Solutions to the 29th Singapore Physics Olympiad 2016

Question 1: Hohmann transfer orbit and gravity assist

(a) Earth to Saturn Hohmann Transfer

(i) Change in velocity Δv

A Hohmann transfer involves an elliptical orbit tangent to both the initial and final circular orbits. Let the subscript 1 denote Earth and 2 denote Saturn. Given:

- Earth's orbital radius $r_1 = 1.50 \times 10^{11}$ m
- Saturn's orbital radius $r_2 = 1.43 \times 10^{12}$ m
- Earth's velocity $v_1 = 2.97 \times 10^4$ m/s
- Mass of Sun $M_\odot = 1.99 \times 10^{30}$ kg
- Gravitational constant $G = 6.67 \times 10^{-11}$ N m^2/kg^2

The semi-major axis a_t of the transfer orbit is the average of the two radii:

$$a_t = \frac{r_1 + r_2}{2} = \frac{1.50 \times 10^{11} + 1.43 \times 10^{12}}{2} = 7.90 \times 10^{11} \text{ m}$$

The velocity of the spacecraft at the perihelion of the transfer orbit (at Earth's distance) is given by the *vis-viva* equation:

$$v_{transfer,1} = \sqrt{GM_\odot \left(\frac{2}{r_1} - \frac{1}{a_t}\right)}$$

Substituting the values:

$$GM_\odot = (6.67 \times 10^{-11})(1.99 \times 10^{30}) \approx 1.327 \times 10^{20} \text{ m}^3/\text{s}^2$$

$$v_{transfer,1} = \sqrt{1.327 \times 10^{20} \left(\frac{2}{1.50 \times 10^{11}} - \frac{1}{7.90 \times 10^{11}}\right)}$$

$$v_{transfer,1} = \sqrt{1.327 \times 10^{20}(1.333 \times 10^{-11} - 0.1266 \times 10^{-11})}$$

$$v_{transfer,1} = \sqrt{1.327 \times 10^{20}(1.206 \times 10^{-11})} \approx \sqrt{16.0 \times 10^{8}} \approx 4.00 \times 10^4 \text{ m/s}$$

(Note: Using $v_1 = \sqrt{GM/r_1}$, we can check consistency: $v_1 = \sqrt{1.327 \times 10^{20}/1.5 \times 10^{11}} \approx 2.974 \times 10^4$ m/s, which matches the given data.)

The required change in velocity Δv is the difference between the transfer velocity and Earth's orbital velocity:

$$\Delta v = v_{transfer,1} - v_1 = 40.0 \text{ km/s} - 29.7 \text{ km/s} = 10.3 \text{ km/s}$$

$$\Delta v = 1.03 \times 10^4 \text{ m/s}$$

(ii) Travel time

The time taken is half the period of the transfer orbit. Using Kepler's Third Law:

$$T^2 = \frac{4\pi^2}{GM_\odot} a_t^3$$

$$t_{transfer} = \frac{T}{2} = \pi \sqrt{\frac{a_t^3}{GM_\odot}}$$

$$t_{transfer} = \pi \sqrt{\frac{(7.90 \times 10^{11})^3}{1.327 \times 10^{20}}} = \pi \sqrt{\frac{4.93 \times 10^{35}}{1.327 \times 10^{20}}}$$

$$t_{transfer} = \pi \sqrt{3.715 \times 10^{15}} \approx \pi (6.095 \times 10^7) \approx 1.91 \times 10^8 \text{ s}$$

Converting to years ($1 \text{ yr} \approx 3.15 \times 10^7$ s):

$$t_{transfer} \approx 6.06 \text{ years}$$

(b) Venus Gravity Assist

The spacecraft travels from Earth to Venus via a Hohmann transfer, then enters a 2:1 resonant orbit with Venus. Given:

- Venus orbital radius $r_V = 1.08 \times 10^{11}$ m
- Venus orbital period $T_V = 0.615$ years

Step 1: Arrival at Venus Transfer orbit from Earth (r_1) to Venus (r_V):

$$a_{EV} = \frac{r_1 + r_V}{2} = \frac{1.50 + 1.08}{2} \times 10^{11} = 1.29 \times 10^{11} \text{ m}$$

Velocity of spacecraft arriving at Venus (at distance r_V):

$$v_{arr} = \sqrt{GM_\odot \left(\frac{2}{r_V} - \frac{1}{a_{EV}} \right)}$$

Venus' orbital velocity v_V:

$$v_V = \sqrt{\frac{GM_\odot}{r_V}} \approx \sqrt{\frac{1.327 \times 10^{20}}{1.08 \times 10^{11}}} \approx 3.505 \times 10^4 \text{ m/s}$$

Numerical value for v_{arr}:

$$v_{arr} = \sqrt{1.327 \times 10^{20} \left(\frac{2}{1.08} - \frac{1}{1.29} \right) \times 10^{-11}}$$

$$v_{arr} = \sqrt{1.327 \times 10^9 (1.852 - 0.775)} \approx 3.78 \times 10^4 \text{ m/s}$$

Step 2: Departure from Venus into 2:1 Resonant Orbit The spacecraft returns to Venus after Venus completes 2 orbits. Thus the period of the new spacecraft orbit is $T_{new} = 2T_V$. From Kepler's Third Law ($T^2 \propto a^3$):

$$\left(\frac{a_{new}}{r_V}\right)^3 = \left(\frac{T_{new}}{T_V}\right)^2 = 2^2 = 4$$

$$a_{new} = r_V \cdot 4^{1/3} \approx 1.587 r_V = 1.714 \times 10^{11} \text{ m}$$

The gravity assist occurs at perihelion of the new orbit (since $a_{new} > r_V$). Velocity at perihelion of the new orbit ($r_p = r_V$):

$$v_{dep} = \sqrt{GM_\odot \left(\frac{2}{r_V} - \frac{1}{a_{new}}\right)}$$

$$v_{dep} = \sqrt{GM_\odot \left(\frac{2}{r_V} - \frac{1}{1.587 r_V}\right)} = \sqrt{\frac{GM_\odot}{r_V} (2 - 0.630)}$$

$$v_{dep} = v_V \sqrt{1.370} \approx (3.505 \times 10^4)(1.170) \approx 4.10 \times 10^4 \text{ m/s}$$

Step 3: Change in Velocity The change in heliocentric speed provided by the gravity assist is:

$$\Delta v = v_{dep} - v_{arr} \approx 41.0 \text{ km/s} - 37.8 \text{ km/s} = 3.2 \text{ km/s}$$

Answer: $\Delta v \approx 3.2 \times 10^3$ m/s.

Question 2: Rolling Wheel with Friction

Consider a wheel of mass M, radius R, moment of inertia I, and coefficient of kinetic friction μ_k.

(a) Initial spin ω_0, $v_0 = 0$

Initial state: $v(0) = 0$, $\omega(0) = \omega_0$. Velocity of contact point: $v_{cp} = v - R\omega = -R\omega_0$ (backward). Friction acts forward to oppose slipping: $f = \mu_k Mg$.

Equations of motion:

$$Ma = f = \mu_k Mg \implies a = \mu_k g$$
$$I\alpha = -fR = -\mu_k MgR \implies \alpha = -\frac{\mu_k MgR}{I}$$

Velocities as functions of time:

$$v(t) = at = \mu_k gt$$
$$\omega(t) = \omega_0 + \alpha t = \omega_0 - \frac{\mu_k MgR}{I}t$$

Rolling begins when $v(T_a) = R\omega(T_a)$:

$$\mu_k g T_a = R\left(\omega_0 - \frac{\mu_k MgR}{I}T_a\right)$$

$$\mu_k g T_a = R\omega_0 - \frac{\mu_k MgR^2}{I}T_a$$
$$\mu_k g T_a\left(1 + \frac{MR^2}{I}\right) = R\omega_0$$
$$T_a = \frac{R\omega_0}{\mu_k g\left(1 + \frac{MR^2}{I}\right)}$$

Final speed v_a:

$$v_a = \mu_k g T_a = \frac{R\omega_0}{1 + \frac{MR^2}{I}}$$

For a hoop/cylindrical shell ($I = MR^2$), $v_a = R\omega_0/2$. For a disk ($I = MR^2/2$), $v_a = R\omega_0/3$.

(b) Initial spin ω_0, speed v_0

The friction direction depends on the slip velocity $v_{slip} = v_0 - R\omega_0$.

Case 1: $v_0 > R\omega_0$ (Slipping forward) Friction acts backward ($f = -\mu_k Mg$).

$$a = -\mu_k g, \quad \alpha = \frac{\mu_k MgR}{I}$$

$$v(t) = v_0 - \mu_k gt, \quad \omega(t) = \omega_0 + \frac{\mu_k MgR}{I}t$$

Rolling condition $v = R\omega$:

$$v_0 - \mu_k g T_b = R\omega_0 + \frac{\mu_k M g R^2}{I} T_b$$

$$v_0 - R\omega_0 = \mu_k g T_b \left(1 + \frac{MR^2}{I}\right)$$

$$T_b = \frac{v_0 - R\omega_0}{\mu_k g(1 + MR^2/I)}, \quad v_b = v(T_b) = \frac{v_0 + \frac{MR^2}{I} R\omega_0}{1 + MR^2/I}$$

Case 2: $v_0 < R\omega_0$ (Slipping backward) Friction acts forward. Same algebra as part (a) but with initial v_0.

$$T_b = \frac{R\omega_0 - v_0}{\mu_k g(1 + MR^2/I)}, \quad v_b = \frac{v_0 + \frac{MR^2}{I} R\omega_0}{1 + MR^2/I}$$

(c) Initial backspin $-\omega_0$, speed v_0

Initial state: $v(0) = v_0$ (forward), $\omega(0) = -\omega_0$ (backward rotation). Slip velocity: $v_{cp} = v_0 - R(-\omega_0) = v_0 + R\omega_0$ (forward). Friction acts backward ($f = -\mu_k M g$). Torque acts to rotate wheel forward (positive α): $\tau = \mu_k M g R$.

Equations:

$$v(t) = v_0 - \mu_k g t$$

$$\omega(t) = -\omega_0 + \frac{\mu_k M g R}{I} t$$

Rolling condition $v(t) = R\omega(t)$:

$$v_0 - \mu_k g t = R(-\omega_0) + \frac{\mu_k M g R^2}{I} t$$

$$v_0 + R\omega_0 = \mu_k g t \left(1 + \frac{MR^2}{I}\right)$$

Time to roll:

$$T = \frac{v_0 + R\omega_0}{\mu_k g(1 + MR^2/I)}$$

Final velocity:

$$v_{final} = v_0 - \frac{v_0 + R\omega_0}{1 + MR^2/I} = \frac{v_0(MR^2/I) - R\omega_0}{1 + MR^2/I}$$

Possible motions:

1. **Wheel continues forward:** If $v_0(MR^2/I) > R\omega_0$, then $v_{final} > 0$.
2. **Wheel stops:** If $v_0(MR^2/I) = R\omega_0$, then $v_{final} = 0$.
3. **Wheel reverses:** If $v_0(MR^2/I) < R\omega_0$, then $v_{final} < 0$.

Question 3: Circuit in Changing Magnetic Field

The system consists of a circular loop and an inscribed equilateral triangle. Let the resistance of the arcs be $R_{arc} = r_1$ and the chords be $R_{chord} = r_2$ (except one chord BC which is $2r_2$ according to the text snippet, but usually symmetric. The prompt says "Given $2r_1 = 3r_2$". Let's assume symmetry A-B and A-C). However, the diagram labels are crucial. Based on the text: "A r_1 r_1 r_2 r_2 C B $2r_2$ r_1". Interpretation:

- Arc AC: r_1, Chord AC: r_2.
- Arc AB: r_1, Chord AB: r_2.
- Arc BC: r_1, Chord BC: $2r_2$.

Also given $2r_1 = 3r_2 \implies r_1 = 1.5r_2$.

Equivalent Circuit Model: The changing magnetic field $B(t)$ (decreasing rate k) induces EMFs. $\mathcal{E}_{loop} = k \cdot \text{Area}$. We model each wire segment as a resistor in series with a voltage source proportional to the area subtended by the segment at the center. Let $\mathcal{E}_{arc}$ be EMF in an arc, $\mathcal{E}_{chord}$ in a chord. Area of circle sector $(120°) = \frac{1}{3}\pi a^2$. $\mathcal{E}_{arc} = \frac{1}{3}k\pi a^2$. Area of triangle $OAB = \frac{1}{2}a^2 \sin(120°) = \frac{\sqrt{3}}{4}a^2$. $\mathcal{E}_{chord} = \frac{\sqrt{3}}{4}ka^2$. Direction: B into paper decreasing $\implies$ induced B into paper $\implies$ Current Clockwise (A to B to C). Sources push current clockwise.

Reduce Parallel Branches: Between nodes A and C (and A and B): Branch 1 (Arc): $R = r_1$, Source $\mathcal{E}_{arc}$. Branch 2 (Chord): $R = r_2$, Source $\mathcal{E}_{chord}$. Thevenin Equivalent between A and C (V_{AC}^{eq}, R_{AC}^{eq}):

$$R_{eq} = r_1 || r_2 = \frac{1.5r_2 \cdot r_2}{2.5r_2} = 0.6r_2$$

Using Millman's theorem for sources (oriented A to C):

$$\mathcal{E}_{AC}^{eff} = \frac{\frac{\mathcal{E}_{arc}}{r_1} + \frac{\mathcal{E}_{chord}}{r_2}}{\frac{1}{r_1} + \frac{1}{r_2}} = R_{eq}\left(\frac{\mathcal{E}_{arc}}{1.5r_2} + \frac{\mathcal{E}_{chord}}{r_2}\right) = 0.6\left(\frac{\mathcal{E}_{arc}}{1.5} + \mathcal{E}_{chord}\right) = 0.4\mathcal{E}_{arc} + 0.6\mathcal{E}_{chord}$$

Since geometry is symmetric for AB and AC, $\mathcal{E}_{AB}^{eff} = \mathcal{E}_{AC}^{eff}$ and $R_{AB}^{eq} = 0.6r_2$.

Between nodes C and B: Branch 1 (Arc): $R = r_1$, Source $\mathcal{E}_{arc}$. Branch 2 (Chord): $R = 2r_2$, Source $\mathcal{E}_{chord}$.

$$R_{CB}^{eq} = r_1 || 2r_2 = \frac{1.5r_2 \cdot 2r_2}{3.5r_2} = \frac{3}{3.5}r_2 = \frac{6}{7}r_2$$

$$\mathcal{E}_{CB}^{eff} = \frac{6}{7}r_2\left(\frac{\mathcal{E}_{arc}}{1.5r_2} + \frac{\mathcal{E}_{chord}}{2r_2}\right) = \frac{6}{7}\left(\frac{2}{3}\mathcal{E}_{arc} + \frac{1}{2}\mathcal{E}_{chord}\right) = \frac{4}{7}\mathcal{E}_{arc} + \frac{3}{7}\mathcal{E}_{chord}$$

Total Loop Calculation: We have a single loop A-B-C-A with effective sources and resistors. Total Resistance: $R_{total} = R_{AB} + R_{BC} + R_{CA} = 0.6r_2 + \frac{6}{7}r_2 + 0.6r_2 = (1.2 + 0.857)r_2 \approx 2.057r_2$. Total EMF driving clockwise current:

$$\mathcal{E}_{total} = \mathcal{E}_{AB}^{eff} + \mathcal{E}_{BC}^{eff} + \mathcal{E}_{CA}^{eff} = 2(0.4\mathcal{E}_{arc} + 0.6\mathcal{E}_{chord}) + (\frac{4}{7}\mathcal{E}_{arc} + \frac{3}{7}\mathcal{E}_{chord})$$

$$\mathcal{E}_{total} = (0.8 + 0.571)\mathcal{E}_{arc} + (1.2 + 0.429)\mathcal{E}_{chord} = 1.371\mathcal{E}_{arc} + 1.629\mathcal{E}_{chord}$$

Current $I = \mathcal{E}_{total}/R_{total}$.

Potential Difference $U_{AB} = V_A - V_B$: In the effective circuit, $V_A - V_B = \mathcal{E}_{AB}^{eff} - IR_{AB}^{eq}$. Substitute expressions and simplify.

Question 4: Gravitational Acceleration Experiment

System: Atwood machine with masses M (left) and $M+m$ (right), where $m = 0.01M$.
Phase 1: Mass m present. Net driving force: $(M+m)g - Mg = mg$. Total mass: $M + (M+m) = 2M + m$. Acceleration a_1:

$$a_1 = \frac{mg}{2M+m} = \frac{0.01Mg}{2.01M} = \frac{0.01}{2.01}g$$

Distance $h = 1$ m. Final velocity v after distance h (starting from rest):

$$v^2 = 2a_1 h = 2\left(\frac{0.01g}{2.01}\right)(1)$$

Phase 2: Mass m removed. Masses are M and M. Net force 0. Acceleration 0. The system moves with constant velocity v. Given distance $H = 0.312$ m in time $t = 1$ s.

$$v = \frac{H}{t} = 0.312 \text{ m/s}$$

Calculate g:

$$v^2 = (0.312)^2 = 0.097344$$

$$0.097344 = \frac{0.02}{2.01}g$$

$$g = \frac{0.097344 \times 2.01}{0.02} = \frac{0.19566}{0.02} \approx 9.78 \text{ m/s}^2$$

Answer: $g \approx 9.78$ m/s^2.

Question 5: Alpha Decay

Decay: $^{228}\text{Th} \to {}^{224}\text{Ra} + \alpha$. $Q = K_\alpha + K_{\text{Ra}}$.

(a) Percentage of energy

Conservation of momentum (parent at rest): $p_\alpha = p_{\text{Ra}}$. Kinetic energy $K = p^2/2m$.

$$\frac{K_\alpha}{K_{\text{Ra}}} = \frac{m_{\text{Ra}}}{m_\alpha} \implies K_{\text{Ra}} = \frac{m_\alpha}{m_{\text{Ra}}} K_\alpha$$

$$Q = K_\alpha \left(1 + \frac{m_\alpha}{m_{\text{Ra}}}\right)$$

Fraction carried by alpha:

$$\frac{K_\alpha}{Q} = \frac{1}{1 + \frac{m_\alpha}{m_{\text{Ra}}}} = \frac{m_{\text{Ra}}}{m_{\text{Ra}} + m_\alpha} \approx \frac{224}{228}$$

$$\% = \frac{224}{228} \times 100 \approx 98.2\%$$

(b) First excited state energy

The highest energy alpha ($K_1 = 5.423$ MeV) corresponds to decay to the ground state. $Q_{ground} = 5.423 \times \frac{228}{224}$ MeV. The next highest ($K_2 = 5.341$ MeV) corresponds to decay to the first excited state. $Q_{excited} = 5.341 \times \frac{228}{224}$ MeV. The energy of the excited state E^* is the difference in Q values:

$$E^* = Q_{ground} - Q_{excited} = \frac{228}{224}(5.423 - 5.341) \text{ MeV}$$

$$E^* = \frac{228}{224}(0.082) \approx 1.018 \times 0.082 \approx 0.0835 \text{ MeV} = 83.5 \text{ keV}$$

Question 6: Rainbows and Bubbles

(a) Zeroth order rainbow

Diagram: Ray enters sphere, refracts, crosses sphere, refracts out. Deviation $\delta = (i - r) + (i' - r')$. For sphere $i' = r$, $r' = i$.

$$\delta = 2(i - r)$$

(b) Non-existence

Rainbow occurs at extremum of δ.

$$\frac{d\delta}{di} = 2(1 - \frac{dr}{di}) = 0 \implies \frac{dr}{di} = 1$$

Snell's Law: $\sin i = n \sin r \implies \cos i = n \cos r \frac{dr}{di}$. Substituting $dr/di = 1$: $\cos i = n \cos r$. Squaring: $\cos^2 i = n^2 \cos^2 r = n^2(1 - \sin^2 r) = n^2 - \sin^2 i$. $1 - \sin^2 i = n^2 - \sin^2 i \implies n = 1$. Since $n \neq 1$ for water, no solution exists.

(c) and (d) Bubblebows

Bubble index $n \approx 1$, surroundings $n_w \approx 1.33$. Relative index $m = 1/n_w < 1$. Snell's Law: $n_w \sin i = \sin r \implies \sin r = n_w \sin i$.

Case 1: No internal reflection Same math as (b). Condition $\cos i = m \cos r$. No solution for $m \neq 1$.

Case 2: One internal reflection Path: Enter -¿ Refract -¿ Reflect off internal surface -¿ Refract out. Deviation $\delta = (i - r) + (\pi - 2r) + (i - r) = 2i - 4r + \pi$. Extremum: $\frac{d\delta}{di} = 2 - 4\frac{dr}{di} = 0 \implies \frac{dr}{di} = \frac{1}{2}$. From Snell's law: $\frac{dr}{di} = \frac{n_w \cos i}{\cos r}$.

$$\frac{n_w \cos i}{\cos r} = \frac{1}{2} \implies 2n_w \cos i = \cos r$$

Square: $4n_w^2(1 - \sin^2 i) = 1 - \sin^2 r = 1 - n_w^2 \sin^2 i$.

$$4n_w^2 - 4n_w^2 \sin^2 i = 1 - n_w^2 \sin^2 i$$

$$3n_w^2 \sin^2 i = 4n_w^2 - 1$$

$$\sin^2 i = \frac{4n_w^2 - 1}{3n_w^2}$$

For $n_w = 1.33$, $\sin^2 i \approx \frac{4(1.77)-1}{3(1.77)} = \frac{6.08}{5.31} > 1$. No solution. Thus, no rainbow exists for air bubbles in water.

Question 7: Rotating Charged Cylinders

(a) Electric Field

Static charge distribution. Use Gauss's Law with cylindrical symmetry. Total charge enclosed per length.

- $r < a$: $Q_{enc} = 0 \implies \mathbf{E} = 0$.
- $a < r < 4a$: $Q_{enc} = +\lambda$. $\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$.
- $r > 4a$: $Q_{enc} = \lambda - \lambda = 0$. $\mathbf{E} = 0$.

(b) Magnetic Field

Rotating inner cylinder constitutes a surface current $K = \sigma v$. $\lambda = 2\pi a\sigma \implies \sigma = \lambda/2\pi a$. $v = \omega a$. $K = \frac{\lambda}{2\pi a}(\omega a) = \frac{\lambda\omega}{2\pi}$. This acts like a solenoid current.

- $r < a$: Uniform field $\mathbf{B} = \mu_0 K\hat{z} = \frac{\mu_0\lambda\omega}{2\pi}\hat{z}$.
- $r > a$: $\mathbf{B} = 0$.

(c) Poynting Vector

$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$. We check regions:

- $r < a$: $\mathbf{E} = 0 \implies \mathbf{S} = 0$.
- $a < r < 4a$: $\mathbf{B} = 0 \implies \mathbf{S} = 0$.
- $r > 4a$: $\mathbf{E} = 0, \mathbf{B} = 0 \implies \mathbf{S} = 0$.

Thus, $\mathbf{S} = 0$ everywhere.

Question 8: Relativistic Spaceship

Proper acceleration a.

(a) Relative speed $v(t')$

Using standard relativistic kinematics for constant proper acceleration:

$$v(t') = c \tanh\left(\frac{at'}{c}\right)$$

Limit $at' \ll c$: $\tanh(x) \approx x$.

$$v(t') \approx c\left(\frac{at'}{c}\right) = at'$$

This recovers the Newtonian result $v = at$.

(b) Relation between t and t'

Time dilation relation $dt = \gamma dt'$.

$$dt = \cosh\left(\frac{at'}{c}\right) dt'$$

Integrating from 0 to t':

$$t = \int_0^{t'} \cosh\left(\frac{a\tau}{c}\right) d\tau = \left[\frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)\right]_0^{t'}$$

$$t = \frac{c}{a}\sinh\left(\frac{at'}{c}\right)$$

Limit $at' \gg c$: $\sinh(x) \approx \frac{1}{2}e^x$.

$$t \approx \frac{c}{2a}e^{at'/c}$$

The lab time grows exponentially with proper time.

Question 9: Electron Bubble

(a) Pressure relation

Mechanical equilibrium of the bubble interface (Young-Laplace equation):

$$P_e - P_{He} = \frac{2\sigma}{R}$$

(b) Relation between E_K and P_e

The electron gas exerts pressure. Using thermodynamics $dE = -PdV$ (at T=0, S=0).

$$P_e = -\frac{dE_K}{dV} = -\frac{dE_K}{dR}\frac{dR}{dV}$$

$$V = \frac{4}{3}\pi R^3 \implies \frac{dV}{dR} = 4\pi R^2$$

$$P_e = -\frac{1}{4\pi R^2}\frac{dE_K}{dR}$$

(c) Estimate $E_0(R)$

Using Heisenberg Uncertainty Principle for a confined particle: $\Delta x \sim R$, so $\Delta p \sim \frac{h}{4R}$ (or similar estimate). Kinetic Energy $E_K \approx \frac{p^2}{2m}$. Using the particle in a box/sphere model ground state energy:

$$E_0 \approx \frac{h^2}{8mR^2}$$

(The exact pre-factor depends on the model chosen, e.g., spherical box gives $\frac{h^2}{8mR^2}$).

(d) Equilibrium radius R_e

Set $P_{He} = 0$. Then $P_e = 2\sigma/R$. Compute P_e from E_0:

$$P_e = -\frac{1}{4\pi R^2}\frac{d}{dR}\left(\frac{h^2}{8mR^2}\right) = -\frac{1}{4\pi R^2}\left(-2\frac{h^2}{8mR^3}\right) = \frac{h^2}{16\pi m R^5}$$

Equating pressures:

$$\frac{2\sigma}{R} = \frac{h^2}{16\pi m R^5}$$

$$R^4 = \frac{h^2}{32\pi m\sigma}$$

$$R_e = \left(\frac{h^2}{32\pi m\sigma}\right)^{1/4}$$

Calculation: $h = 6.63 \times 10^{-34}$, $m = 9.11 \times 10^{-31}$, $\sigma = 3.75 \times 10^{-4}$.

$$R_e \approx \left(\frac{(6.63 \times 10^{-34})^2}{32\pi(9.11 \times 10^{-31})(3.75 \times 10^{-4})}\right)^{1/4} \approx 1.9 \text{ nm}$$