2018 Bulgarian IPhO Team Selection Test – Solutions

Short Exam 1

Problem. Falling ladder. The two ends of a rod of mass m and length l sit on a horizontal floor and on a vertical wall, respectively. The rod lies in a plane which is perpendicular both to the floor and the wall. The acceleration due to gravity is g. Initially the rod is at rest and it makes an angle α_0 with the floor. The rod is let go and starts falling. Ignore friction. The moment of inertia of a rod about an axis passing through its centre of mass perpendicularly to the rod is $I = \frac{1}{12}ml^2$.

(a) Find the velocity of the centre of mass of the rod v, as well as its angular velocity ω, as a function of the angle α which the rod makes with the floor during its descent.

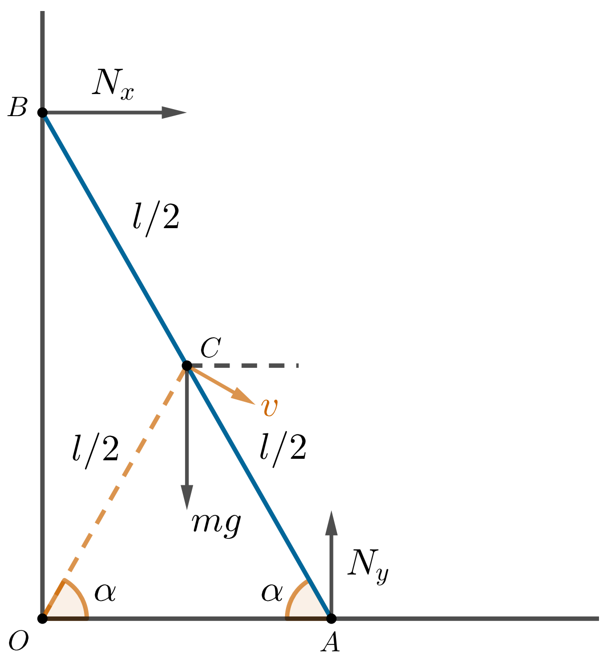
(b) At what angle α_1 does the rod lose contact with the wall?

(c) Find the velocity v_∞ with which the rod will slide on the floor after it has fallen down.

Solution. (a) This is a repeat of Problem 1 from the 2015 Spring Physics Competition. The (anticlockwise) angular velocity of the rod is minus the rate of change of the angle which it makes with the horizon, i.e. $\omega = -\mathrm{d}\alpha/\mathrm{d}t$. Referring to the diagram, the segment which connects the centre-of-mass C to the corner O is a median of a right triangle, so its length is constant and equal to $l/2$. We also see that $\angle AOC = \alpha$, so the segment rotates with ω, same as the rod. Hence $v = \omega l/2$. The kinetic energy of the rod is $T = mv^2/2 + I\omega^2/2 = m\omega^2 l^2/6$, so conservation of energy gives us

$$mg\frac{l}{2}\sin\alpha_0 = mg\frac{l}{2}\sin\alpha + \frac{1}{6}m\omega^2 l^2,$$

$$\boxed{\omega = \sqrt{\frac{3g}{l}}\,(\sin\alpha_0 - \sin\alpha),} \qquad \boxed{v = \sqrt{\frac{3}{4}gl}\,(\sin\alpha_0 - \sin\alpha).}$$



(b) For no reason at all we'll first find the angular acceleration of the rod ε:

$$\varepsilon = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{2\omega}\cdot\frac{3g}{l}\cdot(-\cos\alpha)\cdot\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{3g}{2l}\cos\alpha.$$

Now onto the problem. We lose contact with the wall when $N_x = 0$. We know that $N_x = m(\mathrm{d}v_x/\mathrm{d}t)$, so we need to look for an expression for v_x. From the diagram we have $v_x = v\sin\alpha$, and so $N_x = 0$ when

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{l}{2}\frac{\mathrm{d}(\omega\sin\alpha)}{\mathrm{d}t} = \frac{l}{2}\left(\varepsilon\sin\alpha - \omega^2\cos\alpha\right) = 0 \quad \Leftrightarrow \quad \omega^2 = \frac{3g}{2l}\sin\alpha.$$

Therefore,

$$\frac{3g}{l}\left(\sin\alpha_0 - \sin\alpha_1\right) = \frac{3g}{2l}\sin\alpha_1 \quad \Rightarrow \quad \boxed{\sin\alpha_1 = \frac{2}{3}\sin\alpha_0.}$$

(c) After the ladder loses contact with the wall, there won't be any horizontal forces acting on it, so its horizontal velocity remains equal to

$$v_x = v\sin\alpha_1 = \sqrt{\frac{3}{4}gl\left(\sin\alpha_0 - \sin\alpha_1\right)}\sin\alpha_1 = \frac{1}{3}\sqrt{gl}(\sin\alpha_0)^{3/2}.$$

Conversely, after the ladder strikes the ground, it loses all of its vertical velocity. The total velocity v_∞ is then just v_x, or

$$\boxed{v_\infty = \frac{1}{3}\sqrt{gl}(\sin\alpha_0)^{3/2}.}$$

Short Exam 2

Problem. Induction motor. Figure 1 shows an asynchronous motor. The rotor is made up of two metal rings attached to the axis, and a large number of rods N which connect the rings. A system of inductors (not shown on the figure) creates a homogeneous magnetic field B perpendicular to the axis of the motor. The inductors are powered by three-phase power, and as a result the magnetic field vector rotates around the axis of the motor with an angular velocity ω_0.

The radius of the rings is a and the length of the rods is l. Each rod has a resistance R and the resistance of the rings is negligible.

(a) Obtain an expression for the torque M acting on the rotor when it stays fixed.

(b) Assume the rotor powers some mechanical device, as a result of which it rotates with an angular velocity ω ($0 < \omega < \omega_0$). Find an expression for the torque in terms of ω. What is the maximum possible mechanical power of the motor $P_{\max}$?

Hint: You may want to work in another frame of reference.

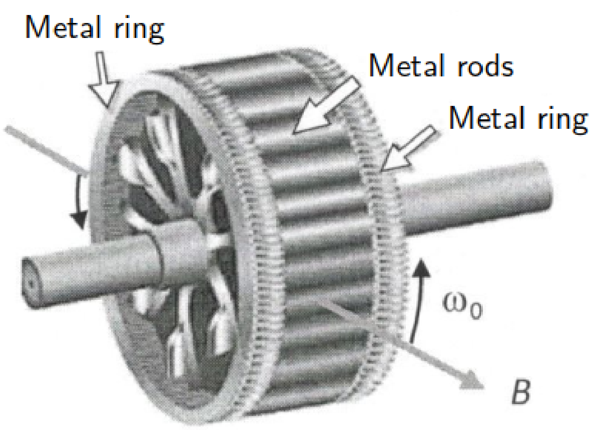


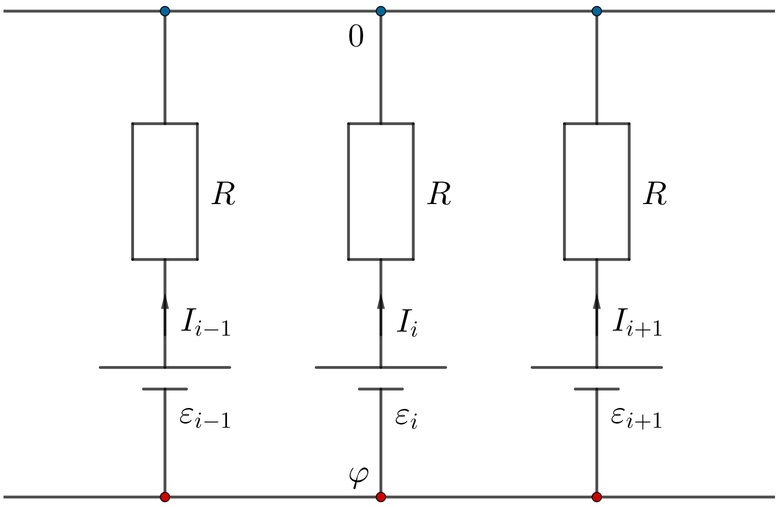
Figure 1

Solution. (a) Between the two rings, the rods have N gaps of length l and width of about $2\pi a/N$ each. As the magnetic field vector rotates, we get an induced voltage for each gap. These voltages are all different, and they'll give rise to some arbitrary current distribution in the motor. However, the gaps are not exactly rectangular, and the rods might be thick, so working with Faraday's law seems like a formidable task.

Let's try something else instead. We switch to a rotating reference frame in which the magnetic field vector stays put. In this frame, the motor rotates with angular velocity ω_0 and so the rods all have a linear velocity $\omega_0 a$. A charge carrier q in a rod with velocity $\mathbf{v}$ experiences a force $\mathbf{f} = q\mathbf{v} \times \mathbf{B}$ along the rod. The velocity vectors of the N rods will all make a different angle with $\mathbf{B}$. Labelling the rods with $0, 1, \ldots, N-1$, the angle for rod i is given by $\alpha_i = \alpha_0 + 2\pi i/N$. Therefore, the force along the rod is $f_i = q\omega_0 aB \sin(\alpha_0 + 2\pi i/N)$. If a charge covers distance l, a work $A_i = f_i l$ will be done on it. But recall now that an electromotive force (EMF) is nothing but work per unit charge. Thus, each rod is associated with an EMF

$$\varepsilon_i = \omega_0 aBl \sin\left(\alpha_0 + \frac{2\pi i}{N}\right).$$

Next we ought to find the current I_i in the rods. The motor essentially reduces to an AC circuit with N sources and N resistors:



The trick here is that all the nodes at the top are equipotential (at zero voltage, without loss of generality). The same holds for all the nodes at the bottom (at some fixed potential φ). Then for any I_i we have $\varepsilon_i - I_i R = \varphi$. To determine the value of φ, we'll also make use of Kirchhoff's first law. The net current into the metal rings has to be zero, because charge can't accumulate. So, after summing the equations, we get $\sum \varepsilon_i = N\varphi$. The sum of voltages can be expressed in terms of the average via $\sum \varepsilon_i = N\langle\varepsilon_i\rangle$, and then $\langle\varepsilon_i\rangle = \varphi$. But the average value of the sine function in ε_i, when sampled uniformly, is just zero. We conclude that $\varphi = 0$ and $I_i = \varepsilon_i/R$.

We'll represent the direction of the rods in space using a vector $\mathbf{l}$, and we'll denote the position vector of each rod with respect to the rotation axis by $\mathbf{a}_i$. The force on each rod is $\mathbf{F}_i = I_i(\mathbf{l} \times \mathbf{B})$, and the torque it gives rise to is $\mathbf{M}_i = \mathbf{a}_i \times \mathbf{F}_i$. From the triple product rule we have $\mathbf{a}_i \times (\mathbf{l} \times \mathbf{B}) = \mathbf{l}(\mathbf{a}_i \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{a}_i \cdot \mathbf{l})} = aB\mathbf{l}\cos(\alpha_i - 90^\circ)$. We see that the torque is along the rotation axis, and it has magnitude

$$M_i = I_i aBl \sin\alpha_i = \frac{\omega_0 (aBl)^2}{R} \sin^2\left(\alpha_0 + \frac{2\pi i}{N}\right).$$

Since α_0 is determined by rotation via $\alpha_0 = \omega_0 t$, the torque is, of course, time-dependent. The time average of $\sin^2 x$ is $1/2$, so rod i contributes an average torque $\langle M_i\rangle = \omega_0 (aBl)^2/2R$. After summing, the total is

$$\boxed{M = \frac{N\omega_0 (aBl)^2}{2R}.}$$

The rotating reference frame doesn't accelerate with respect to the lab frame, so the torque is the same in both frames (here you should think about angular acceleration in terms of $M = I\epsilon$).

(b) The derivation here is very similar. We work in the reference frame which rotates with ω_0. This time, the motor rotates with angular velocity $\omega_0 - \omega$, but everything else will stay the same. Immediately we have

$$\boxed{M = \frac{N(\omega_0 - \omega)(aBl)^2}{2R}.}$$

Working in the lab frame, the mechanical power is given by $P = M\omega$. This is a quadratic function of ω which is maximised at $\omega = \omega_0/2$. In that case,

$$\boxed{P_{\max} = \frac{N(\omega_0 aBl)^2}{8R}.}$$

Short Exam 3

Problem. Bose-Einstein condensate. An ideal monatomic gas ($^4_2$He) composed of bosons is cooled down at constant volume V and constant particle number N. As its temperature decreases, we reach a temperature T_0 below which the properties of the gas arise from the quantum properties of the bosons – i.e. their wavelike nature and their indistinguishability.

(a) Find T_0. The wavelike properties become significant when the de Broglie wavelength at the average thermal energy is approximately equal to the mean distance between the particles. Provide a numerical estimate for the number density $n = N/V$ if $T_0 = 4\,\mathrm{K}$.

At temperatures $T < T_0$ the particles of the gas can be separated into two groups, each encompassing a nonnegligible number of particles. The first group consists of N_0 particles at the lowest energy level ($\varepsilon = 0$), which do not take part in the thermal motion. The second group consists of N^* particles distributed across various energy levels (with $\varepsilon > 0$). These do take part in the thermal motion, and their number is given by

$$N^* = N\left(\frac{T}{T_0}\right)^{3/2}.$$

This is called a degenerate Bose gas.

(b) Find the heat capacity of the gas C_V when $T < T_0$. For this temperature range, find the equation of a reversible adiabatic process in the variables T and V.

(c) Find the pressure of the gas P when $T < T_0$. What is interesting about this result?

Apart from the thermodynamic variables T, V, and N, your results must include the Planck constant h, the Boltzmann constant k_B, and the mass of the Helium atom, $m = 6.7 \times 10^{-27}\,\mathrm{kg}$.

Solution. (a) The average distance between the particles is $d = (1/n)^{1/3} = (V/N)^{1/3}$. Then

$$\left(\frac{V}{N}\right)^{1/3} = \frac{h}{mv} = \frac{h}{m\sqrt{3k_BT_0/m}}$$

This gives us the following expression for T_0:

$$\boxed{T_0 = \frac{h^2}{3k_Bm}\left(\frac{N}{V}\right)^{2/3}.}$$

Meanwhile, the estimate for n is as follows:

$$n = \left(\frac{\sqrt{3mk_BT}}{h} \right)^3 = \boxed{4.0 \times 10^{27}\,\mathrm{m}^{-3}.}$$

(b) The heat input is $\mathrm{d}Q = \mathrm{d}A + \mathrm{d}U$, but at constant volume we have $\mathrm{d}A = P\mathrm{d}V = 0$, so $C_V = \mathrm{d}U/\mathrm{d}T$. We'll assume that all particles outside the ground state, N^* in total, form a classical ideal gas, while those in the ground state contribute zero energy. In this case, $U = \frac{3}{2}N^*k_BT$, so

$$C_V = \frac{\mathrm{d}U}{\mathrm{d}T} = \frac{3}{2}k_B\frac{\mathrm{d}}{\mathrm{d}T}(N^*T) = \boxed{\frac{15}{4}Nk_B\left(\frac{T}{T_0}\right)^{3/2}.}$$

Note that $C_V \to 0$ as $T \to 0$, which is consistent with the third law of thermodynamics.

As for the second question, the adiabatic process obeys $P\mathrm{d}V + \mathrm{d}U = 0$. We'll divide this by $\mathrm{d}T$ because that would allow us to reuse our result for C_V. Also, given that only N^* molecules take part in the thermal motion, we can write down $PV = N^*k_BT$. Thus,

$$N^*k_B\frac{(\mathrm{d}V/V)}{(\mathrm{d}T/T)} + \frac{15}{4}N^*k_B = 0 \quad\Rightarrow\quad \mathrm{d}(\ln V) = -\frac{15}{4}\mathrm{d}(\ln T) \quad\Rightarrow\quad \boxed{TV^{4/15} = \mathrm{const.}}$$

(c) We see that

$$P = \frac{N^*k_BT}{V} = \frac{Nk_BT^{5/2}}{VT_0^{3/2}} = \boxed{3^{3/2}\frac{m^{3/2}}{h^3}(k_BT)^{5/2}.}$$

Rather surprisingly, this depends only on T. The exact result for a low-temperature ideal Bose gas involves the Riemann zeta function:

$$P = (2\pi)^{2/3}\zeta(5/2)\frac{m^{3/2}}{h^3}(k_BT)^{5/2}.$$

The prefactor in our simple model evaluates to 5.20, while the real prefactor is 4.57. Not bad!

Theoretical Exam

Problem 1. Oscillations. Two bodies, each of mass $m = 100\,\mathrm{g}$, are attached to the two ends of a spring with relaxed length $l_0 = 5.00\,\mathrm{cm}$ and spring constant $k = 100\,\mathrm{N/m}$. The bodies are placed on a horizontal surface, where their coefficient of friction with the surface is $\mu = 1.00$. Initially Body 2 is at rest and the spring is relaxed. Body 1 is imparted a velocity $v_0 = 1.00\,\mathrm{m/s}$ directed towards Body 2.

(a) Find the maximum deformation of the spring Δl during the subsequent motion of the system.

(b) Find the displacement x_2 of Body 2 between its initial position and its position at the instant of maximum spring deformation.

Work with $g = 10\,\mathrm{m/s^2}$.

Solution. (a) Initially, only Body 1 is in motion. Body 2 will start moving when the elastic force from the spring overcomes the maximum static friction $F_{\max} = \mu mg$. This happens when the spring has contracted by $\Delta x = \mu mg/k = 0.01\,\mathrm{m}$. To find the velocity $v_{1\mathrm{a}}$ of Body 1 at that instant, we'll use energy conservation:

$$\frac{mv_0^2}{2} - \mu mg\Delta x = \frac{mv_{1\mathrm{a}}^2}{2} + \frac{k\Delta x^2}{2} \quad\Rightarrow\quad v_{1\mathrm{a}} = \sqrt{v_0^2 - \frac{3m}{k}(\mu g)^2} = 0.837\,\mathrm{m/s}.$$

We'll set this as time $t = 0$. Let's look at the motion that follows. Denote the displacements of the two bodies from their initial positions by x_1 and x_2. We'll also write their velocities as v_1 and v_2. We've already established that $x_1(0) = 0.01\,\mathrm{m}$, $x_2(0) = 0$, $v_1(0) = 0.837\,\mathrm{m/s}$, and $v_2(0) = 0$. At the start, both bodies will move to the right, so both friction forces are directed to the left. Then we can write

$$m\ddot{x}_1 = k(x_2 - x_1) - \mu mg,$$
$$m\ddot{x}_2 = -k(x_2 - x_1) - \mu mg.$$

We subtract these equations to get the following dependence for the spring extension $x_2 - x_1$:

$$m(\ddot{x}_2 - \ddot{x}_1) = -2k(x_2 - x_1).$$

The general solution of this differential equation is

$$x_2 - x_1 = A\cos(\omega t) + B\sin(\omega t), \quad \text{where } \omega = \sqrt{\frac{2k}{m}} = 44.72\,\mathrm{rad/s}.$$

We can find the constants A and B from the initial conditions. From $x_1(0)$ and $x_2(0)$ we get $A = -\Delta x = -0.01\,\mathrm{m}$, and after differentiating, $v_1(0)$ and $v_2(0)$ will give us $B\omega = -v_{1\mathrm{a}}$, or $B = -0.0187\,\mathrm{m}$. This dependence for x_1 and x_2 holds as long as both v_1 and v_2 are positive.

Thinking about the spring, we know that v_1 is initially bound to decrease, while v_2 will increase. When they are equal, the spring will reach its maximum compression. We'll thus set

$$v_2 - v_1 = \omega\Delta x\sin(\omega t) - v_{1\mathrm{a}}\cos(\omega t) = 0.$$

And so the time at maximum compression is

$$t_a = \frac{1}{\omega}\arctan\left(\frac{v_{1\mathrm{a}}}{\omega\Delta x}\right) = 0.02415\,\mathrm{s}.$$

We substitute t_a into the equation for the displacement, and we get a compression

$$\Delta l = |x_2 - x_1| = \boxed{0.0212\,\mathrm{m}.}$$

(b) To find x_2, we'll need one more thing. After adding the equations of motion for the two bodies, we reach

$$\ddot{x}_1 + \ddot{x}_2 = -2\mu g.$$

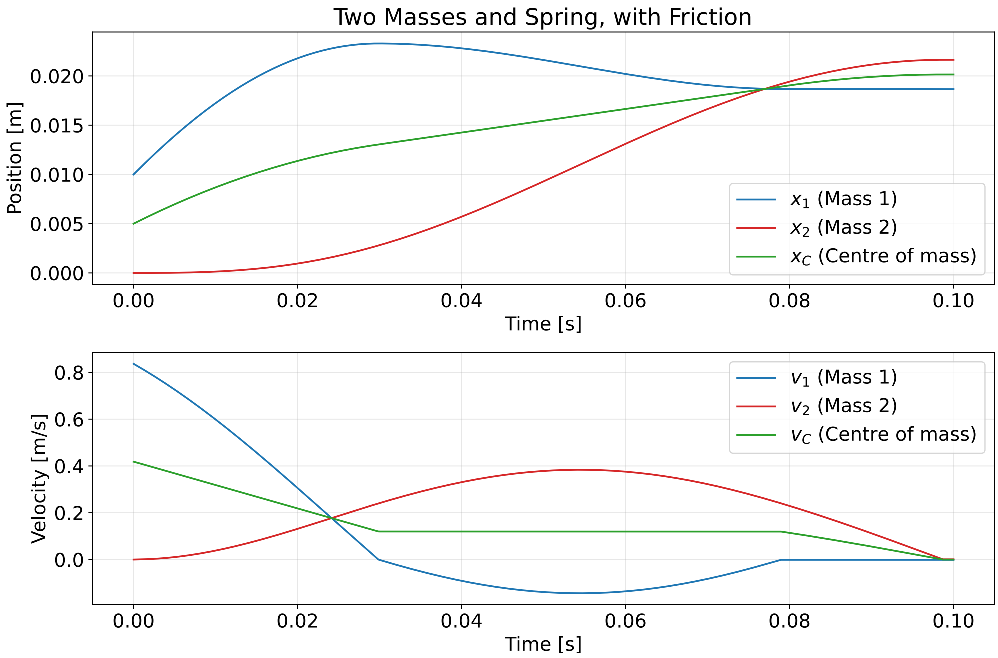
This is the same as stating that the centre of mass has acceleration $-\mu g$. Anyway, now we can integrate this and apply the initial conditions:

$$x_1 + x_2 = \Delta x + v_{1\mathrm{a}}t - (\mu g)t^2.$$

At time t_a, the result is $x_1 + x_2 = 0.0244\,\mathrm{m}$. Previously we got $x_2 - x_1 = -0.0212\,\mathrm{m}$. So, the answer is $x_2 = \boxed{0.0016\,\mathrm{m}.}$

Unfortunately, we're not *really* done. We also need to show calculations for all further instances of extremal spring deformation, because it might happen that one of those exceeds our Δl in magnitude. To make a start, let's find the moment when v_1 turns to zero. From the above equations, this corresponds to

$$v_{1\mathrm{a}}(1 + \cos(\omega t)) - \omega\Delta x\sin(\omega t) - \mu gt = 0.$$



This is to be solved numerically, and the result is $t_b = 0.0299\,\mathrm{s}$. After that point, the friction force on Body 1 points to the right, so the net friction force on the system is zero, i.e. the centre of mass moves with constant velocity $v_C(t_b) = v_2(t_b)/2 = 0.120\,\mathrm{m/s}$. The spring extension at time t_b is $x_2(t_b) - x_1(t_b) = -0.0205\,\mathrm{m}$. The equation for the extension will change to

$$m(\ddot{x}_2 - \ddot{x}_1) = -2k(x_2 - x_1) - 2\mu mg.$$

And now we do more of the same. The general solution is

$$x_2 - x_1 = -\frac{\mu mg}{k} + C\cos(\omega t) + D\sin(\omega t).$$

This time, the boundary conditions at t_b yield $C = -0.0077\,\mathrm{m}$ and $D = -0.0090\,\mathrm{m}$. This regime terminates when either $v_1 = 0$ or $v_2 = 0$. Since we know v_C and we have an expression for $v_2 - v_1$, this is easy to check. It turns out that $v_1 = 0$ comes first, at $t_c = 0.0790\,\mathrm{s}$. Since $v_C = \mathrm{const}$, we know that $v_2(t_c) = v_2(t_b) = 0.240\,\mathrm{m/s}$.

At t_c the deformation of the spring is $x_2(t_c) - x_1(t_c) = 0.0005\,\mathrm{m}$. This is less than Δx, so from now on Body 1 will stay put. We also note that everywhere between t_b and t_c, we have $v_2 - v_1 > 0$, meaning that the spring deformation will smoothly change from $-0.0205\,\mathrm{m}$ to $0.0005\,\mathrm{m}$, never exceeding Δl in magnitude.

Now we've reached a situation where Body 1 is at rest, while Body 2 is still moving to the right. We ask the following question: will Body 2 be stopped by the friction before the spring extends enough to provoke Body 1 into moving? To answer this, let's find the extension in the final state (at time t_d), assuming that Body 1 stays put. Using energy conservation,

$$\frac{mv_2(t_c)^2}{2} + \frac{k[x_2(t_c) - x_1(t_c)]^2}{2} - \mu mg[(x_2(t_d) - x_1(t_d)) - (x_2(t_c) - x_1(t_c))] = \frac{k[x_2(t_d) - x_1(t_d)]^2}{2}.$$

Hence $x_2(t_d) - x_1(t_d) = 0.0030\,\mathrm{m}$. Indeed, this is less than Δx, so our assumption was justified. With this, the whole system is at rest, so we've finally covered everything.

Problem 2. Spray. A long horizontal cylindrical pipe of length l and diameter $d \ll l$ rotates with an angular velocity ω about a vertical axis passing through one of its ends. The pipe contains some ideal incompressible fluid of density ρ which forms a column of length h due to the rotation. There are small holes at both ends of the pipe.

(a) Find a formula for the velocity with which the fluid exits the pipe. Do not account for gravity in this part of the problem.

(b) Calculate the distance from the axis at which the stream strikes the floor. The pipe is 2 m long and the fluid column is 1 m long. The pipe rotates with a period $T = 0.5\,\mathrm{s}$ at a height $H = 2.0\,\mathrm{m}$ above the floor. Neglect air drag.

Solution. (a) The point here is to apply Bernoulli's principle. This requires us to switch to a reference frame which rotates along with the pipe, because the fluid flow has to be steady. The holes at the end of the rotating pipe are small, so what happens is that the water column is nearly static, with the water rapidly accelerating near the holes. In the rotating frame, a block of water Δm at distance r from the rotation axis will experience a centrifugal force $F = \Delta m\omega^2 r$, which must be balanced by a pressure difference in the steady state:

$$\mathrm{d}p\, S = (\rho S \mathrm{d}r)\, \omega^2 r.$$

We can integrate this to find the pressure $p(l)$ right before the water approaches the holes:

$$\int_{p(l-h)}^{p(l)} \mathrm{d}p = \rho\omega^2 \int_{l-h}^{l} r\, \mathrm{d}r.$$

The pressure $p(l-h)$ is equal to the atmospheric pressure p_0, because the water there is in contact with the air. Thus,

$$p(l) = p_0 + \frac{\rho\omega^2}{2}\left(l^2 - (l-h)^2\right).$$

Now we apply Bernoulli's principle for a streamline on both sides of the holes to find the exit velocity u, noting that the water which spews out is again in contact with the atmosphere:

$$p(l) = p_0 + \frac{\rho u^2}{2} \quad \Rightarrow \quad u^2 = \omega^2(2l-h)h.$$

We're still not done, because this radial velocity in the rotating frame must be added to the tangential velocity of the pipe in the lab frame:

$$v^2 = u^2 + (\omega l)^2 \quad \Rightarrow \quad \boxed{v = \omega\sqrt{l^2 + 2lh - h^2}.}$$

(b) We're told that $l = 2\,\mathrm{m}$, $h = 1\,\mathrm{m}$, while $\omega = 2\pi/T = 12.6\,\mathrm{rad/s}$. The water takes time $t = \sqrt{2H/g} = 0.64\,\mathrm{s}$ to strike the floor. In that time it has moved by ut radially and $(\omega l)t$ to the side. The total distance from the axis is therefore

$$x = \sqrt{(l+ut)^2 + (\omega l t)^2} = \boxed{22.7\,\mathrm{m}.}$$

Problem 3. Car suspension. The front axle and the rear axle of a car are at a distance $l = 2.0\,\mathrm{m}$ from each other. The centre of mass of the car is at a distance $h = 0.4\,\mathrm{m}$ above the ground, and it is located midway between the front and the rear wheels. When the car is at rest, the suspension spring of each of the wheels is compressed by $\Delta = 10\,\mathrm{cm}$ with respect to its relaxed length. As the car moves, the driver steps on the brakes and the wheels of the car start slipping along the road. The coefficient of friction between the tyres and the road is $\mu = 1.0$. Calculate the angle at which the car's body would tilt with respect to the horizon. The mass of the wheels is negligible.

Solution. This is Problem 2.25 from MIPT, Volume 1. We'll denote the mass of the car by m, and the stiffness of each spring by k. When the car is at rest, we have $mg = 4k\Delta$. But when the car is braking, the rear springs experience compression Δ_r which is less than the compression Δ_f of the front springs. We'll make the assumption that the springs stay vertical. Then, the tilt of the car is given by

$$\tan\alpha = \frac{\Delta_f - \Delta_r}{l}.$$

When the car is braking, the body of the car has some linear acceleration a, but it doesn't rotate (well, otherwise the question wouldn't make sense). We'll switch to the centre-of-mass frame, which gives rise to a horizontal inertial force ma applied at the CM. We'd rather not deal with it, so we choose to take torques about the CM.

The total normal forces at the rear and the front wheels are $2k\Delta_r$ and $2k\Delta_f$, respectively. From balancing with gravity, it follows that $(\Delta_r + \Delta_f)/2 = \Delta$, meaning that the extra compression of one set of springs is minus that of the other. Consequently, the distance from the CM to the ground stays the same. Then, the total moment from the friction forces is $2\mu k(\Delta_r + \Delta_f)h = 4\mu k\Delta h$. Conversely, the normal forces cause a moment $2k(\Delta_f - \Delta_r)(l/2)$ in the other direction. We equate these to find $\Delta_f - \Delta_r = (4\mu h/l)\Delta$. Thus $\tan\alpha = 4\mu h\Delta/l^2$, or $\boxed{\alpha = 2.3^\circ.}$

Problem 4. Field strength. An infinite sheet is given a uniform charge density σ. A hole of radius a is cut out from the sheet (Figure 2). Find expressions for the electric field at:

(a) A point A lying on the axis of the hole, at a distance z away from the sheet.

(b) A point B lying in the plane of the sheet, at a small distance r ($r \ll a$) away from the centre of the hole.

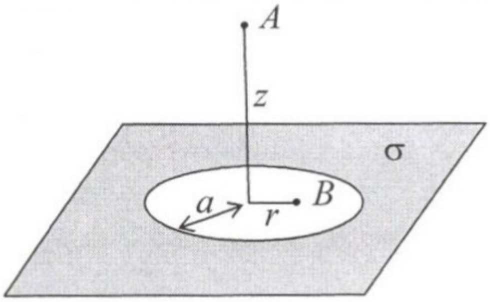


Figure 2

Solution. (a) The configuration is equivalent to a superposition of a complete infinite sheet with charge density σ and a disk of radius a which has a charge density $-\sigma$. From the sheet we have $\mathbf{E}_1 = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}$, and now we want to find the field $\mathbf{E}_2$ from the disk. It's easiest to do this using a trick. Viewed from A, the disk subtends the same solid angle as a spherical cap centered at A and delimited by that disk. Using a well-known formula, this spherical cap corresponds to a solid angle $\Omega = 2\pi(1 - \cos\theta)$, where $\cos\theta = z/(z^2 + a^2)^{1/2}$.

Now, consider a tiny piece of the disk with an area $\mathrm{d}S$ which lies at an angle α from the vertical, such that the distance to it is $x = a/(\cos\alpha)$. The field from this piece is $\mathrm{d}E = k(-\sigma)\mathrm{d}S/x^2$, with a vertical projection of $\mathrm{d}E_z = k(-\sigma)\mathrm{d}S\cos\alpha/x^2$. We now notice that $\mathrm{d}S\cos\alpha/x^2$ is precisely the solid angle $\mathrm{d}\Omega$ which the piece subtends as viewed from A. And after summation, the total field turns out to be $E_z = k(-\sigma)\Omega = -2\pi k\sigma(1-\cos\theta)$, so $\mathbf{E}_2 = -\frac{\sigma}{2\varepsilon_0}(1-\cos\theta)\hat{\mathbf{z}}$. We conclude that

$$\mathbf{E}_A = \mathbf{E}_1 + \mathbf{E}_2 = \boxed{\frac{\sigma}{2\varepsilon_0}\frac{z}{\sqrt{z^2+a^2}}\hat{\mathbf{z}}.}$$

Notice that this checks out in the special cases $a = 0$ and $a \to \infty$.

(b) As an initial observation, note that $\mathbf{E}_B$ points inwards. Now, let's place point A very close to the origin, at $z \ll r$. In the limit of small z, we have $\mathbf{E}_A = \frac{\sigma}{2\varepsilon_0}\frac{z}{a}\hat{\mathbf{z}}$. We'll apply Gauss's law for a cylinder of radius r and height $2z$, such that A is on the upper base and B is on the rim. This is rather devious. Everywhere along the rim, the radial component of the field is equal to E_B. Likewise, the axial field on the bases is equal to E_A (keep in mind that $r \ll a$). The charge inside the cylinder is zero, and so the total flux of the electric field is also zero:

$$2E_A(\pi r^2) - E_B(2\pi r\cdot 2z) = 0 \quad \Rightarrow \quad E_B = \frac{r}{2z}E_A = \frac{\sigma}{2\varepsilon_0}\frac{r}{2a}.$$

Let's write the radial unit vector as $\hat{\mathbf{r}}$. Then

$$\mathbf{E}_B = \boxed{-\frac{\sigma r}{4\varepsilon_0 a}\hat{\mathbf{r}}.}$$

The trick we used here might be familiar from IPhO 2024, Problem 2A-1.

Problem 5. Accelerating ring. A copper ring of cross-section $S = 1\,\mathrm{mm}^2$ and radius $r = 5\,\mathrm{cm}$ starts rotating around its axis with a constant angular acceleration $\alpha = 1000\,\mathrm{rad/s^2}$. Find the magnetic field B at the centre of the ring. The resistivity of copper is $\rho = 1.68\times10^{-8}\,\Omega\mathrm{m}$. *Hint:* You may want to work in the reference frame of the rotating ring. What is the force that gives rise to an EMF in the ring?

Solution. This problem covers the Stewart-Tolman effect, which is also described in Problem 18 from Kevin Zhou's Handout ERev (i.e. MPPP 173). The reason why a current appears in the ring is that the electrons will lag behind the lattice ions by a little, despite the mutual interactions that try to keep them together. This will become clearer if we work in the reference frame of the lattice. So, let's look at the ring at some instant when its angular velocity is ω, and switch to the frame that rotates with ω and accelerates with $\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}$.

In this case, we'll get some inertial forces. The full expression for the force experienced by an object of mass m is

$$\mathbf{F}' = m\mathbf{a}' = \mathbf{F} - m\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\mathbf{r}' - 2m\boldsymbol{\omega}\times\mathbf{v}' - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'),$$

where $\mathbf{F}$ covers the actual physical forces, the second term is the Euler force, the third term is the Coriolis force, and the fourth term is the centrifugal force. Here $\mathbf{r}'$ is measured from the origin, which is somewhere on the rotation axis. This formula is difficult to remember, but the terms can be simplified. Forget about the origin, and denote by ρ the distance from the rotation axis to our object. We'll write the respective unit vector as $\hat{\boldsymbol{\rho}}$, and we'll use $\hat{\boldsymbol{\phi}}$ for the tangential unit vector. After applying the triple product rule, we'll find for the centrifugal term

$$\mathbf{F}_\mathrm{R} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') = (m\omega^2\rho)\hat{\boldsymbol{\rho}}.$$

The Coriolis term is somewhat nicer without the minus:

$$\mathbf{F}_\mathrm{C} = 2m\mathbf{v}'\times\boldsymbol{\omega}.$$

For some reason I always forget whether the $\mathbf{v}'$ or the $\boldsymbol{\omega}$ comes first. In such cases I find it useful to work with mnemonics. For example, I associate the above formula with the phrase "**to view**" (and the m is obvious, because it's a force).

The Euler term reduces to

$$\mathbf{F}_{\mathrm{E}} = -(m\rho\alpha)\hat{\boldsymbol{\phi}}.$$

Considering that $\rho\alpha$ is just the linear acceleration of the object, this force is actually rather familiar. It's the same thing as the inertial force $-ma$ that pushes you against your seat when you're accelerating in a car. You should just imagine that the rectilinear motion of the car is part of a turn with a very large radius ρ and a very small angular acceleration α.

Back to our problem. Here, the centrifugal and the Coriolis forces are both radial, while the Euler force is tangential. This means that if an electron does a lap around the ring, the Euler force will do work on it equal to

$$A = \oint \mathbf{F}_{\mathrm{E}} \cdot \mathrm{d}\mathbf{r} = (-m\alpha r)(2\pi r).$$

But this can be interpreted as an electromotive force $\varepsilon = A/(-e)$. So, we have a current

$$I = \frac{\varepsilon}{R} = \frac{AS}{(-e)2\pi r\rho} = \frac{m\alpha r S}{e\rho}.$$

The magnetic field in the centre of the ring is then

$$B = \frac{\mu_0 I}{2r} = \boxed{\frac{\mu_0 m\alpha S}{2e\rho} = 2.1 \times 10^{-13}\,\mathrm{T}.}$$

This remains the same in the lab frame, because the centre of the ring is motionless in both frames. The reason is that the Lorentz transformations for the EM field are local, meaning that they depend only on the field values *right here*. And in this problem we're talking about a point where the velocity is zero, so we don't expect any changes.

Problem 6. Lens and plate. A source of monochromatic light (of wavelength $\lambda = 532\,\mathrm{nm}$) is placed at a distance $a = 4\,\mathrm{cm}$ from a lens of radius $R = 2\,\mathrm{cm}$ and focal length $f = 3\,\mathrm{cm}$ (Figure 3). A screen is placed in the focal plane on the other side of the lens.

(a) Find the radius r of the illuminated spot on the screen. You can neglect the diffraction from the rim of the lens.

(b) A thin plate of thickness $d = 20\,\mu\mathrm{m}$ and refractive index $n = 1.5$ is put between the source and the lens, perpendicularly to the optical axis of the lens (Figure 4). The illuminated spot on the screen turns into alternating concentric bright and dark rings. Find the number of bright rings N.

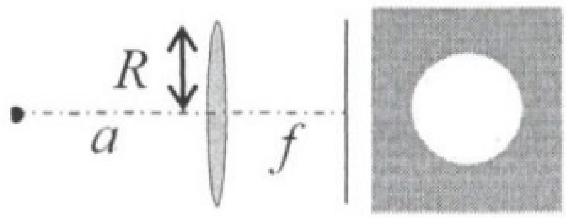


Figure 3

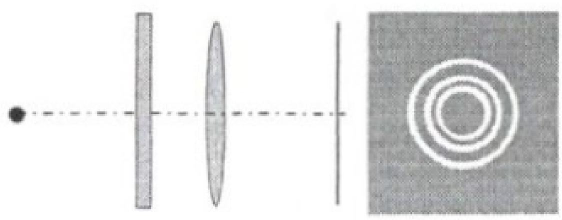


Figure 4

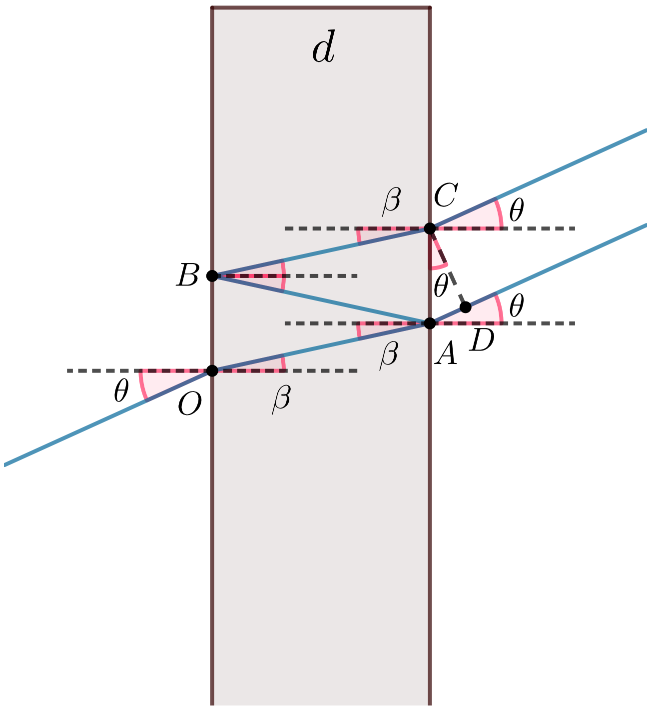
Solution. (a) Consider a ray which makes an angle θ with the optical axis before reaching the lens. Because the screen lies in the focal plane of the lens, our ray will strike the same point on the screen as another imaginary ray which is parallel to ours, but incident at the centre of the lens. It's evident then that the rim of the illuminated spot corresponds to the rays with the largest θ, i.e. those for which $\tan\theta = R/a$. Using the construction outlined above, we get a pair of similar triangles, such that $r/f = R/a$. So $\boxed{r = fR/a = 1.5\,\text{cm}.}$

(b) At first glance this is puzzling. The rays which pass through the thin plate don't change their direction; they only get a tiny translation on the order of d. Still, there must be some interference at play here. For any point on the screen, we need to identify a set of rays which all land there and interfere. There's a lens along the path, so these rays just have to be parallel. Now here's the actual explanation. The thin plate gives us multiple rays for any given θ because the incident light undergoes multiple reflections inside it. We'll work only with the first and the second transmitted rays, as shown on the diagram. They have an optical path difference of

$$\Delta s = s_{ABC} - s_{AD} = n(AB + BC) - AC\sin\theta = \frac{2nd}{\cos\beta} - 2d\tan\beta\sin\theta.$$

We have $\sin\beta = \sin\theta/n$ by Snell's law. Our expression then simplifies as follows:

$$\Delta s = \frac{2d}{n\cos\beta}(n^2 - \sin^2\theta) = 2d\sqrt{n^2 - \sin^2\theta}.$$



Let's measure this in wavelengths, as in $\Delta s = k\lambda$. For $\theta = 0$ we get $k_{\max} = 112.8$, and for $\theta = \arctan(R/a)$ at the rim of the lens we have $k_{\min} = 107.6$. There's constructive interference for those θ for which k is an integer. So, we'll get bright rings for the integers from 108 to 112. That's $\boxed{5}$ rings in total.

Problem 7. Relativistic force. A particle of rest mass m starts moving under a constant force F.

(a) Find the total distance covered by the particle in the lab frame until it reaches a velocity $v = 0.8c$, where c is the speed of light in vacuum.

(b) Find the proper time taken for the particle to reach a velocity $v = 0.8c$.

Hint: $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{1+x^2}\right) + C$

Solution. (a) We will set $c = 1$ to reduce the writing load, and we'll restore the c factors at the end via dimensional analysis. We start with $F = \frac{\mathrm{d}p}{\mathrm{d}t}$. We multiply both sides of $\mathrm{d}p = F\mathrm{d}t$ by v, and we see that the distance covered is

$$x = \int \mathrm{d}x = \frac{1}{F}\int v\,\mathrm{d}p = \frac{1}{F}\left((vp)\Big|_0^{0.8} - \int_0^{0.8} p\,\mathrm{d}v\right) = \frac{m}{F}\left(\frac{v^2}{\sqrt{1-v^2}}\Bigg|_0^{0.8} - \int_0^{0.8}\frac{v}{\sqrt{1-v^2}}\,\mathrm{d}v\right).$$

After evaluating the integral using a substitution, we reach

$$x = \frac{m}{F}\left(\frac{v^2}{\sqrt{1-v^2}} - \sqrt{1-v^2}\right)\Bigg|_0^{0.8} = \frac{m}{F}\left(\frac{1}{\sqrt{1-v^2}} - 1\right) = \frac{2m}{3F} \Leftrightarrow \boxed{\frac{2mc^2}{3F}}.$$

(b) To count the total proper time, we need to sum all the little proper time intervals $\mathrm{d}\tau = \mathrm{d}t/\gamma$. This means we need to compute

$$\tau = \int \sqrt{1-v^2}\,\mathrm{d}t.$$

This isn't as bad as it looks, because we can find the dependence $v(t)$ from $\mathrm{d}p = F\mathrm{d}t$. After integrating, we have $v/\sqrt{1-v^2} = Ft/m$, which is the same as $1 - v^2 = \frac{1}{1+(Ft/m)^2}$. So:

$$\tau = \frac{m}{F}\int_0^{Ft_0/m} \frac{1}{\sqrt{1+(Ft/m)^2}}\,\mathrm{d}(Ft/m),$$

where t_0 is the time taken to reach $v = 0.8$. Using the dependence above, we find $t_0 = 4m/3F$, and therefore

$$\tau = \frac{m}{F}\int_0^{4/3}\frac{1}{\sqrt{1+u^2}}\,\mathrm{d}u = \frac{m}{F}\ln\left(u + \sqrt{1+u^2}\right)\Bigg|_0^{4/3} = \frac{m}{F}(\ln 3 - \ln 1) \Leftrightarrow \boxed{\frac{mc}{F}\ln 3.}$$

Problem 8. Fast and slow. A mole of ideal gas is put in a vertical cylinder under a light freely moving piston. The pressure of the gas is p_0 and its temperature is T_0. Compare the final temperatures T_1 and T_2 of the gas at the end of the following processes:

1. The external pressure increases (or decreases) from p_0 to p instantaneously.
2. The external pressure increases (or decreases) from p_0 to p slowly.

The gas is thermally insulated from the surroundings and it has an adiabatic index of γ.

Solution. This is a repeat of Problem 2B from the 2003 Bulgarian National Round (which has a wrong solution). You can also find this one in many Russian books. It's Problem 1.71 in MIPT, Volume 1 – but the solution there is also wrong. We'll write $p/p_0 \equiv \kappa$, and we'll denote $\nu = 1\,\mathrm{mol}$.

Let's start with T_1. After the outside pressure is changed from p_0 to p, the gas will expand violently. Eventually it has to reach an equilibrium with the surroundings, and its pressure then must be p. This is a nonquasistatic process, and we know nothing about it except for the parameters in the initial and the final state. Thus, we're only allowed to use the first law of thermodynamics. There's no external heat input ($Q_{\text{in}} = 0$), or in other words, the process is adiabatic:

$$A_{\text{gas}} + \Delta U = -A_{\text{surroundings}} + \nu C_v(T_1 - T_0) = p(V_{\text{final}} - V_{\text{initial}}) + \frac{1}{\gamma - 1}\nu R(T_1 - T_0) = 0.$$

Next, we have $p_0 V_{\text{initial}} = \nu R T_0$ and $pV_{\text{final}} = \nu R T_1$. We reach

$$(T_1 - \kappa T_0) + \frac{1}{\gamma - 1}(T_1 - T_0) = 0 \quad \Rightarrow \quad T_1 = \frac{\kappa(\gamma-1)+1}{\gamma} T_0 = \left(1 + \frac{\gamma-1}{\gamma}\frac{(p-p_0)}{p_0}\right) T_0.$$

Now for T_2. This process is quasistatic and adiabatic, so the expansion of the gas will obey $pV^\gamma = \text{const}$. Using the ideal gas equation, we see that this is the same as $p^{\frac{1-\gamma}{\gamma}} T = \text{const}$. It follows that

$$T_2 = \kappa^{\frac{\gamma-1}{\gamma}} T_0 = \left(1 + \frac{p-p_0}{p_0}\right)^{\frac{\gamma-1}{\gamma}} T_0.$$

We'll introduce $x \equiv \frac{p-p_0}{p_0}$ and $a \equiv \frac{\gamma-1}{\gamma}$, noting that $x > -1$ and $a \in (0,1)$. The expressions for the temperature are then $T_1 = (1 + ax)$ and $T_2 = (1+x)^a$. They are equal when $x = 0$. To compare them for all x, we'll need to look at their derivatives:

$$\frac{\mathrm{d}T_1}{\mathrm{d}x} = a, \qquad \frac{\mathrm{d}T_2}{\mathrm{d}x} = \frac{a}{(1+x)^{1-a}}.$$

Since $1-a$ is a positive number, the derivative of T_2 will be smaller than that of T_1 for $x > 0$, and larger than that of T_1 for $x < 0$. This means that the graph of $T_1(x)$ always lies above that of $T_2(x)$, even though they touch at $x = 0$. Thus $\boxed{T_1 \geq T_2}$ (always). In case you're curious, we just proved a special case of the so-called Bernoulli inequality.

Problem 9. Click-clack. A monatomic ideal gas is subjected to a process where the number of collisons Z between the atoms per unit volume per unit time remains constant.

(a) Find how Z depends on the pressure of the gas p and the temperature of the gas T.

(b) Find the equation of the process in terms of p and V.

(c) Find the molar heat capacity of the process C.

Solution. (a) Each atom has a characterisic cross-section σ. If the centre of any other atom is incident on this cross-section, then this counts as a collision. We'll look at the rate of collisions experienced by a single atom, working in its rest frame. Imagine that the quadratic mean velocity of the atoms is v in the lab frame. In the rest frame, however, we have

$$\left\langle \mathbf{v}_{\text{relative}}^2 \right\rangle = \left\langle (\mathbf{v}_1 - \mathbf{v}_2)^2 \right\rangle = \left\langle \mathbf{v}_1^2 + \mathbf{v}_2^2 - 2\mathbf{v}_1 \cdot \mathbf{v}_2 \right\rangle = 2\left\langle \mathbf{v}_1^2 \right\rangle,$$

because $\mathbf{v}_1$ and $\mathbf{v}_2$ are uncorrelated. We see that in the rest frame, the RMS speed is $\sqrt{2}v$ instead of just v. This multiplier $\sqrt{2}$ will also carry over when considering the change of the arithmetic mean velocity $\bar{v}$.

Our atom is bombarded by other atoms with arithmetic mean velocity $\sqrt{2}\bar{v}$ from all directions. However, since it offers the same cross-section σ no matter what the angle of incidence is, nothing will change in terms of the collision rate if we imagine that all the atoms are flying in from one fixed direction. And so the number of collisions in time $\mathrm{d}t$ can be found from counting the atoms contained within a cylinder of base area σ:

$$\mathrm{d}N = \sqrt{2} n \bar{v} \sigma \, \mathrm{d}t.$$

Here n is the number density of the atoms in the gas. Note that by using the arithmetic mean velocity, we've applied appropriate averaging for the fact that atoms of different speeds will fly in at different rates.

Now we know that one atom experiences $\sqrt{2} n \bar{v} \sigma$ collisions per unit time. But there are n atoms per unit volume, and collisions happen in pairs (divide by two!). Hence $Z = n^2\bar{v}\sigma/\sqrt{2}$. Next we use that $n = p/k_BT$ and $\bar{v} = \sqrt{8k_BT/\pi m}$. This yields

$$Z = \left(\frac{2\sigma}{\sqrt{\pi k_B^3 m}}\right)\left(\frac{p^2}{\sqrt{T^3}}\right) \sim \boxed{p^2T^{-3/2}}.$$

The examiner doesn't care about the prefactors, just the p and T dependence, so we could've been less fussy.

(b) Using $p^2T^{-3/2} = \text{const}$ and $pV/T = \text{const}$, we find $\boxed{pV^{-3} = \text{const.}}$

(c) We have $C = \mathrm{d}Q/\nu\mathrm{d}T$, where ν is the number of moles. From $\mathrm{d}Q = \mathrm{d}U + p\mathrm{d}V$ and $\mathrm{d}U = \nu C_v\mathrm{d}T$ we get

$$C = C_v + \frac{p\mathrm{d}V}{\nu\mathrm{d}T} = \frac{3}{2}R + \frac{p\mathrm{d}V}{\nu\mathrm{d}T}.$$

To obtain $p\mathrm{d}V$, we need to look at the equation of the process. We take its logarithm and differentiate, yielding

$$\frac{\mathrm{d}p}{p} - 3\frac{\mathrm{d}V}{V} = 0 \quad\Rightarrow\quad 3p\mathrm{d}V = V\mathrm{d}p.$$

This enables us to find $\mathrm{d}(pV) = p\mathrm{d}V + V\mathrm{d}p = 4p\mathrm{d}V$. But $\mathrm{d}(pV) = \nu R\mathrm{d}T$, and so

$$C = \frac{3}{2}R + \frac{1}{4}\frac{\nu R\mathrm{d}T}{\nu\mathrm{d}T} = \frac{3}{2}R + \frac{1}{4}R = \boxed{\frac{7}{4}R.}$$

Problem 10. Electron beam. Using Heisenberg's uncertainty principle, estimate the minimum diameter d of the spot which an electron beam makes on a screen, given that the electrons take $\tau = 10^{-8}\,\text{s}$ to get from the collimator (a circular opening) to the screen.

Solution. This is Problem 2.29 from MIPT, Volume 3. We'll assume that the collimator has a diameter D. In a world with no diffraction, we'd have $d = D$. Alas, things are not so simple. We'll denote the distance to the screen by l. The electrons will leave the collimator with some fixed momentum p_y along its axis, and also a small orthogonal component p_x that can be found from the uncertainty principle, $p_xD \sim \hbar$. The angular spread θ of the beam is then given by

$$\tan\theta = \frac{p_x}{p} = \frac{\hbar}{mvD} = \frac{\hbar\tau}{mlD}.$$

The diameter of the spot is

$$d = D + 2l\tan\theta = D + \frac{2\hbar\tau}{mD}.$$

We want to pick the D which minimises this expression. Using either derivatives or the AM-GM inequality, we find that the best possible choice is $D = \sqrt{2\hbar\tau/m}$. Substituting this back into our result for d, we get

$$d = 2\sqrt{\frac{2\hbar\tau}{m}} \approx \boxed{3\,\mu\text{m}.}$$

Experimental Exam

Problem 1. Measuring the density of irregular-shaped bodies.

Equipment:
Kitchen scale ($m < 500\,\mathrm{g}$!), stand, binder clip, glass cylinder, plastic cup, stopwatch, ruler, tape measure, bottle with 1.5 l of tap water, scissors, funnel, string, graph paper, and the following five bodies:

1. Fishing sinker (grey ball with a channel through the diameter)
2. Hinge from a cupboard (grey, rectangular, with 4 holes)
3. White piece of metal
4. Reddish piece of metal
5. Bouncy ball with a smiley face

Some of the equipment is shown on Figures 5 and 6. Record all measurements in tables. Write down your results in the answer sheet.

Figure 5

Figure 6

Task 1. Measurements with a scale.

(a) Devise a method for measuring the density ρ_b of irregular-shaped bodies (without having to calculate their volume in advance) using the readings of the scale in the following cases:

1. The body is placed directly on the scale (m_s).
2. The body is left at the bottom of a plastic cup filled with water (m_b).
3. The body is tied on a string and is fully submerged in water, but it does not touch the bottom of the cup (m_s).

The density of water is ρ_w. Find the formula $\rho_b = f(\rho_w, m_s, m_b, m_s)$. **(1.0 pt)**

(b) Take enough good measurements and calculate the density of each body. **(4.0 pt)**

Task 2. Measurements with a stopwatch.

Make a simple pendulum of length $l \approx 0.7\,\text{m}$ using the stand, the clip, the sinker, and some string.

(a) Measure the periods of oscillation T_a and T_w when the sinker is in the air and when the sinker is underwater (inside the glass cylinder). **(1.5 pt)**

(b) Assume that the drag forces in both media have no effect on the oscillation periods. Find a formula for the density of the sinker $\rho_\text{b} = f(\rho_\text{w}, T_\text{a}, T_\text{w})$. Calculate this density and compare it to your result in **1(b)**. **(1.0 pt)**

Task 3. Measurements with a stopwatch and a ruler.

(a) Measure the dependence of the amplitude of the pendulum $A(t)$ on time when it oscillates in air and in water, $A_\text{a}(t)$ and $A_\text{w}(t)$ respectively. Assume that this dependence is of the form $A(t) = A(0)e^{-\gamma t}$. Using appropriate plots, find the damping coefficients γ_a and γ_w. **(5.0 pt)**

(b) For a given pendulum, let the period of damped oscillations with coefficient γ be T, and the period of undamped oscillations be T_0. These are related by

$$T_0 = \frac{1}{\sqrt{\frac{1}{T^2} + \left(\frac{\gamma}{2\pi}\right)^2}}.$$

Use the data from **2(a)** and **3(a)** to find $T_{\text{a}0}$ and $T_{\text{w}0}$, the oscillation periods of the pendulum in air and in water if there were no damping. Calculate the density of the body $\rho_{\text{b}0} = f(\rho_\text{w}, T_{\text{a}0}, T_{\text{w}0})$ again. Has your result improved compared to the one you obtained in **2(b)**? **(1.0 pt)**

Task 4. Working with a different model.

(a) The oscillations are exponentially damped only if the drag force is proportional to the velocity, which is typical of low Reynolds numbers, $\text{Re} < 1$. At large Reynolds numbers ($\text{Re} \gg 1$) the drag force is proportional to the square of the velocity instead. Then, assuming weak damping, the angular amplitude of a simple pendulum depends on time as follows:

$$\alpha(t) = \frac{\alpha(0)}{1 + \alpha(0)\delta t}.$$

Using your measurements for oscillations in water from **3(a)**, make an appropriate plot and use it to find δ. **(1.0 pt)**

(b) For quadratic drag $F_\text{dr} = bv^2$ acting on a ball of cross-section S in a medium of density ρ, the coefficient b is given by $b = \frac{1}{2}C\rho S$. The constant C is called a drag coefficient. If b and δ from **4(a)** are related by $\delta = \frac{8}{3}\frac{bl}{mT}$ (l is the length of the string, m is the mass of the bob, T is the oscillation period), find C for your experimental setup. **(0.5 pt)**

Problem 2. Black box.

Equipment:

1. Black box with three numbered terminals (Figure 7)
2. Multimeter ($\times 1$)
3. Stopwatch ($\times 1$)
4. Graph paper ($\times 2$)
5. Ruler, blank paper

The black box contains three components in a Y-connection – a battery, a resistor, and a capacitor, as shown on Figure 8. Each terminal of the black box leads to the free end of a component. You do not have access to the centre of the Y-connection. Initially the capacitor is either completely discharged or left with a voltage less than 0.5 V.

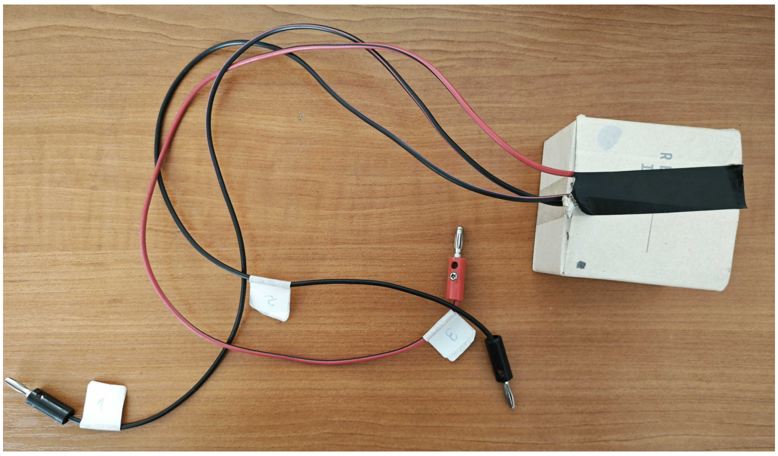


Figure 7

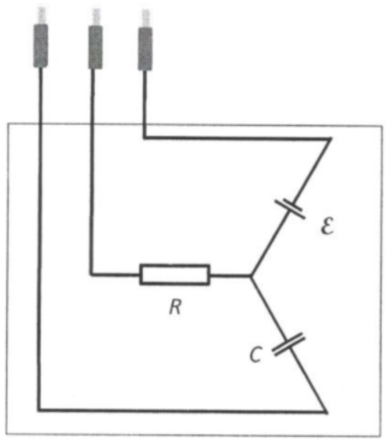


Figure 8

Tasks:

(a) Describe the measurements that you will have to make in order to find out which terminal is connected to which component.

(b) Take the appropriate measurements and sketch the circuit inside the black box, indicating the numbers of the terminals corresponding to each component. Also indicate which end of the battery connects to the terminal.

(c) Write down the relevant theory and take the appropriate measurements so as to find values for:

- the EMF of the battery $\mathcal{E}$
- the capacitance C
- the resistance R.

(d) Estimate the errors in $\mathcal{E}$, C, and R.

You can neglect the internal resistance of the battery and the internal resistance of the multimeter in ammeter mode. The internal resistance of the multimeter in voltmeter mode is not infinitely large and you must treat it as a separate load in the circuit.