A Heavy Vehicle on an Inclined Road

A road roller is modelled as a body of mass $5M$ attached via axles to two identical cylindrical wheels, each with mass M. The centre of mass of the body is located symmetrically with respect to the rear and the front axles; it's separated from each of them by a horizontal distance l and a vertical distance h_1, as shown on Figure 1. There is no rolling resistance between the axles and the cylindrical wheels.

The wheels are modelled as shown on Figure 2. Most of their mass is concentrated in a uniform cylindrical shell of outer radius R and inner radius $\frac{4}{5}R$. The inner surface of the shell is connected to the axle using a number of uniform radial spokes placed at equal angles to each other. The total mass of the spokes on one wheel is $\frac{1}{5}M$.

The road roller is now placed on an inclined road making an angle θ with the horizon. The coefficients of static and kinetic friction between the wheels and the road are μ_s and μ_k, respectively. The acceleration due to gravity is g.

(a) Find an expression for the moment of inertia I of a wheel with respect to its axle in terms of M and R. **[1.5 pt]**

To simplify your working in the following subparts, denote the numerical prefactor in the expression above by k rather than writing it down explicitly.

(b) Draw a free body diagram for the body, the front cylinder, and the rear cylinder. Write down the equations of motion for each of these elements in the x and y directions. Write down expressions for the friction forces between the wheels and the road depending on the type of motion. **[2.5 pt]**

(c) The vehicle is let go from rest, and starts moving down the road due to gravity. List all the possible types of motion of the system. For each case, find the acceleration of the system a. Also find the conditions on the slope θ for which each case is observed. **[4.0 pt]**

(d) Assume that the coefficients of friction are such that both wheels roll without slipping after the vehicle has been let go from rest. After the vehicle has travelled a distance d, it enters a section of the road where the coefficients of friction have some smaller values μ'_s and μ'_k, whereupon both wheels start slipping. Find the angular velocities ω and the linear velocities v of the cylinders after the vehicle has travelled a distance $s > d$ as measured from its initial position. The distance between the axles can be neglected with respect to d and s. You are not required to simplify your final expressions. **[2.0 pt]**

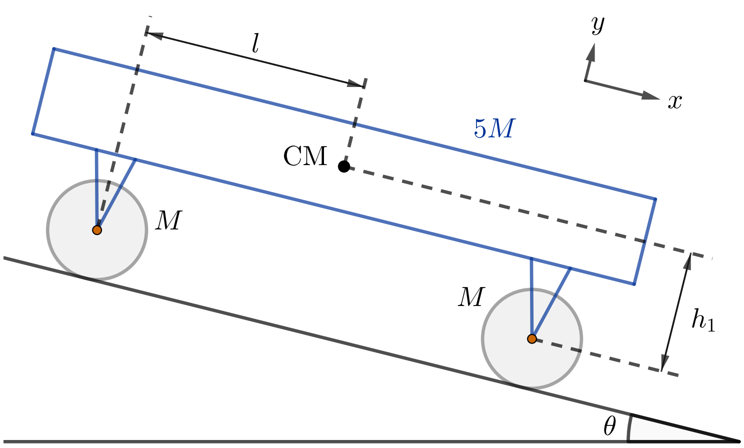


Figure 1

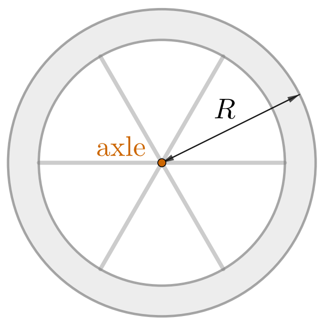


Figure 2

Solution.

(a) The surface density for the cylindrical shell is $\sigma = \frac{(4/5)M}{\pi(R^2-((4/5)R)^2)} = \frac{20M}{9\pi R^2}$. We can represent it as a superposition of two solid cylinders, one with surface density σ and radius R, and the other with surface density $-\sigma$ and radius $\frac{4}{5}R$. For any given cylinder of mass m and radius r, the moment of inertia is $\frac{1}{2}mr^2 = \frac{1}{2}\sigma\pi r^4$. Hence, the moment of inertia for the shell is $I_1 = I_\sigma + I_{-\sigma} = \frac{1}{2}\sigma\pi(R^4 - (\frac{4}{5}R)^4) = \frac{82}{125}\pi MR^2$.

Next, the mass distribution of the spokes is the same as that of a single rod of mass $\frac{1}{5}M$ stretching from the axle to $\frac{4}{5}R$. Its moment of inertia is therefore $I_2 = \frac{1}{3}(\frac{1}{5}M)(\frac{4}{5}R)^2 = \frac{16}{375}MR^2$. We then arrive at $I = I_1 + I_2 = \boxed{\frac{262}{375}MR^2} \equiv kMR^2$.

(b) We'll index the body with 1, the rear wheel with 2, and the front wheel with 3. The free body diagram for the front wheel is shown on Figure 4. There, $f_{3\mathrm{h}}$ and T_3 represent the push from the body via the axle in the x and y directions, respectively, while N_3 and f_3 are the normal force and the friction from the ground. Apart from that, we have gravity Mg pointing downwards. The free body diagram for the rear wheel would be exactly the <u>same</u>, except for the indices. This means we can write the equations of motion for the wheels in tandem. For x and y we have

$$f_{2\mathrm{h}} + Mg\sin\theta - f_2 = Ma, \qquad (1a) \qquad\qquad f_{3\mathrm{h}} + Mg\sin\theta - f_3 = Ma, \qquad (1b)$$

$$Mg\cos\theta + T_2 = N_2, \qquad (2a) \qquad\qquad Mg\cos\theta + T_3 = N_3. \qquad (2b)$$

If a wheel rolls without slipping (implying angular acceleration $\frac{a}{R}$), its rotation will be driven by the friction force with the ground, i.e. $fR = I\frac{a}{R}$, or

$$f_2 = kMa, \qquad (3a) \qquad\qquad f_3 = kMa. \qquad (3b)$$

Conversely, whenever a wheel slips, we write

$$f_2 = \mu_\mathrm{k} N_2, \qquad (4a) \qquad\qquad f_3 = \mu_\mathrm{k} N_3. \qquad (4b)$$

The force diagram for the body is given on Figure 3. From Newton's second law we have

$$5Mg\sin\theta - f_{2\mathrm{h}} - f_{3\mathrm{h}} = 5Ma, \qquad (5)$$

$$T_2 + T_3 = 5Mg\cos\theta. \qquad (6)$$

Additionally, torque balance gives us

$$(f_{2\mathrm{h}} + f_{3\mathrm{h}})h_1 = (T_3 - T_2)l. \qquad (7)$$

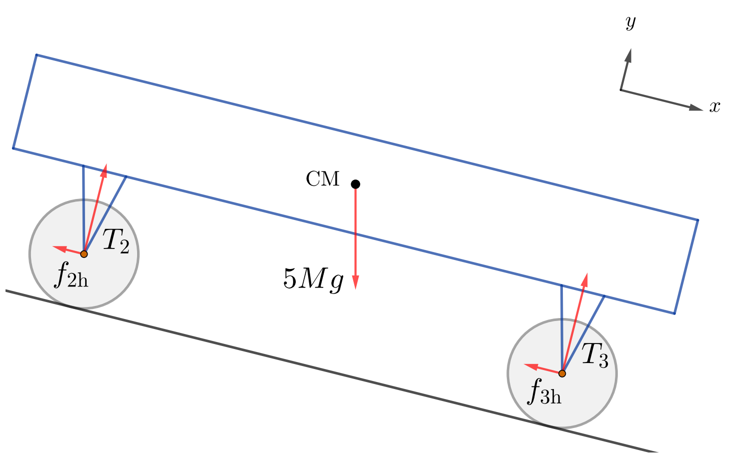


Figure 3

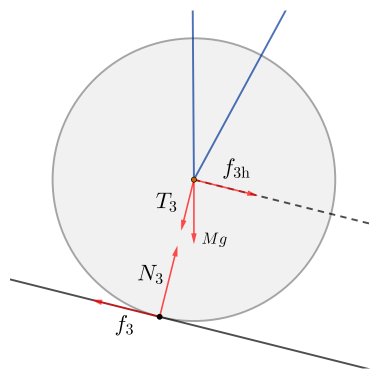


Figure 4

(c) Before we cover the separate cases, let's first derive general equations for N_2 and N_3 in terms of the acceleration a. These will come into play a bit later. Since N_2 and N_3 must counteract gravity, we have $N_2 + N_3 = 7Mg\cos\theta$. Apart from that, Equations (2a), (2b), (5), and (7) give us $N_3 - N_2 = \frac{h_1}{l}(5Mg\sin\theta - 5Ma)$. Therefore

$$N_2 = \frac{7}{2}Mg\cos\theta - \frac{5}{2}\frac{h_1}{l}Mg\sin\theta + \frac{5}{2}\frac{h_1}{l}Ma, \tag{8}$$

$$N_3 = \frac{7}{2}Mg\cos\theta + \frac{5}{2}\frac{h_1}{l}Mg\sin\theta - \frac{5}{2}\frac{h_1}{l}Ma. \tag{9}$$

Note that N_2 is smaller than N_3, because a cannot exceed $g\sin\theta$ (you can't do better than free fall). We certainly want the wheels to remain in contact with the ground, and to that end, it's enough to impose the condition $N_2 > 0$. This is more easily satisfied for higher values of a. But there's an issue. As long as we're holding the vehicle still, its acceleration is $a = 0$, and once we release it, a takes on a distinctly nonzero value. In real life, such changes happen continuously, so we need to ensure $N_2 > 0$ even at $a = 0$. This implies

$$\boxed{\tan\theta < \frac{7}{5}\frac{l}{h_1} \equiv A.} \tag{10}$$

Assuming that this is satisfied, let's look at the possible types of motion.

1. Both wheels roll without slipping.

In this case $f_2 = f_3 = kMa$, and so the equation of motion is $7Mg\sin\theta - 2kMa = 7Ma$. Hence,

$$\boxed{a = \frac{1}{1+\frac{2}{7}k}g\sin\theta.} \tag{11}$$

This regime is feasible only when both the friction forces with the ground are less than their respective thresholds for slipping. Since $N_2 < N_3$, it's sufficient to demand that $f_2 < \mu_s N_2$. Notice that, starting from this state, it's impossible for the front wheel to begin slipping unless the rear wheel slips first. After expanding the expressions for the forces, this reduces to

$$\left(k - \frac{5}{2}\frac{\mu_s h_1}{l}\right)Ma < \mu_s\left(\frac{7}{2}Mg\cos\theta - \frac{5}{2}\frac{h_1}{l}Mg\sin\theta\right). \tag{12}$$

We plug in Equation (11) and divide by $\cos\theta$ to find

$$\boxed{\tan\theta < \left(\frac{1}{k} + \frac{2}{7}\right)\frac{\frac{7}{2}\mu_s}{1+\frac{5}{7}\frac{\mu_s h_1}{l}} \equiv B.} \tag{13}$$

When this doesn't hold, the system transitions to a different regime. Depending on the value of μ_s, Equation (13) might be either more or less strict than Equation (10).

2. The front wheel rolls without slipping and the rear wheel slips.

Start from $7Mg\sin\theta - f_2 - f_3 = 7Ma$ and use $f_2 = \mu_k N_2$, with N_2 taken from Equation (8). As for the front wheel, we still have $f_3 = kMa$, and so

$$7Mg\sin\theta - \left(\frac{7}{2}\mu_k Mg\cos\theta - \frac{5}{2}\frac{\mu_k h_1}{l}Mg\sin\theta + \frac{5}{2}\frac{\mu_k h_1}{l}Ma\right) - kMa = 7Ma, \tag{14}$$

$$\left(7\sin\theta - \frac{7}{2}\mu_k\cos\theta + \frac{5}{2}\frac{\mu_k h_1}{l}\sin\theta\right)g = \left(7 + k + \frac{5}{2}\frac{\mu_k h_1}{l}\right)a. \tag{15}$$

Note that there's a danger of the left-hand side being negative. Let's check. For it to be positive, we need

$$\tan\theta > \frac{\frac{7}{2}\mu_k}{7+\frac{5}{2}\frac{\mu_k h_1}{l}}. \tag{16}$$

We already know that

$$\tan\theta > \left(\frac{1}{k}+\frac{2}{7}\right)\frac{\frac{7}{2}\mu_\mathrm{s}}{1+\frac{5}{7}\frac{\mu_\mathrm{s}h_1}{l}}, \tag{17}$$

so we seek to show that

$$\left(\frac{1}{k}+\frac{2}{7}\right)\frac{\mu_\mathrm{s}}{1+\frac{5}{7}\frac{\mu_\mathrm{s}h_1}{l}} > \frac{\mu_\mathrm{k}}{7+\frac{5}{2}\frac{\mu_\mathrm{k}h_1}{l}} \quad\Leftrightarrow\quad \left(\frac{1}{k}+\frac{2}{7}\right)\left(7+\frac{5}{2}\frac{\mu_\mathrm{k}h_1}{l}\right)\mu_\mathrm{s} = \left(1+\frac{5}{7}\frac{\mu_\mathrm{s}h_1}{l}\right)\mu_\mathrm{k}. \tag{18}$$

You can find something on the left-hand side to match each of the terms on the right-hand side, so we're in the clear. Here we made use of the fact that $\mu_\mathrm{s} > \mu_\mathrm{k}$. After all, if μ_s were less than μ_k, then μ_k would take on the role of the new μ_s.

Anyway, we can now safely conclude that

$$\boxed{a = \frac{7\sin\theta - \frac{7}{2}\mu_\mathrm{k}\cos\theta + \frac{5}{2}\frac{\mu_\mathrm{k}h_1}{l}\sin\theta}{7+k+\frac{5}{2}\frac{\mu_\mathrm{k}h_1}{l}}g.} \tag{19}$$

Now we'll find the conditions under which this regime is possible. We still want the front wheel to roll without slipping, so we need $kMa < \mu_\mathrm{s}N_3$. After substituting Equations (9) and (19), dividing by $Mg\sin\theta$, and lots of rearranging, we get

$$\begin{aligned}\left[\left(k+\frac{5}{2}\frac{\mu_\mathrm{s}h_1}{l}\right)\left(7+\frac{5}{2}\frac{\mu_\mathrm{k}h_1}{l}\right) - \left(7+k+\frac{5}{2}\frac{\mu_\mathrm{k}h_1}{l}\right)\left(\frac{5}{2}\frac{\mu_\mathrm{s}h_1}{l}\right)\right]\tan\theta \\ < \frac{7}{2}\mu_\mathrm{k}\left(k+\frac{5}{2}\frac{\mu_\mathrm{s}h_1}{l}\right) + \frac{7}{2}\mu_\mathrm{s}\left(7+k+\frac{5}{2}\frac{\mu_\mathrm{k}h_1}{l}\right)\end{aligned} \tag{20}$$

Many of the terms on the left-hand side cancel, and we reach

$$k\left(7-\frac{5}{2}\frac{(\mu_\mathrm{s}-\mu_\mathrm{k})h_1}{l}\right)\tan\theta < \frac{35}{2}\mu_\mathrm{k}\mu_\mathrm{s}\left(\frac{h_1}{l}\right)^2 + \frac{7}{2}(\mu_\mathrm{s}+\mu_\mathrm{k})k + \frac{49}{2}\mu_\mathrm{s}. \tag{21}$$

Now there are two distinct situations. First, if the difference between the friction coefficients is

$$\boxed{\mu_\mathrm{s}-\mu_\mathrm{k} > \frac{14}{5}\frac{l}{h_1},} \tag{22}$$

the inequality is automatically satisfied for any θ. If not, we'll need

$$\boxed{\tan\theta < 7\cdot\frac{5\mu_\mathrm{k}\mu_\mathrm{s}\left(\frac{h_1}{l}\right)^2 + (\mu_\mathrm{k}+\mu_\mathrm{s})k + 7\mu_\mathrm{s}}{k\left(14-5(\mu_\mathrm{s}-\mu_\mathrm{k})\frac{h_1}{l}\right)} \equiv C.} \tag{23}$$

If the angle θ is larger than this, the only remaining option is that both wheels slip. As a warning, it's entirely possible that $C < B$. For example, in the limit where $\frac{h_1}{l}\to 0$, we have

$$B = \frac{7\mu_\mathrm{s}+2\mu_\mathrm{s}k}{14k} \quad\text{and}\quad C = \frac{7\mu_\mathrm{s}+(\mu_\mathrm{k}+\mu_\mathrm{s})k}{14k}.$$

When $C < B$, the case where only one wheel slips cannot be realised.

3. Both wheels slip.

Since $N_2+N_3 = 7Mg\cos\theta$, the equation of motion is just $7Mg - \mu_\mathrm{k}(7Mg\cos\theta) = 7Ma$. Thus

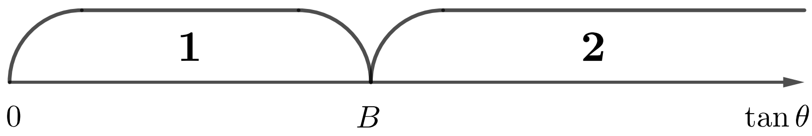
$$\boxed{a = (\sin\theta - \mu_\mathrm{k}\cos\theta)g.} \tag{24}$$

There's again a troubling possibility that the acceleration is negative. We hope that in this regime, $\tan\theta$ will be larger than μ_k. We know that $\tan\theta > C$, and we want to demonstrate that $C > \mu_\mathrm{k}$. Since $\mu_\mathrm{s} > \mu_\mathrm{k} > 0$ and $k < 1$, we can get a lower bound on C like this:

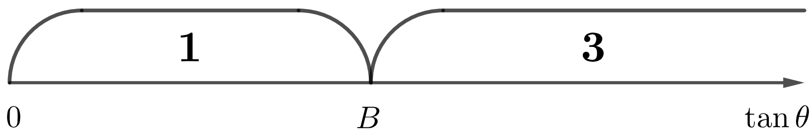
$$7 \cdot \frac{5\mu_\mathrm{k}\mu_\mathrm{s}\left(\frac{h_1}{l}\right)^2 + (\mu_\mathrm{k} + \mu_\mathrm{s})k + 7\mu_\mathrm{s}}{k\left(14 - 5(\mu_\mathrm{s} - \mu_\mathrm{k})\frac{h_1}{l}\right)} > \frac{7(7+k)\mu_\mathrm{s}}{14k} > \frac{7\mu_\mathrm{s}}{2k} > \frac{7}{2}\mu_\mathrm{s} > \mu_\mathrm{s} > \mu_\mathrm{k}. \tag{25}$$

So we're safe. Let's now summarise how the type of motion depends on the value of $\tan\theta$ and whether the inequalities (10) and (22) are satisfied.

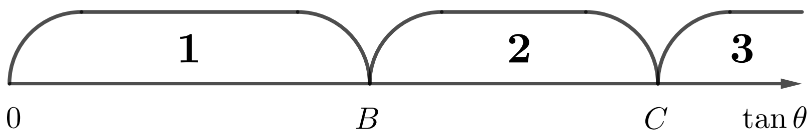
If $\mu_\mathrm{s} - \mu_\mathrm{k} > \frac{14}{5}\frac{l}{h_1}$, the available regimes are as follows:



If $\mu_\mathrm{s} - \mu_\mathrm{k} < \frac{14}{5}\frac{l}{h_1}$ and $C < B$, we have the following:



If $\mu_\mathrm{s} - \mu_\mathrm{k} < \frac{14}{5}\frac{l}{h_1}$ and $C > B$, all the regimes are accessible:



Apart from all that, Equation (10) provides a strict cut-off in the form of $\tan\theta < A$. However, the exact placement of B and C with respect to A will depend on the specific values of μ_k and μ_s.

(d) We are told that the road roller immediately switches from Regime 1 to Regime 3 because the values of B and C have decreased. The vehicle first covers a segment of length d with acceleration $a_1 = \frac{1}{1+(2/7)k} g \sin\theta$, and then covers a segment of length $s - d$ with acceleration $a_2 = (\sin\theta - \mu'_\mathrm{k}\cos\theta)g$.

At the end of the first segment, the velocity of the centres of the wheels v' is given by

$$v' = \sqrt{2a_1 d}. \tag{26}$$

Then, the acceleration changes, and the final velocity of the centres v can be found from standard kinematics, $v^2 - v'^2 = 2a_2(s-d)$. We conclude that

$$\boxed{v = \sqrt{2a_1 d + 2a_2(s-d)}.} \tag{27}$$

We still need to find the angular velocities of the wheels. Since we initially have rolling without slipping, at the end of the first segment both wheels will have an angular velocity

$$\omega' = \frac{v'}{R}. \tag{28}$$

For the second segment, the normal forces on the wheels are

$$N_{2,3} = \left(\frac{7}{2} \mp \frac{5}{2}\frac{\mu_{\rm k}' h_1}{l}\right) Mg\cos\theta, \tag{29}$$

These will give rise to angular accelerations

$$\varepsilon_{2,3} = \frac{\mu_{\rm k}' N_{2,3} R}{I} = \frac{\mu_{\rm k}' N_{2,3}}{kMR}. \tag{30}$$

The time it takes for the vehicle to cover the second region is $\tau = \frac{v-v'}{a_2}$, and then the final angular velocities are $\omega_{2,3} = \omega' + \varepsilon_{2,3}\tau$, or

$$\boxed{\omega_{2,3} = \frac{v'}{R} + \left(\frac{\mu_{\rm k}' N_{2,3}}{kMa_2}\right)\frac{v-v'}{R}.} \tag{31}$$

Last updated: February 17, 2026
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Marking scheme.

Part (a)	**Points**
Idea for superposition of the moments of inertia	0.4
Moment of inertia of cylindrical shell	0.5
Moment of inertia of spokes	0.5
Correct final answer	0.1
Total	**1.5**

Part (b)	**Points**
Free body diagram for the wheels (Figure 4)	0.5
Equations (1)	0.3
Equations (2)	0.3
Equations (3)	0.2
Equations (4)	0.2
Free body diagram for the vehicle's body (Figure 3)	0.4
Equation (5)	0.2
Equation (6)	0.2
Equation (7)	0.2
Total	**2.5**

Part (c)	**Points**
Preliminary work	–
Equations (8) and (9)	0.3
Regime 1	–
Equation of motion with appropriate f_2 and f_3	0.5
Expression for the acceleration (Equation (11))	0.2
Regime 2	–
Equation of motion with appropriate f_2 and f_3	0.5
Expression for the acceleration (Equation (19))	0.5
Regime 3	–
Equation of motion with appropriate f_2 and f_3	0.5
Expression for the acceleration (Equation (24))	0.2
Casework	–
Condition for staying on the ground (Equation (10))	0.2
Stating the conditions for Regime 1 ($f_2 < \mu_s N_2$)	0.1
Inequality for $\tan\theta$ in Regime 1 (Equation (13))	0.1
Stating the conditions for Regime 2 (not in Regime 1 and $kMa < \mu_s N_3$)	0.2
Inequality for $\tan\theta$ in Regime 2 (Equations (21) and (23))	0.2
Distinguishing between the cases with respect to Equation (22)	0.2
Distinguishing between the cases with respect to the ordering of B and C	0.2
Stating the conditions for Regime 3 (not in Regime 1 or 2)	0.1
Total	**4.0**

Part (d)	**Points**
Equation (26)	0.5
Equation (27)	0.4
Equation (28)	0.5
Equation (31)	0.6
Total	**2.0**

Note. I have tried to adhere to the (rather unfair) original marking scheme where possible.