Conflict Driven Clause Learning

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- Resolution: Learning based on the rule $(l \lor c_1) \land (\neg l \lor c_2) \Rightarrow (c_1 \lor c_2)$
- Can be seen as a CP Solver (Search, propagation) augmented by clause learning
- But also:
 - Activity-based branching
 - Lazy data structures (2-Watched Literals)
 - Clause Database Reduction
 - Simplifications
 - Restarts
 - . . .

Exercise: Propose a filtering algorithm for clauses. The algorithm takes as input a clause and has access (read and write) for the variables domains.

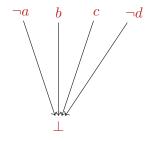
Unit Propagation

Given a clause C of arity n. If n-1 literals are false then set the last one to be true.

Example: $(a \lor \neg b \lor \neg c \lor d)$



$$\neg a \land b \land \neg d \Rightarrow \neg c$$



$$\neg a \land b \land c \land \neg d \Rightarrow \bot$$

- Unit propagation is implemented with an "intelligent" data structure called Two-watched literals
- Observe first that propagation happens only in two cases:
 - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
 - All literals are instantiated and none of them satisfy the clause

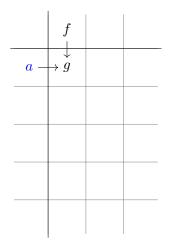
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- The idea of the Two-watched literals is to keep 2 literals for every clause that are not instantiated. Those literals will "watch the clause" and guarantee that no propagation is needed.
- If a literal watching a clause C becomes false, look for replacement. If no replacement found, then perform propagation

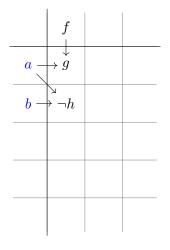
f	

$\neg a \lor \neg f \lor g$	$c \vee h \vee n \vee \neg m$
$\neg a \lor \neg b \lor \neg h$	$c \vee l$
$a \lor c$	$d \vee \neg k \vee l$
$a \vee \neg i \vee \neg l$	$d \vee \neg g \vee l$
$a \vee \neg k \vee \neg j$	$\neg g \vee n \vee o$
$b \lor d$	$h \vee \neg o \vee \neg j \vee r$
$b \vee g \vee \neg n$	$\neg i \lor j$
$b \vee \neg f \vee n \vee k$	$\neg d \vee \neg l \vee \neg m$
$\neg c \lor k$	$\neg e \vee m \vee \neg n$
$\neg c \vee \neg k \vee \neg i \vee l$	$\neg f \vee h \vee i$



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$b \lor d$	$h \vee \neg o \vee \neg j \vee n$
$b \vee g \vee \neg n$	$\neg i \vee j$
$b \vee \neg f \vee n \vee k$	$\neg d \vee \neg l \vee \neg m$
$\neg c \lor k$	$\neg e \vee m \vee \neg n$
$\neg c \vee \neg k \vee \neg i \vee l$	$\neg f \lor h \lor i$

n



$$c \lor h \lor n \lor \neg m$$

$$c \lor l$$

$$d \lor \neg k \lor l$$

$$d \lor \neg g \lor l$$

$$\neg g \lor n \lor o$$

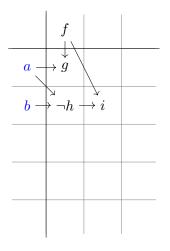
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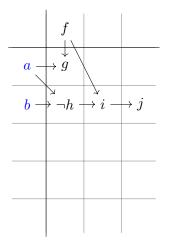
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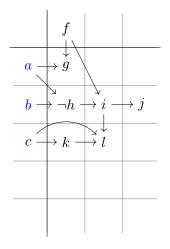
$$\neg d \lor \neg l \lor \neg m$$

$$\neg e \lor m \lor \neg n$$

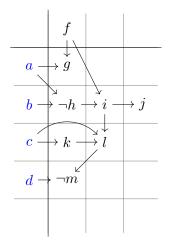
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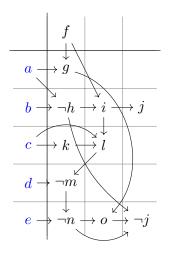
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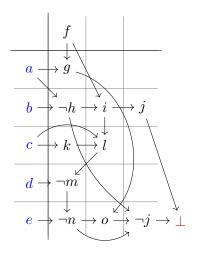


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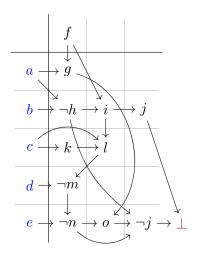
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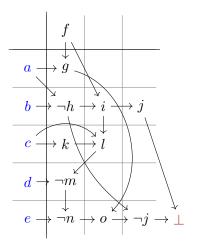
$$\neg e \lor m \lor \neg n$$

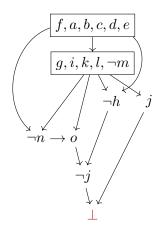
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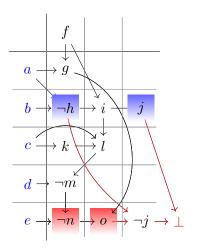


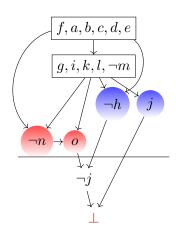
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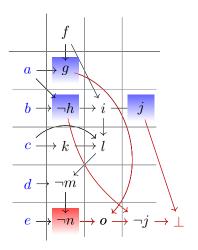
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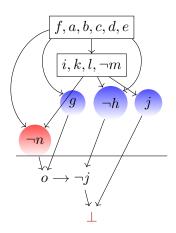


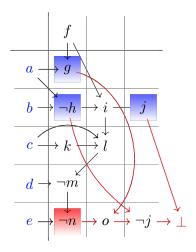


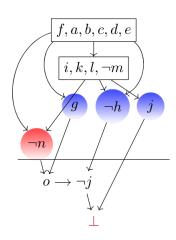


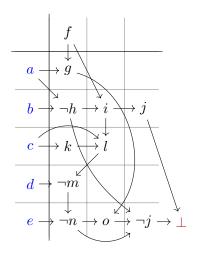






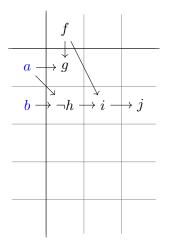






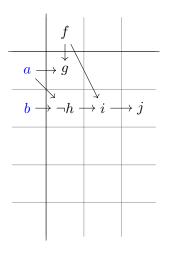
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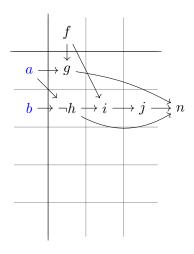
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- When there is only one literal uip propagated in the last level in the current explanation, learn the associated new clause C, backjump (to the last level of propagated literals in C), propagate $\neg uip$ via the new clause, and continue the exploration

Heavy-tail phenomena (SAT and CP)

At any time during the experiment there is a non-negligible probability of hitting a problem that requires exponentially more time to solve than any that has been encountered before.

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- Randomization: breaking ties, random decision between k best choices, . . .
- Restarts: Geometric/Luby

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SAT Solvers (Few examples)

- MiniSat: http://minisat.se/
- Glucose: http://www.labri.fr/perso/lsimon/glucose/
- LingeLing http://fmv.jku.at/lingeling
- Any Solver by Armin Biere http://fmv.jku.at/software/index.html
- Any winner from past and future SAT competitions: https://www.satcompetition.org/

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