1 Simple Harmonic Motion

1.1 (a)(i) Determine the period of the motion.

The relationship between velocity v and displacement x for Simple Harmonic Motion (SHM) is given by:

$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

where ω is the angular frequency and x_0 is the amplitude.

From the provided $v - x$ graph, we identify:

- Maximum displacement (amplitude), $x_0 = 3$ m.
- Maximum velocity, $v_{\max} = 2$ m s^{-1}.

In SHM, the maximum velocity is $v_{\max} = \omega x_0$.

$$2 \text{ m s}^{-1} = \omega \times 3 \text{ m} \implies \omega = \frac{2}{3} \text{ rad s}^{-1}$$

The period of motion T is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2/3} = 3\pi \text{ s}$$

The period is 3π **s** (approximately 9.42 s).

1.2 (a)(ii) Calculate the time taken for the particle to move from point A to point B.

Let the motion be described by $x(t) = 3\sin(\frac{2}{3}t + \phi)$ and $v(t) = 2\cos(\frac{2}{3}t + \phi)$.

State A: $v_A = 1$ m s^{-1}. The corresponding displacement x_A is found using the ellipse equation $\frac{v^2}{v_{\max}^2} + \frac{x^2}{x_0^2} = 1$:

$$\frac{1^2}{2^2} + \frac{x_A^2}{3^2} = 1 \implies x_A = \frac{3\sqrt{3}}{2} \text{ m}$$

The phase $\theta_A = \frac{2}{3}t_A + \phi$ must satisfy:

$$x_A = \frac{3\sqrt{3}}{2} = 3\sin(\theta_A) \implies \sin(\theta_A) = \frac{\sqrt{3}}{2}$$
$$v_A = 1 = 2\cos(\theta_A) \implies \cos(\theta_A) = \frac{1}{2}$$

This gives $\theta_A = \frac{\pi}{3}$.

State B: $x_B = 0$ m and $v_B = -2$ m s^{-1}. The phase θ_B must satisfy:

$$x_B = 0 = 3\sin(\theta_B) \implies \sin(\theta_B) = 0$$
$$v_B = -2 = 2\cos(\theta_B) \implies \cos(\theta_B) = -1$$

This gives $\theta_B = \pi$.

The phase change is $\Delta\theta = \theta_B - \theta_A = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. The time taken is:

$$\Delta t = \frac{\Delta\theta}{\omega} = \frac{2\pi/3}{2/3} = \pi \text{ s}$$

The time taken is π **s** (approximately 3.14 s).

1.3 (b)(i) Show that the potential energy is $E_p = \frac{1}{2}m\omega^2 x^2$.

Total energy $E_T = K + E_p = \frac{1}{2}mv^2 + E_p$. Total energy is also the maximum kinetic energy, $E_T = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}m(\omega x_0)^2$. Using $v^2 = \omega^2(x_0^2 - x^2)$, the kinetic energy is $K(x) = \frac{1}{2}m\omega^2(x_0^2 - x^2)$. The potential energy is $E_p(x) = E_T - K(x)$:

$$E_p(x) = \frac{1}{2}m\omega^2 x_0^2 - \frac{1}{2}m\omega^2(x_0^2 - x^2) = \frac{1}{2}m\omega^2 x^2$$

1.4 (b)(ii) Show that the potential energy of the pendulum is $E = mgL - mg\sqrt{L^2 - x^2}$.

Let the equilibrium position be the reference for potential energy ($E_p = 0$). When displaced by a horizontal distance x, the bob is raised by a vertical height $h = L - \sqrt{L^2 - x^2}$. The potential energy E is mgh:

$$E = mg(L - \sqrt{L^2 - x^2}) = mgL - mg\sqrt{L^2 - x^2}$$

1.5 (b)(iii) Show that the period of a pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$ for small-angle oscillations.

For small angles, $x \ll L$. We use the binomial approximation $\sqrt{1-u} \approx 1 - \frac{u}{2}$ for small u.

$$E = mgL\left(1 - \sqrt{1 - \left(\frac{x}{L}\right)^2}\right) \approx mgL\left(1 - \left(1 - \frac{x^2}{2L^2}\right)\right) = \frac{mg}{2L}x^2$$

Comparing with $E_p = \frac{1}{2}m\omega^2 x^2$, we get:

$$\frac{1}{2}m\omega^2 x^2 = \frac{mg}{2L}x^2 \implies \omega^2 = \frac{g}{L} \implies \omega = \sqrt{\frac{g}{L}}$$

The period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$.