SPhO 2024 Theory Paper
Comprehensive Solutions

Contents

1 Simple Harmonic Motion

1.1 (a)(i) Determine the period of the motion.

The relationship between velocity v and displacement x for Simple Harmonic Motion (SHM) is given by:

$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

where ω is the angular frequency and x_0 is the amplitude.

From the provided $v - x$ graph, we identify:

- Maximum displacement (amplitude), $x_0 = 3$ m.
- Maximum velocity, $v_{\max} = 2$ m s^{-1}.

In SHM, the maximum velocity is $v_{\max} = \omega x_0$.

$$2 \text{ m s}^{-1} = \omega \times 3 \text{ m} \implies \omega = \frac{2}{3} \text{ rad s}^{-1}$$

The period of motion T is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2/3} = 3\pi \text{ s}$$

The period is 3π **s** (approximately 9.42 s).

1.2 (a)(ii) Calculate the time taken for the particle to move from point A to point B.

Let the motion be described by $x(t) = 3\sin(\frac{2}{3}t + \phi)$ and $v(t) = 2\cos(\frac{2}{3}t + \phi)$.

State A: $v_A = 1$ m s^{-1}. The corresponding displacement x_A is found using the ellipse equation $\frac{v^2}{v_{\max}^2} + \frac{x^2}{x_0^2} = 1$:

$$\frac{1^2}{2^2} + \frac{x_A^2}{3^2} = 1 \implies x_A = \frac{3\sqrt{3}}{2} \text{ m}$$

The phase $\theta_A = \frac{2}{3}t_A + \phi$ must satisfy:

$$x_A = \frac{3\sqrt{3}}{2} = 3\sin(\theta_A) \implies \sin(\theta_A) = \frac{\sqrt{3}}{2}$$
$$v_A = 1 = 2\cos(\theta_A) \implies \cos(\theta_A) = \frac{1}{2}$$

This gives $\theta_A = \frac{\pi}{3}$.

State B: $x_B = 0$ m and $v_B = -2$ m s^{-1}. The phase θ_B must satisfy:

$$x_B = 0 = 3\sin(\theta_B) \implies \sin(\theta_B) = 0$$
$$v_B = -2 = 2\cos(\theta_B) \implies \cos(\theta_B) = -1$$

This gives $\theta_B = \pi$.

The phase change is $\Delta\theta = \theta_B - \theta_A = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. The time taken is:

$$\Delta t = \frac{\Delta\theta}{\omega} = \frac{2\pi/3}{2/3} = \pi \text{ s}$$

The time taken is π **s** (approximately 3.14 s).

1.3 (b)(i) Show that the potential energy is $E_p = \frac{1}{2}m\omega^2 x^2$.

Total energy $E_T = K + E_p = \frac{1}{2}mv^2 + E_p$. Total energy is also the maximum kinetic energy, $E_T = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}m(\omega x_0)^2$. Using $v^2 = \omega^2(x_0^2 - x^2)$, the kinetic energy is $K(x) = \frac{1}{2}m\omega^2(x_0^2 - x^2)$. The potential energy is $E_p(x) = E_T - K(x)$:

$$E_p(x) = \frac{1}{2}m\omega^2 x_0^2 - \frac{1}{2}m\omega^2(x_0^2 - x^2) = \frac{1}{2}m\omega^2 x^2$$

1.4 (b)(ii) Show that the potential energy of the pendulum is $E = mgL - mg\sqrt{L^2 - x^2}$.

Let the equilibrium position be the reference for potential energy ($E_p = 0$). When displaced by a horizontal distance x, the bob is raised by a vertical height $h = L - \sqrt{L^2 - x^2}$. The potential energy E is mgh:

$$E = mg(L - \sqrt{L^2 - x^2}) = mgL - mg\sqrt{L^2 - x^2}$$

1.5 (b)(iii) Show that the period of a pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$ for small-angle oscillations.

For small angles, $x \ll L$. We use the binomial approximation $\sqrt{1-u} \approx 1 - \frac{u}{2}$ for small u.

$$E = mgL\left(1 - \sqrt{1 - \left(\frac{x}{L}\right)^2}\right) \approx mgL\left(1 - \left(1 - \frac{x^2}{2L^2}\right)\right) = \frac{mg}{2L}x^2$$

Comparing with $E_p = \frac{1}{2}m\omega^2 x^2$, we get:

$$\frac{1}{2}m\omega^2 x^2 = \frac{mg}{2L}x^2 \implies \omega^2 = \frac{g}{L} \implies \omega = \sqrt{\frac{g}{L}}$$

The period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$.

2 Rotational Motion and Friction

2.1 (a) Calculate the maximum static frictional force that can act on the block.

The normal force $N = mg = 2.5 \text{ kg} \times 9.81 \text{ m/s}^2 = 24.525 \text{ N}$. The maximum static frictional force $f_{s,max}$ is:

$$f_{s,max} = \mu_s N = 0.4 \times 24.525 \text{ N} = 9.81 \text{ N}$$

2.2 (b) Determine whether the block will slide or stay in place.

The required centripetal force is $F_c = m\omega^2 r$:

$$F_c = 2.5 \text{ kg} \times (3 \text{ rad/s})^2 \times 0.3 \text{ m} = 6.75 \text{ N}$$

Since $F_c = 6.75 \text{ N} < f_{s,max} = 9.81 \text{ N}$, the block will **stay in place**.

2.3 (c) Calculate the minimum coefficient of friction required for the block to stay in place at the edge.

At the edge, $r_{\text{edge}} = 0.5 \text{ m}$. The required centripetal force is:

$$F_{c,\text{edge}} = m\omega^2 r_{\text{edge}} = 2.5 \times (3)^2 \times 0.5 = 11.25 \text{ N}$$

For the block to stay, $f_{s,max,min} = \mu_{s,min} mg \geq F_{c,edge}$.

$$\mu_{s,min} \geq \frac{11.25 \text{ N}}{2.5 \text{ kg} \times 9.81 \text{ m/s}^2} \approx 0.4587$$

The minimum coefficient is approximately **0.459**.

2.4 (d) Determine the speed of the block when it begins to slip.

Slipping begins when $\frac{mv^2}{r} = f_{s,max} = \mu_s mg$.

$$v = \sqrt{\mu_s g r} = \sqrt{0.4 \times 9.81 \times 0.5} = \sqrt{1.962} \approx 1.40 \text{ m/s}$$

3 Spring and Conservation Laws

3.1 (a) Derive an expression for the speed of the sphere when the block is fixed.

By conservation of energy (initial spring potential energy to final kinetic energy):

$$\frac{1}{2}kd^2 = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{k}{m}}d$$

3.2 (b) Derive an expression for the speed of the sphere when the block is free to move.

Let v_m be the sphere's velocity and v_M be the block's velocity. Conservation of Momentum: $mv_m + Mv_M = 0 \implies v_M = -\frac{m}{M}v_m$. Conservation of Energy: $\frac{1}{2}kd^2 = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2$. Substituting for v_M:

$$\begin{aligned} kd^2 &= mv_m^2 + M\left(-\frac{m}{M}v_m\right)^2 = mv_m^2 + \frac{m^2}{M}v_m^2 \\ kd^2 &= v_m^2\left(m + \frac{m^2}{M}\right) = v_m^2\frac{m(M+m)}{M} \\ v_m &= d\sqrt{\frac{kM}{m(M+m)}} \end{aligned}$$

3.3 (c) Derive an expression for the distance the block has travelled.

The center of mass (CM) of the system remains stationary. Let the block's displacement be D_M and the sphere's be D_m. $mD_m + MD_M = 0$. The initial compression is the sum of the magnitudes of their displacements relative to the uncompressed position: $d = |D_m| + |D_M|$. From the first equation, $|D_m| = \frac{M}{m}|D_M|$. Substituting this into the second:

$$d = \frac{M}{m}|D_M| + |D_M| = |D_M|\left(\frac{M+m}{m}\right)$$

The distance travelled by the block is $|D_M| = \frac{md}{m+M}$.

4 Electromagnetic Induction

1. **Induced EMF ($\mathcal{E}$):** At distance x, the rod length is $L = 2x\tan(\frac{\alpha}{2})$. The loop area is $A = x^2\tan(\frac{\alpha}{2})$. The magnetic flux is $\Phi_B = Bx^2\tan(\frac{\alpha}{2})$.

$$|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right| = B\tan(\frac{\alpha}{2})\cdot 2x\frac{dx}{dt} = 2Bxv\tan(\frac{\alpha}{2})$$

2. **Induced Current (I):** The rod resistance is $R_{\text{rod}} = R\cdot L = 2Rx\tan(\frac{\alpha}{2})$.

$$I = \frac{|\mathcal{E}|}{R_{\text{rod}}} = \frac{2Bxv\tan(\frac{\alpha}{2})}{2Rx\tan(\frac{\alpha}{2})} = \frac{Bv}{R}$$

3. **Magnetic Braking Force (F_m):**

$$F_m = ILB = \left(\frac{Bv}{R}\right)\left(2x\tan(\frac{\alpha}{2})\right)B = \frac{2B^2vx\tan(\frac{\alpha}{2})}{R}$$

4. **Equation of Motion:** $ma = -F_m$. Use $a = v\frac{dv}{dx}$:

$$mv\frac{dv}{dx} = -\frac{2B^2vx\tan(\frac{\alpha}{2})}{R} \implies m\frac{dv}{dx} = -\frac{2B^2x\tan(\frac{\alpha}{2})}{R}$$

5. **Integration:** Integrate from initial state ($x = x_0, v = v_0$) to final state ($x = x_f, v = 0$).

$$\begin{aligned}
\int_{v_0}^{0} m\,dv &= \int_{x_0}^{x_f} -\frac{2B^2\tan(\frac{\alpha}{2})}{R}x\,dx \\
[mv]_{v_0}^{0} &= -\frac{2B^2\tan(\frac{\alpha}{2})}{R}\left[\frac{x^2}{2}\right]_{x_0}^{x_f} \\
-mv_0 &= -\frac{B^2\tan(\frac{\alpha}{2})}{R}(x_f^2 - x_0^2) \\
x_f^2 - x_0^2 &= \frac{mv_0R}{B^2\tan(\frac{\alpha}{2})} \\
x_f &= \sqrt{x_0^2 + \frac{mv_0R}{B^2\tan\left(\frac{\alpha}{2}\right)}}
\end{aligned}$$

5 Orbital Mechanics and Rotational Dynamics

5.1 (a)(i) Determine an expression for the minimum speed required to enter the transfer orbit.

The Hohmann transfer orbit has semi-major axis $a = \frac{R_1+R_2}{2}$. The speed v_p at periapsis ($r = R_1$) is given by the vis-viva equation:

$$v_p^2 = GM\left(\frac{2}{R_1} - \frac{1}{a}\right) = GM\left(\frac{2}{R_1} - \frac{2}{R_1+R_2}\right)$$

$$v_p^2 = 2GM\left(\frac{R_1+R_2-R_1}{R_1(R_1+R_2)}\right) = \frac{2GMR_2}{R_1(R_1+R_2)}$$

$$v_p = \sqrt{\frac{2GMR_2}{R_1(R_1+R_2)}}$$

5.2 (a)(ii) Show that the increase in total energy for the first burn is as given.

Initial energy in circular orbit: $E_1 = -\frac{GMm}{2R_1}$. Energy in transfer orbit: $E_t = -\frac{GMm}{2a} = -\frac{GMm}{R_1+R_2}$. Increase in energy:

$$\Delta E = E_t - E_1 = -\frac{GMm}{R_1+R_2} + \frac{GMm}{2R_1}$$

$$= GMm\left(\frac{1}{2R_1} - \frac{1}{R_1+R_2}\right) = GMm\left(\frac{R_2-R_1}{2R_1(R_1+R_2)}\right)$$

5.3 (a)(iii) Determine an expression for the time of flight for the transfer.

The transfer takes half the period T of the transfer orbit. By Kepler's Third Law:

$$T^2 = \frac{4\pi^2}{GM}a^3 = \frac{4\pi^2}{GM}\left(\frac{R_1+R_2}{2}\right)^3 = \frac{\pi^2(R_1+R_2)^3}{2GM}$$

The time of flight is:

$$t_{\text{flight}} = \frac{T}{2} = \frac{\pi}{2}\sqrt{\frac{(R_1+R_2)^3}{2GM}} = \pi\sqrt{\frac{(R_1+R_2)^3}{8GM}}$$

6 Electromagnetism and Dynamics

6.1 (a)(i) Determine the magnitude of the magnetic dipole moment.

The magnitude of the magnetic dipole moment μ is:

$$\mu = NIA = 10 \times 5.0 \text{ A} \times (0.80 \text{ m} \times 0.50 \text{ m}) = 20 \text{ A m}^2$$

6.2 (a)(ii) Determine the magnitude of the initial magnetic torque on the loop.

The torque is $\tau = \mu B \sin\theta$. Initially, the loop is horizontal, so its moment $\vec{\mu}$ is vertical. The field $\vec{B}$ is horizontal, so $\theta = 90°$.

$$\tau_{\text{mag}} = \mu B \sin(90°) = (20 \text{ A m}^2) \times (0.50 \text{ T}) = 10 \text{ N m}$$

6.3 (b) Calculate the net torque acting on the pulley.

Torque from the mass: $\tau_{\text{mass}} = rF_g = r(mg) = (0.10 \text{ m})(5.0 \text{ kg}\times 9.81 \text{ m/s}^2) = 4.905 \text{ N m}$. The torques are in opposite directions.

$$\tau_{\text{net}} = \tau_{\text{mag}} - \tau_{\text{mass}} = 10 - 4.905 = 5.095 \text{ N m} \approx 5.10 \text{ N m}$$

6.4 (c)(i) Calculate the change in potential energy of the magnetic dipole.

Potential energy is $U_{\text{mag}} = -\mu B \cos\theta$. Initial state: $\theta_i = 90° \implies U_i = 0$. Final state (upright): $\vec{\mu}$ aligns with $\vec{B}$, so $\theta_f = 0° \implies U_f = -\mu B \cos(0°) = -10$ J.

$$\Delta U_{\text{mag}} = U_f - U_i = -10 \text{ J}$$

6.5 (c)(ii) Calculate the speed of the mass at this position.

Use conservation of energy: $E_i = E_f$. $E_i = K_i + U_{g,i} + U_{\text{mag},i} = 0 + 0 + 0 = 0$. The loop rotates 90°, so the mass moves up by $h = r\Delta\theta = 0.10 \times \frac{\pi}{2} = 0.05\pi$ m. $E_f = K_f + U_{g,f} + U_{\text{mag},f} = \frac{1}{2}mv^2 + mgh + U_f$.

$$0 = \frac{1}{2}mv^2 + mgh + (-\mu B)$$
$$\frac{1}{2}(5.0)v^2 = 10 - (5.0)(9.81)(0.05\pi)$$
$$2.5v^2 \approx 10 - 7.705 = 2.295$$
$$v = \sqrt{\frac{2.295}{2.5}} \approx 0.958 \text{ m/s}$$

The speed is approximately **0.96 m/s**.

7 RLC Circuit

7.1 (a) Show $V = Ae^{-\alpha t}\cos(\omega t)$ is a solution and find ω.

Given: $\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0$. Substituting $V(t) = Ae^{-\alpha t}\cos(\omega t)$ and its derivatives into the equation yields:

$$\cos(\omega t)\left[\alpha^2 - \omega^2 - \frac{R}{L}\alpha + \frac{1}{LC}\right] + \sin(\omega t)\left[2\alpha\omega - \frac{R}{L}\omega\right] = 0$$

For this to be true for all t, the coefficients must be zero. From the $\sin(\omega t)$ term: $2\alpha\omega - \frac{R}{L}\omega = 0 \implies \alpha = \frac{R}{2L}$. From the $\cos(\omega t)$ term, substituting $\alpha = R/2L$:

$$\left(\frac{R}{2L}\right)^2 - \omega^2 - \frac{R}{L}\left(\frac{R}{2L}\right) + \frac{1}{LC} = 0$$

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \implies \omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

7.2 (b)(i) Calculate oscillation frequency and decay time.

Given: $C = 15.0\times10^{-9}$ F, $L = 0.22\times10^{-3}$ H, $R = 75.0\Omega$. $\alpha = \frac{R}{2L} = \frac{75.0}{2\times0.22\times10^{-3}} \approx 1.705\times 10^5\ \text{s}^{-1}$. $\omega = \sqrt{\frac{1}{(0.22\times10^{-3})(15.0\times10^{-9})} - (1.705\times10^5)^2} \approx 5.234\times10^5$ rad/s. Oscillation frequency $f = \frac{\omega}{2\pi} \approx \frac{5.234\times10^5}{2\pi} \approx 8.33\times10^4$ Hz $=$ **83.3 kHz**. Decay to 10%: $e^{-\alpha t} = 0.10 \implies t = \frac{\ln(10)}{\alpha} \approx \frac{2.3026}{1.705\times10^5} \approx 1.35\times10^{-5}$ s $=$ **13.5** μ**s**.

7.3 (b)(ii) Calculate the value of R for a critically damped circuit.

Critical damping occurs when $\omega = 0$.

$$\frac{1}{LC} = \left(\frac{R_{\text{crit}}}{2L}\right)^2 \implies R_{\text{crit}} = 2\sqrt{\frac{L}{C}}$$

$$R_{\text{crit}} = 2\sqrt{\frac{0.22\times10^{-3}}{15.0\times10^{-9}}} \approx 242.2\Omega$$

The resistance is approximately **242** Ω.

8 Special Relativity

8.1 (a) Calculate the time it takes for the spacecraft to travel from A to B as measured by the observer on Earth.

$$\Delta t_{\text{Earth}} = \frac{L_0}{v} = \frac{10 \text{ light-years}}{0.8c} = 12.5 \text{ years}$$

8.2 (b) Determine the time experienced by a clock on the spacecraft.

This is the proper time Δt_0. First, find the Lorentz factor γ:

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-0.8^2}} = \frac{1}{0.6} = \frac{5}{3}$$

$$\Delta t_0 = \frac{\Delta t_{\text{Earth}}}{\gamma} = \frac{12.5 \text{ years}}{5/3} = 7.5 \text{ years}$$

8.3 (c) How much time does it take for the signal to reach point B as measured by:

i. The observer on Earth. The light signal travels 10 light-years at speed c. Time taken is **10 years**.

ii. The astronaut on the spacecraft. Use the Lorentz transformation for time. Earth frame is S, spacecraft is S'. Event 1 (Signal sent from A): $(t_1, x_1) = (0, 0)$ in S. Event 2 (Signal reaches B): $(t_2, x_2) = (10 \text{ yr}, 10 \text{ ly})$ in S. Time of Event 2 in S':

$$t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) = \frac{5}{3}\left(10 \text{ yr} - \frac{(0.8c)(10 \text{ ly})}{c^2}\right) = \frac{5}{3}(10-8) \text{ yr} = \frac{10}{3} \text{ years}$$

Time taken is **3.33 years**.

8.4 (d) Do the events "spacecraft reaches point B" and "signal reaches point A" occur simultaneously according to the observer on Earth?

No.

- Spacecraft reaches B at $t_1 = 12.5$ years.
- The signal is sent from B at $t = 12.5$ years and travels 10 light-years back to A, which takes another 10 years. The signal reaches A at $t_2 = 12.5 + 10 = 22.5$ years.

Since $t_1 \neq t_2$, the events are not simultaneous.

9 Thermodynamics

9.1 (a) Calculate the total work done by the gas.

State A: $P_A = 101.3$ kPa, $T_A = 278.15$ K, $n = 1$ mole. Initial volume $V_A = \frac{nRT_A}{P_A} = \frac{1\times8.31\times278.15}{101.3\times10^3} \approx 0.02283\ \text{m}^3$.

Process A → B (isochoric): Volume is constant, so $W_{AB} = 0$. **Process B → C (isobaric):** $P_B = P_A/2$, $V_C = 2V_B = 2V_A$.

$$W_{BC} = P_B(V_C - V_B) = P_B V_B = \left(\frac{P_A}{2}\right) V_A = \frac{1}{2} nRT_A$$

$$W_{BC} = \frac{1}{2}(1)(8.31)(278.15) \approx 1155.7\ \text{J}$$

Total work $W_{\text{total}} = W_{AB} + W_{BC} = 1155.7$ J $\approx$ **1160 J**.

9.2 (b) Determine the work done by the gas for a process from state A to state C at constant temperature.

State C has $P_C = P_A/2$ and $V_C = 2V_A$, which implies $T_C = T_A$. Work done during an isothermal process:

$$W = \int_{V_A}^{V_C} P dV = nRT_A \ln\left(\frac{V_C}{V_A}\right)$$

$$W = nRT_A \ln(2) = (1)(8.31)(278.15)\ln(2) \approx 1601.9\ \text{J} \approx \mathbf{1600\ J}$$

10 Wave Optics

10.1 (a) Find an expression for the total path difference and the condition for maximum intensity.

Path difference before slits: $\Delta L_1 = d \sin \alpha$. Path difference after slits: $\Delta L_2 = d \sin \beta$. Total path difference ΔL:

$$\Delta L = \Delta L_2 - \Delta L_1 = d(\sin \beta - \sin \alpha)$$

Condition for maximum intensity (constructive interference):

$$d(\sin \beta - \sin \alpha) = m\lambda, \quad \text{where } m = 0, \pm 1, \pm 2, \ldots$$

10.2 (b) Show that the angular separation between adjacent maxima is independent of α for small β.

For the m-th and $(m+1)$-th maxima:

$$\sin \beta_m = \sin \alpha + \frac{m\lambda}{d}$$
$$\sin \beta_{m+1} = \sin \alpha + \frac{(m+1)\lambda}{d}$$

The difference is:

$$\sin \beta_{m+1} - \sin \beta_m = \frac{\lambda}{d}$$

For small angles β, we can use the approximation $\sin \beta_{m+1} - \sin \beta_m \approx \Delta\beta \cos \beta$, where $\Delta\beta = \beta_{m+1} - \beta_m$.

$$\Delta\beta \cos \beta \approx \frac{\lambda}{d}$$

For small values of β, $\cos \beta \approx 1$. Therefore, the angular separation is:

$$\Delta\beta \approx \frac{\lambda}{d}$$

This expression is independent of the angle of incidence α.