Solutions for SPhO 2023 Theory Paper 2

Question 1: Projectile Motion

(a) Stationary Target

The motion of the projectile is described by the following equations:

$$x(t) = (v_0 \cos\theta)t$$

$$y(t) = y_0 + (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

Given values are $v_0 = 180\,\mathrm{m/s}$, $y_0 = 200\,\mathrm{m}$, $x = 2500\,\mathrm{m}$, and $g = 9.81\,\mathrm{m/s^2}$. The shell hits the target at sea level, so the final vertical position is $y = 0$.

From the horizontal motion equation, the time of flight t is:

$$t = \frac{x}{v_0 \cos\theta} = \frac{2500}{180\cos\theta}$$

Substitute this into the vertical motion equation:

$$0 = y_0 + (v_0 \sin\theta)\left(\frac{x}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2$$

$$0 = y_0 + x\tan\theta - \frac{gx^2}{2v_0^2}\sec^2\theta$$

Using the identity $\sec^2\theta = 1 + \tan^2\theta$:

$$0 = y_0 + x\tan\theta - \frac{gx^2}{2v_0^2}(1 + \tan^2\theta)$$

Let $u = \tan\theta$. This gives a quadratic equation in u:

$$\left(\frac{gx^2}{2v_0^2}\right)u^2 - xu - \left(y_0 - \frac{gx^2}{2v_0^2}\right) = 0$$

Calculating the coefficient $\frac{gx^2}{2v_0^2}$:

$$\frac{(9.81)(2500)^2}{2(180)^2} = 946.18\,\mathrm{m}$$

The equation becomes:

$$946.18u^2 - 2500u - (200 - 946.18) = 0$$

$$946.18u^2 - 2500u + 746.18 = 0$$

Solving for u using the quadratic formula:

$$u = \frac{2500 \pm \sqrt{(-2500)^2 - 4(946.18)(746.18)}}{2(946.18)} = \frac{2500 \pm \sqrt{3425816}}{1892.36} = \frac{2500 \pm 1850.9}{1892.36}$$

This yields two possible values for $u = \tan\theta$:

$$u_1 = \frac{4350.9}{1892.36} \approx 2.30 \implies \theta_1 = \arctan(2.30) \approx 66.5^\circ$$

$$u_2 = \frac{649.1}{1892.36} \approx 0.343 \implies \theta_2 = \arctan(0.343) \approx 18.9^\circ$$

Answer: The angle of elevation can be $\mathbf{66.5^\circ}$ or $\mathbf{18.9^\circ}$.

(b) Moving Target

The target moves with speed $v_T = 10\,\text{m/s}$. We use the smaller angle, $\theta = 18.9°$. Let the new muzzle speed be v_0' and the new time of flight be t'. At time t', the horizontal positions must match:

$$(v_0' \cos\theta)t' = 2500 + v_T t'$$

The vertical position of the shell is:

$$0 = y_0 + (v_0' \sin\theta)t' - \frac{1}{2}g(t')^2$$

$$0 = y_0 + (v_0' \cos\theta \tan\theta)t' - \frac{1}{2}g(t')^2$$

Substitute $(v_0' \cos\theta)t'$ from the horizontal equation into the vertical one:

$$0 = y_0 + (2500 + v_T t')\tan\theta - \frac{1}{2}g(t')^2$$

$$\frac{1}{2}g(t')^2 - (v_T \tan\theta)t' - (y_0 + 2500\tan\theta) = 0$$

Substitute the values: $\tan(18.9°) \approx 0.343$:

$$\frac{1}{2}(9.81)(t')^2 - (10 \times 0.343)t' - (200 + 2500 \times 0.343) = 0$$

$$4.905(t')^2 - 3.43t' - 1057.5 = 0$$

Solving for t':

$$t' = \frac{3.43 \pm \sqrt{(-3.43)^2 - 4(4.905)(-1057.5)}}{2(4.905)} = \frac{3.43 \pm 144.08}{9.81}$$

Since $t' > 0$, we take the positive root: $t' = \frac{147.51}{9.81} \approx 15.04\,\text{s}$. Now, we find v_0':

$$v_0' = \frac{2500 + v_T t'}{t' \cos\theta} = \frac{2500 + 10(15.04)}{15.04\cos(18.9°)} = \frac{2650.4}{14.23} \approx 186.2\,\text{m/s}$$

The required change in muzzle speed is $\Delta v_0 = v_0' - v_0 = 186.2 - 180 = 6.2\,\text{m/s}$. **Answer:** The muzzle speed must be increased by approximately **6.2** m/s.

Question 2: Orbital Mechanics and Wave Interference

(a) Satellites in Orbit

For a satellite in a circular orbit, gravitational force equals centripetal force:

$$\frac{GM_E m}{r^2} = m\omega^2 r \implies \omega = \sqrt{\frac{GM_E}{r^3}}$$

Using $g = \frac{GM_E}{R_E^2}$, we get $\omega = R_E\sqrt{\frac{g}{r^3}}$. The radii are: $r_X = R_E + h_X = 6.36 \times 10^6 + 0.5 \times 10^6 = 6.86 \times 10^6\,\text{m}$ $r_Y = R_E + h_Y = 6.36 \times 10^6 + 1.0 \times 10^6 = 7.36 \times 10^6\,\text{m}$ The angular velocities are:

$$\omega_X = (6.36 \times 10^6)\sqrt{\frac{9.81}{(6.86 \times 10^6)^3}} \approx 1.135 \times 10^{-3}\,\text{rad/s}$$

$$\omega_Y = (6.36 \times 10^6)\sqrt{\frac{9.81}{(7.36 \times 10^6)^3}} \approx 1.002 \times 10^{-3}\,\text{rad/s}$$

The satellites are aligned again when the difference in their angular displacement is 2π:

$$(\omega_X - \omega_Y)t = 2\pi$$

$$t = \frac{2\pi}{\omega_X - \omega_Y} = \frac{2\pi}{(1.135 - 1.002) \times 10^{-3}} = \frac{2\pi}{0.133 \times 10^{-3}} \approx 47242\,\text{s}$$

In minutes, $t = \frac{47242}{60} \approx 787.4\,\text{minutes}$. **Answer:** The value of t is approximately **787.4** minutes.