

SAT: Modelling and Implementations

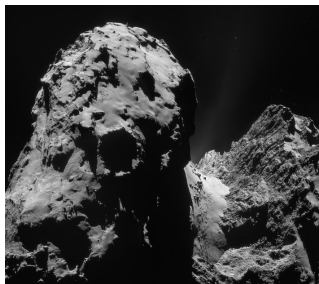
Mohamed Siala
<https://siala.github.io>

INSA-Toulouse & LAAS-CNRS

January 18, 2022

Context: Solving (Very) Hard Combinatorial Problems

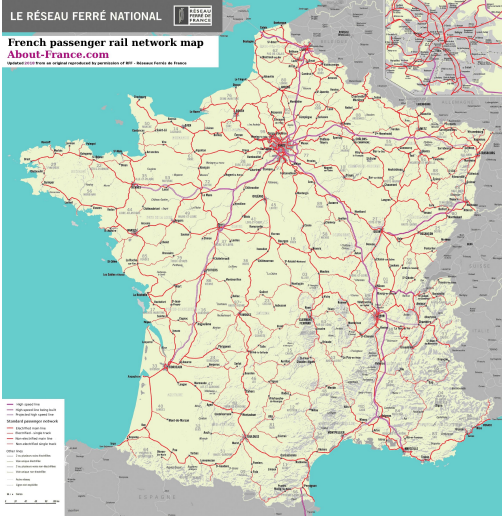
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<https://homepages.laas.fr/ehebrard/rosetta.html>

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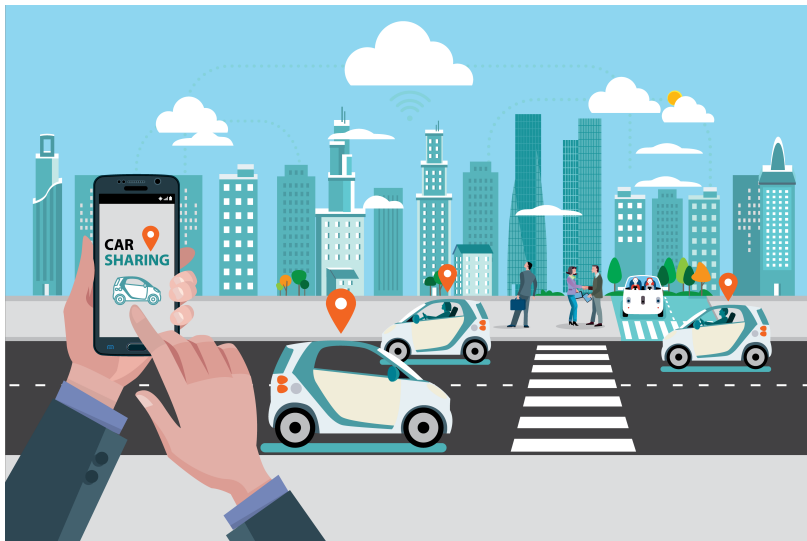
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- Resources: many.. a good start would be the online course on discrete optimisation
<https://www.coursera.org/learn/discrete-optimization>

Solving Methodologies

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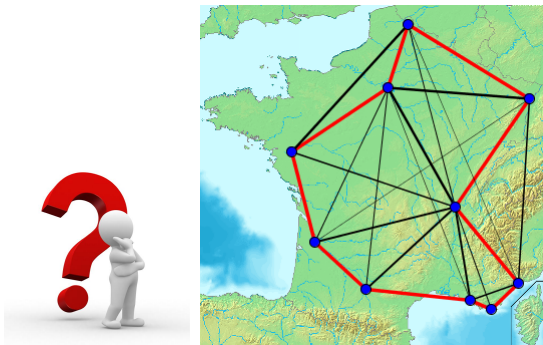
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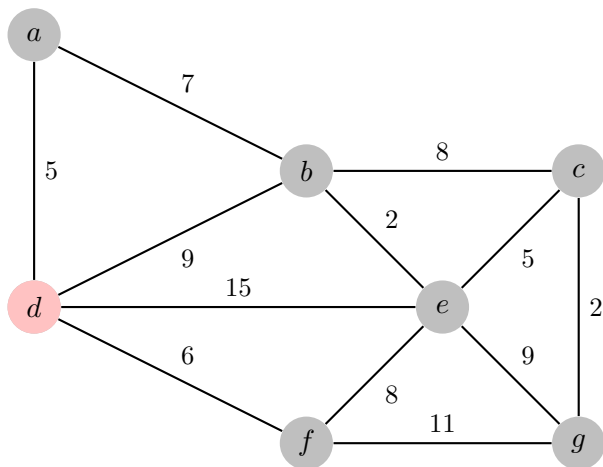
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- Very active community

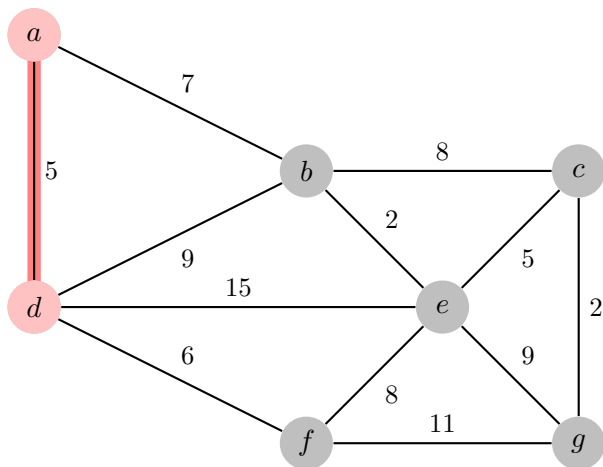
Travelling Salesman Problem



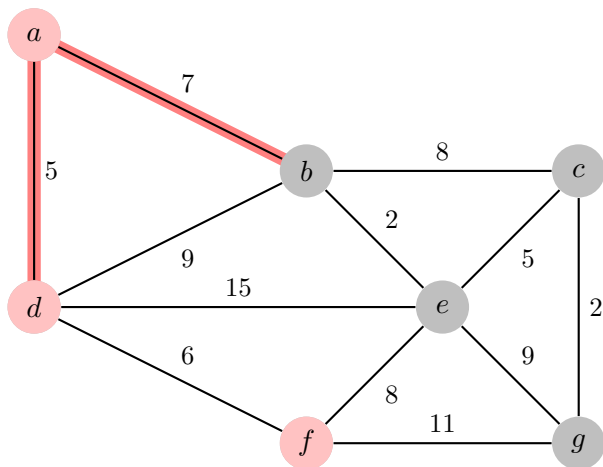
Exemple



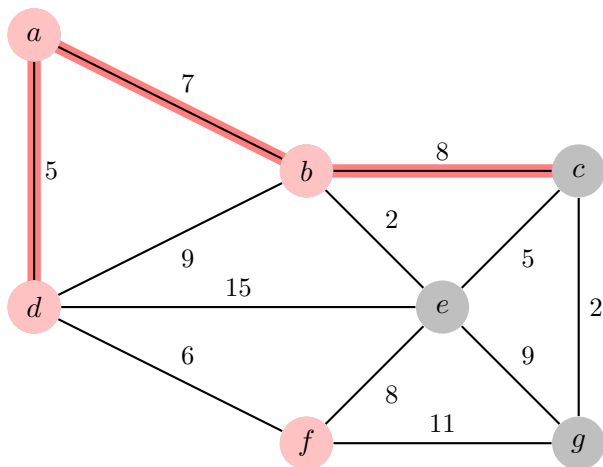
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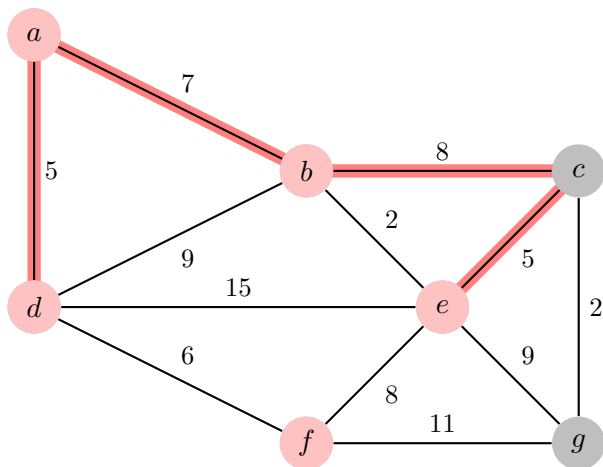
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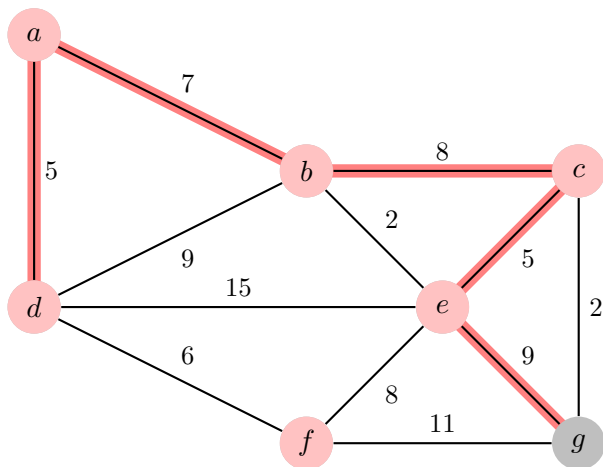
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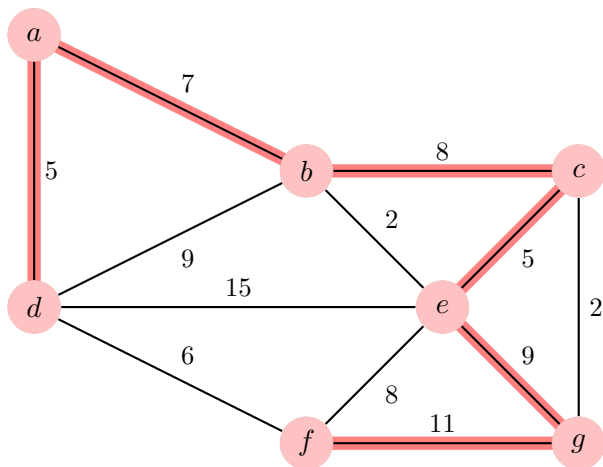
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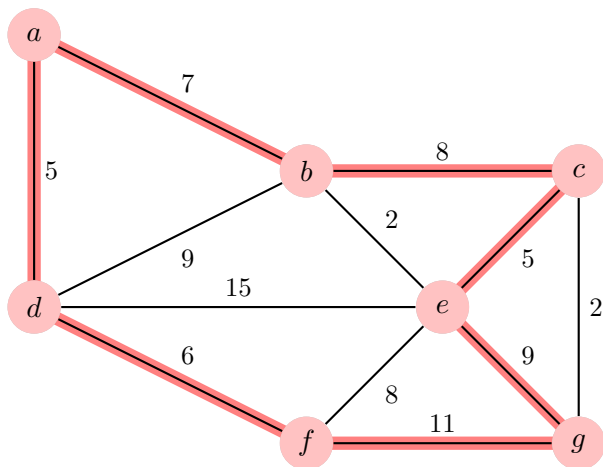
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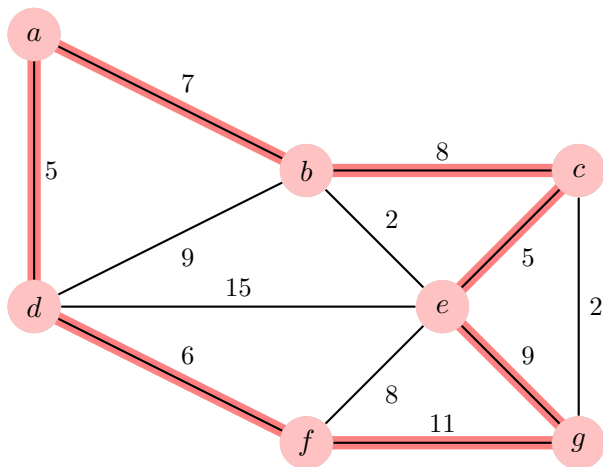
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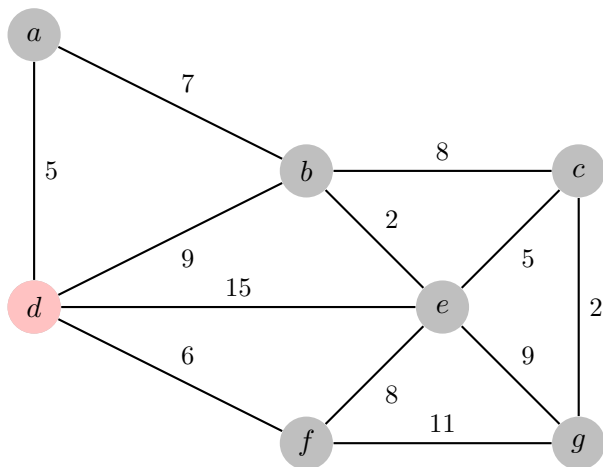


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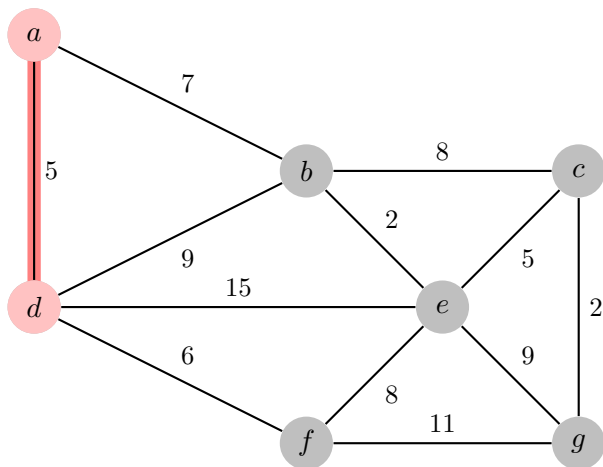


-- > $Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53Km$

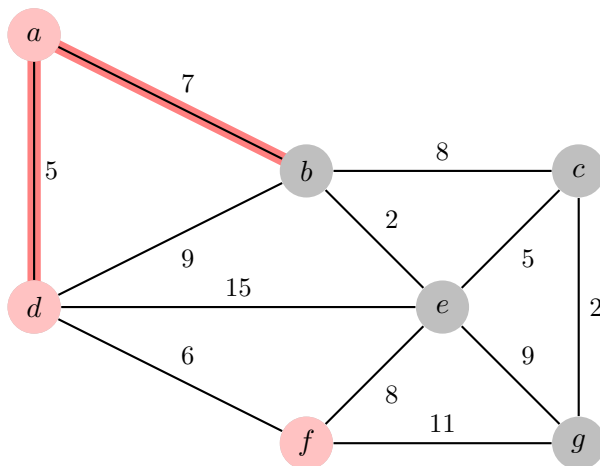
Example



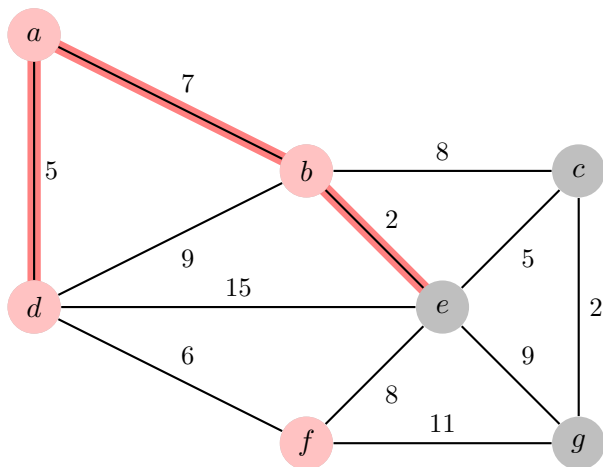
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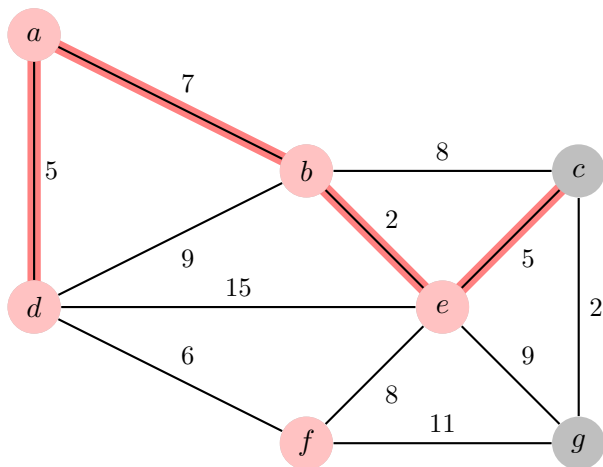
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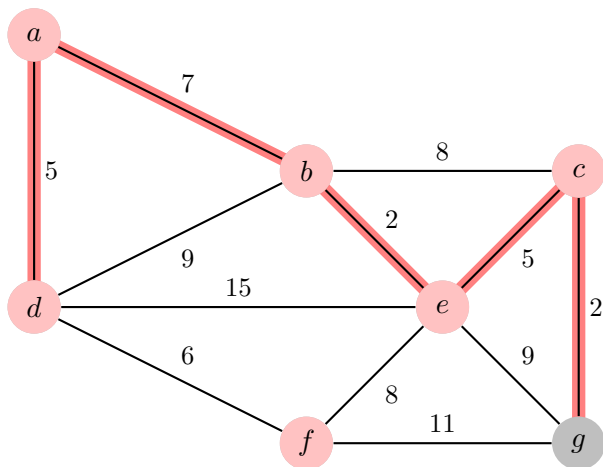
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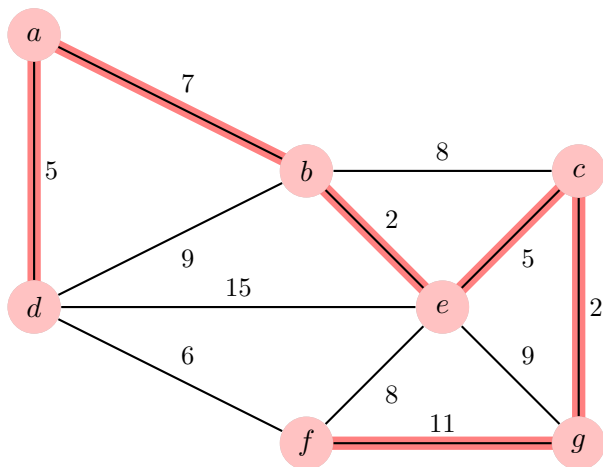
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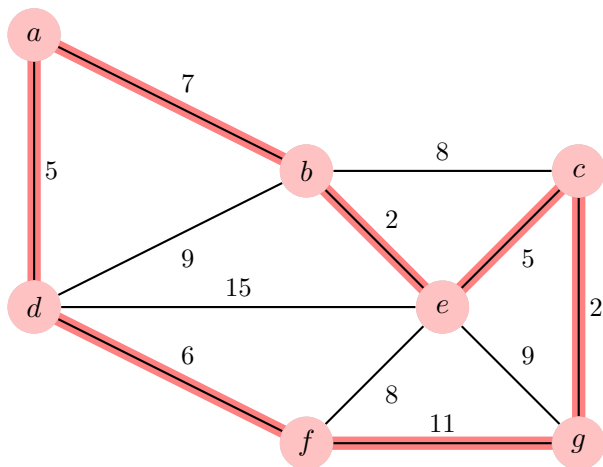
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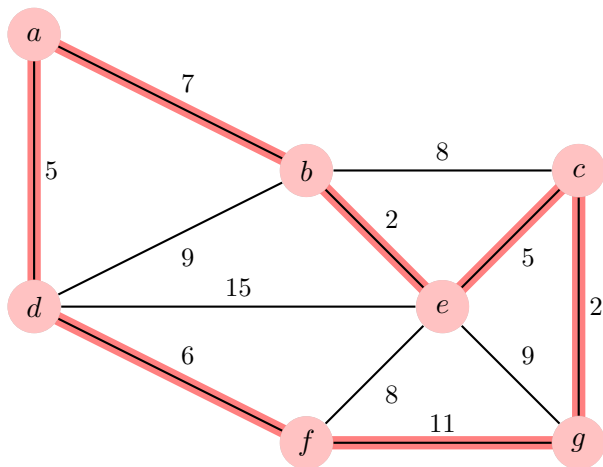
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-- > $Cost : 5 + 7 + 2 + 5 + 2 + 11 + 6 = 38Km$

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The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

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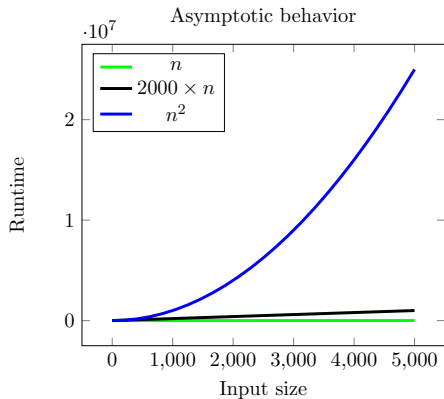
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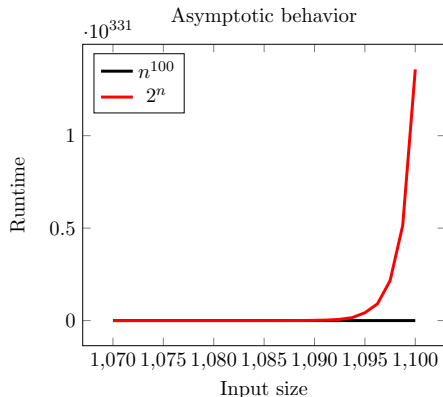
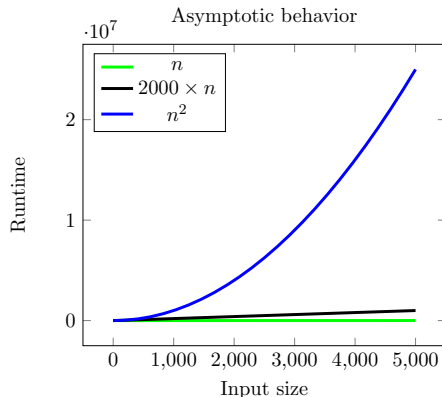
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- What if we don't know if a problem has a polynomial time algorithm?

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- **1 Million \$** question: Is $P=NP$?

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Given a set of Boolean variables x_1, \dots, x_n and a CNF formulae Φ over x_1, \dots, x_n , the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

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- How to use it to solve problems (Modelling)
- Discover some efficient implementations

Example

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$$x \vee \neg y \vee z$$

$$\neg x \vee \neg z$$

$$y \vee w$$

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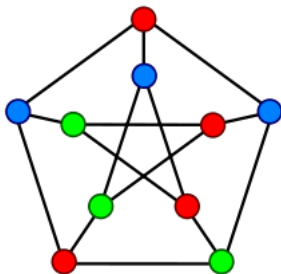
$$\neg w \vee \neg x$$

A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

The example of Graph Colouring

- Graph Coloring is a well know combinatorial problem that has many applications (in particular in scheduling problems)
- Let $G = (V, E)$ be an undirected graph where V is a set of n vertices and E is a set of m edges. Is it possible to colour the graph with k colours such that no two adjacent nodes share the same colour?



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- Propose a SAT model for this problem
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(This is a translation of $x_i^a \rightarrow \neg x_i^b$)

The Example of Graph Coloring: A Possible Model

Let x_i^k be the Boolean variable that is True iff node i is coloured with the colour k .

- Each node has to be colored with at least one color:

$$\forall i \in [1, n], x_i^1 \vee x_i^2 \dots x_i^k$$

- If a node is coloured with a colour a , the other colours are forbidden:

$$\forall i \in [1, n], \forall a \neq b \in [1, k], : \neg x_i^a \vee \neg x_i^b$$

(This is a translation of $x_i^a \rightarrow \neg x_i^b$)

- Forbid two nodes that share an edge to be coloured with the same colour

$$\forall \{i, j\} \in E, \forall a \in [1, k] : \neg x_i^a \rightarrow \neg x_j^a$$

(This is a translation of $x_i^a \rightarrow \neg x_j^a$)

The Example of Graph Coloring: The Model Size

What is the (space) size of the model?

- $n \times k$ Boolean variables
- Constraints form 1: n clauses with k literals each
- Constraints form 2: $n \times k^2$ binary clauses
- Constraints form 3: $m \times k$ binary clauses

The Example of Graph Coloring: The Minimization Version

- Propose a method that uses SAT for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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 - **Decreasing linear Search:** Run iteratively $SAT(V, E, UB - 1), SAT(V, E, UB - 2), \dots$ until the problem is unsatisfiable. The last satisfiable value of k is the optimal value

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 - **Binary search:** Run iteratively $SAT(V, E, z)$ as long as $UB > LB$ where $z = \lceil (UB - LB)/2 \rceil$. If the result is satisfiable, then and $UB \leftarrow z$ otherwise $LB \leftarrow z$

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- An alternative approach is to look for valid theoretical bounds in the literature.

Modelling Cardinality Constraints

- The general form of cardinality constraints is the following:

$$a \leq \sum_{i=1}^n x_i \leq b$$

where a and b are positive integers and $x_1 \dots x_n$ are Boolean variables

- Cardinality constraints are everywhere!
- Many ways to encode such constraints. See for instance <https://www.carstensinz.de/papers/CP-2005.pdf>

Quadratic encoding for $\sum_1^n x_i = 1$

- At least one constraint:

$$x_1 \vee x_2 \dots x_n$$

- at most one constraints:

$$\forall i, j : \neg x_i \vee \neg x_j$$

This generates one clause of size n and (n^2) binary clauses without introducing additional variables.

Linear encoding for $\sum_1^n x_i = 1$

New variables are added as follows: for $i \in [1, n]$, y_i is a new variable that is true iff $\sum_{l=1}^{l=i} x_l = 1$.

$$x_1 \vee x_2 \dots x_n$$

$$y_n^1$$

$$\forall i \in [1, n-1] : y_i \rightarrow y_{i+1}$$

$$\forall i \in [1, n-1] : y_i \rightarrow \neg x_{i+1}$$

$$\forall i \in [1, n] : x_i \rightarrow y_i$$

Size: n new variables, 1 n -ary clause and $3 \times n$ binary clauses,

Linear encoding for $\sum_1^n x_i \geq k$

New variables: $\forall z \in [0, k], \forall i \in [1, n], y_i^z \iff \sum_{l=1}^{l=i} x_l \geq z$

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$$y_{i-1}^z \wedge x_i \rightarrow y_i^{z+1}$$

Linear encoding for $\sum_1^n x_i \geq k$

Size of the encoding:

- $\Theta(n \times k)$ variables
- $\Theta(n + k)$ unary clauses
- $\Theta(n \times k)$ binary clauses
- $\Theta(n \times k)$ ternary clauses

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- Check the MaxSAT competition

Example of applications for MaxSAT

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. In the (decision version of the) graph colouring problem, we are given k colours to colour the graph such that no two adjacent nodes share the same colour.

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The Example of Graph Coloring: A Possible MaxSAT Model

- We shall extend the previous model:
- Consider the previous model $SAT(V, E, k)$ with k an upper bound.
- All the previous clauses are hard.
- Add the following hard clauses:

$$\forall i \in [1, n], \forall a \in [1, k] : \neg u_a \rightarrow \neg x_i^a$$

- Eventually we can add symmetry constraints: $u_a \rightarrow u_{a-1}$
- Then add the soft clauses:

$$\forall a \in [1, k] : \neg u_a$$

- A MaxSAT Optimal solution satisfies all the hard coloring clauses (valid colouring) and maximizes the number of non used colours.

Extensions: Quantified Boolean Formula (QBF)

- A QBF has the form $Q.F$, where F is a CNF-SAT formulae, and Q is a sequence of quantified variables ($\forall x$ or $\exists x$).
- Example $\forall x, \exists y, \exists z, (x \vee \neg y) \wedge (\neg y \vee z)$
- QBF Solver Competition:
https://www.qbflib.org/solvers_list.php
- QBF is less used in practice

Other Extensions

- Satisfiability Modulo Theories
- Answer Set Programming
- More generally: Automated reasoning community
- Check the SAT/SMT summer schools
<http://satassociation.org/sat-smt-school.html>

Modern SAT Solvers: Conflict Driven Clause Learning (CDCL)

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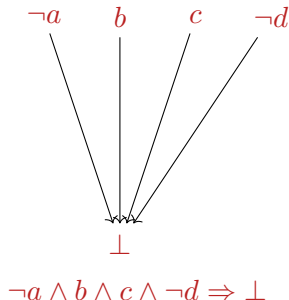
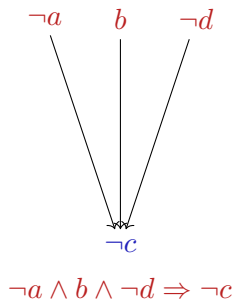
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- But also :
 - Activity-based branching
 - Lazy data structures (2-Watched Literals)
 - Clause Database Reduction
 - Simplifications
 - Restarts
 - ...

Exercise: Propose a filtering algorithm for clauses. The algorithm takes as input a clause and has access (read and write) for the variables domains.

Unit Propagation

Given a clause C of arity n . If $n - 1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$



Two Watched Literals

- Unit propagation is implemented with an “intelligent” data structure called Two-watched literals
- Observe first that propagation happens only in two cases:
 - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
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- If a literal watching a clause C becomes *false*, look for replacement. If no replacement found, then perform propagation

Implication Graph

	f		

$$\neg a \vee \neg f \vee g$$

$$\neg a \vee \neg b \vee \neg h$$

$$a \vee c$$

$$a \vee \neg i \vee \neg l$$

$$a \vee \neg k \vee \neg j$$

$$b \vee d$$

$$b \vee g \vee \neg n$$

$$b \vee \neg f \vee n \vee k$$

$$\neg c \vee k$$

$$\neg c \vee \neg k \vee \neg i \vee l$$

$$c \vee h \vee n \vee \neg m$$

$$c \vee l$$

$$d \vee \neg k \vee l$$

$$d \vee \neg g \vee l$$

$$\neg g \vee n \vee o$$

$$h \vee \neg o \vee \neg j \vee n$$

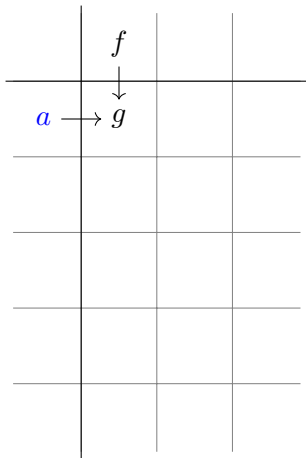
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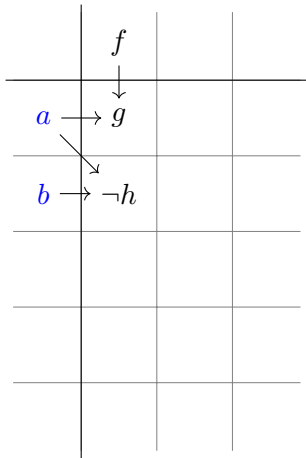
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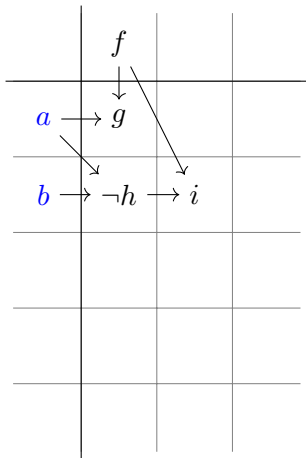
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$$\neg f \vee h \vee i$$

Implication Graph



$$\neg a \vee \neg f \vee g$$

$$\neg a \vee \neg b \vee \neg h$$

$$a \vee c$$

$$a \vee \neg i \vee \neg l$$

$$a \vee \neg k \vee \neg j$$

$$b \vee d$$

$$b \vee g \vee \neg n$$

$$b \vee \neg f \vee n \vee k$$

$$\neg c \vee k$$

$$\neg c \vee \neg k \vee \neg i \vee l$$

$$c \vee h \vee n \vee \neg m$$

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$$\neg g \vee n \vee o$$

$$h \vee \neg o \vee \neg j \vee n$$

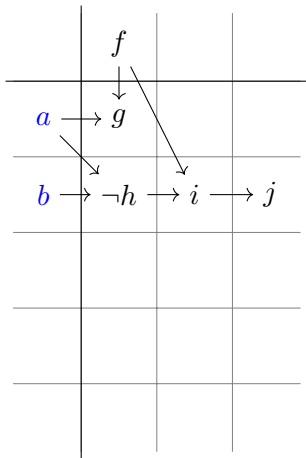
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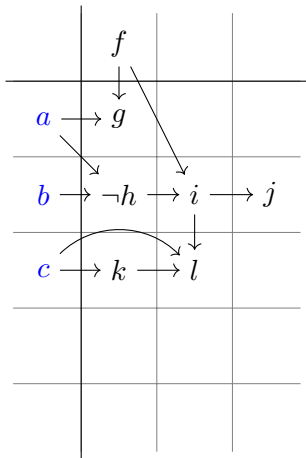
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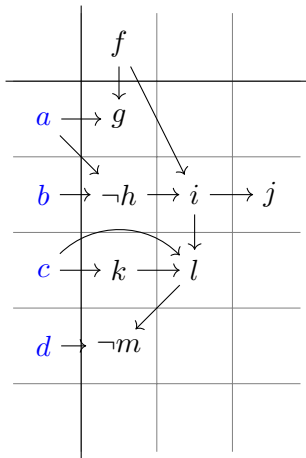
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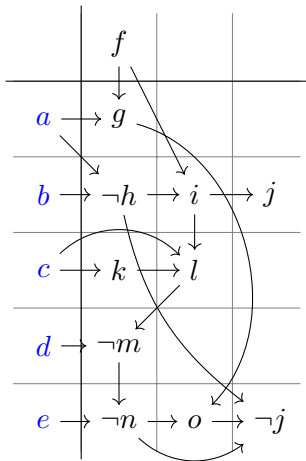
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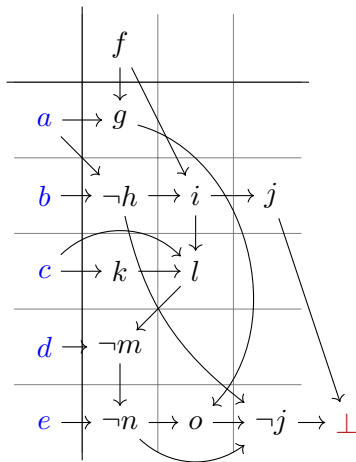
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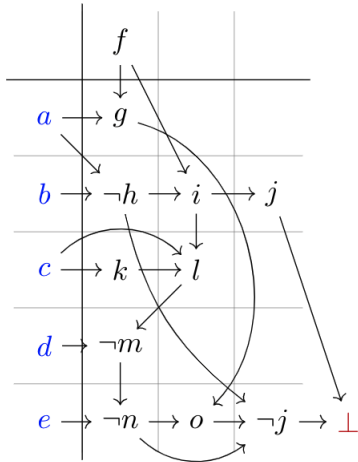
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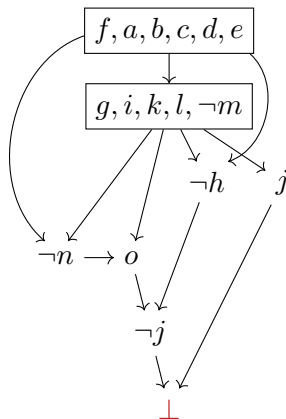
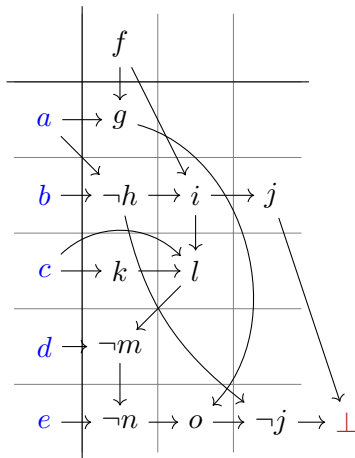
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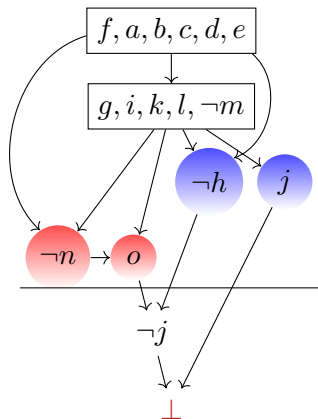
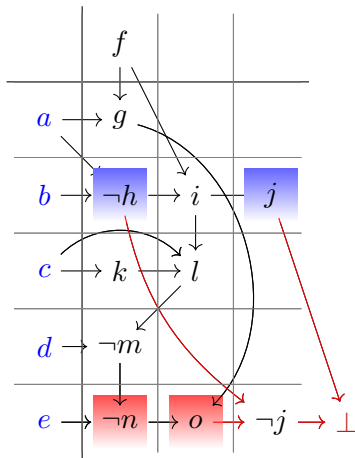
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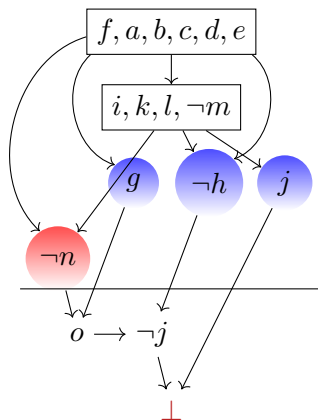
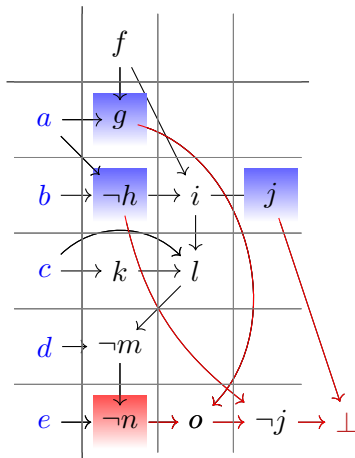
Conflict Analysis



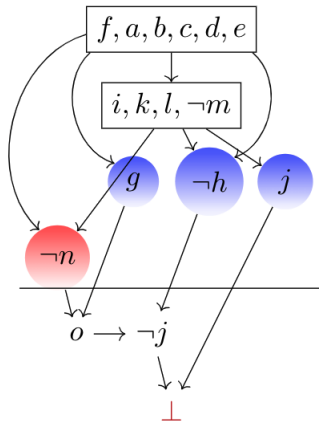
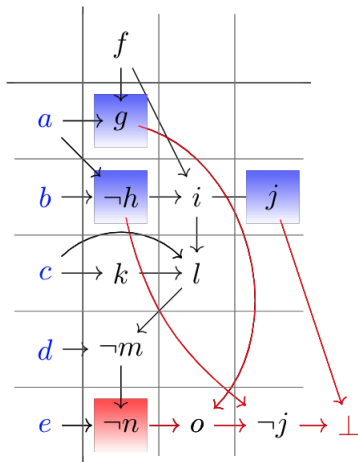
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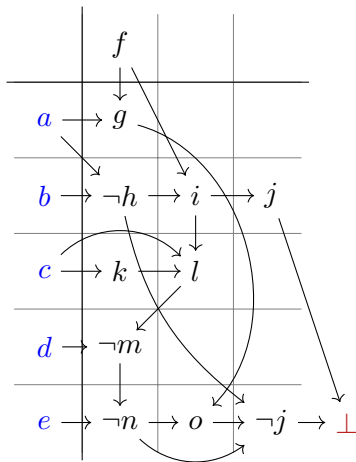
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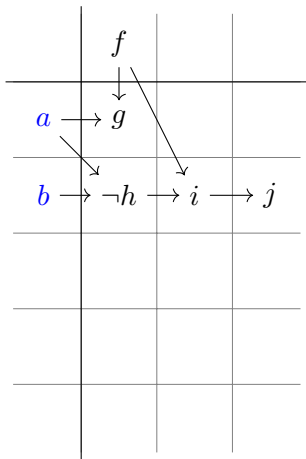
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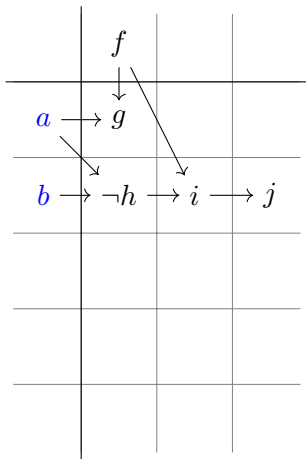
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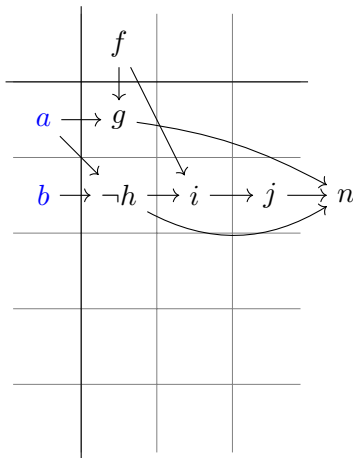


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- Randomization: breaking ties, random decision between k best choices, ...
- Restarts: Geometric/Luby

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SAT Solvers (Few examples)

- MiniSat: <http://minisat.se/>
- Glucose: <http://www.labri.fr/perso/lsimon/glucose/>
- Lingeling <http://fmv.jku.at/lingeling>
- Any Solver by Armin Biere
<http://fmv.jku.at/software/index.html>
- Any winner from past and future SAT competitions:
<https://www.satcompetition.org/>

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- Mostly solvable by backtracking algorithms (Search and Filtering)

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‘Succeed-first’ [Geelen, 1992]:

“Follow the best chances leading to a solution”

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C is Arc Consistent (AC) iff for every variable x in the scope of C , for every value $v \in D(x)$, there exists an assignment w in D satisfying C in which v is assigned to x

Filtering algorithm

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- A Filtering algorithm associated to a constraint C takes as input a list of domains (for the variables in the scope of C) and returns a list of domains that are smaller or identical to the original domains.

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CP vs. SAT

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 - CP vs. SAT: a fundamental difference is the presence of global reasoning in CP.

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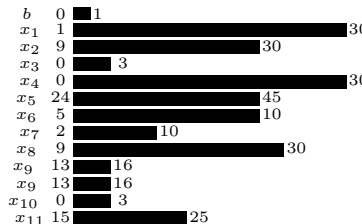
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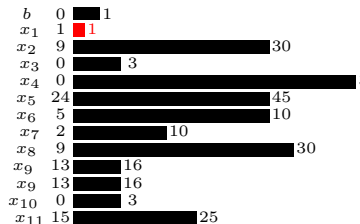
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Learning in CP

$$\begin{aligned}
 &x_1 + x_7 \geq 4 \wedge \\
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Learning in CP

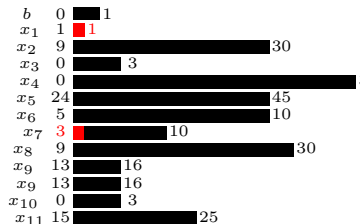
$\llbracket x_1 = 1 \rrbracket$

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Learning in CP

$$\llbracket x_1 = 1 \rrbracket \rightarrow \llbracket x_7 \geq 3 \rrbracket$$

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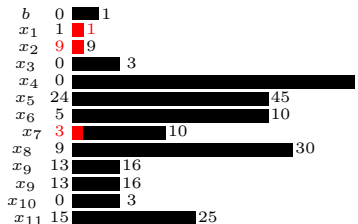


Learning in CP

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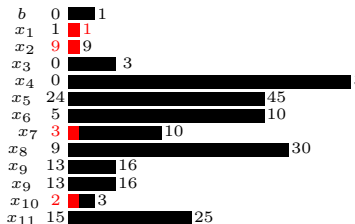


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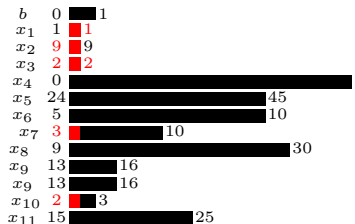
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b	0		1
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x_2	9		9
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x_6	5		10
x_7	3		 10
x_8	9		30
x_9	14		14
x_9	13		16
x_{10}	2		 3
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Learning in CP














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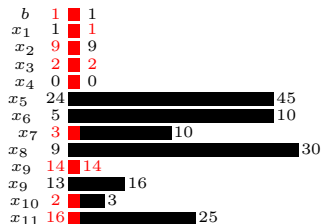
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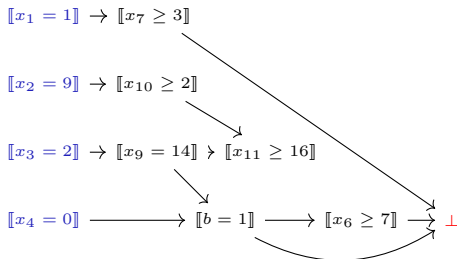
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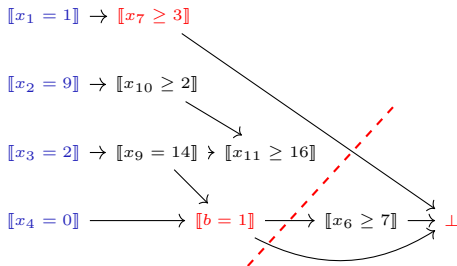
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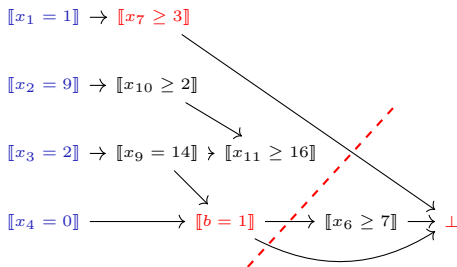


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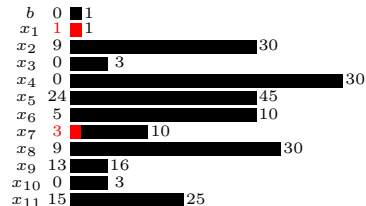


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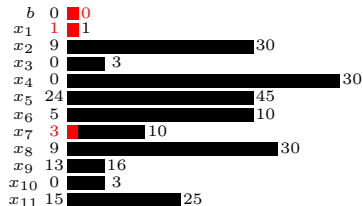


Learning in CP

$$\llbracket x_1 = 1 \rrbracket \rightarrow \llbracket x_7 \geq 3 \rrbracket \longrightarrow \llbracket b = 0 \rrbracket$$

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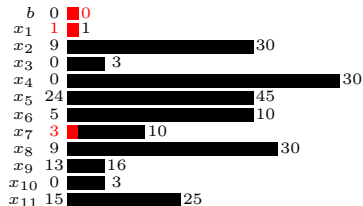


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- Propagate the learnt clause
- Continue exploration

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Conflict analysis

Algorithm 1: 1-UIP-with-Propagators

```

1  $\Psi \leftarrow \text{explain}(\perp)$  ;
2 while  $|\{q \in \Psi \mid \text{level}(q) = \text{current level}\}| > 1$  do
     $p \leftarrow \arg \max_q (\{\text{rank}(q) \mid \text{level}(q) = \text{current level} \wedge q \in \Psi\})$  ;
3    $\Psi \leftarrow \Psi \cup \{q \mid q \in \text{explain}(p) \wedge \text{level}(q) > 0\} \setminus \{p\}$  ;
   return  $\Psi$  ;

```

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Explaining constraints

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- We distinguish two types of explanations:
 - Explaining Failure
 - Explaining Domain filtering
- Example: Explain the constraint $X \leq Y$ with two scenarios (failure and propagation).

Exercise

- Let (x_1, \dots, x_n) be a sequence of Boolean variables, and let d be a positive integer.
- The $\text{CARDINALITY}(x_1, \dots, x_n, d)$ constraint holds iff exactly d variables from the sequence (x_1, \dots, x_n) are true.
- Write a filtering algorithm for CARDINALITY .
- What is the time complexity?
- Does it enforce arc consistency?
- Explain the CARDINALITY filtering.

Correction

Algorithm 4: $\text{CARDINALITY}([x_1, \dots, x_n], d)$

```

if  $|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}| > d$  then
1   $\mathcal{D} \leftarrow \perp$  ;
if  $|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}| > n - d$  then
2   $\mathcal{D} \leftarrow \perp$  ;
if  $|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}| = d$  then
    foreach  $i \in \{1..n\}$  do
        if  $\mathcal{D}(x_i) = \{0, 1\}$  then
3       $\mathcal{D}(x_i) \leftarrow \{0\}$  ;
    else
        if  $|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}| = n - d$  then
            foreach  $i \in \{1..n\}$  do
                if  $\mathcal{D}(x_i) = \{0, 1\}$  then
4       $\mathcal{D}(x_i) \leftarrow \{1\}$  ;
return  $\mathcal{D}$  ;

```

Explaining The Cardinality Constraint

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- Failure 1:

$$x^1 \wedge x^2 \wedge x^{d+1} \rightarrow \perp$$

Where $D(x^i) = \{1\}$

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$$x^1 \wedge x^2 \wedge x^{d+1} \rightarrow \perp$$

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- Failure 2:

$$\neg x^1 \wedge \neg x^2 \wedge \neg x^{n-d+1} \rightarrow \perp$$

Where $D(x^i) = \{0\}$

- Explaining the propagating the value 1: the conjunction of all the assigned variables

Explaining The Cardinality Constraint

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$$x^1 \wedge x^2 \wedge x^{d+1} \rightarrow \perp$$

Where $D(x^i) = \{1\}$

- Failure 2:

$$\neg x^1 \wedge \neg x^2 \wedge \neg x^{n-d+1} \rightarrow \perp$$

Where $D(x^i) = \{0\}$

- Explaining the propagating the value 1: the conjunction of all the assigned variables
- Explaining the propagating the value 0: the conjunction of all the assigned variables

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Take Away Message

- SAT, CP, MIP, (also, MaxSAT, SMT, QBF, ASP, Pseudo-Boolean) are efficient tools to solve hard combinatorial problems
- When you master one or few techniques, it opens the door to work on diverse problems. The more you apply to different problems, the more you learn
- The choice depends on the problem at hand (is it easy to linearise? what is the size of the SAT encoding? Can we use/invent global constraints?, etc)

Take Away Message

- SAT, CP, MIP, (also, MaxSAT, SMT, QBF, ASP, Pseudo-Boolean) are efficient tools to solve hard combinatorial problems
- When you master one or few techniques, it opens the door to work on diverse problems. The more you apply to different problems, the more you learn
- The choice depends on the problem at hand (is it easy to linearise? what is the size of the SAT encoding? Can we use/invent global constraints?, etc)
- You don't need to implement a solver: use existing ones! check the different solver competitions

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