

A Quadratic Algorithm for the Linearization Problem

July 2022

The linearization problem that we consider in this note is the following: we are given $x_1 \dots x_n, l_1 \dots, l_n, u_1 \dots u_n \in \mathbb{R}$, the purpose is to find a function $f(x) = ax + b$ such that $l_j \leq ax_j + b \leq u_j$ for all $j = 1 \dots n$.

Consider $\alpha_j = f(x_j) - l_j$ the slack of x_j . Given a solution f , we call x_i a pivot of f if $\alpha_i = \min\{\alpha_j \mid j = 1 \dots n\}$. By construction we have $g(x) = f(x) - \alpha_i$ a valid solution. Observe that in this case that $g(x_i) = l_i$. Hence, any solution f with a pivot x_i can be transformed to solution g such that $g(x_i) = l_i$. This can be seen as a dominance relationship. Therefore, the linearization problem admits a solution iff there exists a solution g that has a pivot x_i where $g(x_i) = l_i$. We first define a mathematical formulation \mathcal{M} that captures precisely all dominant solutions. Then, we show that \mathcal{M} can be solved in a quadratic time.

We introduce n Boolean variables $d_1 \dots d_n$ where $d_i \Leftrightarrow (ax_i + b = l_i)$. For each i , we define $y_i = \max\{\frac{l_j - l_i}{x_j - x_i} \mid j \neq i\}$ and $z_i = \min\{\frac{u_j - l_i}{x_j - x_i} \mid j \neq i\}$. Our mathematical model \mathcal{M} is defined as follows:

$$\bigvee_{i=1 \dots n} d_i \tag{1}$$

$$\bigwedge_{i=1 \dots n} d_i \Leftrightarrow (ax_i + b = l_i) \tag{2}$$

$$\bigwedge_{i=1 \dots n} d_i \implies y_i \leq a \leq z_i \tag{3}$$

Proposition 1. \mathcal{M} is satisfiable iff the linearization problem admits a solution

Proof. \Rightarrow Suppose that the model is satisfiable. Then $\exists d_i = 1$ such that

$$\begin{aligned} y_i &\leq a \leq z_i \\ \implies \forall j \neq i, \frac{l_j - l_i}{x_j - x_i} &\leq a \leq \frac{u_j - l_i}{x_j - x_i} \\ \implies \forall j \neq i, l_j - l_i &\leq a(x_j - x_i) \leq u_j - l_i \\ \implies \forall j \neq i, l_j &\leq ax_j + b \leq u_j \end{aligned}$$

Hence, the linearization problem admits a solution.

\Leftarrow Consider a dominant solution $g(x) = ax + b$ with a pivot x_i . Constraint 1 is satisfied by setting $d_i = 1$. Constraints 2 are satisfied since x_i is a pivot. For the third set of constraints, observe that they are fired only for the pivots. For any $d_i = 1$, we have $\forall j \neq i, l_j \leq ax_j + b \leq u_j$. Therefore $\forall j \neq i, l_j - l_i \leq ax_j + b - (ax_i + b) \leq u_j - l_i$. Hence $y_i \leq a \leq z_i$ and by consequence \mathcal{M} is satisfiable. \square

In the following, we propose an algorithm to solve \mathcal{M} . For that, one have to find a certain i such that $y_i \leq z_i$. Indeed, by doing so, it is enough to pick $a = y_i$ since $y_i \leq a \leq z_i$, fix $b = ax_i - l_i$, $d_i = 1$, and $d_{j \neq i} = 0$. Our proposed algorithm is given below.

Algorithm 1: An Algorithm for \mathcal{M}

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for  $i \in [1 \dots n]$  do
   $y \leftarrow -\infty$  ;
   $z \leftarrow \infty$  ;
  for  $j \in [1 \dots n] \setminus \{i\}$  do
     $w = \frac{l_j - l_i}{x_j - x_i}$  ;
    if  $y < w$  then
       $y \leftarrow w$  ;
    end
     $w = \frac{u_j - l_i}{x_j - x_i}$  ;
    if  $z > w$  then
       $z \leftarrow w$  ;
    end
  end
  if  $y \leq z$  then
     $a \leftarrow y$  ;
     $b \leftarrow ax_i - l_i$  ;
     $d_i \leftarrow 1$  ;
    return  $(a, b, d_i)$  ;
  end
end
return  $\emptyset$  ;

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The overall complexity is $O(n^2)$. I'm very sad that I couldn't make it linear. In an online setting, the approach can be adapted as a linear algorithm.