Solutions for the 34th Singapore Physics Olympiad Theory Paper

1. Parallel-Plate Capacitor

This problem can be solved by modeling the capacitor as a combination of three separate capacitors.

- The left half of the device can be treated as a single capacitor, C_1.
- The right half consists of two capacitors stacked vertically, in a **series** combination, C_2 and C_3.
- The left half (C_1) and the right half (the series combination of C_2 and C_3) are in **parallel**.

The total capacitance C_{eq} is the sum of the capacitance of the left half and the equivalent capacitance of the right half (C_{23}).

$$C_{eq} = C_1 + C_{23}$$

where C_{23} is given by the formula for capacitors in series:

$$\frac{1}{C_{23}} = \frac{1}{C_2} + \frac{1}{C_3}$$

Step-by-step Calculation

1. **Calculate the capacitance of each section.** The formula for a parallel-plate capacitor is $C = \frac{\kappa\epsilon_0 A_{\text{plate}}}{d_{\text{separation}}}$.

   - For C_1: The area is $A/2$, the separation is $2d$, and the dielectric constant is κ_1.

   $$C_1 = \frac{\kappa_1\epsilon_0(A/2)}{2d} = \frac{\kappa_1\epsilon_0 A}{4d}$$

   - For C_2: The area is $A/2$, the separation is d, and the dielectric constant is κ_2.

   $$C_2 = \frac{\kappa_2\epsilon_0(A/2)}{d} = \frac{\kappa_2\epsilon_0 A}{2d}$$

   - For C_3: The area is $A/2$, the separation is d, and the dielectric constant is κ_3.

   $$C_3 = \frac{\kappa_3\epsilon_0(A/2)}{d} = \frac{\kappa_3\epsilon_0 A}{2d}$$

2. **Calculate the equivalent capacitance of the right half (C_{23}).** Since C_2 and C_3 are in series:

   $$\begin{aligned}\frac{1}{C_{23}} &= \frac{1}{C_2} + \frac{1}{C_3} = \frac{2d}{\kappa_2\epsilon_0 A} + \frac{2d}{\kappa_3\epsilon_0 A} \\ &= \frac{2d}{\epsilon_0 A}\left(\frac{1}{\kappa_2} + \frac{1}{\kappa_3}\right) = \frac{2d}{\epsilon_0 A}\left(\frac{\kappa_3+\kappa_2}{\kappa_2\kappa_3}\right) \\ C_{23} &= \frac{\epsilon_0 A}{2d}\left(\frac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right)\end{aligned}$$

3. **Calculate the total capacitance (C_{eq}).** Since C_1 and C_{23} are in parallel:

   $$C_{eq} = C_1 + C_{23} = \frac{\kappa_1\epsilon_0 A}{4d} + \frac{\epsilon_0 A}{2d}\left(\frac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right)$$

   We can factor out $\frac{\epsilon_0 A}{2d}$:

   $$C_{eq} = \frac{\epsilon_0 A}{2d}\left(\frac{\kappa_1}{2} + \frac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right)$$

4. **Substitute the given values.** Given: $A = 10.5\,\mathrm{cm}^2 = 10.5 \times 10^{-4}\,\mathrm{m}^2$, $2d = 7.12\,\mathrm{mm} = 7.12 \times 10^{-3}\,\mathrm{m}$, $\kappa_1 = 21.0$, $\kappa_2 = 42.0$, $\kappa_3 = 58.0$, $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}$.

$$\begin{aligned}
\frac{\kappa_1}{2} &= \frac{21.0}{2} = 10.5 \\
\frac{\kappa_2\kappa_3}{\kappa_2 + \kappa_3} &= \frac{42.0 \times 58.0}{42.0 + 58.0} = \frac{2436}{100} = 24.36 \\
\frac{\epsilon_0 A}{2d} &= \frac{(8.85 \times 10^{-12})(10.5 \times 10^{-4})}{7.12 \times 10^{-3}} \approx 1.305 \times 10^{-12}\,\mathrm{F} \\
C_{eq} &= (1.305 \times 10^{-12})(10.5 + 24.36) = (1.305 \times 10^{-12})(34.86) \\
C_{eq} &\approx 4.55 \times 10^{-11}\,\mathrm{F} = 45.5\,\mathrm{pF}
\end{aligned}$$