5 Multiple Financial Time Series

Ch 8 from the textbook
In order to obtain the relationship between the series and improve the accuracy of forecasts (by using more information), we can model multiple series jointly. Knowing how the markets are interrelated is of great importance in finance: Economic globalization and Internet have accelerated the integration of world financial markets in recent years. Price movements in one market can spread easily to another market. For an investor or a financial institution holding multiple assets, the dynamic relationships between returns of the assets play an important role in decision making.

A multivariate time series consists of multiple single series referred to as components. Examples are U.S. quarterly GDP and unemployment rate series, or quarterly GDP growth rates of Canada, United Kingdom, and United States. Concepts of vector and matrix are useful in understanding multivariate time series analysis. Here, we focus on two series (i.e., the bivariate case) so the time series can be written as $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_T$ where

$$\mathbf{x}_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

Note that we use boldface notation to indicate vectors and matrices. See Figure 42 for the log prices of the two energy funds, Oil Service Holdings(OIH) and Energy Select Section SPDR (XLE). The prices seem to move in unison.

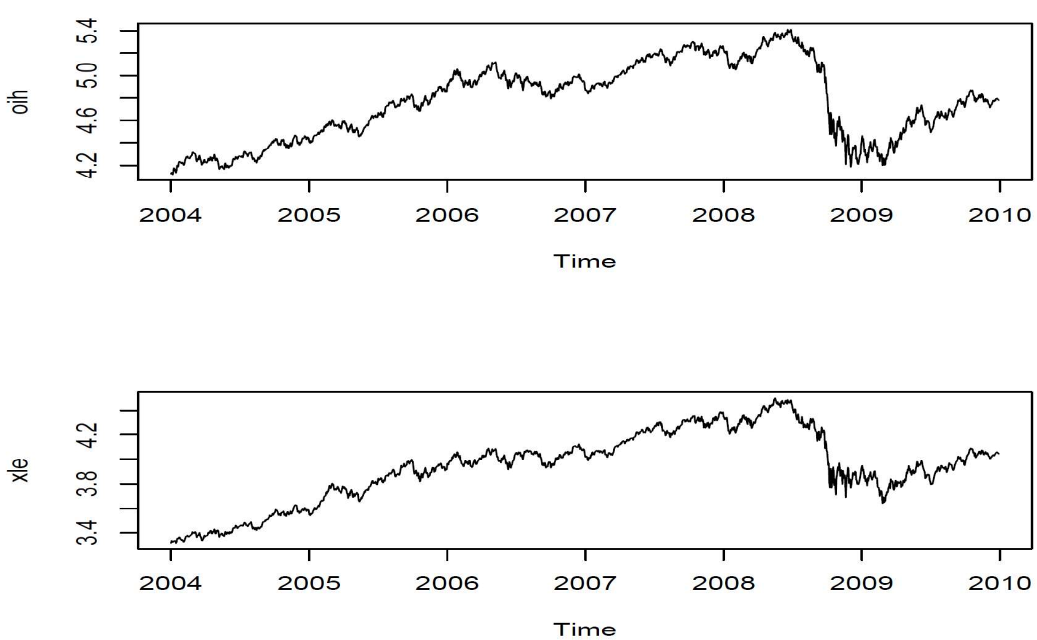

Figure 42: Daily log prices of OIH and XLE funds from January 2004 to December 2009.

5.1 Weak Stationarity and Cross-Correlation Matrices

The series $\mathbf{x}_t$ is weakly stationary if its first and second moments are time invariant. In particular, the mean vector and covariance matrix of a weakly stationary series are constant over time and can be written as

$$E[\mathbf{x}_t] = \begin{bmatrix} E[x_{1,t}] \\ E[x_{2,t}] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \boldsymbol{\mu} \tag{5.1}$$

$$E[(\mathbf{x}_t - \boldsymbol{\mu})(\mathbf{x}_t - \boldsymbol{\mu})'] = \begin{bmatrix} Var[x_{1,t}] & Cov[x_{1,t}, x_{2,t}] \\ Cov[x_{1,t}, x_{2,t}] & Var[x_{2,t}] \end{bmatrix} = \boldsymbol{\Gamma}_0 \tag{5.2}$$

The mean $\boldsymbol{\mu}$ is a 2-dimensional vector consisting of the unconditional expectations of the components of $\mathbf{x}_t$. The covariance matrix $\boldsymbol{\Gamma}_0$ is a 2×2 matrix. The first and second diagonal element of $\boldsymbol{\Gamma}_0$ is the variance of $x_{1,t}$ and $x_{2,t}$, respectively, whereas the off-diagonal element of $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_{12}(0)$, is the covariance between $x_{1,t}$ and $x_{2,t}$.

Besides the mean vector and covariance matrix, all lag-ℓ cross-covariance matrices of $\mathbf{x}_t$ are time-invariant.

$$\begin{aligned} \boldsymbol{\Gamma}_\ell &= E[(\mathbf{x}_t - \boldsymbol{\mu})(\mathbf{x}_{t-\ell} - \boldsymbol{\mu})'] \qquad (5.3) \\ &= \begin{bmatrix} Cov[x_{1,t}, x_{1,t-\ell}] & Cov[x_{1,t}, x_{2,t-\ell}] \\ Cov[x_{2,t}, x_{1,t-\ell}] & Cov[x_{2,t}, x_{2,t-\ell}] \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{\Gamma}_{11}[\ell] & \boldsymbol{\Gamma}_{12}[\ell] \\ \boldsymbol{\Gamma}_{21}[\ell] & \boldsymbol{\Gamma}_{22}[\ell] \end{bmatrix} \qquad (5.4) \end{aligned}$$

Note: For a weakly stationary series, the cross-covariance matrix $\boldsymbol{\Gamma}_\ell$ is a function of ℓ, not the time index t.

Note: $\boldsymbol{\Gamma}_\ell$ is not necessarily symmetric for $\ell \neq 0$:

- $\boldsymbol{\Gamma}_{12}[\ell] = Cov[x_{1,t}, x_{2,t-\ell}]$ shows how $x_{1,t}$ depends on past $x_{2,t}$
- $\boldsymbol{\Gamma}_{21}[\ell] = Cov[x_{2,t}, x_{1,t-\ell}]$ shows how $x_{2,t}$ depends on past $x_{1,t}$

Note: using $Cov(x, y) = Cov(y, x)$ and the weak stationarity assumption, we have

$$\boldsymbol{\Gamma}_{ij}[\ell] = \boldsymbol{\Gamma}_{ji}[-\ell]$$

The lag-ℓ cross-correlation matrix (CCM) of $\mathbf{x}_t$ is defined as

$$\boldsymbol{\rho}_\ell = \begin{bmatrix} \frac{\boldsymbol{\Gamma}_{11}[\ell]}{\boldsymbol{\Gamma}_{11}[0]} & \frac{\boldsymbol{\Gamma}_{12}[\ell]}{\sqrt{\boldsymbol{\Gamma}_{11}[0]\boldsymbol{\Gamma}_{22}[0]}} \\ \frac{\boldsymbol{\Gamma}_{21}[\ell]}{\sqrt{\boldsymbol{\Gamma}_{22}[0]\boldsymbol{\Gamma}_{11}[0]}} & \frac{\boldsymbol{\Gamma}_{22}[\ell]}{\boldsymbol{\Gamma}_{22}[0]} \end{bmatrix}$$

Note: In general, $\boldsymbol{\rho}_{12}[\ell] \neq \boldsymbol{\rho}_{21}[\ell]$ because the two correlation coefficients measure different linear relationships between $x_{1,t}$ and $x_{2,t}$.

Note: The diagonal elements $\boldsymbol{\rho}_{11}[0]$ are 1.

Note: The off-diagonal elements $\boldsymbol{\rho}_{12}[0] = \boldsymbol{\rho}_{21}[0]$ measure the concurrent linear relationship between $x_{1,t}$ and $x_{2,t}$.

Note: The diagonal element $\boldsymbol{\rho}_{11}[\ell]$ and $\boldsymbol{\rho}_{22}[\ell]$ are simply the lag-ℓ autocorrelation coefficients of $x_{1,t}$ and $x_{2,t}$.

Note: $\boldsymbol{\rho}_{12}[\ell]$ is the correlation coefficient between $x_{1,t}$ and $x_{2,t-\ell}$, respectively.

Note: If $\boldsymbol{\rho}_{12}[\ell] \neq 0$, $\ell > 0$, we say that the series $x_{2,t}$ *leads* the series $x_{1,t}$ at lag ℓ.

Note: If $\boldsymbol{\rho}_{12}[\ell] = 0$ for all ℓ, then $x_{1,t}$ does not depend linearly on any past value $x_{2,t-\ell}$ of the $x_{2,t}$ series.

Note: If $\boldsymbol{\rho}_{12}[\ell] = 0$ and $\boldsymbol{\rho}_{21}[\ell] = 0$ for all $\ell > 0$, we say the two series are uncoupled.

Note: $\boldsymbol{\rho}_{\ell} = \boldsymbol{\rho}'_{-\ell}$. For instance, corr$(x_{1,t}, x_{2,t-1})$=corr$(x_{2,t}, x_{1,t+1})$.

You know how to calculate sample cross-covariance matrices. Right? e.g.,

$$\hat{\boldsymbol{\Gamma}}_{\ell} = \frac{1}{T} \sum_{t=\ell+1}^{T} (\mathbf{x_t} - \bar{\mathbf{x}})(\mathbf{x_{t-\ell}} - \bar{\mathbf{x}})'$$

where $\bar{\mathbf{x}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{x_t}$.

5.2 Testing for Serial Linear Dependence

Multivariate version of Ljung-Box $Q(m)$ statistics available.

$$H_0 : \boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = ... = \boldsymbol{\rho}_m = 0 \text{ vs } H_a : \boldsymbol{\rho}_i \neq 0, \text{ for some } i$$

The test statistic is

$$Q_2(m) = T^2 \sum_{\ell=1}^{m} \frac{1}{T-\ell} tr(\hat{\boldsymbol{\Gamma}}'_{\ell} \hat{\boldsymbol{\Gamma}}_0^{-1} \hat{\boldsymbol{\Gamma}}_{\ell} \hat{\boldsymbol{\Gamma}}_0^{-1})$$

and follows distribution $x^2_{k^2 m}$ under the null hypothesis.

Note: tr is the sum of diagonal elements, T is the sample size, and $k = 2$ is the dimension of the time series..

Remark: Analysis of multiple financial time series can be carried out in R via the package *MTS*.

R Demonstration: Consider the quarterly series of U.S. GDP and unemployment data

```
require(MTS)
x=read.table("q-gdpun.txt",header=T)
dim(x)
[1] 228 5
x[1,]
year mon day gdp unemp
1 1948 1 1 7.3878 3.7333
z=x[,4:5]
mq(z,10)
     "m,    Q(m)     p-value:"
[1] 1.0000 434.0739 0.0000
[1] 2.0000 827.5327 0.0000
```

```
[1] 3.000 1176.616 0.000
[1] 4.000 1486.840 0.000
[1] 5.000 1767.619 0.000
[1] 6.000 2026.774 0.000
[1] 7.000 2268.947 0.000
[1] 8.000 2496.995 0.000
[1] 9.000 2713.950 0.000
[1] 10.000 2921.077 0.000
dz=diffM(z) ### Take difference of individual series
mq(dz,10)
[1]    "m,   Q(m)   p-value:"
[1] 1.0000 105.3880 0.0000
[1] 2.0000 153.2457 0.0000
[1] 3.0000 176.7565 0.0000
[1] 4.0000 196.1902 0.0000
[1] 5.0000 207.9687 0.0000
[1] 6.0000 212.5574 0.0000
[1] 7.0000 215.8745 0.0000
[1] 8.0000 221.8316 0.0000
[1] 9.0000 225.8715 0.0000
[1] 10.0000 228.1209 0.0000
```

The results show that the bivariate series is strongly serially correlated.

5.3 Vector Autoregressive Models (VAR)

VAR(1) model for two return series:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{10} \\ \phi_{20} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix},$$

where $\mathbf{a}_t = (a_{1,t}, a_{2,t})'$ is a sequence of i.i.d bivariate normal random vectors with mean zero and covariance matrix

$$\text{cov}(\mathbf{a}_t) = \text{cov}(a_{1,t}, a_{2,t}) = \mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

where $\sigma_{21} = \sigma_{12}$.

Rewrite the model as

$$\begin{aligned} x_{1,t} &= \phi_{10} + \phi_{11} x_{1,t-1} + \phi_{12} x_{2,t-1} + a_{1,t} \\ x_{2,t} &= \phi_{20} + \phi_{21} x_{1,t-1} + \phi_{22} x_{2,t-1} + a_{2,t} \end{aligned}$$

Thus, ϕ_{11} and ϕ_{12} denote the dependence of $x_{1,t}$ on the past returns $x_{1,t-1}$ and $x_{2,t-1}$, respectively.

For the VAR(1) model, if $\phi_{12} = 0$, but $\phi_{21} \neq 0$, then $x_{1,t}$ does not depend on $x_{2,t-1}$, but $x_{2,t}$ depends on $x_{1,t-1}$, implying that knowing $x_{1,t-1}$ is helpful in predicting $x_{2,t}$, but $x_{2,t-1}$ is not helpful in forecasting $x_{1,t}$. Here $\{x_{1,t}\}$ is an input, $\{x_{2,t}\}$ is the output variable. This is an example of unidirectional relationship.

If $\phi_{12} = \phi_{21} = 0$, then $x_{1,t}$ and $x_{2,t}$ are uncoupled. If $\phi_{12} \neq 0$ and $\phi_{21} \neq 0$, then there is a feedback relationship between the two series.

If $\sigma_{12} = 0$, then there is no concurrent linear relationship between the two component series.

In the econometric literature, the VAR(1) model is called a reduced-form model because it does not show explicitly the concurrent dependence between the component series. In time series analysis, the reduced-form model is commonly used for two reasons. The first reason is ease in estimation. The second and main reason is that the concurrent correlations cannot be used in forecasting.

Stationarity condition: Generalization of 1-dimensional case. Write the VAR(1) model as

$$\mathbf{x}_t = \boldsymbol{\phi_0} + \boldsymbol{\Phi}\mathbf{x}_{t-1} + \mathbf{a}_t$$

$\mathbf{x}_t$ is stationary if and only if zeros of the polynomial $|I - \boldsymbol{\Phi}x|$ are greater than 1 in modulus. $|A|$ illustrates the determinant of matrix A.

Taking expectation of the model and using $E(\mathbf{a}_t) = 0$, mean of $\mathbf{x}_t$ satisfies

$$(I - \boldsymbol{\Phi})\boldsymbol{\mu} = \boldsymbol{\phi_0}$$

or

$$\boldsymbol{\mu} = E[\mathbf{x}_t] = (I - \boldsymbol{\Phi})^{-1}\boldsymbol{\phi_0}$$

if the inverse exists.

Covariance matrices of VAR(1) models:

$$\text{cov}(\mathbf{x}_t) = \sum_{i=0}^{\infty} \boldsymbol{\Phi}^i \Sigma (\boldsymbol{\Phi}^i)' (\text{using the MA representation})$$

and

$$\boldsymbol{\Gamma}_\ell = \boldsymbol{\Phi}\boldsymbol{\Gamma}_{\ell-1}, \quad \text{for } \ell > 0.$$

where $\boldsymbol{\Gamma}_\ell$ is the lag-ℓ cross-covariance matrix of $\mathbf{x}_t$. This result is a generalization of that of a univariate AR(1) process. By repeated substitutions, we can shows that

$$\boldsymbol{\Gamma}_\ell = \boldsymbol{\Phi}^\ell \boldsymbol{\Gamma}_0, \quad \text{for } \ell > 0.$$

VAR(1) models can be generalized to higher order models.

Building VAR models:

- Order selection: use AIC or BIC.
- Estimation: use ordinary least-squares method or the ML method. The two methods are asymptotically equivalent.
- Model checking: similar to the univariate case. The $Q_k(m)$ statistic can be applied to the residual series to check the assumption that there are no serial or cross correlations in the residuals. For a fitted VAR(p) model, the $Q_k(m)$ statistic of the residuals is asymptotically a chi-squared distribution with $k^2m - g$ degrees of freedom, where g is the number of estimated parameters in the AR coefficient matrices.
- Forecasting: similar to the univariate case

Commands for VAR modeling

– *adf.Test()*: tests for unit root presence in series.
– Box.Test: performs various tests for residual dependence presence (*qm*).
– *VARorder*: compute various information criteria for a vector time series.
– *VAR*: estimates a VAR model.
– *refVAR*: refines an estimated VAR model by fixing insignificant estimates to zero.
– *MTSdiag*: model checking.
– *VARpred*: predicts a fitted VAR model.

Note: We can also have multivariate MA models or ARMA models. Here, we focus on AR models.

5.4 Cointegration

When modeling several unit-root non-stationary time series jointly, one may encounter the case of *cointegration* where

– $x_{1,t}$ and $x_{2,t}$ are unit-root non-stationary
– a linear combination of $x_{1,t}$ and $x_{2,t}$ is stationary

That is, $x_{1,t}$ and $x_{2,t}$ share a single unit root!

Why is it of interest?

Stationary series is mean reverting. Long term forecasts of the linear combination converge to a mean value, implying that the long-term forecasts of $x_{1,t}$ and $x_{2,t}$ must be linearly related.

Example. Consider the the time plots of the daily log-prices of the two stocks (adjusted closing prices), shown in Figure 43. From the plots, the prices of the two stocks exhibit certain characteristics of comovement.

What can be said about the two prices? Is there any arbitrage opportunity between the two funds? The two series have a unit root (based on ADF test). Are they co-integrated?

Figure 44 shows the time plot of the residual series. The plot shows that the residual series has certain characteristics of a stationary time series. In particular, it has mean zero and fluctuates around its mean within a fixed range. Figure 45 gives the sample ACF of the residual. The ACFs decay exponentially, supporting that the residuals series is indeed stationary. You can also use the augmented Dickey–Fuller unit-root test to see the residual series is stationary.

This mean-reverting property has many applications. For instance, pairs trading in Finance. Pairs trading is a market-neutral trading strategy. The general theme for trading in the equity markets is to buy undervalued stocks and sell overvalued ones. However, the true price of a stock is hard to assess. Pairs trading attempts to resolve this difficulty using the idea of relative pricing. Based on the arbitrage pricing theory (APT) in finance, if two stocks have similar characteristics, then the prices of both stocks must be more or less the same. If the prices differ, then

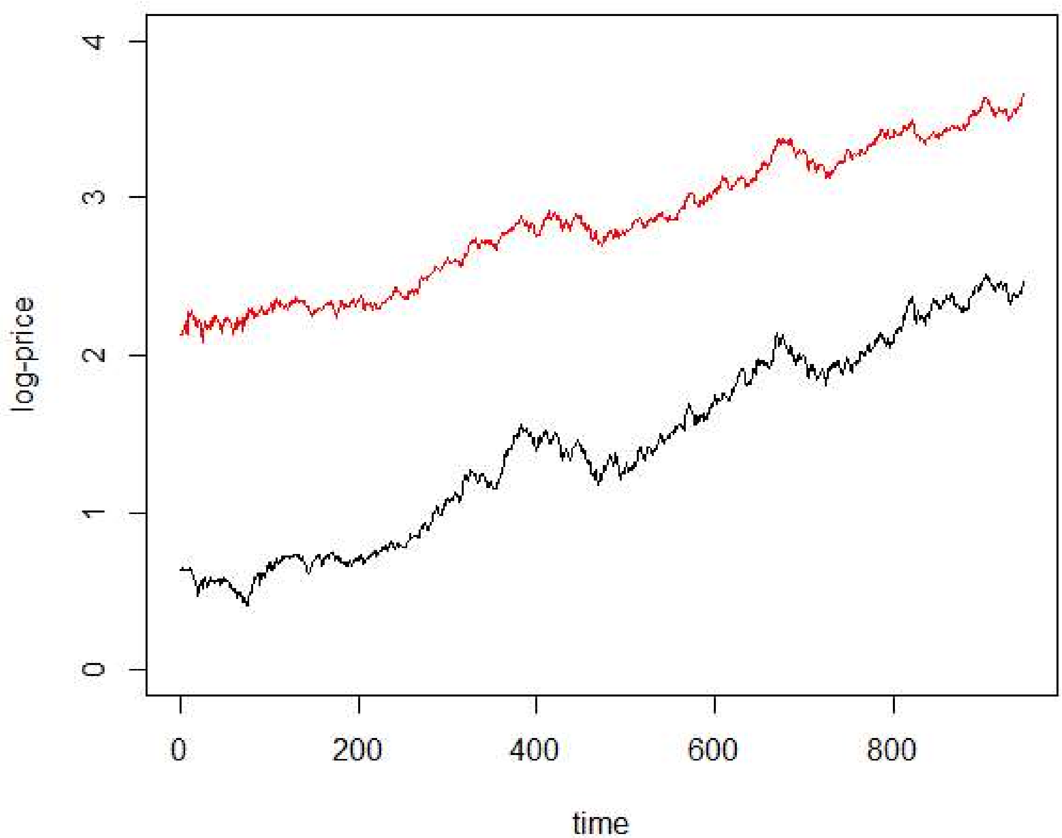

Figure 43: Daily log (adjusted) closing prices of BHP(p_{1t}) and VALE(p_{2t}) stocks from July 1, 2002, to March 31, 2006. Upper plot is for BHP stock.

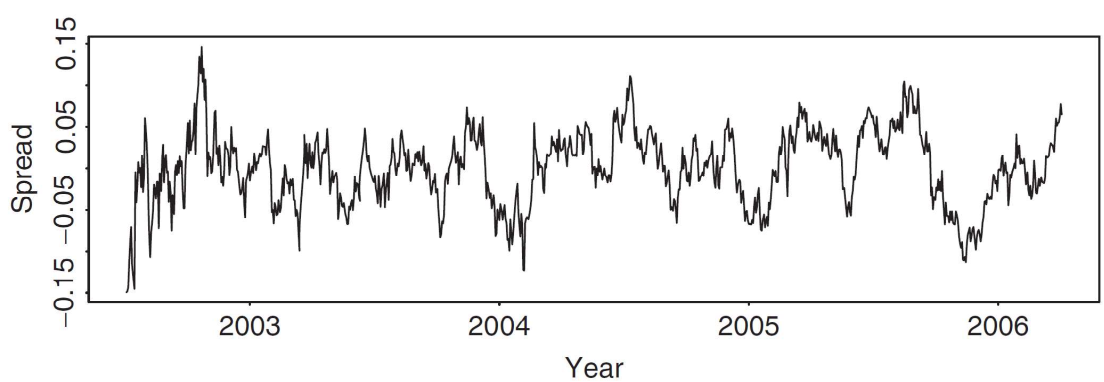

Figure 44: Time plot of the estimated spread between BHP and VALE daily log stock prices, i.e $\epsilon_t = p_{1t} - \gamma p_{2t}, \gamma = 1$

it is likely that one of the stocks is overpriced and the other underpriced. Pairs trading involves selling the higher priced stock and buying the lower priced stock

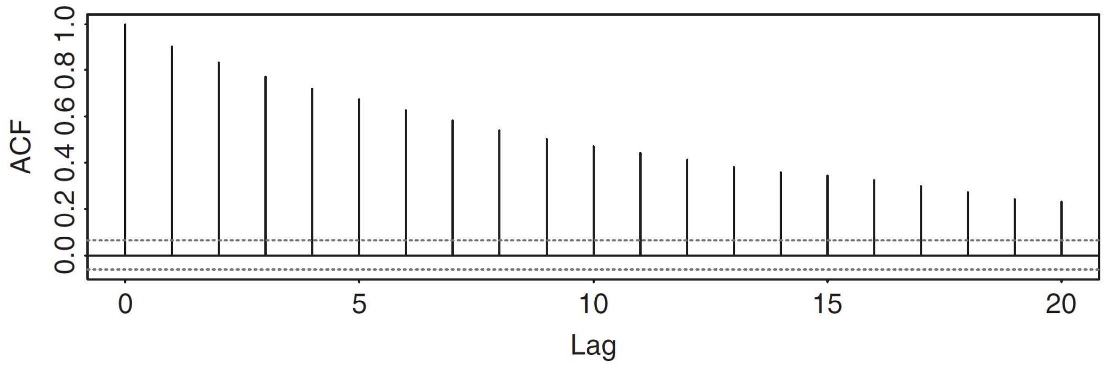

Figure 45: Sample autocorrelation functions of estimated spread between BHP and VALE daily log stock prices.

with the hope that the mispricing will correct itself in the future. The gap (properly scaled) between the two observed prices is called the spread. For pairs trading, the greater the spread, the larger the magnitude of mispricing and the greater the profit potential. In practice, how big the deviation needs to be in order for the trading to be profitable depends on several factors. Trading costs, marginal interest rates, and bid–ask spreads of the two stocks are three obvious factors.

One of many possible trading strategies:

Consider two stocks. Stock 1 and Stock 2. Let p_{it} be the log price of Stock i at time t. It is reasonable to assume that the time series of the prices contain a unit root when they are analyzed individually.

Assume that the two log-price series are co-integrated, that is, there exists a linear combination $c_1 p_{1t} - c_2 p_{2t}$ that is stationary. Dividing the linear combination by c_1, we have

$$\omega_t = p_{1t} - \gamma p_{2t}$$

which is stationary. The stationarity implies that ω_t is mean-reverting.

Now, form the portfolio Z by buying 1 share of Stock 1 and selling short on γ shares of Stock 2. The return of the portfolio for a given period h is (ignore the time value of money)

$$r(h) = (p_{1,t+h} - p_{1t}) - \gamma(p_{2,t+h} - p_{2t}) = \omega_{t+h} - \omega_t$$

which is the increment of the stationary series $\{\omega_t\}$ from t to $t+h$. Since ω_t is stationary, we have obtained a direct link of the portfolio to a stationary time series whose forecasts we can predict.

Assume that $E(\omega_t) = \mu$. Select a threshold Δ.

- Buy a share of stock 1 and short γ shares of stock 2 at time t if

$$p_{1t} - \gamma p_{2t} < \mu - \Delta$$

- Unwind the position at time $t+h$ when $p_{1,t+h} - \gamma p_{2,t+h} > \mu + \Delta$.
- Profit: $r(h) = \omega_{t+h} - \omega_t > 2\Delta$.

There are many ways available to search for co-integrating pairs of stocks. For example, via fundamentals, risk factors, etc.

For co-integration tests, see the textbook and references therein (not covered here).

5.5 Multivariate Volatility Models

See Chapter 10 of the textbook.

How do the correlations between asset returns change over time?

Focus on two series (Bivariate)

Let $I_{1:t-1}$ denote the information available at time $t-1$. Partition the return as

$$\mathbf{x}_t = \boldsymbol{\mu}_t + \mathbf{a}_t, \quad \mathbf{a}_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\epsilon}_t$$

where $\boldsymbol{\mu}_t = \mathrm{E}(\mathbf{x}_t | I_{1:t-1})$ is the predictable component, and

$$\mathrm{cov}(\mathbf{x}_t | I_{1:t-1}) = \boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{22,t} \end{bmatrix},$$

$\boldsymbol{\epsilon}_t$ are i.i.d 2-dimensional random vectors with mean zero and identity covariance matrix.

Here, we study time evolution of $\{\boldsymbol{\Sigma}_t\}$:

- $\boldsymbol{\Sigma}_t$ is symmetric.
- There are 3 variables in $\boldsymbol{\Sigma}_t$.
- $\boldsymbol{\Sigma}_t$ must be positive definite for all t:

$$\sigma_{11,t} > 0, \quad \sigma_{22,t} > 0, \quad \sigma_{11,t}\sigma_{22,t} - \sigma_{12,t}^2 > 0$$

BEKK model of Engle and Kroner (1995) model:

$$\boldsymbol{\Sigma}_t = A_0' A_0 + A_1' \boldsymbol{a}_{t-1} \boldsymbol{a}_{t-1}' A_1 + B_1' \boldsymbol{\Sigma}_{t-1} B_1. \tag{5.5}$$

where A_0 is a lower triangular matrix, A_1 and B_1 are square matrices without restrictions.

Pros: positive definite

Cons: Many parameters, dynamic relations require further study

Estimation: *BEKK11* command in *MTS* package can be used for k =2 and 3 only.