Extending Probability Generating Function Semantics to Negative Variable Valuations

Bachelor Colloquium

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12.04.2016

Probabilistic Programs

- Different output on same input
- Probabilistic operations
- Solve hard problems faster
- Mathematical model to prove usefulness

Formal Power Series

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- $\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i X^i = \frac{1}{1 - \frac{1}{2}X}$
- Multivariate FPS: $\sum_{s \in \mathbb{N}^k} \mu_s X_1^{s_1} \cdots X_k^{s_k}$

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- Analogous for multivariate FPS

PGFs and Programs

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- *skip*
- *diverge*
- $X := e$

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- $X := e$
- $P_1; P_2$
- $\mathit{if}(B)\{\ P_1\ \}\ \mathit{else}\{\ P_2\ \}$
- $\{P_1\}[p]\{P_2\}$
- $\mathit{while}(B)\{P\}$

Challenges

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- Usage of closed forms (requires a ring)

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- Solution: Use PGFs for partitions of the state space.

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  $$\vdots$$
  $$\sum_{s \in S_{2k}} \mu_s X_1^{s_1} \cdots X_k^{s_k}$$

Semantic Tuples

Definition (Semantic Tuple)

Let P be a program of k variables. A *semantic tuple* T_G is an object of the form

$$T_G = \left(\sum_{s \in S_1} \mu_s X_1^{s_1} \cdots X_k^{s_k}, \ldots, \sum_{s \in S_{2^k}} \mu_s X_1^{s_1} \cdots X_k^{s_k} \right)$$

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- Number of entries grows exponentially.
- Use extended PGFs: $\sum_{s \in \mathbb{Z}^k} \mu_s X_1^{s_1} \cdots X_k^{s_k}$

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- $\left\langle 0.4 X^1 Y^0 + 0.6 X^4 Y^4 \right\rangle_{X=1} = 0.4 X^1 Y^0$

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- $[\![P_1; P_2]\!]\,(G) = [\![P_2]\!]\,([\![P_1]\!]\,(G))$

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  { X := X - 1 }[0.5]{ F := 1 }
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- $G = 1X^0F^0 = 1$

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- Actually: $\left(0, \frac{F}{2-X^{-1}}, 0, 0\right)$

Termination Probability

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Equivalent Programs

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F := 0;
while ( F = 0 ) {
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- Proven equivalent in distribution for $p = \frac{1}{2}$ and $q = \frac{2}{3}$.
- Expected value of X is the same if $q = \frac{1}{2-p}$. (Gretz et al.)

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  while ( F = 0 ) {
    { X := X + 1 }[p]{ F := 1 }
  }

} else {
  F := 0;
  while ( F = 0 ) {
    X := X − 1;
    { skip }[q]{ F := 1 }
  }
}
```

- Introduced by Kiefer et al. in 2012.
- Proven equivalent in distribution for $p = \frac{1}{2}$ and $q = \frac{2}{3}$.
- Expected value of X is the same if $q = \frac{1}{2-p}$. (Gretz et al.)
- Today: equivalence in distribution if $q = \frac{1}{2-p}$.

Equivalent Programs

- First program's semantics:

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- Equivalent in distribution if $q = \frac{1}{2-p}$.

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- Comparison of PGF and wp semantics (McIver and Morgan)

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Theorem

Given a program P, a measure μ and an expectation f, the following holds:

$$\mathbb{E}_{[\![P]\!](\mu)}(f) = \mathbb{E}_{\mu}(\mathsf{wp}\,(P, f))$$

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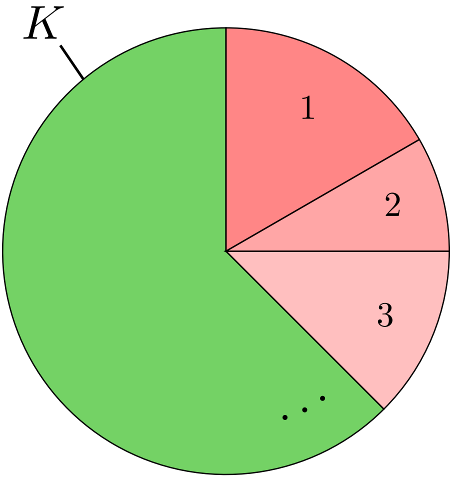

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Definition (Weak Bisimulation)

Let R be a binary relation. If $(K, G) \in R$, then

Let $G' = \mathrm{wh}_{B,P}(G)$

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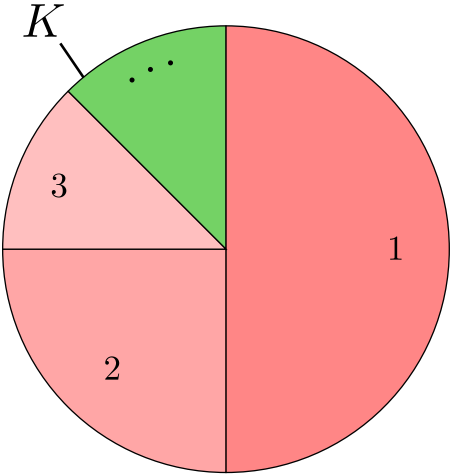

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Let R be a strong bisimulation. Then

$$(K, G) \in R \wedge G \models B \implies [\![\mathit{while}(B)\{P\}]\!](G) = K$$

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  {X := X + 1}[0.5]{F := 1}
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- $\llbracket \mathit{while}(F = 0)\{\ldots\} \rrbracket (1) \sqsubseteq K$
- $\llbracket \mathit{while}(F = 0)\{\ldots\} \rrbracket (1) = K$

Contribution

- Semantics for negative variable valuations with closed forms
- Deeper understanding of two relevant programs
- Comparison to an established semantics
- Novel approach for finding loop semantics

Thank you for your attention!

Wp-semantics

1. Expectations: $E = \{\mathbb{S} \to \mathbb{R}_{\geq 0}\}$
2. $\mathrm{wp}\colon E \to E$
3. $\mathrm{wp}\,(\mathit{skip}, f) = f$.
4. $\mathrm{wp}\,(\mathit{diverge}, f) = 0$.
5. $\mathrm{wp}\,(X := e, f) = f[X/e]$.
6. $\mathrm{wp}\,(P; Q, f) = \mathrm{wp}\,(P, \mathrm{wp}\,(Q, f))$.
7. $\mathrm{wp}\,(\mathit{if}(B)\,\{\,P\,\}\,\mathit{else}\,\{\,Q\,\}, f) = [B]\cdot\mathrm{wp}\,(P, f) + [\neg B]\cdot\mathrm{wp}\,(Q, f)$.
8. $\mathrm{wp}\,(\{P\}[p]\{Q\}, f) = p \cdot \mathrm{wp}\,(P, f) + (1 - p) \cdot \mathrm{wp}\,(Q, f)$.
9. $\mathrm{wp}\,(\mathit{while}\,(B)\{P\}, f) = \mu X.\,([B] \cdot \mathrm{wp}\,(P, X) + [\neg B] \cdot f)$.