Negation

The negation of a statement P is the statement that P is not true. Some ways to phrase this are

Not P.
It is false that P.

Examples:

Statement	Negation
It rained on 1 September 2005.	It did not rain on 1 September 2005.
All teachers are female.	Not all teachers are female.
Mike's dog has a black tail.	Mike's dog does not have a black tail.
2 + 2 = 4	2 + 2 ≠ 4.
Triangle ABC is equilateral.	Triangle ABC is not equilateral.

Negation inverts the truth or falsehood of logical statements. In other words, not P is False when P is True, and Not P is True when P is False. In tabular form:

P	Not P
True	False
False	True

The logical symbol for negation is "¬", so you can write ¬ P for Not P.

Even though "Not" is the simplest logical operator, the negation of statements is important when trying to prove that certain objects have or do not have certain properties. It makes the skill of being able to correctly negate statements an important one.

Conjunction

The conjunction of two statements P and Q is the statement that P and Q are both True. Some ways to phrase this are

P and Q.
P but Q.
P however Q.

Note that phrasing in English can sometimes include meaning that is not captured by the word 'and'. For example the statement

We had a good time even though it rained.

captures the idea that the fact that it rained would lead you to expect that it would be difficult to have a good time. Logically though, the statement is equivalent to

We had a good time and it rained.

since both combine the statements

We had a good time.

and

It rained.

Examples:

First statement	Second statement	Conjunction
The hall was long.	The hall was dark.	The hall was long and dark.
All teachers are female.	All teachers are humans.	All teachers are female humans.
Mike's dog has a black tail.	Mike's dog has a wet nose.	Mike's dog does has a black tail and a wet nose.
4 is even.	6 is odd.	4 is even and 6 is odd.
Triangle ABC is equilateral.	Triangle ABC is equiangular.	Triangle ABC is equilateral and equiangular.

Conjunction combines the assertions of two statements into a single statement. It's difficult to be more specific without being circular, but you might say P and Q is True when both P and Q are True, and False when either P or Q are False. In tabular form:

P	Q	P and Q
True	True	True
True	False	False
False	True	False
False	False	False

The logical symbol for conjunction is "∧", so you can write $P \wedge Q$ for P and Q.

Disjunction

The disjunction of two statements P and Q is the statement that at least one of P and Q are True. Some ways to phrase this are

P or Q.
P unless Q.

In mathematics the exclusive or is never used, so

P or Q.

always means

P or Q or both.

This contrasts with English where the exclusive or is often implied by context, as in

You can choose either the Big Box or whatever is behind Curtain #2.

In the rare cases where exclusive or is needed in mathematics, the phrase "but not both" can be added to make it clear.

Examples:

First statement	Second statement	Disjunction
The hall was long.	The hall was dark.	The hall was either long or dark.

Mike's dog has a black tail.	Dave's dog has a black tail.	Either Mike's dog or Dave's dog has a black tail.
4 is even.	6 is odd.	4 is even or 6 is odd.
Triangle ABC is isosceles.	Triangle ABC is scalene.	Triangle ABC is either isosceles or scalene.

Disjunction offers two possibilities which are given by the two statements. Again, it's difficult to be more specific without being circular, but you might say P or Q is True when either P or Q (or both) are True, and False when both P and Q are False. In tabular form:

P	Q	P or Q
True	True	True
True	False	True
False	True	True
False	False	False

The logical symbol for disjunction is "V", so you can write P V Q for P or Q.

Properties

Properties of Logical Equivalence. Denote by TT and FF a tautology and a contradiction, respectively. We have the following properties for any propositional variables pp, qq, and rr.

1. ***Commutative properties***:

   $p \vee q \equiv q \vee p$,

   $p \wedge q \equiv q \wedge p$.

$p\vee q\equiv q\vee p$,

$p\wedge q\equiv q\wedge p$.

2. ***Associative properties***: $(p\vee q)\vee r\equiv p\vee(q\vee r)$,

$(p\wedge q)\wedge r\equiv p\wedge(q\wedge r)$.

$(p\vee q)\vee r\equiv p\vee(q\vee r)$,

$(p\wedge q)\wedge r\equiv p\wedge(q\wedge r)$.

3. ***Distributive laws***: $p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$,

$p\wedge(q\vee r)\equiv(p\wedge q)\vee(p\wedge r)$.

$p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$,

$p\wedge(q\vee r)\equiv(p\wedge q)\vee(p\wedge r)$.

4. ***Idempotent laws***: $p\vee p\equiv p$,

$p\wedge p\equiv p$.

$p\vee p\equiv p$,

$p\wedge p\equiv p$.

5. ***De Morgan's laws***: $\overline{p\vee q}\equiv\overline{p}\wedge\overline{q}$,

$\overline{p\wedge q}\equiv\overline{p}\vee\overline{q}$.

$\overline{p\vee q}\equiv\overline{p}\wedge\overline{q}$,

$\overline{p\wedge q}\equiv\overline{p}\vee\overline{q}$.

6. ***Laws of the excluded middle***, or ***inverse laws***: $p\vee\overline{p}\equiv T$,

$p\wedge\overline{p}\equiv F$.

$p\vee\overline{p}\equiv T$,

$p\wedge\overline{p}\equiv F$.

7. ***Identity laws***: $p\vee F\equiv p$,

   $p\wedge T\equiv p$.

   $p\vee F\equiv p$,

   $p\wedge T\equiv p$.

8. ***Domination laws***: $p\vee T\equiv T$,

   $p\wedge F\equiv F$.

   $p\vee T\equiv T$,

   $p\wedge F\equiv F$.

9. Equivalence of an implication and its contrapositive: $p\Rightarrow q\equiv\overline{q}\Rightarrow\overline{p}$

   $p\Rightarrow q\equiv\overline{q}\Rightarrow\overline{p}$.

10. Writing an implication as a disjunction: $p\Rightarrow q\equiv\overline{p}\vee q$$p\Rightarrow q\equiv\overline{p}\vee q$.
11. The negation of an implication: $\overline{p\Rightarrow q}\equiv p\wedge\overline{q}$$\overline{p\Rightarrow q}\equiv p\wedge\overline{q}$

Be sure you understand and memorize the last three equivalences, because we will use them frequently in the rest of the course.

It may not be easy to memorize the names of all these properties; however, they should all make sense to you. The important name is De Morgan's laws. Let us explain them in words, and compare them to similar operations on the real numbers,

1. ***Commutative properties***: In short, they say that "the order of operation does not matter." It does not matter which of the two logical statements comes first, the result from conjunction and disjunction always produces the same truth value. Compare this to addition of real numbers: $x+y=y+x$$x+y=y+x$. Subtraction is not commutative, because it is not always true that $x-y=y-x$$x-y=y-x$. This explains why we have to make sure that an operation is commutative.
2. ***Associative properties***: Roughly speaking, these properties also say that "the order of operation does not matter." However, there is a key difference between them and the commutative properties.

- Commutative properties apply to operations on *two* logical statements, but associative properties involves *three* logical statements.
  Since $\wedge$ and $\vee$ are *binary* operations, we can only work on a pair of statements at a time. Given the three statements p, q, and r, appearing in that order, which pair of statements should we operate on first? The answer is: it does not matter. It is the order of *grouping* (hence the term associative) that does not matter in associative properties.
- The important consequence of the associative property is: since it does not matter on which pair of statements we should carry out the operation first, we can eliminate the parentheses and write, for example,

$$p\vee q\vee r \tag{2.5.13}$$

without worrying about any confusion.

- Not all operations are associative. Subtraction is not associative. Given three numbers 5, 7, and 4, in that order, how should we carry out two subtractions? Which interpretation should we use:

$$(5-7)-4, \text{or } 5-(7-4)? \tag{2.5.14}$$

Since they lead to different results, we have to be careful where to place the parentheses.

3. ***Distributive laws***: When we mix two *different* operations on three logical statements, one of them has to work on a pair of statements first, forming an "inner" operation. This is followed by the "outer" operation to complete the compound statement. Distributive laws say that we can distribute the "outer" operation over the inner one.
4. ***Idempotent laws***: When an operation is applied to a pair of identical logical statements, the result is the same logical statement. Compare this to the equation $x^2=x$, where x is a real number. It is true only when $x=0$ or $x=1$. But the logical equivalences $p\vee p\equiv p$ and $p\wedge p\equiv p$ are true for all p.
5. ***De Morgan's laws***: When we negate a disjunction (respectively, a conjunction), we have to negate the two logical statements, and change the operation from disjunction to conjunction (respectively, from conjunction to a disjunction).
6. ***Laws of the excluded middle***, or ***inverse laws***: Any statement is either true or false, hence $p\vee\overline{p}$ is always true. Likewise, a statement cannot be both true and false at the same time, hence $p\wedge\overline{p}$ is always false.
7. ***Identity laws***: Compare them to the equation $x\cdot 1=x$: the value of x is unchanged after multiplying by 1. We call the number 1 the multiplicative identity. For logical operations, the identity for disjunction is F, and the identity for conjunction is T.

8. ***Domination laws***: Compare them to the equation $x\cdot 0=0$x·0=0 for real numbers: the result is always 0, regardless of the value xx. The "zero" for disjunction is T, and the "zero" for conjunction is F.

Duality

The principle of duality is a type of pervasive property of algebraic structure in which two concepts are interchangeable only if all results held in one formulation also hold in another. This concept is known as dual formulation. We will interchange unions(∪) into intersections(∩) or intersections() into the union() and also interchange universal set into the null set(∅) or null set into universal(U) to get the dual statement. If we interchange the symbol and get this statement itself, it will be known as the self-dual statement.

The dual of $(X \cap Y) \cup Z$ is $(X \cup Y) \cap Z$

Duality can also be described as a property that belongs to the branch of algebra. This theory can be called lattice theory. This theory has the ability to involve order and structure, which are common to different mathematical systems. If the mathematical system has the order in a specified way, this structure will be known as lattice.

The principle of duality concept should not be avoided or underestimated. It has the ability to provide several sets of theorems, concepts, and identities. To explain the duality principle of sets, we will assume S be any identity that involves sets, and operation complement, union, intersection. Suppose we obtain the S* from S with the help of substituting $\cup \rightarrow \cap\ \Phi$. In this case, the statement S* will also be true, and S* can also be known as dual statement S.

Duality Principle

- According to the duality principle, if we have postulates or if we have theorems of Boolean Algebra for any one type of operation then the operation can be converted into another type of operation.
- In other words AND can be converted to OR and OR can be converted into AND
- We can interchange '0 with 1', '1 with 0', '(+) sign with (.) sign' and '(.) sign with (+) sign' to perform dual operation. T
- This principle ensures that if a theorem is proved using postulates of Boolean algebra, then the dual of this theorem automatically holds and there is no requirement of proving it separately.

The dual of a Boolean expression can easily be obtained by interchanging sums and products and interchanging 0 as well as 1. Let’s know how to find the dual of any expression.

For example, the dual of $x\bar{y} + 1$ is equal to $(x + y) \cdot 0$

Duality Principle: The Duality principle states that when both sides are replaced by their duals the Boolean identity remains valid.

Some Boolean expressions and their corresponding duals are given in the table below:

Boolean Expressions and Their Corresponding Duals

Given Expression	**Dual**	**Given Expression**	**Dual**
0 = 1	1 = 0	A. (A+B) = A	A + A.B = A
0.1 = 0	1 + 0 = 1	AB = A + B	A+B = A.B
A.0 = 0	A + 1 = 1	(A+C) (A +B) = AB + AC	AC + AB = (A+B).(A+C)
A.B = B. A	A + B = B + A	A+B = AB + AB +AB	AB = (A+B).(A+B).(A+B)
A.A = 0	A + A = 1	AB + A + AB = 0	((A+B)).A.(A+B) = 1
A. (B.C) = (A.B). C	A+(B+C) = (A+B) + C		

Functional Completeness

• A set of logical connectives is called functionally complete if every boolean expression is equivalent to one involving only these connectives.

• The set { ¬, ∨, ∧ } is functionally complete. – Every boolean expression can be turned into a CNF, which involves only ¬, ∨, and ∧.

• The sets { ¬, ∨ } and { ¬, ∧ } are functionally complete. – By the above result and de Morgan's laws.

• { nand } and { nor } are functionally complete.

Functionally Complete Operations

A switching function is expressed by binary variables, the logic operation symbols, and constants 0 and 1. When every switching function can be expressed by means of operations in it, then only a set of operation is said to be functionally complete.

1. The set (AND, OR, NOT) is a functionally complete set.
2. The set (AND, NOT) is said to be functionally complete.
3. The set (OR, NOT) is also said to be functionally complete.

Here,
The set (AND, NOT) is said to be functionally complete as (OR) can be derived using 'AND' and 'NOT' operations.