SAT: Introduction

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Context: Solving (Very) Hard Combinatorial Problems

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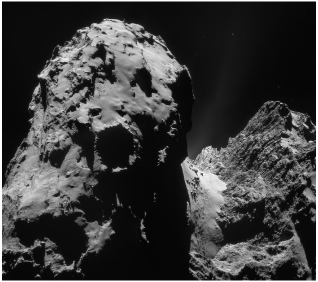

https://homepages.laas.fr/ehebrard/rosetta.html

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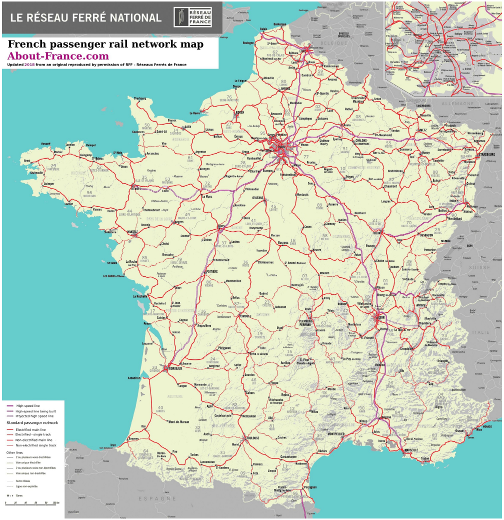

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- Resources: many.. a good start would be the online course on discrete optimisation `https://www.coursera.org/learn/discrete-optimization`

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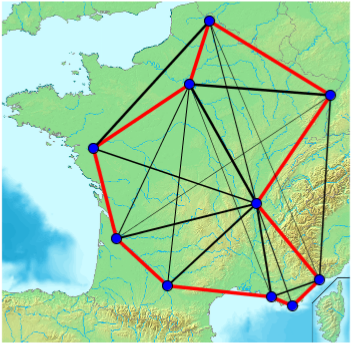
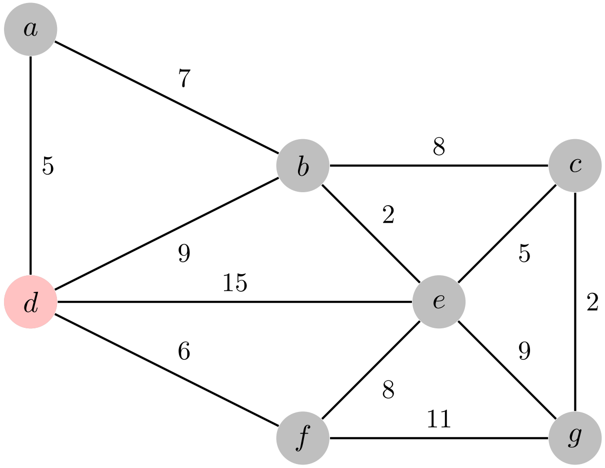
- Very active community

Travelling Salesman Problem



Exemple



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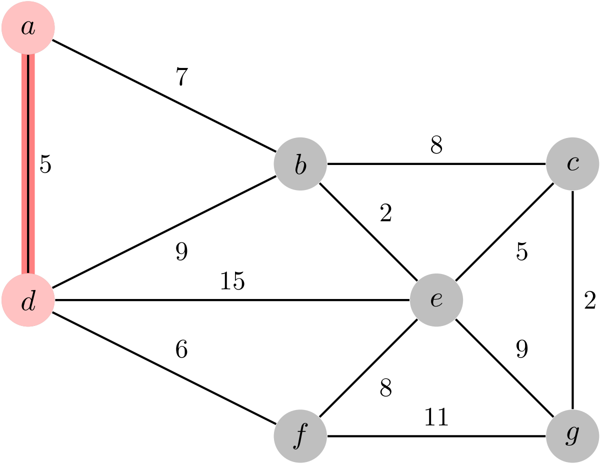

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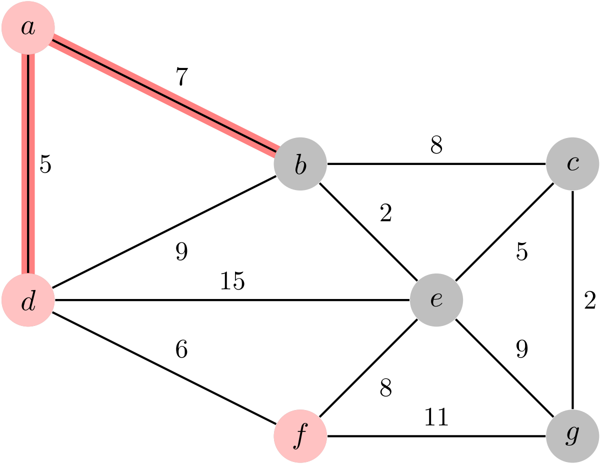

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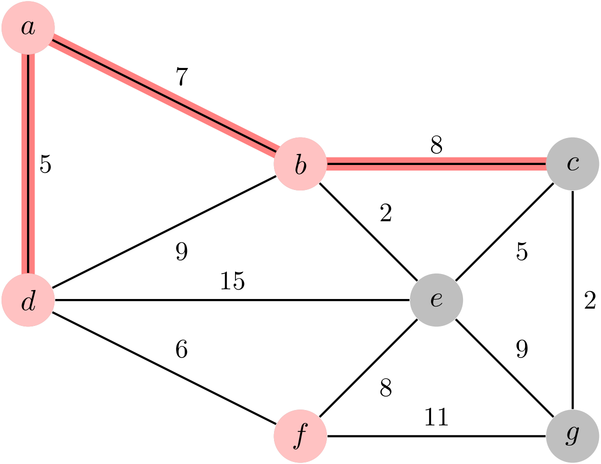

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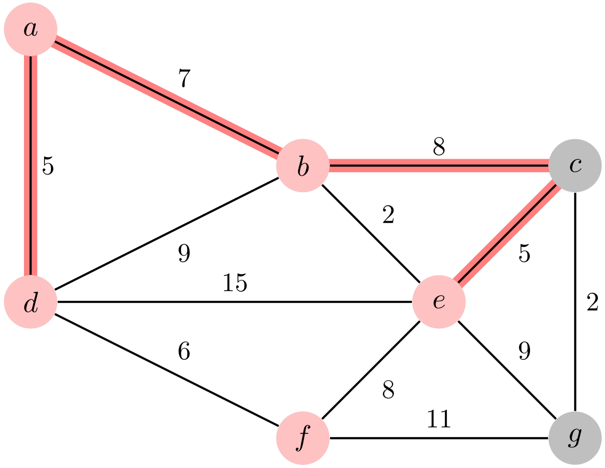

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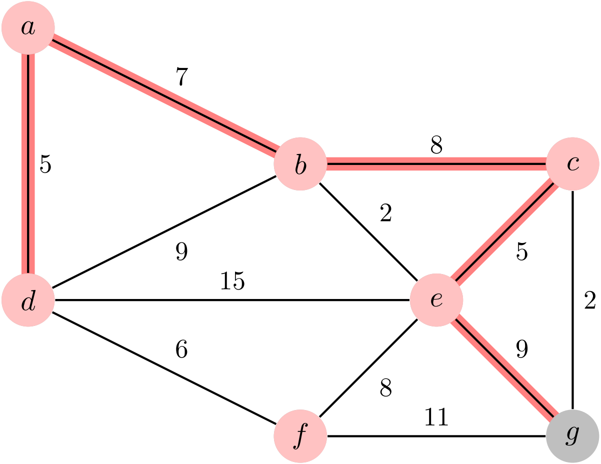

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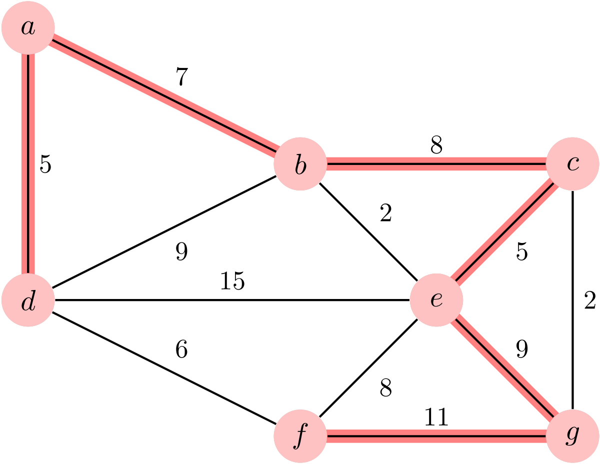

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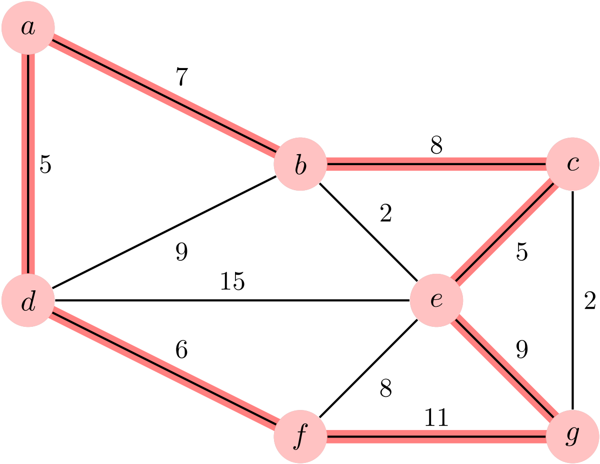

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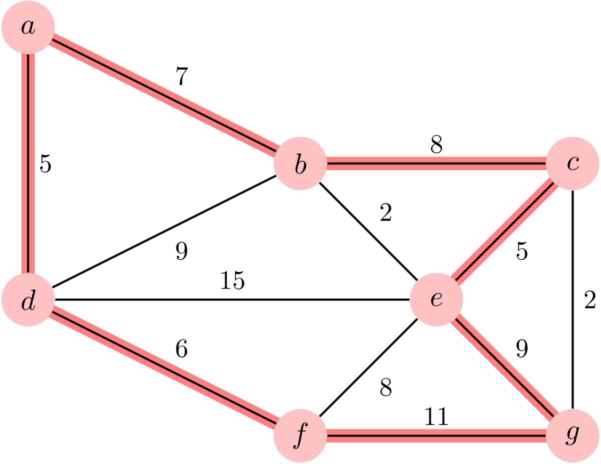
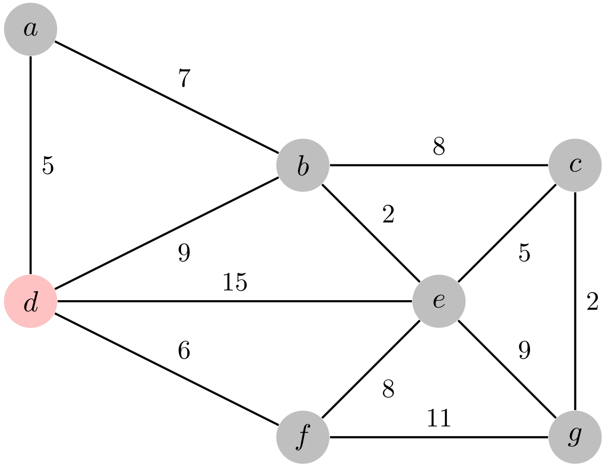


$--> Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53Km$

Example



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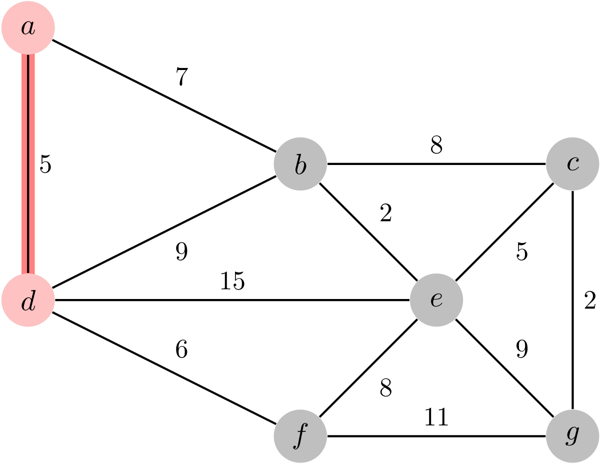

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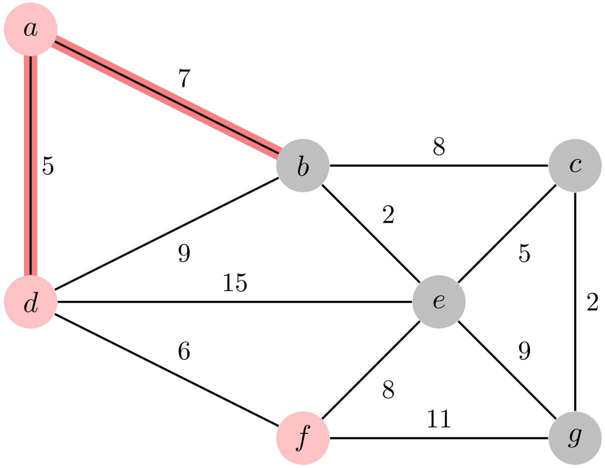

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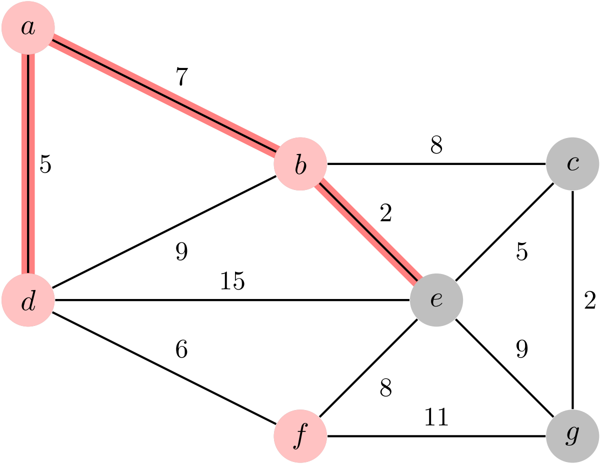

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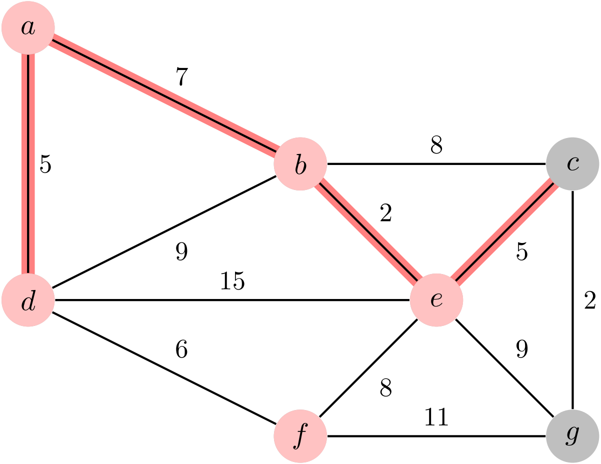

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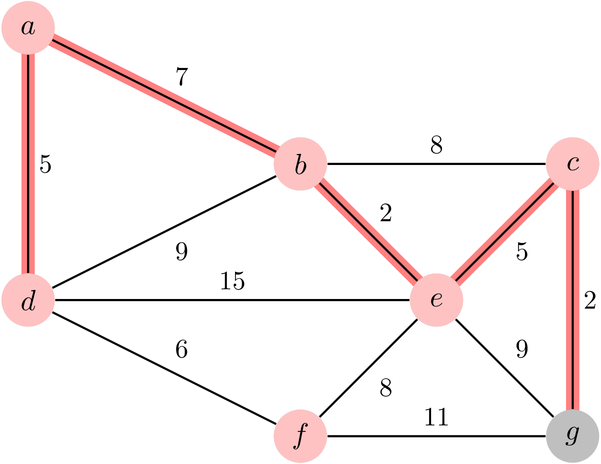

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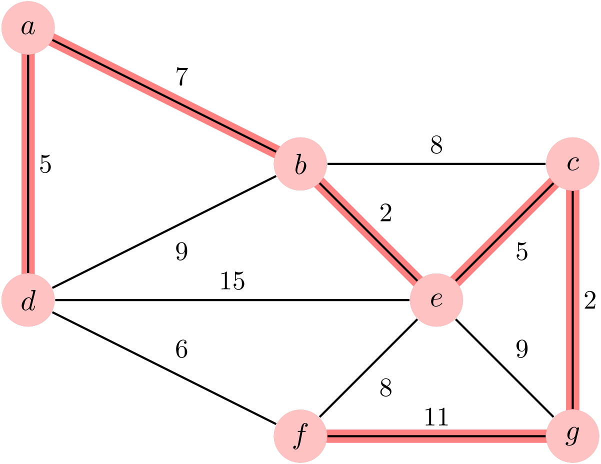

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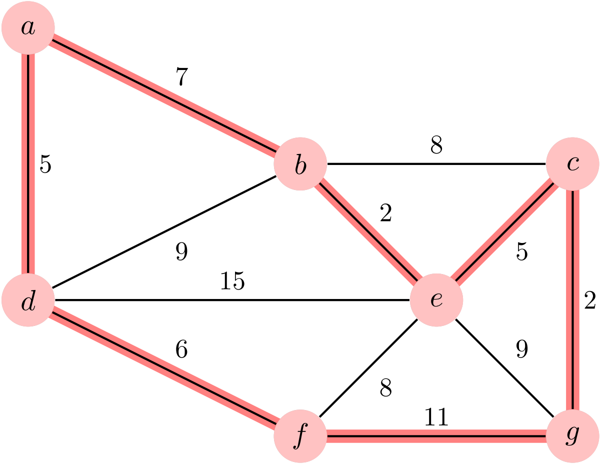

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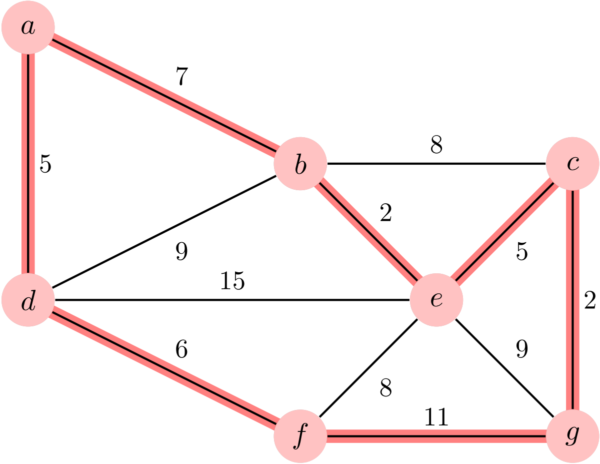

$$--> Cost : 5 + 7 + 2 + 5 + 2 + 11 + 6 = 38Km$$

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The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

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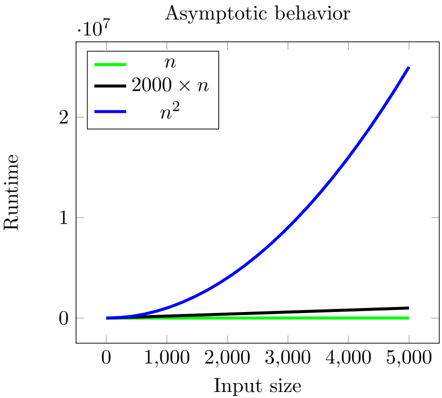

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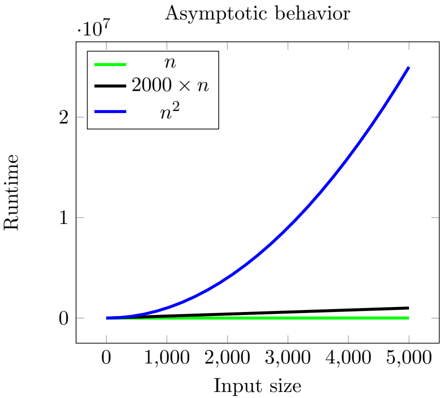

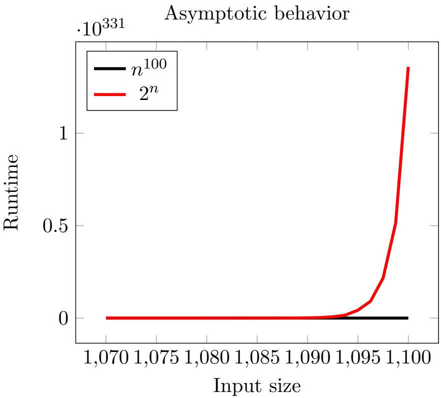

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- What if we don't know if a problem has a polynomial time algorithm?

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Given a set of Boolean variables $x_1, \ldots x_n$ and a CNF formulae Φ over $x_1, \ldots x_n$, the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

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- Discover some efficient implementations

Example

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$x \vee \neg y \vee z$

$\neg x \vee \neg z$

$y \vee w$

$\neg w \vee \neg x$

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$x \vee \neg y \vee z$

$\neg x \vee \neg z$

$y \vee w$

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A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

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