# Conflict Driven Clause Learning

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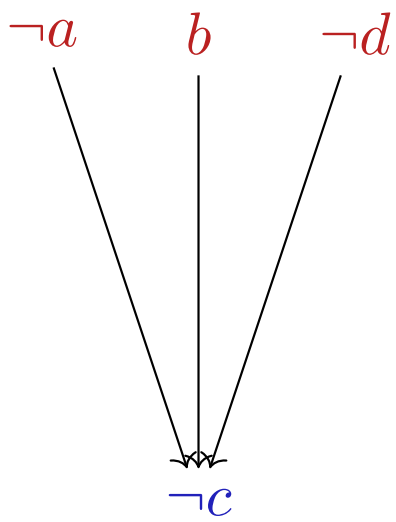
- **Can be seen as a CP Solver (Search, propagation) augmented by clause learning**
- But also :
  - Activity-based branching
  - Lazy data structures (2-Watched Literals)
  - Clause Database Reduction
  - Simplifications
  - Restarts
  - ...

**Exercise:** Propose a filtering algorithm for clauses. The algorithm takes as input a clause and has access (read and write) for the variables domains.

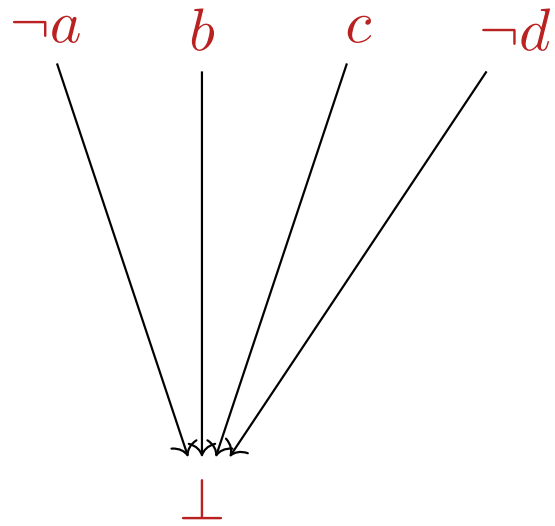
# Unit Propagation

Given a clause $C$ of arity $n$. If $n-1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$




$\neg a \wedge b \wedge \neg d \Rightarrow \neg c$

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# Two Watched Literals

- Unit propagation is implemented with an "intelligent" data structure called Two-watched literals
- Observe first that propagation happens only in two cases:
  - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
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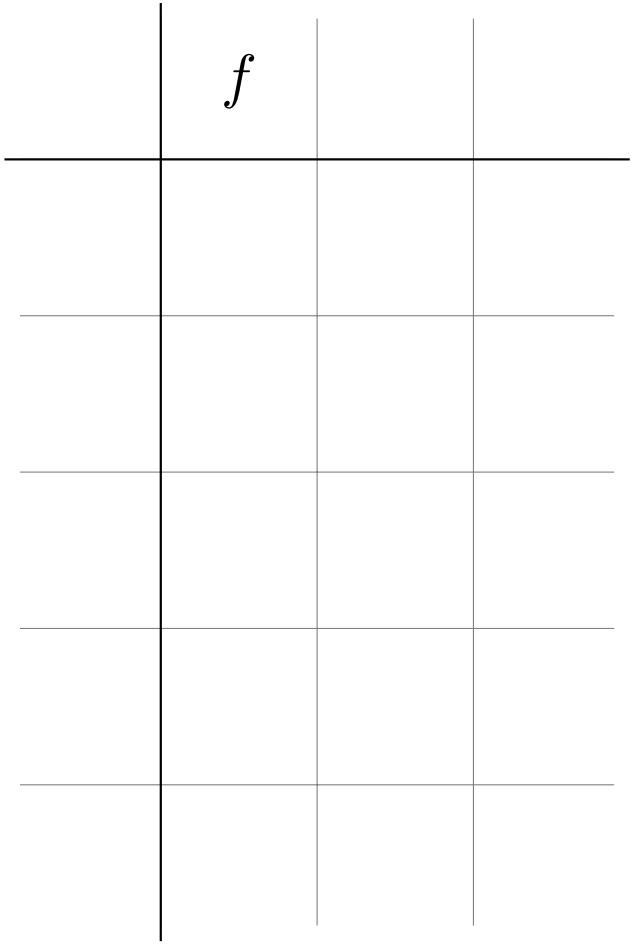
- If a literal watching a clause $C$ becomes $false$, look for replacement. If no replacement found, then perform propagation

# Implication Graph

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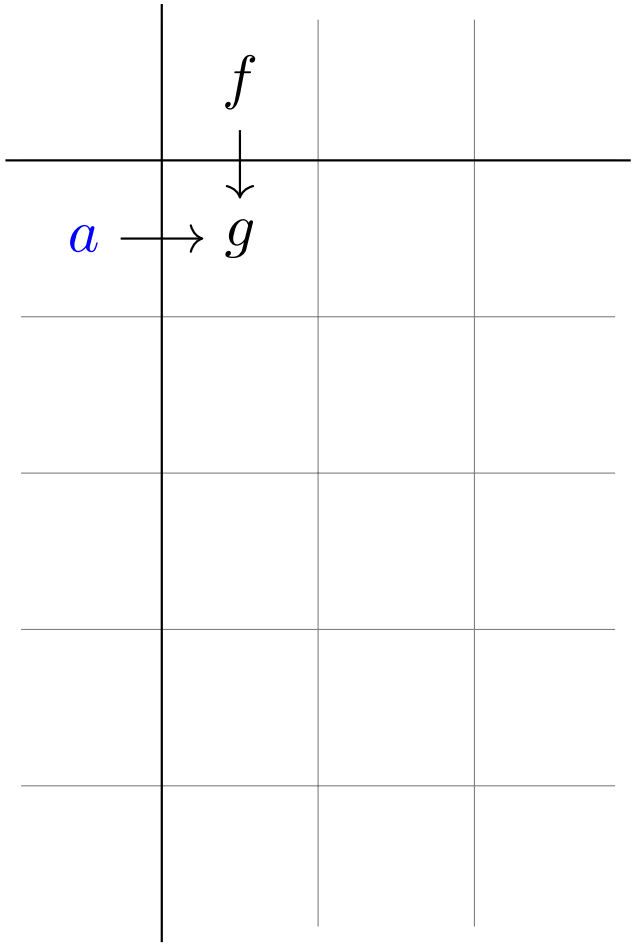


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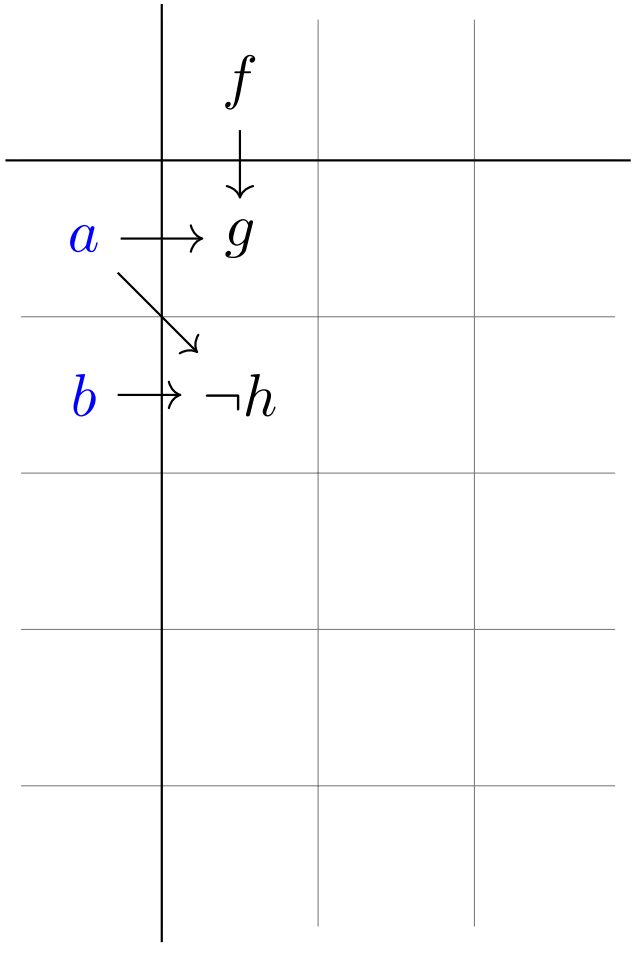


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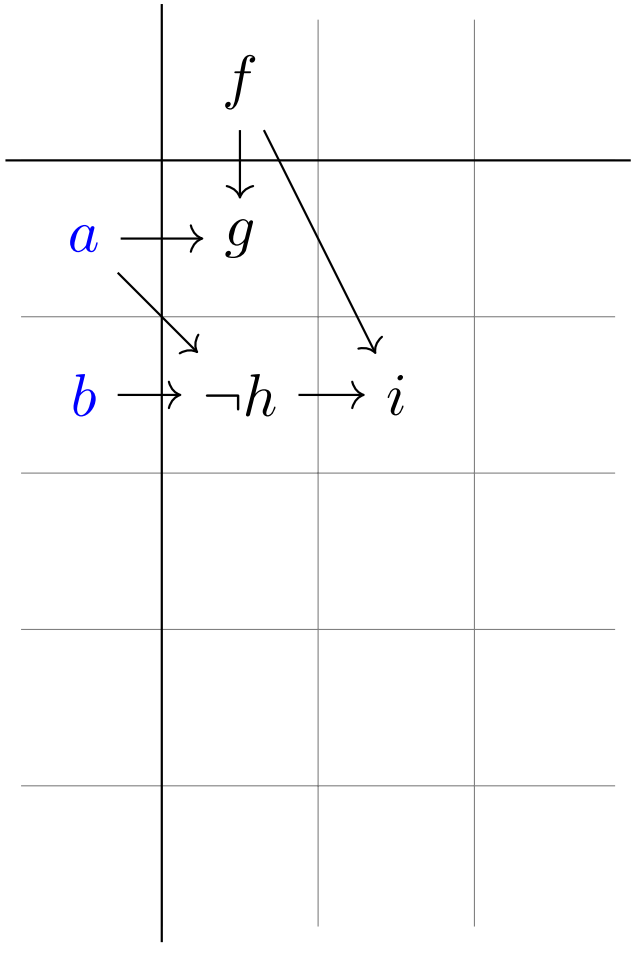


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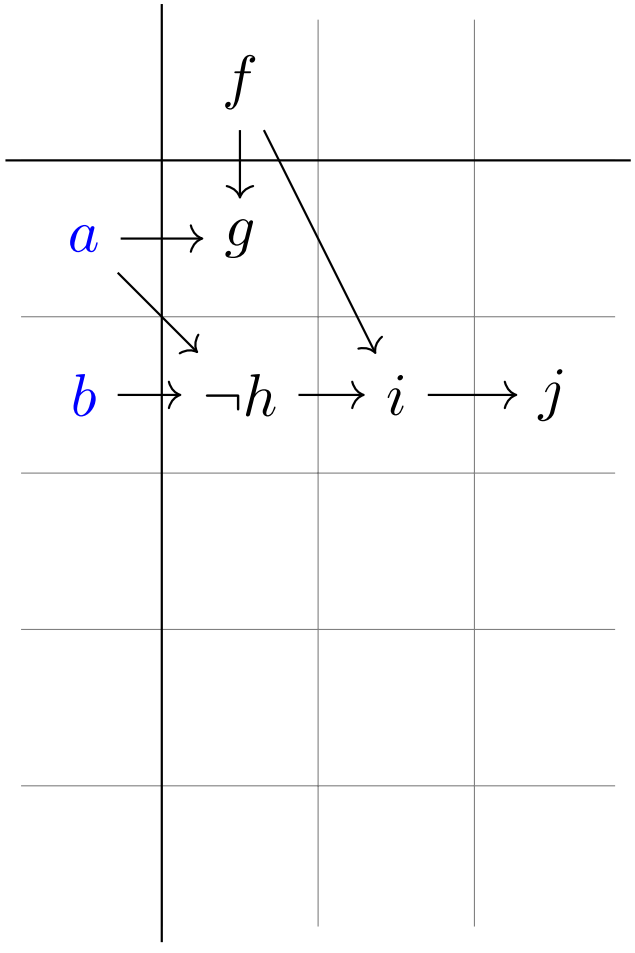


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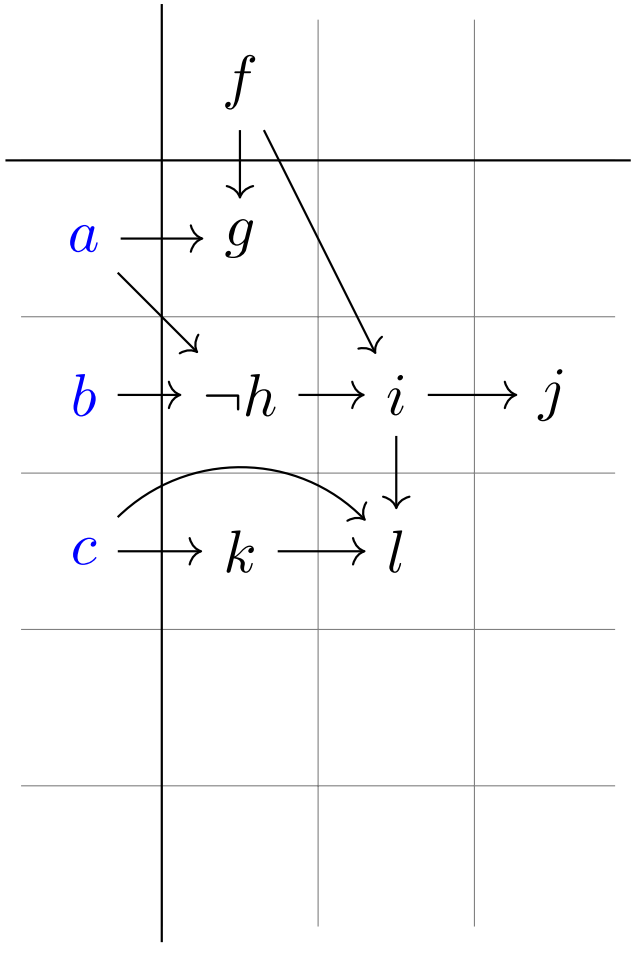


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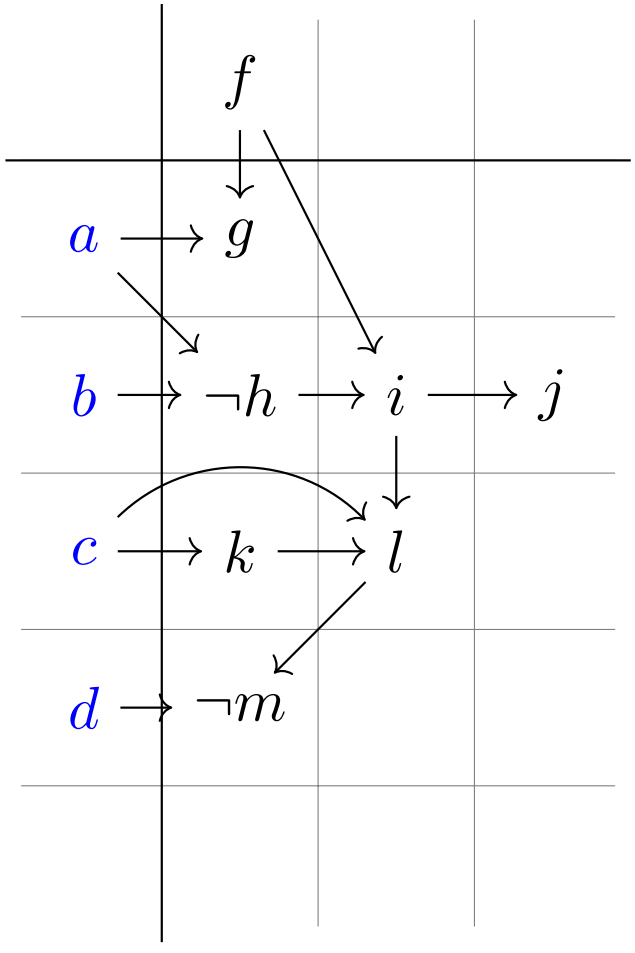


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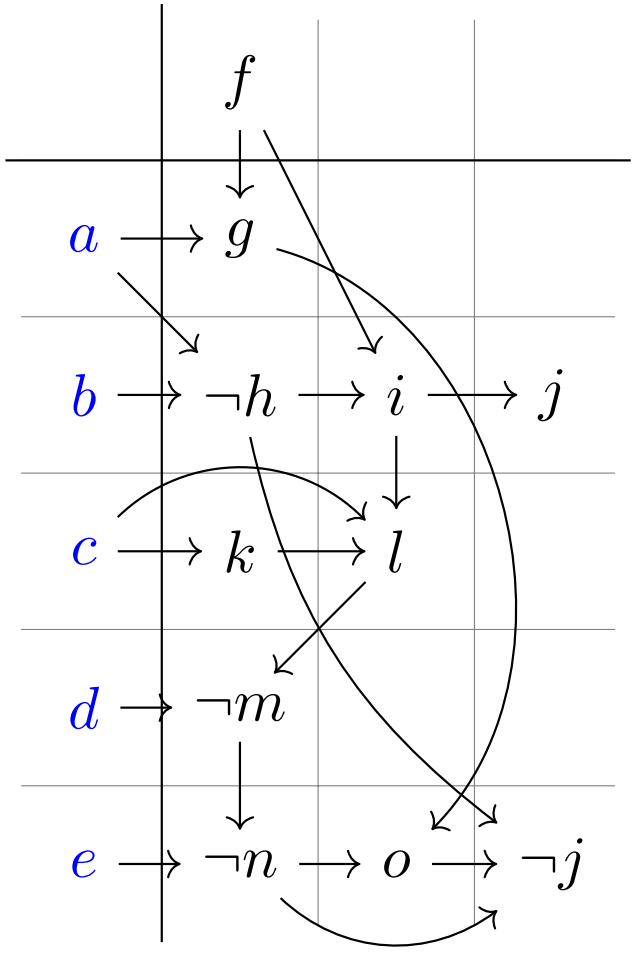


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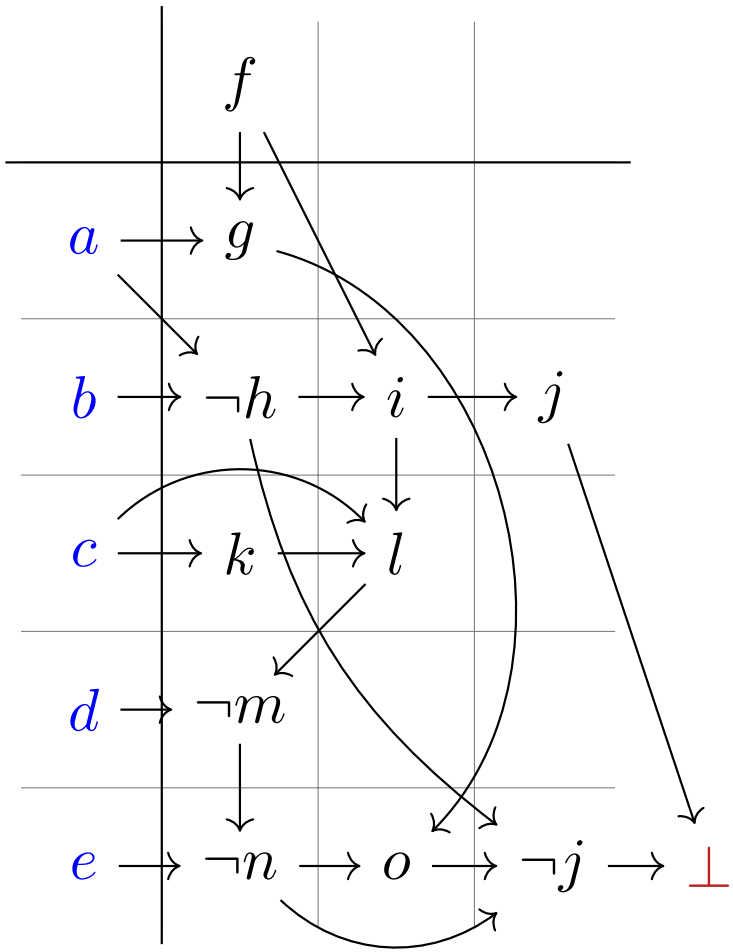


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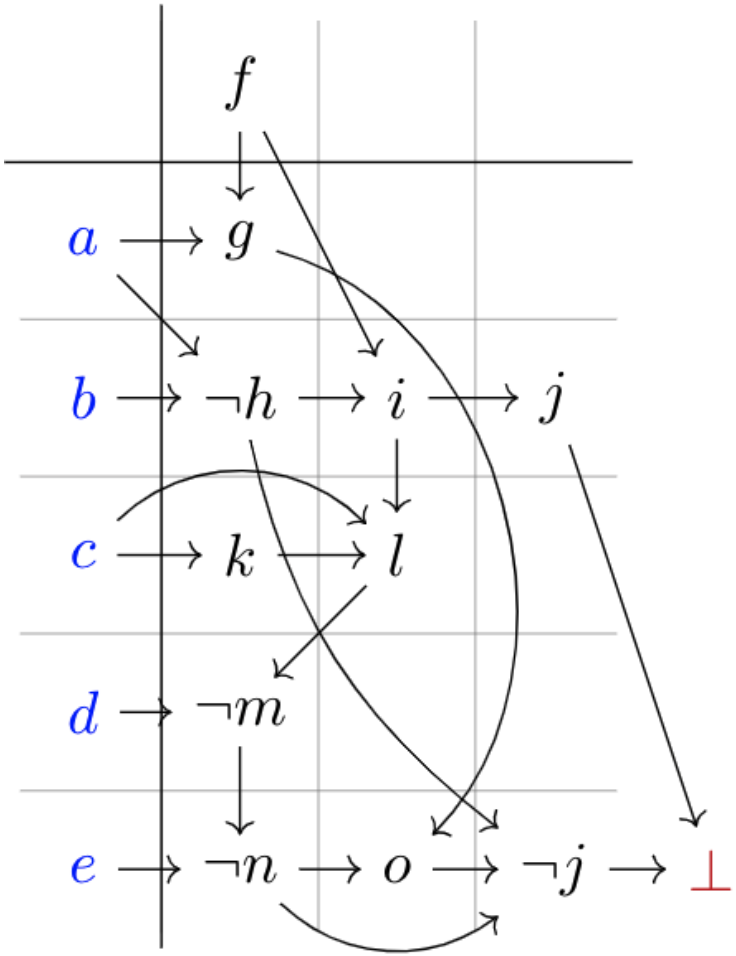


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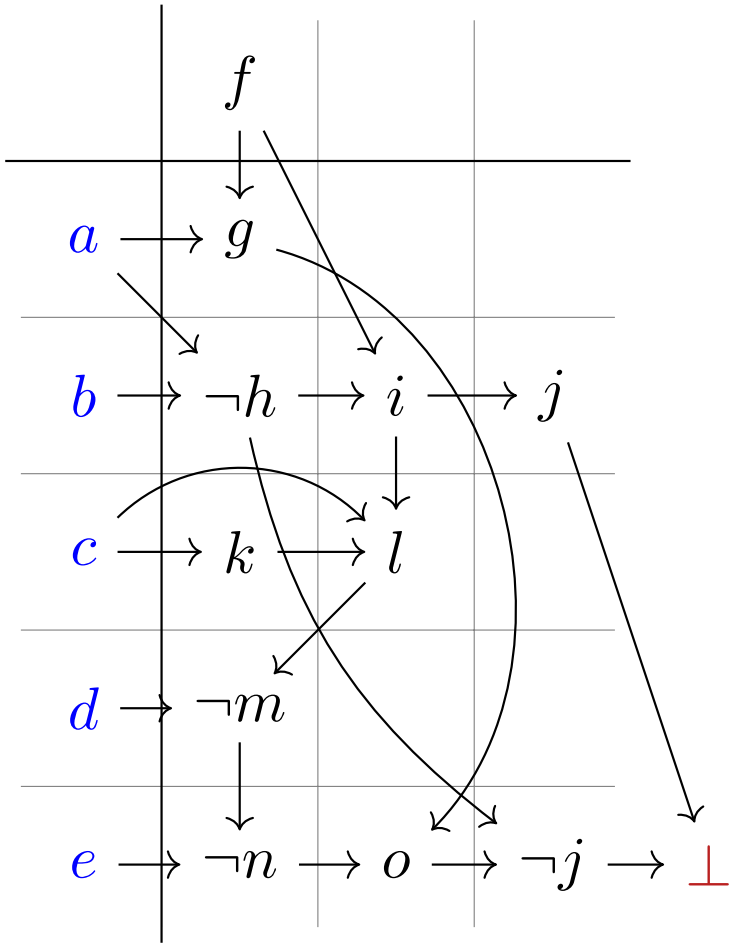


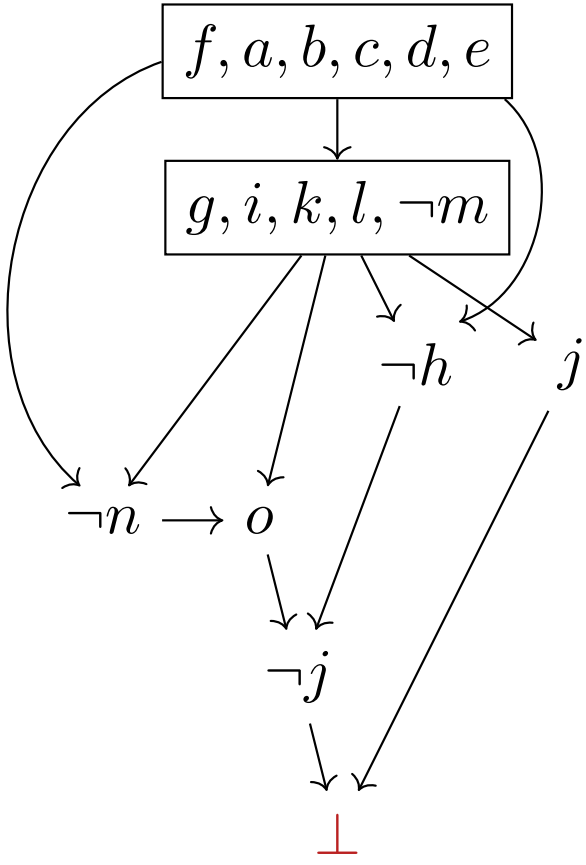

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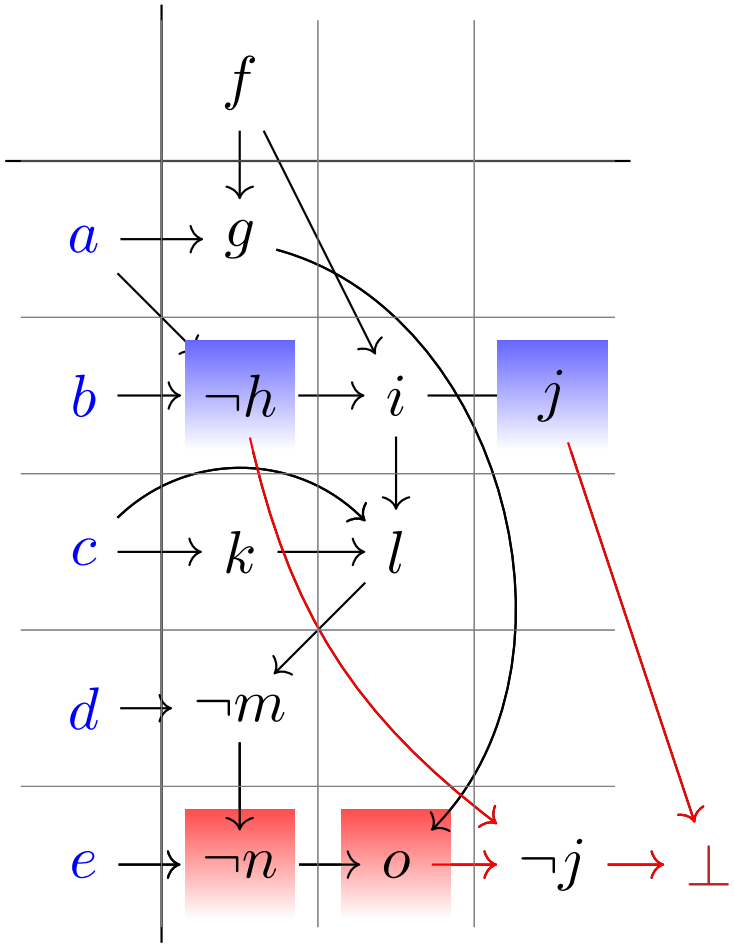


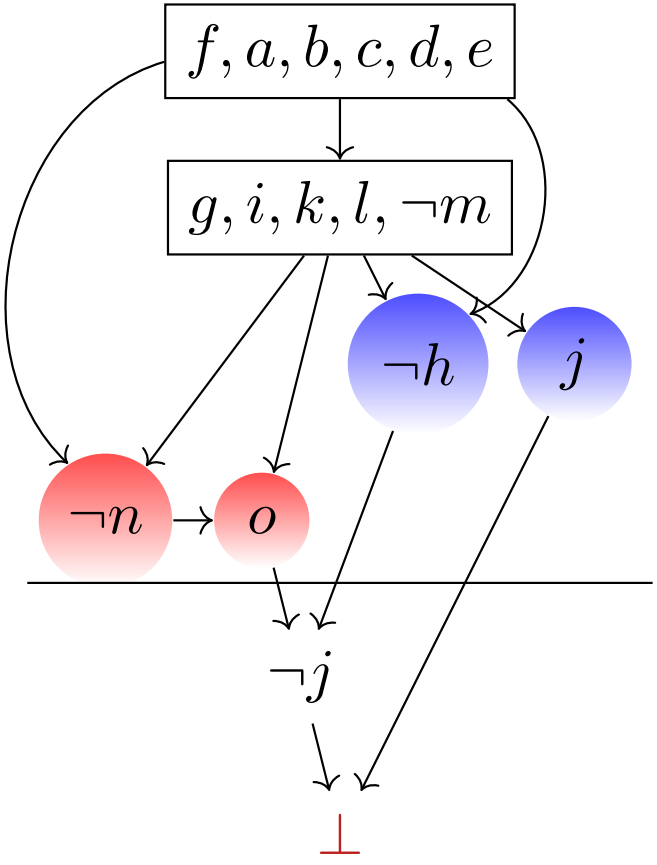

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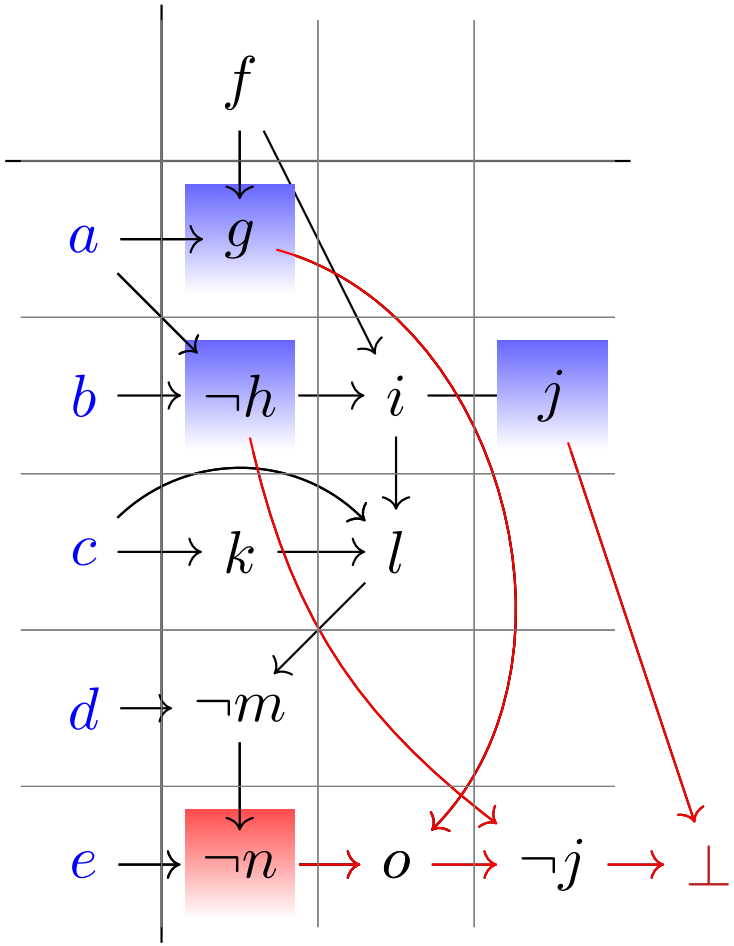


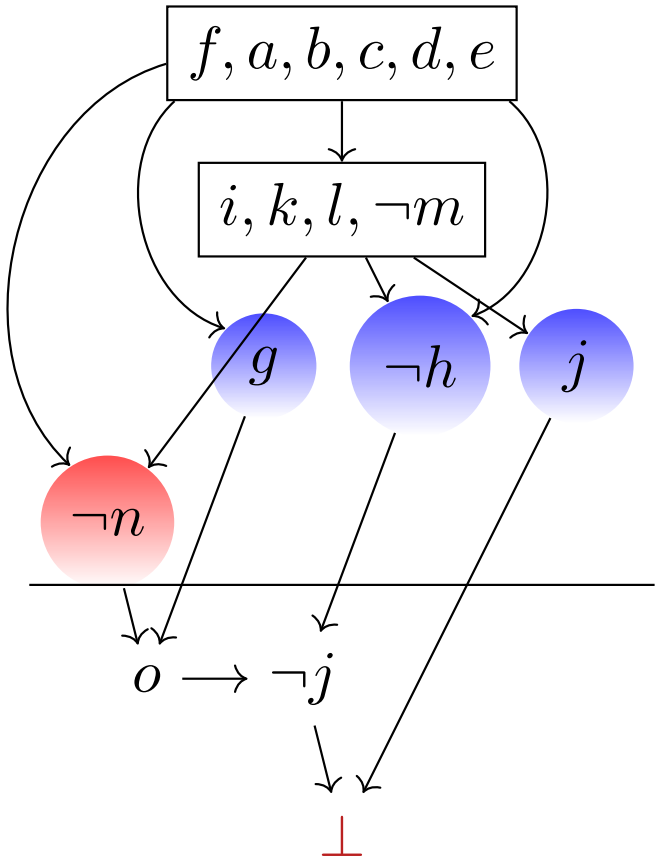

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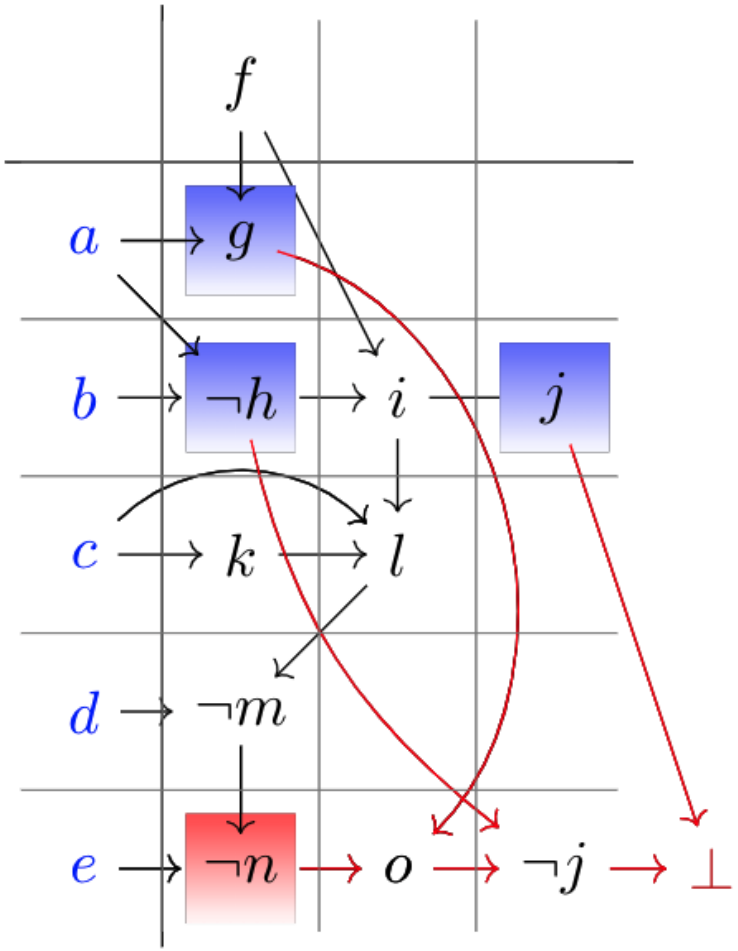


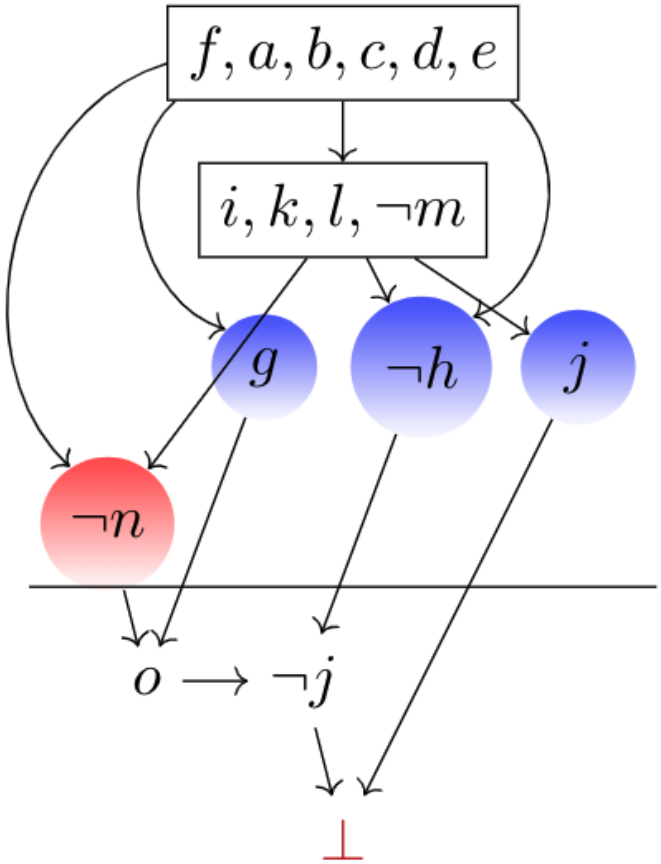

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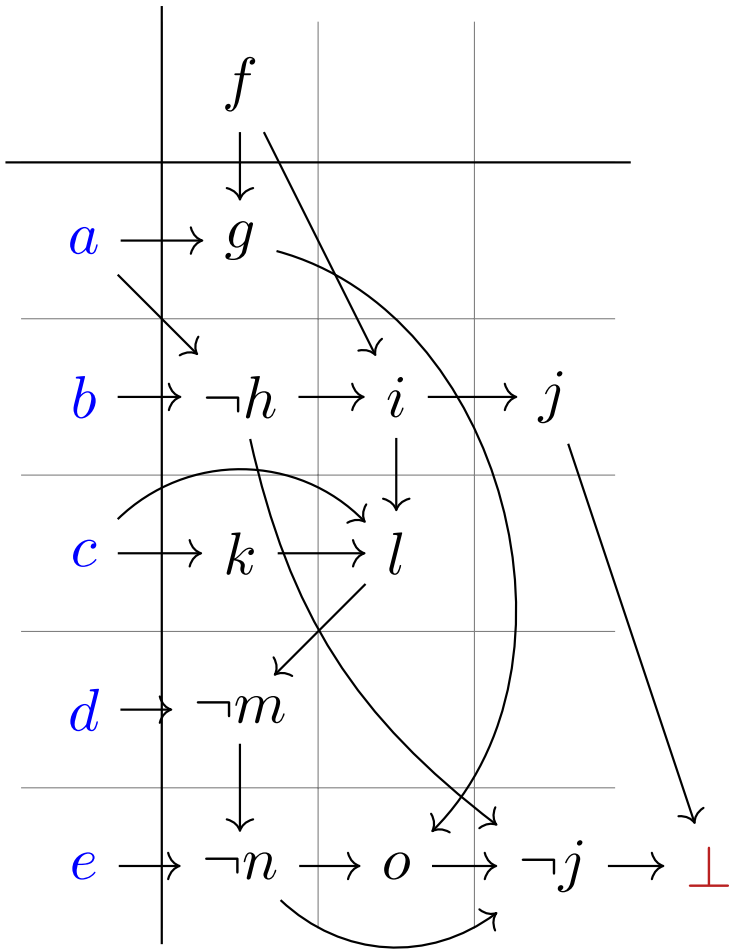


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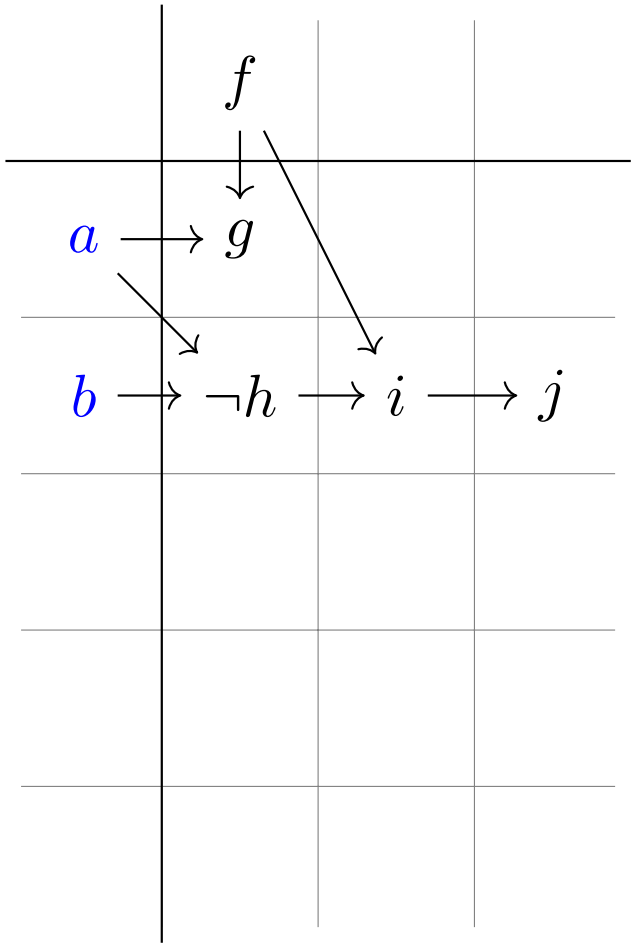


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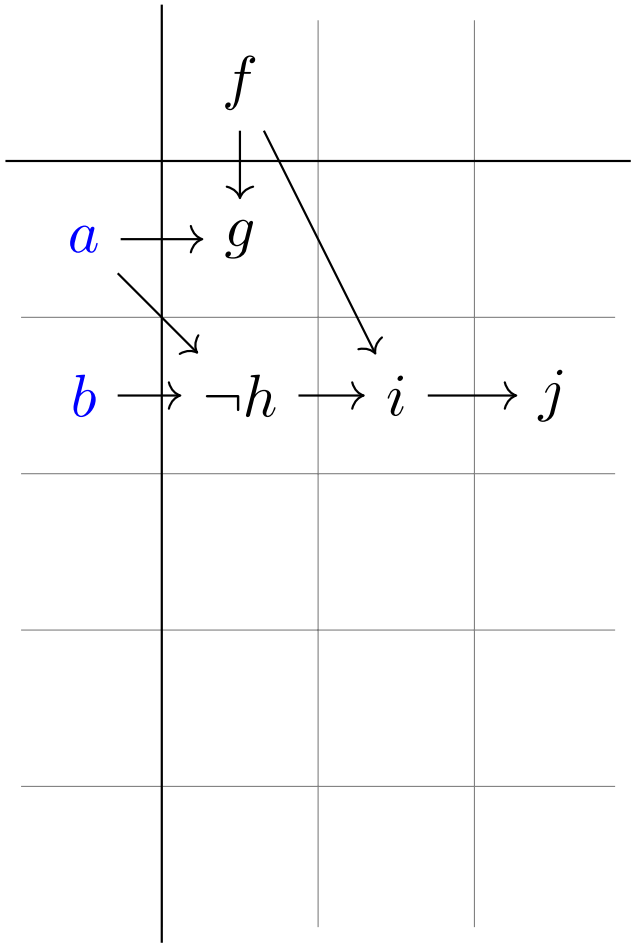


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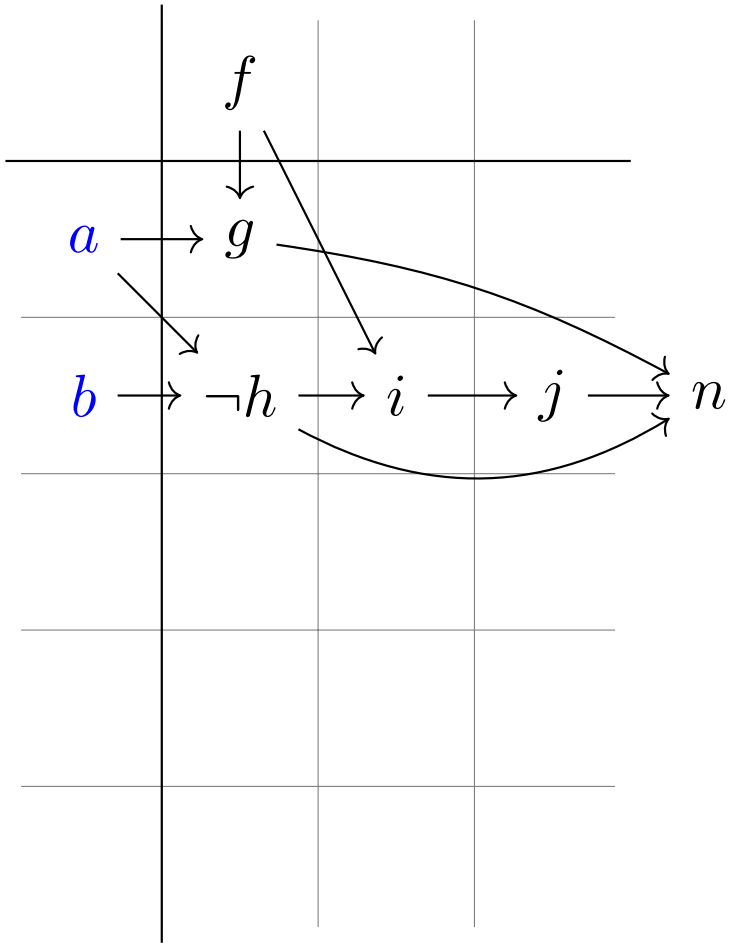
$\neg f \vee h \vee i$
$\boxed{\neg g \vee h \vee \neg j \vee n}$

# Conflict analysis




$\neg a \vee \neg f \vee g$
$\neg a \vee \neg b \vee \neg h$
$a \vee c$
$a \vee \neg i \vee \neg l$
$a \vee \neg k \vee \neg j$
$b \vee d$
$b \vee g \vee \neg n$
$b \vee \neg f \vee n \vee k$
$\neg c \vee k$
$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$
$c \vee l$
$d \vee \neg k \vee l$
$d \vee \neg g \vee l$
$\neg g \vee n \vee o$
$h \vee \neg o \vee \neg j \vee n$
$\neg i \vee j$
$\neg d \vee \neg l \vee \neg m$
$\neg e \vee m \vee \neg n$
$\neg f \vee h \vee i$
$\boxed{\neg g \vee h \vee \neg j \vee n}$

# Boosting Search through Randomization and Restarts [Gomes et al., 1998]

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## Heavy-tail phenomena (SAT and CP)

At any time during the experiment there is a non-negligible probability of hitting a problem that requires exponentially more time to solve than any that has been encountered before.

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Hardness = Instance $\oplus$ deterministic algorithm.

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Hardness = Instance $\oplus$ deterministic algorithm.

- Randomization: breaking ties, random decision between $k$ best choices, ...
- Restarts: Geometric/Luby

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# SAT Solvers (Few examples)

- MiniSat: http://minisat.se/
- Glucose: `http://www.labri.fr/perso/lsimon/glucose/`
- LingeLing `http://fmv.jku.at/lingeling`
- Any Solver by Armin Biere
  `http://fmv.jku.at/software/index.html`
- Any winner from past and future SAT competitions:
  https://www.satcompetition.org/

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