Solutions to 31st Singapore Physics Olympiad 2018

Theory Paper 2

Question 1: Inelastic Neutron-Helium Collision

(a) Kinetic energy of the singly ionized helium atom

1. Define System Parameters:

- Mass of neutron: $m_n = m$
- Mass of Helium ion (He^+): $m_{He} \approx 4m$
- Initial kinetic energy of neutron: $K_{n,i} = 68.2\,\text{eV}$
- Scattering angle of neutron: $90°$

2. Conservation of Momentum: Let the initial velocity of the neutron be along the x-axis, $\vec{v}_{n,i} = v_0\hat{i}$. After the collision, the neutron moves along the $-y$ direction, $\vec{v}_{n,f} = -v_1\hat{j}$. The Helium ion moves with velocity $\vec{v}_{He}$.

Conservation in x-direction:

$$mv_0 = 4mv_{He,x} \implies v_{He,x} = \frac{v_0}{4} \tag{1}$$

Conservation in y-direction:

$$0 = -mv_1 + 4mv_{He,y} \implies v_{He,y} = \frac{v_1}{4} \tag{2}$$

3. Kinetic Energy Relationships: The kinetic energy is given by $K = \frac{p^2}{2M}$. The final kinetic energy of the Helium ion is:

$$K_{He} = \frac{1}{2}(4m)(v_{He,x}^2 + v_{He,y}^2) = 2m\left(\frac{v_0^2}{16} + \frac{v_1^2}{16}\right) \tag{3}$$

$$K_{He} = \frac{1}{4}\left(\frac{1}{2}mv_0^2 + \frac{1}{2}mv_1^2\right) = \frac{K_{n,i} + K_{n,f}}{4} \tag{4}$$

4. Conservation of Energy: The collision is inelastic. The Helium ion is excited by an energy ΔE.

$$K_{n,i} = K_{n,f} + K_{He} + \Delta E \tag{5}$$

Substituting the expression for K_{He}:

$$K_{n,i} = K_{n,f} + \frac{K_{n,i} + K_{n,f}}{4} + \Delta E \tag{6}$$

Rearranging to solve for $K_{n,f}$:

$$\frac{3}{4}K_{n,i} - \Delta E = \frac{5}{4}K_{n,f} \implies K_{n,f} = \frac{3}{5}K_{n,i} - \frac{4}{5}\Delta E \tag{7}$$

5. Determining the Excitation Energy ΔE: The energy levels of He^+ are $E_n = -54.4/n^2$ eV.

- Ground state ($n = 1$): $E_1 = -54.4\,\text{eV}$
- First excited state ($n = 2$): $E_2 = -13.6\,\text{eV}$
- Transition energy $\Delta E = E_2 - E_1 = -13.6 - (-54.4) = 40.8\,\text{eV}$

Check if this transition is allowed ($K_{n,f} > 0$):

$$K_{n,f} = 0.6(68.2) - 0.8(40.8) = 40.92 - 32.64 = 8.28\,\text{eV} \tag{8}$$

Since $K_{n,f} > 0$, this transition is physically possible. Higher transitions ($n = 3$) would result in $K_{n,f} \approx 2.2\,\text{eV}$, but $n = 2$ is the most probable dominant excitation. We assume excitation to $n = 2$.

6. Calculate K_{He}:

$$K_{He} = \frac{68.2 + 8.28}{4} = \frac{76.48}{4} = 19.12\,\text{eV} \tag{9}$$

Answer: The kinetic energy of the helium ion is **19.1 eV**.

(b) Kinetic energy of the neutron after collision

Using the calculation from part (a):

$$K_{n,f} = 8.28\,\text{eV} \tag{10}$$

Answer: The kinetic energy of the neutron is **8.28 eV**.

(c) Calculate the angle θ

Let θ be the angle of the Helium ion's velocity vector with respect to the x-axis.

$$\tan\theta = \frac{p_{He,y}}{p_{He,x}} \tag{11}$$

From momentum conservation, $p_{He,x} = p_{n,i}$ and $p_{He,y} = p_{n,f}$.

$$\tan\theta = \frac{p_{n,f}}{p_{n,i}} = \frac{\sqrt{2mK_{n,f}}}{\sqrt{2mK_{n,i}}} = \sqrt{\frac{K_{n,f}}{K_{n,i}}} \tag{12}$$

$$\tan\theta = \sqrt{\frac{8.28}{68.2}} \approx \sqrt{0.1214} \approx 0.3484 \tag{13}$$

$$\theta = \arctan(0.3484) \approx 19.2^\circ \tag{14}$$

Answer: The angle is **19.2°**.

Question 2: Split Lens Interference

(a) Describe the image observed by the camera

1. Lens Analysis: The original lens has focal length $f = 10\,\text{cm}$. The object distance is $u = 30\,\text{cm}$. Using the lens formula $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$:

$$\frac{1}{v} = \frac{1}{10} - \frac{1}{30} = \frac{2}{30} \implies v = 15\,\text{cm} \tag{15}$$

The magnification is $M = -\frac{v}{u} = -\frac{15}{30} = -0.5$.

2. Effect of Splitting: The lens is split and separated by $d = 1.0\,\text{cm}$.

- Upper half (Lens A) moves up by $0.5\,\text{cm}$. Its optical axis is at $y = 0.5$. The object is at $y = 0$, so relative to axis A, $u_y = -0.5$. The image forms at $y'_{i,A} = Mu_y = (-0.5)(-0.5) = +0.25$ relative to axis. Absolute position: $y_A = 0.5 + 0.25 = 0.75\,\text{cm}$.
- Lower half (Lens B) moves down by $0.5\,\text{cm}$. By symmetry, the image forms at absolute position $y_B = -0.75\,\text{cm}$.

3. Interference: We effectively have two coherent point sources separated by $a = 0.75 - (-0.75) = 1.5\,\text{cm}$. These sources are located $15\,\text{cm}$ from the lens. The CCD screen is $90\,\text{cm}$ from the lens, so the distance from the sources to the screen is $D = 90 - 15 = 75\,\text{cm}$.

Answer: The two images act as coherent sources producing a **pattern of parallel interference fringes** (bright and dark bands) on the CCD camera.

(b) Calculate the fringe spacing

The fringe spacing Δy in a double-slit experiment configuration is given by:

$$\Delta y = \frac{\lambda D}{a} \tag{16}$$

Given:

- $\lambda = 632.8\,\text{nm} = 6.328 \times 10^{-7}$ m

- $D = 75\,\text{cm} = 0.75\,\text{m}$
- $a = 1.5\,\text{cm} = 0.015\,\text{m}$

Calculation:

$$\Delta y = \frac{(6.328 \times 10^{-7})(0.75)}{0.015} = (6.328 \times 10^{-7})(50) \tag{17}$$

$$\Delta y = 3.164 \times 10^{-5}\ \text{m} = 31.6\,\mu\text{m} \tag{18}$$

Answer: The fringe spacing is $\mathbf{31.6\,\mu m}$.

(c) Number of fringes on the CCD

1. Determine the Overlap Region: We must check if the light cones from the two lens halves overlap sufficiently to cover the CCD. Let's trace the marginal rays for Lens A (top half, shifted to $y \in [0.5, 2.5]$? No, usually split implies the gap is empty, or geometric centers move. Assuming gap $d = 1$ means edges are at ± 0.5).

- Bottom edge of Lens A ($y = 0.5$): Ray passes through image ($y = 0.75$ at $x = 15$) to screen ($x = 90$). By similar triangles, $y_{screen} = 2.0\,\text{cm}$.
- Top edge of Lens A ($y = 2.5$): Ray passes through image to screen. $y_{screen} = -8.0\,\text{cm}$.

Range of light from A: $[-8.0, 2.0]$ cm. By symmetry, Range of light from B: $[-2.0, 8.0]$ cm. The **Overlap Region** is $[-2.0, 2.0]$ cm (Width $= 4.0\,\text{cm}$).

2. Compare with CCD: The CCD width is $2.4\,\text{cm}$. Since $2.4 < 4.0$, the CCD is completely filled with interference fringes.

3. Calculate Count:

$$N = \frac{\text{Width of CCD}}{\text{Fringe Spacing}} = \frac{2.4 \times 10^{-2}}{3.164 \times 10^{-5}} \tag{19}$$

$$N \approx 758.5 \tag{20}$$

Answer: Approximately **758 fringes** are observed.

Question 3: Numerical Simulation of Particle Motion

(a) Describe the path of the particle

The force is $\vec{F} = q\vec{v} \times \vec{B}$. The magnetic force is always perpendicular to the velocity.

- The speed remains constant ($W = \Delta K = 0$).
- The particle undergoes uniform circular motion in the plane perpendicular to the magnetic field.

Answer: The path is a **circle**.

(b) State the shape of the path using Method 1

Method 1 uses: $\vec{v}_{new} = \vec{v}_{old} + \frac{q}{m}(\vec{v}_{old} \times \vec{B})\Delta t$. Since the acceleration term is perpendicular to $\vec{v}_{old}$, we have a right-angled vector addition.

$$|\vec{v}_{new}|^2 = |\vec{v}_{old}|^2 + \left|\frac{q}{m} vB\Delta t\right|^2 > |\vec{v}_{old}|^2 \tag{21}$$

The speed increases at every step. Since the radius of gyration $r \propto v$, the radius increases.

Answer: The path is an **outward spiral**.

(c) Equations for Method 2

Let $\alpha = \frac{qB}{m}$. The equations given are: 1. $\frac{v_x(t+\Delta t) - v_x(t)}{\Delta t} = \alpha v_y(t)$ 2. $\frac{v_y(t+\Delta t) - v_y(t)}{\Delta t} = -\alpha v_x(t + \Delta t)$

Rearranging for the update rule:

$$v_x(n+1) = v_x(n) + \alpha\Delta t\, v_y(n) \tag{22}$$
$$v_y(n+1) = v_y(n) - \alpha\Delta t\, v_x(n+1) \tag{23}$$

Substituting the new v_x into the second equation:

$$v_y(n+1) = v_y(n) - \alpha\Delta t[v_x(n) + \alpha\Delta t\, v_y(n)] \tag{24}$$

$$v_y(n+1) = (1 - (\alpha\Delta t)^2)v_y(n) - \alpha\Delta t\, v_x(n) \tag{25}$$

Answer:

$$v_x(n+1) = v_x(n) + (\alpha\Delta t)v_y(n) \tag{26}$$

$$v_y(n+1) = (1 - (\alpha\Delta t)^2)v_y(n) - (\alpha\Delta t)v_x(n) \tag{27}$$

(d) Kinetic Energy Ratio

(See derivation in thought process: Method 2 is a symplectic integrator. It preserves phase space area but the energy fluctuates slightly around a mean value for stable step sizes). The ratio is generally not 1, but for $\alpha\Delta t < 2$, the motion is stable. **Answer:** The ratio varies but remains bounded near 1.

(e, f, g) Calculation of B, Radius, and Period

Given $K = 1\,\text{keV} = 1.6 \times 10^{-16}$ J and $m_e = 9.11 \times 10^{-31}$ kg.

$$v = \sqrt{\frac{2K}{m}} \approx 1.87 \times 10^7 \text{ m/s} \tag{28}$$

Note: The problem text provided does not define the radius or B-field magnitude explicitly. We provide the algebraic expressions.

- **(e) B-field:** $B = \frac{\sqrt{2mK}}{er}$
- **(f) Radius:** $r = \frac{\sqrt{2mK}}{eB}$
- **(g) Period:** $T = \frac{2\pi m}{eB}$

(h) Sketch of Trajectories

- **Real:** A perfect circle.
- **Method 1:** A spiral starting at the real radius and growing larger outwards.

- **Method 2:** A closed orbit (elliptical or polygon-like) that stays very close to the real circular path without spiraling away.

[Sketch a circle, a spiral, and a closed loop overlaying the circle].

Question 4: Relativistic Rocket Radar

(a) Distance of the rocket at first reflection

1. Parameters:

- Proper length of rocket: $L_0 = 600\,\text{m}$.
- Round trip time for first pulse (back end): $t_{total} = 5.00\,\text{min} = 300\,\text{s}$.

2. Calculation: Assuming the pulse is sent at $t = 0$, hits the rocket at t_A, and returns at $t_1 = 300$. For a radar measurement, the distance at the instant of reflection is:

$$d = c \times \frac{t_{total}}{2} \tag{29}$$

$$d = (3.00 \times 10^8 \text{ m/s}) \times \frac{300 \text{ s}}{2} = 1.50 \times 10^8 \times 300 \tag{30}$$

$$d = 4.50 \times 10^{10} \text{ m} \tag{31}$$

Answer: The distance is $\mathbf{4.50 \times 10^{10}}$ m.

(b) Velocity of the rocket

1. Setup: Let the rocket velocity be $v = \beta c$. Pulse 1 hits the back at time t_A and returns at $t_1 = 2t_A$. Pulse 2 (part of same beam) passes the back, travels length L (contracted), reflects off front, travels back through length L, and returns to Earth.

2. Derivation of Delay: Let L be the contracted length: $L = L_0\sqrt{1-\beta^2}$. The time taken for the light to traverse the rocket from back to front (while rocket moves away) is Δt_{pass}. Distance light travels $= L + v\Delta t_{pass} = c\Delta t_{pass}$.

$$\Delta t_{pass} = \frac{L}{c - v} \tag{32}$$

The time for light to return from front to back (while rocket moves away) is Δt_{ret}. Light travels distance L'? No, relative speed approach. Actually, let's use the standard result for the time delay between reflections from front and back of a moving object observed at source. The extra time for the second pulse is the time to cover the rocket length and back, accounting for rocket motion.

$$\Delta t_{delay} = \Delta t_{pass} + \Delta t_{return\_trip\_lag} \tag{33}$$

Using the logic derived in the thought process:

$$\Delta t_{delay} = \frac{2L}{c}\frac{1}{1-\beta^2} = \frac{2L_0\sqrt{1-\beta^2}}{c(1-\beta^2)} = \frac{2L_0}{c\sqrt{1-\beta^2}} \times \dots \tag{34}$$

Correction: Let's use the explicit result derived:

$$\Delta t_{delay} = \frac{2L_0}{c}\sqrt{\frac{1+\beta}{1-\beta}} \tag{35}$$

3. Calculation: Given $\Delta t_{delay} = 12.0\,\mu\text{s} = 12.0 \times 10^{-6}$ s.

$$12.0 \times 10^{-6} = \frac{2(600)}{3 \times 10^8}\sqrt{\frac{1+\beta}{1-\beta}} \tag{36}$$

$$12.0 \times 10^{-6} = 4.0 \times 10^{-6}\sqrt{\frac{1+\beta}{1-\beta}} \tag{37}$$

$$3 = \sqrt{\frac{1+\beta}{1-\beta}} \implies 9 = \frac{1+\beta}{1-\beta} \tag{38}$$

$$9 - 9\beta = 1 + \beta \implies 10\beta = 8 \implies \beta = 0.8 \tag{39}$$

$$v = 0.8c = 2.40 \times 10^8 \text{ m/s} \tag{40}$$

Answer: The velocity is $\mathbf{2.40 \times 10^8}$ m/s.

(c) Time interval in the rocket frame

In the rocket's rest frame, the length is $L_0 = 600\,\text{m}$. The light simply travels from back to front at speed c.

$$\Delta t' = \frac{L_0}{c} \tag{41}$$

$$\Delta t' = \frac{600}{3.00 \times 10^8} = 2.00 \times 10^{-6} \text{ s} \tag{42}$$

Answer: The time interval is $\mathbf{2.00}\,\mu\mathbf{s}$.