\gamma$ , note that  $\angle ODR_1 = \angle O_u S_1 O = \theta$ , and then write  $OR_1 = \gamma f = \theta L$ . Thus, the rays from  $S_2$  have a phase difference of

$$\Delta\phi = \frac{2\pi a L d}{\lambda f^2}.$$

An object of size  $\delta$  should then supply a range of phase differences between 0 and  $\frac{2\pi a L \delta}{\lambda f^2}$  at  $D$ . This smears out the minima and the maxima. For example, at the point where rays used to interfere at a phase difference  $\phi$  (back when we had a point source  $S$ ), we now have a distribution of phase differences  $\left[\phi - \frac{\pi a L \delta}{\lambda f^2}, \phi + \frac{\pi a L \delta}{\lambda f^2}\right]$  centered at  $\phi$ . For the interference pattern to vanish, the maxima should be indistinguishable from the minima. To check this, we will use something akin to the Rayleigh criterion. We will assume that a minimum and a maximum become indistinguishable when the tail of the distribution for the minimum lies at the centre of the distribution for the maximum:



In that case,  $\frac{\pi a L \delta}{\lambda f^2} = \pi$ , or

$$\delta = \frac{\lambda f^2}{a L} = 0.1 \text{ mm.}$$

This answer seems to make sense – as we increase  $L$ , phase differences accumulate more easily, so the setup fails at a lesser  $\delta$ . However, I'd still like someone to confirm it, because I couldn't find anything similar in the literature. Please email me if you have read my solution and you are confident that it is correct.

(d) The distance between adjacent maxima is always  $\Delta x$ , so the number of maxima is largest when the zone of interference is at its widest. As we found previously, this corresponds to a distance of

$$L_{\max} = \frac{(D - a)f}{2a} = 99.5 \text{ m.}$$

The width of the zone at that point is  $\frac{D-a}{2}$ . We reuse the general formula for the number of maxima to find

$$N_{\max} = 1 + 2 \left\lfloor \frac{(D-a)a}{2\lambda f} \right\rfloor = 99.$$

If we had approximated  $(D - a)$  to  $D$  too early, we would have instead got  $N_{\max} = 101$ .

# Experimental Exam

## Problem 1. Seiche.

*Equipment:*

Rectangular box, 3 l of water, measuring vessel (in ml), stopwatch, ruler, graph paper.

Consider a harmonic wave of wavelength  $\lambda$  propagating along the surface of an infinite layer of liquid of depth  $h$ . Neglecting the viscosity and the surface tension of the liquid, and assuming a wave amplitude  $A \ll h$ , the propagation speed is given by

$$v = \sqrt{\frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right)}, \quad (3)$$

where  $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$  and  $g$  is the acceleration due to gravity. This formula yields approximations for the propagation speed  $v$  in the cases of ‘shallow’ and ‘deep’ water, respectively:

$$v_{\text{shallow}} = \sqrt{gh}, \quad (4)$$

$$v_{\text{deep}} = \sqrt{\frac{g\lambda}{2\pi}}. \quad (5)$$

A seiche is a standing wave formed on the surface of an enclosed body of water. The surface of the water remains nearly flat, however the water level at the two ends of the vessel oscillates in antiphase with a period  $T$ . The seiche can be considered as a standing wave formed by the superposition of two harmonic travelling waves of equal amplitudes that propagate in opposite directions.

In shallow enough water, instead a seiche one could create a travelling nonlinear wave with a single peak, called a soliton. The velocity of this type of wave can also be assumed to obey (3).

The aim of this problem is to study the period  $T$  of a seiche and of a soliton for different values of the depth  $h$ , i.e. both for ‘shallow’ water and ‘deep’ water.

Before you start taking measurements, play around with the setup at different depths. Induce waves along both sides of the box and see when you can reliably measure the timescales. Hence decide on your experimental procedure and the ranges in which you will measure.

- (a) Induce a seiche or a soliton at different depths  $h$  and measure the period of the wave  $T$ . Explain how you induce the waves. (0.5 pt)
- Present your results in a table. (3.5 pt)
- (b) Write down the depths  $h$  at which you can create a seiche and those at which you can create a soliton. (0.5 pt)
- (c) Plot a graph of the period  $T$  against the depth  $h$ . (3.0 pt)
- (d) For what depths is the period  $T$  constant (or nearly constant)? (0.5 pt)
- (e) Assuming that in ‘deep’ water the seiche has some effective wavelength  $\lambda_{\text{eff}}$ , find the ratio  $n = \frac{\lambda_{\text{eff}}}{2L}$ , where  $L$  is the length of the box in the direction of propagation of the seiche. (0.5 pt)
- (f) Assuming that in ‘shallow’ water the wave velocity is given by

$$v \propto h^k, \quad (6)$$

plot this dependence in appropriate coordinates so that  $k$  can be determined from the graph. (3.0 pt)

Also present the data in the graph in tabular form. (0.5 pt)

- (g) Calculate  $k$ . (1.0 pt)
- (h) Find the values of the ratio  $h/L$  for which (6) holds. (1 pt)
- (i) Explain qualitatively why (3) also holds for a soliton. (0.5 pt)
- (j) At what depths does viscosity become significant for your experiment? (0.5 pt)

Call the examiner in case of any technical difficulties.