SPhO 2022 Theory Paper 2 Solutions

1 Mechanics

1.1 Compound Pendulum Collision

This problem involves an inelastic collision followed by a pendulum swing. We apply the conservation of angular momentum for the collision and the conservation of mechanical energy for the swing.

1.1.1 1. Moment of Inertia Calculation

The total moment of inertia (I_{total}) of the system (rod + disc + object) about the pivot point A after the collision is calculated.

- **Rod:** $I_{\text{rod}} = \frac{1}{3}M_{\text{rod}}l^2 = \frac{1}{3}(1.5 \text{ kg})(0.50 \text{ m})^2 = 0.125 \text{ kg m}^2$

- **Disc (Parallel Axis Theorem):** $I_{\text{disc}} = I_{\text{cm}} + M_{\text{disc}}d^2 = \frac{1}{2}M_{\text{disc}}R^2 + M_{\text{disc}}l^2$ $I_{\text{disc}} = \frac{1}{2}(3.0)(0.075)^2 + (3.0)(0.50)^2 = 0.0084375 + 0.75 = 0.7584375 \text{ kg m}^2$

- **Object:** $I_{\text{obj}} = m_{\text{obj}}l^2 = (0.1 \text{ kg})(0.50 \text{ m})^2 = 0.025 \text{ kg m}^2$

- **Total Moment of Inertia:** $I_{\text{total}} = I_{\text{rod}} + I_{\text{disc}} + I_{\text{obj}} = 0.125 + 0.7584375 + 0.025 = 0.9084375 \text{ kg m}^2$

1.1.2 2. Conservation of Angular Momentum

Angular momentum about pivot A is conserved during the collision.

$$L_{\text{initial}} = L_{\text{final}}$$
$$m_{\text{obj}}vl = I_{\text{total}}\omega$$
$$(0.1)(30)(0.50) = (0.9084375)\omega$$
$$\omega = \frac{1.5}{0.9084375} \approx 1.6511 \text{ rad/s}$$

1.1.3 3. Conservation of Energy

The rotational kinetic energy is converted into gravitational potential energy.

- **Center of Mass (from pivot A):** $y_{\text{cm}} = \frac{M_{\text{rod}}(l/2) + M_{\text{disc}}(l) + m_{\text{obj}}(l)}{M_{\text{total}}} = \frac{(1.5)(0.25) + (3.0)(0.5) + (0.1)(0.5)}{4.6} \approx 0.4185 \text{ m}$

- **Energy Conservation:**

$$\frac{1}{2}I_{\text{total}}\omega^2 = M_{\text{total}}g\Delta h_{\text{cm}} = M_{\text{total}}gy_{\text{cm}}(1 - \cos\theta)$$
$$\frac{1}{2}(0.9084375)(1.6511)^2 = (4.6)(9.81)(0.4185)(1 - \cos\theta)$$
$$1.2396 = 18.88(1 - \cos\theta)$$
$$1 - \cos\theta = \frac{1.2396}{18.88} \approx 0.06565$$
$$\cos\theta \approx 0.93435$$
$$\theta = \arccos(0.93435) \approx 20.9°$$

Note: The calculated answer is $20.9°$. The provided answer in the paper is $17.5°$. This discrepancy may be due to a typo in the problem's given values.

1.2 Alloy Composition

Using Archimedes' principle to find the composition of the alloy.

- **Buoyant Force and Volume:** Buoyant force $F_B = W_{\text{air}} - W_{\text{sol}} = 10.0 - 9.326 = 0.674$ kgf. This corresponds to a displaced mass of $m_{\text{sol}} = 0.674$ kg. Total volume of the block: $V_{\text{total}} = \frac{m_{\text{sol}}}{\rho_{\text{sol}}} = \frac{0.674}{1230} \approx 5.4797 \times 10^{-4}$ m^3.

- **System of Equations:**

$$m_{\text{Au}} + m_{\text{Cu}} = 10.0 \quad \text{(Mass Conservation)}$$
$$\frac{m_{\text{Au}}}{\rho_{\text{Au}}} + \frac{m_{\text{Cu}}}{\rho_{\text{Cu}}} = V_{\text{total}} \quad \text{(Volume Conservation)}$$

- **Solving for Masses:** Substitute $m_{\text{Cu}} = 10.0 - m_{\text{Au}}$ into the volume equation:

$$\frac{m_{\text{Au}}}{19300} + \frac{10.0 - m_{\text{Au}}}{8960} = 5.4797 \times 10^{-4}$$
$$m_{\text{Au}}\left(\frac{1}{19300} - \frac{1}{8960}\right) = 5.4797 \times 10^{-4} - \frac{10.0}{8960}$$
$$m_{\text{Au}}(-5.9797 \times 10^{-5}) = -5.6813 \times 10^{-4}$$
$$m_{\text{Au}} \approx 9.50 \text{ kg}$$

Then, $m_{\text{Cu}} = 10.0 - 9.50 = 0.50$ kg.

- **Percentage by Mass:** Gold: $\frac{9.50}{10.0} \times 100\% = \mathbf{95}\%$. Copper: $\frac{0.50}{10.0} \times 100\% = \mathbf{5}\%$.

2 Waves and Oscillations

2.1 Beats from a Vibrating Wire

- **Initial State:** $f_1 = 256 \pm 5$ Hz $\implies f_1 = 261$ Hz or $f_1 = 251$ Hz.

- **Final State:** $f_2 = 256 \pm 3$ Hz $\implies f_2 = 259$ Hz or $f_2 = 253$ Hz.

- **Condition:** Tension increases, so frequency increases ($f_2 > f_1$). This implies the initial frequency must be $f_1 = 251$ Hz. Both $f_2 = 259$ Hz and $f_2 = 253$ Hz are possible final frequencies.

- **Ratio Calculation:** Since $f \propto \sqrt{F}$, we have $\frac{F_2}{F_1} = \left(\frac{f_2}{f_1}\right)^2$.

  - Case 1: $\frac{F_2}{F_1} = \left(\frac{259}{251}\right)^2 \approx \mathbf{1.065}$
  - Case 2: $\frac{F_2}{F_1} = \left(\frac{253}{251}\right)^2 \approx \mathbf{1.016}$

2.2 Springs System

2.2.1 (i) Bar in Horizontal Equilibrium

- **Equilibrium Conditions:** $\sum F_y = F_1 + F_2 - W = 0 \implies k_1 \Delta l + k_2 \Delta l = W \implies (30 + 50)\Delta l = 20 \implies \Delta l = 0.25$ m. $\sum \tau_A = F_2(0.6 - x) - W(0.3) = 0$.

- **Solving for x:** $F_2 = k_2 \Delta l = 50(0.25) = 12.5$ N. $12.5(0.6 - x) - 20(0.3) = 0 \implies 12.5(0.6 - x) = 6 \implies 0.6 - x = 0.48 \implies x = \mathbf{0.12}$ m.

2.2.2 (ii) Particle Oscillating Between Springs

- **Equilibrium Position:** $k_1(l_1 - l_0) = k_2(l_2 - l_0)$ and $l_1 + l_2 = 0.5$. $30(l_1 - 0.2) = 50(0.5 - l_1 - 0.2) \implies 30l_1 - 6 = 15 - 50l_1 \implies 80l_1 = 21 \implies l_1 = \mathbf{0.2625}$ m. $l_2 = 0.5 - 0.2625 = \mathbf{0.2375}$ m.

- **Frequency of Oscillation:** The system behaves as two springs in parallel, so $k_{\text{eff}} = k_1 + k_2 = 80$ N/m. The angular frequency is $\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{80}{0.05}} = 40$ rad/s. The frequency is $f = \frac{\omega}{2\pi} = \frac{40}{2\pi} \approx \mathbf{6.37}$ Hz.

3   Electromagnetism

3.1   Wire in a Magnetic Field

For zero tension, the magnetic force must balance the gravitational force.

$$F_m = F_g$$
$$ILB = mg$$
$$I(0.5)(0.04) = (0.02)(9.81)$$
$$I = \frac{0.1962}{0.02} = \textbf{9.81 A}$$

Using the right-hand rule, for an upward force with $\vec{B}$ into the page, the current must flow **from A to B**. The potential difference is $V = IR = (9.81)(1.2) = \textbf{11.77 V}$.

3.2   Force on a Point Charge from a Disc

- **Electric Field of an Infinitesimal Ring:** The charge on a ring of radius r and thickness dr is $dQ = \sigma(r)dA = (kr)(2\pi r dr) = 2\pi kr^2 dr$. The axial electric field from this ring is $dE = \frac{dQ}{4\pi\epsilon_0}\frac{x}{(r^2+x^2)^{3/2}} = \frac{kx}{2\epsilon_0}\frac{r^2}{(r^2+x^2)^{3/2}}dr$.

- **Total Electric Field (Integration):**

$$E = \int_0^R \frac{kx}{2\epsilon_0}\frac{r^2}{(r^2+x^2)^{3/2}}dr$$
$$= \frac{kx}{2\epsilon_0}\left[\ln(r+\sqrt{r^2+x^2}) - \frac{r}{\sqrt{r^2+x^2}}\right]_0^R$$
$$\text{For } x = 0.5, R = 0.1: \quad E \approx \frac{(1.05\times10^{-6})(0.5)}{2(8.85\times10^{-12})}(0.0026) \approx 77.1 \text{ N/C}$$

- **Electrostatic Force:** $\vec{F} = q\vec{E} = (10\times10^{-9}\text{ C})(77.1\text{ N/C})\hat{i} \approx \textbf{7.71}\times\textbf{10}^{-7}\hat{\textbf{i}}$ **N**.

Note: The calculated force is 7.71×10^{-7} N. The provided answer in the paper is 1.523×10^{-6} N, which is nearly double this value. This may indicate a typo.

4   Thermodynamics

4.1   Blackbody Radiation

4.1.1   (i) Maximum Temperature

At equilibrium, power in equals power out (Stefan-Boltzmann law).

$$P_{\text{in}} = \sigma\epsilon A(T_{\max}^4 - T_{\text{env}}^4)$$
$$1800 = (5.67\times10^{-8})(1)(4\pi(0.3)^2)(T_{\max}^4 - 293.15^4)$$
$$T_{\max}^4 = \frac{1800}{(5.67\times10^{-8})(1.131)} + 293.15^4 \approx 3.544\times10^{10}$$
$$T_{\max} = (3.544\times10^{10})^{1/4} \approx 433.8 \text{ K}$$

Note: The calculated temperature is 433.8 K. The provided answer is 318.0 K, which would correspond to a heater power of $\sim$180 W, suggesting a typo in the problem (1.8 kW).

4.1.2   (ii) Initial Rate of Fall of Temperature

The rate of heat loss is $\frac{dQ}{dt} = -mc\frac{dT}{dt}$.

- **Mass:** $m = \rho V = (8940)\left(\frac{4}{3}\pi(0.3)^3\right) \approx 1011.1$ kg.

- At the moment the heater is switched off, the rate of heat loss is equal to the heater power, $\frac{dQ}{dt} = 1800$ W.

- Rate of temperature fall:

$$-\frac{dT}{dt} = \frac{1}{mc}\frac{dQ}{dt} = \frac{1800 \text{ W}}{(1011.1 \text{ kg})(389 \text{ J kg}^{-1}\text{K}^{-1})} \approx \mathbf{4.58 \times 10^{-3}} \text{ K/s}$$

4.2 Gas Pistons System

4.2.1 (i) Changes in Argon Gas

- Oxygen (isothermal): $p_{b,f} = 2p_0$. Argon (adiabatic): $p_{a,f} = p_{b,f} = 2p_0$.

- **Pressure Change (Argon):** $p_a V_a^\gamma = p_{a,f} V_{a,f}^\gamma \implies p_a V_a^{5/3} = (2p_0)(8V_a)^{5/3} \implies p_a = 64p_0$. $\Delta p = p_{a,f} - p_a = 2p_0 - 64p_0 = \mathbf{-62p_0}$.

- **Temperature Change (Argon):** $T_a V_a^{\gamma-1} = T_{a,f} V_{a,f}^{\gamma-1} \implies T_a V_a^{2/3} = T_{a,f}(8V_a)^{2/3} \implies T_{a,f} = T_a/4$. $\Delta T = T_{a,f} - T_a = T_a/4 - T_a = -\frac{3}{4}\mathbf{T_a}$.

4.2.2 (ii) Final Pressure of Mixture

- Total volume: $V_{\text{total}} = V_{a,f} + V_{b,f} = 8V_a + 7V_a = 15V_a$.

- Moles of Oxygen: $n = \frac{|Q|}{RT_0 \ln(2)}$.

- Total moles: $n_{\text{total}} = n_a + n = 8 + \frac{|Q|}{RT_0 \ln(2)}$.

- Final Pressure (Ideal Gas Law): $p_{\text{final}} V_{\text{total}} = n_{\text{total}} RT_0$.

$$p_{\text{final}}(15V_a) = \left(8 + \frac{|Q|}{RT_0 \ln(2)}\right) RT_0$$

$$p_{\text{final}} = \frac{8RT_0 + \frac{|Q|}{\ln(2)}}{15V_a}$$

5 Relativity

5.1 Relativistic Density

5.1.1 Method 1: Using Relativistic Mass

$\rho = \frac{m}{V} = \frac{\gamma m_0}{V_0/\gamma} = \gamma^2 \rho_0 = \frac{\rho_0}{1-v^2/c^2}$.

$$1.25 = \frac{1}{1-\beta^2} \implies 1 - \beta^2 = 0.8 \implies \beta^2 = 0.2$$

$$v = \sqrt{0.2}c \approx \mathbf{0.447c}$$

5.1.2 Method 2: Using Invariant Mass

$\rho = \frac{m_0}{V} = \frac{m_0}{V_0/\gamma} = \gamma\rho_0 = \frac{\rho_0}{\sqrt{1-v^2/c^2}}$.

$$1.25 = \frac{1}{\sqrt{1-\beta^2}} \implies \sqrt{1-\beta^2} = 0.8 \implies 1 - \beta^2 = 0.64$$

$$\beta^2 = 0.36 \implies v = \mathbf{0.6c}$$

5.2 Time Intervals Between Spaceships

5.2.1 Part 1: Time Interval in Earth Frame (Δt_S)

- Contracted lengths in Earth frame: $L_A = L_{0A}\sqrt{1 - 0.8^2} = 200(0.6) = 120$ m. $L_B = L_{0B}\sqrt{1 - (-0.6)^2} = 150(0.8) = 120$ m.

- Time for tails to pass: $\Delta t_S(v_A - v_B) = L_A + L_B$ $\Delta t_S = \frac{120+120}{0.8c-(-0.6c)} = \frac{240}{1.4c} \approx \mathbf{5.71 \times 10^{-7}}$ s.

5.2.2 Part 2: Time Interval in Spaceship A's Frame (Δt_A)

- Use Lorentz transformation $t' = \gamma_A(t - v_A x/c^2)$ with $v_A = 0.8c$ and $\gamma_A = 5/3$.

- Event 1 (noses pass) in S: $(t_1, x_1) = (0, 0)$.

- Event 2 (tails pass) in S: $t_2 = \Delta t_S$, $x_2 = v_A t_2 - L_A = \frac{120}{7}$ m.

- Transforming event times to A's frame (S'): $t'_1 = 0$.

$$t'_2 = \gamma_A(t_2 - v_A x_2/c^2) = \frac{5}{3}\left(\frac{240}{1.4c} - \frac{0.8c(120/7)}{c^2}\right)$$

$$= \frac{5}{3}\left(\frac{1200}{7c} - \frac{96}{7c}\right) = \frac{5}{3}\frac{1104}{7c} = \frac{1840}{7c}$$

- Time interval in A's frame: $\Delta t_A = t'_2 - t'_1 = \frac{1840}{7(3\times10^8)} \approx \mathbf{8.76 \times 10^{-7}}$ s.

Note: The calculated time in frame S matches the provided answer, but the time in frame A (8.76×10^{-7} s) differs from the paper's answer of 1.715×10^{-7} s.