Solutions to 32nd Singapore Physics Olympiad 2019

Theory Paper 1

Question 1: Power Cable Sag

Problem Statement

A power cable with mass M and length L is stretched between two points at the same vertical height. The cable has a small sag d at the midpoint ($d \ll L$). Determine the tension T in the cable at the midpoint and at the ends.

Solution

1. Model the Cable: Let the cable take the shape of a curve $y(x)$. Since the sag is small, we can approximate the shape as a parabola or analyse the forces on half of the cable. Let the two support points be at $x = -L/2$ and $x = L/2$, with $y = d$. The midpoint is at $x = 0, y = 0$. Alternatively, consider the forces on the right half of the cable (length $L/2$, mass $M/2$).

2. Forces on Half the Cable: Consider the right half of the cable from the lowest point (midpoint) to the support. Forces acting on this segment:

- Tension at the midpoint T_0 acting horizontally to the left.
- Tension at the support T_{end} acting at an angle θ upwards.
- Weight of the half-cable $W = \frac{M}{2}g$ acting downwards.

3. Equilibrium Conditions:

- **Horizontal Equilibrium:** The horizontal component of tension is constant.

$$T_{end} \cos\theta = T_0 \tag{1}$$

- **Vertical Equilibrium:** The vertical component of tension at the support supports the weight.

$$T_{end} \sin\theta = \frac{Mg}{2} \tag{2}$$

4. Geometric Constraint (Small Sag): For a small sag $d \ll L$, the angle θ is small. We can relate the geometry to the forces by taking moments about the support or using the slope approximation. Assuming a parabolic shape $y = kx^2$ for small sag: At $x = L/2$, $y = d$, so $d = k(L/2)^2 \implies k = \frac{4d}{L^2}$. The slope at the support is $\tan\theta = \frac{dy}{dx}\Big|_{x=L/2} = 2kx = 2(\frac{4d}{L^2})(\frac{L}{2}) = \frac{4d}{L}$.

Alternatively, using torque about the support point for the half-string is complex because the center of mass isn't exactly at $L/4$. A better approximation for small sag uses the force triangle:

$$\tan\theta = \frac{\text{Vertical Force}}{\text{Horizontal Force}} = \frac{Mg/2}{T_0} \tag{3}$$

Equating the geometric slope to the force ratio:

$$\frac{4d}{L} \approx \frac{Mg}{2T_0} \tag{4}$$

$$T_0 = \frac{MgL}{8d} \tag{5}$$

5. Tension at the Ends: Using the Pythagorean theorem for the force components:

$$T_{end}^2 = T_0^2 + \left(\frac{Mg}{2}\right)^2 \tag{6}$$

$$T_{end} = \sqrt{\left(\frac{MgL}{8d}\right)^2 + \left(\frac{Mg}{2}\right)^2} \tag{7}$$

Since $d \ll L$, the term $(\frac{MgL}{8d})^2$ is much larger than $(\frac{Mg}{2})^2$.

$$T_{end} \approx T_0 = \frac{MgL}{8d} \tag{8}$$

Answer:

- Tension at the midpoint: $T_{mid} = \dfrac{MgL}{8d}$
- Tension at the ends: $T_{end} \approx \dfrac{MgL}{8d}$

Question 2: Thermodynamic Cycle

Problem Statement

One mole of an ideal monatomic gas undergoes a cycle consisting of: 1. Isothermal expansion at temperature T_h from volume V_1 to V_2. 2. Isochoric cooling to temperature T_c. 3. Isothermal compression at temperature T_c back to volume V_1. 4. Isochoric heating to temperature T_h. Determine the efficiency of this cycle.

Solution

1. Heat Transfers in Each Step: We use the First Law of Thermodynamics $Q = \Delta U + W$ and the ideal gas properties $U = \frac{3}{2}nRT$ (monatomic).

- **Process** $1 \to 2$ **(Isothermal Expansion):** Temperature constant at T_h. $\Delta U = 0$. Work done by gas $W_{12} = \int PdV = nRT_h \ln(V_2/V_1)$. Heat absorbed:

$$Q_{in,1} = W_{12} = nRT_h \ln\left(\frac{V_2}{V_1}\right) \tag{9}$$

- **Process** $2 \to 3$ **(Isochoric Cooling):** Volume constant at V_2. $W_{23} = 0$. Heat released:

$$Q_{out,1} = |\Delta U| = \frac{3}{2}nR(T_h - T_c) \tag{10}$$

- **Process** $3 \to 4$ **(Isothermal Compression):** Temperature constant at T_c. $\Delta U = 0$. Work done on gas. Heat released:

$$Q_{out,2} = |W_{34}| = nRT_c \ln\left(\frac{V_2}{V_1}\right) \tag{11}$$

- **Process** $4 \to 1$ **(Isochoric Heating):** Volume constant at V_1. $W_{41} = 0$. Heat absorbed:

$$Q_{in,2} = \Delta U = \frac{3}{2}nR(T_h - T_c) \tag{12}$$

2. Calculate Efficiency: The efficiency η is defined as the net work done divided by the total heat absorbed.

$$\eta = \frac{W_{net}}{Q_{in,total}} \tag{13}$$

Net Work:

$$W_{net} = W_{12} - |W_{34}| = nR(T_h - T_c) \ln\left(\frac{V_2}{V_1}\right) \tag{14}$$

Total Heat Absorbed:

$$Q_{in,total} = Q_{in,1} + Q_{in,2} = nRT_h \ln\left(\frac{V_2}{V_1}\right) + \frac{3}{2}nR(T_h - T_c) \tag{15}$$

Efficiency:

$$\eta = \frac{nR(T_h - T_c)\ln(V_2/V_1)}{nRT_h \ln(V_2/V_1) + \frac{3}{2}nR(T_h - T_c)} \tag{16}$$

Canceling nR:

$$\eta = \frac{(T_h - T_c)\ln(V_2/V_1)}{T_h \ln(V_2/V_1) + \frac{3}{2}(T_h - T_c)} \tag{17}$$

Answer: The efficiency is:

$$\eta = \frac{(T_h - T_c)\ln(V_2/V_1)}{T_h \ln(V_2/V_1) + \frac{3}{2}(T_h - T_c)}$$

Question 3: Particle in a Box

Problem Statement

Consider a quantum mechanical particle of mass m confined in a 1D infinite potential well of width L ($0 < x < L$). (a) Write down the normalized wavefunctions and energy levels. (b) Calculate the probability of finding the particle in the middle third of the box ($L/3 < x < 2L/3$) for the ground state.

Solution

(a) Wavefunctions and Energies Inside the well ($V = 0$), the Schrödinger equation is $-\frac{\hbar^2}{2m}\psi'' = E\psi$. The solutions satisfying boundary conditions $\psi(0) = \psi(L) = 0$ are:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \tag{18}$$

The corresponding energy levels are:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{19}$$

(b) Probability for Ground State ($n = 1$) We need to calculate the probability P for $L/3 \leq x \leq 2L/3$:

$$P = \int_{L/3}^{2L/3} |\psi_1(x)|^2 dx = \int_{L/3}^{2L/3} \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right) dx \tag{20}$$

Use the identity $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$:

$$P = \frac{2}{L} \int_{L/3}^{2L/3} \frac{1}{2} \left[1 - \cos\left(\frac{2\pi x}{L}\right)\right] dx \tag{21}$$

$$P = \frac{1}{L} \left[x - \frac{L}{2\pi} \sin\left(\frac{2\pi x}{L}\right)\right]_{L/3}^{2L/3} \tag{22}$$

Evaluate at the limits: Upper limit ($2L/3$): $\frac{2L}{3} - \frac{L}{2\pi}\sin\left(\frac{4\pi}{3}\right)$ Lower limit ($L/3$): $\frac{L}{3} - \frac{L}{2\pi}\sin\left(\frac{2\pi}{3}\right)$

Difference:

$$P = \frac{1}{L} \left[\left(\frac{2L}{3} - \frac{L}{3}\right) - \frac{L}{2\pi}\left(\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3}\right)\right] \tag{23}$$

$$P = \frac{1}{L}\left[\frac{L}{3} - \frac{L}{2\pi}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right] \tag{24}$$

$$P = \frac{1}{3} - \frac{1}{2\pi}(-\sqrt{3}) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} \tag{25}$$

Numerical value: $P \approx 0.333 + \frac{1.732}{6.283} \approx 0.333 + 0.276 = 0.609$

Answer: (a) $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$, $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ (b) Probability $P = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} \approx 0.61$

Question 4: Electric Potential and Mass Spectrometer

(a) Combined Droplet Potential

Problem: Two spherical oil drops, each with radius R and potential $V_0 = 1000$ V, merge into a single spherical drop. Find the new potential V_{new}. Assume charge and volume are conserved.

Solution: Let the initial charge on each drop be Q. Potential of one drop:

$$V_0 = \frac{kQ}{R} \implies Q = \frac{V_0 R}{k} \tag{26}$$

When they merge:

- **Total Charge:** $Q_{new} = 2Q$.
- **Total Volume:** Volume is conserved.

$$V_{vol,new} = 2V_{vol,old} \implies \frac{4}{3}\pi R_{new}^3 = 2\left(\frac{4}{3}\pi R^3\right) \tag{27}$$

$$R_{new}^3 = 2R^3 \implies R_{new} = 2^{1/3}R \tag{28}$$

New Potential:

$$V_{new} = \frac{kQ_{new}}{R_{new}} = \frac{k(2Q)}{2^{1/3}R} \tag{29}$$

Substitute $kQ/R = V_0$:

$$V_{new} = \frac{2}{2^{1/3}}V_0 = 2^{2/3}V_0 \tag{30}$$

Calculation: $2^{2/3} \approx 1.587$. $V_{new} = 1.587 \times 1000 = 1587$ V.

Answer: The new potential is approx **1590 V**.

(b) Bainbridge Mass Spectrometer

Problem: A velocity selector has $E = 100\,\mathrm{V/cm} = 10^4\,\mathrm{V/m}$ and $B = 0.2\,\mathrm{T}$. (i) Calculate the speed of the selected ion. (ii) Can it resolve ^{3}He and ^{4}He if the exit slit is 1 mm wide?

Solution (i): Velocity Selector Speed The condition for passing through the velocity selector undeflected is $qE = qvB$.

$$v = \frac{E}{B} \tag{31}$$

$$v = \frac{10^4\,\mathrm{V/m}}{0.2\,\mathrm{T}} = 50,000\,\mathrm{m/s} = 5.0 \times 10^4\,\mathrm{m/s} \tag{32}$$

Solution (ii): Resolution After the velocity selector, ions enter a region with magnetic field B' (assume same $B = 0.2$ T) and move in a semi-circle. Radius of path:

$$qvB = \frac{mv^2}{r} \implies r = \frac{mv}{qB} \tag{33}$$

Diameter of path (position on detector): $D = 2r = \frac{2mv}{qB}$.

Calculate diameter for each isotope: Charge $q = +e = 1.60 \times 10^{-19}$ C. Mass of ^{3}He $\approx 3 \times 1.67 \times 10^{-27}$ kg $= 5.01 \times 10^{-27}$ kg. Mass of ^{4}He $\approx 4 \times 1.67 \times 10^{-27}$ kg $= 6.68 \times 10^{-27}$ kg.

For ^{3}He:

$$D_3 = \frac{2(5.01 \times 10^{-27})(5.0 \times 10^4)}{(1.60 \times 10^{-19})(0.2)} = \frac{5.01 \times 10^{-22}}{3.2 \times 10^{-20}} \approx 1.56 \times 10^{-2} \text{ m} = 15.6 \text{ mm} \tag{34}$$

For ^{4}He:

$$D_4 = \frac{2(6.68 \times 10^{-27})(5.0 \times 10^4)}{(1.60 \times 10^{-19})(0.2)} \approx 2.09 \times 10^{-2} \text{ m} = 20.9 \text{ mm} \tag{35}$$

Separation: $\Delta D = D_4 - D_3 = 20.9 - 15.6 = 5.3$ mm.

Since the separation (5.3 mm) is significantly larger than the slit width (1 mm), the two beams are spatially separated enough to be resolved.

Answer: (i) $v = 5.0 \times 10^4$ m/s. (ii) Yes, the separation (5.3 mm) $>$ slit width (1 mm).

Question 5: Radiation Pressure and Positronium

(a) Radiation Pressure

Problem: Light intensity $I = 50\,\mathrm{W/m^2}$ incident normally on a perfectly reflecting surface. Find the pressure P.

Solution: When a photon is reflected, its momentum change is $\Delta p = 2p = 2E/c$. The force exerted is the rate of change of momentum. Total power $W = IA$. Total force $F = \frac{2W}{c} = \frac{2IA}{c}$. Pressure $P = F/A$:

$$P = \frac{2I}{c} \tag{36}$$

$$P = \frac{2(50)}{3.00 \times 10^8} = \frac{100}{3.00 \times 10^8} \approx 3.33 \times 10^{-7}\,\mathrm{Pa} \tag{37}$$

Answer: The pressure is $\mathbf{3.33 \times 10^{-7}}$ Pa.

(b) Positronium

Problem: Positronium consists of an electron (e^-) and a positron (e^+). It is hydrogen-like. Determine the ground state energy and the radius compared to Hydrogen.

Solution: This is a two-body problem. We must use the reduced mass μ. Masses: $m_1 = m_e$, $m_2 = m_{positron} = m_e$. Reduced mass:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_e m_e}{2m_e} = \frac{m_e}{2} \tag{38}$$

1. Energy Levels: For Hydrogen, $E_n \propto m_e$ (strictly $\mu \approx m_e$ since proton is heavy). $E_{1,H} = -13.6$ eV. For Positronium, replace m_e with $\mu = m_e/2$.

$$E_{1,Pos} = \frac{\mu}{m_e} E_{1,H} = \frac{1}{2}(-13.6 \text{ eV}) = -6.8 \text{ eV} \tag{39}$$

2. Radius (Bohr Radius): For Hydrogen, $a_0 \propto 1/m_e$. For Positronium, replace m_e with $\mu = m_e/2$.

$$a_{Pos} = \frac{m_e}{\mu} a_0 = 2a_0 \tag{40}$$

$a_0 \approx 0.0529$ nm. $a_{Pos} \approx 0.106$ nm.

Answer: Ground State Energy: **-6.8 eV**. Orbital Radius: **Double the Bohr radius** (≈ 0.106 nm).