- **Methods based on Extreme Value Theory**

Review of Extreme Value Theory (EVT)

So far, we have looked at distributions for the entire sample such as $N(\mu, \sigma_a^2)$. We know these work well for the first and second moments i.e near the central region around the mean. However in Finance, fat tails are highly prominent! Although events in the tail region happen rarely, when they do they create severe (sometimes even catastrophic) impacts. So it makes sense to focus on the tail regions instead. Extreme Value Theory is an ingenious technique, introduced by Fisher (1920) to study the same. EVT only focuses on the tails, which is essential to study extreme loss cases. But a drawback is that it converges very slowly.

Let $x_{(n)}$ be the sample maximum of a loss variable (maximum order statistic): $x_{(1)}, x_{(2)}, ..., x_{(n)}$. EVT is concerned with the limiting distribution of $x_{(n)}$, after some proper normalizing, as $n \longrightarrow \infty$.

Let α_n and β_n be the scale and location parameter, respectively. For independent samples, the limiting distribution of the normalized maximum $x^*_{(n)} = \frac{x_{(n)} - \beta_n}{\alpha_n}$ is given by

$$F_*(x) = \begin{cases} \exp[-(1+\xi x)^{-1/\xi}] & \text{if } \xi \neq 0 \\ \\ \exp[-\exp(-x)] & \text{if } \xi = 0 \end{cases} \tag{6.9}$$

for $x < -1/\xi$ if $\xi < 0$ and for $x > -1/\xi$ if $\xi > 0$, where the subscript $*$ signifies the normalized maximum. There are 3 parameters in total to be estimated: the location parameter β_n, the scale parameter α_n, and the shape parameter that defines the nature of the tail ξ. This is the generalized extreme value distribution (GEV). GEV distribution nests all limiting models for the sample maxima for the continuous loss functions.

The parameter ξ is referred to as the shape parameter that governs the tail behavior of the limiting distribution. The parameter $1/\xi$ is called the *tail index* of the distribution.

The GEV distribution encompasses three types of limiting distribution of Gnedenko (1943):

- Type I: $\xi = 0$, the Gumbel family. The CDF is

$$F_*(x) = \exp[-\exp(-x)], \qquad -\infty < x < \infty \tag{6.10}$$

  The right tail will decline exponentially for the Gumbel family of distribution.

- Type II: $\xi > 0$, the Frechet family. The CDF is

$$F_*(x) = \begin{cases} \exp[-(1+\xi x)^{-1/\xi}] & \text{if } x > -1/\xi \\ \\ 1 & \text{otherwise} \end{cases} \tag{6.11}$$

  Here, the right tail declines by a power function. The Frechet family of limiting distributions is especially important in Finance (when dealing with returns) due to the heavy tail presence in the distributions ($\xi > 0$).

– Type III: $\xi < 0$, the Weibull family. The CDF here is

$$F_*(x) = \begin{cases} \exp[-(1+\xi x)^{-1/\xi}] & \text{if } x < -1/\xi \\ 0 & \text{otherwise} \end{cases}$$

For the Weibull family, the right tail is finite.

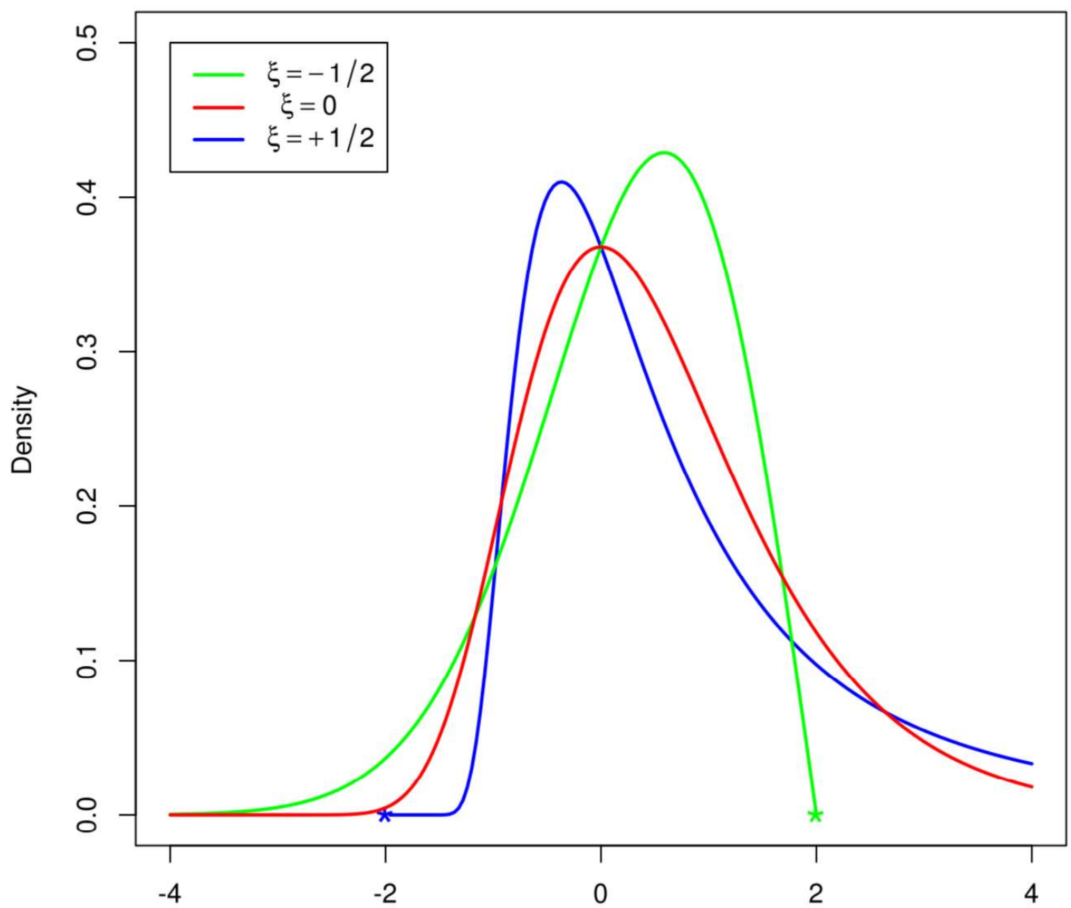

Figure 53: Probability density functions of extreme value distributions for normalized maximum. The red line is for a Gumbel distribution, the green line is for the Weibull distribution with $\xi = -0.5$, and the blue line is for the Frechet distribution with $\xi = 0.5$.

Two important implications of EVT:

1. The tail behavior of the CDF $F(x)$ of x_t determines the limiting distribution $F_*(x)$ of the normalized maximum.

   The sequences $\{\beta_n\}$ and $\{\alpha_n\}$ depend on the CDF $F(x)$. See McNeil, Frey and Embtrechts (2005, Chapter 7).

2. The tail index ξ does not depend on the time interval of x_t. That is, the tail index is invariant under time aggregation; handy in the VaR calculation.

Empirical Estimation and Demonstration: EVT contains three parameters: ξ (shape), β_n (location) and α_n (scale). For a given sample, there is only a single maximum, and we cannot estimate the three parameters with only an extreme observation. One approach is to divide the sample into subsamples and apply the extreme value theory to each subsample. Assume that there are T returns $\{r_t\}_{t=1}^{T}$ available. We divide the sample to g non-overlapping subsamples each with n observations. With n sufficiently large, we hope that the extreme value theory applies to each subsample. Let $x_{n,i}$ be the maximum of the ith subsample. We use the collection of subsample maxima $\{x_{n,i}\}_{i=1}^{g}$ to estimate the EVT parameters $\{\alpha_n, \beta_n, \xi_n\}$. Keep in mind that:

$$\underbrace{T}_{\text{total no. of observations}} = \underbrace{n}_{\text{size of subsample}} \times \underbrace{g}_{\text{no. of subsamples}}$$

Note:

1. if T is not a multiple of n, ignore the first few observation.
2. There is a fight between n and g.
3. For daily data: $n = 21$ or $n = 63$.

There are different techniques to EVT estimation:

1. Block Maxima Method
2. Maximum Likelihood Method
3. The Nonparametric Approach: Estimating the shape parameter ξ by Hill estimator or Pickands estimator.

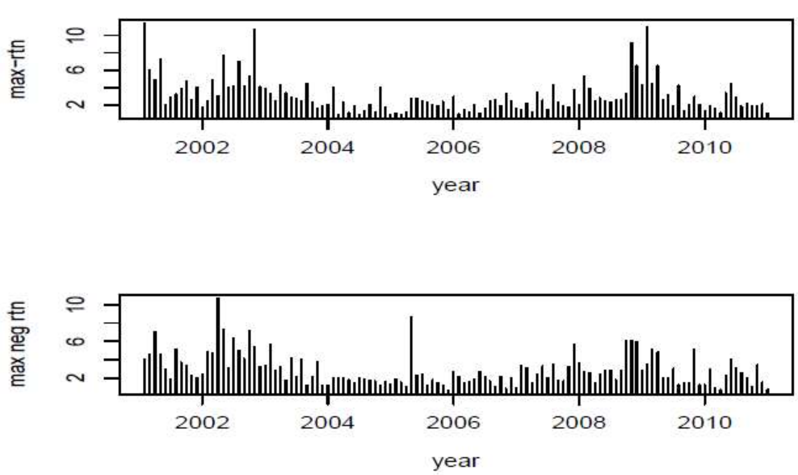

Figure 54: Block maximum of daily log returns of IBM stock, in percentages, when the subperiod is 21 trading days. The data span is from January 2, 2001 to December 31, 2010 so that there are 120 blocks. The upper plot is for positive returns and the lower plot for negative returns.

Application to risk measures:

For a given small upper tail probability p, the VaR of a financial position with loss variable x_t is

$$\mathrm{VaR}_{1-p} = \begin{cases} \beta_n - \frac{\alpha_n}{\xi_n}\{1 - [-n\ln(1-p)]^{-\xi_n}\} & \text{if } \xi_n \neq 0 \\ \\ \beta_n - \alpha_n \ln[-n\ln(1-p)] & \text{if } \xi_n = 0 \end{cases} \tag{6.12}$$

where n is the length of sub-periods.

Summary

We summarize the approach of applying the traditional extreme value theory to VaR calculation as follows:

1. Select the length of the sub-period n and obtain sub-period maxima $\{x_{n,i}\}_{i=1}^{g}$ where $g = [T/n]$.
2. Obtain the maximum likelihood estimates of β_n, α_n, and ξ_n.
3. Check the adequacy of the fitted extreme value model; Define the residuals as

$$\omega_i = (1 + \xi_n \frac{r_{n,i} - \beta_n}{\alpha_n})^{-1/\xi_n}$$

   $\{\omega_i\}$ should form an i.i.d random sample of exponential distribution if the fitted model is correctly specified.
4. If the extreme value model is adequate, apply Equation (6.12) to calculate VaR.

Multi-period VaR under EVT:

$$\mathrm{VaR}(\ell) = \ell^{\xi}\mathrm{VaR}$$

Example: Consider the daily log returns, in percentage, of IBM stock.

We have $\hat{\alpha}_n = 1.029$, $\hat{\beta}_n = 1.966$, and $\hat{\xi}_n = 0.251$ for $n = 21$. Therefore, for the left-tail probability $p = 0.05$, the corresponding VaR is

$$\mathrm{VaR}_{0.95} = 1.966 - \frac{1.029}{0.251}\{1 - [-21\ln(1-0.05)]^{-0.251}\} = 1.8902$$

Thus, for negative daily log returns of the stock, the upper 5% quantile is 1.8902%. Consequently, we have $\mathrm{VaR}_{0.95} = \$1,000,000 \times 0.018902 = \$18,902$. If the probability is 0.01, then the corresponding $\mathrm{VaR}_{0.99}$ is $39,242.

If we chose $n = 42$ (i.e., approximately 2 months), then $\hat{\alpha}_n = 1.1$, $\hat{\beta}_n = 2.489$, and $\hat{\xi}_n = 0.287$. The upper 1% quantile of the loss variable based on the extreme value distribution is

$$\text{VaR}_{0.99} = 2.489 - \frac{1.1}{0.287}\{1 - [-42\ln(1-0.01)]^{-0.287}\} = 3.5655$$

Therefore, for a long position of \$1,000,000, the corresponding 1-day horizon VaR is \$35,655 at the 1% risk level. If the probability is 0.05, then the corresponding $\text{VaR}_{0.95}$ is \$17,313. In this particular case, the choice of $n = 21$ gives higher VaR values.

Discussion: Applications using daily log returns of IBM stock. If the tail probability is 5%, then the VaR is

1. \$11,730 for the RiskMetrics,
2. \$12,262 for a Gaussian GARCH(1,1) model,
3. \$12,399 for a GARCH(1,1) model with a standardized Student-t distribution with 5.75 degrees of freedom
4. \$26,540 for using the empirical quantile
5. \$13,385 for using quantile regression, and
6. \$17,313 for applying the traditional extreme value theory using $n = 21$ for the length of sub-periods.

If the tail probability is 1%, then the VaR is

1. \$16,590 for the RiskMetrics, a
2. \$17,592 for a Gaussian GARCH(1,1) model,
3. \$20,448 for a GARCH(1,1) model with a standardized Student-t distribution with 5.75 degrees of freedom,
4. \$50,132 for using the empirical quantile, and
5. \$35,655 for applying the traditional extreme value theory using $n = 21$.

■ **Peaks Over Thresholds (POT):** A two-dimensional framework (exceedance and exceeding times)

Calculating VaR using EVT encounters some difficulty; the choice of subperiod length n is not clearly defined. Here, we look at the Peak over Threshold (POT) approach that does not require the choice of a sub-period length, but it requires the specification of threshold, η. In this new approach based on EVT, we focus on the exceedances of the loss over some high threshold η and the number of times these exceedances occur. The basic theory of the POT approach is to consider the conditional distribution of $x = y + \eta$ given $x > \eta$. η is a threshold.

The conditional distribution of $x \leq y + \eta$ given $x > \eta$ is

$$p(x \leq y + \eta | x > \eta) = \frac{p(\eta \leq x \leq y + \eta)}{p(x > \eta)} \tag{6.13}$$

Using the CDF

$$F_*(x) = \begin{cases} \exp[-(1+\xi x)^{-1/\xi}] & \text{if } \xi \neq 0 \\ \\ \exp[-\exp(-x)] & \text{if } \xi = 0 \end{cases}$$

and the approximation $e^{-z} \sim (1-z)$ and after some algebra, we obtain that

$$p(x \leq y + \eta | x > \eta) \sim 1 - (1 + \frac{\xi y}{\alpha + \xi(\eta - \beta)})^{-1/\xi} \tag{6.14}$$

where $y > 0$ and $1 + \frac{\xi(\eta-\beta)}{\alpha} > 0$. The case of $\xi = 0$ is taken as the limit of $\xi \longrightarrow 0$ so that

$$p(x \leq y + \eta | x > \eta) \sim 1 - e^{-y/\alpha} \tag{6.15}$$

- **Generalized Pareto Distribution:**

  If we define $\psi(\eta) = \alpha + \xi(\eta - \beta)$, then we see that Equations 6.14 and 6.15 can be formulated as the generalized Pareto distribution. The probability distribution with cumulative distribution function

$$G_{\xi,\psi(\eta)} = \begin{cases} 1 - [1 + \frac{\xi y}{\psi(\eta)}]^{-1/\xi} & \text{if } \xi \neq 0 \\ \\ 1 - \exp[\frac{-y}{\psi(\eta)}] & \text{if } \xi = 0 \end{cases} \tag{6.16}$$

is called the *generalized Pareto distribution* (GPD). We can calculate VaR and ES using

$$\text{VaR}_{1-p} = \eta - \frac{\psi(\eta)}{\xi}\{1 - [\frac{T}{N_\eta}p]^{-\xi}\} \tag{6.17}$$

$$\text{ES}_{1-p} = \frac{\text{VaR}_{1-p}}{1-\xi} + \frac{\psi(\eta) - \xi\eta}{1-\xi} \tag{6.18}$$

where N_η is the number of observations greater than threshold.

Estimation:

For a given high threshold, use GPD to obtain parameter estimates. For the long position of $1 million on IBM stock, we have

$$\text{VaR}_{0.95} = \$25,855 \quad \text{and} \quad \text{ES}_{0.95} = \$39,625,$$

for the first trading day of 2011 when the threshold of 1% is used. If the threshold is 1.2%, we have

$$\text{VaR}_{0.95} = \$26,115 \quad \text{and} \quad \text{ES}_{0.95} = \$39,603,$$

Finally, for threshold of 0.8%, we have

$$\text{VaR}_{0.95} = \$25,866 \quad \text{and} \quad \text{ES}_{0.95} = \$39,620.$$

An Alternative Parameterization: Use $\psi(\eta) = \alpha + \xi(\eta - \beta)$. The parameter of GPD becomes $(\xi, \psi(\eta))$.

Thr. η	n.exceed	Shape ξ	Scale	Location	$\psi(\eta)$
1	504	0.107(0.042)	0.009(0.001)	−0.006(0.001)	0.011
1.2	410	0.075(0.044)	0.010(0.001)	−0.007(0.002)	0.011
0.8	610	0.106(0.039)	0.009(0.001)	−0.006(0.001)	0.010

Figure 55: Maximum Likelihood Estimates of the Generalized Pareto Distribution For Negative Daily Log Returns of IBM Stock from January 2, 2001 to December 31, 2010. Standard errors are in parentheses and n.exceed denotes the number of exceedances.

R Demonstration:

```
> da=read.table("d-ibm-0110.txt",header=T)
> head(da)
date return
1 20010102 -0.002206
.....
> ibm=log(da[,2]+1)*100
> source("Igarch.R")
> mm=Igarch(ibm)
Coefficient(s):
Estimate Std. Error t value Pr(>|t|)
alpha 0.942857 0.007172 131.464 < 2.22e-16 ***
 ### You may use the result to calculate volatility forecast and VaR ###
> fvariance=m1$par*m1$volatility[2515]^2+(1-m1$par)*ibm[2515]^2
> fvol=sqrt(fvariance)
> fvol
beta
0.7133031
> VaR_0.95=qnorm(.95)*fvol
> VaR_0.95
beta
1.173279
 ### A summary of the calculation is given below
Risk measure based on RiskMetrics:
prob VaR ES
[1,] 0.950 1.173279 1.471339
[2,] 0.990 1.659391 1.901105
[3,] 0.999 2.204272 2.401756
#### GARCH
> xt=-log(da$return+1) % calculate negative log returns.
> library(fGarch)
> m1=garchFit(~garch(1,1),data=xt,trace=F)
> m1
Title: GARCH Modelling
Call: garchFit(formula = ~garch(1, 1), data = xt, trace = F)
Mean and Variance Equation:
data ~ garch(1, 1) [data = xt]
Conditional Distribution: norm
Error Analysis:
Estimate Std. Error t value Pr(>|t|)
mu -6.010e-04 2.393e-04 -2.511 0.012044 *
omega 4.378e-06 1.160e-06 3.774 0.000161 ***
alpha1 1.011e-01 1.851e-02 5.463 4.67e-08 ***
beta1 8.841e-01 1.991e-02 44.413 < 2e-16 ***
```

```
---
> predict(m1,3)
meanForecast meanError standardDeviation
1 -0.0006009667 0.007824302 0.007824302
2 -0.0006009667 0.008043298 0.008043298
3 -0.0006009667 0.008253382 0.008253382
> source("RMeasure.R")
> m11=RMeasure(-.000601,.0078243)
Risk Measures for selected probabilities:
prob VaR ES
[1,] 0.950 0.01226883 0.01553828
[2,] 0.990 0.01760104 0.02025244
[3,] 0.999 0.02357790 0.02574412
>
> m2=garchFit(~garch(1,1),data=xt,trace=F,cond.dist="std")
> m2
Title: GARCH Modelling
Call: garchFit(formula =~garch(1,1), data=xt,cond.dist="std", trace=F)
Mean and Variance Equation:
data ~ garch(1, 1) [data = xt]
Conditional Distribution: std
Error Analysis:
Estimate Std. Error t value Pr(>|t|)
mu -4.113e-04 2.254e-04 -1.824 0.06811 .
omega 1.922e-06 7.417e-07 2.592 0.00954 **
alpha1 6.448e-02 1.323e-02 4.874 1.09e-06 ***
beta1 9.286e-01 1.407e-02 65.993 < 2e-16 ***
shape 5.751e+00 6.080e-01 9.459 < 2e-16 ***
---
> predict(m2,3)
meanForecast meanError standardDeviation
1 -0.0004112738 0.008100872 0.008100872
2 -0.0004112738 0.008191119 0.008191119
3 -0.0004112738 0.008279772 0.008279772
> m22=RMeasure(-.0004113,.0081009,cond.dist="std",df=5.751)
Risk Measures for selected probabilities:
prob VaR ES
[1,] 0.950 0.01545311 0.02184843
[2,] 0.990 0.02542061 0.03294803
[3,] 0.999 0.04289786 0.05332908
##### Empirical quantiles
> ibm=-log(da[,2]+1)
> prob1=c(0.9,0.95,0.99,0.999) % probabilities of interest
> quantile(ibm,prob1)
90% 95% 99% 99.9%
0.01736836 0.02653783 0.05013151 0.07198369
> sibm=sort(ibm) % Sorting into increasing order
> 0.95*2515
[1] 2389.25
> es=sum(sibm[2390:2515])/(2515-2389)
> es
[1] 0.03994857
### Quantile regression
> dd=read.table("d-ibm-rq.txt",header=T) % Load data
> head(dd)
nibm vol vix
1 -0.109478400 0.01700121 29.99
2 0.015308580 0.01614694 26.60
.....
6 -0.009408600 0.03211091 27.99
> dim(dd)
[1] 2514 3
> dd[,3]=dd[,3]/100
> library(quantreg)
> mm=rq(nibm~vol+vix,tau=0.95,data=dd) % Quantile regression
> summary(mm)
Call: rq(formula = nibm ~ vol + vix, tau = 0.95, data = dd)
tau: [1] 0.95 % probability
Coefficients:
Value Std. Error t value Pr(>|t|)
(Intercept) -0.00104 0.00257 -0.40317 0.68686
vol 1.17724 0.22268 5.28660 0.00000
vix 0.02809 0.01615 1.73977 0.08202
```

```
> names(mm)
[1] "coefficients" "x" "y" "residuals"
[5] "dual" "fitted.values" "formula" "terms"
[9] "xlevels" "call" "tau" "rho"
[13] "method" "model"
> fit=mm$fitted.values
> tdx=c(2:2515)/252+2001
> plot(tdx,dd$nibm,type='l',xlab='year',ylab='neg-log-rtn')
> lines(tdx,fit,col='red')
> v1[2515]
[1] 0.008018202
> vix[2515]
[1] 17.75
> vfit=-.00104+1.17724*v1[2515]+0.02809*vix[2515]/100
> vfit
[1] 0.01338532
> mm=rq(xt~vol+vix,tau=0.99,data=dd) % 99th quantile
> summary(mm)
Call: rq(formula = xt ~ vol + vix, tau = 0.99, data = dd)
tau: [1] 0.99
Coefficients:
Value Std. Error t value Pr(>|t|)
(Intercept) 0.01182 0.00831 1.42190 0.15518
vol 1.03129 0.73125 1.41031 0.15857
vix 0.04409 0.05335 0.82641 0.40865
### Extreme value theory
> library(evir) % Load package
> par(mfcol=c(2,1))
> m1=gev(xt,block=21)
> m1
$n.all
[1] 2515
$n
[1] 120
$data
[1] 4.0335654 4.6038703 6.9818569 ......
$block
[1] 21
$par.ests
xi sigma mu
0.251353 1.028910 1.965850
$par.ses
xi sigma mu
0.08847742 0.09013351 0.10932034
$varcov
[,1] [,2] [,3]
[1,] 0.007828254 -0.001080741 -0.003453668
[2,] -0.001080741 0.008124049 0.006145413
[3,] -0.003453668 0.006145413 0.011950936
$converged
[1] 0
> plot(m1)
Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 1
### POT method
> da=read.table("d-ibm-0110.txt",header=T)
> ibm=log(da[,2]+1)
> xt=-ibm
> m1=pot(xt,threshold=0.01)
> m1
$n
[1] 2515
$period
[1] 1 2515
$data
[1] 0.01530858 0.01074553 0.01139063 .....
attr(,"times")
[1] 3 6 10 .....
$span
[1] 2514
```

```
$threshold
[1] 0.01
$p.less.thresh
[1] 0.7996024
$n.exceed
[1] 504
$par.ests
xi sigma mu beta
0.107268254 0.008914461 -0.005634968 0.010591597
$par.ses
xi sigma mu
0.0415025597 0.0009052881 0.0012156539
$intensity
[1] 0.2004773
$converged
[1] 0
> plot(m1)
Make a plot selection (or 0 to exit):
1: plot: Point Process of Exceedances
2: plot: Scatterplot of Gaps
3: plot: Qplot of Gaps
4: plot: ACF of Gaps
5: plot: Scatterplot of Residuals
6: plot: Qplot of Residuals
7: plot: ACF of Residuals
8: plot: Go to GPD Plots
Selection: 0
> riskmeasures(m1,c(0.95,0.99))
p quantile sfall
[1,] 0.950 0.02585540 0.03962479
[2,] 0.990 0.04744964 0.06381374
> riskmeasures(m2,c(0.95,0.99)) % Threshold=0.012
p quantile sfall
[1,] 0.950 0.02611524 0.03960353
[2,] 0.990 0.04745886 0.06267327
> riskmeasures(m3,c(0.95,0.99)) % Threshold=0.008
p quantile sfall
[1,] 0.950 0.02586561 0.03962012
[2,] 0.990 0.04744180 0.06376612
### Generalized Pareto distribution
> library(evir)
> da=read.table("d-ibm-0110.txt",header=T)
> ibm=log(da[,2]+1)
> xt=-ibm
> m1gpd=gpd(xt,threshold=0.01)
> m1gpd
$n
[1] 2515
$data
[1] 0.01530858 0.01074553 0.01139063 .....
$threshold
[1] 0.01
$p.less.thresh
[1] 0.7996024
$n.exceed
[1] 504
$method
[1] "ml"
$par.ests
xi beta
0.10703752 0.01059601
$par.ses
xi beta
0.0544269528 0.0007255951
$converged
[1] 0
$nllh.final
[1] -1733.994
> names(m1gpd)
[1] "n" "data" "threshold" "p.less.thresh"
[5] "n.exceed" "method" "par.ests" "par.ses"
[9] "varcov" "information" "converged" "nllh.final"
```

```
> par(mfcol=c(2,2))
> plot(m1gpd)
Make a plot selection (or 0 to exit):
1: plot: Excess Distribution
2: plot: Tail of Underlying Distribution
3: plot: Scatterplot of Residuals
4: plot: QQplot of Residuals
Selection: 0
> riskmeasures(m1gpd,c(0.95,0.99))
p quantile sfall
[1,] 0.950 0.02585941 0.03962658
[2,] 0.990 0.04745161 0.06380699
$
```

■ Credit Risk

Reference: Credit Risk Measurement: New Approaches to Value at Risk and Other Paradigms, 2nd Edition, by Anthony Saunders and Linda Allen, Wiley, 2002.

Credit risk is defined as the potential that a borrower will fail to meet its contractual obligations in accordance with agreed terms. It refers to the risk that a lender may not receive the owed principal and interest, which results in an interruption of cash flows and increased costs for collection.

Some techniques for credit risk measurement:

1. Long-term credit rating (High to Low)

S&P	Moody	Fitch
AAA	Aaa	AAA
AA	Aa	AA
A	A	A
BBB	Baa	BBB
BB	Ba	BB
B	B	B
CCC	Caa	CCC
CC	Ca	CC
C	C	C
D	D	D

Figure 56: Credit Ratings.

2. Z score (Mainly in U.S.)
   Z = 3.3(Earnings before Interest and Taxes [EBIT]/Totl Assets)
   + 1.0(Sales/Total Assets)
   + 0.6(Market Value of Equity/Book Value of Debt)
   + 1.4(Retained Earnings/Total Assets)
   + 1.2(Working Capital/Total Assets)

3. CreditMetrics (JP Morgan and other sponsors)

CreditMetrics: Developed by J.P. Morgan and other sponsors in 1997.

Simply put, CreditMetrics addresses the question:

"How much will one lose on his loans and loan portfolios next year for a given confidence level?"

From the assessment of market risk, the current market value and its volatility of a financial position play an important role in VaR calculation. Application of VaR methodology to *non-tradable* loans encounters some immediate problems:

1. The current market value of the loan is not directly observable, because most loans are not traded.
2. No time-series data available to estimate the volatility.

To overcome the difficulties, we make use of

1. Available data on a borrower's credit rating
2. The probability that the rating will change over the next year (the rating transition matrix)
3. Recovery rates on defaulted loans
4. Credit spreads and yields in the bond (or loan) market.

Note that the VaR calculations when dealing with credit risk involve factors such as:

1. Credit Rating
2. Migration Likelihoods
3. PV of Bond Revaluations

The essence of this approach is to combine two elements-

- A credit rating transition matrix
- Forward curves to price the exposures.

Example: Consider a five-year fixed-rate loan of $100 million made at 6% annual interest, and the borrower is rated BBB.

Note: The numerical numbers used in this example are from Chapter 6 of the reference book cited above.

This example shows the credit Value-at-Risk for a BBB bond. Our first step is to specify the one year transition matrix that's shown here. The transition matrix contains the marginal probabilities for a bond to stay or change this rating. Based on historical data on publicly traded bonds (or loans) collected by Standard and Poor's (S&P), Moody's, KMV, or other bond or loan analysts, the probability that a BBB borrower will migrate to other rating categories is given by a transition matrix. One-year transition probabilities for BBB-rated borrower is:

AAA	AA	A	BBB	BB	B	CCC	Defaulty
0.02	0.33	5.95	86.93	5.30	1.17	0.12	0.18

Since we are dealing specifically with a BBB bond for this example, the data above is the row of the transition matrix containing the probabilities for a BBB bond to switch its rating. The sum of the elements will be 1, as it should be since it implies the bond will be one of those ratings by the end of the year. It is a probability distribution of all possible ratings the BBB bond can be.

The data shows that probability that a BBB bond will stay at BBB over the next year is estimated at 86.93 percent. There is also some probability that the borrower will be upgraded (e.g., 5.95%) or will be downgraded (e.g., 0.12% to CCC or even 0.18% to default, D). Given the whole transition matrix, we can notice that the highest elements tend to be on the diagonals, because it is most likely that a bond rating does not switch. The first step is now complete.

Valuation Rating change (upgrades and downgrades) will affect the required credit risk spreads or premiums on the loan's remaining cash flows and, hence, the implied market value of the loan.

Downgrade $\longrightarrow$ credit spread premium rises $\longrightarrow$ present value of the loan should fall.

Upgrade has the opposite effect.

Our next step is the specification of the forward curve. If we have the spot rate curves, we can extract the forward rates out of that.

For each rating category (7 in this case), we can get the 1-year, 2-year, 3-year and 4-year spot rates. The lowest rating, CCC, has the highest forward rates.

- We have a 5-year fixed rate loan of 100$.
- Our BBB bond has the following CF's: 6$ for each year and 106$ in total in the final year.
- Remember, we are looking at a 1-year holding period!

At the end of the holding period, our bond could transmit to any one of these rating categories. The price at the end of of the year will be formulated as:

$$P = 6 + \frac{6}{1 + r_{1,1} + s_1} + \frac{6}{(1 + r_{1,2} + s_2)^2} + \frac{6}{(1 + r_{1,3} + s_3)^3} + \frac{106}{(1 + r_{1,4} + s_4)^4}$$

where $r_{1,i}$ are the risk-free rates on zero-coupon U.S. Treasury bonds expected to exist one year into the future and s_i is the annual credit spread on loans of a particular rating class of 1-year, 2-year, 3-year and 4-year maturities (derived from observed spreads in the corporate bond market over Treasuries). One-year forward zero curves plus credit spreads by credit rating category are as follows:

Category	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

Suppose that, during the first year, the borrower gets upgraded from BBB to A. Then the forward rates plus spread will be: 3.72%, 4.32%, 4.93%, 5.32% for Year 1 to Year 4, therefore the price of the bond (loan) at the end of year 1 is

$$P = 6 + \frac{6}{1.0372} + \frac{6}{(1.0432)^2} + \frac{6}{(1.0493)^3} + \frac{106}{(1.0532)^4} = \$108.66.$$

Value of the loan at the end of Year 1, under different rating changes (including first-year coupon):

Rating	AAA	AA	A	BBB	BB	B	CCC	Defaulty
value	109.37	109.19	108.66	107.55	102.02	98.10	83.64	51.13

So, the value of our BBB bond at the end of the year depends on the rating category the bond ends up with. It could end up with AAA where the quality is higher and the discount rates are lower (and thereby having higher present value of cash flows). This gives us the real essence of the Credit Metrics approach which is a set of price outcomes on this bond in the future. We also input an estimate for the case of Default(51.13) using the recovery rate. So we have obtained the prices and probabilities (the row on the transition matrix) of each outcome below.

Calculation of VaR

To calculate VaR, we need to define a loss function. This could be the revalued price minus the current price. Or the revalued price minus the average expected price. Here we assume that our loss function is

"the new price at the end of the holding period minus the weighted average of the possible prices at the end of the holding period"

Year-end Rating	Probability of State(%)	New Loan Value Plus Coupon (millions)	Probability Weighted Value($)	Difference of Value from Mean ($)	Probability Weighted Diff. Squared
AAA	0.02	109.37	0.02	2.28	0.0010
AA	0.33	109.19	0.36	2.10	0.0146
A	5.95	108.66	6.47	1.57	0.1474
BBB	86.93	107.55	93.49	0.46	0.1853
BB	5.30	102.02	5.41	(5.06)	1.3592
B	1.17	98.10	1.15	(8.99)	0.9446
CCC	0.12	83.64	1.10	(23.45)	0.6598
Default	0.18	51.13	0.09	(55.96)	5.6358

Form the table, the mean value of the loan is \$107.09 (sum of the 4-th column). The variance of the value is 8.9477 (sum of the last column). Consequently, the standard deviation is $\sigma = \sqrt{8.9477} = 2.99$.

If normal distribution of the loan value is used,

- $\text{VaR}_{0.95} = 1.65 \times \sigma = \$4.93$
- $\text{VaR}_{0.99} = 2.33 \times \sigma = \$6.97$

However, the assumption of a normal distribution for returns on some tradable assets is a rough approximation, and the approximation becomes even rougher when applied to the possible distribution of values for loan (loans have both severely truncated upside returns and long downside risks).

If actual distribution is used,

Rate	AAA	AA	A	BBB	BB	B	C	D
Loss	-2.28	-2.1	-1.57	-0.46	5.07	8.99	23.45	55.96
Probability	0.02%	0.33%	5.95%	86.93%	5.3%	1.17%	0.12%	0.18%
CDF (of loss)	0.02%	0.35%	6.3%	93.23%	98.53%	99.7%	99.82%	100%

- $F(5.07) = 0.9853$, therefore, $\text{VaR}_{0.9853} = \$5.07$
- $F(8.99) = 0.9907$, therefore, $\text{VaR}_{0.9907} = \$8.99$
- $\text{VaR}_{0.99} = \$8.48$, by interpolation.

The 1% number 8.48 is obtained by interpolation of the two points (98.53%, \$5.07) and (99.7%, \$8.99):

$$\text{VaR}_{0.99} = 8.99 + \frac{5.07 - 8.99}{98.53 - 99.07} * (-0.07) = 8.48$$