review: frequentist vs. bayesian inference

- p-value
- posterior

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Image source: http://commons.wikimedia.org/wiki/File:Plain-M%26Ms-Pile.jpg

M&Ms

- We have a population of M&Ms.
- The percentage of yellow M&Ms is either 10% or 20%.
- You have been hired as a statistical consultant to decide whether the true percentage of yellow M&Ms is 10%.
- You are being asked to make a decision, and there are associated payoff/losses that you should consider.

Decision Table

	True state of the population	
Decision	**% yellow = 10%**	**%yellow = 20%**
% yellow = 10%	Your boss gives you a bonus :)	You lose your job :(
%yellow = 20%	You lose your job :(	Your boss gives you a bonus :)

Data

- You can "buy" a random sample from the population
- You pay $200 for each M&M, and you must buy in $1,000 increments (5 M&Ms at a time)
- You have a total of $4,000 to spend (you may buy 5, 10, 15, or 20 M&Ms)

Frequentist Inference

- Hypotheses:
  - H_0: 10% yellow M&Ms
  - H_A: 20% yellow M&Ms
- Your test statistic is the number of yellow M&Ms you observe in the sample
- The p-value is the probability of observing this many or more yellow M&Ms given that the null hypothesis is true

Sample Size

How many M&Ms would you buy?

- 5, 10, 15, or 20?
- What is the cost/benefit of buying fewer/more?

$n = 5$

Significance Level

What significance level would you use?

- 5% or something else?
- What are the pros/cons of using a lower/higher significance level?

$\alpha = 0.05$

Frequentist Inference

H_0: 10% yellow M&Ms
H_A: 20% yellow M&Ms

sample: RGYBO

number of yellows in the first 5 draws: 1

Frequentist Inference

p-value: P(1 or more yellows | n = 5, p = 10%)

$$\text{Binomial}(p = 0.10,\ n = 5)$$

$$P(k \geq 1) = 1 - P(k = 0)$$

$$= 1 - 0.9^5$$

$$\approx 0.41$$

$$\text{p-value} \approx 0.41 > 0.05 \rightarrow \text{Fail to reject } H_0$$

	Frequentist
Number of yellow M&Ms in first	P(k or more | 10% yellow)
n = 5: 1	0.41
n = 10:	
n = 15:	
n = 20:	

H_0: 10% yellow M&Ms
H_A: 20% yellow M&Ms

Frequentist Inference

n = 10

RGYBO BBGOY

p-value: P(2 or more yellows | n = 10, p = 10%)

$Binomial(p = 0.10, n = 10)$

$P(k \geq 2) \approx 0.26$

R

```
> sum(dbinom(2:10, 10, 0.1))
[1] 0.2639011
```

p-value $\approx 0.26 > 0.05 \rightarrow$ Fail to reject H_0

	FREQUENTIST
Number of yellow M&Ms in first	P(k or more | 10% yellow)
n = 5: 1	0.41
n = 10: 2	0.26
n = 15:	
n = 20:	

H_0: 10% yellow M&Ms

H_A: 20% yellow M&Ms

Frequentist Inference

n = 15

RGYBO BBGOY YRBRR

p-value: P(3 or more yellows | n = 15, p = 10%)

$Binomial(p = 0.10, n = 15)$

$P(k \geq 3) \approx 0.18$

R

```
> sum(dbinom(3:15, 15, 0.1))
[1] 0.1840611
```

p-value $\approx 0.18 > 0.05 \rightarrow$ Fail to reject H_0

	FREQUENTIST
Number of yellow M&Ms in first	P(k or more | 10% yellow)
n = 5: 1	0.41
n = 10: 2	0.26
n = 15: 3	0.18
n = 20:	

H_0: 10% yellow M&Ms
H_A: 20% yellow M&Ms

Frequentist Inference

n = 20

RGYBO BBGOY YRBRR GORBY

p-value: P(4 or more yellows | n = 20, p = 10%)

$$\text{Binomial}(p = 0.10,\ n = 20)$$

$$P(k \geq 4) \approx 0.13$$

R

```
> sum(dbinom(4:20, 20, 0.1))
[1] 0.1329533
```

$$\text{p-value} \approx 0.13 > 0.05 \rightarrow \text{Fail to reject } H_0$$

	FREQUENTIST
Number of yellow M&Ms in first	P(k or more | 10% yellow)
n = 5: 1	0.41
n = 10: 2	0.26
n = 15: 3	0.18
n = 20: 4	0.13

H_0: 10% yellow M&Ms
H_A: 20% yellow M&Ms

RGYBO BBGOY YRBRR GORBY

$p = 20\%$

FREQUENTIST	TRUE STATE OF THE POPULATION	
DECISION	% yellow = 10%	%yellow = 20%
% yellow = 10%	Your boss gives you a bonus :)	You lose your job :(
%yellow = 20%	You lose your job :(	Your boss gives you a bonus :)

Bayesian Inference

- Start over, with 1:1 odds that the percentage of yellows is 10%:20% (the prior probabilities)
  - P(10% yellow) = 0.5
  - P(20% yellow) = 0.5
- Use the same data and Bayes' theorem to calculate the probability of either of the hypotheses being true given the observed data (the posterior probabilities)

Bayes' theorem: $$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(10\% \text{ yellow} \mid \text{data}) = \frac{P(\text{data} \;\&\; 10\% \text{ yellow})}{P(\text{data})}$$

$$= \frac{P(\text{data} \mid 10\% \text{ yellow}) \times P(10\% \text{ yellow})}{P(\text{data})}$$

$$P(20\% \text{ yellow} \mid \text{data}) = \frac{P(\text{data} \mid 20\% \text{ yellow}) \times P(20\% \text{ yellow})}{P(\text{data})}$$

$$= 1 - P(10\% \text{ yellow} \mid \text{data})$$

Bayesian Inference

n = 5

RGYBO

prior: P(p = 10%) = 0.5; P(p = 20%) = 0.5

posterior: P(p = 10% | data)

data = (n = 5, k = 1)

$$P(p=10\% \mid data) = \frac{P(data \mid p=10\%) \times P(p=10\%)}{P(data \mid p=10\%) \times P(p=10\%) + P(data \mid p=20\%) \times P(p=20\%)}$$

$$= \frac{0.33 \times 0.5}{0.33 \times 0.5 + 0.41 \times 0.5} \approx 0.44$$

R

```
> dbinom(1, 5, 0.1)
[1] 0.32805
```

	Bayesian	
Number of yellow M&Ms in first	P(10% yellow | n,k)	P(20% yellow | n,k)
n = 5: 1	0.44	0.56
n = 10:		
n = 15:		
n = 20:		

H_1: 10% yellow M&Ms
H_2: 20% yellow M&Ms

Bayesian Inference

n = 10

RGYBO BBGOY

prior: P(p = 10%) = 0.5; P(p = 20%) = 0.5

posterior: P(p = 10% | data)

data = (n = 10, k = 2)

$$P(p=10\% \mid data) = \frac{P(data \mid p=10\%) \times P(p=10\%)}{P(data \mid p=10\%) \times P(p=10\%) + P(data \mid p=20\%) \times P(p=20\%)}$$

$$= \frac{0.19 \times 0.5}{0.19 \times 0.5 + 0.3 \times 0.5} \approx 0.39$$

	Bayesian	
Number of yellow M&Ms in first	P(10% yellow | n,k)	P(20% yellow | n,k)
n = 5: 1	0.44	0.56
n = 10: 2	0.39	0.61
n = 15:		
n = 20:		

H_1: 10% yellow M&Ms
H_2: 20% yellow M&Ms

Bayesian Inference

n = 15

RGYBO BBGOY YRBRR

prior: P(p = 10%) = 0.5; P(p = 20%) = 0.5

posterior: P(p = 10% | data)

data = (n = 15, k = 3)

$$P(p=10\% \mid data) = \frac{P(data \mid p=10\%) \times P(p=10\%)}{P(data \mid p=10\%) \times P(p=10\%) + P(data \mid p=20\%) \times P(p=20\%)}$$

$$= \frac{0.13 \times 0.5}{0.13 \times 0.5 + 0.25 \times 0.5} \approx 0.34$$

	Bayesian	
Number of yellow M&Ms in first	P(10% yellow | n,k)	P(20% yellow | n,k)
n = 5: 1	0.44	0.56
n = 10: 2	0.39	0.61
n = 15: 3	0.34	0.66
n = 20:		

H_1: 10% yellow M&Ms
H_2: 20% yellow M&Ms

Bayesian Inference

n = 20

RGYBO BBGOY YRBRR GORBY

prior: P(p = 10%) = 0.5; P(p = 20%) = 0.5

posterior: P(p = 10% | data)

data = (n = 20, k = 4)

$$P(p=10\% \mid data) = \frac{P(data \mid p=10\%) \times P(p=10\%)}{P(data \mid p=10\%) \times P(p=10\%) + P(data \mid p=20\%) \times P(p=20\%)}$$

$$= \frac{0.08 \times 0.5}{0.08 \times 0.5 + 0.22 \times 0.5} \approx 0.29$$

	Bayesian	
Number of yellow M&Ms in first	P(10% yellow | n,k)	P(20% yellow | n,k)
n = 5: 1	0.44	0.56
n = 10: 2	0.39	0.61
n = 15: 3	0.34	0.66
n = 20: 4	0.29	0.71

H_1: 10% yellow M&Ms

H_2: 20% yellow M&Ms

H_0: 10% yellow M&Ms
H_A: 20% yellow M&Ms

H_1: 10% yellow M&Ms
H_2: 20% yellow M&Ms

	Frequentist	Bayesian	
Number of yellow M&Ms in first	P(k or more | 10% yellow)	P(10% yellow | n,k)	P(20% yellow | n,k)
n = 5: 1	0.41	0.44	0.56
n = 10: 2	0.26	0.39	0.61
n = 15: 3	0.18	0.34	0.66
n = 20: 4	0.13	0.29	0.71

RGYBO BBGOY YRBRR GORBY

$p = 20\%$

Bayesian	True state of the population	
Decision	% yellow = 10%	%yellow = 20%
%yellow = 10%	Your boss gives you a bonus :)	You lose your job :(
%yellow = 20%	You lose your job :(	Your boss gives you a bonus :)