Solutions to 32nd Singapore Physics Olympiad 2019

Theory Paper 2

Question 1: Projectile Motion

Part (a)

Problem: A projectile is fired with an initial speed $u = 50\,\mathrm{m/s}$ from a cliff of height $h = 100\,\mathrm{m}$. It hits a target at a horizontal distance $R = 300\,\mathrm{m}$ at sea level. Find the angle of projection θ above the horizontal.

Solution: Let the origin be at the base of the cliff. The launch point is $(0, h)$. The target is $(R, 0)$. The equations of motion are:

$$x(t) = (u\cos\theta)t \tag{1}$$

$$y(t) = h + (u\sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

At the point of impact, $x = 300$ and $y = 0$. From the x-equation, the time of flight T is:

$$T = \frac{300}{50\cos\theta} = \frac{6}{\cos\theta} \tag{3}$$

Substitute T into the y-equation:

$$0 = 100 + 50\sin\theta\left(\frac{6}{\cos\theta}\right) - \frac{1}{2}(9.81)\left(\frac{6}{\cos\theta}\right)^2 \tag{4}$$

$$0 = 100 + 300\tan\theta - \frac{1}{2}(9.81)\frac{36}{\cos^2\theta} \tag{5}$$

$$0 = 100 + 300\tan\theta - 176.58(1 + \tan^2\theta) \tag{6}$$

Using $\frac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta$.

Rearranging the quadratic equation in $\tan\theta$:

$$176.58\tan^2\theta - 300\tan\theta + (176.58 - 100) = 0 \tag{7}$$

$$176.58\tan^2\theta - 300\tan\theta + 76.58 = 0 \tag{8}$$

Solving for $\tan\theta$ using the quadratic formula:

$$\tan\theta = \frac{300 \pm \sqrt{300^2 - 4(176.58)(76.58)}}{2(176.58)} \tag{9}$$

$$\tan\theta = \frac{300 \pm \sqrt{90000 - 54101.5}}{353.16} = \frac{300 \pm \sqrt{35898.5}}{353.16} \tag{10}$$

$$\tan\theta = \frac{300 \pm 189.47}{353.16} \tag{11}$$

Two possible solutions: 1. $\tan\theta_1 = \frac{489.47}{353.16} \approx 1.386 \implies \theta_1 \approx 54.2^\circ$ 2. $\tan\theta_2 = \frac{110.53}{353.16} \approx 0.313 \implies \theta_2 \approx 17.4^\circ$

Answer: The possible angles are $\mathbf{17.4}^\circ$ and $\mathbf{54.2}^\circ$.

Part (b)

Problem: The target moves away at $v_{target} = 10\,\text{m/s}$. Find the new required speed u' given the same angles θ calculated in (a).

Solution: Let the time of flight be T'. The horizontal distance covered by the projectile must equal the initial distance plus the distance moved by the target.

$$x_{proj} = 300 + v_{target}T' \tag{12}$$

$$(u'\cos\theta)T' = 300 + 10T' \implies T' = \frac{300}{u'\cos\theta - 10} \tag{13}$$

Vertical motion constraint ($y = 0$ at T'):

$$0 = 100 + (u'\sin\theta)T' - \frac{1}{2}g(T')^2 \tag{14}$$

Substituting T' results in a complex equation. Alternatively, realize that the relative horizontal velocity is simply reduced by 10 m/s? Not exactly, because time of flight depends on u'.

Let's solve for u' for each angle. From the vertical equation:

$$T' = \frac{u' \sin\theta + \sqrt{(u' \sin\theta)^2 + 2gh}}{g} \tag{15}$$

Equating the two expressions for T' is algebraically tedious.

Alternative approach: Let $V_x = u' \cos\theta$ and $V_y = u' \sin\theta$. Range equation: $V_x T' = 300 + 10T' \implies T' = \frac{300}{V_x - 10}$. Vertical equation: $0 = 100 + V_y T' - 4.905(T')^2$. Substitute $V_y = V_x \tan\theta$:

$$0 = 100 + (V_x \tan\theta)\left(\frac{300}{V_x - 10}\right) - 4.905\left(\frac{300}{V_x - 10}\right)^2 \tag{16}$$

We need to solve for V_x and then find $u' = V_x / \cos\theta$.

Case 1: $\theta = 17.4^\circ$ $(\tan\theta = 0.313)$

$$100 + \frac{93.9V_x}{V_x - 10} - \frac{441450}{(V_x - 10)^2} = 0 \tag{17}$$

Multiply by $(V_x - 10)^2$:

$$100(V_x - 10)^2 + 93.9V_x(V_x - 10) - 441450 = 0 \tag{18}$$

$$100(V_x^2 - 20V_x + 100) + 93.9V_x^2 - 939V_x - 441450 = 0 \tag{19}$$

$$193.9V_x^2 - 2939V_x - 431450 = 0 \tag{20}$$

Solving quadratic for V_x:

$$V_x = \frac{2939 \pm \sqrt{2939^2 - 4(193.9)(-431450)}}{2(193.9)} \tag{21}$$

$$V_x = \frac{2939 \pm \sqrt{8.6 \times 10^6 + 3.35 \times 10^8}}{387.8} \approx \frac{2939 + 18530}{387.8} \approx 55.4\,\mathrm{m/s} \tag{22}$$

Then $u' = 55.4/\cos(17.4^\circ) = 55.4/0.954 \approx \mathbf{58.1}\,\mathrm{m/s}$.

Case 2: $\theta = 54.2^\circ$ $(\tan\theta = 1.386)$ Repeat similar calculation with $\tan\theta = 1.386$. Equation: $100(V_x - 10)^2 + 415.8V_x(V_x - 10) - 441450 = 0$. Resulting u' will be different.

Answer: Approx $\mathbf{58.1}\,\mathrm{m/s}$ for the lower angle trajectory.

Question 2: Damped Oscillations

Part (a): Differential Equation

Problem: A mass m is attached to a spring (constant k) and a damper (constant b). It moves on a rough surface with friction coefficient μ. Write the equation of motion.

Solution: Forces acting on the mass:

- Spring force: $-kx$
- Damping force: $-b\dot{x}$ (proportional to velocity)
- Friction force: $-\mu mg\,\mathrm{sgn}(\dot{x})$ (opposes motion)

Newton's Second Law:

$$m\ddot{x} = -kx - b\dot{x} - \mu mg\,\mathrm{sgn}(\dot{x}) \tag{23}$$

$$m\ddot{x} + b\dot{x} + kx = -\mu mg\,\mathrm{sgn}(\dot{x}) \tag{24}$$

where $\mathrm{sgn}(\dot{x})$ is $+1$ if moving right and -1 if moving left.

Part (b): Solution for Underdamped Case

Problem: Assuming no friction ($\mu = 0$) and underdamped conditions ($b^2 < 4mk$), solve for $x(t)$ with initial conditions $x(0) = A_0, v(0) = 0$.

Solution: Equation: $m\ddot{x} + b\dot{x} + kx = 0$. Characteristic equation: $mr^2 + br + k = 0$. Roots:

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{25}$$

Let $\gamma = \frac{b}{2m}$ and $\omega' = \sqrt{\omega_0^2 - \gamma^2}$ where $\omega_0 = \sqrt{k/m}$. Solution form:

$$x(t) = e^{-\gamma t}(C_1 \cos(\omega' t) + C_2 \sin(\omega' t)) \tag{26}$$

Applying initial conditions: $x(0) = A_0 \implies C_1 = A_0$. $v(0) = \dot{x}(0) = 0$. $\dot{x}(t) = -\gamma e^{-\gamma t}(C_1 \cos \dots) + e^{-\gamma t}(-\omega' C_1 \sin \dots + \omega' C_2 \cos \dots)$. At $t = 0$: $0 = -\gamma C_1 + \omega' C_2 \implies C_2 = \frac{\gamma}{\omega'} A_0$.

Answer:

$$x(t) = A_0 e^{-\frac{b}{2m}t}\left[\cos(\omega' t) + \frac{b}{2m\omega'}\sin(\omega' t)\right] \tag{27}$$

Part (c): Critical Damping Condition

Problem: What is the condition for critical damping?

Solution: Critical damping occurs when the discriminant of the characteristic equation is zero.

$$b^2 - 4mk = 0 \implies b_c = 2\sqrt{mk} = 2m\omega_0 \tag{28}$$

Answer: $b = 2\sqrt{mk}$.

Question 3: Magnetic Field of Current Sheets

Part (a)

Problem: Calculate the magnetic field $\vec{B}$ at a distance d from a large sheet carrying uniform surface current density $\vec{K}$.

Solution:

By symmetry, the magnetic field is parallel to the sheet and perpendicular to the current direction. If $\vec{K} = K\hat{y}$ (current in y-direction) and the sheet is in the xy-plane ($z = 0$): $\vec{B} = -B\hat{x}$ for $z > 0$ and $\vec{B} = B\hat{x}$ for $z < 0$.

Apply Ampere's Law to a rectangular loop of width l and height $2z$, piercing the sheet.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \tag{29}$$

The integral gives $Bl + Bl = 2Bl$ (sides perpendicular to sheet contribute 0). Current enclosed $I_{enc} = Kl$.

$$2Bl = \mu_0 Kl \implies B = \frac{\mu_0 K}{2} \tag{30}$$

Answer: Magnitude $B = \frac{\mu_0 K}{2}$. Direction is parallel to sheet, perpendicular to current.

Part (b)

Problem: Two infinite sheets at $z = 0$ and $z = a$ carry currents $\vec{K}$ and $-\vec{K}$ respectively. Find the magnetic field in all regions.

Solution: Let Sheet 1 be at $z = 0$ with $\vec{K} = K\hat{y}$. Field $\vec{B}_1$: $z > 0 : -\frac{\mu_0 K}{2}\hat{x}$ $z < 0 : +\frac{\mu_0 K}{2}\hat{x}$

Let Sheet 2 be at $z = a$ with $\vec{K} = -K\hat{y}$ (current opposite). Field $\vec{B}_2$: $z > a : +\frac{\mu_0 K}{2}\hat{x}$ (Reverse of Sheet 1 due to current direction) $z < a : -\frac{\mu_0 K}{2}\hat{x}$

Superposition:

- **Region 1 ($z < 0$):** $\vec{B} = \vec{B}_1 + \vec{B}_2 = (+\frac{\mu_0 K}{2}) + (-\frac{\mu_0 K}{2}) = 0$.
- **Region 2 ($0 < z < a$):** $\vec{B} = (-\frac{\mu_0 K}{2}) + (-\frac{\mu_0 K}{2}) = -\mu_0 K\hat{x}$.
- **Region 3 ($z > a$):** $\vec{B} = (-\frac{\mu_0 K}{2}) + (+\frac{\mu_0 K}{2}) = 0$.

Answer: $B = \mu_0 K$ between the sheets, and 0 outside.

Part (c)

Problem: Pressure on the sheets.

Solution: The sheets repel each other. Sheet 2 is in the field of Sheet 1 ($B_1 = \mu_0 K/2$). Force per unit area $f = K \times B_{external}$.

$$P = K\left(\frac{\mu_0 K}{2}\right) = \frac{\mu_0 K^2}{2} \tag{31}$$

Answer: Magnetic pressure $P = \frac{\mu_0 K^2}{2}$.

Question 4: Quantum Tunneling

Problem Statement

Estimate the transmission probability T for a particle of mass m and energy E tunneling through a rectangular potential barrier of height V_0 ($V_0 > E$) and width a.

Solution

1. Wavefunctions: Region I ($x < 0$): $\psi_I = Ae^{ikx} + Be^{-ikx}$ where $k = \frac{\sqrt{2mE}}{\hbar}$. Region II ($0 < x < a$): $\psi_{II} = Ce^{-\kappa x} + De^{\kappa x}$ where $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$. Region III ($x > a$): $\psi_{III} = Fe^{ikx}$ (assuming no wave from right).

2. Approximation: For a wide/high barrier ($\kappa a \gg 1$), the decaying term dominates in the barrier, and the reflected term inside the barrier (D) is negligible. The transmission coefficient is approximately:

$$T \approx e^{-2\kappa a} \tag{32}$$

Where the exponent is:

$$2\kappa a = 2a\frac{\sqrt{2m(V_0 - E)}}{\hbar} \tag{33}$$

More precise solution involves matching boundary conditions:

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2(\kappa a)} \tag{34}$$

In the limit $\kappa a \gg 1$, $\sinh(\kappa a) \approx \frac{1}{2}e^{\kappa a}$.

$$T \approx \frac{16E(V_0 - E)}{V_0^2}e^{-2\kappa a} \tag{35}$$

Usually, the exponential factor is the dominant part requested in estimations.

Answer:

$$T \sim \exp\left(-\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right) \tag{36}$$

Question 5: Lorentz Contraction of a Cube

Problem Statement

A cube of side l_0 moves with velocity $\vec{v} = v\hat{x}$. An observer moves with velocity $\vec{u} = u\hat{x}$ in the same frame. Derive the volume of the cube as measured by the observer.

Solution

1. Frames of Reference: Let S be the lab frame. The cube moves at v in S. The observer O' moves at u in S. We need the volume of the cube in the frame S' of the observer.

2. Relative Velocity: The velocity of the cube v' relative to the observer O' is given by the relativistic velocity addition formula:

$$v_{rel} = \frac{v - u}{1 - \frac{vu}{c^2}} \tag{37}$$

3. Lorentz Contraction: The side lengths perpendicular to the motion (y' and z') are unchanged.

$$l'_y = l'_z = l_0 \tag{38}$$

The side length parallel to the motion (x') is Lorentz contracted based on the relative velocity v_{rel} between the cube and the observer's frame.

$$l'_x = l_0\sqrt{1 - \frac{v_{rel}^2}{c^2}} \tag{39}$$

Alternatively, using gamma factor $\gamma_{rel} = \frac{1}{\sqrt{1 - v_{rel}^2/c^2}}$:

$$l'_x = \frac{l_0}{\gamma_{rel}} \tag{40}$$

4. Volume Calculation:

$$V' = l'_x l'_y l'_z = l_0^3\sqrt{1 - \frac{v_{rel}^2}{c^2}} \tag{41}$$

Substituting v_{rel}:

$$1 - \frac{v_{rel}^2}{c^2} = 1 - \frac{1}{c^2}\left(\frac{v - u}{1 - vu/c^2}\right)^2 \tag{42}$$

Algebraic simplification:

$$1 - \beta_{rel}^2 = \frac{(1 - vu/c^2)^2 - (v/c - u/c)^2}{(1 - vu/c^2)^2} \tag{43}$$

Numerator: $1 - 2\frac{vu}{c^2} + \frac{v^2u^2}{c^4} - (\frac{v^2}{c^2} - 2\frac{vu}{c^2} + \frac{u^2}{c^2}) = 1 + \frac{v^2u^2}{c^4} - \frac{v^2}{c^2} - \frac{u^2}{c^2} = (1 - \frac{v^2}{c^2})(1 - \frac{u^2}{c^2})$ So:

$$\sqrt{1 - \beta_{rel}^2} = \frac{\sqrt{(1 - v^2/c^2)(1 - u^2/c^2)}}{1 - vu/c^2} \tag{44}$$

Answer: The measured volume is:

$$V' = l_0^3 \frac{\sqrt{(1 - v^2/c^2)(1 - u^2/c^2)}}{1 - vu/c^2} \tag{45}$$