Solutions for SPhO 2019 Theory Paper 1

1. Projectile Motion

(a) Expression for Horizontal Separation

Let's establish a coordinate system where the launch point is the origin (0,0), the x-axis is horizontal, and the y-axis is vertical. The initial velocity is v_0 at an angle θ to the horizontal.

The equations of motion for the projectile are:

$$x(t) = (v_0 \cos\theta)t$$
$$y(t) = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

We want to find the times, t_1 and t_2, when the projectile is at a height h. We set $y(t) = h$:

$$h = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

Rearranging this gives a quadratic equation for t:

$$\frac{1}{2}gt^2 - (v_0 \sin\theta)t + h = 0$$

The two solutions to this equation, t_1 and t_2, are the times when the projectile passes the height h. Using the quadratic formula, $t = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$:

$$t = \frac{v_0 \sin\theta \pm \sqrt{(-v_0 \sin\theta)^2 - 4(\frac{1}{2}g)(h)}}{2(\frac{1}{2}g)} = \frac{v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta - 2gh}}{g}$$

Let t_1 be the earlier time and t_2 be the later time. The time interval between passing these two points is:

$$\Delta t = t_2 - t_1 = \frac{2\sqrt{v_0^2 \sin^2\theta - 2gh}}{g}$$

The horizontal separation, D, is the horizontal distance traveled during this time interval. Since the horizontal velocity $v_x = v_0 \cos\theta$ is constant:

$$D = v_x \Delta t = (v_0 \cos\theta)\left(\frac{2\sqrt{v_0^2 \sin^2\theta - 2gh}}{g}\right)$$

Thus, the expression for the horizontal separation is:

$$D = \frac{2v_0 \cos\theta}{g}\sqrt{v_0^2 \sin^2\theta - 2gh}$$

(b) Calculation for Maximum Range Conditions

Maximum range on a horizontal plane is achieved at a launch angle of $\theta = 45°$. Interpreting the "two points" as the start and end points of the trajectory on the horizontal plane means $h = 0$. Setting $h = 0$ in our derived expression for D:

$$D = \frac{2v_0 \cos\theta}{g}\sqrt{v_0^2 \sin^2\theta - 0} = \frac{2v_0 \cos\theta}{g}(v_0 \sin\theta) = \frac{v_0^2(2\sin\theta\cos\theta)}{g} = \frac{v_0^2 \sin(2\theta)}{g}$$

Now, we substitute the given values $v_0 = 200\,\text{m s}^{-1}$, $\theta = 45°$, and $g = 9.81\,\text{m s}^{-2}$:

$$D = R_{max} = \frac{v_0^2 \sin(2 \times 45°)}{g} = \frac{(200)^2 \sin(90°)}{9.81}$$

$$D = \frac{40000 \times 1}{9.81} \approx 4077.5\,\text{m}$$

The value of **D** is **4080 m** (to 3 significant figures).

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2. Thermodynamics

(a) Entropy Change of the Surroundings

The mass comes to rest at the equilibrium position, where the spring force balances the gravitational force. Let this downward displacement be x_{eq}.

$$kx_{eq} = mg \implies x_{eq} = \frac{mg}{k}$$

The change in mechanical energy, ΔE, is dissipated as heat Q into the surroundings.

$$\Delta E = E_f - E_i = \left(-\frac{m^2g^2}{k} + \frac{1}{2}\frac{m^2g^2}{k}\right) - 0 = -\frac{1}{2}\frac{m^2g^2}{k}$$

The heat transferred is $Q = |\Delta E| = \frac{1}{2}\frac{m^2g^2}{k}$. Using the given values $m = 0.25$ kg, $k = 20\,\text{N m}^{-1}$, $g = 9.81\,\text{m s}^{-2}$:

$$Q = \frac{1}{2}\frac{(0.25)^2(9.81)^2}{20} \approx 0.1504\,\text{J}$$

The entropy change of the surroundings at $T = 27°\text{C} = 300.15\,\text{K}$ is:

$$\Delta S_{surr} = \frac{Q}{T} = \frac{0.1504\,\text{J}}{300.15\,\text{K}} \approx 5.01 \times 10^{-4}\,\text{J K}^{-1}$$

(b) Surface Temperature of the Sun and Mars

(i) Surface Temperature of the Sun

The intensity I_{earth} at Earth's orbit is related to the Sun's temperature T_{sun} by:

$$I_{earth} = \sigma T_{sun}^4 \left(\frac{R_{sun}}{d_{earth}}\right)^2$$

Solving for T_{sun}:

$$T_{sun} = \left[\frac{I_{earth}}{\sigma}\left(\frac{d_{earth}}{R_{sun}}\right)^2\right]^{1/4}$$

Substituting values $I_{earth} = 1.37 \times 10^3\,\text{W m}^{-2}$, $\sigma = 5.67 \times 10^{-8}\,\text{W m}^{-2}\text{K}^{-4}$, $R_{sun} = 6.957 \times 10^8\,\text{m}$, $d_{earth} = 1.496 \times 10^{11}\,\text{m}$:

$$T_{sun} = \left[\frac{1.37 \times 10^3}{5.67 \times 10^{-8}}\left(\frac{1.496 \times 10^{11}}{6.957 \times 10^8}\right)^2\right]^{1/4} \approx 5780\,\text{K}$$

(ii) Equilibrium Temperature of Mars

At equilibrium, power absorbed equals power radiated. For Mars, assuming it's a blackbody:

$$I_{mars}(\pi R_{mars}^2) = \sigma(4\pi R_{mars}^2)T_{mars}^4$$

where $I_{mars} = I_{earth} \left(\frac{d_{earth}}{d_{mars}}\right)^2$.

$$I_{earth} \left(\frac{d_{earth}}{d_{mars}}\right)^2 = 4\sigma T_{mars}^4$$

Solving for T_{mars}:

$$T_{mars} = \left[\frac{I_{earth}}{4\sigma} \left(\frac{d_{earth}}{d_{mars}}\right)^2\right]^{1/4}$$

Substituting $d_{mars} = 2.280 \times 10^8\,\mathrm{km} = 2.280 \times 10^{11}$ m:

$$T_{mars} = \left[\frac{1.37 \times 10^3}{4 \times 5.67 \times 10^{-8}} \left(\frac{1.496 \times 10^{11}}{2.280 \times 10^{11}}\right)^2\right]^{1/4} \approx 226\,\mathrm{K}$$

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3. Simple Harmonic Motion

(a) Oscillating Particle

The force $F = -10x$ implies an effective spring constant $k = 10\,\mathrm{N/m}$. The angular frequency is:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{0.1}} = 10\,\mathrm{rad\ s^{-1}}$$

Initial conditions at $t = 0$: $x(0) = 0.05$ m, $v(0) = +\frac{\sqrt{3}}{2}$ m s^{-1}.

(i) Amplitude of the motion

Using the relation $v^2 = \omega^2(A^2 - x^2)$:

$$\left(\frac{\sqrt{3}}{2}\right)^2 = (10)^2(A^2 - (0.05)^2) \implies \frac{3}{4} = 100(A^2 - 0.0025)$$

$$0.0075 = A^2 - 0.0025 \implies A^2 = 0.01 \implies A = 0.1\,\mathrm{m}$$

(ii) Initial phase angle

Using $x(t) = A\cos(\omega t + \phi)$ and $v(t) = -A\omega\sin(\omega t + \phi)$:

$$0.05 = 0.1\cos(\phi) \implies \cos(\phi) = \frac{1}{2}$$

$$\frac{\sqrt{3}}{2} = -(0.1)(10)\sin(\phi) \implies \sin(\phi) = -\frac{\sqrt{3}}{2}$$

The angle is in the fourth quadrant, so $\phi = -\frac{\pi}{3}$ rad.

(iii) Maximum speed and acceleration

$$v_{max} = A\omega = (0.1)(10) = 1\,\mathrm{m\ s^{-1}}$$
$$a_{max} = A\omega^2 = (0.1)(10)^2 = 10\,\mathrm{m\ s^{-2}}$$

(b) Floating Rod

At equilibrium, weight equals buoyant force. Let the rod have length L, area A, density ρ_r, and the liquid density be ρ_l. The submerged length is $L - h$.

$$(AL\rho_r)g = A(L-h)\rho_l g$$

When displaced down by x, the net restoring force is:

$$F_{net} = -(A\rho_l g)x$$

This is SHM with $k_{eff} = A\rho_l g$. The period is:

$$T = 2\pi\sqrt{\frac{m_{rod}}{k_{eff}}} = 2\pi\sqrt{\frac{AL\rho_r}{A\rho_l g}} = 2\pi\sqrt{\frac{L\rho_r}{g\rho_l}}$$

From equilibrium, $\frac{\rho_r}{\rho_l} = \frac{L-h}{L}$. Substituting this in:

$$T = 2\pi\sqrt{\frac{L}{g}\left(\frac{L-h}{L}\right)} = 2\pi\sqrt{\frac{L-h}{g}}$$

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4. Electromagnetism

(a) Merging Oil Drops

For a single drop with radius r and charge q, the potential is $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} = 1000\,\mathrm{V}$. When two drops merge, the new charge is $Q_{new} = 2q$. The new volume is doubled, so the new radius is $R = 2^{1/3}r$. The new potential is:

$$V_{new} = \frac{1}{4\pi\epsilon_0}\frac{Q_{new}}{R} = \frac{1}{4\pi\epsilon_0}\frac{2q}{2^{1/3}r} = \frac{2}{2^{1/3}}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)$$

$$V_{new} = 2^{2/3}V = 2^{2/3} \times 1000\,\mathrm{V} \approx 1587\,\mathrm{V}$$

(b) Bainbridge Mass Spectrometer

(i) Speed of the ion

For an ion to pass through the velocity filter undeflected, $qE = qvB$.

$$v = \frac{E}{B} = \frac{100\,\mathrm{V\ cm^{-1}}}{0.2\,\mathrm{T}} = \frac{10000\,\mathrm{V\ m^{-1}}}{0.2\,\mathrm{T}} = 50000\,\mathrm{m\ s^{-1}}$$

(ii) Resolution of isotopes

In the magnetic field, the ions travel in a semicircle of radius $r = \frac{mv}{qB}$. They strike the detector at a distance $d = 2r = \frac{2mv}{qB}$. For singly ionized ^{3}He ($m_3 \approx 3\times1.67\times10^{-27}$ kg) and ^{4}He ($m_4 \approx 4\times1.67\times10^{-27}$ kg):

$$d_3 = \frac{2(5.01\times10^{-27})(5.0\times10^4)}{(1.60\times10^{-19})(0.2)} \approx 0.01566\,\mathrm{m} = 15.66\,\mathrm{mm}$$

$$d_4 = \frac{2(6.68\times10^{-27})(5.0\times10^4)}{(1.60\times10^{-19})(0.2)} \approx 0.02088\,\mathrm{m} = 20.88\,\mathrm{mm}$$

The separation is $\Delta d = d_4 - d_3 = 5.22\,\mathrm{mm}$. Since the separation (5.22 mm) is greater than the slit width (1 mm), yes, the machine can resolve the two isotopes.

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5. Radiation and Atomic Physics

(a) Radiation Pressure

For a perfectly reflecting surface, the change in momentum of light is doubled. The force exerted is $F = \frac{2IA}{c}$, and the pressure is:

$$P = \frac{F}{A} = \frac{2I}{c}$$

Given $I = 50\,\text{W m}^{-2}$ and $c = 3.00 \times 10^8\,\text{m s}^{-1}$:

$$P = \frac{2 \times 50}{3.00 \times 10^8} = \frac{100}{3.00 \times 10^8} \approx 3.33 \times 10^{-7}\,\text{Pa}$$

(b) Positronium Lyman Series

Positronium consists of an electron and a positron, so its reduced mass is $\mu_p = \frac{m_e \cdot m_e}{m_e + m_e} = \frac{m_e}{2}$. The energy levels are proportional to the reduced mass, so they are half that of hydrogen:

$$E_{n,p} = \frac{1}{2}E_{n,H} = \frac{1}{2}\left(-\frac{13.6\,\text{eV}}{n^2}\right) = -\frac{6.8\,\text{eV}}{n^2}$$

The Lyman series involves transitions to the $n = 1$ state. The shortest wavelength corresponds to the highest energy transition, from $n_i = \infty$ to $n_f = 1$. The energy of the emitted photon is:

$$\Delta E = E_\infty - E_1 = 0 - (-6.8\,\text{eV}) = 6.8\,\text{eV}$$

The wavelength is calculated from $\Delta E = \frac{hc}{\lambda}$:

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.63 \times 10^{-34}\,\text{J s})(3.00 \times 10^8\,\text{m s}^{-1})}{6.8\,\text{eV} \times (1.60 \times 10^{-19}\,\text{J/eV})}$$

$$\lambda \approx 1.828 \times 10^{-7}\,\text{m} = 182.8\,\text{nm}$$

The shortest wavelength is **183 nm**.