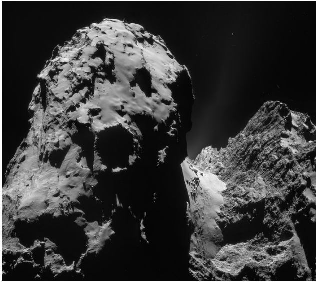
SAT: Modelling and Implementations

Mohamed Siala
https://siala.github.io

INSA-Toulouse & LAAS-CNRS

January 13, 2022

Context: Solving (Very) Hard Combinatorial Problems



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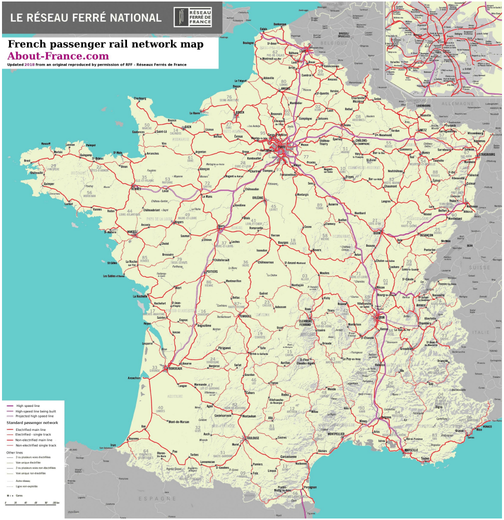
Context: Solving (Very) Hard Combinatorial Problems



Context: Solving (Very) Hard Combinatorial Problems

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Why this Lecture?

- I noticed that most graduate students are doing software development.
- We are missing job opportunities in optimisation!
- Resources: many.. a good start would be the online course on discrete optimisation
  `https://www.coursera.org/learn/discrete-optimization`

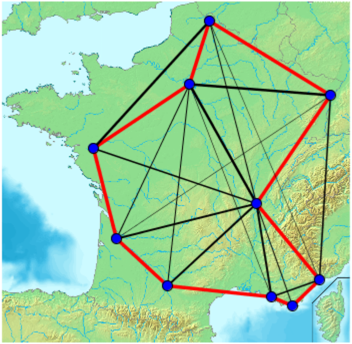
Solving Methodologies

1. Adhoc methods
   1. Specific exact algorithm
   2. Heuristic method
   3. Meta-heuristic (genetic algorithms, ant colony, ..)
2. Declarative Approached
   1. (Mixed) Integer Programming,
   2. Constraint Programming
   3. Boolean Satisfiability (SAT)
   4. ...

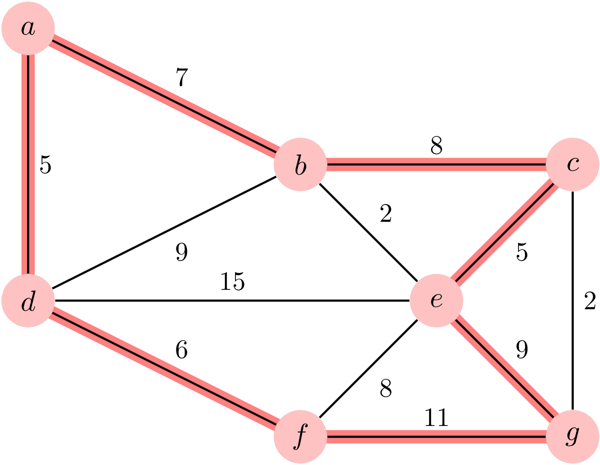
Why Declarative Approaches?

- They are problem independent! The user models the problem in a specific language and the solver do the job!
- Very active community

Travelling Salesman Problem

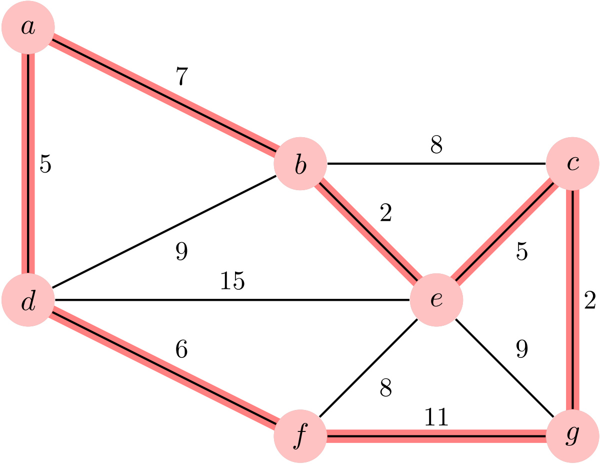


Exemple



$$--> Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53 Km$$

Example



$-- > Cost: 5 + 7 + 2 + 5 + 2 + 11 + 6 = 38Km$

What if we check all possibilities?

- 2 Cities →1
- 5 Cities →24
- 8 Cities →4032
- 40 Cities →2.10^{46} (with a modern machine: 3.10^{27} years!)
- 95 Cities, if we use a Plack (the shortest possible time interval that can be measured) processor and fill the universe with a processor per mm^3, we need $3\times$ the age of the universe

The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

A step back: Problems, Instances, and Algorithms

- A problem is a question that associates an input of an output
- Many instances (instantiation of the input) for the same problem
- Many algorithms (methodologies) to solve the same problem

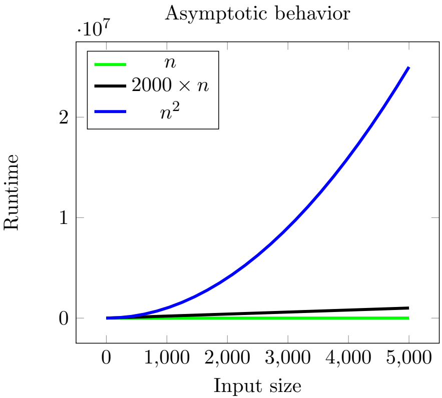
Example: The Sorting Integers problem

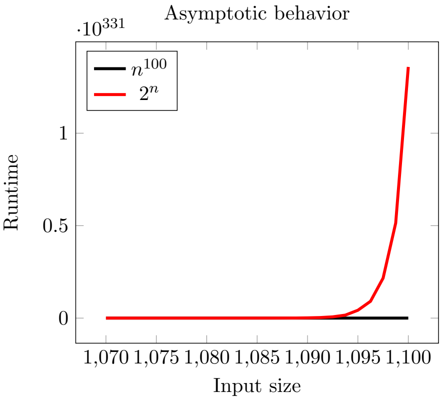
- Problem: sort a given sequence of n integers.
- Instance: a sequence of n integers
- A simple algorithm:
  - Scan the list to look for the smallest element
  - Swap it with the first position
  - Repeat for the list of remaining elements
- Example with the instance : 9, 3, 8, 7, 2
  - 2, 9, 3, 8, 7
  - 2, 3, 9, 8, 7
  - 2, 3, 7, 9, 8
  - 2, 3, 7, 8, 9
  - 2, 3, 7, 8, 9

Complexity

- Complexity: a measure to analyze/classify algorithms based on the amount of resource required (Time and Memory)
- Time Complexity: number of operations as a function of the size of the input
- Space Complexity: memory occupied by the algorithm as a function of the size of the input
- The evaluation is made usually by reasoning about the worst case.
- The analysis is given with regard with the asymptotic behaviour

Asymptotic behaviour





- If f is a polynomial and g is exponential then $f \in O(g)$.
  For instance $n^{10000} \in O(2^n)$
- Convention:
  - Easy/Tractable Problem: We know a polynomial time algorithm to solve the problem
  - Hard/Intractable: No known polynomial algorithm
- Example: Th sorting problem is easy because we have an algorithm that runs in the worst case in $O(n^2)$ (and actually the same for memory consumption)
- What if we don't know if a problem has a polynomial time algorithm?

Classes of problems

- **P** is the class of problems that are **solvable** in polynomial time (easy problems)
- **NP** is the class of problems that are **verifiable** in polynomial time algorithm
- We know that $P \in NP$ (if you can solve then you can verify)
- For many Problems in NP, we don't know if a polynomial time algorithm exists.
- **1 Million $** question: Is P=NP?

The Boolean Satisfiability Problem (SAT)

Definitions

- Atoms (Boolean variables): x_1, x_2, ...
- Literal: $x_1, \neg x_1$
- Clauses: a clause is a disjunction of literals
- Example of clause: $(\neg x_1 \vee \neg x_4 \vee x_7)$
- Propositional formula Φ given in a `Conjunctive Normal Form` (`CNF`) $\Phi : c_1 \wedge .. \wedge c_n$

Given a set of Boolean variables $x_1, \ldots x_n$ and a CNF formulae Φ over $x_1, \ldots x_n$, the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

Why SAT?

- SAT is the first problem that is shown to be in the class NP-Complete (the hardest problems in NP)
- Many theoretical properties
- Huge practical improvements in the past 2 decades
- Is considered today as a powerful technology to solve computational problems

In this lecture, we focus on the practical side

- How to use it to solve problems (Modelling)
- Discover some efficient implementations

Example

$x \vee \neg y \vee z$

$\neg x \vee \neg z$

$y \vee w$

$\neg w \vee \neg x$

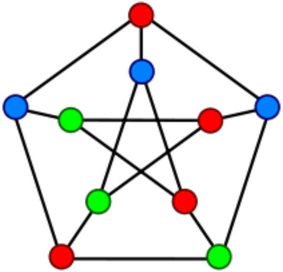
A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

Modelling in SAT: The example of Graph Coloring

Graph Coloring is a well know combinatorial problem that has many applications (in particular in scheduling problems).

Let $G = (V, E)$ be an undirected graph where V is a set of n vertices and E is a set of m edges. Is it possible to colour the graph with k colours such that no two adjacent nodes share the same colour?



Modelling in SAT: The example of Graph Coloring

- Propose a SAT model for this problem
  (hint $x \to y$ is equivalent to $\neg y \to x$ and both are translated into the clause $\neg x \vee y$).

The Example of Graph Coloring: A Possible Model

Let x_i^k be the Boolean variable that is True iff node i is coloured with the colour k.

- Each node has to be colored with at least one color:

$$\forall i \in [1, n], x_i^1 \vee x_i^2 \dots x_i^k$$

- If a node is coloured with a colour a, the other colours are forbidden:

$$\forall i \in [1, n], \forall a \neq b \in [1, k], : \neg x_i^a \vee \neg x_i^b$$

  (This is a translation of $x_i^a \rightarrow \neg x_i^b$)

- Forbid two nodes that share an edge to be coloured with the same colour

$$\forall \{i, j\} \in E, \forall a \in [1, k] : \neg x_i^a \rightarrow \neg x_j^a$$

  (This is a translation of $x_i^a \rightarrow \neg x_j^a$)

The Example of Graph Coloring: The Model Size

What is the (space) size of the model?

- $n \times k$ Boolean variables
- Constraints form 1: n clauses with k literals each
- Constraints form 2: $n \times k^2$ binary clauses
- Constraints form 3: $m \times k$ binary clauses

The Example of Graph Coloring: The Minimization Version

- Propose a method that uses SAT for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

A Straightforward Approach

- Find a valid upper bound UB and a lower bound LB for k
- Run iteratively the decision version until converging to the optimal value
- Let's call $SAT(V, E, K)$ the SAT model of the decision version of the problem (i.e., can we find a valid colouring of $G(V, E)$ with k colours). Use $SAT(V, E, K)$ as an oracle within an iterative search. For instance:
  - **Decreasing linear Search:** Run iteratively $SAT(V, E, UB-1), SAT(V, E, UB-2), \ldots$ until the problem is unsatisfiable. The last satisfiable value of k is the optimal value
  - **Binary search:** Run iteratively $SAT(V, E, z)$ as long as $UB > LB$ where $z = \lceil (UB - LB)/2 \rceil$. If the result is satisfiable, then and $UB \leftarrow z$ otherwise $LB \leftarrow z$

Upper/Lower Bound?

- Upper bound: For instance, we can run the following iterative greedy algorithm:
  - Each vertex v is considered non-coloured and has a portfolio S_v of available colours. The set is initially $\{1, 2, \ldots n\}$ for each vertex
  - At leach iteration, look for a non-coloured vertex v that has the greatest number of non coloured neighbours. Colour it with the smallest colour in S_v and remove its colour from all its neighbours.
  - The resulting colouring is valid and the the upper bound is the number of different colours used.
  - The run time complexity is $O(n^2 \times m)$
- Lower bound: Well, we can simply consider 2 as long as there is an edge. A more advanced one is to look for a clique in the graph.
- An alternative approach is to look for valid theoretical bounds in the literature.

Modelling Cardinality Constraints

- The general form of cardinality constraints is the following:

$$a \leq \sum_{1}^{n} x_i \leq b$$

  where a and b are positive integers and $x_1 \dots x_n$ are Boolean variables
- Cardinality constraints are everywhere!
- Many ways to encode such constraints. See for instance `https://www.carstensinz.de/papers/CP-2005.pdf`

Quadratic encoding for $\sum_1^n x_i = 1$

- At least one constraint:

$$x_1 \vee x_2 \ldots x_n$$

- at most one constraints:

$$\forall i, j : \neg x_i \vee \neg x_j$$

This generates one clause of size n and (n^2) binary clauses without introducing additional variables.

Linear encoding for $\sum_1^n x_i = 1$

New variables are added as follows: for $i \in [1, n]$, y_i is a new variable that is true iff $\sum_{l=1}^{l=i} x_l = 1$.

$$x_1 \vee x_2 \dots x_n$$

$$y_n^1$$

$$\forall i \in [1, n-1] : y_i \to y_{i+1}$$

$$\forall i \in [1, n-1] : y_i \to \neg x_{i+1}$$

$$\forall i \in [1, n] : x_i \to y_i$$

Size: n new variables, 1 $n-$ary clause and $3 \times n$ binary clauses,

Linear encoding for $\sum_1^n x_i \geq k$

New variables: $\forall z \in [0,k], \forall i \in [1,n], y_i^z \iff \sum_{l=1}^{l=i} x_l \geq z$

$$\forall i \in [0,n] : y_i^0 \leftarrow 1$$

$$y_1^1 \leftarrow x_1 \ and \ \forall z \in [2,k], y_1^z \leftarrow 0$$
$$y_n^k \leftarrow 1$$

$$\forall i \in [1,n], \forall z \in [1,k-1] : y_i^{z+1} \rightarrow y_i^z$$

$$\forall i \in [1,n-1], \forall z \in [1,k] : y_i^z \rightarrow y_{i+1}^z$$

$$\neg y_{i-1}^z \rightarrow \neg y_i^{z+1}$$

$$y_{i-1}^z \wedge x_i \rightarrow y_i^{z+1}$$

Linear encoding for $\sum_1^n x_i \geq k$

Size of the encoding:

- $\Theta(n \times k)$ variables
- $\Theta(n + k)$ unary clauses
- $\Theta(n \times k)$ binary clauses
- $\Theta(n \times k)$ ternary clauses

Linear encoding for $\sum_1^n x_i = k$?

- Encode $\sum_1^n x_i \geq k+1$
- Force y_n^{k+1} to be false and y_n^k to be true

Size of the encoding: Same as $\sum_1^n x_i \geq k$ (asymptotically)

Linear encoding for $\sum_1^n x_i \leq k$?

- Encode $\sum_1^n x_i \geq k+1$
- Force y_n^{k+1} to be false

Size of the encoding: Same as $\sum_1^n x_i \geq k$ (asymptotically)

Linear encoding for $a \leq \sum_1^n x_i \leq b$?

- Encode $\sum_1^n x_i \leq b$
- Force y_n^a to be true

Size of the encoding: Same as $\sum_1^n x_i \geq b$ (asymptotically)

Extensions: MaxSAT

- MaxSAT is an optimisation extension of SAT where some clauses are "hard" (must be satisfied) and others are "soft" (can be violated).
- The task is to find an assignment of the variables that satisfy the hard clauses and maximises the number of "soft" clauses
- MaxSAT:
  - Variables: Booleans, Clauses: hard and soft clauses
  - Maximisation problem: Is there an assignment of the variables that satisfy all the hard clauses, and maximises the number of satisfied soft clauses?
- Weighted MaxSAT: Extension of MaxSAT where every soft clause is associated with a weight
- Objective: satisfy hard clauses and maximizes the weighted sum of satisfied soft clauses.
- Check the MaxSAT competition

Example of applications for MaxSAT

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. In the (decision version of the) graph colouring problem, we are given k colours to colour the graph such that no two adjacent nodes share the same colour.

- Propose a MaxSAT model for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

The Example of Graph Coloring: A Possible MaxSAT Model

- We shall extend the previous model:
- Consider the previous model $SAT(V, E, k)$ with k an upper bound.
- All the previous clauses are hard.
- Add the following hard clauses:

$$\forall i \in [1, n], \forall a \in [1, k] : \neg u_a \rightarrow \neg x_i^a$$

- Eventually we can add symmetry constraints: $u_a \rightarrow u_{a-1}$
- Then add the soft clauses:

$$\forall a \in [1, k] : \neg u_a$$

- A MaxSAT Optimal solution satisfies all the hard coloring clauses (valid colouring) and maximizes the number of non used colours.

Extensions: Quantified Boolean Formula (QBF)

- A QBF has the form Q.F , where F is a CNF-SAT formulae, and Q is a sequence of quantified variables ($\forall x$ or $\exists x$).
- Example $\forall x, \exists y, \exists z,\ (x \vee \neg y) \wedge (\neg y \vee z)$
- QBF Solver Competition: `https://www.qbflib.org/solvers_list.php`
- QBF is less used in practice

Other Extensions

- Satisfiability Modulo Theories
- Answer Set Programming
- More generally: Automated reasoning community
- Check the SAT/SMT summer schools
  `http://satassociation.org/sat-smt-school.html`

Modern SAT Solvers: Conflict Driven Clause Learning (CDCL)

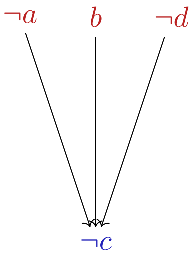
- [Silva and Sakallah, 1999, Moskewicz et al., 2001]
- DPLL [Davis et al., 1962] $\oplus$ Resolution [Robinson, 1965]
- DPLL: Backtracking + Unit Propagation
- Resolution: Learning based on the rule $(l \vee c_1) \wedge (\neg l \vee c_2) \Rightarrow (c_1 \vee c_2)$
- **Can be seen as a CP Solver (Search, propagation) augmented by clause learning**
- But also :
  - Activity-based branching
  - Lazy data structures (2-Watched Literals)
  - Clause Database Reduction
  - Simplifications
  - Restarts
  - ...

Exercise: Propose a filtering algorithm for clauses. The algorithm takes as input a clause and has access (read and write) for the variables domains.

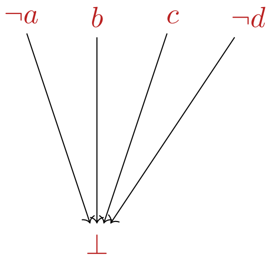
Unit Propagation

Given a clause C of arity n. If $n-1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$



$\neg a \wedge b \wedge \neg d \Rightarrow \neg c$

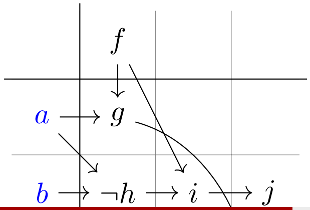


$\neg a \wedge b \wedge c \wedge \neg d \Rightarrow \perp$

Two Watched Literals

- Unit propagation is implemented with an “intelligent” data structure called Two-watched literals
- Observe first that propagation happens only in two cases:
  - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
  - All literals are instantiated and none of them satisfy the clause
- Therefore for each clause C, as long as there are two literals non instantiated in C, nothing happens!
- The idea of the Two-watched literals is to keep 2 literals for every clause that are not instantiated. Those literals will “watch the clause” and guarantee that no propagation is needed.
- If a literal watching a clause C becomes $false$, look for replacement. If no replacement found, then perform propagation

Implication Graph



$\neg a \lor \neg f \lor g$

$\neg a \lor \neg f \lor g$

$\neg a \lor \neg b \lor \neg h$

$\neg a \lor \neg b \lor \neg h$

$\neg a \lor \neg b \lor \neg h$

$a \lor c$

$a \lor c$

$a \lor c$

$a \lor \neg i \lor \neg l$

$a \lor \neg i \lor \neg l$

$a \lor \neg i \lor \neg l$

$a \lor \neg i \lor \neg l$

$a \lor \neg k \lor \neg j$

$a \lor \neg k \lor \neg j$

$a \lor \neg k \lor \neg j$

$c \lor h \lor n \lor \neg m$

$c \lor h \lor n \lor \neg m$

$c \lor h \lor n \lor \neg m$

$c \lor h \lor n \lor \neg m$

$c \lor h \lor n \lor \neg m$

$c \lor l$

$c \lor l$

$d \lor \neg k \lor l$

$d \lor \neg k \lor l$

$d \lor \neg k \lor l$

$d \lor \neg g \lor l$

$d \lor \neg g \lor l$

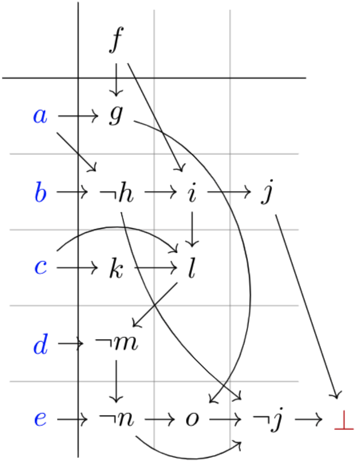
$d \lor \neg g \lor l$

$d \lor \neg g \lor l$

$\neg g \lor n \lor o$

$\neg g \lor n \lor o$

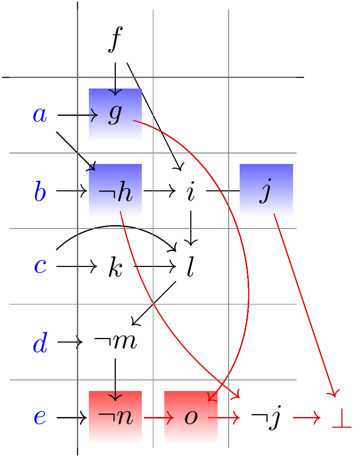
Implication Graph

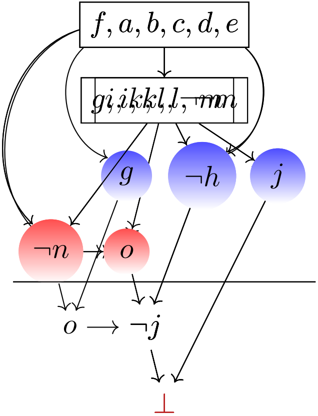


$\neg a \lor \neg f \lor g$
$\neg a \lor \neg b \lor \neg h$
$a \lor c$
$a \lor \neg i \lor \neg l$
$a \lor \neg k \lor \neg j$
$b \lor d$
$b \lor g \lor \neg n$
$b \lor \neg f \lor n \lor k$
$\neg c \lor k$
$\neg c \lor \neg k \lor \neg i \lor l$

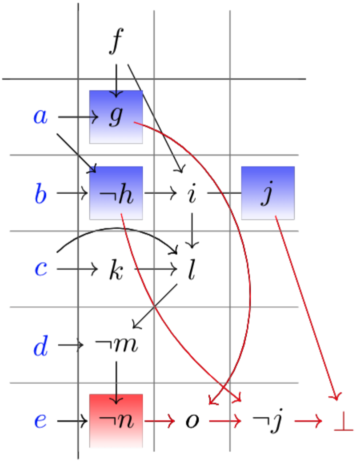
$c \lor h \lor n \lor \neg m$
$c \lor l$
$d \lor \neg k \lor l$
$d \lor \neg g \lor l$
$\neg g \lor n \lor o$
$\boxed{h \lor \neg o \lor \neg j \lor n}$
$\neg i \lor j$
$\neg d \lor \neg l \lor \neg m$
$\neg e \lor m \lor \neg n$
$\neg f \lor h \lor i$

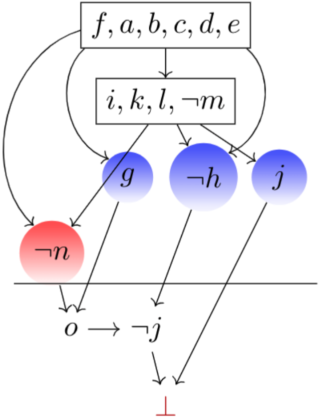
Conflict Analysis



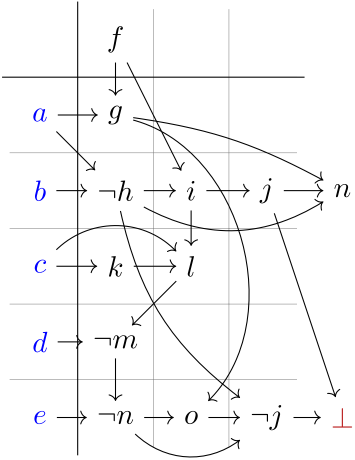


Conflict Analysis





Conflict analysis



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

$\boxed{\neg g \vee h \vee \neg j \vee n}$

$\boxed{\neg g \vee h \vee \neg j \vee n}$

Boosting Search through Randomization and Restarts [Gomes et al., 1998]

Heavy-tail phenomena (SAT and CP)

At any time during the experiment there is a non-negligible probability of hitting a problem that requires exponentially more time to solve than any that has been encountered before.

Hardness = Instance $\oplus$ deterministic algorithm.

- Randomization: breaking ties, random decision between k best choices, ...
- Restarts: Geometric/Luby

Other techniques

- Forgetting clauses: The number of the learnt clauses can be exponential, we sometimes need to free some space by forgetting some clauses.
- VSIDS (Variable State Independent Decaying Sum): VSIDS is a popular variable ordering heuristic that is based on the notion of activity. The activity of a variable is measured by the number of times it participates in the conflict analysis. Each time we use a variable x during conflict analysis, we increment its activity. From time to time, we divide the counters by a constant (to diminish the effect of early conflicts).

SAT Solvers (Few examples)

- MiniSat: http://minisat.se/
- Glucose: `http://www.labri.fr/perso/lsimon/glucose/`
- LingeLing `http://fmv.jku.at/lingeling`
- Any Solver by Armin Biere
  `http://fmv.jku.at/software/index.html`
- Any winner from past and future SAT competitions:
  https://www.satcompetition.org/

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