SAT: Modelling and Implementations

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Context: Solving (Very) Hard Combinatorial Problems

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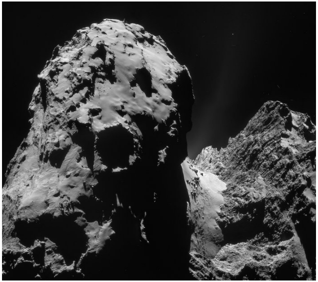

https://homepages.laas.fr/ehebrard/rosetta.html

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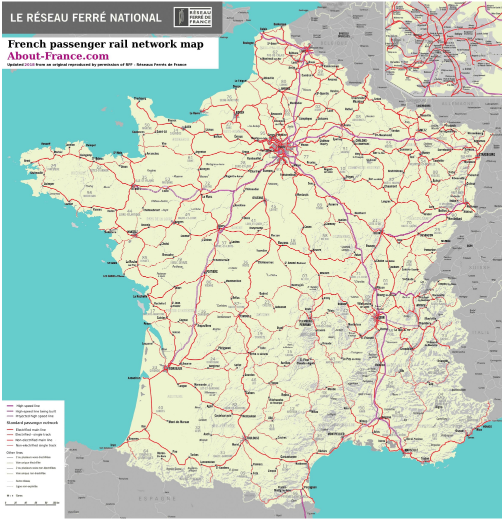

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- Resources: many.. a good start would be the online course on discrete optimisation
  `https://www.coursera.org/learn/discrete-optimization`

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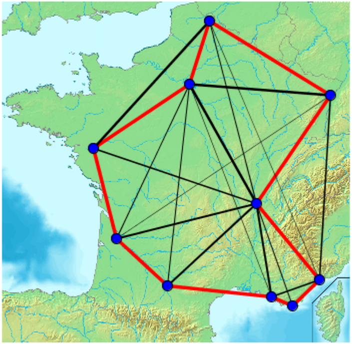
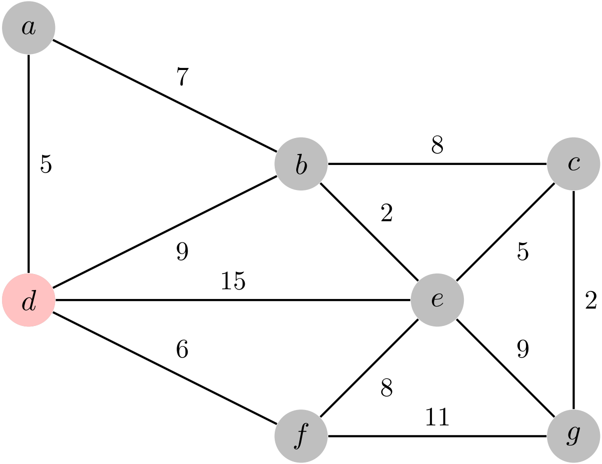
- Very active community

Travelling Salesman Problem



Exemple



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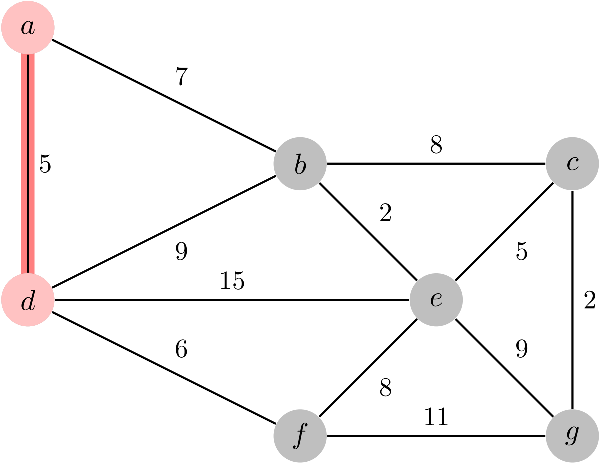

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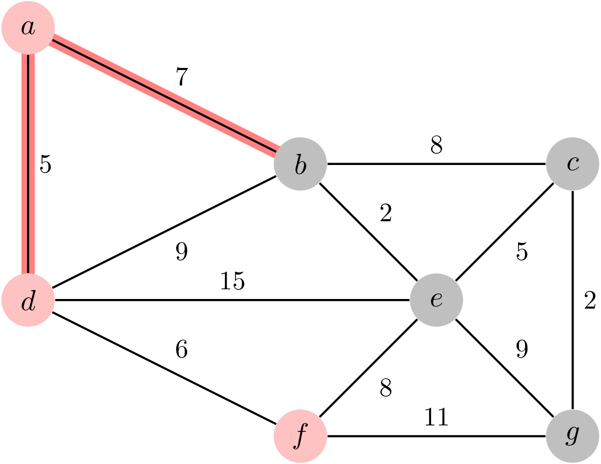

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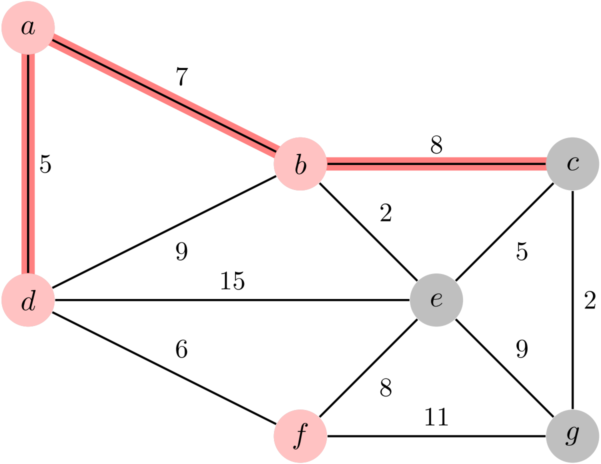

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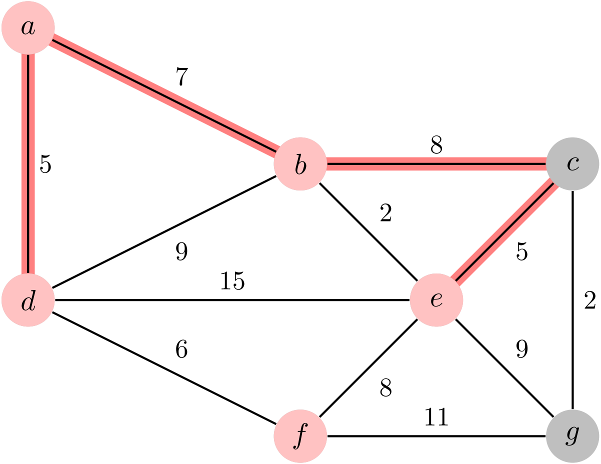

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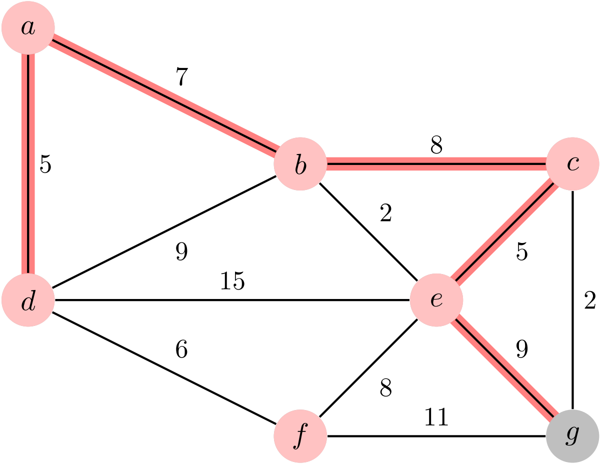

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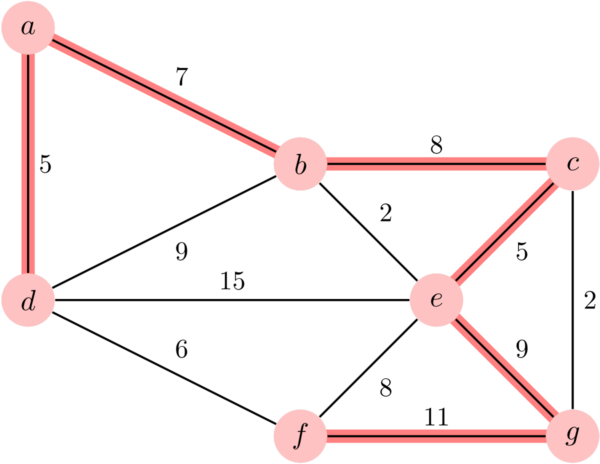

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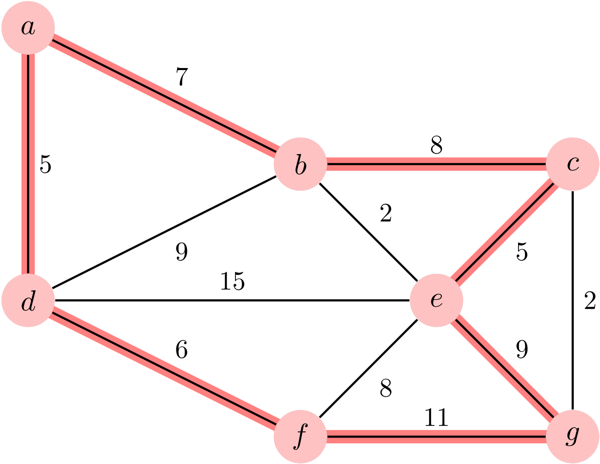

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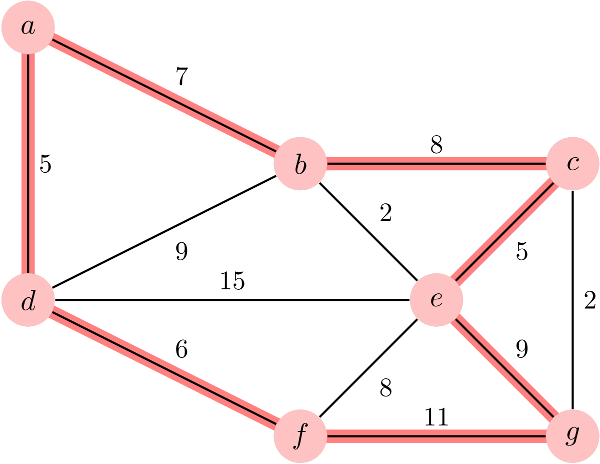
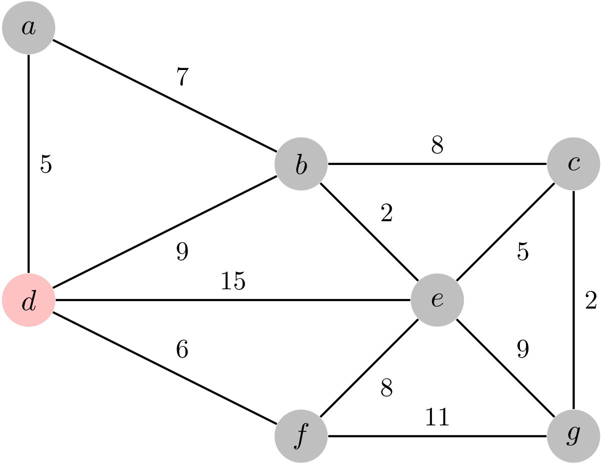


$-- > Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53Km$

Example



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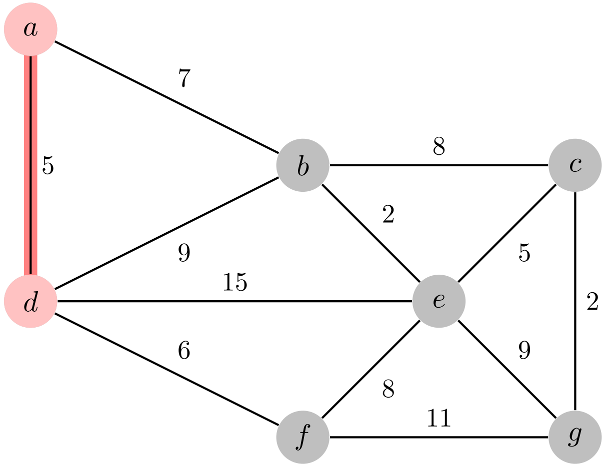

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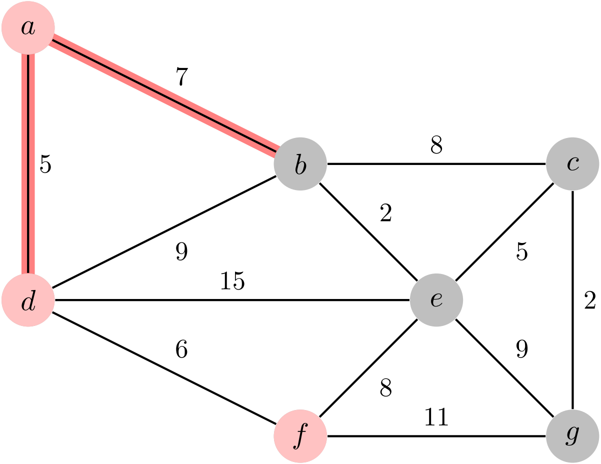

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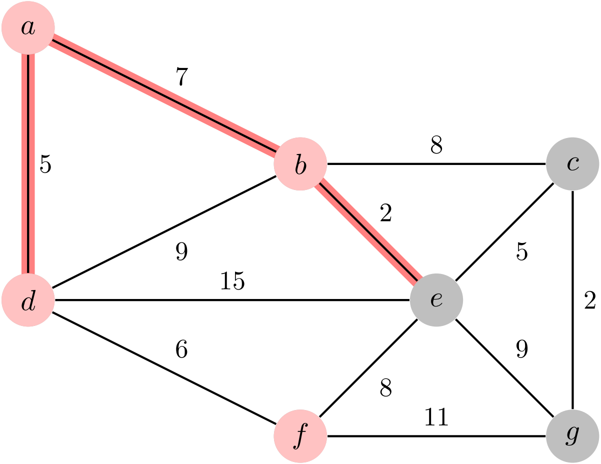

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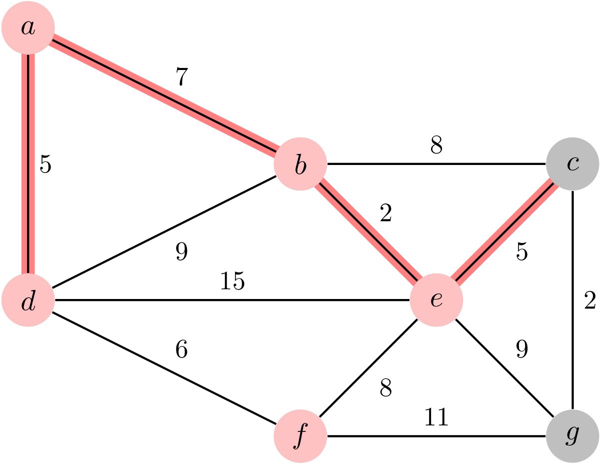

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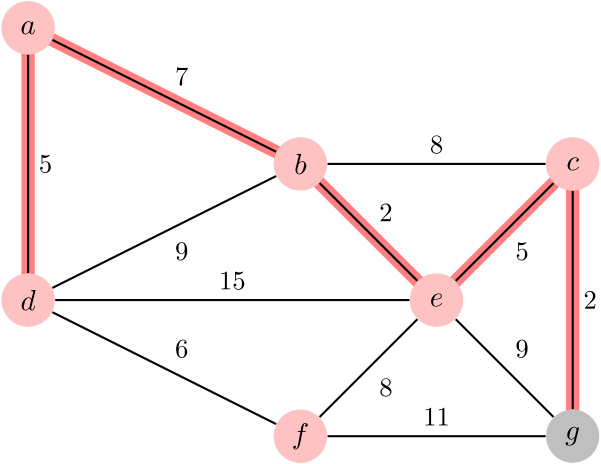

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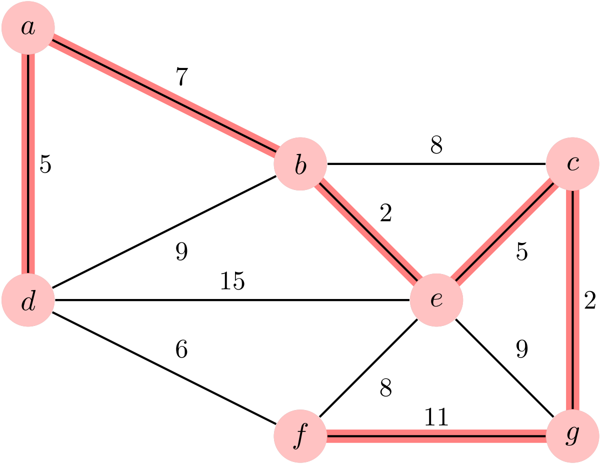

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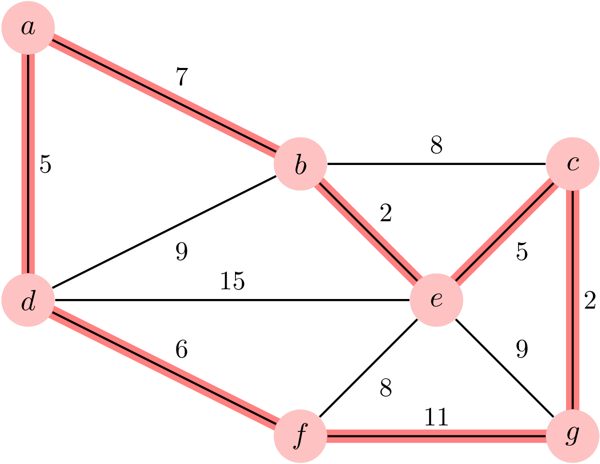

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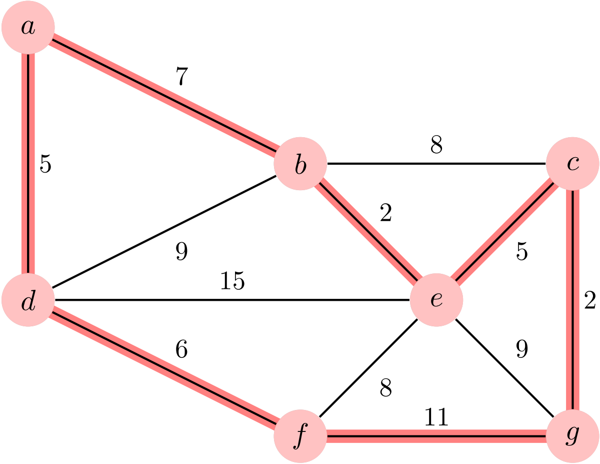

$--> Cost : 5 + 7 + 2 + 5 + 2 + 11 + 6 = 38Km$

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The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

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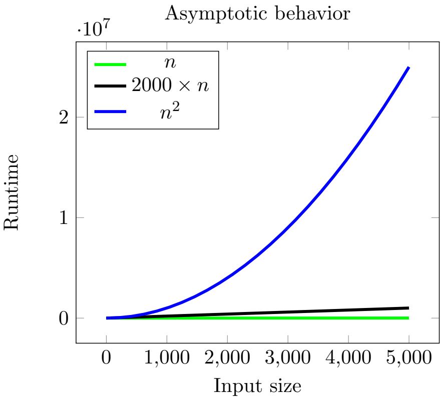

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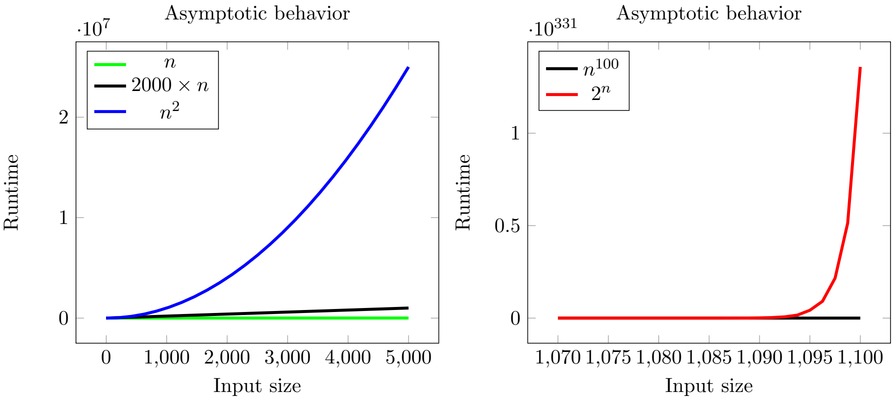

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- What if we don't know if a problem has a polynomial time algorithm?

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Given a set of Boolean variables $x_1, \ldots x_n$ and a CNF formulae Φ over $x_1, \ldots x_n$, the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

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- How to use it to solve problems (Modelling)
- Discover some efficient implementations

Example

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$x \vee \neg y \vee z$

$\neg x \vee \neg z$

$y \vee w$

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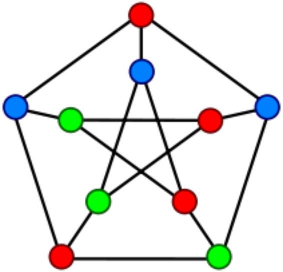
A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

The example of Graph Colouring

- Graph Coloring is a well know combinatorial problem that has many applications (in particular in scheduling problems)
- Let $G = (V, E)$ be an undirected graph where V is a set of n vertices and E is a set of m edges. Is it possible to colour the graph with k colours such that no two adjacent nodes share the same colour?



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- If a node is coloured with a colour a, the other colours are forbidden:

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The Example of Graph Coloring: The Model Size

What is the (space) size of the model?

- $n \times k$ Boolean variables
- Constraints form 1: n clauses with k literals each
- Constraints form 2: $n \times k^2$ binary clauses
- Constraints form 3: $m \times k$ binary clauses

The Example of Graph Coloring: The Minimization Version

- Propose a method that uses SAT for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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  - **Binary search:** Run iteratively $SAT(V, E, z)$ as long as $UB > LB$ where $z = \lceil (UB - LB)/2 \rceil$. If the result is satisfiable, then and $UB \leftarrow z$ otherwise $LB \leftarrow z$

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- Lower bound: Well, we can simply consider 2 as long as there is an edge. A more advanced one is to look for a clique in the graph.
- An alternative approach is to look for valid theoretical bounds in the literature.

Modelling Cardinality Constraints

- The general form of cardinality constraints is the following:

$$a \leq \sum_{1}^{n} x_i \leq b$$

  where a and b are positive integers and $x_1 \ldots x_n$ are Boolean variables
- Cardinality constraints are everywhere!
- Many ways to encode such constraints. See for instance `https://www.carstensinz.de/papers/CP-2005.pdf`

Quadratic encoding for $\sum_1^n x_i = 1$

- At least one constraint:

$$x_1 \vee x_2 \ldots x_n$$

- at most one constraints:

$$\forall i, j : \neg x_i \vee \neg x_j$$

This generates one clause of size n and (n^2) binary clauses without introducing additional variables.

Linear encoding for $\sum_1^n x_i = 1$

New variables are added as follows: for $i \in [1, n], y_i$ is a new variable that is true iff $\sum_{l=1}^{l=i} x_l = 1$.

$$x_1 \vee x_2 \dots x_n$$

$$y_n^1$$

$$\forall i \in [1, n-1] : y_i \rightarrow y_{i+1}$$

$$\forall i \in [1, n-1] : y_i \rightarrow \neg x_{i+1}$$

$$\forall i \in [1, n] : x_i \rightarrow y_i$$

Size: n new variables, 1 $n-$ary clause and $3 \times n$ binary clauses,

Linear encoding for $\sum_1^n x_i \geq k$

New variables: $\forall z \in [0, k], \forall i \in [1, n], y_i^z \iff \sum_{l=1}^{l=i} x_l \geq z$

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$$y_{i-1}^z \wedge x_i \rightarrow y_i^{z+1}$$

Linear encoding for $\sum_1^n x_i \geq k$

Size of the encoding:

- $\Theta(n \times k)$ variables
- $\Theta(n + k)$ unary clauses
- $\Theta(n \times k)$ binary clauses
- $\Theta(n \times k)$ ternary clauses

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- Check the MaxSAT competition

Example of applications for MaxSAT

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. In the (decision version of the) graph colouring problem, we are given k colours to colour the graph such that no two adjacent nodes share the same colour.

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The Example of Graph Coloring: A Possible MaxSAT Model

- We shall extend the previous model:
- Consider the previous model $SAT(V, E, k)$ with k an upper bound.
- All the previous clauses are hard.
- Add the following hard clauses:

$$\forall i \in [1, n], \forall a \in [1, k] : \neg u_a \rightarrow \neg x_i^a$$

- Eventually we can add symmetry constraints: $u_a \rightarrow u_{a-1}$
- Then add the soft clauses:

$$\forall a \in [1, k] : \neg u_a$$

- A MaxSAT Optimal solution satisfies all the hard coloring clauses (valid colouring) and maximizes the number of non used colours.

Extensions: Quantified Boolean Formula (QBF)

- A QBF has the form Q.F , where F is a CNF-SAT formulae, and Q is a sequence of quantified variables ($\forall x$ or $\exists x$).
- Example $\forall x, \exists y, \exists z,\ (x \vee \neg y) \wedge (\neg y \vee z)$
- QBF Solver Competition: `https://www.qbflib.org/solvers_list.php`
- QBF is less used in practice

Other Extensions

- Satisfiability Modulo Theories
- Answer Set Programming
- More generally: Automated reasoning community
- Check the SAT/SMT summer schools
  `http://satassociation.org/sat-smt-school.html`

Modern SAT Solvers: Conflict Driven Clause Learning (CDCL)

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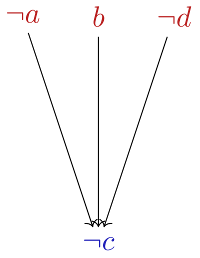
- **Can be seen as a CP Solver (Search, propagation) augmented by clause learning**
- But also :
  - Activity-based branching
  - Lazy data structures (2-Watched Literals)
  - Clause Database Reduction
  - Simplifications
  - Restarts
  - ...

Exercise: Propose a filtering algorithm for clauses. The algorithm takes as input a clause and has access (read and write) for the variables domains.

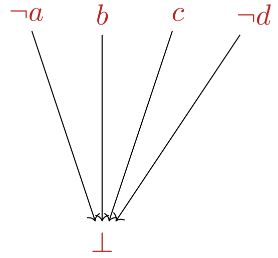
Unit Propagation

Given a clause C of arity n. If $n-1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$



$\neg a \wedge b \wedge \neg d \Rightarrow \neg c$



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Two Watched Literals

- Unit propagation is implemented with an “intelligent” data structure called Two-watched literals
- Observe first that propagation happens only in two cases:
  - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
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- If a literal watching a clause C becomes $false$, look for replacement. If no replacement found, then perform propagation

Implication Graph

	f		

$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

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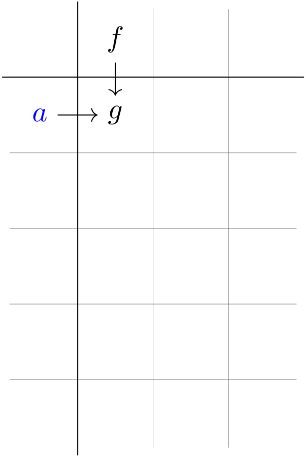

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$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

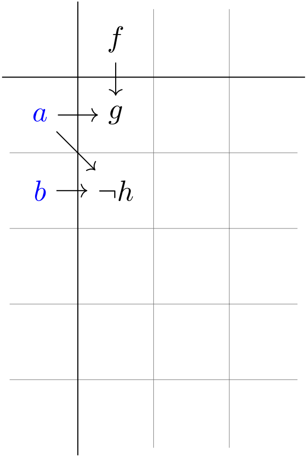
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$
$\neg a \vee \neg b \vee \neg h$
$a \vee c$
$a \vee \neg i \vee \neg l$
$a \vee \neg k \vee \neg j$
$b \vee d$
$b \vee g \vee \neg n$
$b \vee \neg f \vee n \vee k$
$\neg c \vee k$
$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$
$c \vee l$
$d \vee \neg k \vee l$
$d \vee \neg g \vee l$
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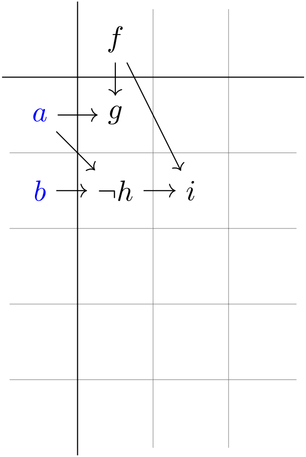

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$\neg a \vee \neg b \vee \neg h$
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$b \vee g \vee \neg n$
$b \vee \neg f \vee n \vee k$
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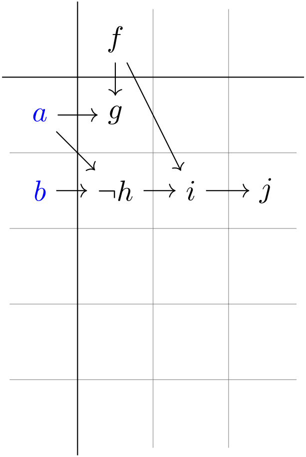

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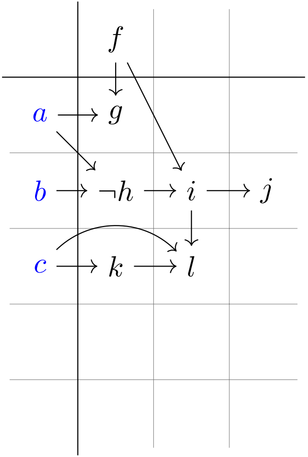

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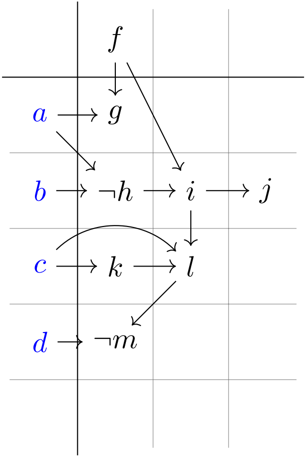

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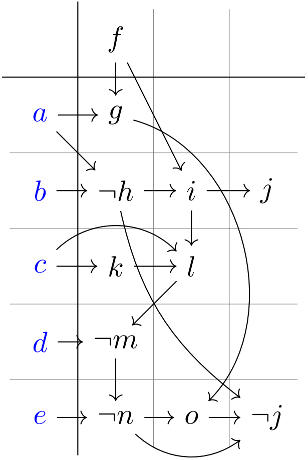

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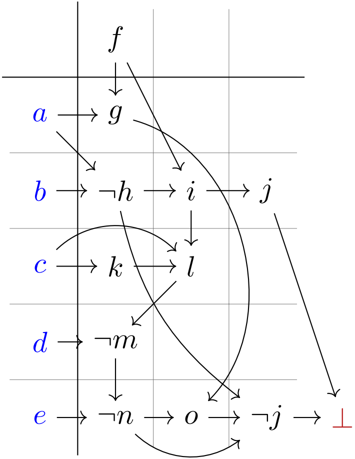

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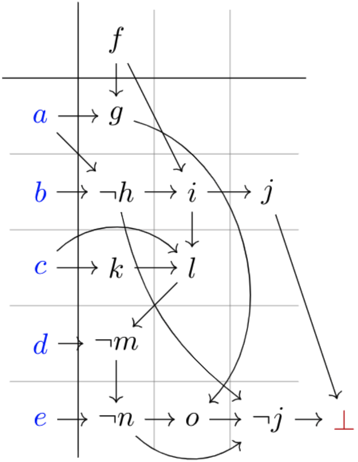

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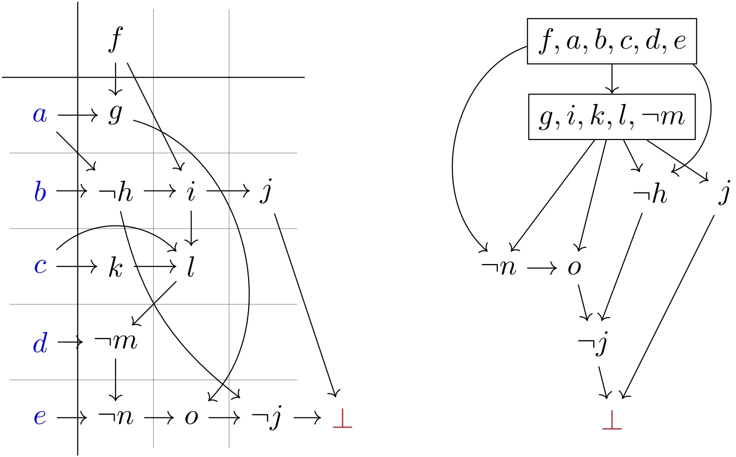
Conflict Analysis



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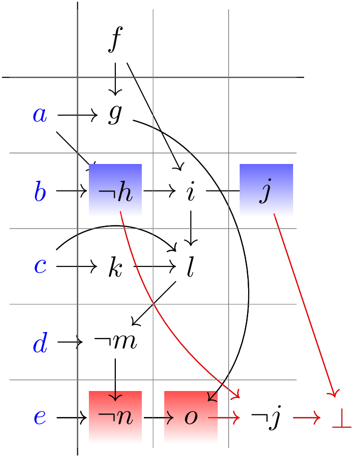

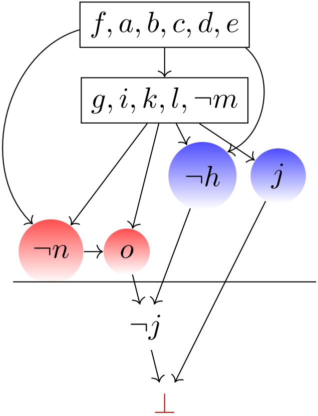

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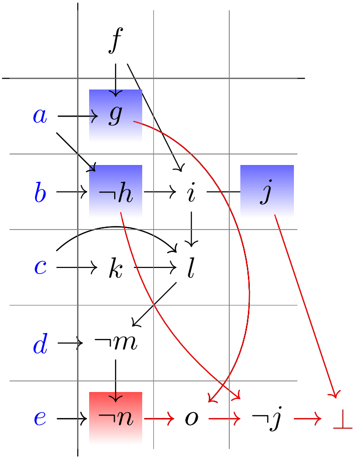

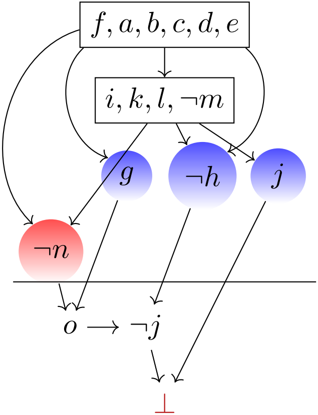

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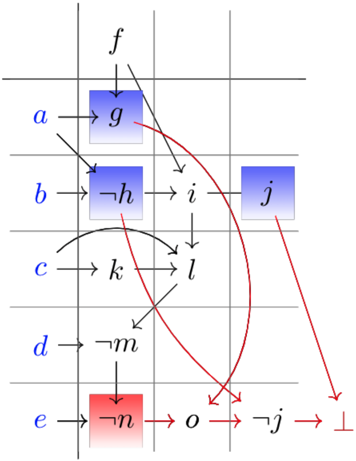

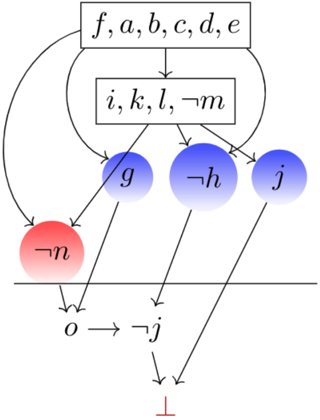

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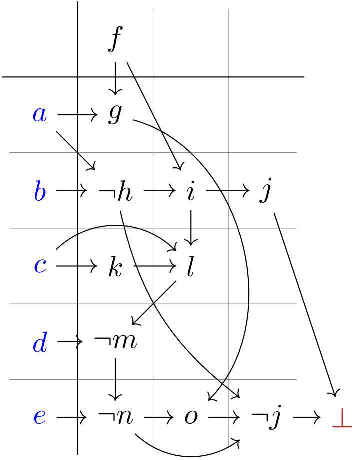

$\neg a \vee \neg f \vee g$
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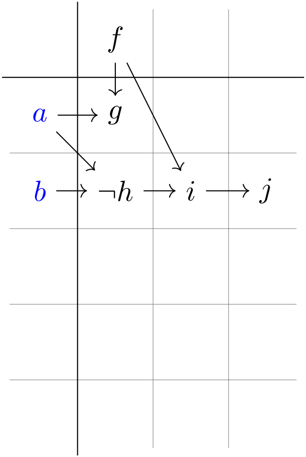

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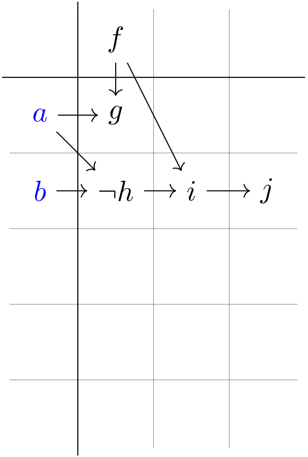

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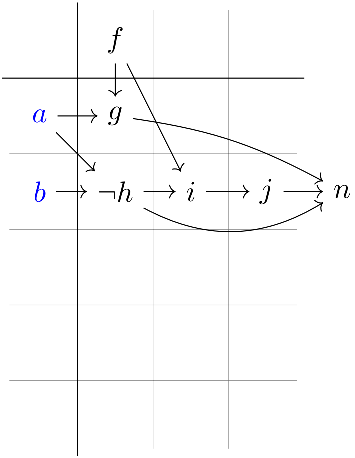
Conflict analysis



$\neg a \vee \neg f \vee g$
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Learning and Backjumping

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- Definition: Explaining a failure: $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ where $\neg l_1 \vee \ldots \vee \neg l_n$ is the clause triggering failute

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- When there is only one literal uip propagated in the last level in the current explanation, learn the associated new clause C, backjump (to the last level of propagated literals in C), propagate $\neg uip$ via the new clause, and continue the exploration

Boosting Search through Randomization and Restarts [Gomes et al., 1998]

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Heavy-tail phenomena (SAT and CP)

At any time during the experiment there is a non-negligible probability of hitting a problem that requires exponentially more time to solve than any that has been encountered before.

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- Randomization: breaking ties, random decision between k best choices, ...
- Restarts: Geometric/Luby

Other techniques

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SAT Solvers (Few examples)

- MiniSat: http://minisat.se/
- Glucose: `http://www.labri.fr/perso/lsimon/glucose/`
- LingeLing `http://fmv.jku.at/lingeling`
- Any Solver by Armin Biere `http://fmv.jku.at/software/index.html`
- Any winner from past and future SAT competitions: https://www.satcompetition.org/

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