

# 1. Projectile Motion

## (a) Angle of Inclination

Let the initial velocity be  $v_0 = 50 \text{ m/s}$  and the angle of inclination be  $\theta$ . The equations for horizontal and vertical motion are:

$$x(t) = (v_0 \cos \theta)t$$

$$y(t) = y_0 + (v_0 \sin \theta)t - \frac{1}{2}gt^2$$

Given  $y_0 = 100 \text{ m}$ ,  $g = 9.81 \text{ m/s}^2$ , and the target is at  $(300, 0)$ . From the horizontal motion,  $300 = (50 \cos \theta)t$ , which gives the time of flight  $t = \frac{6}{\cos \theta}$ . Substituting this into the vertical motion equation:

$$0 = 100 + (50 \sin \theta) \left( \frac{6}{\cos \theta} \right) - \frac{1}{2}(9.81) \left( \frac{6}{\cos \theta} \right)^2$$

$$0 = 100 + 300 \tan \theta - 176.58 \sec^2 \theta$$

Using the identity  $\sec^2 \theta = 1 + \tan^2 \theta$ :

$$0 = 100 + 300 \tan \theta - 176.58(1 + \tan^2 \theta)$$

Let  $u = \tan \theta$ . This leads to the quadratic equation:

$$176.58u^2 - 300u + 76.58 = 0$$

Solving for  $u$ :

$$u = \frac{300 \pm \sqrt{300^2 - 4(176.58)(76.58)}}{2(176.58)} = \frac{300 \pm 189.62}{353.16}$$

This yields two solutions:

- $u_1 = 1.386 \implies \theta_1 = \arctan(1.386) \approx 54.2^\circ$
- $u_2 = 0.3126 \implies \theta_2 = \arctan(0.3126) \approx 17.4^\circ$

## (b) Speed of Projection for Moving Target

The target's position is now  $x_T(t) = 300 + 10t$ . Let the new projectile speed be  $v'$ . A hit occurs at time  $t_{hit}$  when:

$$(v' \cos \theta)t_{hit} = 300 + 10t_{hit}$$

$$0 = 100 + (v' \sin \theta)t_{hit} - \frac{1}{2}gt_{hit}^2$$

From the first equation,  $v' \sin \theta = \tan \theta(v' \cos \theta) = \tan \theta \left( \frac{300}{t_{hit}} + 10 \right)$ . Substituting this into the second equation gives a quadratic for  $t_{hit}$ :

$$4.905t_{hit}^2 - (10 \tan \theta)t_{hit} - (100 + 300 \tan \theta) = 0$$

**Case 1:**  $\theta = 54.2^\circ$  ( $\tan \theta = 1.386$ )

$$4.905t_{hit}^2 - 13.86t_{hit} - 515.8 = 0 \implies t_{hit} \approx 11.76 \text{ s}$$

The required speed is  $v' = \left( \frac{300}{11.76} + 10 \right) \frac{1}{\cos(54.2^\circ)} \approx 60.7 \text{ m/s}$ .

**Case 2:**  $\theta = 17.4^\circ$  ( $\tan \theta = 0.3126$ )

$$4.905t_{hit}^2 - 3.126t_{hit} - 193.78 = 0 \implies t_{hit} \approx 6.61 \text{ s}$$

The required speed is  $v' = \left( \frac{300}{6.61} + 10 \right) \frac{1}{\cos(17.4^\circ)} \approx 58.1 \text{ m/s}$ .

## 2. Sound Waves

### (a) Interference of Sound Waves

The path difference is  $\Delta P = 2\sqrt{60^2 + d^2} - 120$ . For the first maximum to recur after moving a distance  $h$ , the path difference must change by one wavelength  $\lambda = 1.33 \text{ m}$ .

$$\begin{aligned}\Delta P_2 - \Delta P_1 &= \lambda \\ \left(2\sqrt{60^2 + (90+h)^2} - 120\right) - \left(2\sqrt{60^2 + 90^2} - 120\right) &= 1.33 \\ \sqrt{3600 + (90+h)^2} &= \sqrt{11700} + \frac{1.33}{2} \approx 108.832 \\ 3600 + (90+h)^2 &\approx 11844.4 \implies (90+h)^2 \approx 8244.4 \\ 90+h &\approx 90.80 \implies h \approx 0.80 \text{ m}\end{aligned}$$

### (b) Sonometer Wire

The frequency of the second harmonic ( $n = 2$ ) is  $f_2 = \frac{1}{L} \sqrt{\frac{T}{\mu}}$ . The linear mass density is  $\mu = \pi r_w^2 \rho_w = \pi (2.55 \times 10^{-4})^2 (8.885 \times 10^3) \approx 1.820 \times 10^{-3} \text{ kg/m}$ . The tensions are  $T_1 = (\rho_c - 0.5\rho_l)V_cg$  (half immersed) and  $T_2 = (\rho_c - \rho_l)V_cg$  (fully immersed). The ratio of frequencies gives a relationship between the densities:

$$\frac{\rho_c - 0.5\rho_l}{\rho_c - \rho_l} = \left(\frac{f_1}{f_2}\right)^2 = \left(\frac{118.4}{114.7}\right)^2 \approx 1.0656$$

This simplifies to  $\rho_c \approx 8.622\rho_l$ . From the first case,  $T_1 = \mu(f_1 L)^2 \approx 9.185 \text{ N}$ .

$$T_1 = (\rho_c - 0.5\rho_l)V_cg \implies \rho_c - 0.5\rho_l = \frac{T_1}{V_cg} \approx 4769 \text{ kg/m}^3$$

Solving the system of two equations:

$$\begin{aligned}8.622\rho_l - 0.5\rho_l &= 4769 \implies \rho_l \approx 587 \text{ kg/m}^3 \\ \rho_c &= 8.622 \times 587 \approx 5061 \text{ kg/m}^3\end{aligned}$$

## 3. Gravitation and Thermodynamics

### (a) Gravitational Potential of a Hollow Sphere

The potential on the outer surface is  $V_{outer} = -\frac{GM}{R} = -\frac{G}{R} \frac{4}{3}\pi\rho(R^3 - r^3)$ . The potential on the inner surface is constant and equal to the potential at the center,  $V_{inner} = \int_r^R -G\frac{dM}{x} = \int_r^R -4\pi G\rho x dx = -2\pi G\rho(R^2 - r^2)$ . The ratio is:

$$\frac{V_{outer}}{V_{inner}} = \frac{-\frac{4\pi G\rho}{3R}(R^3 - r^3)}{-2\pi G\rho(R^2 - r^2)} = \frac{2(R^3 - r^3)}{3R(R^2 - r^2)} = \frac{2(R^2 + Rr + r^2)}{3R(R + r)}$$

### (b) Thermodynamics and Atomic Physics

#### (i) Rate of Change of Entropy

The rate of heat flow is  $P = kA \frac{T_H - T_C}{L} = (400)(\pi \times 10^{-4}) \frac{200}{0.30} \approx 83.78 \text{ W}$ . The rate of change of entropy for the system (reservoirs + rod) is:

$$\begin{aligned}\frac{dS_{sys}}{dt} &= \frac{dS_H}{dt} + \frac{dS_C}{dt} + \frac{dS_{rod}}{dt} = -\frac{P}{T_H} + \frac{P}{T_C} + 0 \\ \frac{dS_{sys}}{dt} &= 83.78 \left( \frac{1}{273.15} - \frac{1}{473.15} \right) \approx 0.130 \text{ J K}^{-1}\text{s}^{-1}\end{aligned}$$

#### (ii) Wavelengths Emitted by Hydrogen Atom

The electron's kinetic energy is  $K_e = \frac{1}{2}m_e v^2 \approx 12.3 \text{ eV}$ . Assuming the H atom is in the ground state ( $n = 1, E_1 = -13.6 \text{ eV}$ ), the collision can excite it to states where the excitation energy  $\Delta E \leq 12.3 \text{ eV}$ .

- $\Delta E_{1 \rightarrow 2} = 10.2 \text{ eV}$  (Possible)
- $\Delta E_{1 \rightarrow 3} = 12.09 \text{ eV}$  (Possible)
- $\Delta E_{1 \rightarrow 4} = 12.75 \text{ eV}$  (Not possible)

The possible photon energies emitted during de-excitation are 12.09 eV ( $3 \rightarrow 1$ ), 10.2 eV ( $2 \rightarrow 1$ ), and 1.89 eV ( $3 \rightarrow 2$ ). The corresponding wavelengths ( $\lambda = hc/E$ ) are:

- $\lambda_1 = \frac{1240}{12.09} \approx 102.6 \text{ nm}$
- $\lambda_2 = \frac{1240}{10.2} \approx 121.6 \text{ nm}$
- $\lambda_3 = \frac{1240}{1.89} \approx 656.1 \text{ nm}$

## 4. Electromagnetism

### (a) Oscillation of a Charge in a Ring's Field

The force on the charge  $-q$  is  $F_x = -qE_x = -\frac{qQx}{4\pi\epsilon_0(a^2+x^2)^{3/2}}$ . For small displacements ( $x \ll a$ ), this becomes a linear restoring force  $F_x \approx -\left(\frac{qQ}{4\pi\epsilon_0 a^3}\right)x = -kx$ . The frequency of this Simple Harmonic Motion is  $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ . Substituting  $k$  and  $Q = 2\pi a \lambda$ :

$$f = \frac{1}{2\pi}\sqrt{\frac{q(2\pi a \lambda)}{4\pi\epsilon_0 m a^3}} = \frac{1}{2\pi}\sqrt{\frac{q\lambda}{2\epsilon_0 m a^2}}$$

### (b) Magnetic Fields

#### (i) Magnetic Field at the Proton in a Hydrogen Atom

The orbiting electron creates a current  $I = \frac{ev}{2\pi r}$ . The magnetic field at the center is:

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 ev}{4\pi r^2}$$

$$B = \frac{(4\pi \times 10^{-7})(1.60 \times 10^{-19})(6.91 \times 10^5)}{4\pi(5.3 \times 10^{-10})^2} \approx 0.0394 \text{ T}$$

#### (ii) Magnetic Field at the Center of a Spinning Disc

A ring of radius  $r$  and width  $dr$  carries a current  $dI = (dq)n = (\sigma 2\pi r dr)n$ . This produces a field  $dB = \frac{\mu_0 dI}{2r} = \mu_0 \pi \sigma n dr$ . Integrating from  $r = 0$  to  $R$ :

$$B = \int_0^R \mu_0 \pi \sigma n dr = \mu_0 \pi \sigma n R$$

## 5. Special Relativity

### (a) Events in Different Inertial Frames

#### (i) Velocity of S' frame

For two events to occur at the same location in S',  $\Delta x' = x'_2 - x'_1 = 0$ . Using the Lorentz transformation  $x' = \gamma(x - vt)$ :

$$0 = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) = \gamma(4c - 5v)$$

This implies  $4c - 5v = 0$ , so  $v = \frac{4}{5}c = 0.8c$ .

#### (ii) Time Interval in S' frame

The Lorentz factor is  $\gamma = (1 - 0.8^2)^{-1/2} = 5/3$ . The time interval in S' is:

$$\Delta t' = t'_2 - t'_1 = \gamma \left( t_2 - \frac{vx_2}{c^2} \right) - \gamma \left( t_1 - \frac{vx_1}{c^2} \right)$$

$$\Delta t' = \frac{5}{3} \left( 5 - \frac{(0.8c)(4c)}{c^2} \right) = \frac{5}{3}(5 - 3.2) = 3 \text{ s}$$

### (b) Volume of a Moving Cube

The velocity of the cube relative to the observer is  $w = \frac{u-v}{1-uv/c^2}$ . The observer measures a volume affected by length contraction in the direction of motion:  $V' = l^3 \sqrt{1 - w^2/c^2}$ . The term under the square root simplifies as:

$$1 - \frac{w^2}{c^2} = 1 - \frac{(u-v)^2}{c^2(1-uv/c^2)^2} = \frac{(1-u^2/c^2)(1-v^2/c^2)}{(1-uv/c^2)^2}$$

Therefore, the measured volume is:

$$V' = l^3 \frac{\sqrt{(1-u^2/c^2)(1-v^2/c^2)}}{1-uv/c^2}$$