Solutions for SPhO 2023 Theory Paper 2

Question 1: Projectile Motion

(a) Stationary Target

The motion of the projectile is described by the following equations:

$$x(t) = (v_0 \cos\theta)t$$

$$y(t) = y_0 + (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

Given values are $v_0 = 180\,\text{m/s}$, $y_0 = 200\,\text{m}$, $x = 2500\,\text{m}$, and $g = 9.81\,\text{m/s}^2$. The shell hits the target at sea level, so the final vertical position is $y = 0$.

From the horizontal motion equation, the time of flight t is:

$$t = \frac{x}{v_0 \cos\theta} = \frac{2500}{180\cos\theta}$$

Substitute this into the vertical motion equation:

$$0 = y_0 + (v_0 \sin\theta)\left(\frac{x}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2$$

$$0 = y_0 + x\tan\theta - \frac{gx^2}{2v_0^2}\sec^2\theta$$

Using the identity $\sec^2\theta = 1 + \tan^2\theta$:

$$0 = y_0 + x\tan\theta - \frac{gx^2}{2v_0^2}(1 + \tan^2\theta)$$

Let $u = \tan\theta$. This gives a quadratic equation in u:

$$\left(\frac{gx^2}{2v_0^2}\right)u^2 - xu - \left(y_0 - \frac{gx^2}{2v_0^2}\right) = 0$$

Calculating the coefficient $\frac{gx^2}{2v_0^2}$:

$$\frac{(9.81)(2500)^2}{2(180)^2} = 946.18\,\text{m}$$

The equation becomes:

$$946.18u^2 - 2500u - (200 - 946.18) = 0$$

$$946.18u^2 - 2500u + 746.18 = 0$$

Solving for u using the quadratic formula:

$$u = \frac{2500 \pm \sqrt{(-2500)^2 - 4(946.18)(746.18)}}{2(946.18)} = \frac{2500 \pm \sqrt{3425816}}{1892.36} = \frac{2500 \pm 1850.9}{1892.36}$$

This yields two possible values for $u = \tan\theta$:

$$u_1 = \frac{4350.9}{1892.36} \approx 2.30 \implies \theta_1 = \arctan(2.30) \approx 66.5^\circ$$

$$u_2 = \frac{649.1}{1892.36} \approx 0.343 \implies \theta_2 = \arctan(0.343) \approx 18.9^\circ$$

Answer: The angle of elevation can be $\mathbf{66.5^\circ}$ or $\mathbf{18.9^\circ}$.

(b) Moving Target

The target moves with speed $v_T = 10\,\text{m/s}$. We use the smaller angle, $\theta = 18.9°$. Let the new muzzle speed be v_0' and the new time of flight be t'. At time t', the horizontal positions must match:

$$(v_0' \cos\theta)t' = 2500 + v_T t'$$

The vertical position of the shell is:

$$0 = y_0 + (v_0' \sin\theta)t' - \frac{1}{2}g(t')^2$$

$$0 = y_0 + (v_0' \cos\theta \tan\theta)t' - \frac{1}{2}g(t')^2$$

Substitute $(v_0' \cos\theta)t'$ from the horizontal equation into the vertical one:

$$0 = y_0 + (2500 + v_T t')\tan\theta - \frac{1}{2}g(t')^2$$

$$\frac{1}{2}g(t')^2 - (v_T \tan\theta)t' - (y_0 + 2500\tan\theta) = 0$$

Substitute the values: $\tan(18.9°) \approx 0.343$:

$$\frac{1}{2}(9.81)(t')^2 - (10 \times 0.343)t' - (200 + 2500 \times 0.343) = 0$$

$$4.905(t')^2 - 3.43t' - 1057.5 = 0$$

Solving for t':

$$t' = \frac{3.43 \pm \sqrt{(-3.43)^2 - 4(4.905)(-1057.5)}}{2(4.905)} = \frac{3.43 \pm 144.08}{9.81}$$

Since $t' > 0$, we take the positive root: $t' = \frac{147.51}{9.81} \approx 15.04\,\text{s}$. Now, we find v_0':

$$v_0' = \frac{2500 + v_T t'}{t' \cos\theta} = \frac{2500 + 10(15.04)}{15.04\cos(18.9°)} = \frac{2650.4}{14.23} \approx 186.2\,\text{m/s}$$

The required change in muzzle speed is $\Delta v_0 = v_0' - v_0 = 186.2 - 180 = 6.2\,\text{m/s}$. **Answer:** The muzzle speed must be increased by approximately **6.2** m/s.

Question 2: Orbital Mechanics and Wave Interference

(a) Satellites in Orbit

For a satellite in a circular orbit, gravitational force equals centripetal force:

$$\frac{GM_E m}{r^2} = m\omega^2 r \implies \omega = \sqrt{\frac{GM_E}{r^3}}$$

Using $g = \frac{GM_E}{R_E^2}$, we get $\omega = R_E\sqrt{\frac{g}{r^3}}$. The radii are: $r_X = R_E + h_X = 6.36\times10^6 + 0.5\times10^6 = 6.86\times10^6\,\text{m}$ $r_Y = R_E + h_Y = 6.36\times10^6 + 1.0\times10^6 = 7.36\times10^6\,\text{m}$ The angular velocities are:

$$\omega_X = (6.36\times10^6)\sqrt{\frac{9.81}{(6.86\times10^6)^3}} \approx 1.135\times10^{-3}\,\text{rad/s}$$

$$\omega_Y = (6.36\times10^6)\sqrt{\frac{9.81}{(7.36\times10^6)^3}} \approx 1.002\times10^{-3}\,\text{rad/s}$$

The satellites are aligned again when the difference in their angular displacement is 2π:

$$(\omega_X - \omega_Y)t = 2\pi$$

$$t = \frac{2\pi}{\omega_X - \omega_Y} = \frac{2\pi}{(1.135 - 1.002)\times10^{-3}} = \frac{2\pi}{0.133\times10^{-3}} \approx 47242\,\text{s}$$

In minutes, $t = \frac{47242}{60} \approx 787.4\,\text{minutes}$. **Answer:** The value of t is approximately **787.4** minutes.

(b) Sound Wave Interference

The path difference between the direct and reflected waves is $\Delta P = \sqrt{L^2 + (2d)^2} - L$. At position 1 ($d_1 = 90\,\text{m}$), there is constructive interference, so $\Delta P_1 = n\lambda$.

$$\Delta P_1 = \sqrt{120^2 + (180)^2} - 120 = \sqrt{46800} - 120 \approx 96.333\,\text{m}$$

At position 2 ($d_2 = 90 + h$), there is destructive interference for the first time. The path difference must have increased by $\lambda/2$.

$$\Delta P_2 = \Delta P_1 + \frac{\lambda}{2} = 96.333 + \frac{1.33}{2} = 97.00\,\text{m}$$

Also, $\Delta P_2 = \sqrt{120^2 + (2(90+h))^2} - 120$.

$$\sqrt{120^2 + (180 + 2h)^2} = 120 + 97.00 = 217.00$$

$$14400 + (180 + 2h)^2 = 217.00^2 = 47089$$

$$(180 + 2h)^2 = 32689$$

$$180 + 2h = \sqrt{32689} \approx 180.80$$

$$2h \approx 0.80 \implies h \approx 0.40\,\text{m}$$

Answer: The value of h is approximately **0.40 m**.

Question 3: Electrical Circuits and Electrostatics

(a) Maximum Power

For a single resistor: $R = 3\,\Omega$, $P_{max} = 108\,\text{W}$. Max current: $I_{max} = \sqrt{P_{max}/R} = \sqrt{108/3} = 6\,\text{A}$. Max voltage: $V_{max} = \sqrt{P_{max} \cdot R} = \sqrt{108 \times 3} = 18\,\text{V}$.

Fig. 2(a): Two resistors in series ($R_{top} = 6\,\Omega$) are in parallel with one resistor ($R_{bot} = 3\,\Omega$). The voltage is the same across both. The bottom resistor is the limiting component. The max voltage across it is 18 V. At this voltage: $P_{bot} = V^2/R_{bot} = 18^2/3 = 108\,\text{W}$. $P_{top} = V^2/R_{top} = 18^2/6 = 54\,\text{W}$. Total power: $P_{total} = P_{top} + P_{bot} = 54 + 108 = 162\,\text{W}$. **Answer:** Maximum power for Fig. 2(a) is **162 W**.

Fig. 2(b): Two resistors in parallel ($R_{par} = 1.5\,\Omega$) are in series with one resistor ($R_{ser} = 3\,\Omega$). Total resistance is $R_{eq} = 1.5 + 3 = 4.5\,\Omega$. The single series resistor is the limiting component. The max current through it is $I_{max} = 6\,\text{A}$. This is the total current. Total power: $P_{total} = I_{total}^2 R_{eq} = 6^2 \times 4.5 = 36 \times 4.5 = 162\,\text{W}$. **Answer:** Maximum power for Fig. 2(b) is **162 W**.

(b) Motion in an Electric Field

Inside the sphere, the electric field at distance r from the center is found using Gauss's Law:

$$E = \frac{\rho r}{3\epsilon_0}$$

The force on the particle with charge q is $F = qE$. Since q is negative, the force is a restoring force directed towards the center:

$$F = q\frac{\rho r}{3\epsilon_0} = -\left(\frac{|q|\rho}{3\epsilon_0}\right) r$$

This is Simple Harmonic Motion, $F = -k_{eff} r$, with $k_{eff} = \frac{|q|\rho}{3\epsilon_0}$. The angular frequency is:

$$\omega = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{|q|\rho}{3\epsilon_0 m}}$$

$$\omega = \sqrt{\frac{(0.05 \times 10^{-9})(5 \times 10^{-9})}{3(8.85 \times 10^{-12})(1.0 \times 10^{-9})}} = \sqrt{9.416} \approx 3.068\,\text{rad/s}$$

(i) The time to move from one end to the other is half a period:

$$t = \frac{T}{2} = \frac{1}{2}\left(\frac{2\pi}{\omega}\right) = \frac{\pi}{\omega} = \frac{\pi}{3.068} \approx 1.024\,\mathrm{s}$$

(ii) The speed is maximum at the center, $v_{max} = A\omega$, where amplitude $A = R = 0.5\,\mathrm{m}$.

$$v_{max} = (0.5)(3.068) \approx 1.534\,\mathrm{m/s}$$

Answers: (i) The time taken is $\mathbf{1.024\,s}$. (ii) The speed at the center is $\mathbf{1.534\,m/s}$.

Question 4: Thermodynamics and Statistical Mechanics

(a) Entropy Change

For an isobaric (constant pressure) process, the change in entropy is:

$$\Delta S = nC_P \ln\left(\frac{T_2}{T_1}\right)$$

From the ideal gas law at constant pressure, $\frac{V_1}{T_1} = \frac{V_2}{T_2}$, so $\frac{T_2}{T_1} = \frac{V_2}{V_1}$. Given $V_2 = 2V_1$, we have $\frac{T_2}{T_1} = 2$. For a diatomic ideal gas, $C_P = \frac{7}{2}R$.

$$\Delta S = nC_P \ln\left(\frac{V_2}{V_1}\right) = (1)\left(\frac{7}{2}R\right)\ln(2)$$

$$\Delta S = \frac{7}{2}(8.31)\ln(2) \approx 20.16\,\mathrm{J/K}$$

Answer: The change in entropy is $\mathbf{20.2\,J/K}$.

(b) Number of Gas Particles

The probability of a particle having energy E is $p(E) \propto e^{-E/kT}$. Taking the ratio of probabilities for the first two states:

$$\frac{p(E_2)}{p(E_1)} = \frac{e^{-E_2/kT}}{e^{-E_1/kT}} = e^{-(E_2-E_1)/kT}$$

$$\frac{0.23}{0.63} = e^{-(0.0129-0.0043)/kT} = e^{-0.0086/kT}$$

$$\ln\left(\frac{0.23}{0.63}\right) = -1.0076 = -\frac{0.0086}{kT}$$

$$kT = \frac{0.0086}{1.0076} \approx 0.008535\,\mathrm{eV}$$

Now find the probability for the third state, $p(E_3)$:

$$\frac{p(E_3)}{p(E_1)} = e^{-(E_3-E_1)/kT}$$

$$p(E_3) = p(E_1)e^{-(0.0215-0.0043)/kT} = 0.63 \cdot e^{-0.0172/0.008535}$$

$$p(E_3) = 0.63 \cdot e^{-2.015} \approx 0.63 \times 0.1333 \approx 0.0840$$

For 1 mole of gas, $N_{total} = N_A = 6.02 \times 10^{23}$ particles.

$$N_3 = N_{total} \cdot p(E_3) = (6.02 \times 10^{23})(0.0840) \approx 5.06 \times 10^{22}$$

Answer: The number of particles is approximately $\mathbf{5.06 \times 10^{22}}$.

Question 5: Modern Physics

(a) Recoil of a Hydrogen Atom

The energy of the emitted photon is approximately the energy difference between the states:

$$\Delta E = E_3 - E_1 = -13.6\text{eV}\left(\frac{1}{3^2} - \frac{1}{1^2}\right) = -13.6\left(\frac{1}{9} - 1\right) = 13.6 \times \frac{8}{9} \approx 12.09\,\text{eV}$$

$$E_\gamma \approx \Delta E = 12.09\,\text{eV} = 1.934 \times 10^{-18}\,\text{J}$$

The photon's momentum is $p_\gamma = E_\gamma / c$:

$$p_\gamma = \frac{1.934 \times 10^{-18}}{3.00 \times 10^8} \approx 6.447 \times 10^{-27}\,\text{kg m/s}$$

By conservation of momentum, the recoil momentum of the atom is $p_H = p_\gamma$. The recoil speed is:

$$v_H = \frac{p_H}{M_H} = \frac{6.447 \times 10^{-27}}{1.66 \times 10^{-27}} \approx 3.88\,\text{m/s}$$

The recoil energy is:

$$E_{recoil} = \frac{p_H^2}{2M_H} = \frac{(6.447 \times 10^{-27})^2}{2(1.66 \times 10^{-27})} \approx 1.25 \times 10^{-26}\,\text{J}$$

Answers: Recoil speed is $\mathbf{3.88\,m/s}$. Recoil energy is $\mathbf{1.25 \times 10^{-26}\,J}$.

(b) Special Relativity

The Lorentz transformation for time is $t' = \gamma(t - vx/c^2)$. The Lorentz factor is $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0.6^2}} = \frac{1}{0.8} = 1.25$. Event 1 (Red flash): $(x_1, t_1) = (0, 0)$. In S', $t_1' = \gamma(0 - 0) = 0$. Event 2 (Blue flash): $(x_2, t_2) = (x_B, 9.50 \times 10^{-6}\,\text{s})$. In S',

$$t_2' = \gamma\left(t_2 - \frac{vx_B}{c^2}\right) = 1.25\left(9.50 \times 10^{-6} - \frac{0.6c \cdot x_B}{c^2}\right)$$

B sees the blue flash before the red flash, so $t_2' < t_1'$.

$$1.25\left(9.50 \times 10^{-6} - \frac{0.6x_B}{c}\right) < 0$$

$$9.50 \times 10^{-6} < \frac{0.6x_B}{c}$$

$$x_B > \frac{(9.50 \times 10^{-6})c}{0.6} = \frac{(9.50 \times 10^{-6})(3.00 \times 10^8)}{0.6} = \frac{2850}{0.6} = 4750\,\text{m}$$

So, $x_B > 4.75\,\text{km}$. **Answer:** The range of values is $\mathbf{x_B > 4.75}\,\text{km}$.