Sets and Subsets

Set - A collection of objects. The specific objects within the set are called the *elements* or *members* of the set. *Capital letters* are commonly used to name sets.

Examples: $Set\ A = \{a, b, c, d\}$ or $Set\ B = \{1, 2, 3, 4\}$

Set Notation - Braces { } can be used to list the members of a set, with each member separated by a comma. This is called the "Roster Method." A description can also be used in the braces. This is called "Set-builder" notation.

Example: *Set A:* The natural numbers from 1 to 10.

Roster Method

Members of A: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Set Notation: $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Set Builder Not.: $\{x \mid x\ is\ a\ natural\ number\ from\ 1\ to\ 10\}$

Ellipsis - Three dots (...) used within the braces to indicate that the list continues in the established pattern. This is helpful notation to use for *long lists* or *infinite lists*. If the dots come at the end of the list, they indicate that the list goes on indefinitely (i.e. an infinite set).

Examples: *Set A:* Lowercase letters of the English alphabet

Set Notation: $\{a, b, c, \ldots, z\}$

Cardinality of a Set – The number of *distinct* elements in a set.

Example: Set A: The days of the week

Members of Set A: Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday

Cardinality of Set $A = \boldsymbol{n(A)} = 7$

Equal Sets – Two sets that contain exactly the same elements, regardless of the order listed or possible repetition of elements.

Example: $A = \{1, 1, 2, 3, 4\}$, $B = \{4, 3, 2, 1, 2, 3, 4, \}$.

Sets $A\ and\ B$ are equal because they contain exactly the same elements (i.e. 1, 2, 3, & 4). This can be written as $\boldsymbol{A = B}$.

Union of Sets – The Union of Sets *A* and *B* is the set of elements that are members of Set *A*, Set *B*, or both Sets. It can be written as $\boldsymbol{A} \cup \boldsymbol{B}$.

Example: Find the Intersection and the Union for the Sets *A* and *B*.

$$Set\ A = \{Red, Blue, Green\}$$

$$Set\ B = \{Yellow, Orange, Red, Purple, Green\}$$

Set *A* and *B* only have 2 elements in common.

Intersection: $\boldsymbol{A} \cap \boldsymbol{B} = \{Red, Green\}$

Union: $\boldsymbol{A} \cup \boldsymbol{B} = \{Red, Blue, Green, Yellow, Orange, Purple\}$

List each distinct element only once, even if it appears in both Set A and Set B.

Complement of a Set - The Complement of Set *A*, written as ***A'***, is the set of all elements in the given Universal Set (*U*), that are not in Set *A*.

Example: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1,3,5,7,9\}$

Find A'.

Cross off everything in U that is also in A. What is left over will be the elements that are in A'

$$U = \{\cancel{1}, 2, \cancel{3}, 4, \cancel{5}, 6, \cancel{7}, 8, \cancel{9}, 10\}$$

So, $A' = \{2, 4, 6, 8, 10\}$

Try these on your own!

Given the set descriptions below, answer the following questions.

$U = All\ Integers\ from\ 1\ to\ 10.$ $A = Odd\ Integers\ from\ 1\ to\ 10,$

$B = Even\ Integers\ from\ 1\ to\ 10,$ $C = Multiples\ of\ 2\ from\ 1\ to\ 10.$

1. Write each of the sets in roster notation. $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,3,5,7,9\}$, $B = \{2,4,6,8,10\}$, $C = \{2,4,6,8,10\}$
2. What is the *cardinality* of Sets U and *A*? *Cardinality: U→10, A →5*
3. Are Set *B* and Set *C Equal*? *Yes, they are Equal*
4. Are Set *A* and Set *C Equivalent*? *Yes, they are Equivalent*
5. How many *Proper Subsets* of Set *U* are there? $2^{10} - 1 = 1023$
6. Find $\boldsymbol{B}'\ and\ \boldsymbol{C}'$ $B' = C' = \{1, 3, 5, 7, 9\}$
7. Find $\boldsymbol{A} \cup \boldsymbol{C}'$ $A \cup C' = \{1, 3, 5, 7, 9\}$
8. Find $\boldsymbol{B}' \cap \boldsymbol{C}$ $B' \cap C = \{\ \}$ or $\emptyset$

Subsets - For Sets *A* and *B*, Set *A* is a **Subset** of Set *B* if every element in Set *A* is also in Set *B*. It is written as $\boldsymbol{A \subseteq B}$.

Proper Subsets - For Sets *A* and *B*, Set *A* is a **Proper Subset** of Set *B* if every element in Set *A* is also in Set *B*, ***but*** *Set A does **not** equal Set B.* $(\boldsymbol{A \neq B})$ It is written as $\boldsymbol{A \subset B}$.

Example: $Set\ A = \{2, 4, 6\}$ $\qquad\qquad$ $Set\ B = \{0, 2, 4, 6, 8\}$

$\{2, 4, 6\} \subseteq \{0, 2, 4, 6, 8\}$ **and** $\{2, 4, 6\} \subset \{0, 2, 4, 6, 8\}$

Set *A* is a **Subset** of Set *B* because every element in *A* is also in *B*. $\boldsymbol{A \subseteq B}$	Set *A* is a **Proper Subset** of Set *B* because every element in *A* is also in B, but A $\neq B$. $\boldsymbol{A \subset B}$

Note: *The Empty Set is a Subset of every Set.*
The Empty Set is also a Proper Subset of every Set except the Empty Set.

Number of Subsets – The number of distinct subsets of a set containing n elements is given by $\boldsymbol{2^n}$.

Number of Proper Subsets – The number of distinct proper subsets of a set containing n elements is given by $\boldsymbol{2^n - 1}$.

Example: How many Subsets and Proper Subsets does Set *A* have?

$$Set\ A = \{bananas, oranges, strawberries\}$$
$$n = 3$$

Subsets = $2^n = 2^3 = 8$ $\qquad$ **Proper Subsets** = $2^n - 1 = 7$

Example: List the **Proper Subsets** for the Example above.

1. $\{bananas\}$
2. $\{oranges\}$
3. $\{strawberries\}$
4. $\{bananas, oranges\}$
5. $\{bananas, strawberries\}$
6. $\{oranges, strawberries\}$
7. $\emptyset$

Intersection of Sets – The Intersection of Sets *A* and *B* is the set of elements that are in both *A* and *B*, *i.e.* *what they have in common*. It can be written as $\boldsymbol{A \cap B}$.

Equivalent Sets – Two sets that contain the same number of distinct elements.

Example: $A = \{Football, Basketball, Baseball, Soccer\}$
$B = \{penny, nickel, dime, quarter\}$

Both Sets have 4 elements

$n(A) = 4 \quad and \quad n(B) = 4$

$A\ and\ B\ are\ Equivalent\ Sets,\ meaning\ n(A) = n(B).$

Note: If two sets are Equal, they are also Equivalent!

Example: $Set\ A = \{a, b, c, d\}$ $\quad Set\ B = \{d, d, c, c, b, b, a, a\}$

Are Sets A and B Equal?

Sets A and B have the exact same elements! $\{a, b, c, d\}$

→ **Yes!**

Are Sets A and B Equivalent?

Sets A and B have the exact same number of distinct elements! $n(A) = n(B) = 4$

→ **Yes!**

The Empty Set ***(or Null Set)*** – The set that contains **no elements**.
It can be represented by either $\{\ \ \}$ or $\emptyset$.

Note: Writing the empty set as $\{\emptyset\}$ is **not correct**!

Symbols commonly used with Sets –

$\in \rightarrow$ *indicates an object is an* ***element*** *of a set.*

$\notin \rightarrow$ *indicates an object is* ***not*** *an element of a set.*

$\subseteq \rightarrow$ *indicates a set is a* ***subset*** *of another set.*

$\subset \rightarrow$ *indicates a set is a* ***proper subset*** *of another set.*

$\cap \rightarrow$ *indicates the* ***intersection*** *of sets.*

$\cup \rightarrow$ *indicates the* ***union*** *of sets.*

Cartesian product

Definition: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where s $\in$ S and t $\in$ T. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T\}$.

Examples:

- S = {1,2} and T = {a,b,c}
- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: S x T $\neq$ T x S !!!!

Cardinality of the Cartesian product

- |S x T| = |S| * |T|.

Example:

- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- |A x B| = 9
- |A|=3, |B|=3 → |A| |B|= 9

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Infinite set

Definition: A set is **infinite** if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- N = {1, 2, 3, ... }

- The set of reals is an infinite set.

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set $\varnothing$
- What is the power set of $\varnothing$? $P(\varnothing) = \{ \varnothing \}$
- What is the cardinality of $P(\varnothing)$? $|P(\varnothing)| = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

Power set

- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{ \varnothing, \{1\}, \{2\}, \{1,2\} \}$
- $|P(\{1,2\})| =4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$
- $|P(\{1,2,3\} | = 8$

- **If S is a set with $|S| = n$ then $| P(S) | = ?$**

Power set

- $P(\{1\}) = \{\varnothing, \{1\}\}$
- $|P(\{1\})| = 2$

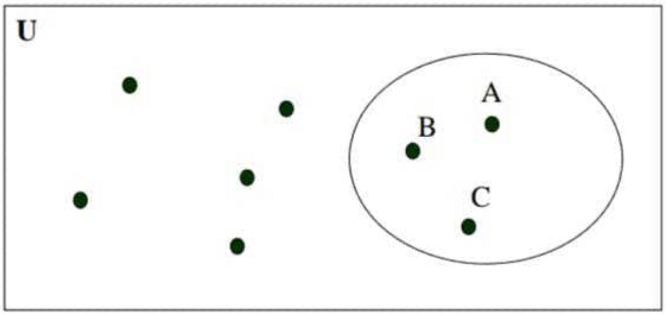
- Assume $\{1,2\}$
- $P(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$
- $|P(\{1,2\})| = 4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $|P(\{1,2,3\}| = 8$

- **If S is a set with $|S| = n$ then $|P(S)| = 2^n$**

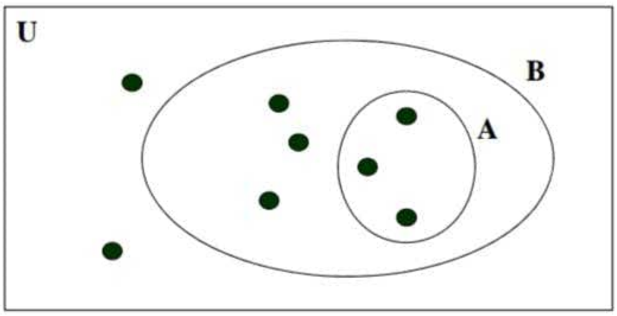
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }



A Subset

- **Definition**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use **$A \subseteq B$** to indicate **A is a subset of B**.



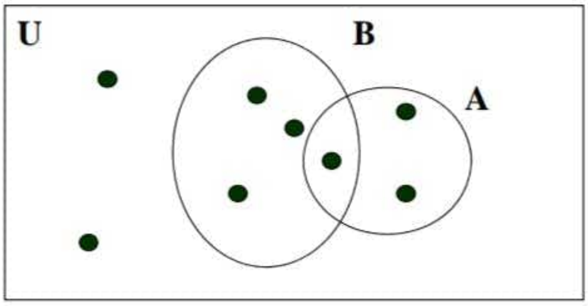
- Alternate way to define A is a subset of B:

  $\forall x \, (x \in A) \rightarrow (x \in B)$

Set operations

Definition: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.

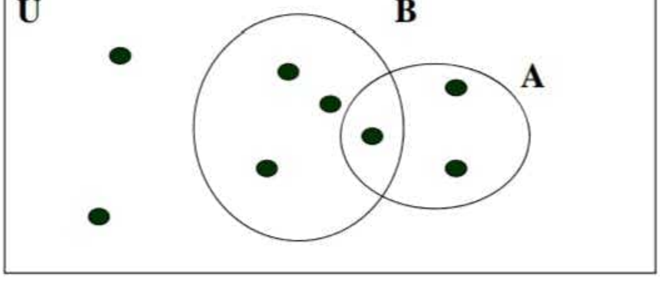


- **Example:**
- $A = \{1,2,3,6\}$ $\quad B = \{ 2,4,6,9\}$
- $A \cup B = \{ 1,2,3,4,6,9 \}$

Set operations

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



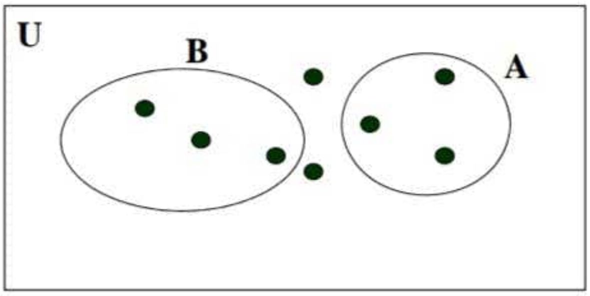
Example:

- $A = \{1,2,3,6\}$ $\quad B = \{ 2, 4, 6, 9\}$
- $A \cap B = \{ 2, 6 \}$

Disjoint sets

Definition: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \emptyset$.



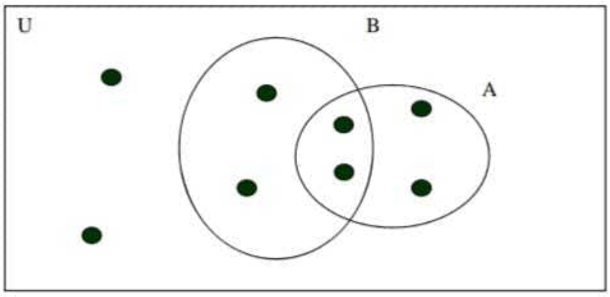
Example:

- $A=\{1,2,3,6\}$ $B=\{4,7,8\}$ Are these disjoint?
- Yes.
- $A \cap B = \emptyset$

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

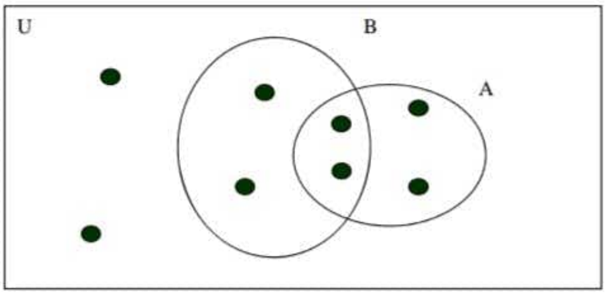


- Why this formula?

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

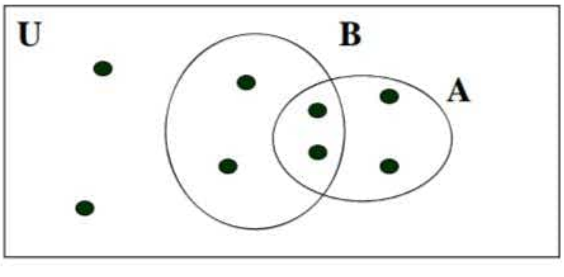


- Why this formula? Correct for an over-count.
- More general rule:
  - **The principle of inclusion and exclusion.**

Set difference

Definition: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \wedge x \notin B \}$.



Example: A= {1,2,3,5,7} B = {1,5,6,8}

- A - B ={2,3,7}

Question 2: Let A and B be two finite sets such that $n(A) = 20$, $n(B) = 28$ and $n(A \cup B) = 36$, find $n(A \cap B)$.

Solution: Using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

then $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 20 + 28 - 36$

$= 48 - 36$

$= 12$

Question 3: In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?

Solution: Let A = Set of people who like cold drinks B = Set of people who like hot drinks

Given,

$(A \cup B) = 60$ $n(A) = 27$ $n(B) = 42$ then;

$n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 27 + 42 - 60$

$= 69 - 60 = 9$

$= 9$

Question: In a class of 100 students, 35 like science and 45 like math. 10 like both. How many like either of them and how many like neither?

Solution:

Total number of students, $n(\mu) = 100$

Number of science students, $n(S) = 35$

Number of math students, $n(M) = 45$

Number of students who like both, $n(M \cap S) = 10$

Number of students who like either of them,

$$n(M \cup S) = n(M) + n(S) - n(M \cap S)$$

Total number of students, $n(\mu) = 100$

Number of science students, $n(S) = 35$

Number of math students, $n(M) = 45$

Number of students who like both, $n(M \cap S) = 10$

Number of students who like either of them,

$n(M \cup S) = n(M) + n(S) - n(M \cap S)$

$\rightarrow 45 + 35 - 10 = 70$

Number of students who like neither $= n(\mu) - n(M \cup S) = 100 - 70 = 30$