

An Introduction to Boolean Satisfiability

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siala.github.io

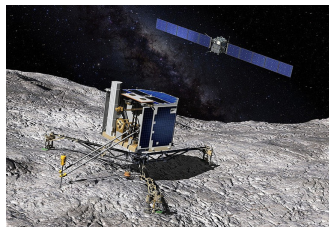
INSA-Toulouse & LAAS-CNRS

January 15, 2023

Context & Introduction

Context: Solving (Very) Hard Combinatorial Problems

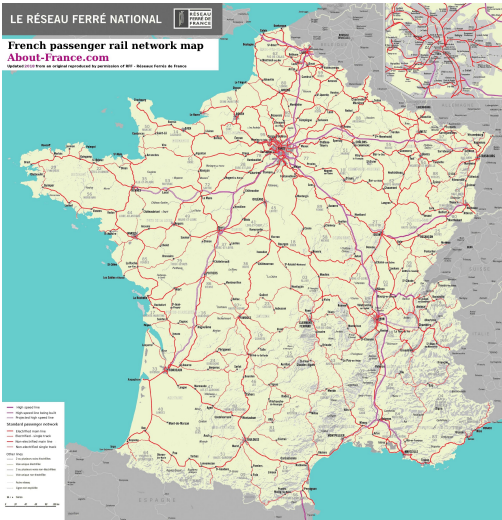
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<https://homepages.laas.fr/ehebrard/rosetta.html>

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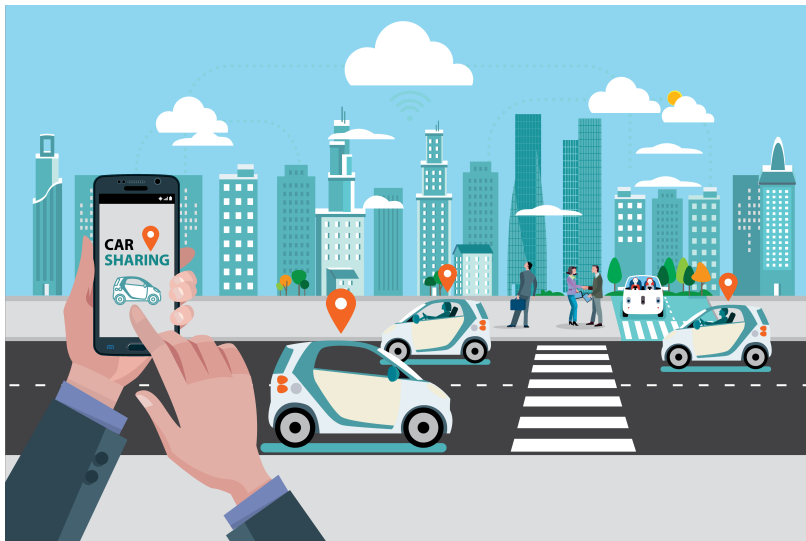
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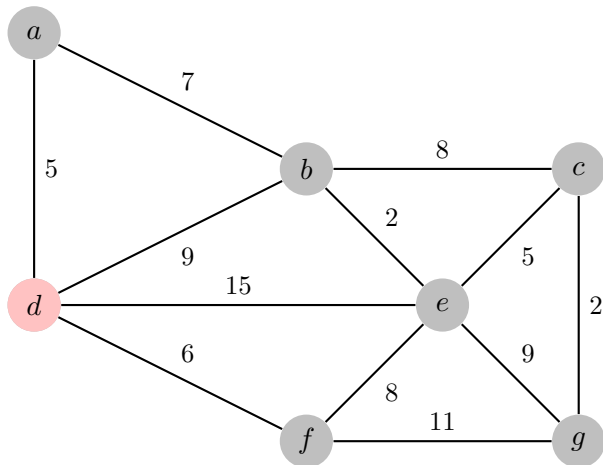
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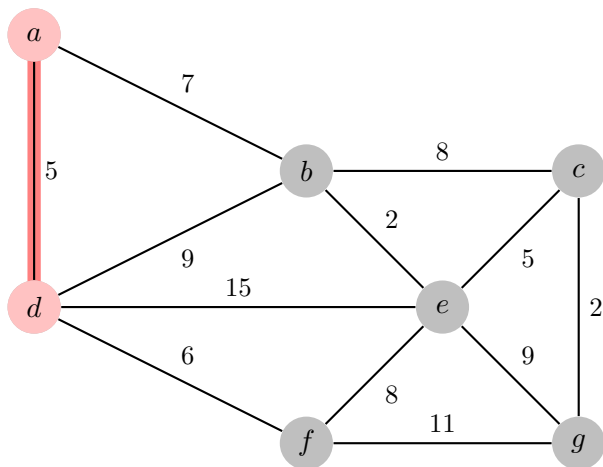
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- SAT as a tool to tackle combinatorial problems
- We focus in this course on the modelling aspect
- Resources for combinatorial optimisation: Many! a good start would be the online course on discrete optimisation
<https://www.coursera.org/learn/discrete-optimization>
- Handbook of Satisfiability - Second Edition - IOS Press, 2021

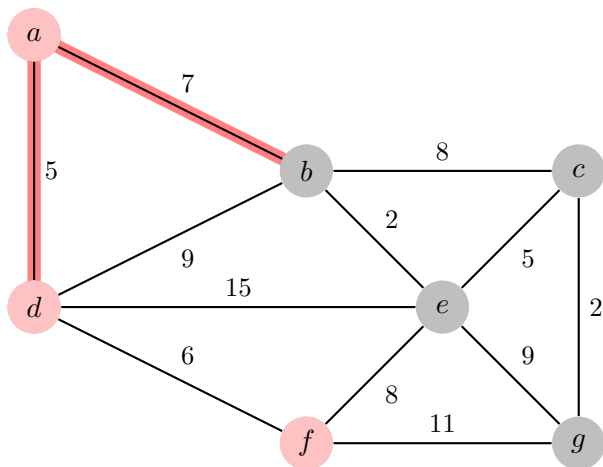
Exemple



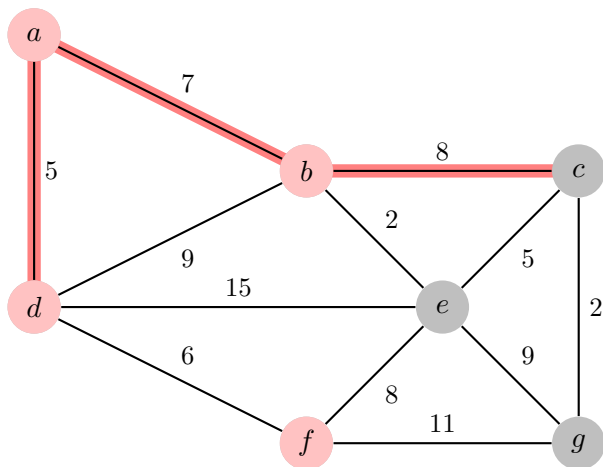
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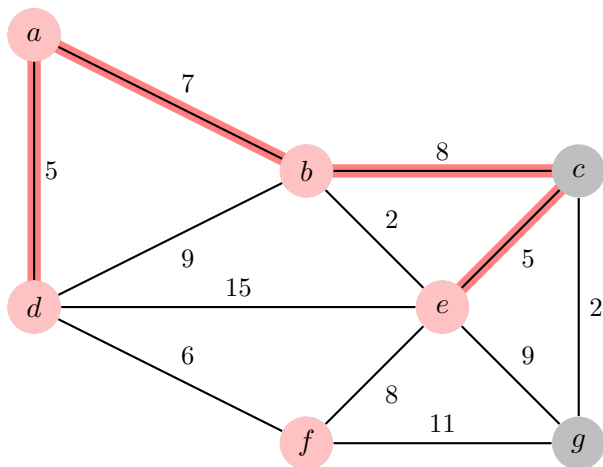
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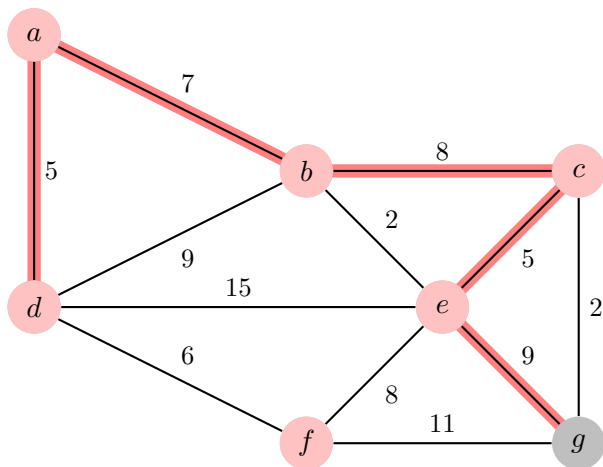
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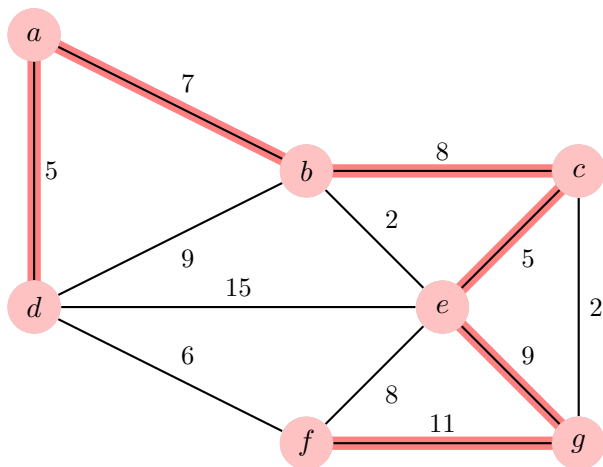
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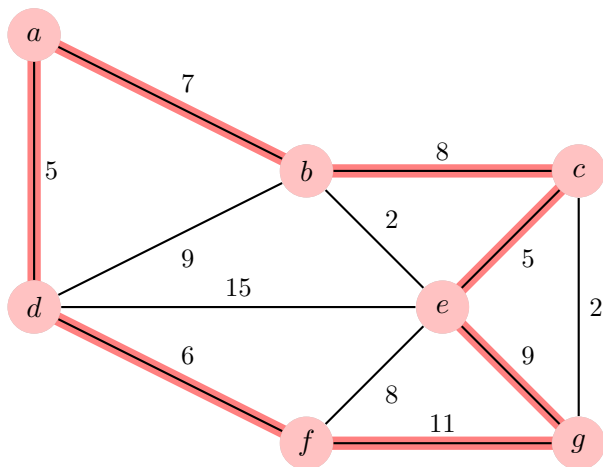
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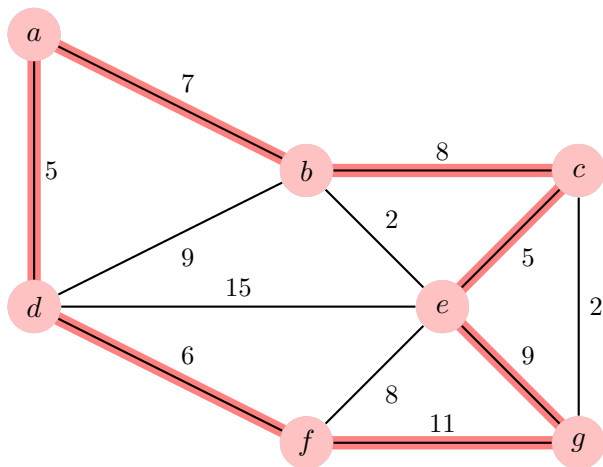
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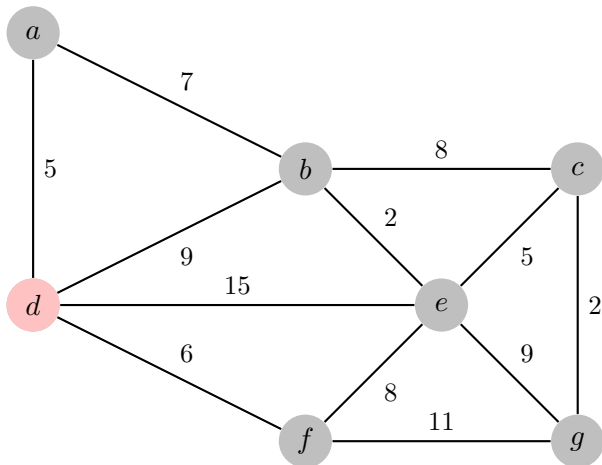


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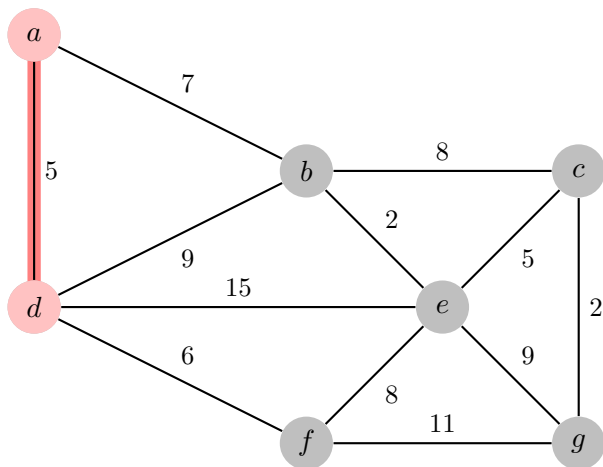


-- > $Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53Km$

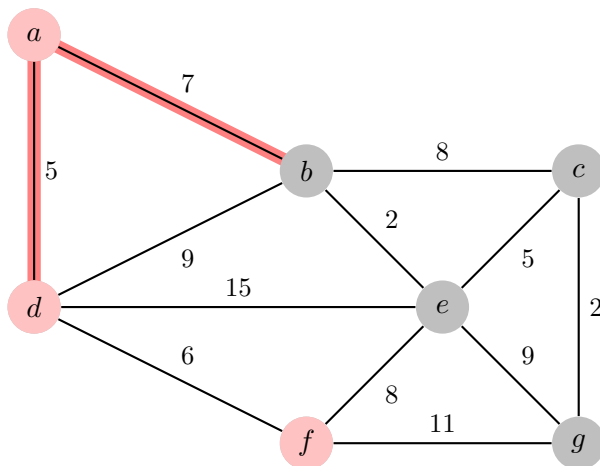
Example



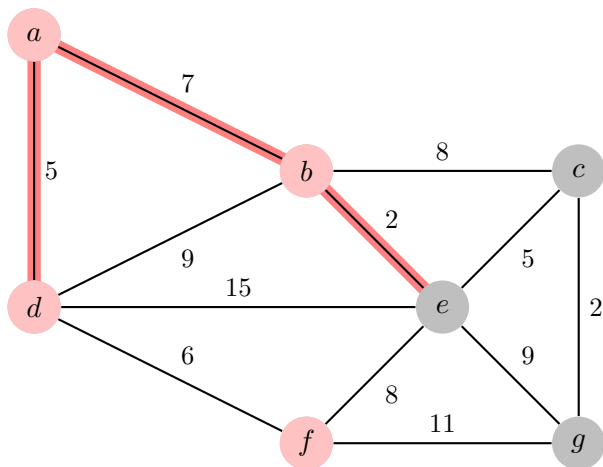
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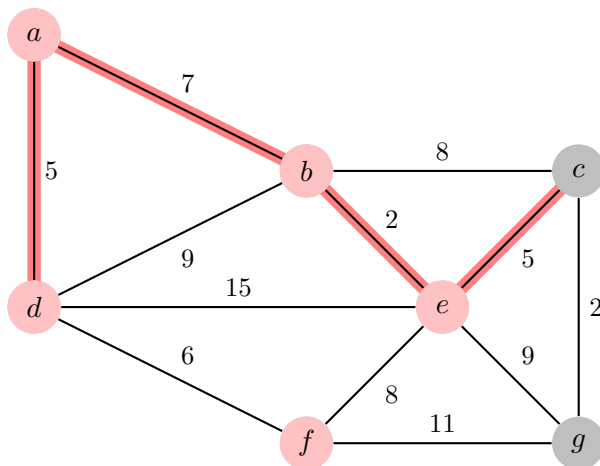
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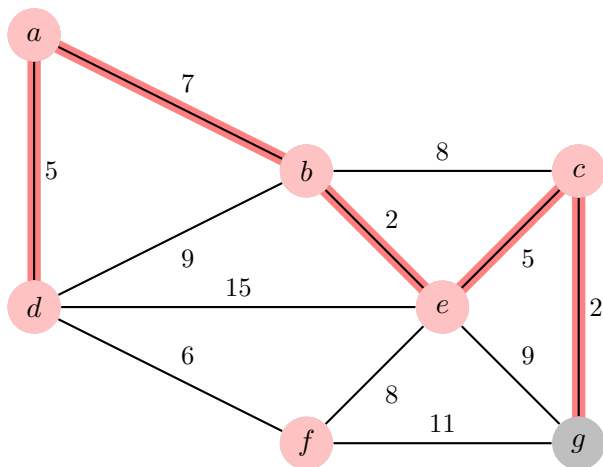
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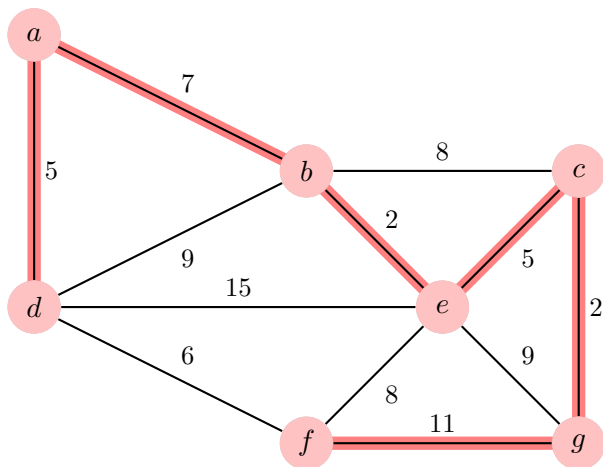
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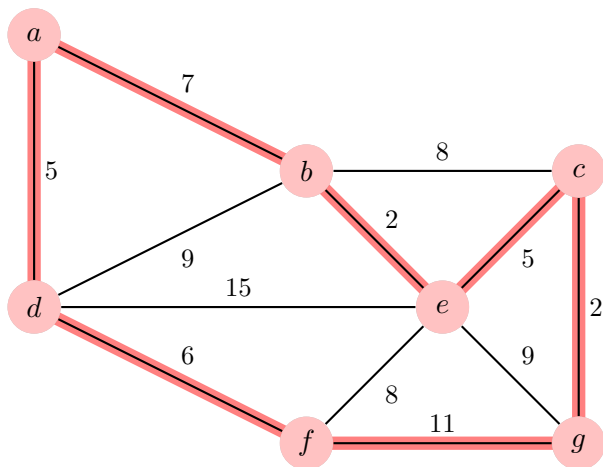
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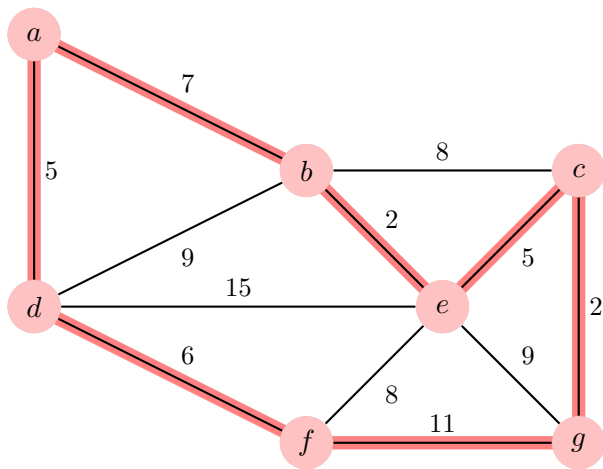
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-- > $Cost : 5 + 7 + 2 + 5 + 2 + 11 + 6 = 38Km$

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The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

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Solving Methodologies

- ① Adhoc methods
 - ① Specific exact algorithm
 - ② Heuristic method
 - ③ Meta-heuristic (genetic algorithms, ant colony, ..)
- ② Declarative Approaches
 - ① (Mixed) Integer Programming,
 - ② Constraint Programming
 - ③ Boolean Satisfiability (SAT)
 - ④ ...

Why Declarative Approaches?

- They are problem independent! The user models the problem in a specific language and the solver does the job!
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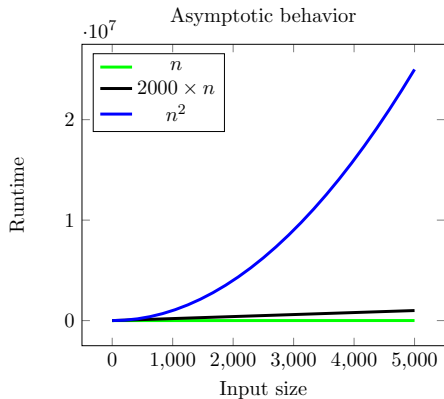
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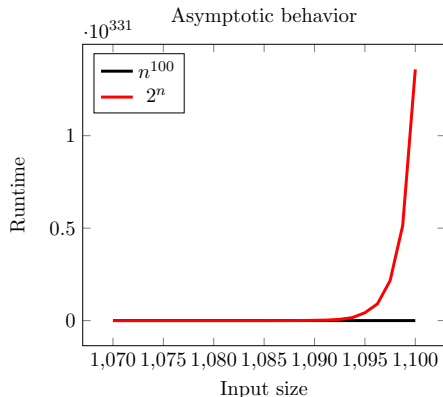
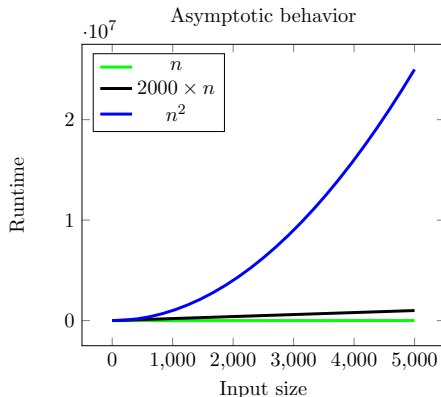
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- What if we don't know if a problem has a polynomial time algorithm?

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- For many Problems in NP , we don't know if a polynomial time algorithm exists. Is $P=NP$?

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Given a set of Boolean variables x_1, \dots, x_n and a CNF formula Φ over x_1, \dots, x_n , the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

Example

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$$x \vee \neg y \vee z$$

$$\neg x \vee \neg z$$

$$y \vee w$$

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A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

Why SAT?

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- SAT is the first problem that is shown to be in the class NP-Complete (the class of the 'hardest' problems in NP):
 - Any problem in NP can be reduced polynomially to SAT
 - If you can solve SAT in polynomial time, call me straight away!
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- It is considered today as a powerful technology to solve computational problems
- Huge practical improvements in the past 2 decades or so

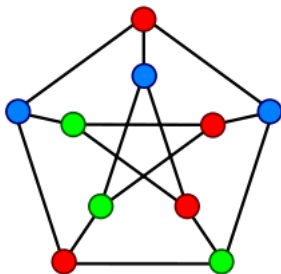
Examples of Applications

- AI Planning
- Scheduling
- Software verification
- Machine learning
 - Robustness
 - Synthesis
 - Verification
- Mathematical Proofs!
`https://news.cnrs.fr/articles/
the-longest-proof-in-the-history-of-mathematics`
- Timetabling
- ...

Modelling in SAT

The example of Graph Colouring

- Graph Coloring is a well known combinatorial problem that has many applications (in particular in scheduling problems)
- Let $G = (V, E)$ be an undirected graph where V is a set of n vertices and E is a set of m edges. Is it possible to colour the graph with k colours such that no two adjacent nodes share the same colour?



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(This is a translation of $x_i^a \rightarrow \neg x_i^b$)

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- Forbid two nodes that share an edge to be coloured with the same colour

$$\forall \{i, j\} \in E, \forall a \in [1, k] : \neg x_i^a \vee \neg x_j^a$$

(This is a translation of $x_i^a \rightarrow \neg x_j^a$)

The Example of Graph Coloring: The Model Size

What is the (space) size of the model?

- $n \times k$ Boolean variables
- Constraints form 1: n clauses with k literals each
- Constraints form 2: $n \times k^2$ binary clauses
- Constraints form 3: $m \times k$ binary clauses

The Example of Graph Coloring: The Minimization Version

- Propose a method that uses SAT for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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- Let's call $SAT(V, E, K)$ the SAT model of the decision version of the problem (i.e., can we find a valid colouring of $G(V, E)$ with k colours). Use $SAT(V, E, K)$ as an oracle within an iterative search. For instance:
 - **Decreasing linear Search:** Run iteratively $SAT(V, E, UB - 1), SAT(V, E, UB - 2), \dots$ until the problem is unsatisfiable. The last satisfiable value of k is the optimal value
 - **Binary search:** Run iteratively $SAT(V, E, z)$ as long as $UB > LB$ where $z = \lceil (UB - LB)/2 \rceil$. If the result is satisfiable, then and $UB \leftarrow z$ otherwise $LB \leftarrow z$

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- An alternative approach is to look for valid theoretical bounds in the literature.

Modelling Cardinality Constraints

- A cardinality constraint takes as input a sequence of Boolean variables $[x_1, \dots, x_n]$ and an integer k and enforces

$$\sum_{i=1}^n x_i \leq k$$

- Cardinality constraints are everywhere!
- There exist many ways in the literature to encode such constraints. See for instance <https://www.carstensinz.de/papers/CP-2005.pdf>

Quadratic encoding for $\sum_1^n x_i = 1$

- At least one constraint:

$$x_1 \vee x_2 \dots \vee x_n$$

- at most one constraint:

$$\forall i, j : \neg x_i \vee \neg x_j$$

This generates one clause of size n and (n^2) binary clauses without introducing additional variables.

Linear encoding for $\sum_1^n x_i = 1$

A sequence of Boolean variables $[y_1, \dots, y_n]$ is introduced such that $\forall i \in [1, n], y_i$ is true iff $\sum_{l=1}^{l=i} x_l = 1$. The set of clauses for the encoding is the following:

$$x_1 \vee x_2 \dots \vee x_n$$

$$y_n^1$$

$$\forall i \in [1, n-1] : y_i \rightarrow y_{i+1}$$

$$\forall i \in [1, n-1] : y_i \rightarrow \neg x_{i+1}$$

$$\forall i \in [1, n] : x_i \rightarrow y_i$$

Size: n new variables, 1 n -ary clause and $3 \times n$ binary clauses,

Encoding for $\sum_1^n x_i \geq k$

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- Do not Increment: $\neg y_{i-1}^z \wedge \neg x_i \rightarrow \neg y_i^z$

Encoding for $\sum_1^n x_i \geq k$

Size of the encoding:

- $\Theta(n \times k)$ variables
- $\Theta(n + k)$ unary clauses
- $\Theta(n \times k)$ binary clauses
- $\Theta(n \times k)$ ternary clauses

Encoding for $\sum_1^n x_i = k$?

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- Add y_n^k
- Replace y_n^{k+1} by $\neg y_n^{k+1}$
- The size of the encoding is the same as $\sum_1^n x_i \geq k$ (asymptotically)

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- Encode $\sum_1^n x_i \leq b$
- $\sum_1^n x_i \geq a$ with the same additional variables
- The size of the encoding is the same as $\sum_1^n x_i \geq k$ (asymptotically)

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- Check the MaxSAT competition

The Example of Graph Coloring: A Possible MaxSAT Model

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. In the (decision version of the) graph colouring problem, we are given k colours to colour the graph such that no two adjacent nodes share the same colour.

- Propose a MaxSAT model for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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- We shall extend the previous model:
- Let u_a be a Boolean variable that is True iff. the colour $a \in [1, k]$ is used
- Consider the previous model $SAT(V, E, k)$ with k an upper bound.
- All the previous clauses are hard.
- Add the following hard clauses:

$$\forall i \in [1, n], \forall a \in [1, k] : \neg u_a \rightarrow \neg x_i^a$$

- Eventually we can add symmetry breaking constraints: $u_a \rightarrow u_{a-1}$
- Then add the soft clauses:

$$\forall a \in [1, k] : \neg u_a$$

- A MaxSAT Optimal solution satisfies all the hard coloring clauses (valid colouring) and maximizes the number of non used colours.

Extensions: Quantified Boolean Formula (QBF)

- A QBF has the form $Q.F$, where F is a CNF-SAT formulae, and Q is a sequence of quantified variables ($\forall x$ or $\exists x$).
- Example $\forall x, \exists y, \exists z, (x \vee \neg y) \wedge (\neg y \vee z)$
- QBF Solver Competition:
https://www.qbflib.org/solvers_list.php

Extensions: Satisfiability Modulo Theories (SMT)

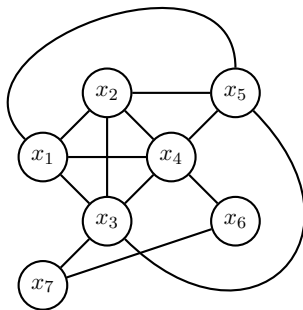
- SMT extends SAT by allowing higher level constraints
- Such constraints belong to certain theories
- Examples of theories include linear integer arithmetic, linear real arithmetic, difference logic, etc
- Check the SAT/SMT summer schools
<http://satassociation.org/sat-smt-school.html>

Exercise: SAT for Machine Learning

- Let $F = [f_1, \dots, f_k]$ be a set of k features and $E = [e_1, \dots, e_n]$ a set of n examples.
- We want to build an undirected acyclic graph for prediction
- Task1: Propose a model for the topology of the graph
- Task 2: Extend the model to make sure that each example is well classified
- Task 3: Adapt the model to maximize the accuracy of the model

Exercise: Clique

A clique in a graph $G(V, E)$ (where V is the set of vertices and E is the set of edges). A clique in G is a set of vertices $C \subseteq V$ such that $\forall a, b \in C, \{a, b\} \in E$. For examples, in the example below: $\{x_1, x_2, x_3, x_4, x_5\}$ is a clique and $\{x_3, x_6, x_7\}$ is not a clique.



- Propose a SAT model to find a clique of size $\geq k$ for a graph $G(V, E)$.

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- A possible solution:
 - x_i true iff v_i is in the clique
 - For each $\{i, j\} \notin E$:

$$\neg x_i \vee \neg x_j$$

- Clique size:

$$\sum x_i \geq k$$

- Implied constraints: If a vertex v_i has less than k edges it shouldn't be part of the clique:

$$\neg x_i$$

MaxSAT

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- Adapt your model into a MaxSAT formulae to find a clique with a maximum size

MaxSAT

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Same model without cardinality constraints, without implies constraints, and each x_i is added as a soft clause

Exercise: Shortest Path

Let $G(V, E)$ be a directed graph (where V is the set of vertices and E is the set of directed edges). Suppose that G has a one source $s \in V$ and one sink $o \in V$.

Propose a SAT model to find a path from s to o .

Adapt your model to find a shortest path

Conflict Driven Clause Learning

Modern SAT Solvers: Conflict Driven Clause Learning (CDCL)

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- [?, ?]

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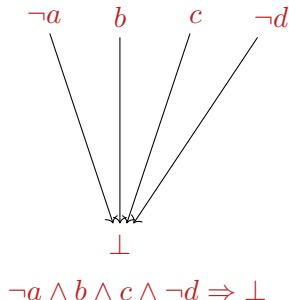
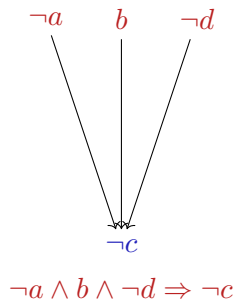
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- **Can be seen as a CP Solver (Search, propagation) augmented by clause learning**
- But also :
 - Activity-based branching
 - Lazy data structures (2-Watched Literals)
 - Clause Database Reduction
 - Simplifications
 - Restarts
 - ...

Exercise: Propose a filtering algorithm to prune the variables domain in a given clause

Unit Propagation

Given a clause C of arity n . If $n - 1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$



Algorithm 1: Unit Propagation

Data: A clause C

if *All literals in C are false* **then**

return Failure ;

else

if *Only one literal $l \in C$ is unassigned and the rest are false*
 then

 | Make l true ;

end

end

Unit Propagation

- Observe first that propagation happens only in two cases:
 - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
 - All literals are instantiated and none of them satisfy the clause

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- If a literal watching a clause C becomes *false*, look for replacement. If no replacement found, then perform propagation

Algorithm 2: Two watched Literals (decision d)

- ▷ Assuming initially that all variables are unassigned and that each clause contain at least 2 literals
 - ▷ For each clause C , $W[C]$ is initialized with a set that contains two variables in C
 - ▷ For each variable x , $B[x]$ is the set of clauses watched by x
 - ▷ d is the latest decision ;

```

 $S \leftarrow \{d\}$  ;
while  $S \neq \emptyset$  do
  Let  $x \in S$  ;
   $S \leftarrow S \setminus \{x\}$  ;
  while  $B[x] \neq \emptyset$  do
    Let  $C \in B[x]$  ;
    if  $x$  does not satisfy  $C$  then
       $W[C] \leftarrow W[C] \setminus \{x\}$  ;
      if  $\exists x' \in C \setminus W[C]$  such that  $x'$  is unassigned then
         $W[C] \leftarrow W[C] \cup \{x'\}$  ;
         $B[x'] \leftarrow B[x'] \cup \{C\}$  ;
      else
        Let  $y \in W[C]$  ;
        if  $y$  is not assigned then
          assign  $y$  to a value that satisfies  $C$  ;
           $S \leftarrow S \cup \{y\}$  ;
           $S \leftarrow \emptyset$  ;
        else
          if  $y$  does not satisfy  $C$  then
            return FAILURE ;
          end
        end
      end
    end
  end
end
end
end
end

```

Learning and Backjumping

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- Definition: Explaining a propagation of l : $l_1 \wedge \dots \wedge l_n \rightarrow l$ where $\neg l_1 \vee \dots \vee \neg l_n \vee \neg l$ is the triggering clause

Learning and Backjumping

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- When there is only one literal *uip* propagated in the last level in the current explanation, learn the associated new clause C , backjump (to the last level of propagated literals in C), propagate $\neg uip$ via the new clause, and continue the exploration

Conflict Analysis

Algorithm 1: 1-UIP-with-Propagators

```

1  $\Psi \leftarrow \text{explain}(\perp)$  ;
2 while  $|\{q \in \Psi \mid \text{level}(q) = \text{current level}\}| > 1$  do
   |  $p \leftarrow \arg \max_q (\{\text{rank}(q) \mid \text{level}(q) = \text{current level} \wedge q \in \Psi\})$  ;
3   |  $\Psi \leftarrow \Psi \cup \{q \mid q \in \text{explain}(p) \wedge \text{level}(q) > 0\} \setminus \{p\}$  ;
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- Why stop with one literal l propagated at the last level ?
- **To make sure that when the algorithm backjumps, propagation takes place by making l true**
- When backjumping using a clause that contains more than one literal propagated at the last level, then no propagation can be performed.

Implication Graph

	f		

$$\neg a \vee \neg f \vee g$$

$$\neg a \vee \neg b \vee \neg h$$

$$a \vee c$$

$$a \vee \neg i \vee \neg l$$

$$a \vee \neg k \vee \neg j$$

$$b \vee d$$

$$b \vee g \vee \neg n$$

$$b \vee \neg f \vee n \vee k$$

$$\neg c \vee k$$

$$\neg c \vee \neg k \vee \neg i \vee l$$

$$c \vee h \vee n \vee \neg m$$

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$$d \vee \neg g \vee l$$

$$\neg g \vee n \vee o$$

$$h \vee \neg o \vee \neg j \vee n$$

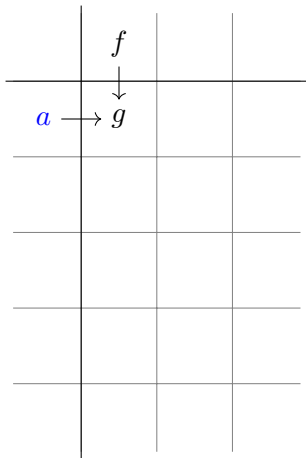
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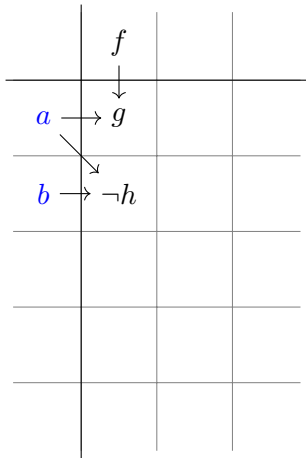
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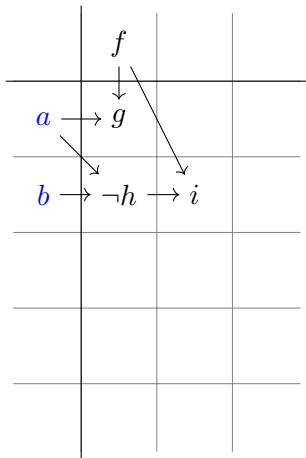
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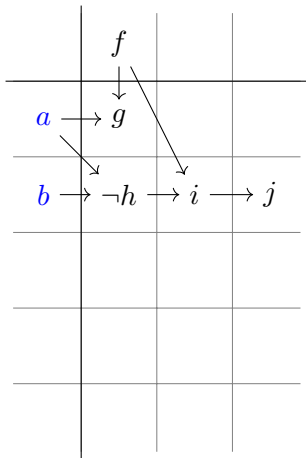
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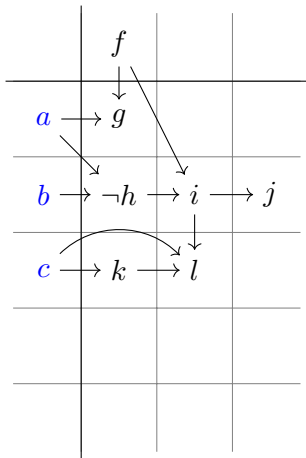
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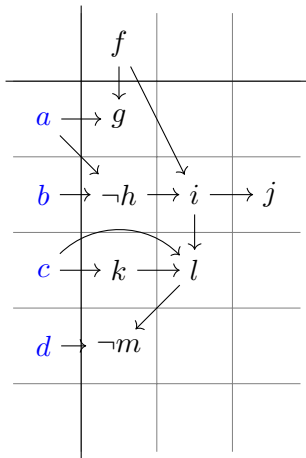
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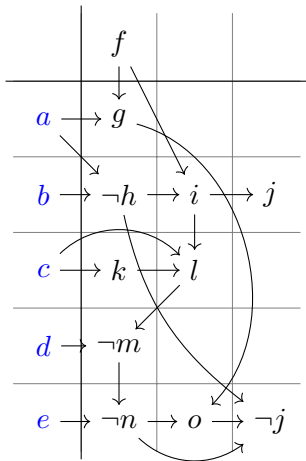
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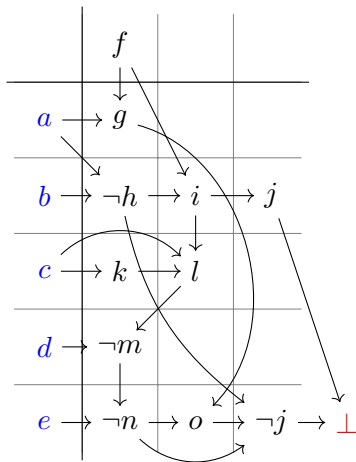
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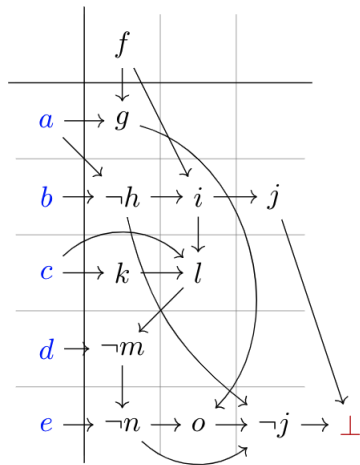
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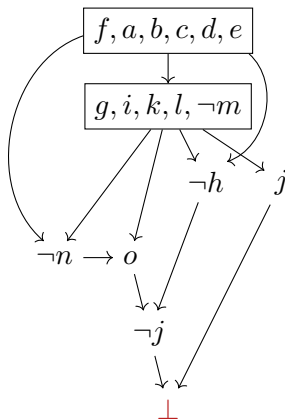
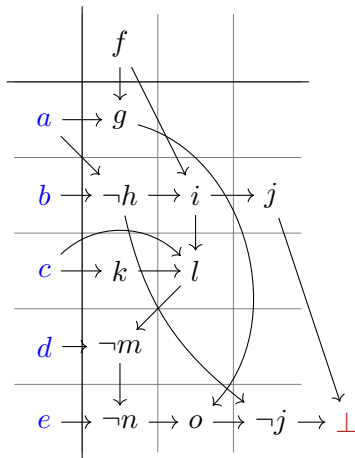
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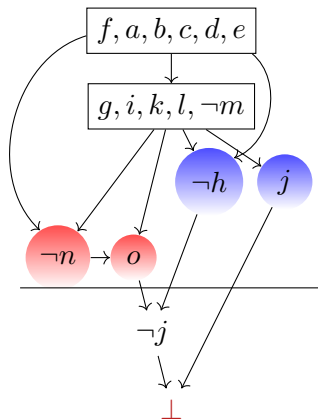
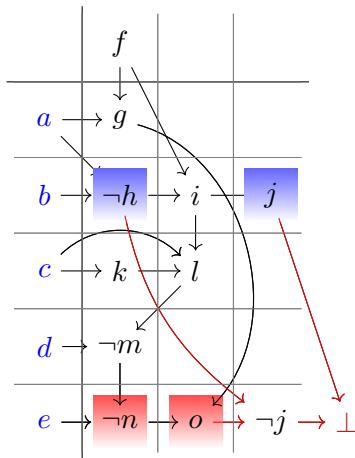
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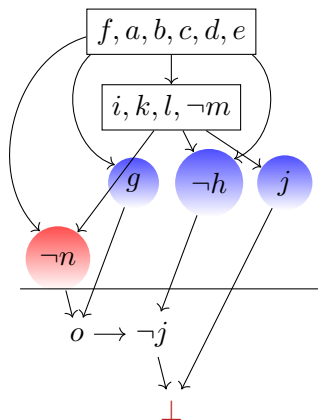
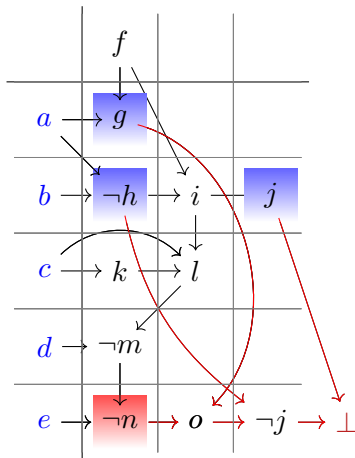
Conflict Analysis



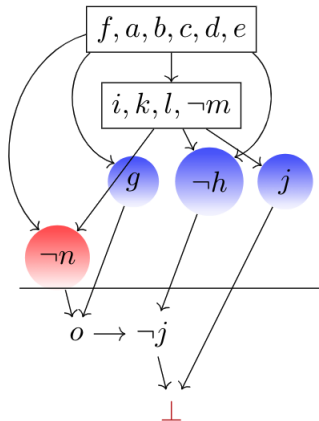
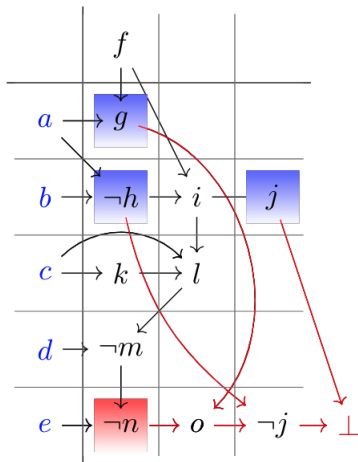
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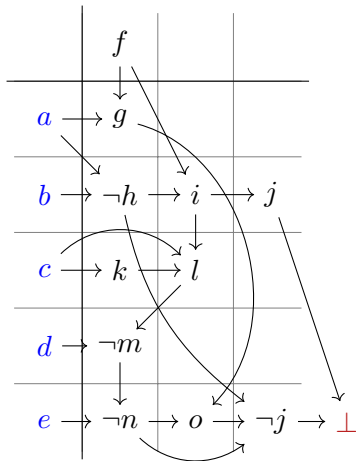
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Conflict analysis



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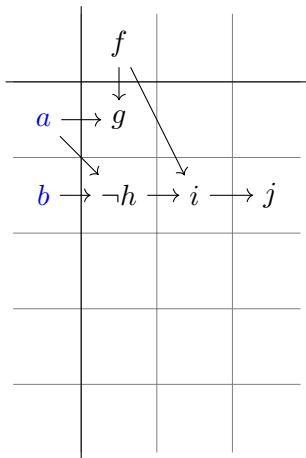
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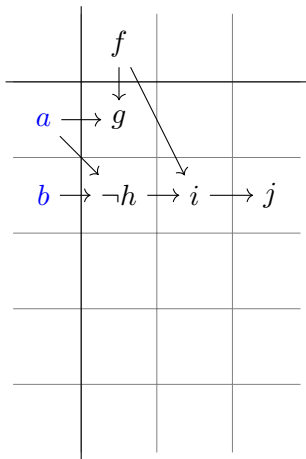
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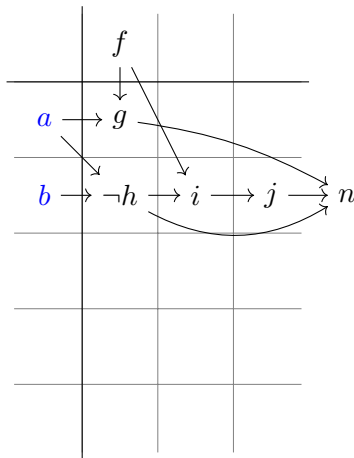
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Boosting Search through Randomization and Restarts [?]

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- Randomization: breaking ties, random decision between k best choices, ...
- Restarts: Geometric/Luby

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SAT Solvers (Few examples)

- MiniSat: <http://minisat.se/>
- Glucose: <http://www.labri.fr/perso/lsimon/glucose/>
- Lingeling <http://fmv.jku.at/lingeling>
- Any Solver by Armin Biere
<http://fmv.jku.at/software/index.html>
- Any winner from past and future SAT competitions:
<https://www.satcompetition.org/>

SAT vs CSP

Back to Constraint Programming

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- The constraint satisfaction problem (CSP) is the problem of deciding if a constraint network has a solution
- Mostly solvable by backtracking algorithms (Search and Filtering)

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Value Ordering

‘Succeed-first’ [Geelen, 1992]:

“Follow the best chances leading to a solution”

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Let C be a constraint and D be a list of domains for the variables in the scope of C .

C is Arc Consistent (AC) iff for every variable x in the scope of C , for every value $v \in D(x)$, there exists an assignment w in D satisfying C in which v is assigned to x

Filtering algorithm

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- If each domain is a singleton, the propagator must be able to check if the assignment corresponds to a solution or not.

CP vs. SAT

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- Every CSP can be encoded into SAT
 - When should we encode to SAT, when shouldn't we?
 - CP vs. SAT: a fundamental difference is the presence of global reasoning in CP and clause learning in SAT.

CP vs. SAT : To decompose or not to decompose?

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Modern Constraint Solvers: Hybrid CP/SAT

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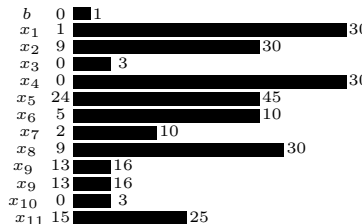
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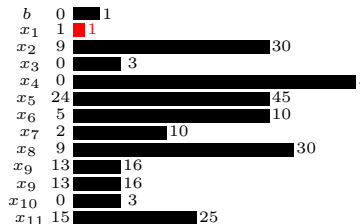
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Learning in CP

$$\begin{aligned}
 &x_1 + x_7 \geq 4 \wedge \\
 &x_2 + x_{10} \geq 11 \wedge \\
 &x_3 + x_9 = 16 \wedge \\
 &x_5 \geq x_8 + x_9 \wedge \\
 &b \leftrightarrow (x_9 - x_4 = 14) \wedge \\
 &b \rightarrow (x_6 \geq 7) \wedge \\
 &b \rightarrow (x_6 + x_7 \leq 9) \wedge \\
 &x_{11} \geq x_9 + x_{10}
 \end{aligned}$$


Learning in CP

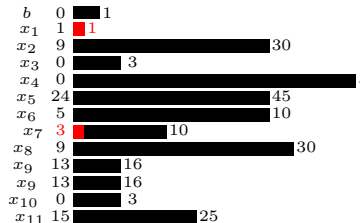
$\llbracket x_1 = 1 \rrbracket$

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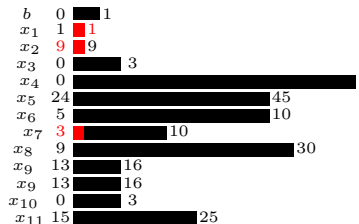


Learning in CP

$$\llbracket x_1 = 1 \rrbracket \rightarrow \llbracket x_7 \geq 3 \rrbracket$$

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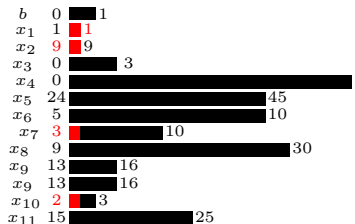


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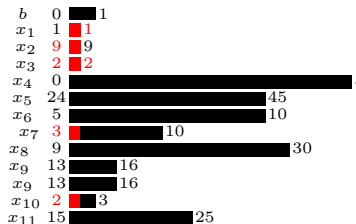
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b	0		1
x_1	1		1
x_2	9		9
x_3	2		2
x_4	0		
x_5	24		45
x_6	5		10
x_7	3		 10
x_8	9		30
x_9	14		14
x_9	13		16
x_{10}	2		 3
x_{11}	15		25

Learning in CP












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Learning in CP

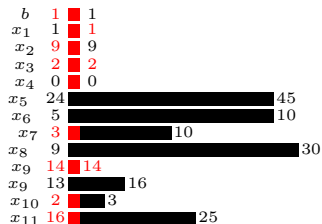
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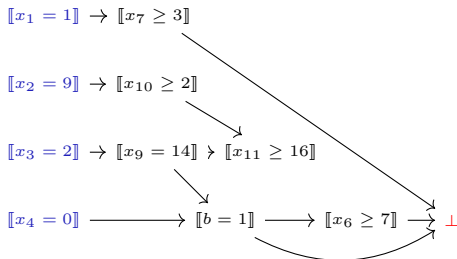
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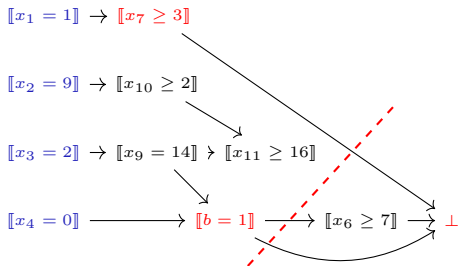
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Learning in CP

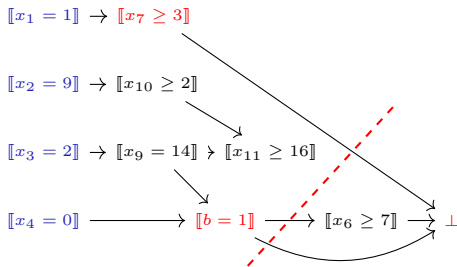


- Conflict analysis: $\llbracket b = 1 \rrbracket \wedge \llbracket x_7 \geq 3 \rrbracket \Rightarrow \perp$

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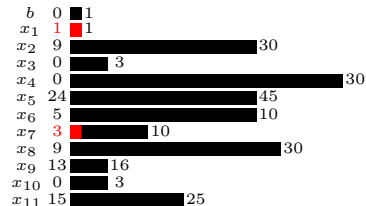


Learning in CP

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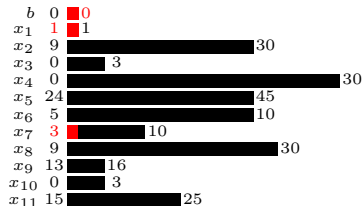


Learning in CP

$$\llbracket x_1 = 1 \rrbracket \rightarrow \llbracket x_7 \geq 3 \rrbracket \longrightarrow \llbracket b = 0 \rrbracket$$

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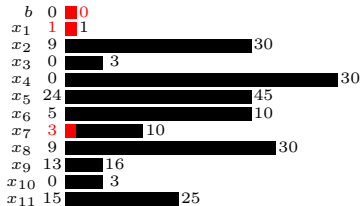


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- Propagate the learnt clause
- Continue exploration

$$\begin{array}{l} x_1 + x_7 \geq 4 \wedge \\ x_2 + x_{10} \geq 11 \wedge \\ x_3 + x_9 = 16 \wedge \\ x_5 \geq x_8 + x_9 \wedge \\ b \leftrightarrow (x_9 - x_4 = 14) \wedge \\ b \rightarrow (x_6 \geq 7) \wedge \\ b \rightarrow (x_6 + x_7 \leq 9) \wedge \\ x_{11} \geq x_9 + x_{10} \end{array}$$



Conflict analysis

Algorithm 1: 1-UIP-with-Propagators

```

1  $\Psi \leftarrow \text{explain}(\perp)$  ;
2 while  $|\{q \in \Psi \mid \text{level}(q) = \text{current level}\}| > 1$  do
     $p \leftarrow \arg \max_q (\{\text{rank}(q) \mid \text{level}(q) = \text{current level} \wedge q \in \Psi\})$  ;
3    $\Psi \leftarrow \Psi \cup \{q \mid q \in \text{explain}(p) \wedge \text{level}(q) > 0\} \setminus \{p\}$  ;
   return  $\Psi$  ;
```

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- To enable clause learning in CP, each propagator must be able to explain its filtering in the form of clauses (“Lazy Clause Generation”).
- We distinguish two types of explanations:
 - Explaining Failure
 - Explaining Domain filtering
- Example: Explain the constraint $X \leq Y$ with two scenarios (failure and propagation).

- Let (x_1, \dots, x_n) be a sequence of Boolean variables, and let d be a positive integer.
- The $\text{CARDINALITY}(x_1, \dots, x_n, d)$ constraint holds iff exactly d variables from the sequence (x_1, \dots, x_n) are true.
- Write a filtering algorithm for CARDINALITY .
- What is the time complexity?
- Does it enforce arc consistency?
- Explain the CARDINALITY filtering.

Correction

Algorithm 4: CARDINALITY($[x_1, \dots, x_n], d$)

```

if  $|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}| > d$  then
1   $\mathcal{D} \leftarrow \perp$  ;
if  $|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}| > n - d$  then
2   $\mathcal{D} \leftarrow \perp$  ;
if  $|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}| = d$  then
    foreach  $i \in \{1..n\}$  do
        if  $\mathcal{D}(x_i) = \{0, 1\}$  then
3       $\mathcal{D}(x_i) \leftarrow \{0\}$  ;
    else
        if  $|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}| = n - d$  then
            foreach  $i \in \{1..n\}$  do
                if  $\mathcal{D}(x_i) = \{0, 1\}$  then
4       $\mathcal{D}(x_i) \leftarrow \{1\}$  ;
return  $\mathcal{D}$  ;

```

Explaining The Cardinality Constraint

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$$x^1 \wedge x^2 \wedge \dots \wedge x^{d+1} \rightarrow \perp$$

Where $D(x^i) = \{1\}$

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- Explaining the propagating the value 1: the conjunction of all the assigned variables

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Where $D(x^i) = \{0\}$

- Explaining the propagating the value 1: the conjunction of all the assigned variables
- Explaining the propagating the value 0: the conjunction of all the assigned variables

Encoding CSP into SAT

- How to encode the variables' domain ?
- How to encode each constraint into a set of clauses ?

Domain Encoding: Quadratic Encoding

- Suppose that $D(x) = \{v_1, \dots, v_n\}$

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- The number of variables is linear

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- As a clause: $\neg x_i \vee \neg x_j$
- The number of variables is linear
- The number of clauses is quadratic

Domain Encoding: Linear Encoding

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- $y_j \rightarrow y_{j+1}$
- $x_i \rightarrow y_i \wedge \neg y_{i-1}$
- The number of variables is linear in the size of the domain
- The number of clauses is linear. However, some clauses are of arity three

Exercise: Constraint encoding ?

- How to encode the AllDifferent constraint ?
- How to encode $\sum_i X_i \leq k$ (X_i is an integer variable)?
- How to encode $\sum_i a_i \times X_i \leq k$?

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