Solutions to the 26th Singapore Physics Olympiad (2012)

Question 1: Atomic Bomb Explosion

Dimensional Analysis

We assume the radius of the shockwave R depends on the energy of the explosion E, the density of the medium ρ, and the time elapsed t.

$$R \propto E^a \rho^b t^c$$

The dimensions of the quantities are:

- R: $[L]$
- E: $[ML^2T^{-2}]$
- ρ: $[ML^{-3}]$
- t: $[T]$

Equating dimensions:

$$[L] = [ML^2T^{-2}]^a [ML^{-3}]^b [T]^c = M^{a+b} L^{2a-3b} T^{-2a+c}$$

This gives a system of linear equations for the exponents: 1. $a + b = 0 \implies b = -a$ 2. $2a - 3b = 1 \implies 2a - 3(-a) = 5a = 1 \implies a = 1/5$ 3. $-2a + c = 0 \implies c = 2a = 2/5$

Thus, $b = -1/5$. The expression for the radius is:

$$R = C\left(\frac{Et^2}{\rho}\right)^{1/5}$$

where C is a dimensionless proportionality constant. The problem states to assume $C = 1$.

Energy Estimation

From the provided image , we have the following data:

- Time $t = 25$ msec $= 0.025$ s.
- Density of air $\rho = 1.2$ kg m^{-3}.
- Scale bar represents 100 m.

Visually estimating the radius of the hemispherical shockwave from Figure 1 , the height of the dome appears to be approximately 1.4 times the 100 m scale bar. Let us estimate $R \approx 140$ m.

Rearranging the formula to solve for Energy E:

$$R^5 = \frac{Et^2}{\rho} \implies E = \frac{\rho R^5}{t^2}$$

Substituting the values:

$$E = \frac{1.2 \times (140)^5}{(0.025)^2}$$

$$E = \frac{1.2 \times 5.378 \times 10^{10}}{6.25 \times 10^{-4}} \approx 1.03 \times 10^{14} \text{ J}$$

Converting to tons of TNT equivalent (1 ton TNT $= 4.184 \times 10^9$ J):

$$E_{\text{TNT}} = \frac{1.03 \times 10^{14}}{4.184 \times 10^9} \approx 24,600 \text{ tons}$$

The estimated energy is approximately **25 kilotons**. (Historical value for Trinity was $\sim 20 - 22$ kt).

Question 2: Adiabatic Invariant of a Pendulum

The adiabatic invariant for a periodic system with slowly varying parameters is the action variable $J = \oint p\,dq$, or equivalently the ratio of the energy to the frequency E/ν (or E/ω) for a harmonic oscillator.

For a simple pendulum of length L and mass M undergoing small oscillations:

- Angular frequency $\omega = \sqrt{\frac{g}{L}}$.
- Energy $E = \frac{1}{2}M\omega^2 A^2$, where A is the linear amplitude of oscillation.

The adiabatic invariant condition states:

$$\frac{E}{\omega} = \text{constant}$$

Substituting E:

$$\frac{\frac{1}{2}M\omega^2 A^2}{\omega} = \frac{1}{2}M\omega A^2 = \text{constant}$$

Since M is constant, we have $\omega A^2 = \text{constant}$. Substituting $\omega = \sqrt{g/L}$:

$$\sqrt{\frac{g}{L}}A^2 = \text{constant} \implies \frac{A^2}{\sqrt{L}} = \text{constant} \implies A \propto L^{1/4}$$

If the string is shortened by a factor of 2 ($L_f = L_i/2$):

$$\frac{A_f}{A_i} = \left(\frac{L_f}{L_i}\right)^{1/4} = \left(\frac{1}{2}\right)^{1/4} \approx 0.84$$

The amplitude decreases by a factor of $2^{-1/4}$ or approximately **0.84**.

Question 3: Exploding Rocket

Let the rocket explode at height h. Just before explosion, velocity is zero. It breaks into three fragments of mass m. Conservation of momentum just after explosion:

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$

Fragment 1 falls straight down, so $\vec{v}_1 = -v_1\hat{j}$. Fragments 2 and 3 land at the same time t_2, which implies by symmetry they have the same initial vertical velocity component u_y. Vertical momentum conservation:

$$m(-v_1) + m(u_y) + m(u_y) = 0 \implies v_1 = 2u_y \implies u_y = \frac{v_1}{2}$$

Using kinematics ($y = y_0 + v_{y0}t - \frac{1}{2}gt^2$) with final height $y = 0$ and $y_0 = h$: For Fragment 1 (downward):

$$0 = h - v_1 t_1 - \frac{1}{2}gt_1^2 \implies h = v_1 t_1 + \frac{1}{2}gt_1^2 \quad \text{— (1)}$$

For Fragments 2 & 3 (upward):

$$0 = h + u_y t_2 - \frac{1}{2}gt_2^2 \implies h = -u_y t_2 + \frac{1}{2}gt_2^2 = -\frac{v_1}{2}t_2 + \frac{1}{2}gt_2^2 \quad \text{— (2)}$$

From (1), $v_1 = \frac{h}{t_1} - \frac{gt_1}{2}$. Substitute into (2):

$$h = -\frac{t_2}{2}\left(\frac{h}{t_1} - \frac{gt_1}{2}\right) + \frac{1}{2}gt_2^2$$

$$h = -\frac{ht_2}{2t_1} + \frac{gt_1t_2}{4} + \frac{gt_2^2}{2}$$

Multiply by $4t_1$:

$$4t_1h = -2ht_2 + gt_1^2t_2 + 2gt_1t_2^2$$
$$h(4t_1 + 2t_2) = gt_2(t_1^2 + 2t_1t_2)$$
$$h(2(2t_1 + t_2)) = gt_1t_2(t_1 + 2t_2)$$
$$h = \frac{gt_1t_2(t_1 + 2t_2)}{2(2t_1 + t_2)}$$

Question 4: Thomson Model of Hydrogen

(i) Restoring Force The positive charge $+e$ is uniformly distributed in a sphere of radius R. Charge density $\rho = \frac{e}{(4/3)\pi R^3}$. By Gauss's Law, the electric field at a distance $r < R$ from the center is determined by the enclosed charge $q_{enc} = \rho \frac{4}{3}\pi r^3 = e\frac{r^3}{R^3}$.

$$E(r)4\pi r^2 = \frac{q_{enc}}{\epsilon_0} \implies E(r) = \frac{er}{4\pi\epsilon_0 R^3}$$

The force on the electron (charge $-e$) at r is:

$$F = -eE(r) = -\frac{e^2}{4\pi\epsilon_0 R^3}r$$

This is of the form $F = -Kr$ with constant $K = \frac{e^2}{4\pi\epsilon_0 R^3}$. At $r = 0$, $F = 0$, so the electron is in equilibrium.

(ii) Frequency of Oscillation The equation of motion is $m_e\ddot{r} = -Kr$, which describes simple harmonic motion with angular frequency $\omega = \sqrt{K/m_e}$. Frequency $f = \frac{\omega}{2\pi}$:

$$f = \frac{1}{2\pi}\sqrt{\frac{e^2}{4\pi\epsilon_0 R^3 m_e}}$$

(iii) Calculation of R Given $f = 2.47 \times 10^{15}$ Hz. Solve for R:

$$f^2 = \frac{1}{4\pi^2}\frac{e^2}{4\pi\epsilon_0 m_e R^3} = \frac{e^2}{16\pi^3\epsilon_0 m_e R^3}$$

$$R^3 = \frac{e^2}{16\pi^3\epsilon_0 m_e f^2}$$

Using constants: $e \approx 1.6 \times 10^{-19}$ C, $\epsilon_0 \approx 8.85 \times 10^{-12}$ F/m, $m_e \approx 9.11 \times 10^{-31}$ kg.

$$R^3 = \frac{(1.6 \times 10^{-19})^2}{16\pi^3(8.85 \times 10^{-12})(9.11 \times 10^{-31})(2.47 \times 10^{15})^2}$$

$$R \approx 1.0 \times 10^{-10} \text{ m} = 1 \text{ Å}$$

Question 5: Antimatter Rocket

Let initial mass be M_0. Final ship mass $m = fM_0$. Ship speed v. Units $c = 1$. **(i) Energy Conservation** Initial energy: $E_i = M_0$. Final energy: Ship energy $E_{ship} = \gamma m$ + Radiation energy E_{rad}.

$$M_0 = \gamma m + E_{rad}$$

(ii) Momentum Conservation Initial momentum: $P_i = 0$. Final momentum: $P_{ship} = \gamma mv$. Radiation momentum P_{rad}. Assuming radiation is directed strictly backward (for max thrust): $P_{rad} = -E_{rad}$ (magnitude E, direction opposite).

$$0 = \gamma mv - E_{rad} \implies E_{rad} = \gamma mv$$

(iii) Relation for f Substitute E_{rad} into energy equation:

$$M_0 = \gamma m + \gamma mv = \gamma m(1 + v)$$

Dividing by M_0 and using $f = m/M_0$:

$$1 = \gamma f(1 + v) = \gamma f + \gamma vf$$

(iv) Quadratic Equation for f We need to express v in terms of γ. Since $\gamma = 1/\sqrt{1 - v^2}$, we have $\gamma^2 - 1 = \gamma^2 v^2$, so $\gamma v = \sqrt{\gamma^2 - 1}$. Substitute into the result from (iii):

$$\gamma f + f\sqrt{\gamma^2 - 1} = 1$$

$$f(\gamma + \sqrt{\gamma^2 - 1}) = 1$$

Invert to find f:

$$f = \frac{1}{\gamma + \sqrt{\gamma^2 - 1}} = \gamma - \sqrt{\gamma^2 - 1}$$

Rearranging $f = \gamma - \sqrt{\gamma^2 - 1}$:

$$\gamma - f = \sqrt{\gamma^2 - 1}$$

Square both sides:

$$\gamma^2 - 2\gamma f + f^2 = \gamma^2 - 1$$

$$f^2 - 2\gamma f + 1 = 0$$

(v) Value for $\gamma = 10$

$$f = 10 - \sqrt{100 - 1} = 10 - \sqrt{99} \approx 10 - 9.95 = 0.05$$

Yes, it is possible to construct such a ship if the payload and shell are 5% of the initial mass (95% fuel).

(vi) Energy of Radiation vs Fuel Mass Mass of fuel consumed $M_{fuel} = M_0 - m = M_0(1 - f)$. Energy of radiation $E_{rad} = \gamma mv = \gamma m\frac{\sqrt{\gamma^2-1}}{\gamma} = m\sqrt{\gamma^2 - 1}$. From conservation of energy, $E_{rad} = M_0 - \gamma m$. We compare $M_0 - m$ (fuel mass) and $M_0 - \gamma m$ (radiation energy). Since $\gamma > 1$ (for $v > 0$), $\gamma m > m$, so $M_0 - \gamma m < M_0 - m$. The energy of the emitted radiation is **less** than the mass of the fuel consumed because a portion of the fuel's mass-energy is converted into the kinetic energy of the remaining spaceship ($\gamma m - m$).

Question 6: Ring in Magnetic Field

Induced Force The magnetic field changes as $B(t) = B_0 + \alpha t$. Flux through the large ring of radius R is $\Phi = \pi R^2(B_0 + \alpha t)$. Induced EMF via Faraday's Law: $\mathcal{E} = -\frac{d\Phi}{dt} = -\pi R^2 \alpha$. The induced electric field E along the ring is tangential:

$$E(2\pi R) = \pi R^2 \alpha \implies E = \frac{R\alpha}{2}$$

The force on the small ring of charge q is $F_t = qE = \frac{qR\alpha}{2}$. This force acts tangentially, accelerating the small ring along the large ring.

Motion Newton's Second Law for tangential motion:

$$ma_t = F_t \implies mR\frac{d\omega}{dt} = \frac{qR\alpha}{2}$$

Angular acceleration $\frac{d\omega}{dt} = \frac{q\alpha}{2m}$ is constant. Angular velocity $\omega(t) = \frac{q\alpha}{2m}t$.

Force on the Big Ring The small ring exerts a force on the big ring (Newton's 3rd Law). The forces are: 1. Tangential reaction force: $F_{tan} = \frac{qR\alpha}{2}$. 2. Radial force: The big ring provides the normal force to keep the small ring in circular motion against magnetic and centripetal effects. Radial force on small ring (inward positive): $F_r = qvB - mR\omega^2$ (direction depends on signs, assuming Lorentz force opposes centripetal or adds). Actually, the net radial force required for motion is $-mR\omega^2$ (inward). Forces applied: Normal force N (inward) and Lorentz force (outward $q\omega RB$). $N - q\omega RB = -mR\omega^2 \implies N = q\omega RB - mR\omega^2$. The force of the small ring on the big ring is $-N$ (outward).

Question 7: Three Polarizers

Using Malus's Law: $I_{out} = I_{in}\cos^2(\Delta\theta)$. Incident light $I_0 = 10.0$ polarized vertically ($0°$).

(i) $\theta_1 = 20°, \theta_2 = 40°, \theta_3 = 60°$. Angle differences are all $20°$.

$$I_1 = I_0 \cos^2(20°)$$

$$I_2 = I_1 \cos^2(40° - 20°) = I_0(\cos^2 20°)^2$$

$$I_3 = I_2 \cos^2(60° - 40°) = I_0(\cos^2 20°)^3$$

Calculation: $\cos 20° \approx 0.9397$. $\cos^2 20° \approx 0.883$. $I_3 = 10.0 \times (0.883)^3 \approx 10.0 \times 0.688 = 6.88$.

(ii) $\theta_1 = 0°, \theta_2 = 30°, \theta_3 = 60°$.

$$I_1 = I_0 \cos^2(0) = I_0$$

$$I_2 = I_1 \cos^2(30°) = I_0(3/4)$$

$$I_3 = I_2 \cos^2(30°) = I_0(3/4)^2 = I_0(9/16)$$

$I_3 = 10.0 \times 0.5625 = 5.625$.

(iii) $\theta_1 = 0°, \theta_3 = 60°$. Maximize I_3 by varying θ_2. $I_3 = I_0 \cos^2(\theta_2)\cos^2(60° - \theta_2)$. Maximum occurs when θ_2 is exactly in the middle of θ_1 and θ_3. $\theta_2 = 30°$.

Question 8: Railgun Circuit

Setup $B = 0.0100$ T. Rail width $L = 0.100$ m. Left rod (10Ω) moves left at $v_1 = 4.00$ m/s. Right rod (15Ω) moves right at $v_2 = 2.00$ m/s. Middle resistor $R_c = 5.00\Omega$.

Induced EMFs Both rods move outward, increasing the flux area. $\mathcal{E}_1 = BLv_1 = 0.01 \times 0.1 \times 4 = 0.004$ V. $\mathcal{E}_2 = BLv_2 = 0.01 \times 0.1 \times 2 = 0.002$ V.

Circuit Analysis We model the rods as voltage sources in series with their resistance. Let the bottom rail be ground (0 V) and the top rail be at potential V. Determining polarity: For expanding area, Lenz's law implies induced current opposes the external field. If B is up, induced B is down, current is Clockwise (viewed from top). Left Loop (Left Rod + Middle Resistor): Clockwise current means Up through Left Rod, Down through Middle Resistor. Right Loop (Right Rod + Middle Resistor): Clockwise current (in the loop defined by the circuit) means Down through Right Rod, Up through Middle Resistor. Alternatively, using Motional EMF vector force $\vec{F} = q\vec{v} \times \vec{B}$. Assume B is out of page ($+z$). Left Rod (v left): Force on $+q$ is Down. Top is negative, Bottom is positive. Right Rod (v right): Force on $+q$ is Down. Top is negative, Bottom is positive. Wait, let's re-check directions. If B is $+z$, $v_1 = -v\hat{x}$. $F = q(-v\hat{x} \times B\hat{z}) = qvB\hat{y}$ (Up). Top is +. If B is $+z$, $v_2 = +v\hat{x}$. $F = q(v\hat{x} \times B\hat{z}) = qvB(-\hat{y})$ (Down). Top is -.

So Left Rod acts as battery 4 mV, + terminal at Top. Right Rod acts as battery 2 mV, + terminal at Bottom (so -2 mV at Top). Nodal Analysis at Top Rail (Voltage V): Currents leaving node V: 1. To Left Rod: $(V - 0.004)/10$ 2. To Middle Resistor: $V/5$ 3. To Right Rod: $(V - (-0.002))/15$ Sum = 0:

$$\frac{V - 0.004}{10} + \frac{V}{5} + \frac{V + 0.002}{15} = 0$$

Multiply by 30:

$$3(V - 0.004) + 6V + 2(V + 0.002) = 0$$

$$3V - 0.012 + 6V + 2V + 0.004 = 0$$

$$11V = 0.008$$

$$V = \frac{0.008}{11} \approx 0.000727 \text{ V}$$

Current in 5.00 Ω Resistor

$$I = \frac{V}{5} = \frac{0.000727}{5} \approx 145 \times 10^{-6} \text{ A} = 145\,\mu\text{A}$$

Direction: From Top to Bottom (since $V > 0$).

Question 9: RL Circuit

(i) Initial Conditions At $t < 0$, switch S is closed. The battery (18 V) is connected. Assuming standard configuration (battery in center branch): Left branch $R_2 = 6$ kΩ. Right branch $R_1 = 2$ kΩ + L. Inductor behaves as short circuit. $I_L(0^-) = \frac{18}{2000} = 9$ mA. Flow is Down. $I_{R2}(0^-) = \frac{18}{6000} = 3$ mA. Flow is Down.

At $t = 0$, S opens. Battery is removed. The circuit becomes a single loop with R_1, L, and R_2 in series. Current in inductor cannot change instantly: $I(0^+) = 9$ mA. This current circulates: Inductor $\to$ Bottom Wire $\to$ R_2 (Up) $\to$ Top Wire $\to$ R_1 $\to$ Inductor. Voltage across L: $V_L + I(R_1 + R_2) = 0$. $V_L = -I(R_1 + R_2) = -(9 \text{ mA})(8 \text{ k}\Omega) = -72$ V. Potential difference $V_a - V_b$. Current flows $a \to b$. The inductor acts as the source driving the current. Inside the source, current flows from $-$ to $+$. So b is at a higher potential than a. Value: **72 V**. b is higher.

(ii) Graphs I_{R1}: Starts at +9 mA, decays exponentially to 0. I_{R2}: Jump from +3 mA (down) to -9 mA (up, loop current). Decays to 0.

(iii) Time Calculation Current decay $I(t) = I_0 e^{-t/\tau}$. $\tau = \frac{L}{R_{eq}} = \frac{0.100}{8000} = 12.5\,\mu$s. Find t when $I_{R2} = 2.00$ mA.

$$2 = 9e^{-t/12.5\mu\text{s}}$$

$$e^{t/\tau} = 4.5 \implies t = \tau \ln(4.5)$$

$$t = 12.5 \times 1.504 \approx 18.8\,\mu\text{s}$$

Question 10: Transparent Cylinder with Mirror

Geometry Radius $R = 2.00$ m. Right half is mirrored. Incident ray is parallel to the exiting ray, separated by distance $d = 2.00$ m. Let the incident ray height be $y = d/2 = 1.00$ m. Exit ray height is $y = -1.00$ m. Path symmetry implies the ray enters at A ($y = 1$), reflects at a point B on the axis ($y = 0$) at the back of the cylinder, and exits at D ($y = -1$).

Angles 1. **Entry at A ($y = 1$):** $\sin\theta_A = y/R = 1/2 \implies \theta_A = 30°$ (angle of radius with horizontal axis). Normal is at 150° from the positive x-axis. Incident ray is horizontal (180°). Angle of incidence $i = 30°$. Let refraction angle be r.

2. **Reflection at B:** For the ray to exit symmetrically at $y = -1$, it must reflect at the vertex of the mirror $B(2, 0)$ on the optical axis. Consider the triangle formed by the Center O, Entry point A, and Mirror point B. $OA = R, OB = R$. Angle $\angle AOB = 150°$. Triangle AOB is isosceles. The base angles are $\angle OAB = \angle OBA = \frac{180-150}{2} = 15°$. The angle $\angle OAB$ is the angle between the radius (normal) and the ray path AB. Thus, the angle of refraction is $r = 15°$.

3. **Calculation of Refractive Index:** Using Snell's Law at point A:

$$\sin i = n \sin r$$

$$\sin 30° = n \sin 15°$$

$$0.5 = n \sin 15°$$

Using half-angle formula: $\sin 15° = \sin(45 - 30) = \frac{\sqrt{6}-\sqrt{2}}{4} \approx 0.2588$.

$$n = \frac{0.5}{0.2588} \approx 1.93$$

Answer The index of refraction is **1.93**.