Brownian Motion and Brownian Bridge Construction
Final Project of Group 1

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1 Brownian Motion and Wiener Process

In financial engineering, stochastic processes play a crucial role in modeling the behavior of stock prices in financial markets. The simplest one is the so-called Brownian motion. It is used in the celebrated Black-Scholes model to model the logarithms of stock prices.

A standardized *Brownian motion* $W(t)$ (called a *Wiener process*) is a continuous-time stochastic process (i.e., a random path) with the property:

1. $W(0) = 0$, a.s.

2. W has independent increments: for every $s, t > 0$, the future increments $W_{t+s} - W_t$ are independent of W_t.

3. $W_{t+s} - W_t$ is normally distributed with mean 0 and variance s, $W_{t+s} - W_t \sim \mathcal{N}(0, s)$.

4. W_t is continuous in t with probability 1.

For μ and $\sigma > 0$, we call a process $X(t)$ a *Brownian motion* with drift μ and diffusion coefficient σ^2 (abbreviated by $X \sim BM(\mu, \sigma^2)$) if

$$\frac{X(t) - \mu t}{\sigma}$$

is a Wiener process. Thus, we construct X from a Wiener process W by:

$$X(t) = \mu t + \sigma W(t).$$

2 Random Walk Construction

When we work with Wiener processes (and Brownian motions), we need numerical approximations in order to sample paths. That is, for a fixed set of points $0 < t_1 < t_2 < \ldots < t_n$, we need to simulate values $(W(t_1), W(t_2), \ldots, W(t_n))$. Since the increments of a Brownian motion are independent and normally distributed, such a simulation is straightforward:

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}$$

where $Z_i \sim \mathcal{N}(0, 1)$ are standard normally distributed variables.

We restrict our simulation to equidistributed points, i.e., we assume that:

$$t_{i+1} - t_i = \delta t \quad \text{is constant for all } i.$$

Thus, we obtain an approximate realization of a Wiener process using the *Euler–Maruyama scheme*, where the increments

$$\delta W(t) = W(t + \delta t) - W(t)$$

are normally distributed with mean 0 and variance δt. This leads to the following procedure for constructing a discrete approximation of the Wiener path at points $t_i = i \, \delta t$:

1. Fix δt.

2. Start with $W(0) \leftarrow 0, t_0 \leftarrow 0, i \leftarrow 0$.

3. Repeat:

    (a) Generate $Z \sim \mathcal{N}(0,1)$.
    (b) $t_{i+1} \leftarrow t_i + \delta t$.
    (c) $W(t_{i+1}) \leftarrow W(t_i) + \sqrt{\delta t} Z$.
    (d) $i \leftarrow i + 1$.

3   Brownian Bridge

Let T be the time horizon for our simulation, i.e., $t_n = T$. Then the stochastic process:

$$B(t) = W(t) - \frac{t}{T} W(T) \tag{1}$$

is a Wiener process that is pinned to value 0 at both $t = 0$ and $t = T$

$$B(t) = \big(W(t) \,|\, W(T) = 0\big), \quad t \in [0, T]$$

It is called a *Brownian bridge*. It has similar properties as a Wiener process. In particular, increments are normal distributed. Notice, however, that increments are not independent. Moreover, one finds $B(t)$ has mean 0 and variance

$$\frac{t(T-t)}{T},$$

i.e.,

$$B(t) \sim \mathcal{N}\left(0, \frac{t(T-t)}{T}\right) \tag{2}$$

In particular, we find:

$$B(T/2) \sim \mathcal{N}\left(0, \frac{T}{4}\right) \tag{3}$$

4   Brownian Bridge Construction

The *Brownian bridge construction* is a way to simulate Brownian motion paths by successively adding finer scale detail.

From relation (1), we can derive a construction for the Wiener process:

$$W(t) = B(t) + tZ,$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normally distributed random variable.

These relations can be used to simulate a Wiener process. We have simulated $W(T)$ first. Now the Wiener process is fixed at $t_0 = 0$ and $t_2 = T$. Now we use the Brownian bridge to sample at $t_1 = T/2$:

$$W(0) = 0, \quad W(T) = \sqrt{T}Z \sim \mathcal{N}(0,T),$$

$$W(T/2) \sim \mathcal{N}\left(0, \frac{T}{4}\right) + \frac{1}{2}W(T).$$

This idea can be extended. Suppose the values $W(t_i) = w_i$ of the Brownian path have already been determined for points t_i. Now we wish to "interpolate" between these points. So let $t \in (t_i, t_{i+1})$. Then we obtain from the above idea that $W(t)$, conditional to $W(t_i) = w_i$ and $W(t_{i+1}) = w_{i+1}$, has distribution:

$$W(t) \,|\, W(t_i) = w_i, \tag{4}$$

$$W(t_{i+1}) = w_{i+1} \sim \mathcal{N}\left(\frac{(t_{i+1} - t)w_i + (t - t_i)w_{i+1}}{t_{i+1} - t_i}, \frac{(t_{i+1} - t)(t - t_i)}{t_{i+1} - t_i}\right) \tag{5}$$

We now can use distribution (5) to construct a Wiener process recursively when the number of constructed time intervals is a power of 2 and represents a so-called "level" of the Brownian Bridge as described by Goodman (2011). Using the midpoint displacement method we iteratively add the midpoint in each interval (t_i, t_{i+1}):
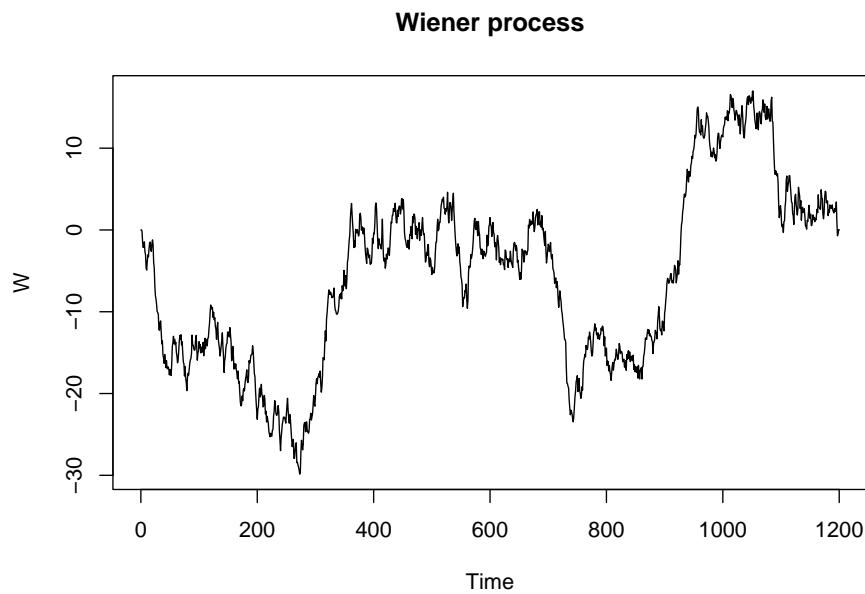
1. $W(0) \leftarrow 0$ and generate $W(T) \sim \mathcal{N}(0,T)$.

2. Repeat:

   (a) For all intervals (t_i, t_{i+1}):
   (b) $t \leftarrow (t_i + t_{i+1})/2$.
   (c) $m \leftarrow \frac{t_{i+1} - t}{t_{i+1} - t_i}W(t_i) + \frac{t - t_i}{t_{i+1} - t_i}W(t_{i+1})$.
   (d) $s \leftarrow \frac{(t_{i+1} - t)(t - t_i)}{t_{i+1} - t_i}$.
   (e) Generate $W(t) \sim \mathcal{N}(m, s)$.

3. Update set of time points $\{t_i\}$ and values $\{W(t_i)\}$.

We further use the implementation of this method in R to show the increasing details when the number of intervals is increased from 2^1 to 2^{10}.

Implementation in R

```r
wiener.path<-function(n,Time,plot=TRUE){
  dt<-Time/n
  X<-c(0,rnorm(n,0,sqrt(dt)))
  W_t<-cumsum(X)

  if (plot){
    plot(seq(0,Time,by=dt),W_t,type="l",
    xlab="Time",ylab="W", main="Wiener process")
  }
}
```

Wiener process


```r
bridge.path<-function(m,Time,plot=TRUE){
  n<-2^m

  t_pts<-c(0,Time)
  W_pts<-c(0,rnorm(1,0,sqrt(Time)))

  while(length(t_pts) < n+1){

    new_t_pts<-c()
```

```r
  new_W_pts<-c()

  for (i in seq_along(t_pts[-length(t_pts)])){
    t1<-t_pts[i]
    t2<-t_pts[i+1]
    W1<-W_pts[i]
    W2<-W_pts[i+1]

    t<-(t1+t2)/2

    mu<-((t2-t)*W1+W2*(t-t1))/(t2-t1)
    s<-sqrt((t2-t)*(t-t1)/(t2-t1))

    Wt<-rnorm(1,mu,s)

    new_t_pts<-c(new_t_pts,t1,t)
    new_W_pts<-c(new_W_pts,W1,Wt)

  }
  t_pts<-c(new_t_pts,tail(t_pts,1))
  W_pts<-c(new_W_pts,tail(W_pts,1))
  }
  if (plot){
    plot(t_pts, W_pts, type = "l", col = "blue",
         main = paste("Brownian Bridge of Level",m),
         xlab = "Time", ylab = "W(t)")
    lines(c(0,tail(t_pts,1)),c(0,tail(W_pts,1)),
    col = "red", lty = 2)
  }

  if (!plot) {
    return(cbind(t_pts, W_pts))
  }

}
```
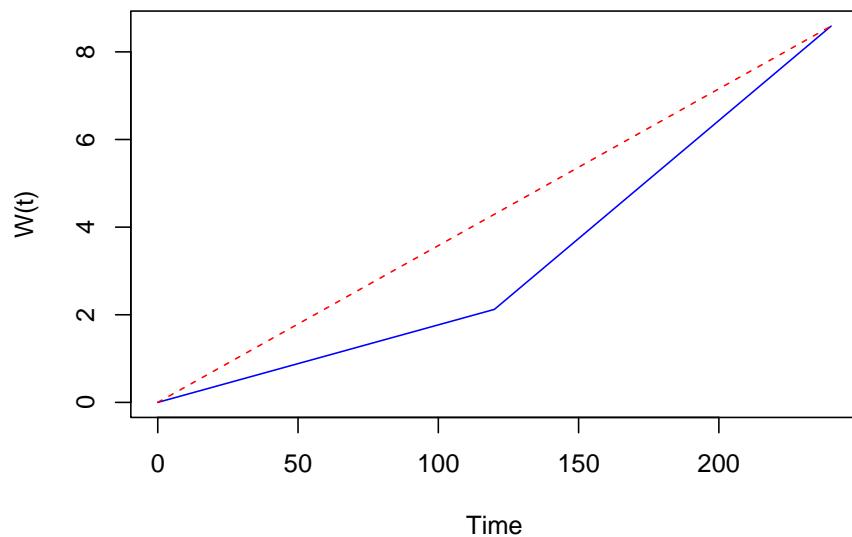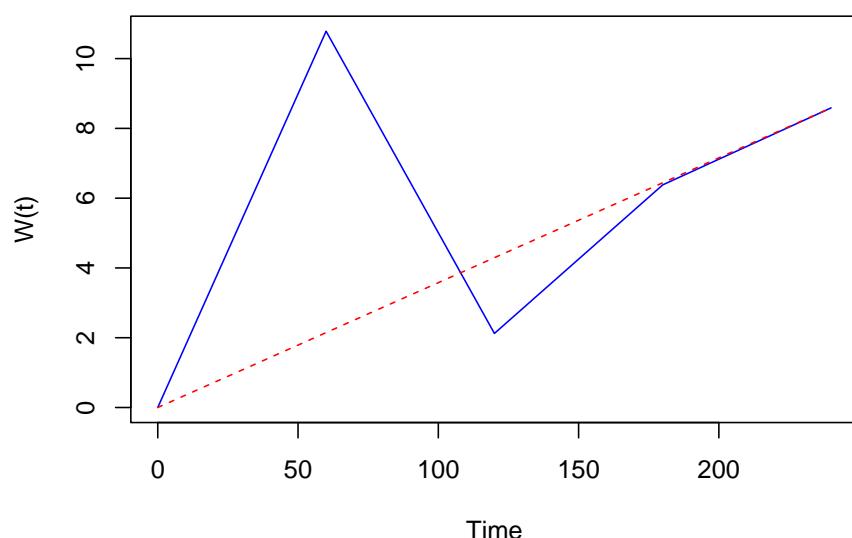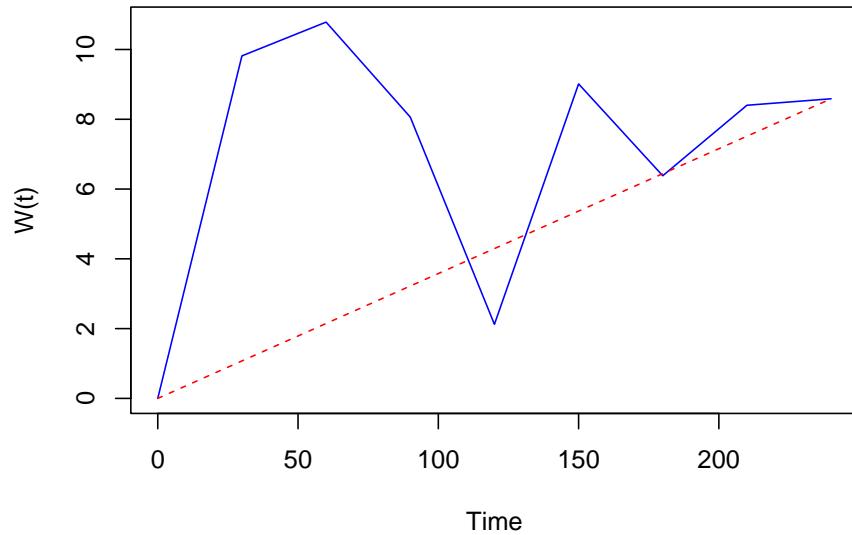
Brownian Bridge of Level 1
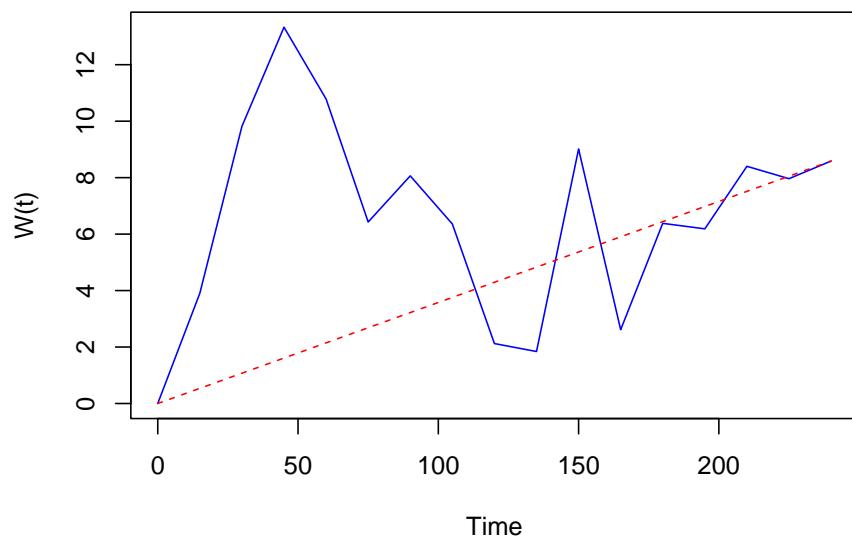

## Brownian Bridge of Level 2
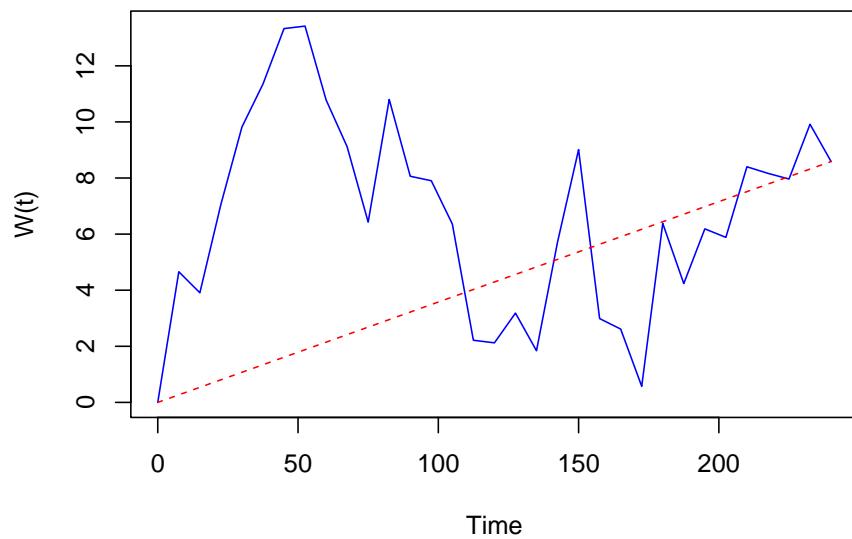
**Brownian Bridge of Level 3**
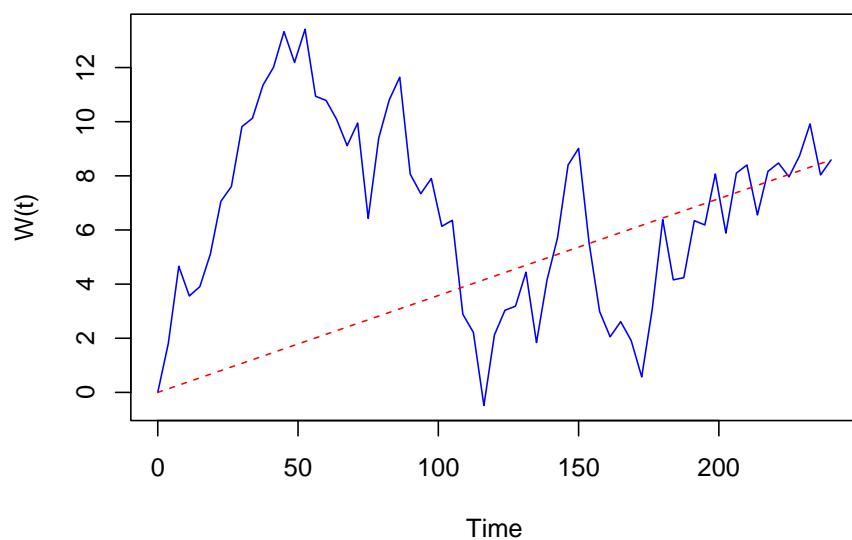
Brownian Bridge of Level 4

8
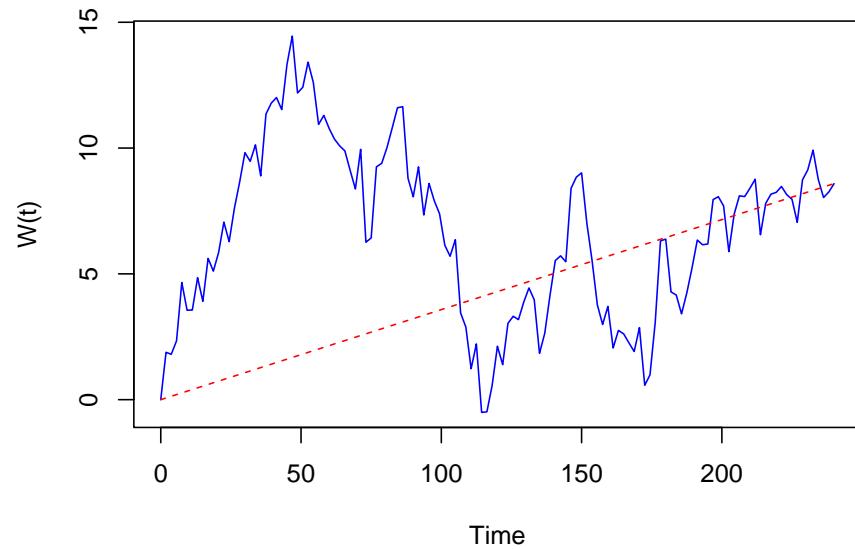
Brownian Bridge of Level 5
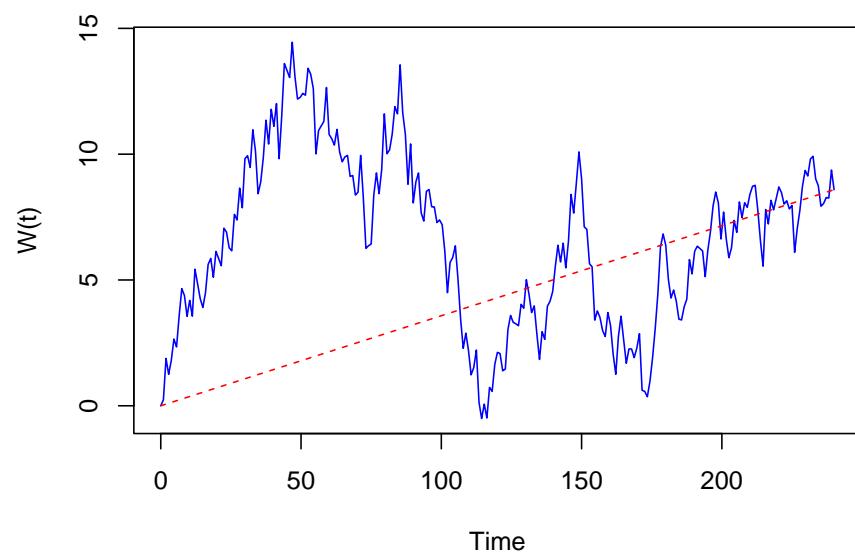
Brownian Bridge of Level 6
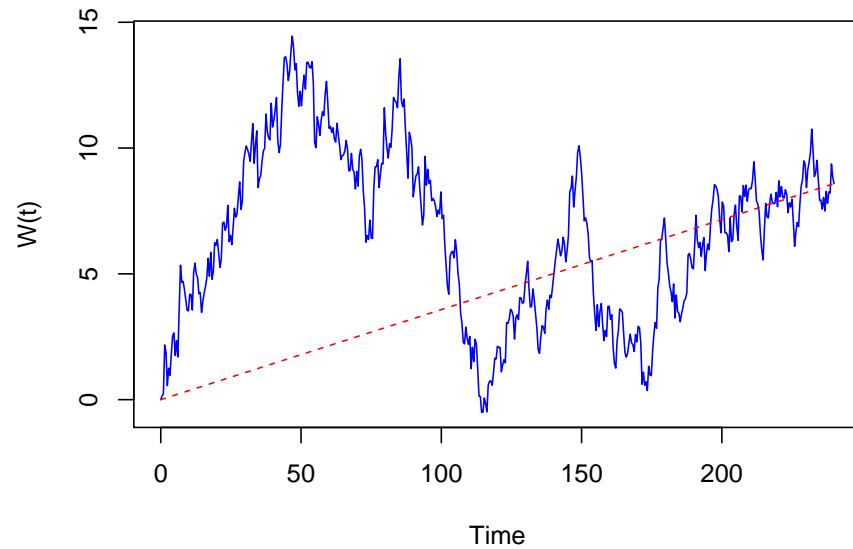
**Brownian Bridge of Level 7**
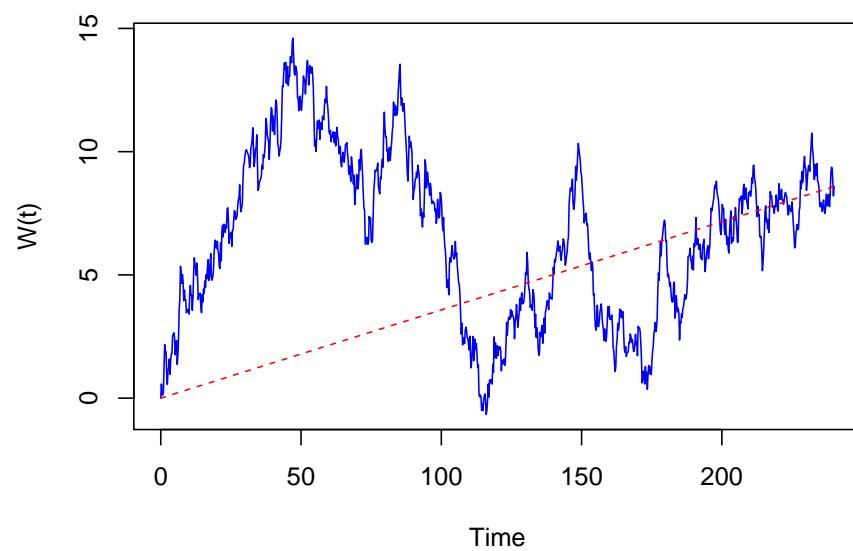
Brownian Bridge of Level 8

Brownian Bridge of Level 9

Brownian Bridge of Level 10

11

References

Goodman, J. (2011). Diffusions 2. *Course on Stochastic calculus, New York University*, pages 1–4.