An Introduction to Boolean Satisfiability

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Context & Introduction

Context: Solving (Very) Hard Combinatorial Problems

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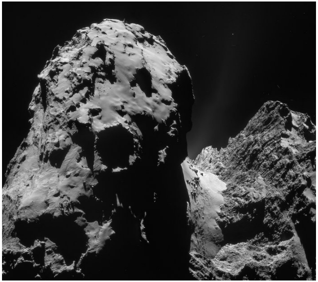

https://homepages.laas.fr/ehebrard/rosetta.html

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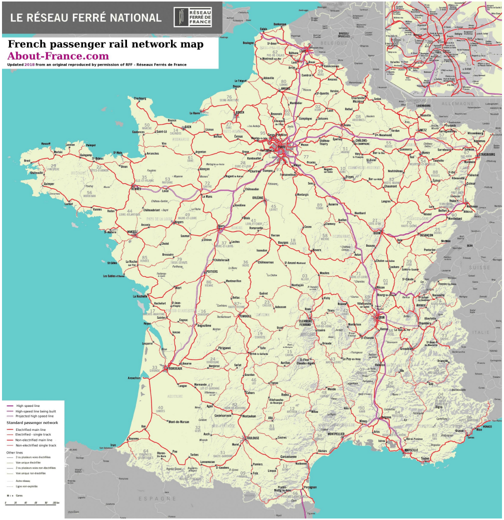

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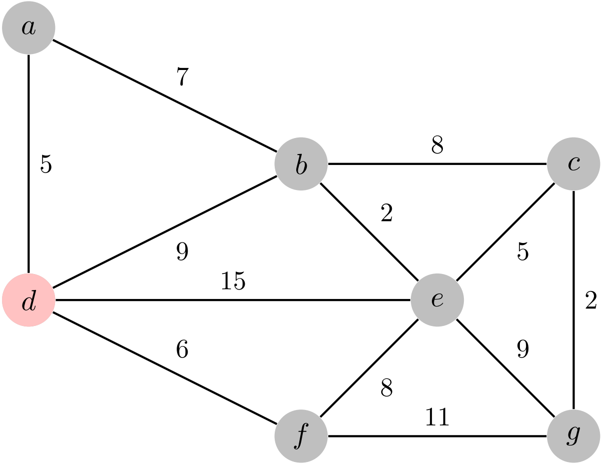
- We focus in this course on the modelling aspect
- Resources for combinatorial optimisation: Many! a good start would be the online course on discrete optimisation `https://www.coursera.org/learn/discrete-optimization`
- Handbook of Satisfiability - Second Edition - IOS Press, 2021

Exemple



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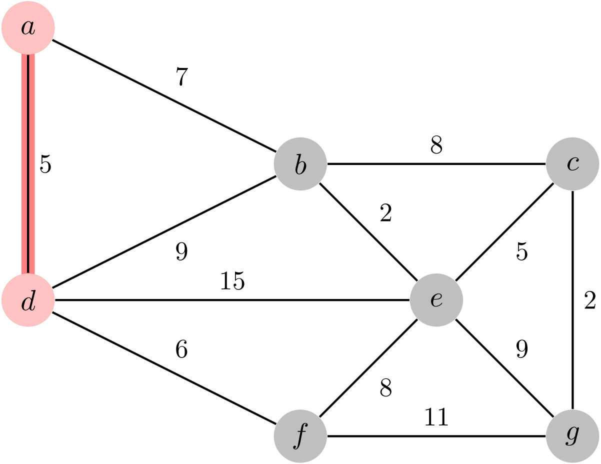

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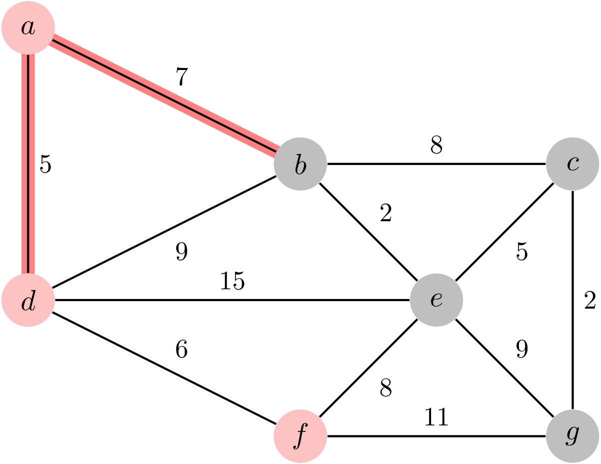

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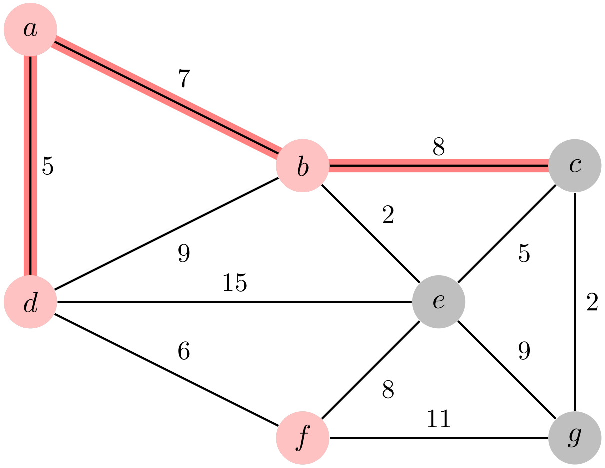

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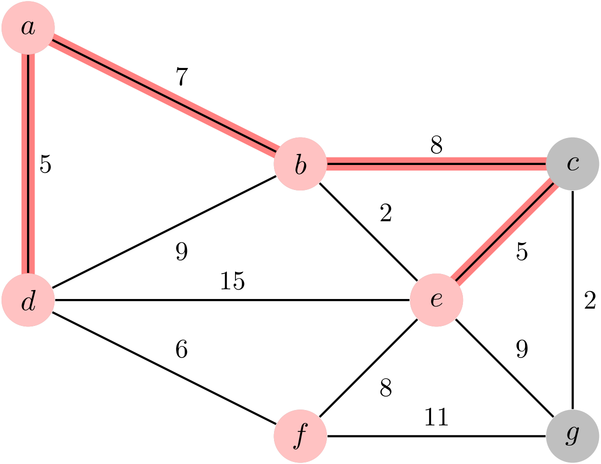

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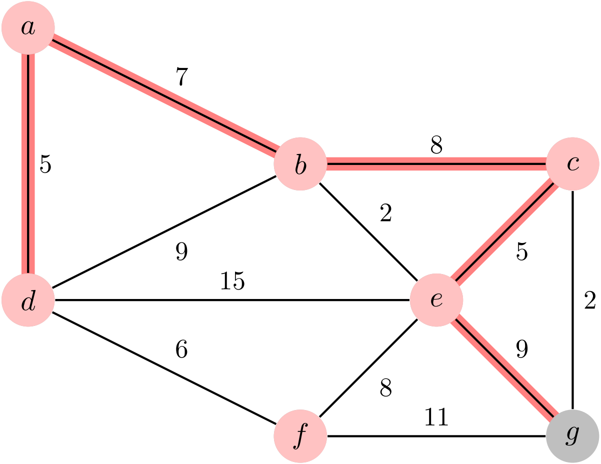

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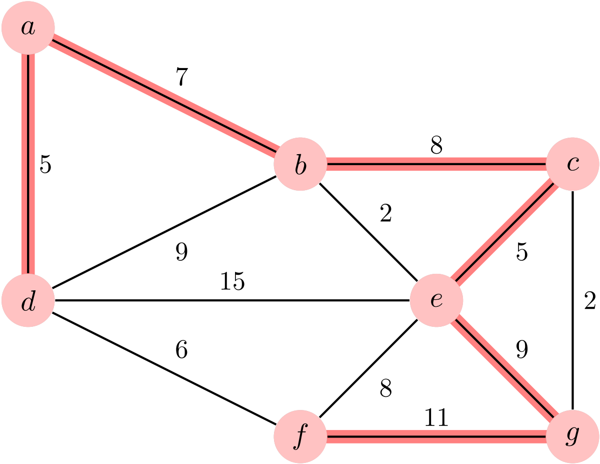

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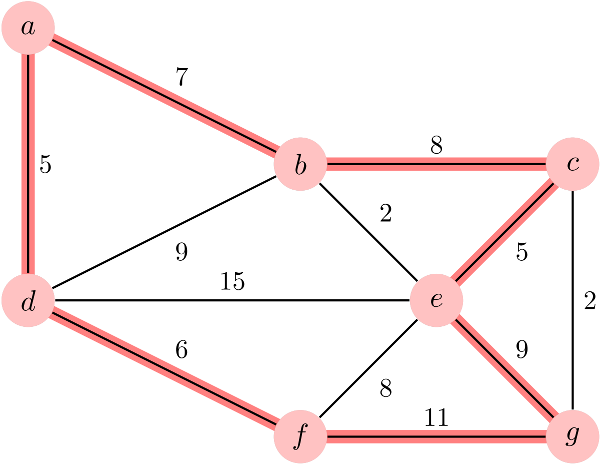

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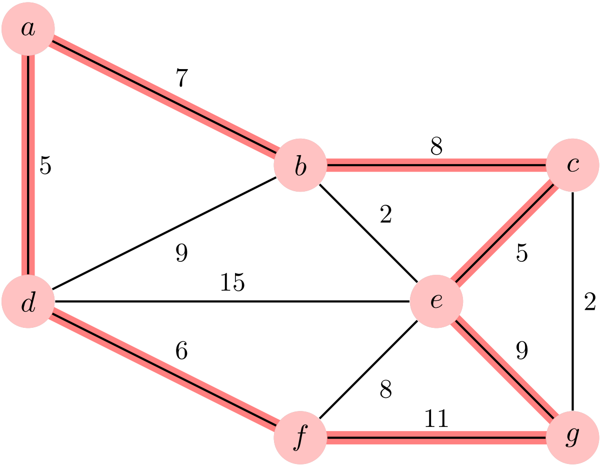

$-- > Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53Km$

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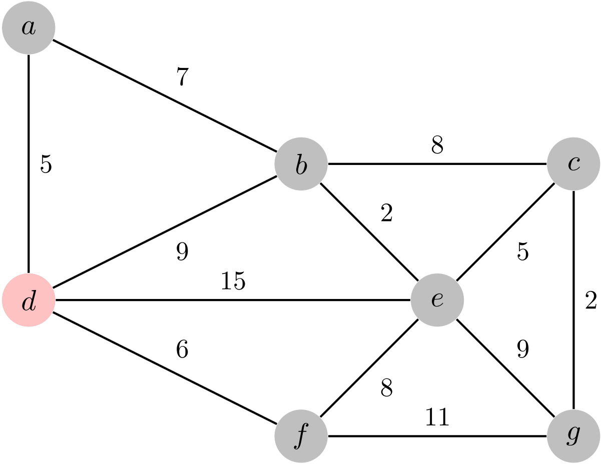

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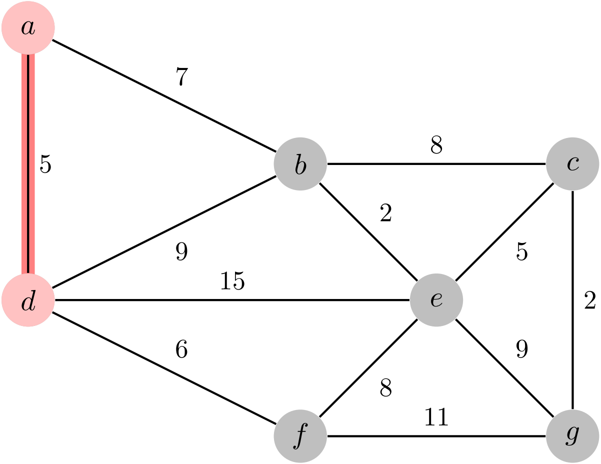

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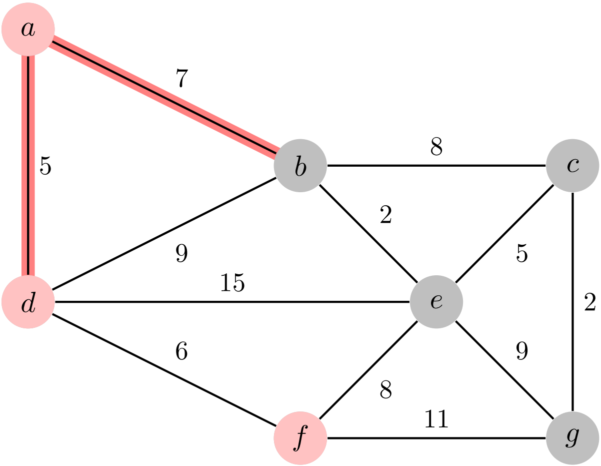

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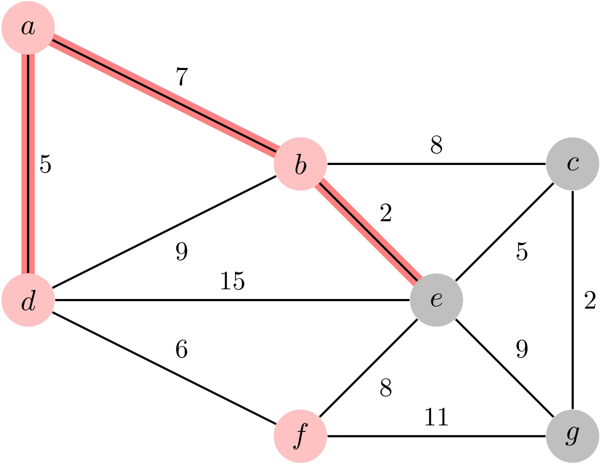

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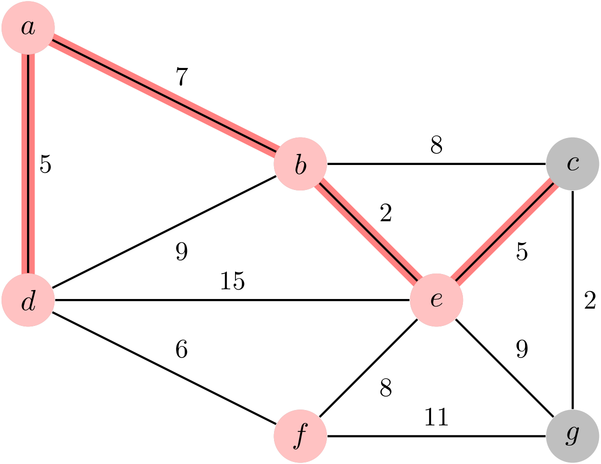

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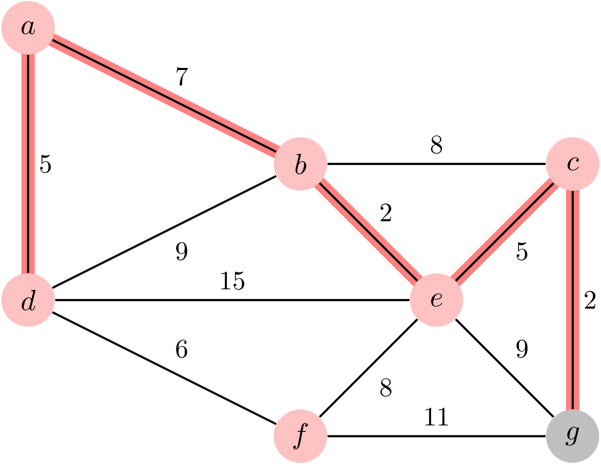

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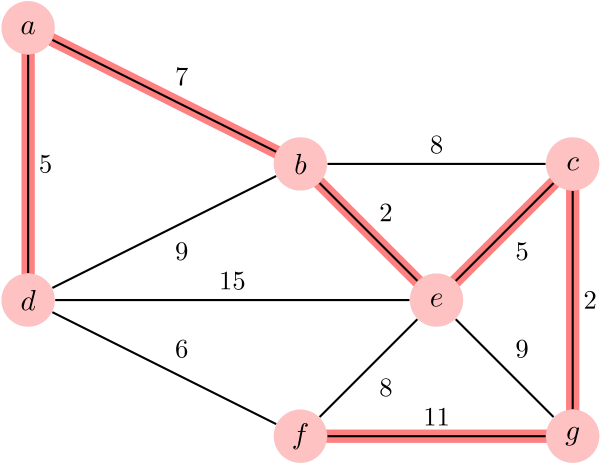

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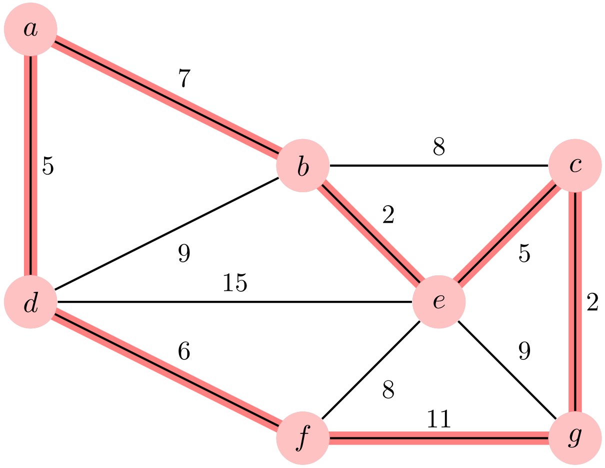

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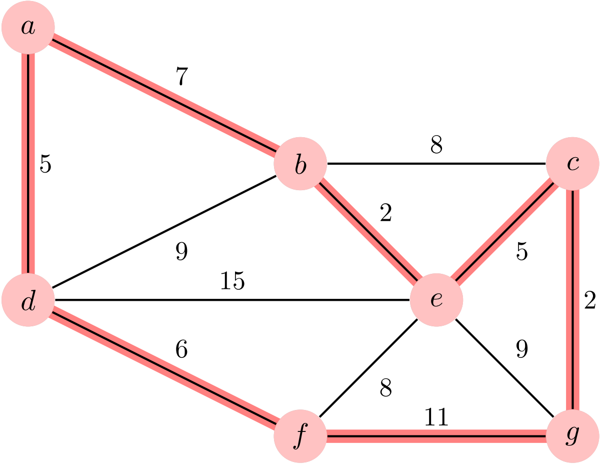

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The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

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1. Adhoc methods
    1. Specific exact algorithm
    2. Heuristic method
    3. Meta-heuristic (genetic algorithms, ant colony, ..)
2. Declarative Approaches
    1. (Mixed) Integer Programming,
    2. Constraint Programming
    3. Boolean Satisfiability (SAT)
    4. ...

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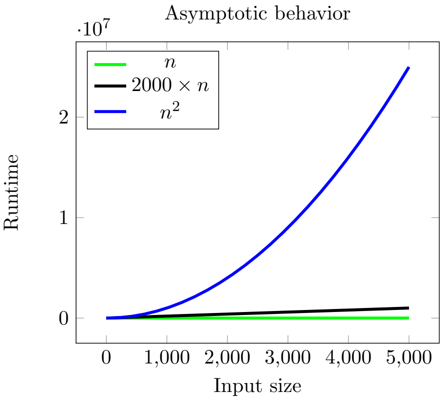

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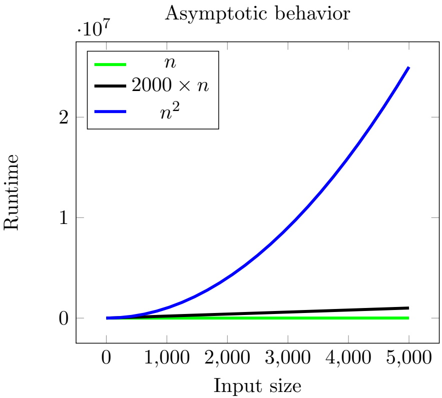

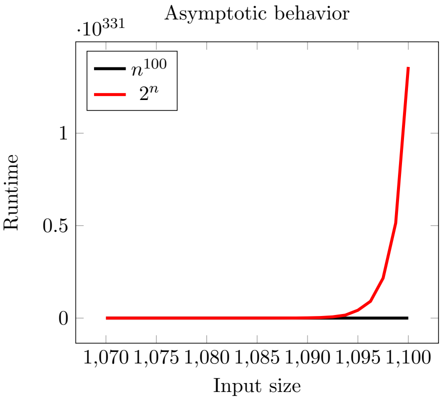

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- For many Problems in NP, we don't know if a polynomial time algorithm exists. Is P=NP?

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Given a set of Boolean variables $x_1, \ldots x_n$ and a CNF formula Φ over $x_1, \ldots x_n$, the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

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$x \vee \neg y \vee z$

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A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

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- Huge practical improvements in the past 2 decades or so

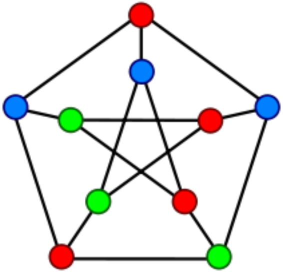
Examples of Applications

- AI Planning
- Scheduling
- Software verification
- Machine learning
  - Robustness
  - Synthesis
  - Verification
- Mathematical Proofs! `https://news.cnrs.fr/articles/the-longest-proof-in-the-history-of-mathematics`
- Timetabling
- ...

Modelling in SAT

The example of Graph Colouring

- Graph Coloring is a well known combinatorial problem that has many applications (in particular in scheduling problems)
- Let $G = (V, E)$ be an undirected graph where V is a set of n vertices and E is a set of m edges. Is it possible to colour the graph with k colours such that no two adjacent nodes share the same colour?



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- Forbid two nodes that share an edge to be coloured with the same colour

$$\forall \{i, j\} \in E, \forall a \in [1, k] : \neg x_i^a \vee \neg x_j^a$$

  (This is a translation of $x_i^a \rightarrow \neg x_j^a$)

The Example of Graph Coloring: The Model Size

What is the (space) size of the model?

- $n \times k$ Boolean variables
- Constraints form 1: n clauses with k literals each
- Constraints form 2: $n \times k^2$ binary clauses
- Constraints form 3: $m \times k$ binary clauses

The Example of Graph Coloring: The Minimization Version

- Propose a method that uses SAT for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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- Run iteratively the decision version until converging to the optimal value
- Let's call $SAT(V, E, K)$ the SAT model of the decision version of the problem (i.e., can we find a valid colouring of $G(V, E)$ with k colours). Use $SAT(V, E, K)$ as an oracle within an iterative search. For instance:

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A Straightforward Approach

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  - **Binary search:** Run iteratively $SAT(V, E, z)$ as long as $UB > LB$ where $z = \lceil (UB - LB)/2 \rceil$. If the result is satisfiable, then and $UB \leftarrow z$ otherwise $LB \leftarrow z$

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- An alternative approach is to look for valid theoretical bounds in the literature.

Modelling Cardinality Constraints

- A cardinality constraint takes as input a sequence of Boolean variables $[x_1, \ldots, x_n]$ and an integer k and enforces

$$\sum_{1}^{n} x_i \leq k$$

- Cardinality constraints are everywhere!
- There exist many ways in the literature to encode such constraints. See for instance `https://www.carstensinz.de/papers/CP-2005.pdf`

Quadratic encoding for $\sum_1^n x_i = 1$

- At least one constraint:

$$x_1 \vee x_2 \ldots \vee x_n$$

- at most one constraint:

$$\forall i, j : \neg x_i \vee \neg x_j$$

This generates one clause of size n and (n^2) binary clauses without introducing additional variables.

Linear encoding for $\sum_1^n x_i = 1$

A sequence of Boolean variables $[y_1, \ldots, y_n]$ is introduced such that $\forall i \in [1, n], y_i$ is true iff $\sum_{l=1}^{l=i} x_l = 1$. The set of clauses for the encoding is the following:

$$x_1 \vee x_2 \ldots \vee x_n$$

$$y_n^1$$

$$\forall i \in [1, n-1] : y_i \rightarrow y_{i+1}$$

$$\forall i \in [1, n-1] : y_i \rightarrow \neg x_{i+1}$$

$$\forall i \in [1, n] : x_i \rightarrow y_i$$

Size: n new variables, 1 $n-$ary clause and $3 \times n$ binary clauses,

Encoding for $\sum_1^n x_i \geq k$

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- $y_1^0 \leftarrow 1$

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- Do not Increment: $\neg y_{i-1}^z \wedge \neg x_i \rightarrow y_i^{z+1}$

Encoding for $\sum_1^n x_i \geq k$

Size of the encoding:

- $\Theta(n \times k)$ variables
- $\Theta(n + k)$ unary clauses
- $\Theta(n \times k)$ binary clauses
- $\Theta(n \times k)$ ternary clauses

Encoding for $\sum_1^n x_i = k$?

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- Encode $\sum_1^n x_i \geq k+1$
- Add y_n^k
- Replace y_n^{k+1} by $\neg y_n^{k+1}$
- The size of the encoding is the same as $\sum_1^n x_i \geq k$ (asymptotically)

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- Encode $\sum_1^n x_i \leq b$
- $\sum_1^n x_i \geq a$ with the same additional variables
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- Check the MaxSAT competition

The Example of Graph Coloring: A Possible MaxSAT Model

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. In the (decision version of the) graph colouring problem, we are given k colours to colour the graph such that no two adjacent nodes share the same colour.

- Propose a MaxSAT model for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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- We shall extend the previous model:
- Let u_a be a Boolean variable that is True iff. the colour $a \in [1, k]$ is used
- Consider the previous model $SAT(V, E, k)$ with k an upper bound.
- All the previous clauses are hard.
- Add the following hard clauses:

$$\forall i \in [1, n], \forall a \in [1, k] : \neg u_a \rightarrow \neg x_i^a$$

- Eventually we can add symmetry breaking constraints: $u_a \rightarrow u_{a-1}$
- Then add the soft clauses:

$$\forall a \in [1, k] : \neg u_a$$

- A MaxSAT Optimal solution satisfies all the hard coloring clauses (valid colouring) and maximizes the number of non used colours.

Extensions: Quantified Boolean Formula (QBF)

- A QBF has the form Q.F , where F is a CNF-SAT formulae, and Q is a sequence of quantified variables ($\forall x$ or $\exists x$).
- Example $\forall x, \exists y, \exists z, \ (x \vee \neg y) \wedge (\neg y \vee z)$
- QBF Solver Competition: `https://www.qbflib.org/solvers_list.php`

Extensions: Satisfiability Modulo Theories (SMT)

- SMT extends SAT by allowing higher level constraints
- Such constraints belong to certain theories
- Examples of theories include linear integer arithmetic, linear real arithmetic, difference logic, etc
- Check the SAT/SMT summer schools
  `http://satassociation.org/sat-smt-school.html`

Exercise: SAT for Machine Learning

- Let $F = [f_1, \ldots f_k]$ be a set of k features and $E = [e_1, \ \ldots e_n]$ a set of n examples.
- We want to build an undirected acyclic graph for prediction
- Task1: Propose a model for the topology of the graph
- Task 2: Extend the model to make sure that each example is well classified
- Task 3: Adapt the model to maximize the accuracy of the model

Conflict Driven Clause Learning

Modern SAT Solvers: Conflict Driven Clause Learning (CDCL)

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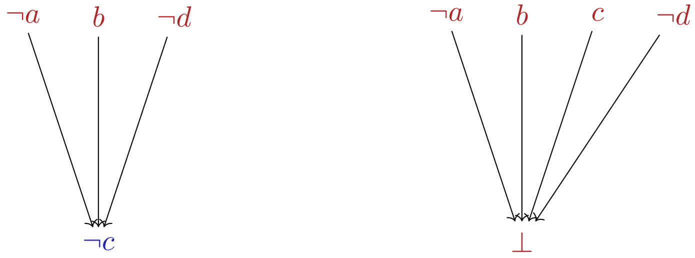
- **Can be seen as a CP Solver (Search, propagation) augmented by clause learning**
- But also :
  - Activity-based branching
  - Lazy data structures (2-Watched Literals)
  - Clause Database Reduction
  - Simplifications
  - Restarts
  - ...

Exercise: Propose a filtering algorithm to prune the variables domain in a given clause

Unit Propagation

Given a clause C of arity n. If $n-1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$



$\neg a \wedge b \wedge \neg d \Rightarrow \neg c$

$\neg a \wedge b \wedge c \wedge \neg d \Rightarrow \bot$

Algorithm 1: Unit Propagation

Data: A clause C

if *All literals in* C *are false* **then**

    return Failure ;

else

    if *Only one literal* $l \in C$ *is unassigned and the rest are false* **then**

        Make l true ;

    end

end

Unit Propagation

- Observe first that propagation happens only in two cases:
  - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
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- If a literal watching a clause C becomes $false$, look for replacement. If no replacement found, then perform propagation

Algorithm 2: Two watched Literals (decision d)

```
▷ Assuming initially that all variables are unassigned and that each clause contain at least 2 literals
▷ For each clause C, W[C] is initialized with a set that contains two variables in C
▷ For each variable x, B[x] is the set of clauses watched by x
▷ d is the latest decision ;
S ← {d} ;
while S ≠ ∅ do
    Let x ∈ S ;
    S ← S \ {x} ;
    while B[x] ≠ ∅ do
        Let C ∈ B[x] ;
        W[C] ← W[C] \ {x} ;
        if ∃x' ∈ C \ W[C] such that x' is unassigned then
            W[C] ← W[C] ∪ {x'} ;
            B[x'] ← B[x'] ∪ {C} ;
        else
            Let y ∈ W[C] ;
            if y is not assigned then
                assign y to a value that satisfies C ;
                S ← S ∪ {y} ;
                S ← ∅
            else
                if y does not satisfy C then
                    return FAILURE ;
                end
            end
        end
    end
end
```

Learning and Backjumping

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- At each conflict learn a new clause as following:

Learning and Backjumping

- Definition: Explaining a failure: $l_1 \land \ldots \land l_n \rightarrow \bot$ where $\neg l_1 \lor \ldots \lor \neg l_n$ is the clause triggering the failure
- Definition: Explaining a propagation of l: $l_1 \land \ldots \land l_n \rightarrow l$ where $\neg l_1 \lor \ldots \lor \neg l_n \lor \neg l$ is the triggering clause
- At each conflict learn a new clause as following:
- Start with the explanation from the clause triggering failure in the form of $l_1 \land \ldots \land l_n \rightarrow \bot$ and let it be the initial explanation

Learning and Backjumping

- Definition: Explaining a failure: $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ where $\neg l_1 \vee \ldots \vee \neg l_n$ is the clause triggering the failure
- Definition: Explaining a propagation of l: $l_1 \wedge \ldots \wedge l_n \rightarrow l$ where $\neg l_1 \vee \ldots \vee \neg l_n \vee \neg l$ is the triggering clause
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- While there is more than one literal propagated in the last level in the current explanation, replace it with it's explanation from the triggering clause
- When there is only one literal *uip* propagated in the last level in the current explanation, learn the associated new clause C, backjump (to the last level of propagated literals in C), propagate $\neg uip$ via the new clause, and continue the exploration

Conflict Analysis

Algorithm 1: 1-UIP-with-Propagators

$\Psi \leftarrow explain(\bot)$;
while $|\{q \in \Psi \mid level(q) = current\ level\}| > 1$ **do**
  $p \leftarrow \arg\max_q(\{rank(q) \mid level(q) = current\ level\ \wedge\ q \in \Psi\})$;
  $\Psi \leftarrow \Psi \cup \{q \mid q \in explain(p) \wedge level(q) > 0\} \setminus \{p\}$;
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- Why stop with one literal l propagated at the last level ?
- **To make sure that when the algorithm backjumps, propagation takes place by making l true**
- When backjumping using a clause that contains more than one literal propagated at the last level, then no propagation can be performed.

Implication Graph

	f		

$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

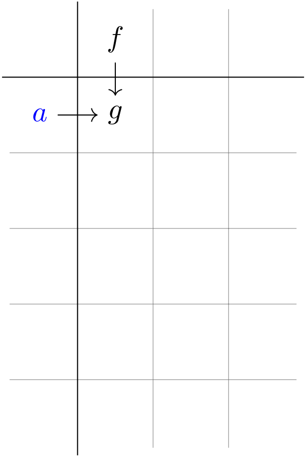
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

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$\neg e \vee m \vee \neg n$

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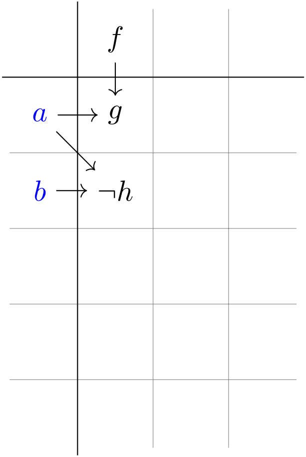
Implication Graph



$\neg a \vee \neg f \vee g$

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$a \vee c$

$a \vee \neg i \vee \neg l$

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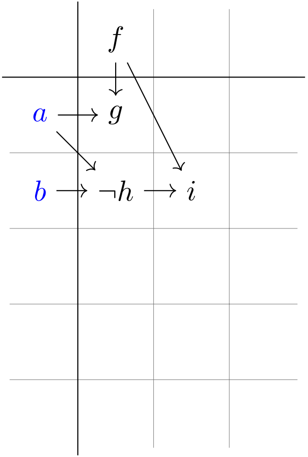
Implication Graph



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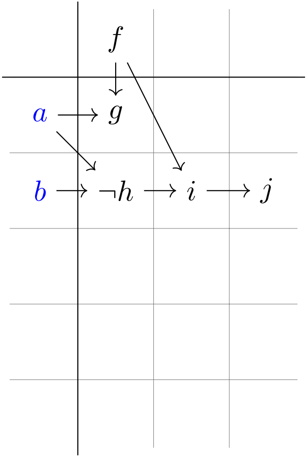
Implication Graph



$\neg a \vee \neg f \vee g$
$\neg a \vee \neg b \vee \neg h$
$a \vee c$
$a \vee \neg i \vee \neg l$
$a \vee \neg k \vee \neg j$
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$b \vee \neg f \vee n \vee k$
$\neg c \vee k$
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$c \vee h \vee n \vee \neg m$
$c \vee l$
$d \vee \neg k \vee l$
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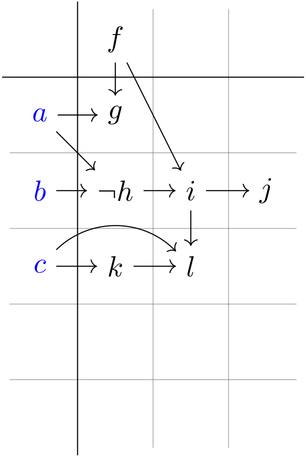
Implication Graph



$\neg a \vee \neg f \vee g$
$\neg a \vee \neg b \vee \neg h$
$a \vee c$
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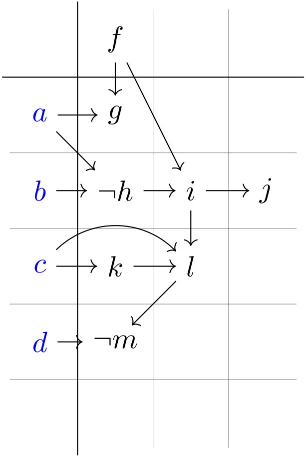

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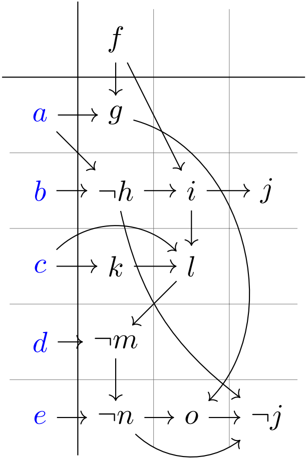

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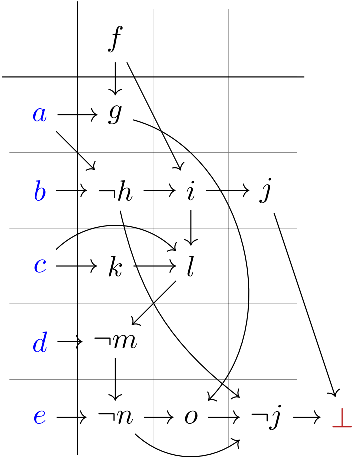

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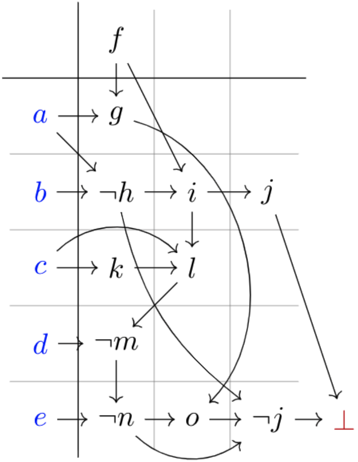

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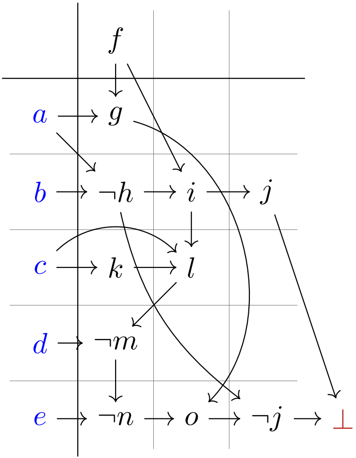
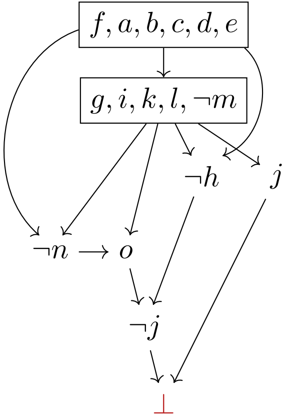
Conflict Analysis





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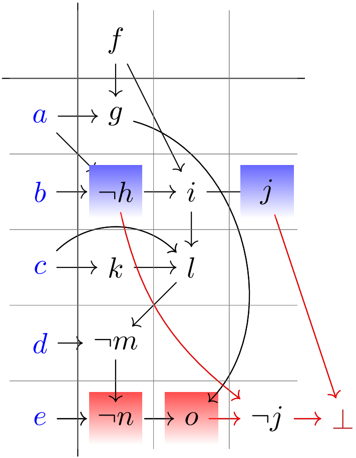

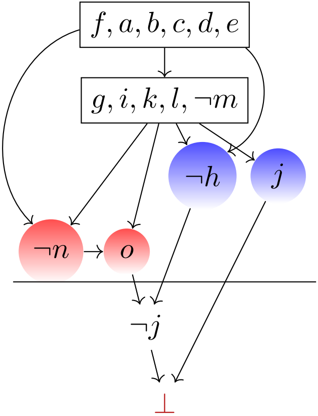

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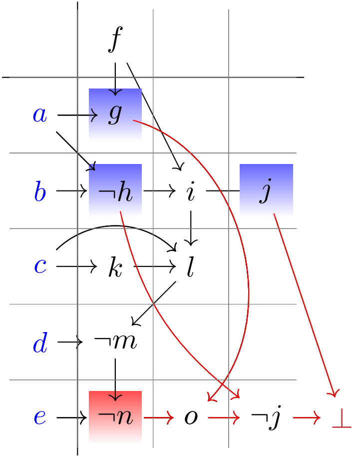

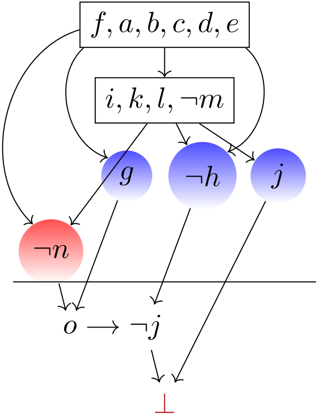

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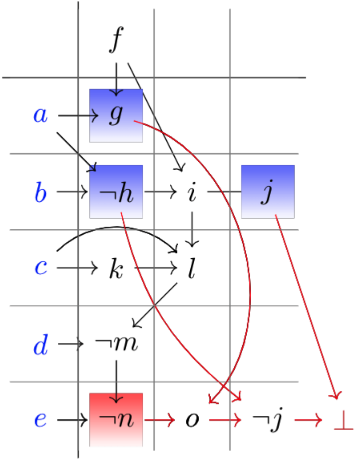

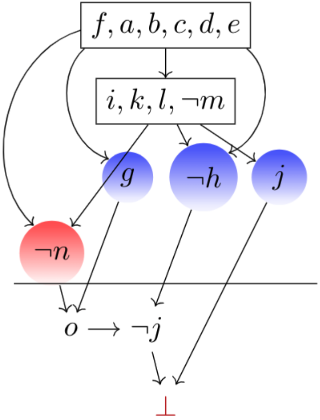

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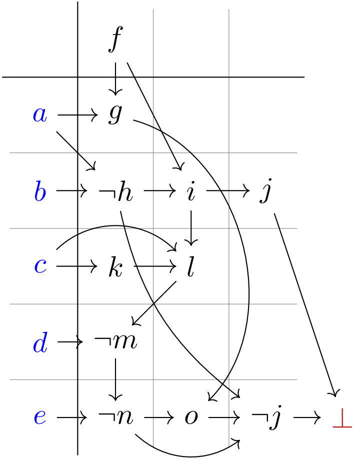

$\neg a \vee \neg f \vee g$
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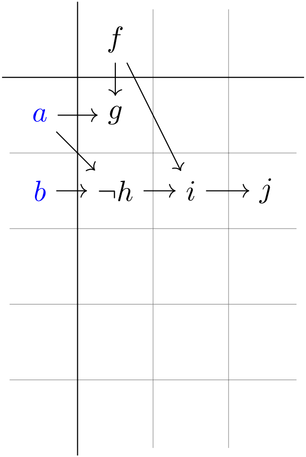
Conflict analysis



$\neg a \lor \neg f \lor g$
$\neg a \lor \neg b \lor \neg h$
$a \lor c$
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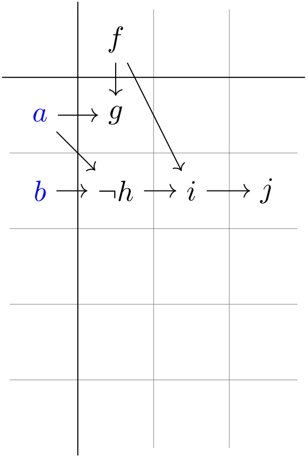

$\neg a \vee \neg f \vee g$

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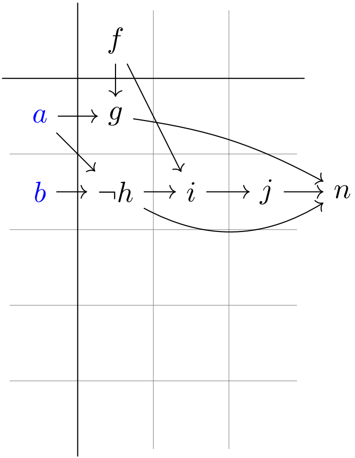
$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

$\boxed{\neg g \vee h \vee \neg j \vee n}$

Conflict analysis



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Boosting Search through Randomization and Restarts [Gomes et al., 1998]

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Heavy-tail phenomena (SAT and CP)

At any time during the experiment there is a non-negligible probability of hitting a problem that requires exponentially more time to solve than any that has been encountered before.

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- Randomization: breaking ties, random decision between k best choices, ...
- Restarts: Geometric/Luby

Other techniques

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- Forgetting clauses: The number of the learnt clauses can be exponential, we sometimes need to free some space by forgetting some clauses.

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- VSIDS (Variable State Independent Decaying Sum): VSIDS is a popular variable ordering heuristic that is based on the notion of activity.

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SAT Solvers (Few examples)

- MiniSat: http://minisat.se/
- Glucose: `http://www.labri.fr/perso/lsimon/glucose/`
- LingeLing `http://fmv.jku.at/lingeling`
- Any Solver by Armin Biere
  `http://fmv.jku.at/software/index.html`
- Any winner from past and future SAT competitions:
  https://www.satcompetition.org/

SAT vs CSP

Back to Constraint Programming

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- Mostly solvable by backtracking algorithms (Search and Filtering)

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Value Ordering

'Succeed-first' [Geelen, 1992]:
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Arc Consistency

Let C be a constraint and D be a list of domains for the variables in the scope of C.
C is Arc Consistent (AC) iff for every variable x in the scope of C, for every value $v \in D(x)$, there exists an assignment w in D satisfying C in which v is assigned to x

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- If each domain is a singleton, the propagator must be able to check if the assignment corresponds to a solution or not.

CP vs. SAT

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- CP vs. SAT: a fundamental difference is the presence of global reasoning win CP.

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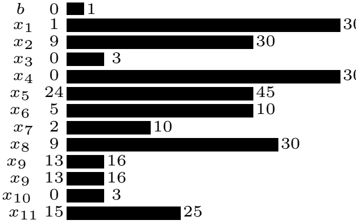

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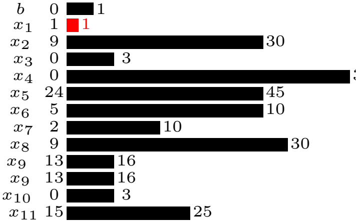

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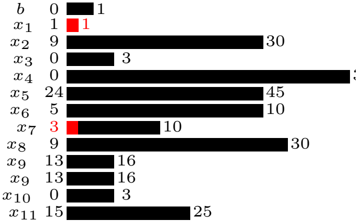

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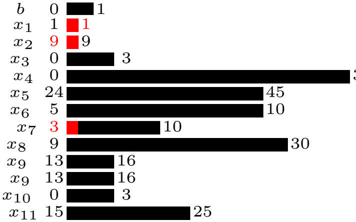

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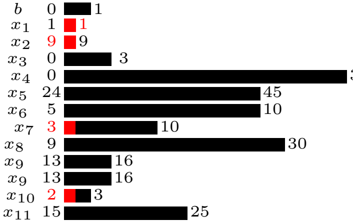

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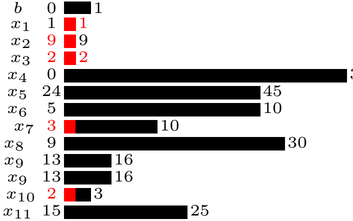

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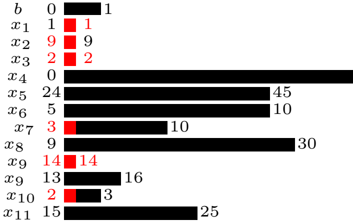

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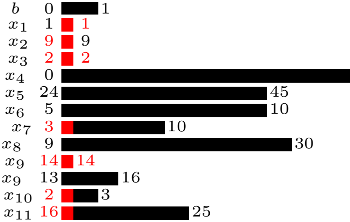

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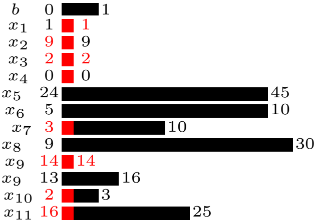

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$\llbracket x_4 = 0 \rrbracket \longrightarrow \llbracket b = 1 \rrbracket$

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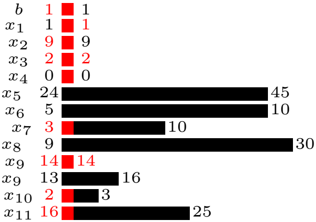

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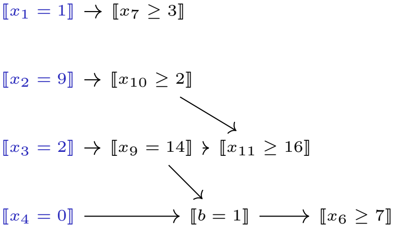

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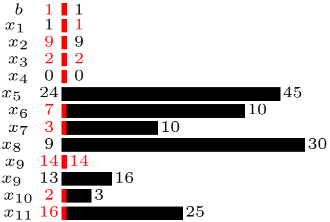

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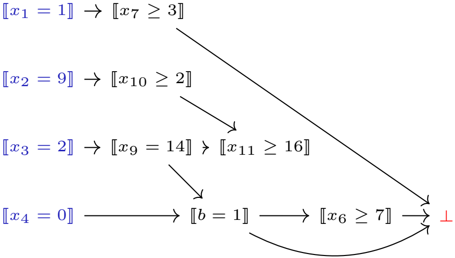

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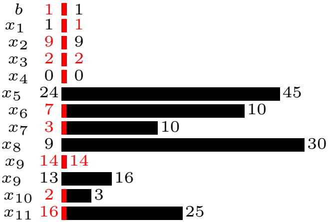

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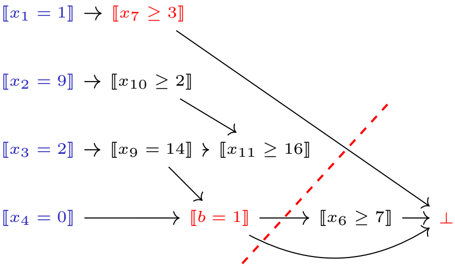

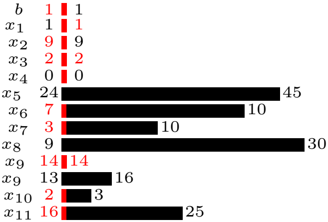

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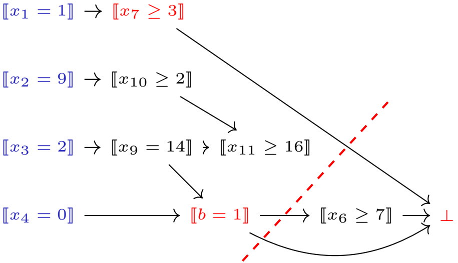
Learning in CP



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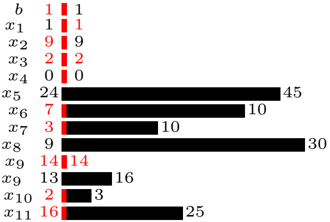

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Learning in CP

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- Conflict analysis: $\llbracket b = 1 \rrbracket \wedge \llbracket x_7 \geq 3 \rrbracket \Rightarrow \bot$
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- Backtrack to level 1

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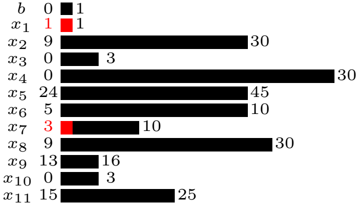

Learning in CP

$[\![x_1 = 1]\!] \rightarrow [\![x_7 \geq 3]\!] \longrightarrow [\![b = 0]\!]$

- Conflict analysis: $[\![b = 1]\!] \wedge [\![x_7 \geq 3]\!] \Rightarrow \bot$
- New clause: $[\![b \neq 1]\!] \vee [\![x_7 \leq 2]\!]$
- Backtrack to level 1
- Propagate the learnt clause

$$
\begin{aligned}
& x_1 + x_7 \geq 4 \wedge \\
& x_2 + x_{10} \geq 11 \wedge \\
& x_3 + x_9 = 16 \wedge \\
& x_5 \geq x_8 + x_9 \wedge \\
& b \leftrightarrow (x_9 - x_4 = 14) \wedge \\
& b \rightarrow (x_6 \geq 7) \wedge \\
& b \rightarrow (x_6 + x_7 \leq 9) \wedge \\
& x_{11} \geq x_9 + x_{10}
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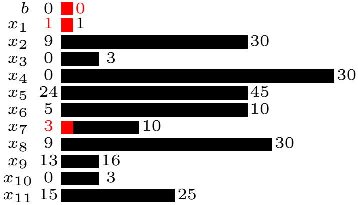
$$



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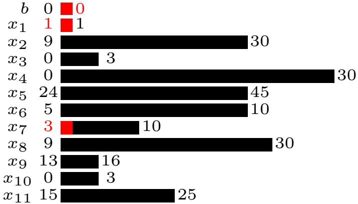

Conflict analysis

Algorithm 1: 1-UIP-with-Propagators

$\Psi \leftarrow explain(\perp)$;
while $|\{q \in \Psi \mid level(q) = current\ level\}| > 1$ **do**
  $p \leftarrow \arg\max_q(\{rank(q) \mid level(q) = current\ level \ \wedge \ q \in \Psi\})$;
  $\Psi \leftarrow \Psi \cup \{q \mid q \in explain(p) \wedge level(q) > 0\} \setminus \{p\}$;
return Ψ ;

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- To enable clause learning in CP, each propagator must be able to explain its filtering in the form of clauses ("Lazy Clause Generation").
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  - Explaining Domain filtering
- Example: Explain the constraint $X \leq Y$ with two scenarios (failure and propagation).

- Let $(x_1, \ldots, x_n)$ be a sequence of Boolean variables, and let d be a positive integer.
- The CARDINALITY$(x_1, \ldots, x_n, d)$ constraint holds iff exactly d variables from the sequence $(x_1, \ldots, x_n)$ are true.
- Write a filtering algorithm for CARDINALITY.
- What is the time complexity?
- Does it enforce arc consistency?
- Explain the CARDINALITY filtering.

Correction

Algorithm 4: $\textsc{Cardinality}([x_1, \ldots, x_n], d)$

if $\left|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}\right| > d$ **then**
1     $\mathcal{D} \leftarrow \perp$;
if $\left|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}\right| > n - d$ **then**
2     $\mathcal{D} \leftarrow \perp$;
if $\left|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}\right| = d$ **then**
    foreach $i \in \{1..n\}$ **do**
        if $\mathcal{D}(x_i) = \{0, 1\}$ **then**
3             $\mathcal{D}(x_i) \leftarrow \{0\}$;
else
    if $\left|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}\right| = n - d$ **then**
        foreach $i \in \{1..n\}$ **do**
            if $\mathcal{D}(x_i) = \{0, 1\}$ **then**
4                 $\mathcal{D}(x_i) \leftarrow \{1\}$;
return $\mathcal{D}$;

Explaining The Cardinality Constraint

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- Failure 1:

$$x^1 \land x^2 \land \ldots \land x^{d+1} \rightarrow \bot$$

Where $D(x^i) = \{1\}$

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$$x^1 \wedge x^2 \wedge \ldots \wedge x^{d+1} \rightarrow \perp$$

  Where $D(x^i) = \{1\}$

- Failure 2:

$$\neg x^1 \wedge \neg x^2 \wedge \neg x^{n-d+1} \rightarrow \perp$$

  Where $D(x^i) = \{0\}$

- Explaining the propagating the value 1: the conjunction of all the assigned variables

Explaining The Cardinality Constraint

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Where $D(x^i) = \{1\}$
- Failure 2:
$$\neg x^1 \wedge \neg x^2 \wedge \neg x^{n-d+1} \rightarrow \bot$$
Where $D(x^i) = \{0\}$
- Explaining the propagating the value 1: the conjunction of all the assigned variables
- Explaining the propagating the value 0: the conjunction of all the assigned variables

Encoding CSP into SAT

- How to encode the variables' domain ?
- How to encode each constraint into a set of clauses ?

Domain Encoding: Quadratic Encoding

- Suppose that $D(x) = \{v_1, \ldots, v_n\}$

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- As a clause: $\neg x_i \vee \neg x_j$
- The number of variables is linear
- The number of clauses is quadratic

Domain Encoding: Linear Encoding

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Domain Encoding: Linear Encoding

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Domain Encoding: Linear Encoding

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- $x_1 \vee \dots \vee x_n$
- $y_j \rightarrow y_{j+1}$
- $x_i \rightarrow y_{v_i} \wedge \neg y_{v_i - 1}$
- The number of variables is linear in the size of the domain
- The number of clauses is linear. However, some clauses are of arity three

Exercise: Constraint encoding ?

- How to encode the AllDifferent constraint ?
- How to encode $\sum_i X_i \leq k$ (X_i is an integer variable)?
- How to encode $\sum_i a_i \times X_i \leq k$?

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