SAT: Modelling and Implementations

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Context: Solving (Very) Hard Combinatorial Problems

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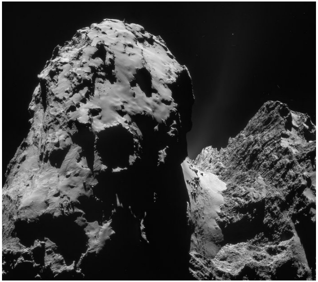

https://homepages.laas.fr/ehebrard/rosetta.html

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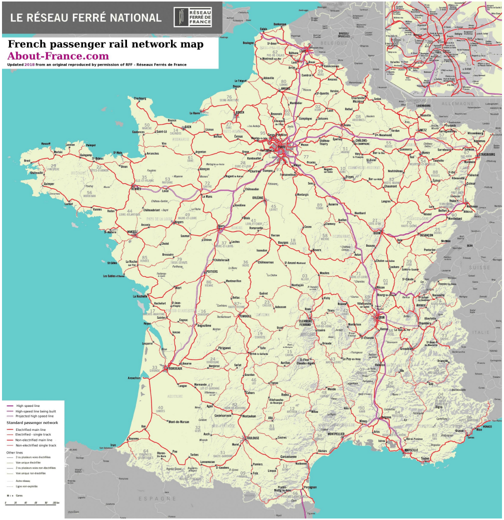

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- Resources: many.. a good start would be the online course on discrete optimisation
  `https://www.coursera.org/learn/discrete-optimization`

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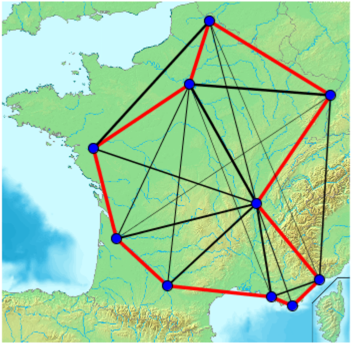
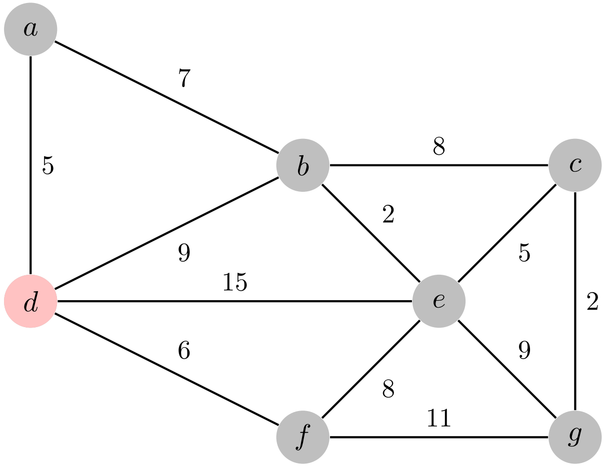
- Very active community

Travelling Salesman Problem



Exemple



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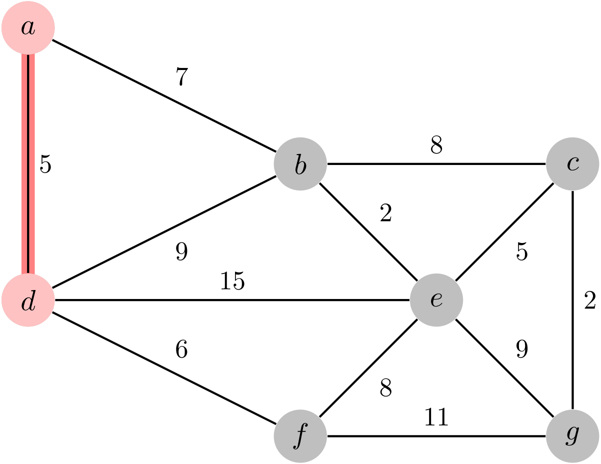

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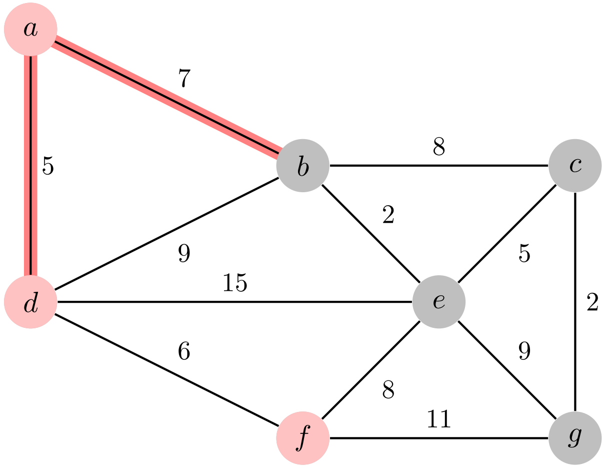

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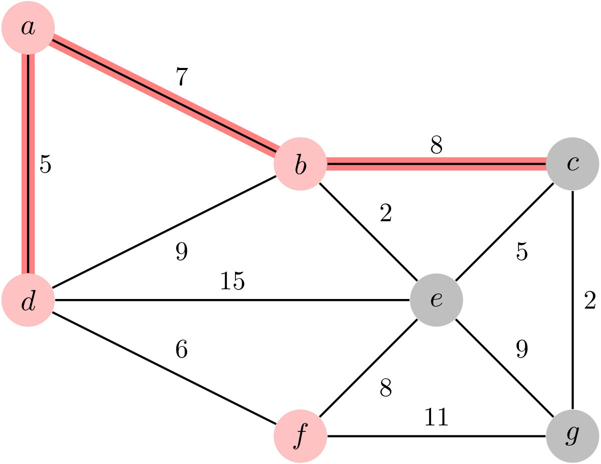

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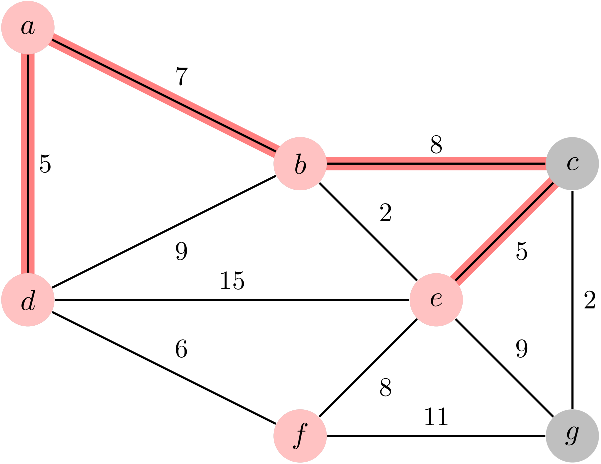

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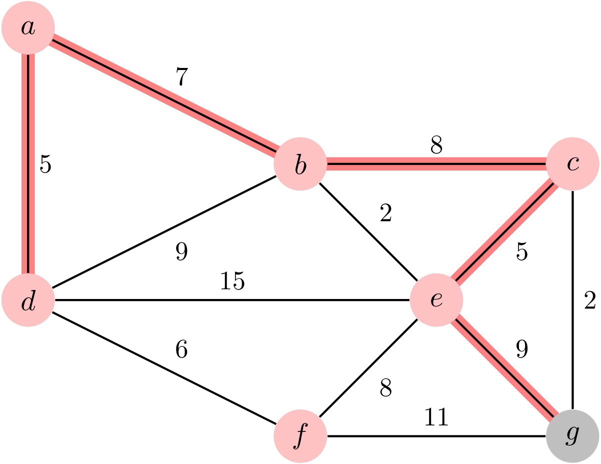

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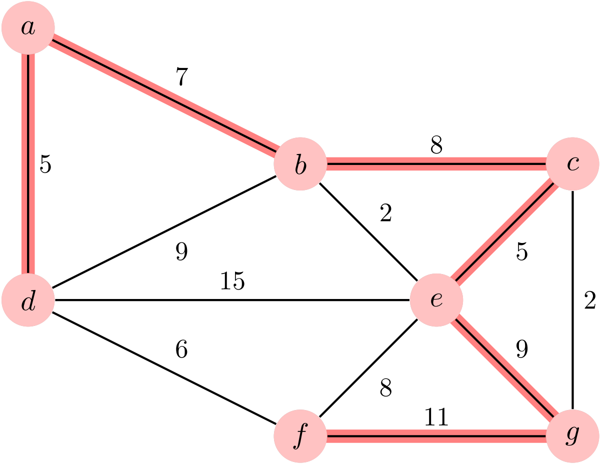

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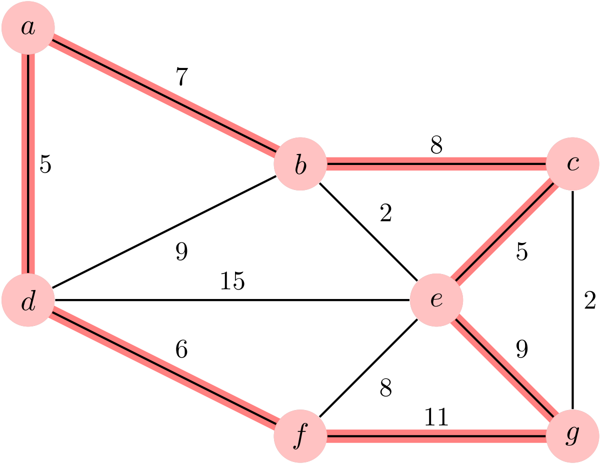

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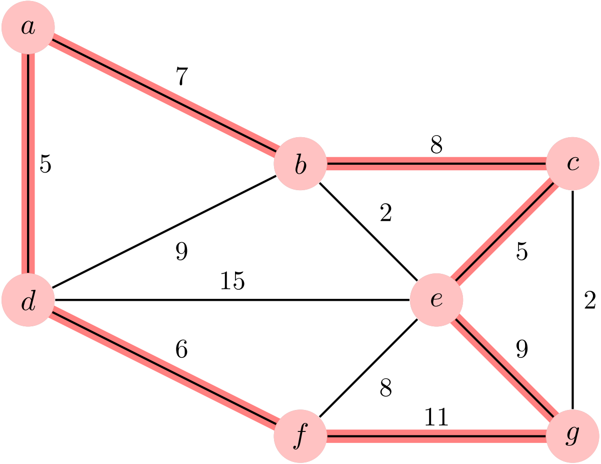
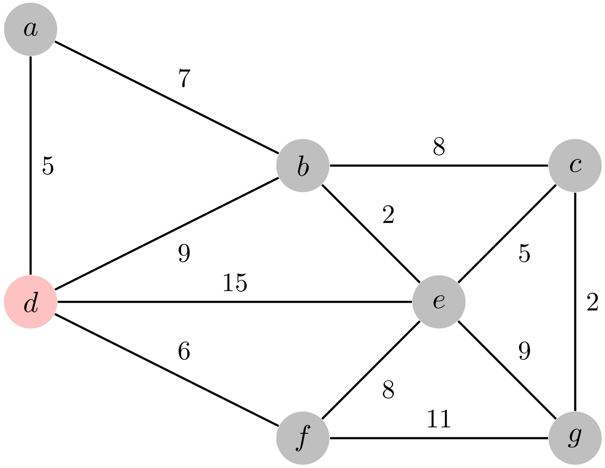


$--> Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53Km$

Example



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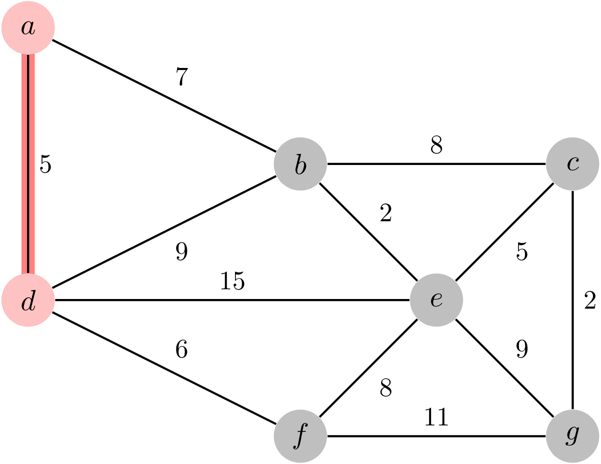

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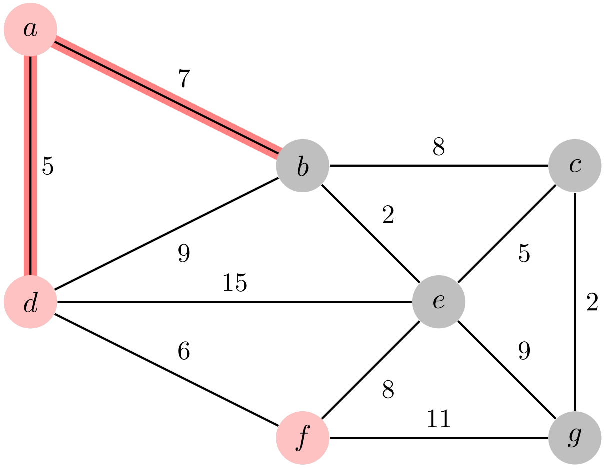

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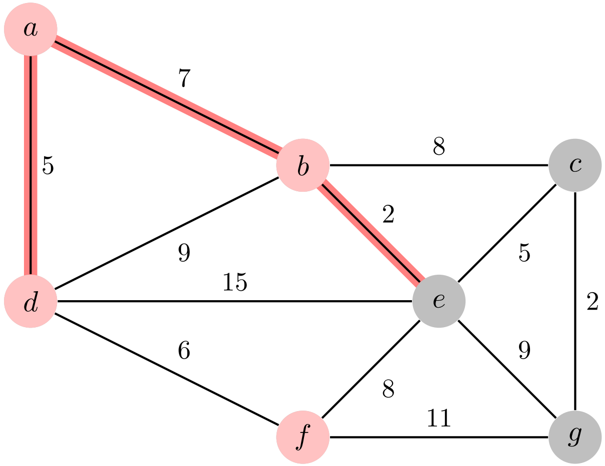

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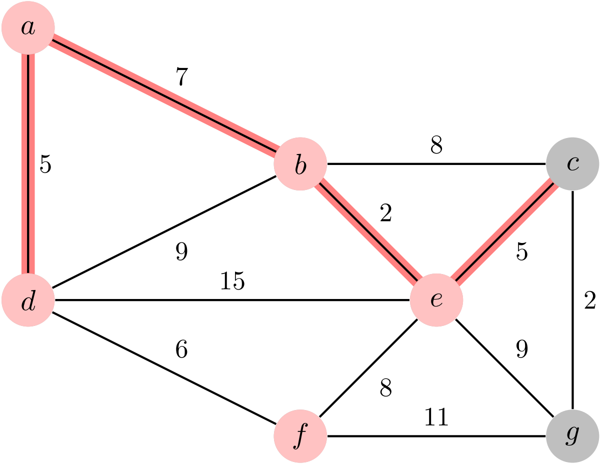

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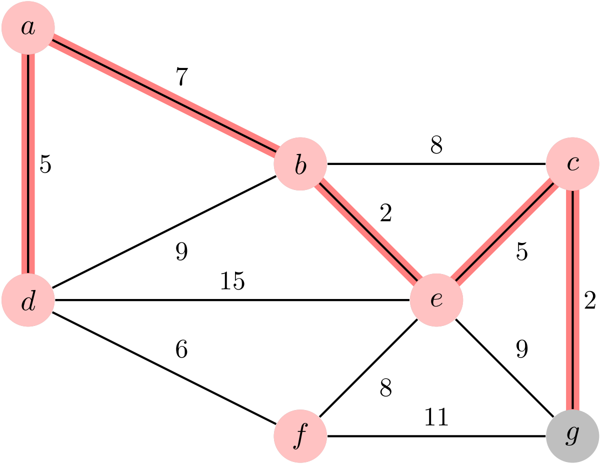

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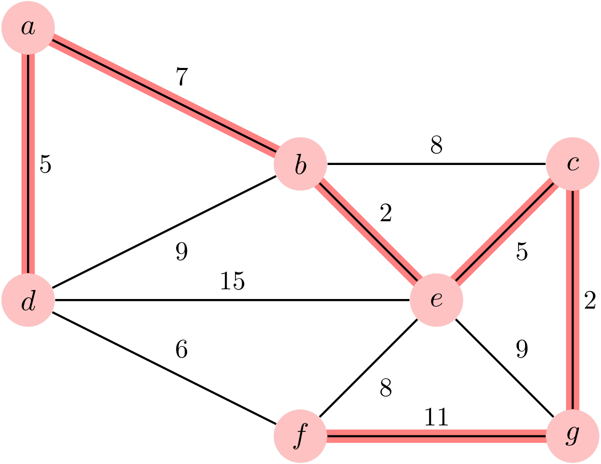

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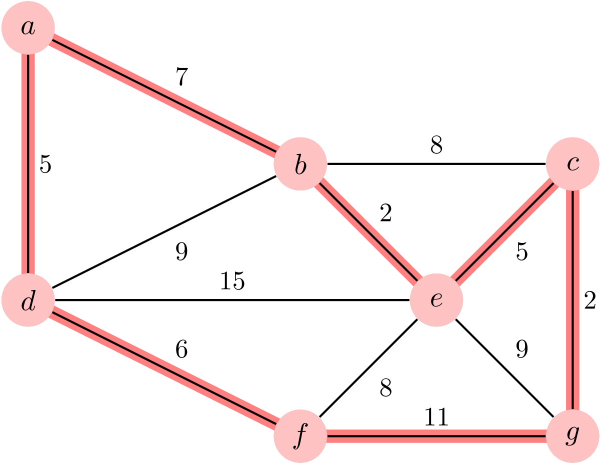

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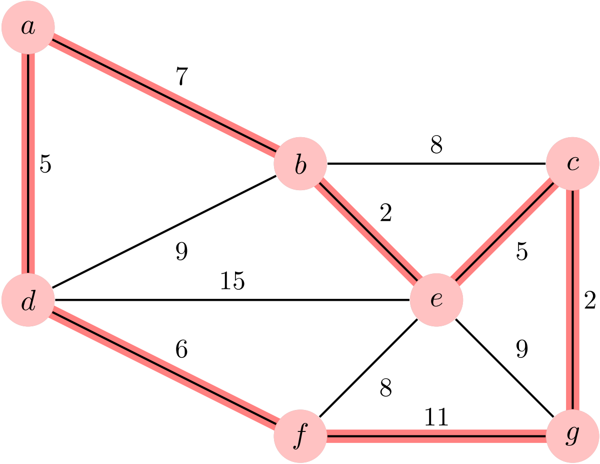

$--> Cost : 5 + 7 + 2 + 5 + 2 + 11 + 6 = 38Km$

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The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

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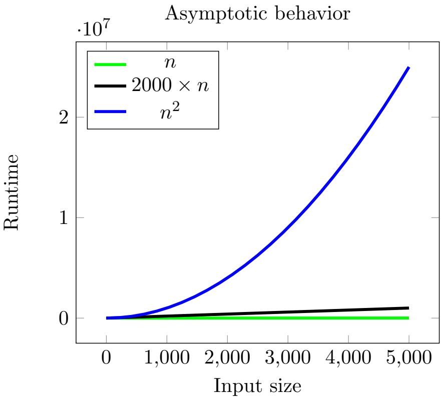

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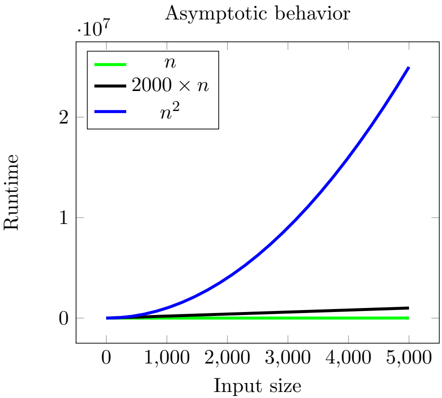

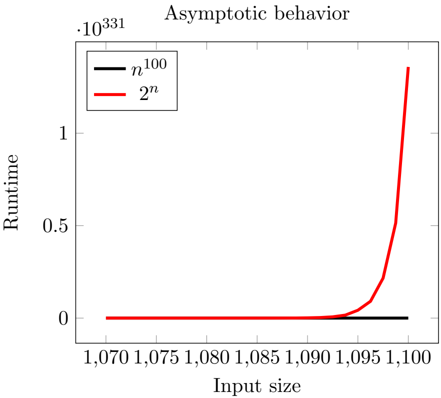

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- What if we don't know if a problem has a polynomial time algorithm?

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- **1 Million $** question: Is P=NP?

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Given a set of Boolean variables $x_1, \ldots x_n$ and a CNF formulae Φ over $x_1, \ldots x_n$, the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

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- How to use it to solve problems (Modelling)
- Discover some efficient implementations

Example

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$x \vee \neg y \vee z$

$\neg x \vee \neg z$

$y \vee w$

$\neg w \vee \neg x$

Example

$x \lor \neg y \lor z$

$\neg x \lor \neg z$

$y \lor w$

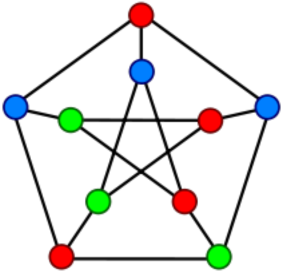
$\neg w \lor \neg x$

A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

The example of Graph Colouring

- Graph Coloring is a well know combinatorial problem that has many applications (in particular in scheduling problems)
- Let $G = (V, E)$ be an undirected graph where V is a set of n vertices and E is a set of m edges. Is it possible to colour the graph with k colours such that no two adjacent nodes share the same colour?



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The Example of Graph Coloring: The Model Size

What is the (space) size of the model?

- $n \times k$ Boolean variables
- Constraints form 1: n clauses with k literals each
- Constraints form 2: $n \times k^2$ binary clauses
- Constraints form 3: $m \times k$ binary clauses

The Example of Graph Coloring: The Minimization Version

- Propose a method that uses SAT for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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    - **Binary search:** Run iteratively $SAT(V, E, z)$ as long as $UB > LB$ where $z = \lceil (UB - LB)/2 \rceil$. If the result is satisfiable, then and $UB \leftarrow z$ otherwise $LB \leftarrow z$

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- An alternative approach is to look for valid theoretical bounds in the literature.

Modelling Cardinality Constraints

- The general form of cardinality constraints is the following:

$$a \leq \sum_{1}^{n} x_i \leq b$$

  where a and b are positive integers and $x_1 \ldots x_n$ are Boolean variables
- Cardinality constraints are everywhere!
- Many ways to encode such constraints. See for instance `https://www.carstensinz.de/papers/CP-2005.pdf`

Quadratic encoding for $\sum_1^n x_i = 1$

- At least one constraint:

$$x_1 \vee x_2 \ldots x_n$$

- at most one constraints:

$$\forall i, j : \neg x_i \vee \neg x_j$$

This generates one clause of size n and (n^2) binary clauses without introducing additional variables.

Linear encoding for $\sum_1^n x_i = 1$

New variables are added as follows: for $i \in [1, n], y_i$ is a new variable that is true iff $\sum_{l=1}^{l=i} x_l = 1$.

$$x_1 \vee x_2 \dots x_n$$

$$y_n^1$$

$$\forall i \in [1, n-1] : y_i \rightarrow y_{i+1}$$

$$\forall i \in [1, n-1] : y_i \rightarrow \neg x_{i+1}$$

$$\forall i \in [1, n] : x_i \rightarrow y_i$$

Size: n new variables, 1 $n-$ary clause and $3 \times n$ binary clauses,

Linear encoding for $\sum_1^n x_i \geq k$

New variables: $\forall z \in [0, k], \forall i \in [1, n], y_i^z \iff \sum_{l=1}^{l=i} x_l \geq z$

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$$y_{i-1}^z \wedge x_i \rightarrow y_i^{z+1}$$

Linear encoding for $\sum_1^n x_i \geq k$

Size of the encoding:

- $\Theta(n \times k)$ variables
- $\Theta(n + k)$ unary clauses
- $\Theta(n \times k)$ binary clauses
- $\Theta(n \times k)$ ternary clauses

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- Check the MaxSAT competition

Example of applications for MaxSAT

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. In the (decision version of the) graph colouring problem, we are given k colours to colour the graph such that no two adjacent nodes share the same colour.

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The Example of Graph Coloring: A Possible MaxSAT Model

- We shall extend the previous model:
- Let u_a be a Boolean variable that is True iff. the colour $a \in [1, k]$ is used
- Consider the previous model $SAT(V, E, k)$ with k an upper bound.
- All the previous clauses are hard.
- Add the following hard clauses:

$$\forall i \in [1, n], \forall a \in [1, k] : \neg u_a \rightarrow \neg x_i^a$$

- Eventually we can add symmetry breaking constraints: $u_a \rightarrow u_{a-1}$
- Then add the soft clauses:

$$\forall a \in [1, k] : \neg u_a$$

- A MaxSAT Optimal solution satisfies all the hard coloring clauses (valid colouring) and maximizes the number of non used colours.

Extensions: Quantified Boolean Formula (QBF)

- A QBF has the form Q.F , where F is a CNF-SAT formulae, and Q is a sequence of quantified variables ($\forall x$ or $\exists x$).
- Example $\forall x, \exists y, \exists z,$ $(x \vee \neg y) \wedge (\neg y \vee z)$
- QBF Solver Competition: `https://www.qbflib.org/solvers_list.php`
- QBF is less used in practice

Other Extensions

- Satisfiability Modulo Theories
- Answer Set Programming
- More generally: Automated reasoning community
- Check the SAT/SMT summer schools
  `http://satassociation.org/sat-smt-school.html`

Modern SAT Solvers: Conflict Driven Clause Learning (CDCL)

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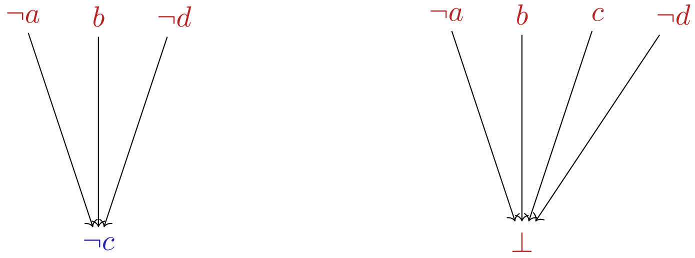
- But also :
  - Activity-based branching
  - Lazy data structures (2-Watched Literals)
  - Clause Database Reduction
  - Simplifications
  - Restarts
  - ...

Exercise: Propose a filtering algorithm for clauses. The algorithm takes as input a clause and has access (read and write) for the variables domains.

Unit Propagation

Given a clause C of arity n. If $n-1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$



$\neg a \wedge b \wedge \neg d \Rightarrow \neg c$

$\neg a \wedge b \wedge c \wedge \neg d \Rightarrow \perp$

Two Watched Literals

- Unit propagation is implemented with an “intelligent” data structure called Two-watched literals
- Observe first that propagation happens only in two cases:
  - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
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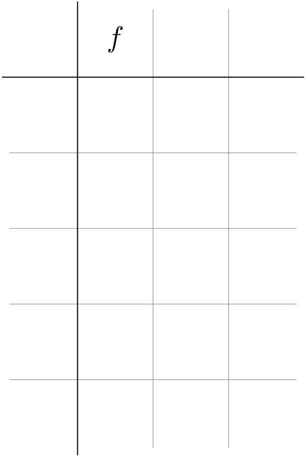
- If a literal watching a clause C becomes $false$, look for replacement. If no replacement found, then perform propagation

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

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$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

$\neg i \vee j$

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$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

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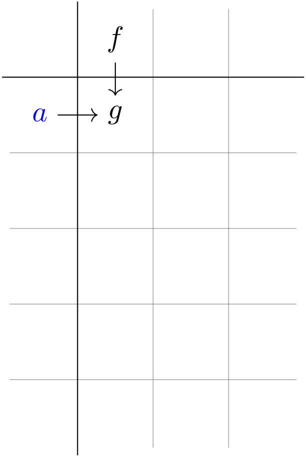

$\neg a \lor \neg f \lor g$
$\neg a \lor \neg b \lor \neg h$
$a \lor c$
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$a \lor \neg k \lor \neg j$
$b \lor d$
$b \lor g \lor \neg n$
$b \lor \neg f \lor n \lor k$
$\neg c \lor k$
$\neg c \lor \neg k \lor \neg i \lor l$

$c \lor h \lor n \lor \neg m$
$c \lor l$
$d \lor \neg k \lor l$
$d \lor \neg g \lor l$
$\neg g \lor n \lor o$
$h \lor \neg o \lor \neg j \lor n$
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$\neg d \lor \neg l \lor \neg m$
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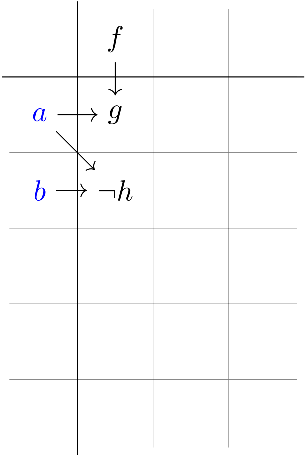
$\neg f \lor h \lor i$

Implication Graph



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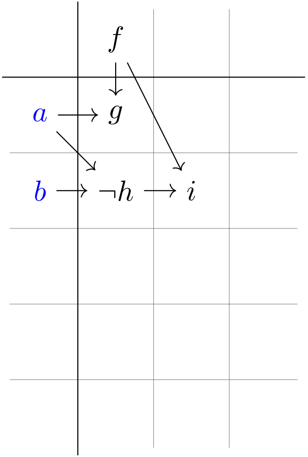

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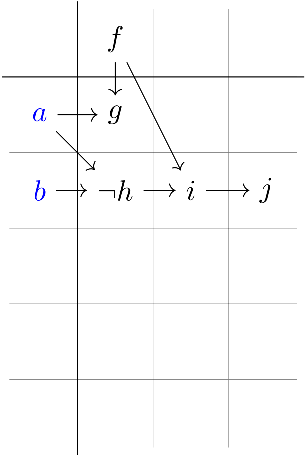

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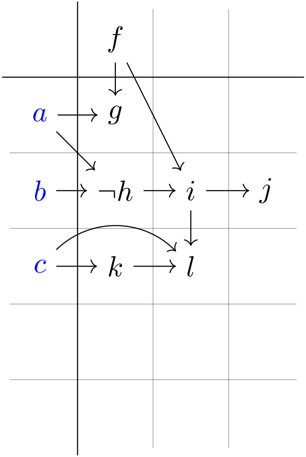

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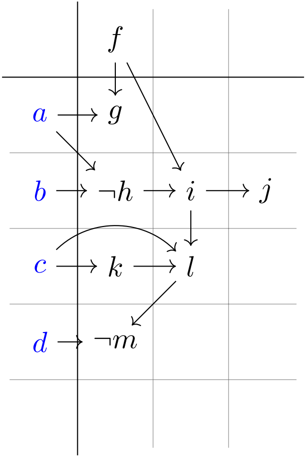

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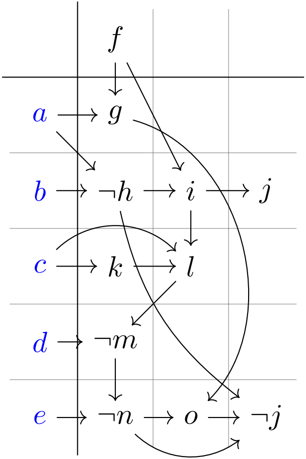

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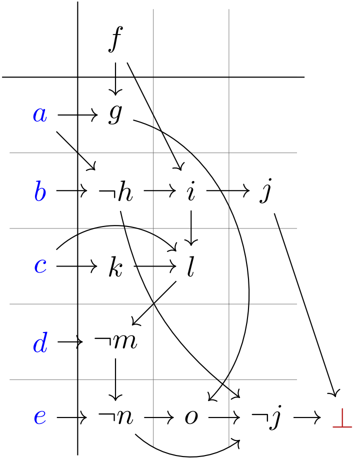

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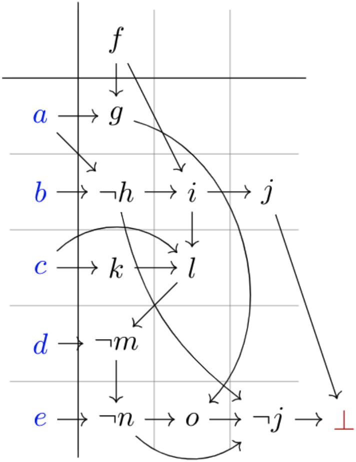

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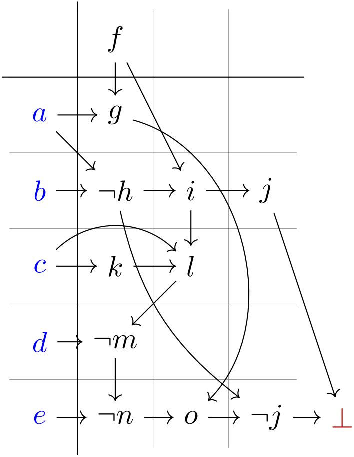
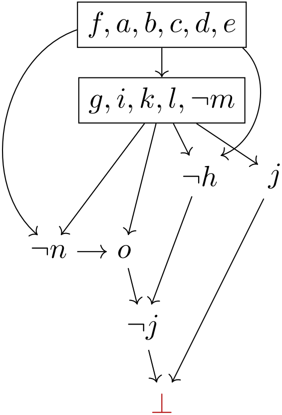
Conflict Analysis





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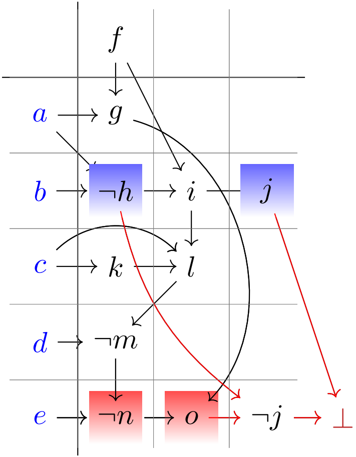

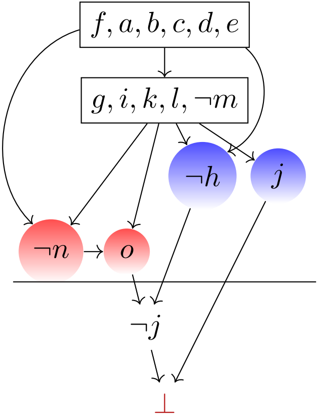

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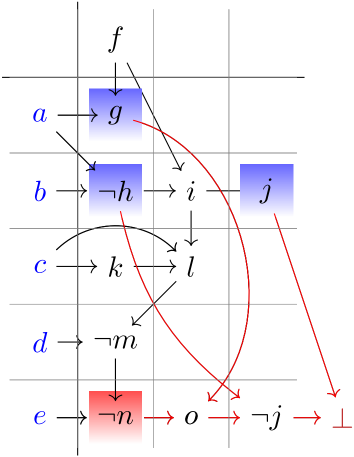

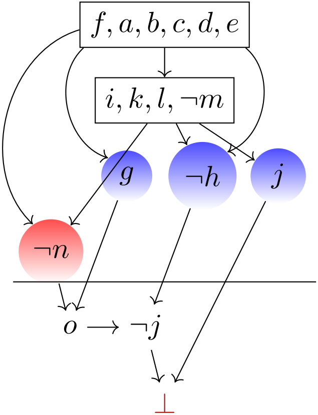

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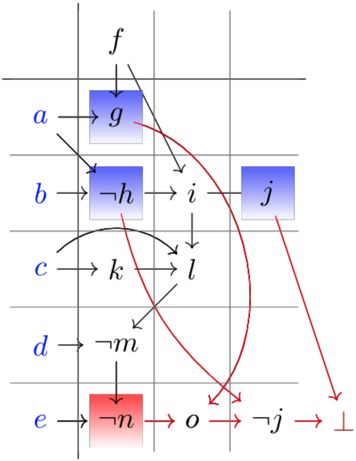

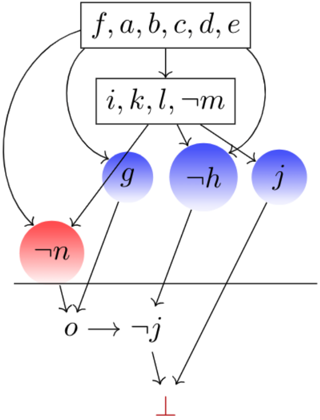

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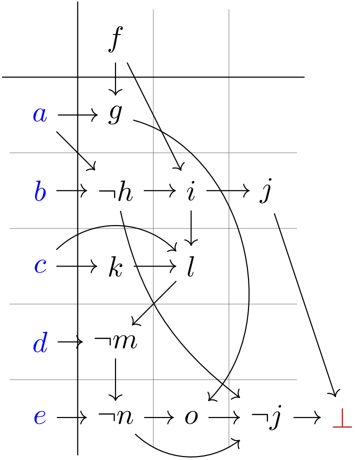

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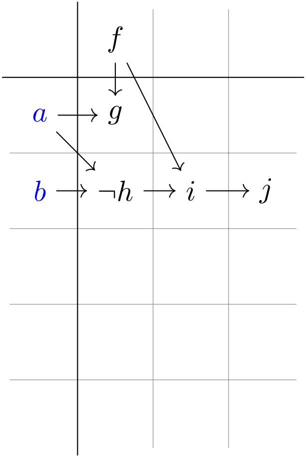

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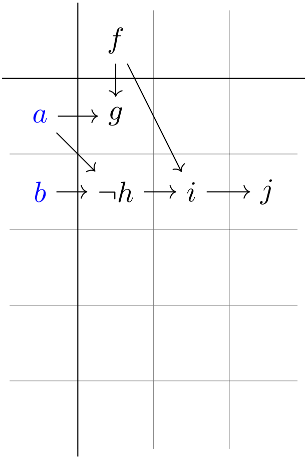

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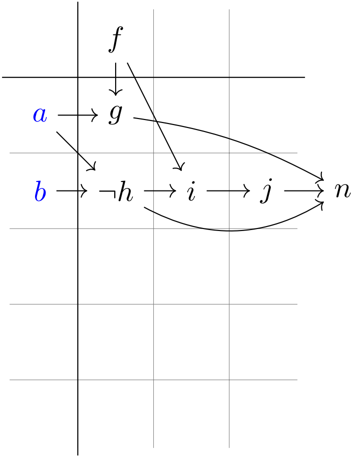

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Learning and Backjumping

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- When there is only one literal *uip* propagated in the last level in the current explanation, learn the associated new clause C, backjump (to the last level of propagated literals in C), propagate $\neg uip$ via the new clause, and continue the exploration

Boosting Search through Randomization and Restarts [Gomes et al., 1998]

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Heavy-tail phenomena (SAT and CP)

At any time during the experiment there is a non-negligible probability of hitting a problem that requires exponentially more time to solve than any that has been encountered before.

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- Randomization: breaking ties, random decision between k best choices, ...
- Restarts: Geometric/Luby

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SAT Solvers (Few examples)

- MiniSat: http://minisat.se/
- Glucose: `http://www.labri.fr/perso/lsimon/glucose/`
- LingeLing `http://fmv.jku.at/lingeling`
- Any Solver by Armin Biere
  `http://fmv.jku.at/software/index.html`
- Any winner from past and future SAT competitions:
  https://www.satcompetition.org/

Back to Constraint Programming

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- Mostly solvable by backtracking algorithms (Search and Filtering)

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Value Ordering

'Succeed-first' [Geelen, 1992]:
"Follow the best chances leading to a solution"

Filtering

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Arc Consistency

Let C be a constraint and D be a list of domains for the variables in the scope of C.
C is Arc Consistent (AC) iff for every variable x in the scope of C, for every value $v \in D(x)$, there exists an assignment w in D satisfying C in which v is assigned to x

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- If all the domains are singleton, the propagator must be able to check if the assignment corresponds to a solution or not.

CP vs. SAT

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- CP vs. SAT: a fundamental difference is the presence of global reasoning win CP.

CP vs. SAT : To decompose or nor to decompose?

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- **Can we find something that takes advantages from both worlds?** $\rightarrow$ **Clause learning in CP**

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Learning in CP

$x_1 + x_7 \geq 4 \wedge$
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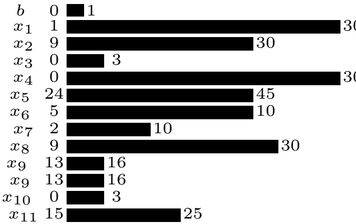

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$[\![x_1 = 1]\!]$

b	0	1
x_1	1	1
x_2	9	30
x_3	0	3
x_4	0	3
x_5	24	45
x_6	5	10
x_7	2	10
x_8	9	30
x_9	13	16
x_9	13	16
x_{10}	0	3
x_{11}	15	25

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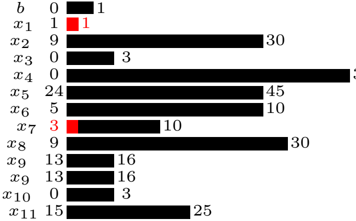

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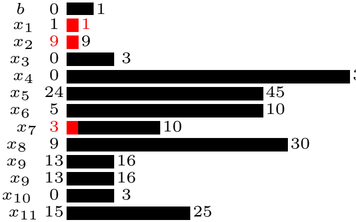

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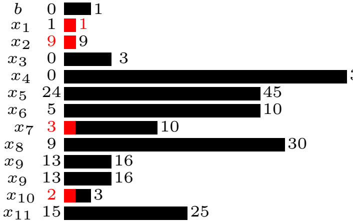

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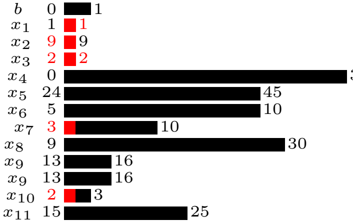

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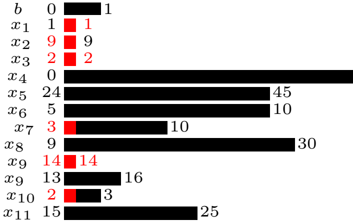

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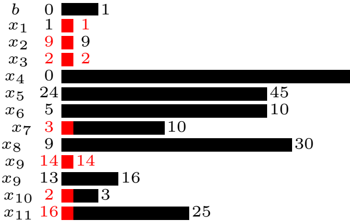

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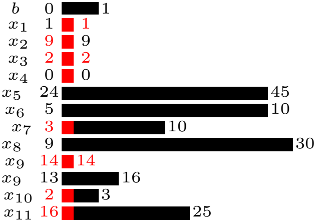

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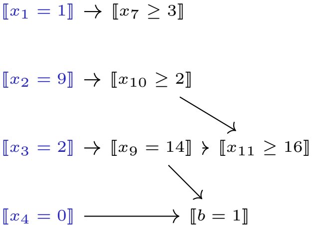

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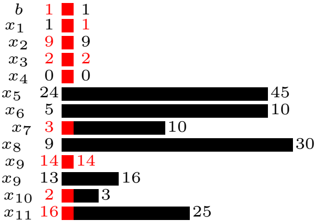

Learning in CP

$$\llbracket x_1 = 1 \rrbracket \rightarrow \llbracket x_7 \geq 3 \rrbracket$$

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$$\llbracket x_3 = 2 \rrbracket \rightarrow \llbracket x_9 = 14 \rrbracket \rightarrow \llbracket x_{11} \geq 16 \rrbracket$$

$$\llbracket x_4 = 0 \rrbracket \longrightarrow \llbracket b = 1 \rrbracket \longrightarrow \llbracket x_6 \geq 7 \rrbracket$$

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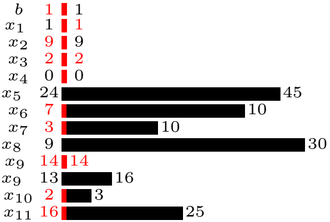

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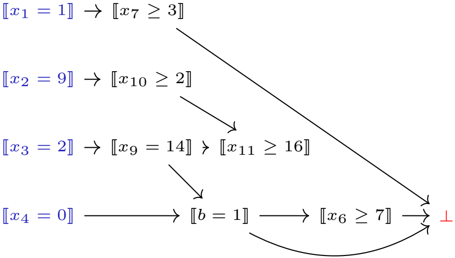

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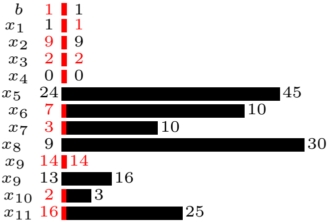

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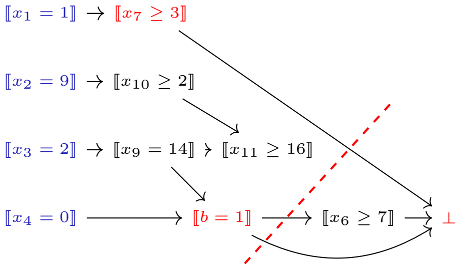

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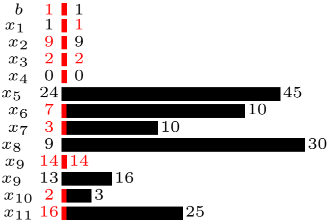
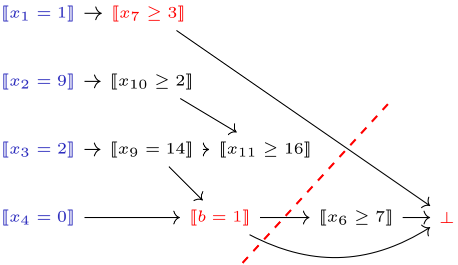


- Conflict analysis: $[\![b = 1]\!] \wedge [\![x_7 \geq 3]\!] \Rightarrow \perp$

Learning in CP



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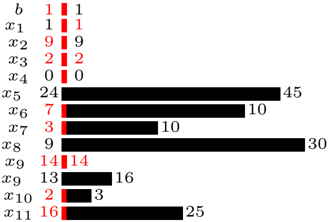

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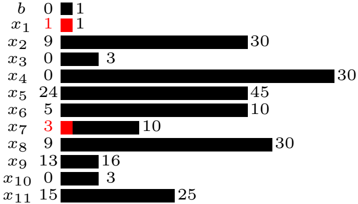

Learning in CP

$[\![x_1 = 1]\!] \rightarrow [\![x_7 \geq 3]\!] \longrightarrow [\![b = 0]\!]$

- Conflict analysis: $[\![b = 1]\!] \wedge [\![x_7 \geq 3]\!] \Rightarrow \bot$
- New clause: $[\![b \neq 1]\!] \vee [\![x_7 \leq 2]\!]$
- Backtrack to level 1
- Propagate the learnt clause

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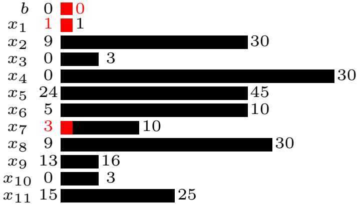
\end{aligned}
$$

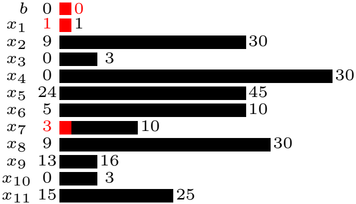


Learning in CP

$[\![x_1 = 1]\!] \rightarrow [\![x_7 \geq 3]\!] \longrightarrow [\![b = 0]\!]$

- Conflict analysis: $[\![b = 1]\!] \wedge [\![x_7 \geq 3]\!] \Rightarrow \bot$
- New clause: $[\![b \neq 1]\!] \vee [\![x_7 \leq 2]\!]$
- Backtrack to level 1
- Propagate the learnt clause
- Continue exploration

$$
\begin{aligned}
& x_1 + x_7 \geq 4 \wedge \\
& x_2 + x_{10} \geq 11 \wedge \\
& x_3 + x_9 = 16 \wedge \\
& x_5 \geq x_8 + x_9 \wedge \\
& b \leftrightarrow (x_9 - x_4 = 14) \wedge \\
& b \rightarrow (x_6 \geq 7) \wedge \\
& b \rightarrow (x_6 + x_7 \leq 9) \wedge \\
& x_{11} \geq x_9 + x_{10}
\end{aligned}
$$



Conflict analysis

Algorithm 1: 1-UIP-with-Propagators

$\Psi \leftarrow explain(\perp)$;
while $|\{q \in \Psi \mid level(q) = current\ level\}| > 1$ **do**
  $p \leftarrow \arg\max_q(\{rank(q) \mid level(q) = current\ level\ \wedge\ q \in \Psi\})$;
  $\Psi \leftarrow \Psi \cup \{q \mid q \in explain(p) \wedge level(q) > 0\} \setminus \{p\}$;
return Ψ ;

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- To enable clause learning in CP, every propagator must be able to explain their filtering in the form of clauses ("Lazy Clause Generation").
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- Example: Explain the constraint $X \leq Y$ with two scenarios (failure and propagation).

Exercise

- Let $(x_1, \ldots, x_n)$ be a sequence of Boolean variables, and let d be a positive integer.
- The CARDINALITY$(x_1, \ldots, x_n, d)$ constraint holds iff exactly d variables from the sequence $(x_1, \ldots, x_n)$ are true.
- Write a filtering algorithm for CARDINALITY.
- What is the time complexity?
- Does it enforce arc consistency?
- Explain the CARDINALITY filtering.

Correction

Algorithm 4: CARDINALITY($[x_1, \dots, x_n], d$)

if $\left|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}\right| > d$ **then**
  $\mathcal{D} \leftarrow \perp$;
if $\left|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}\right| > n - d$ **then**
  $\mathcal{D} \leftarrow \perp$;
if $\left|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}\right| = d$ **then**
  foreach $i \in \{1..n\}$ **do**
    if $\mathcal{D}(x_i) = \{0, 1\}$ **then**
      $\mathcal{D}(x_i) \leftarrow \{0\}$;
else
  if $\left|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}\right| = n - d$ **then**
    foreach $i \in \{1..n\}$ **do**
      if $\mathcal{D}(x_i) = \{0, 1\}$ **then**
        $\mathcal{D}(x_i) \leftarrow \{1\}$;
return $\mathcal{D}$;

Explaining The Cardinality Constraint

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$$x^1 \wedge x^2 \wedge x^{d+1} \rightarrow \bot$$

Where $D(x^i) = \{1\}$

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Where $D(x^i) = \{0\}$

- Explaining the propagating the value 1: the conjunction of all the assigned variables

Explaining The Cardinality Constraint

- Failure 1:
$$x^1 \wedge x^2 \wedge x^{d+1} \rightarrow \perp$$
Where $D(x^i) = \{1\}$
- Failure 2:
$$\neg x^1 \wedge \neg x^2 \wedge \neg x^{n-d+1} \rightarrow \perp$$
Where $D(x^i) = \{0\}$
- Explaining the propagating the value 1: the conjunction of all the assigned variables
- Explaining the propagating the value 0: the conjunction of all the assigned variables

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