Solutions for the 36th Singapore Physics Olympiad Theory Paper 1 (2023)

1. Car on a Sloping Road

(a) Resistive Force

When the car moves at a constant speed, the net force is zero. Let the direction down the slope be positive. The forces along the slope are the driving force (F_{engine}), the component of gravity ($mg\sin(\theta)$), and the resistive force ($F_{\text{resistive}}$). The equation is:

$$F_{\text{engine}} + mg\sin(\theta) - F_{\text{resistive}} = 0$$

The driving force is calculated from the power ($P = 112.5\,\text{kW}$) and speed ($v = 20\,\text{m/s}$):

$$F_{\text{engine}} = \frac{P}{v} = \frac{112500\,\text{W}}{20\,\text{m/s}} = 5625\,\text{N}$$

The gravitational component is:

$$F_g = mg\sin(10^\circ) = (1200\,\text{kg})(9.81\,\text{m/s}^2)\sin(10^\circ) \approx 2043.6\,\text{N}$$

Solving for the resistive force:

$$F_{\text{resistive}} = F_{\text{engine}} + F_g = 5625\,\text{N} + 2043.6\,\text{N} = 7668.6\,\text{N} \approx \mathbf{7670}\,\text{N}$$

(b) Minimum Braking Force

The total stopping distance is 40 m. The process has two stages:

1. **Reaction Time:** Distance covered in $t_{\text{reaction}} = 0.5\,\text{s}$ is $d_{\text{reaction}} = v \times t_{\text{reaction}} = 20 \times 0.5 = 10\,\text{m}$.
2. **Braking:** Remaining distance is $d_{\text{braking}} = 40\,\text{m} - 10\,\text{m} = 30\,\text{m}$.

The required acceleration to stop in 30 m is found using $v_f^2 = v_i^2 + 2ad$:

$$0^2 = (20)^2 + 2a(30) \implies a = -\frac{400}{60} = -\frac{20}{3} \approx -6.67\,\text{m/s}^2$$

Applying Newton's Second Law during braking (engine is off):

$$F_{\text{net}} = mg\sin(10^\circ) - F_{\text{resistive}} - F_{\text{brake}} = ma$$

$$F_{\text{brake}} = mg\sin(10^\circ) - F_{\text{resistive}} - ma$$

$$F_{\text{brake}} = 2043.6 - 7668.6 - (1200)(-\frac{20}{3}) = -5625 + 8000 = \mathbf{2375}\,\text{N}$$

(c) Impulse During Impact

The new braking force is $F'_{\text{brake}} = 0.95 \times 2375\,\text{N} = 2256.25\,\text{N}$. The new acceleration is:

$$a' = \frac{mg\sin(10^\circ) - F_{\text{resistive}} - F'_{\text{brake}}}{m} = \frac{2043.6 - 7668.6 - 2256.25}{1200} \approx -6.568\,\text{m/s}^2$$

The speed just before impact after traveling 30 m is:

$$v_{\text{impact}}^2 = v_i^2 + 2a'd = (20)^2 + 2(-6.568)(30) = 5.92 \implies v_{\text{impact}} \approx 2.43\,\text{m/s}$$

The impulse (J) is the change in momentum. Assuming the car stops after impact ($v_{\text{final}} = 0$):

$$J = \Delta p = mv_{\text{final}} - mv_{\text{impact}} = 0 - (1200\,\text{kg})(2.43\,\text{m/s}) = -2916\,\text{Ns}$$

The magnitude of the impulse is **2920** Ns.

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2. Waves on a Sonometer

(a) Distance Between Wedges

For the fundamental mode, the frequency is $f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$. Tension $T = mg = (2.0\,\text{kg})(9.81\,\text{m/s}^2) = 19.62\,\text{N}$. Linear mass density $\mu = \rho A = \rho(\pi r^2) = (8830)\pi(0.75 \times 10^{-3})^2 \approx 0.0156\,\text{kg/m}$. Rearranging for length L:

$$L = \frac{1}{2f_1}\sqrt{\frac{T}{\mu}} = \frac{1}{2(22.0)}\sqrt{\frac{19.62}{0.0156}} \approx \mathbf{0.806}\,\text{m}$$

(b) Change in Distance with Cylinder in Water

The buoyant force F_B reduces the tension. Volume of cylinder $V = \frac{m}{\rho_{\text{cyl}}} = \frac{2.0}{8400} \approx 2.381 \times 10^{-4}\,\text{m}^3$. Buoyant force $F_B = \rho_{\text{water}} g V = (1000)(9.81)(2.381 \times 10^{-4}) \approx 2.336\,\text{N}$. New tension $T' = T - F_B = 19.62 - 2.336 = 17.284\,\text{N}$. To keep the frequency constant, the new length L' must satisfy $\frac{1}{2L}\sqrt{\frac{T}{\mu}} = \frac{1}{2L'}\sqrt{\frac{T'}{\mu}}$.

$$L' = L\sqrt{\frac{T'}{T}} = 0.806\sqrt{\frac{17.284}{19.62}} \approx 0.757\,\text{m}$$

The change in distance is $\Delta L = L - L' = 0.806 - 0.757 = 0.049\,\text{m}$. The distance must be **decreased by 4.9 cm**.

(c) Beat Frequency

The cylinder is now in brine ($\rho_{\text{brine}} = 1220\,\text{kg/m}^3$). New buoyant force $F_B' = \rho_{\text{brine}} g V = (1220)(9.81)(2.381 \times 10^{-4}) \approx 2.850\,\text{N}$. New tension $T'' = T - F_B' = 19.62 - 2.850 = 16.77\,\text{N}$. New fundamental frequency with length $L = 0.806$ m:

$$f_1'' = \frac{1}{2L}\sqrt{\frac{T''}{\mu}} = \frac{1}{2(0.806)}\sqrt{\frac{16.77}{0.0156}} \approx 20.33\,\text{Hz}$$

The frequency of the second overtone (3rd harmonic) is $f_3'' = 3f_1'' = 3 \times 20.33 \approx 61.0\,\text{Hz}$. The beat frequency with a 64 Hz piano note is:

$$f_{\text{beat}} = |f_3'' - f_{\text{piano}}| = |61.0 - 64| = \mathbf{3.0}\,\text{Hz}$$

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3. Electromagnetism

(a) Charge Flow in a Capacitor

Initial charge: $Q_i = C_i V = (6 \times 10^{-6}\,\text{F})(24\,\text{V}) = 144\,\mu\text{C}$. When the dielectric ($\kappa = 2.5$) fills half the space, the new capacitance is equivalent to two capacitors in parallel: one with dielectric and one with air.

$$C_f = C_1 + C_2 = \frac{\kappa\epsilon_0(A/2)}{d} + \frac{\epsilon_0(A/2)}{d} = \frac{\kappa + 1}{2}C_i$$

$$C_f = \frac{2.5+1}{2}(6\,\mu\text{F}) = 10.5\,\mu\text{F}$$

The battery is still connected, so V is constant. Final charge:

$$Q_f = C_f V = (10.5\times10^{-6}\,\text{F})(24\,\text{V}) = 252\,\mu\text{C}$$

The charge that flowed through the battery is:

$$\Delta Q = Q_f - Q_i = 252\,\mu\text{C} - 144\,\mu\text{C} = \mathbf{108}\,\mu\text{C}$$

(b) Inclination of a Charged Rod

For rotational equilibrium, the net torque about the center of the rod is zero. Electric torque τ_E:

$$\tau_E = \int_0^L (dF_E)(\text{lever arm}) = \int_0^L (\alpha x E\,dx)(x - L/2)\cos\theta$$

$$\tau_E = \alpha E\cos\theta\left[\frac{x^3}{3} - \frac{Lx^2}{4}\right]_0^L = \frac{\alpha E L^3}{12}\cos\theta$$

Spring torque $\tau_s = (F_B - F_A)\frac{L}{2}\cos\theta$. Equating torques: $F_B - F_A = \frac{\alpha E L^2}{6}$. The difference in spring forces is also related to the inclination angle θ: $F_B - F_A = k(\delta_B - \delta_A) = k(y_A - y_B) = kL\sin\theta$. Equating the two expressions for the force difference:

$$kL\sin\theta = \frac{\alpha E L^2}{6} \implies \sin\theta = \frac{\alpha E L}{6k}$$

$$\sin\theta = \frac{(0.9)(24)(0.6)}{6(20)} = 0.108$$

$$\theta = \arcsin(0.108) \approx \mathbf{6.20}^\circ$$

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4. Optics and Thermodynamics

(a) Telescope Focal Lengths

Case 1 (Normal adjustment): $f_o + f_e = 105\,\text{cm}$. Case 2 (Eyepiece moved): Distance between lenses is $L' = 104\,\text{cm}$. The object distance for the eyepiece is $u_e = L' - f_o = 104 - f_o$. The virtual image is at $v_e = -20\,\text{cm}$. Using the lens equation for the eyepiece:

$$\frac{1}{f_e} = \frac{1}{u_e} + \frac{1}{v_e} = \frac{1}{104 - f_o} - \frac{1}{20}$$

Substitute $f_o = 105 - f_e$:

$$\frac{1}{f_e} = \frac{1}{104 - (105 - f_e)} - \frac{1}{20} = \frac{1}{f_e - 1} - \frac{1}{20}$$

$$\frac{20 + f_e}{20 f_e} = \frac{1}{f_e - 1} \implies (20 + f_e)(f_e - 1) = 20 f_e \implies f_e^2 - f_e - 20 = 0$$

Factoring gives $(f_e - 5)(f_e + 4) = 0$. Since the eyepiece is a converging lens, $\mathbf{f_e} = \mathbf{5}\,\text{cm}$. Then, $\mathbf{f_o} = \mathbf{105} - \mathbf{5} = \mathbf{100}\,\text{cm}$.

(b) Surface Temperature of Mars

Solar intensity at Mars (I_M) is found using the inverse square law:

$$I_M = I_E\left(\frac{R_E}{R_M}\right)^2 = 1380\,\text{W/m}^2\left(\frac{1.496\times10^8}{2.280\times10^8}\right)^2 \approx 594\,\text{W/m}^2$$

At thermal equilibrium, power absorbed equals power radiated.

$$P_{\text{in}} = I_M(\pi r_M^2) \quad \text{and} \quad P_{\text{out}} = \sigma T_M^4 (4\pi r_M^2)$$

$$I_M \pi r_M^2 = 4\sigma T_M^4 \pi r_M^2 \implies T_M = \left(\frac{I_M}{4\sigma}\right)^{1/4}$$

$$T_M = \left(\frac{594}{4 \times 5.67 \times 10^{-8}}\right)^{1/4} \approx \mathbf{226}\,\text{K}$$

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5. Modern Physics

(a) Force from a Light Beam

Force is the rate of change of momentum. For a perfectly absorbing (blackened) plate, the momentum transferred per second is the total momentum of the photons arriving per second. Photon momentum $p = E/c$. The total energy per second is the power, P_{power}.

$$F = \frac{\Delta p}{\Delta t} = \frac{P_{\text{power}}}{c}$$

The power is $P_{\text{power}} = I \times A = (24\,\text{W/m}^2)(200 \times 10^{-4}\,\text{m}^2) = 0.48\,\text{W}$.

$$F = \frac{0.48\,\text{W}}{3.00 \times 10^8\,\text{m/s}} = \mathbf{1.6} \times \mathbf{10^{-9}}\,\text{N}$$

Assumption: The plate is a perfect absorber (blackbody).

(b) Radioactive Decay Chain

The quantity of nucleus B is maximum when its rate of change is zero: $\frac{dN_B}{dt} = 0$.

$$\frac{dN_B}{dt} = \lambda_A N_A(t) - \lambda_B N_B(t) = 0$$

At the time of maximum quantity, $t = T$: $\lambda_A N_A(T) = \lambda_B N_B(T)$. The solution for T is given by:

$$T = \frac{\ln(\lambda_A/\lambda_B)}{\lambda_A - \lambda_B}$$

With $t_{1/2,A} = 1$ hr and $t_{1/2,B} = 6$ hr, we have $\lambda_A = \ln(2)/1$ and $\lambda_B = \ln(2)/6$.

$$\frac{\lambda_A}{\lambda_B} = 6 \quad \text{and} \quad \lambda_A - \lambda_B = \frac{5}{6}\ln(2)$$

$$T = \frac{\ln(6)}{\frac{5}{6}\ln(2)} \approx \mathbf{3.10}\,\text{hours}$$

The maximum quantity $X = N_B(T)$. We use the relation $N_B(T) = \frac{\lambda_A}{\lambda_B} N_A(T) = 6N_A(T)$.

$$N_A(T) = N_0 e^{-\lambda_A T} = N_0 e^{-(\ln 2)T} = N_0 e^{-(\ln 2)\frac{6\ln 6}{5\ln 2}} = N_0 e^{-\frac{6}{5}\ln 6} = N_0(6^{-6/5})$$

$$X = 6N_A(T) = 6N_0(6^{-6/5}) = N_0 \cdot 6^{-1/5}$$

Given $N_0 = (3 \times 10^{-3}\,\text{mol}) \times (6.02 \times 10^{23}\,\text{mol}^{-1}) = 1.806 \times 10^{21}$.

$$X = \frac{1.806 \times 10^{21}}{6^{1/5}} \approx \frac{1.806 \times 10^{21}}{1.431} \approx \mathbf{1.26} \times \mathbf{10^{21}}\,\text{nuclei}$$