A Quadratic Algorithm for the Linearization Problem

July 2022

The linearization problem that we consider in this note is the following: we are given $x_1 \dots x_n$, $l_1 \dots, l_n$, $u_1 \dots u_n \in \mathbb{R}$, the purpose is to find a function $f(x) = ax + b$ such that $l_j \leq ax_j + b \leq u_j$ for all $j = 1 \dots n$.

Consider $\alpha_j = f(x_j) - l_j$ the slack of x_j. Given a solution f, we call x_i a pivot of f if $\alpha_i = min\{\alpha_j \mid j = 1 \dots n\}$. By construction we have $g(x) = f(x) - \alpha_i$ a valid solution. Observe that in this case that $g(x_i) = l_i$. Hence, any solution f with a pivot x_i can be transformed to solution g such that $g(x_i) = l_i$. This can be seen as a dominance relationship. Therefore, the linearizartion problem admits a solution iff there exists a solution g that has a pivot x_i where $g(x_i) = l_i$. We first define a mathematical formulation $\mathcal{M}$ that captures precisely all dominant solutions. Then, we show that $\mathcal{M}$ can be solved in a quadratic time.

We introduce n Boolean variables $d_1 \dots d_n$ where $d_i \Leftrightarrow (ax_i + b = l_i)$. For each i, we define $y_i = max\{\frac{l_j - l_i}{x_j - x_i} \mid j \neq i\}$ and $z_i = min\{\frac{u_j - l_i}{x_j - x_i} \mid j \neq i\}$. Our mathematical model $\mathcal{M}$ is defined as follows:

$$\bigvee_{i=1\dots n} d_i \tag{1}$$

$$\bigwedge_{i=1\dots n} d_i \Leftrightarrow (ax_i + b = l_i) \tag{2}$$

$$\bigwedge_{i=1\dots n} d_i \implies y_i \leq a \leq z_i \tag{3}$$

Proposition 1. *$\mathcal{M}$ is satisfiable iff the linearization problem admits a solution*

Proof. $\Rightarrow$ Suppose that the model is satisfiable. Then $\exists\ d_i = 1$ such that

$$y_i \leq a \leq z_i$$

$$\implies \forall j \neq i, \frac{l_j - l_i}{x_j - x_i} \leq a \leq \frac{u_j - l_i}{x_j - x_i}$$

$$\implies \forall j \neq i, l_j - l_i \leq a(x_j - x_i) \leq u_j - l_i$$

$$\implies \forall j \neq i, l_j \leq ax_j + b \leq u_j$$

Hence, the linearization problem admits a solution.

$\Leftarrow$ Consider a dominant solution $g(x) = ax + b$ with a pivot x_i. Constraint 1 is satisfied by setting $d_i = 1$. Constraints 2 are satisfied since x_i is a pivot. For the third set of constraints, observe that they are fired only for the pivots. For any $d_i = 1$, we have $\forall j \neq i,\ l_j \leq ax_j + b \leq u_j$. Therefore $\forall j \neq i,\ l_j - l_i \leq ax_j + b - (ax_i + b) \leq u_j - l_i$. Hence $y_i \leq a \leq z_i$ and by consequence $\mathcal{M}$ is satifiable. $\square$

In the following, we propose an algorithm to solve $\mathcal{M}$. For that, one have to find a certain i such that $y_i \leq z_i$. Indeed, by doing so, it is enough to pick $a = y_i$ since $y_i \leq a \leq z_i$, fix $b = ax_i - l_i$, $d_i = 1$, and $d_{j \neq i} = 0$. Our proposed algorithm is given below.

Algorithm 1: An Algorithm for $\mathcal{M}$

```
for i ∈ [1 ... n] do
    y ← −∞ ;
    z ← ∞ ;
    for j ∈ [1 ... n] \ {i} do
        w = (l_j − l_i)/(x_j − x_i) ;
        if  y < w then
            y ← w ;
        end
        w = (u_j − l_i)/(x_j − x_i) ;
        if  z > w then
            z ← w ;
        end
    end
    if y ≤ z then
        a ← y_i ;
        b ← ax_i − l_i ;
        d_i ← 1 ;
        return (a, b, d_i) ;
    end
end
return ∅ ;
```

The overall complexity is $O(n^2)$. I'm very sad that I couldn't make it linear. In an online setting, the approach can be adapted as a linear algorithm.