Detailed Solutions: 27th Singapore Physics Olympiad (2014)

Question 1: Yo-Yo on a Table

Problem Statement

A yo-yo of mass M, outer radius A, inner radius B, and moment of inertia $I_{cm} = \frac{1}{2}MA^2$ lies on a smooth horizontal table. A string is pulled with force F from the inner radius B at an angle θ.

Solution

(a) Direction of Rolling

To determine the direction of rolling, we analyze the torque τ_P about the instantaneous axis of rotation, which is the contact point P between the yo-yo and the table. The yo-yo rolls in the direction that this torque tries to rotate it.

The torque is defined as:

$$\vec{\tau}_P = \vec{r} \times \vec{F} \tag{1}$$

where $\vec{r}$ is the vector from P to the point of force application.

1. **Case $\theta = 0$ (Horizontal pull to the right):** The force acts horizontally to the right at a height $A - B$ (assuming the string comes from the bottom of the inner hub). The line of action passes *above* the contact point P.
   - The torque about P is **clockwise**.
   - **Result:** The yo-yo rolls to the **RIGHT**.
2. **Case $\theta = \pi/2$ (Vertical pull upwards):** The force acts vertically upwards tangent to the right side of the inner hub. The line of action is at a horizontal distance B to the right of P.
   - The force exerts an upward pull at a lever arm B.
   - The torque about P is **counter-clockwise**.
   - **Result:** The yo-yo rolls to the **LEFT**.
3. **Case $\theta = \pi$ (Horizontal pull to the left):** The force pulls the string to the left. This creates a counter-clockwise torque about the center, and also about the contact point P (the line of action is at height $A + B$ or $A - B$ depending on winding, but generally above P).

- The torque about P is **counter-clockwise**.
- **Result:** The yo-yo rolls to the **LEFT**.

(b) Critical Angle for Sliding

The yo-yo is in rotational equilibrium (on the verge of sliding without rolling) when the net torque about the contact point P is zero. This occurs when the line of action of the force F passes directly through the contact point P.

Consider the geometry:

- The string leaves the inner radius B tangentially.
- The line of the string must pass through the bottom contact point P.
- We form a right-angled triangle with the radius B as the side opposite to the angle at P, and the outer radius A as the hypotenuse connecting the center to P.

The angle θ is measured from the horizontal. The angle of the string relative to the vertical is α. From geometry, the angle θ satisfies:

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{B}{A} \tag{2}$$

Thus, the critical angle is:

$$\theta_{crit} = \arccos\left(\frac{B}{A}\right) \tag{3}$$

(c) Acceleration for Vertical Pull ($\theta = \pi/2$)

We apply Newton's Second Law for rotation about the instantaneous contact point P.

- The force F is applied vertically upwards at a horizontal distance B from the center.
- The contact point P is at a horizontal distance 0 from the center.
- The lever arm for the torque about P is B.

The torque about P is:

$$\tau_P = F \cdot B \tag{4}$$

The moment of inertia about P (using the Parallel Axis Theorem) is:

$$I_P = I_{cm} + MA^2 = \frac{1}{2}MA^2 + MA^2 = \frac{3}{2}MA^2 \tag{5}$$

Using the rotational dynamic equation $\tau_P = I_P\alpha$:

$$FB = \left(\frac{3}{2}MA^2\right)\alpha \tag{6}$$

Solving for angular acceleration α:

$$\alpha = \frac{2FB}{3MA^2} \tag{7}$$

The linear acceleration a of the center of mass (assuming rolling without slipping) is related by $a = A\alpha$:

$$a = A\left(\frac{2FB}{3MA^2}\right) = \frac{2FB}{3MA} \tag{8}$$

Answer: The acceleration is $\mathbf{a} = \mathbf{\frac{2FB}{3MA}}$ directed to the left.

Question 2: Bead on a Semicircle

Problem Statement

A bead of mass m slides from rest from the top of a smooth semicircle of radius R. Find the reaction force N as a function of the angle θ from the vertical.

Solution

1. Velocity via Conservation of Energy

Let the top of the semicircle be the reference point for zero potential energy, or measure height from the center.

- Initial state ($\theta = 0$): $v = 0$. Height $h_i = R$.
- Final state (θ): Velocity v. Height $h_f = R\cos\theta$.

Conservation of Energy:

$$E_{total} = mgR = mg(R\cos\theta) + \frac{1}{2}mv^2 \tag{9}$$

Rearranging to solve for v^2:

$$mgR(1-\cos\theta) = \frac{1}{2}mv^2 \tag{10}$$

$$v^2 = 2gR(1-\cos\theta) \tag{11}$$

2. Force Analysis (Radial Direction)

We apply Newton's Second Law in the radial direction (towards the center of the semicircle). The forces acting radially are:

- The component of gravity: $mg\cos\theta$ (inwards).
- The normal reaction force: N (outwards).

The net radial force provides the centripetal acceleration:

$$\sum F_{rad} = mg\cos\theta - N = \frac{mv^2}{R} \tag{12}$$

Solving for the normal force N:

$$N = mg\cos\theta - \frac{m}{R}(v^2) \tag{13}$$

Substitute the expression for v^2:

$$N = mg\cos\theta - \frac{m}{R}\left[2gR(1-\cos\theta)\right] \tag{14}$$

$$N = mg\cos\theta - 2mg(1-\cos\theta) \tag{15}$$

$$N = mg\cos\theta - 2mg + 2mg\cos\theta \tag{16}$$

Final Expression:

$$N(\theta) = mg(3\cos\theta - 2) \tag{17}$$

3. Point of Losing Contact

The bead loses contact when the normal force drops to zero ($N = 0$):

$$3\cos\theta - 2 = 0 \implies \cos\theta = \frac{2}{3} \tag{18}$$

$$\theta_{separation} = \arccos\left(\frac{2}{3}\right) \approx 48.2^\circ \tag{19}$$

Question 3: Binary Star System

Problem Statement

Two stars with masses M_1 and M_2 orbit their common center of mass.

- Orbital period: T
- Maximum observed orbital velocities: v_1 and v_2.

Determine the total mass of the system.

Solution

1. Center of Mass Frame

In the center of mass frame, the total momentum is zero.

$$M_1 v_1 = M_2 v_2 \implies \frac{M_1}{M_2} = \frac{v_2}{v_1} \tag{20}$$

2. Orbital Radii

Assuming circular orbits, the velocity is related to the orbital radius r and period T by $v = \frac{2\pi r}{T}$.

$$r_1 = \frac{v_1 T}{2\pi}, \quad r_2 = \frac{v_2 T}{2\pi} \tag{21}$$

The total separation R between the stars is:

$$R = r_1 + r_2 = \frac{T}{2\pi}(v_1 + v_2) \tag{22}$$

3. Dynamics (Kepler's Third Law)

The gravitational force provides the centripetal force for star 1:

$$\frac{GM_1M_2}{R^2} = M_1\frac{v_1^2}{r_1} = M_1 r_1 \omega^2 \tag{23}$$

where $\omega = \frac{2\pi}{T}$.

Canceling M_1:

$$\frac{GM_2}{R^2} = r_1\omega^2 \tag{24}$$

Similarly for star 2:

$$\frac{GM_1}{R^2} = r_2\omega^2 \tag{25}$$

Adding these two equations:

$$\frac{G(M_1 + M_2)}{R^2} = (r_1 + r_2)\omega^2 = R\omega^2 \tag{26}$$

$$G(M_1 + M_2) = R^3\omega^2 \tag{27}$$

4. Solving for Total Mass

Substitute $R = \frac{T(v_1+v_2)}{2\pi}$ and $\omega = \frac{2\pi}{T}$:

$$M_1 + M_2 = \frac{1}{G}\left[\frac{T(v_1+v_2)}{2\pi}\right]^3\left[\frac{2\pi}{T}\right]^2 \tag{28}$$

Simplifying the constants:

$$M_1 + M_2 = \frac{1}{G}\frac{T^3(v_1+v_2)^3}{8\pi^3}\frac{4\pi^2}{T^2} \tag{29}$$

$$M_{total} = \frac{T(v_1+v_2)^3}{2\pi G} \tag{30}$$

Question 4: Thermodynamic Cycle

Problem Statement

An ideal gas undergoes a cycle consisting of thermodynamic processes (e.g., isothermal, adiabatic, isobaric). Calculate the efficiency.

Solution

1. General Definition of Efficiency

The thermal efficiency η of a heat engine is defined as the ratio of net work done to the heat absorbed.

$$\eta = \frac{W_{net}}{Q_{in}} = \frac{Q_{in} - |Q_{out}|}{Q_{in}} = 1 - \frac{|Q_{out}|}{Q_{in}} \tag{31}$$

2. Calculating Work and Heat for Processes

For an ideal gas with equation of state $PV = nRT$:

- **Isobaric Process (Constant P):**

$$W = P\Delta V, \quad Q = nC_P \Delta T \tag{32}$$

- **Isochoric Process (Constant V):**

$$W = 0, \quad Q = nC_V \Delta T \tag{33}$$

- **Isothermal Process (Constant T):**

$$W = nRT \ln\left(\frac{V_f}{V_i}\right), \quad \Delta U = 0, \quad Q = W \tag{34}$$

- **Adiabatic Process ($Q = 0$):**

$$PV^\gamma = \text{const}, \quad W = \frac{P_f V_f - P_i V_i}{1 - \gamma} = -\Delta U \tag{35}$$

3. Calculation Procedure

1. Determine P, V, T at all cycle vertices using the Ideal Gas Law. 2. Calculate Q for each segment. Identify segments where $Q > 0$ (Heat In) and $Q < 0$ (Heat Out). 3. Sum the positive heat values to find Q_{in}. 4. Sum the work done over the cycle to find W_{net} (Area enclosed). 5. Compute $\eta = W_{net}/Q_{in}$.

Question 5: Diffraction Grating

Problem Statement

Light containing two wavelengths λ_1 and λ_2 is incident on a diffraction grating with line density N (lines/mm).

Solution

1. Grating Equation

The condition for constructive interference maxima is:

$$d \sin \theta_m = m\lambda \tag{36}$$

where:

- $d = 1/N$ is the grating spacing.
- m is the order of the spectrum $(0, 1, 2, \dots)$.
- θ_m is the diffraction angle.

2. Angular Separation

For a small difference in wavelength $\Delta\lambda = \lambda_2 - \lambda_1$, the difference in angle $\Delta\theta$ can be found by differentiating the grating equation:

$$d(\cos \theta)\, d\theta = m\, d\lambda \tag{37}$$

$$\Delta\theta \approx \frac{m\Delta\lambda}{d \cos \theta} \tag{38}$$

3. Resolving Power

The resolving power R required to distinguish two wavelengths λ and $\lambda + \Delta\lambda$ is:

$$R = \frac{\lambda}{\Delta\lambda} \tag{39}$$

For a diffraction grating of order m with N_{total} total illuminated lines:

$$R_{grating} = mN_{total} \tag{40}$$

Resolution occurs if $R_{grating} \geq \frac{\lambda}{\Delta\lambda}$.

Question 6: Electric Field of a Charged Rod

Problem Statement

Find the electric field E at a point P distance y from the center of a uniformly charged rod of length L and total charge Q. P is on the perpendicular bisector.

Solution

1. Charge Element

Let $\lambda = Q/L$ be the linear charge density. Consider a small element dx at position x along the rod.

$$dq = \lambda dx \tag{41}$$

The distance from this element to P is $r = \sqrt{x^2 + y^2}$.

2. Field Contribution

The magnitude of the field due to dq is:

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} = \frac{k\lambda dx}{x^2 + y^2} \tag{42}$$

3. Symmetry and Components

Due to symmetry, the horizontal components of the field cancel. We only integrate the vertical component dE_y.

$$dE_y = dE\cos\theta \tag{43}$$

where $\cos\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$.

4. Integration

$$E_{net} = \int_{-L/2}^{L/2} \frac{k\lambda dx}{x^2 + y^2}\left(\frac{y}{\sqrt{x^2 + y^2}}\right) \tag{44}$$

$$E_{net} = k\lambda y \int_{-L/2}^{L/2} \frac{dx}{(x^2 + y^2)^{3/2}} \tag{45}$$

Using the integral result $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$:

$$E_{net} = k\lambda y \left[\frac{x}{y^2\sqrt{x^2 + y^2}}\right]_{-L/2}^{L/2} \tag{46}$$

Evaluating the limits:

$$E_{net} = \frac{k\lambda}{y}\left(\frac{L/2}{\sqrt{(L/2)^2 + y^2}} - \frac{-L/2}{\sqrt{(-L/2)^2 + y^2}}\right) \tag{47}$$

$$E_{net} = \frac{k\lambda}{y}\frac{L}{\sqrt{(L/2)^2 + y^2}} \tag{48}$$

Since $Q = \lambda L$:

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{y\sqrt{y^2 + (L/2)^2}} \tag{49}$$

Question 7: Solenoid and Induction

Problem Statement

A long solenoid has n turns per meter and carries current $I(t) = I_0 \cos(\omega t)$. A small coil of N turns, radius r, and resistance R is placed inside.

Solution

1. Magnetic Field

The magnetic field inside a long solenoid is uniform and parallel to the axis:

$$B(t) = \mu_0 n I(t) = \mu_0 n I_0 \cos(\omega t) \tag{50}$$

2. Magnetic Flux

The flux through the small coil is:

$$\Phi_{coil} = N \cdot B \cdot A = N(\mu_0 n I_0 \cos(\omega t))(\pi r^2) \tag{51}$$

3. Induced EMF

Using Faraday's Law:

$$\mathcal{E} = -\frac{d\Phi_{coil}}{dt} \tag{52}$$

$$\mathcal{E} = -N\pi r^2 \mu_0 n I_0 \frac{d}{dt}[\cos(\omega t)] \tag{53}$$

$$\mathcal{E} = -N\pi r^2 \mu_0 n I_0 [-\omega \sin(\omega t)] \tag{54}$$

$$\mathcal{E} = N\pi r^2 \mu_0 n I_0 \omega \sin(\omega t) \tag{55}$$

4. Induced Current

Using Ohm's Law $I_{ind} = \mathcal{E}/R$:

$$I_{ind}(t) = \frac{N\pi r^2 \mu_0 n I_0 \omega}{R} \sin(\omega t) \tag{56}$$

Question 8: Relativistic Decay

Problem Statement

A particle travels a distance L in the lab frame before decaying. Its proper lifetime is τ. Determine its speed v.

Solution

1. Time Dilation

The lifetime of the particle in the laboratory frame, t_{lab}, is dilated compared to the proper time τ measured in the particle's rest frame.

$$t_{lab} = \gamma\tau = \frac{\tau}{\sqrt{1 - v^2/c^2}} \tag{57}$$

2. Distance Relation

The distance traveled in the lab is:

$$L = vt_{lab} = v\gamma\tau = \frac{v\tau}{\sqrt{1 - v^2/c^2}} \tag{58}$$

3. Solving for Speed v

Square both sides:

$$L^2 = \frac{v^2\tau^2}{1 - v^2/c^2} \tag{59}$$

$$L^2 \left(1 - \frac{v^2}{c^2}\right) = v^2\tau^2 \tag{60}$$

$$L^2 = v^2\tau^2 + L^2\frac{v^2}{c^2} = v^2\left(\tau^2 + \frac{L^2}{c^2}\right) \tag{61}$$

Isolating v^2:

$$v^2 = \frac{L^2}{\tau^2 + L^2/c^2} = \frac{c^2L^2}{c^2\tau^2 + L^2} \tag{62}$$

$$v = \frac{cL}{\sqrt{L^2 + c^2\tau^2}} \tag{63}$$

Question 9: Quantum Potential Well

Problem Statement

A particle of mass m is confined in a 1D infinite potential well of width a ($0 < x < a$). Find the energy levels and wavefunctions.

Solution

1. Schrödinger Equation

Inside the well ($V = 0$), the time-independent Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \tag{64}$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0, \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar} \tag{65}$$

2. Boundary Conditions

The potential is infinite outside, so the wavefunction must vanish at the boundaries:

$$\psi(0) = 0 \quad \text{and} \quad \psi(a) = 0 \tag{66}$$

The general solution is $\psi(x) = A\sin(kx) + B\cos(kx)$.

- $\psi(0) = B = 0 \implies \psi(x) = A\sin(kx)$.
- $\psi(a) = A\sin(ka) = 0$.

This implies $ka = n\pi$ for integer $n = 1, 2, 3, \ldots$.

3. Energy Levels

Using $k_n = \frac{n\pi}{a}$:

$$\frac{\sqrt{2mE_n}}{\hbar} = \frac{n\pi}{a} \tag{67}$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{68}$$

4. Normalized Wavefunctions

Normalization condition $\int_0^a |\psi|^2 dx = 1$:

$$\int_0^a A^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = A^2\left(\frac{a}{2}\right) = 1 \tag{69}$$

$$A = \sqrt{\frac{2}{a}} \tag{70}$$

The wavefunctions are:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \tag{71}$$

Question 10: Radioactive Dating

Solution

(a) Decay Reaction

Rubidium-87 undergoes beta-minus decay to Strontium-87. A neutron turns into a proton, emitting an electron and an antineutrino.

$$^{87}_{37}\text{Rb} \to {}^{87}_{38}\text{Sr} + e^- + \bar{\nu}_e \tag{72}$$

(b) The Isochron Equation

The number of daughter nuclei $D(t)$ (^{87}Sr) at time t is given by the initial amount D_0 plus the amount generated by the decay of the parent P (^{87}Rb).

$$D_t = D_0 + (P_0 - P_t) \tag{73}$$

Since $P_t = P_0 e^{-\lambda t}$, we have $P_0 = P_t e^{\lambda t}$. Substituting this:

$$D_t = D_0 + P_t(e^{\lambda t} - 1) \tag{74}$$

Dividing by the constant amount of stable isotope S (^{86}Sr):

$$\frac{D_t}{S} = \frac{D_0}{S} + \frac{P_t}{S}(e^{\lambda t} - 1) \tag{75}$$

This is the equation of a line $y = c + mx$, where:

- $y = (^{87}\text{Sr}/^{86}\text{Sr})_{now}$
- $x = (^{87}\text{Rb}/^{86}\text{Sr})_{now}$
- Slope $m = e^{\lambda t} - 1$

(c) Determining Age

From the slope m of the isochron plot:

$$m = e^{\lambda t} - 1 \implies e^{\lambda t} = m + 1 \tag{76}$$

$$t = \frac{1}{\lambda} \ln(m + 1) \tag{77}$$

Using the half-life $T_{1/2}$, where $\lambda = \ln 2 / T_{1/2}$:

$$t = T_{1/2} \frac{\ln(m + 1)}{\ln 2} \tag{78}$$