Solutions for the 34th Singapore Physics Olympiad Theory Paper

1. Parallel-Plate Capacitor

This problem can be solved by modeling the capacitor as a combination of three separate capacitors.

- The left half of the device can be treated as a single capacitor, C_1.
- The right half consists of two capacitors stacked vertically, in a **series** combination, C_2 and C_3.
- The left half (C_1) and the right half (the series combination of C_2 and C_3) are in **parallel**.

The total capacitance C_{eq} is the sum of the capacitance of the left half and the equivalent capacitance of the right half (C_{23}).

$$C_{eq} = C_1 + C_{23}$$

where C_{23} is given by the formula for capacitors in series:

$$\frac{1}{C_{23}} = \frac{1}{C_2} + \frac{1}{C_3}$$

Step-by-step Calculation

1. **Calculate the capacitance of each section.** The formula for a parallel-plate capacitor is $C = \frac{\kappa\epsilon_0 A_{\text{plate}}}{d_{\text{separation}}}$.
   - For C_1: The area is $A/2$, the separation is $2d$, and the dielectric constant is κ_1.
   $$C_1 = \frac{\kappa_1\epsilon_0(A/2)}{2d} = \frac{\kappa_1\epsilon_0 A}{4d}$$
   - For C_2: The area is $A/2$, the separation is d, and the dielectric constant is κ_2.
   $$C_2 = \frac{\kappa_2\epsilon_0(A/2)}{d} = \frac{\kappa_2\epsilon_0 A}{2d}$$
   - For C_3: The area is $A/2$, the separation is d, and the dielectric constant is κ_3.
   $$C_3 = \frac{\kappa_3\epsilon_0(A/2)}{d} = \frac{\kappa_3\epsilon_0 A}{2d}$$

2. **Calculate the equivalent capacitance of the right half (C_{23}).** Since C_2 and C_3 are in series:
$$\begin{aligned}\frac{1}{C_{23}} &= \frac{1}{C_2} + \frac{1}{C_3} = \frac{2d}{\kappa_2\epsilon_0 A} + \frac{2d}{\kappa_3\epsilon_0 A}\\ &= \frac{2d}{\epsilon_0 A}\left(\frac{1}{\kappa_2} + \frac{1}{\kappa_3}\right) = \frac{2d}{\epsilon_0 A}\left(\frac{\kappa_3 + \kappa_2}{\kappa_2\kappa_3}\right)\\ C_{23} &= \frac{\epsilon_0 A}{2d}\left(\frac{\kappa_2\kappa_3}{\kappa_2 + \kappa_3}\right)\end{aligned}$$

3. **Calculate the total capacitance (C_{eq}).** Since C_1 and C_{23} are in parallel:
$$C_{eq} = C_1 + C_{23} = \frac{\kappa_1\epsilon_0 A}{4d} + \frac{\epsilon_0 A}{2d}\left(\frac{\kappa_2\kappa_3}{\kappa_2 + \kappa_3}\right)$$
   We can factor out $\frac{\epsilon_0 A}{2d}$:
$$C_{eq} = \frac{\epsilon_0 A}{2d}\left(\frac{\kappa_1}{2} + \frac{\kappa_2\kappa_3}{\kappa_2 + \kappa_3}\right)$$

4. **Substitute the given values.** Given: $A = 10.5\,\text{cm}^2 = 10.5 \times 10^{-4}\,\text{m}^2$, $2d = 7.12\,\text{mm} = 7.12 \times 10^{-3}\,\text{m}$, $\kappa_1 = 21.0$, $\kappa_2 = 42.0$, $\kappa_3 = 58.0$, $\epsilon_0 = 8.85 \times 10^{-12}\,\text{F/m}$.

$$\begin{aligned}
\frac{\kappa_1}{2} &= \frac{21.0}{2} = 10.5 \\
\frac{\kappa_2\kappa_3}{\kappa_2+\kappa_3} &= \frac{42.0 \times 58.0}{42.0 + 58.0} = \frac{2436}{100} = 24.36 \\
\frac{\epsilon_0 A}{2d} &= \frac{(8.85 \times 10^{-12})(10.5 \times 10^{-4})}{7.12 \times 10^{-3}} \approx 1.305 \times 10^{-12}\,\text{F} \\
C_{eq} &= (1.305 \times 10^{-12})(10.5 + 24.36) = (1.305 \times 10^{-12})(34.86) \\
C_{eq} &\approx 4.55 \times 10^{-11}\,\text{F} = 45.5\,\text{pF}
\end{aligned}$$

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2. Electric Potential and Work

The electric potential V from a single point charge q at a distance r is $V = k\frac{q}{r}$, where $k \approx 8.99 \times 10^9\,\text{N} \cdot \text{m}^2/\text{C}^2$.

(a) Electric Potential at Corner A

At corner A (0, 0), the distances are $r_{1A} = 0.050\,\text{m}$ and $r_{2A} = 0.15\,\text{m}$.

$$V_A = k\left(\frac{q_1}{r_{1A}} + \frac{q_2}{r_{2A}}\right) = (8.99 \times 10^9)\left(\frac{-5.0 \times 10^{-6}}{0.050} + \frac{+2.0 \times 10^{-6}}{0.15}\right)$$

$$V_A = (8.99 \times 10^9)(-1.0 \times 10^{-4} + 0.1333 \times 10^{-4}) \approx -7.79 \times 10^5\,\text{V}$$

(b) Electric Potential at Corner B

At corner B, the distances are $r_{1B} = 0.15\,\text{m}$ and $r_{2B} = 0.050\,\text{m}$.

$$V_B = k\left(\frac{q_1}{r_{1B}} + \frac{q_2}{r_{2B}}\right) = (8.99 \times 10^9)\left(\frac{-5.0 \times 10^{-6}}{0.15} + \frac{+2.0 \times 10^{-6}}{0.050}\right)$$

$$V_B = (8.99 \times 10^9)(-0.3333 \times 10^{-4} + 0.4 \times 10^{-4}) \approx 5.99 \times 10^4\,\text{V}$$

(c) Work Required to Move Charge q_3

The work W required is $W = q_3(V_A - V_B)$.

$$W = (3.0 \times 10^{-6})(-7.79 \times 10^5 - 5.99 \times 10^4) = (3.0 \times 10^{-6})(-8.389 \times 10^5) \approx -2.52\,\text{J}$$

(d) Change in Potential Energy

The work done by an external force equals the change in potential energy, $W = \Delta U_E$. Since $W < 0$, the potential energy of the system **decreases**.

(e) & (f) Path Independence

The electrostatic force is a **conservative force**, so the work done is independent of the path taken. The work required is the **same**.

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3. Motional EMF and Lenz's Law

(a) Magnitude of the Induced EMF

The motional EMF is $\mathcal{E} = BLv$.

$$\mathcal{E} = (1.2\,\text{T})(0.10\,\text{m})(5.0\,\text{m/s}) = \mathbf{0.60}\,\text{V}$$

(b) Direction of the Induced EMF

By the right-hand rule for Lorentz force, $\vec{F} = q(\vec{v} \times \vec{B})$, positive charges are pushed to the bottom of the rod. The direction of the EMF is **down** the rod.

(c) Magnitude of the Current

By Ohm's Law, $I = \mathcal{E}/R$.

$$I = \frac{0.60\,\text{V}}{0.40\,\Omega} = \mathbf{1.5}\,\text{A}$$

(d) Direction of the Current

The current flows from high to low potential, resulting in a **clockwise** current in the loop.

(e) Rate of Thermal Energy Generation

The power dissipated is $P = I^2 R$.

$$P_{\text{thermal}} = (1.5\,\text{A})^2(0.40\,\Omega) = \mathbf{0.90}\,\text{W}$$

(f) External Force Needed

To maintain constant velocity, an external force must balance the magnetic drag force $\vec{F}_m = I(\vec{L} \times \vec{B})$. This force is directed to the right, so the external force must be to the left with magnitude:

$$F_{\text{ext}} = ILB = (1.5\,\text{A})(0.10\,\text{m})(1.2\,\text{T}) = \mathbf{0.18}\,\text{N}$$

(g) Rate of Work Done by This Force

The rate of work is power, $P_{\text{ext}} = F_{\text{ext}} v$.

$$P_{\text{ext}} = (0.18\,\text{N})(5.0\,\text{m/s}) = \mathbf{0.90}\,\text{W}$$

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4. RL Circuit with Constant Current Source

(a) Current Through the Inductor as a Function of Time

By Kirchhoff's laws:

1. KCL: $I = I_R(t) + I_L(t)$
2. KVL: $V_R = V_L \implies I_R(t)R = L\frac{dI_L}{dt}$

Substituting $I_R = I - I_L$ gives the differential equation: $(I - I_L)R = L\frac{dI_L}{dt}$. Solving this equation with the initial condition $I_L(0) = 0$ yields:

$$I_L(t) = I(1 - e^{-(R/L)t})$$

(b) Time When Currents are Equal

We need to find the time t when $I_R(t) = I_L(t)$, which implies $I_L(t) = I/2$.

$$\begin{aligned} \frac{I}{2} &= I(1 - e^{-(R/L)t}) \\ \frac{1}{2} &= 1 - e^{-(R/L)t} \\ e^{-(R/L)t} &= \frac{1}{2} \\ -\frac{R}{L}t &= \ln\left(\frac{1}{2}\right) = -\ln(2) \\ t &= \frac{L}{R}\ln(2) \end{aligned}$$

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5. Standing Waves on a String

Derivation

The speed of a wave on a string with tension $T = mg$ and linear density μ is $v = \sqrt{mg/\mu}$. For a standing wave with n loops on a string of length L, the wavelength is $\lambda = 2L/n$, and the speed is $v = f\lambda = 2Lf/n$. Equating the two expressions for speed:

$$\sqrt{\frac{mg}{\mu}} = \frac{2Lf}{n} \implies m = \left(\frac{4L^2f^2\mu}{g}\right)\frac{1}{n^2}$$

Let m_1 correspond to $n+1$ loops and m_2 correspond to n loops.

$$\frac{m_1}{m_2} = \frac{n^2}{(n+1)^2} \implies \sqrt{\frac{m_1}{m_2}} = \frac{n}{n+1}$$

$$n = \frac{\sqrt{m_1}}{\sqrt{m_2} - \sqrt{m_1}} = \frac{\sqrt{0.2861}}{\sqrt{0.4470} - \sqrt{0.2861}} = \frac{0.5349}{0.6686 - 0.5349} \approx 4.00$$

So, $n = 4$. Now we solve for μ using the equation for m_2:

$$\mu = \frac{m_2 g n^2}{4L^2f^2} = \frac{(0.4470)(9.80)(4^2)}{4(1.20)^2(120)^2} \approx 8.45 \times 10^{-4}\,\text{kg/m}$$

The linear density is $\mathbf{0.845\,g/m}$.

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6. Relativistic Electron in a Nucleus

Calculation Steps

1. **Estimate Minimum Momentum:** From Heisenberg's Uncertainty Principle, with $\Delta x \approx r = 6 \times 10^{-15}$ m:

$$p \approx \Delta p \approx \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}\,\text{J}\cdot\text{s}}{6 \times 10^{-15}\,\text{m}} \approx 1.757 \times 10^{-20}\,\text{kg}\cdot\text{m/s}$$

2. **Calculate Energy Terms in MeV:**

   - Rest Energy of Electron: $m_e c^2 = 0.511\,\text{MeV}$.
   - Momentum Energy: $pc = (1.757 \times 10^{-20})(3.00 \times 10^8) = 5.271 \times 10^{-12}$ J.

$$pc = \frac{5.271 \times 10^{-12}\,\text{J}}{1.602 \times 10^{-13}\,\text{J/MeV}} \approx 32.9\,\text{MeV}$$

3. **Calculate Kinetic Energy:** Using the relativistic energy-momentum relation $E^2 = (pc)^2 + (m_e c^2)^2$:

$$E = \sqrt{(32.9\,\text{MeV})^2 + (0.511\,\text{MeV})^2} \approx 32.9\,\text{MeV}$$

The minimum kinetic energy is $K = E - m_e c^2$:

$$K = 32.9\,\text{MeV} - 0.511\,\text{MeV} \approx 32.4\,\text{MeV}$$

4. **Convert Back to Joules:**

$$K = (32.4\,\text{MeV})(1.602 \times 10^{-13}\,\text{J/MeV}) \approx \mathbf{5.19 \times 10^{-12}}\,\text{J}$$

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7. Inelastic Collision and Rotational Motion

1. **Block Sliding Down:** By energy conservation, $v = \sqrt{2gh} = \sqrt{2(9.80)(0.20)} \approx 1.98\,\text{m/s}$.
2. **The Collision:** By conservation of angular momentum about the pivot O:

$$L_i = L_f$$
$$L(mv) = I_{total}\omega$$

The total moment of inertia is $I_{total} = I_{rod} + I_{block} = \frac{1}{3}ML^2 + mL^2 = \left(\frac{1}{3}M + m\right)L^2$.

$$\omega = \frac{mv}{\left(\frac{1}{3}M + m\right)L} = \frac{(0.050)(1.98)}{\left(\frac{1}{3}(0.100) + 0.050\right)(0.40)} \approx 2.97\,\text{rad/s}$$

3. **The Swing Up:** By conservation of mechanical energy:

$$K_i = \Delta U_g$$
$$\frac{1}{2}I_{total}\omega^2 = \left(\frac{1}{2}M + m\right)gL(1 - \cos\theta)$$

Calculating the numerical values:

$$I_{total} = \left(\frac{1}{3}(0.1) + 0.05\right)(0.4)^2 \approx 0.01333\,\text{kg}\cdot\text{m}^2$$
$$K_i = \frac{1}{2}(0.01333)(2.97)^2 \approx 0.0588\,\text{J}$$
$$\Delta U_g = \left(\frac{1}{2}(0.1) + 0.05\right)(9.8)(0.4)(1 - \cos\theta) = 0.392(1 - \cos\theta)$$

Equating them: $0.0588 = 0.392(1 - \cos\theta) \implies 1 - \cos\theta \approx 0.15$.

$$\cos\theta = 0.85 \implies \theta = \arccos(0.85) \approx \mathbf{31.8}^\circ$$

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8. Thermodynamic Cycle

For a monatomic ideal gas, $C_V = \frac{3}{2}R$, $C_p = \frac{5}{2}R$, $\gamma = 5/3$.

- **State b:** $p_b = 1.013 \times 10^6$ Pa, $V_b = 1.00 \times 10^{-3}$ m^3, $T_b \approx 121.9$ K.
- **State c:** $V_c = 8V_b$. Adiabatic process gives $p_c = p_b(V_b/V_c)^\gamma \approx 3.166 \times 10^4$ Pa. $T_c \approx 30.48$ K.
- **State a:** $V_a = V_b$, $p_a = p_c$. $T_a \approx 3.81$ K.

(a) Energy Added as Heat (Q_H)

Heat is added during the isochoric process a $\rightarrow$ b.

$$Q_H = Q_{ab} = nC_V(T_b - T_a) = (1.00)\left(\frac{3}{2} \times 8.31\right)(121.9 - 3.81) \approx \mathbf{1472}\,\mathrm{J}$$

(b) Energy Leaving as Heat (Q_C)

Heat leaves during the isobaric process c $\rightarrow$ a.

$$Q_C = |Q_{ca}| = nC_p(T_c - T_a) = (1.00)\left(\frac{5}{2} \times 8.31\right)(30.48 - 3.81) \approx \mathbf{554}\,\mathrm{J}$$

(c) Net Work Done (W_{net})

$$W_{net} = Q_H - Q_C = 1472\,\mathrm{J} - 554\,\mathrm{J} = \mathbf{918}\,\mathrm{J}$$

(d) Efficiency (η)

$$\eta = \frac{W_{net}}{Q_H} = \frac{918}{1472} \approx 0.624 = \mathbf{62.4}\%$$

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9. Mass on a Rotating Tilted Spring System

(a) Effective Spring Constant (k_t) for $\omega = 0$

The two spring segments act in parallel, so the effective constant is $k_t = k_1 + k_2$.

$$k_1 = k\frac{L}{L_1}, \quad k_2 = k\frac{L}{L - L_1}$$

$$k_t = kL\left(\frac{1}{L_1} + \frac{1}{L - L_1}\right) = \frac{kL^2}{L_1(L - L_1)}$$

(b) Equilibrium Position (r_0) with Rotation

At equilibrium, the net inward spring force balances the outward centrifugal force: $F_{spring} = m\omega^2 r_0$. Solving the force balance equation for r_0 gives:

$$r_0 = \frac{kLR_0}{(L - L_1)(k_t - m\omega^2)} = \frac{kLR_0L_1}{kL^2 - m\omega^2 L_1(L - L_1)}$$

(c) Lowest Angular Frequency ω to Reach the Edge

The mass reaches the edge ($r_0 = R_0$) when the total outward force (centrifugal plus the maximum radial gravity component) is balanced by the spring force at that position.

$$F_{spring}(R_0) = F_{centrifugal}(R_0) + F_{gravity,max}$$

$$\frac{kLR_0}{L_1} = m\omega^2 R_0 + mg\sin\phi$$

Solving for ω:

$$\omega = \sqrt{\frac{kL}{mL_1} - \frac{g\sin\phi}{R_0}}$$

Substituting values:

$$\omega = \sqrt{\frac{(10)(0.5)}{(1)(0.2)} - \frac{(9.8)\sin(0.4)}{1}} = \sqrt{25 - 3.816} = \sqrt{21.184} \approx \mathbf{4.60}\,\mathrm{rad/s}$$

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10. Young's Double Slit Experiment

(a) Condition for Interference Pattern

The light waves from the two slits must be **coherent**, meaning they have a constant phase relationship.

(b) Conditions for Interference

For a path difference $\Delta r = d \sin\theta$:

- **Constructive (Maxima):** $d \sin\theta = m\lambda, \quad m = 0, \pm1, \pm2, ...$
- **Destructive (Minima):** $d \sin\theta = (m + \frac{1}{2})\lambda, \quad m = 0, \pm1, \pm2, ...$

(c) Distance Between Adjacent Fringes

Using the small-angle approximation $\sin\theta \approx y/\ell$, the position of the m-th bright fringe is $y_m = \frac{m\lambda\ell}{d}$. The distance between adjacent fringes is:

$$\Delta y = y_{m+1} - y_m = \frac{\lambda\ell}{d}$$

(d) Intensity of the Maxima

The intensity I is proportional to the square of the electric field amplitude. The superposition of two waves with amplitude E_0 and phase difference δ gives a resultant amplitude $A = 2E_0 \cos(\delta/2)$. If I_0 is the intensity from one slit ($I_0 \propto E_0^2$), the intensity distribution is:

$$I(\theta) = 4I_0 \cos^2(\delta/2) = 4I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

At the maxima, $d \sin\theta = m\lambda$, so the argument of the cosine is $m\pi$, and $\cos^2(m\pi) = 1$. Thus, the intensity of the maxima is $I_{maxima} = 4I_0$.