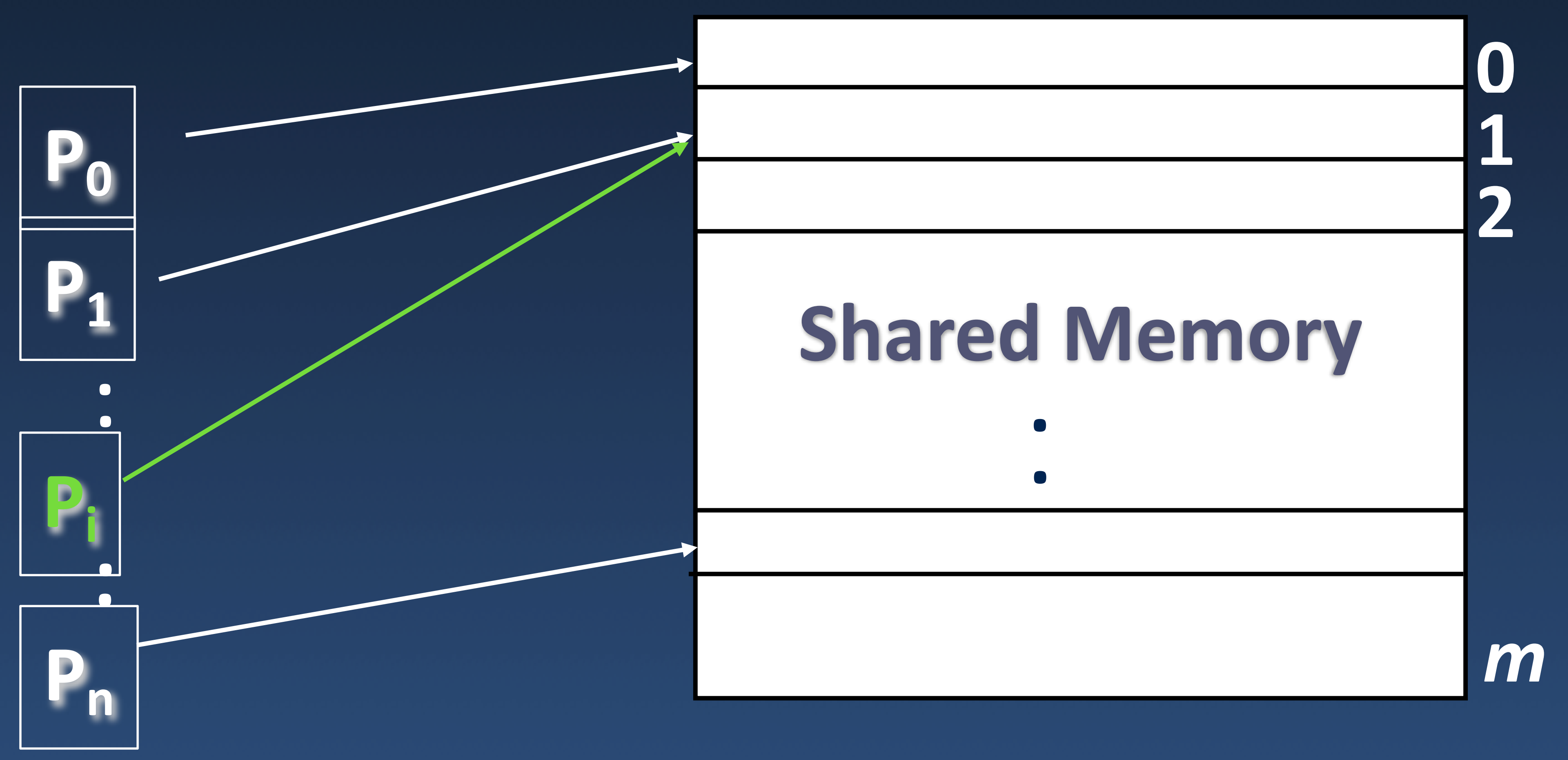
COL380

Introduction to
Parallel & Distributed Programming

PRAM Recap

Recall PRAM



Synchronous steps:

- Read
- Compute
- Write

- EREW
- CREW
- CRCW

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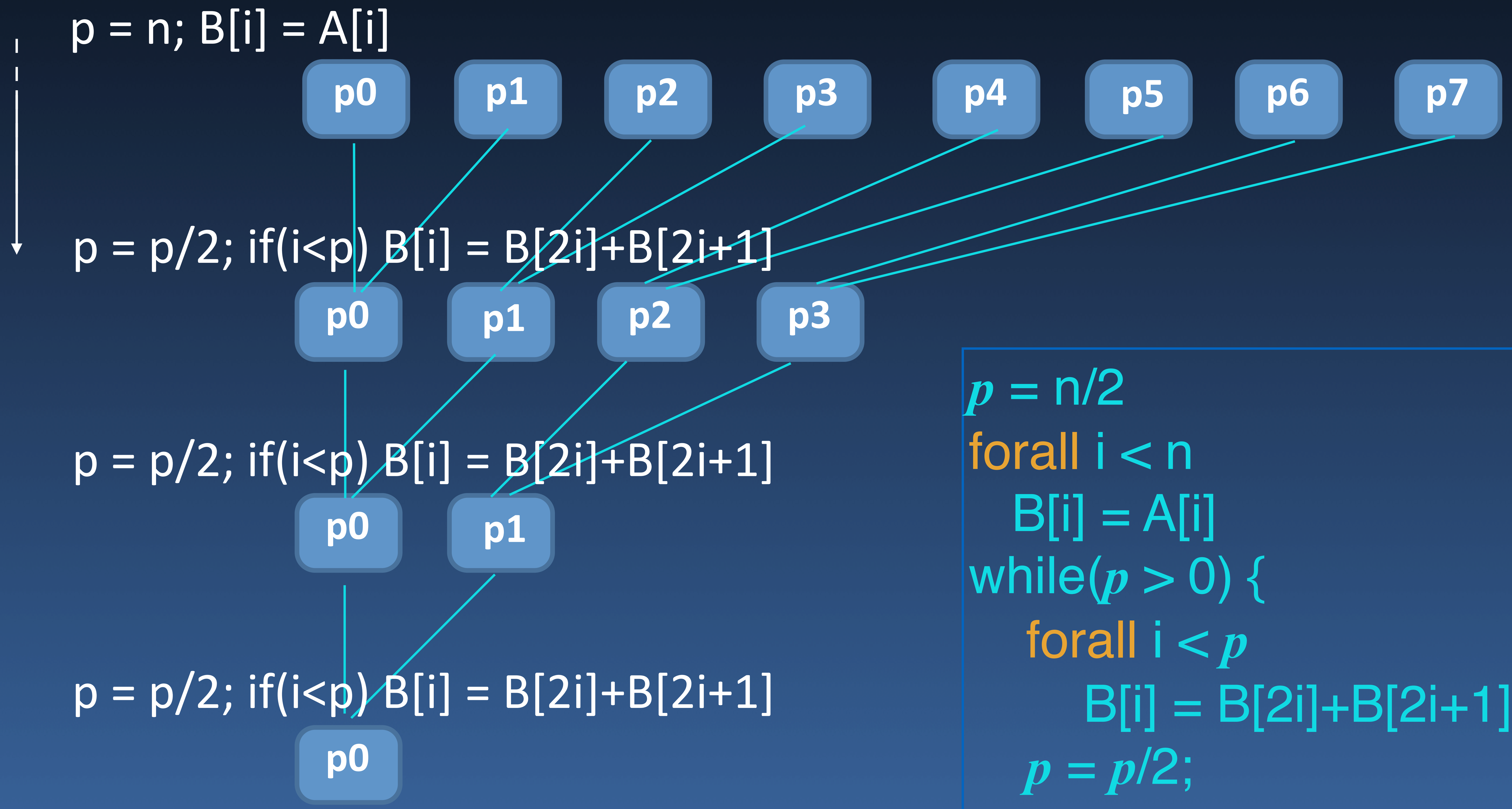
- Maximize concurrency
  - ➡ Reduce dependency
    - OK to sometime recompute data
- Map tasks to processors
  - ➡ Statically or Dynamically
  - ➡ Reduce communication

Parallel Addition




```
p = n/2
forall i < n
   B[i] = A[i]
while(p > 0) {
    forall i < p
        B[i] = B[2i]+B[2i+1]
    p = p/2;
}
```

(assumes n is a power of 2)

- processors: n
- time: O(log n)
- Speed-up: n/(log n)
- Efficiency: 1/log(n)
- Cost: n log n
- Work: n

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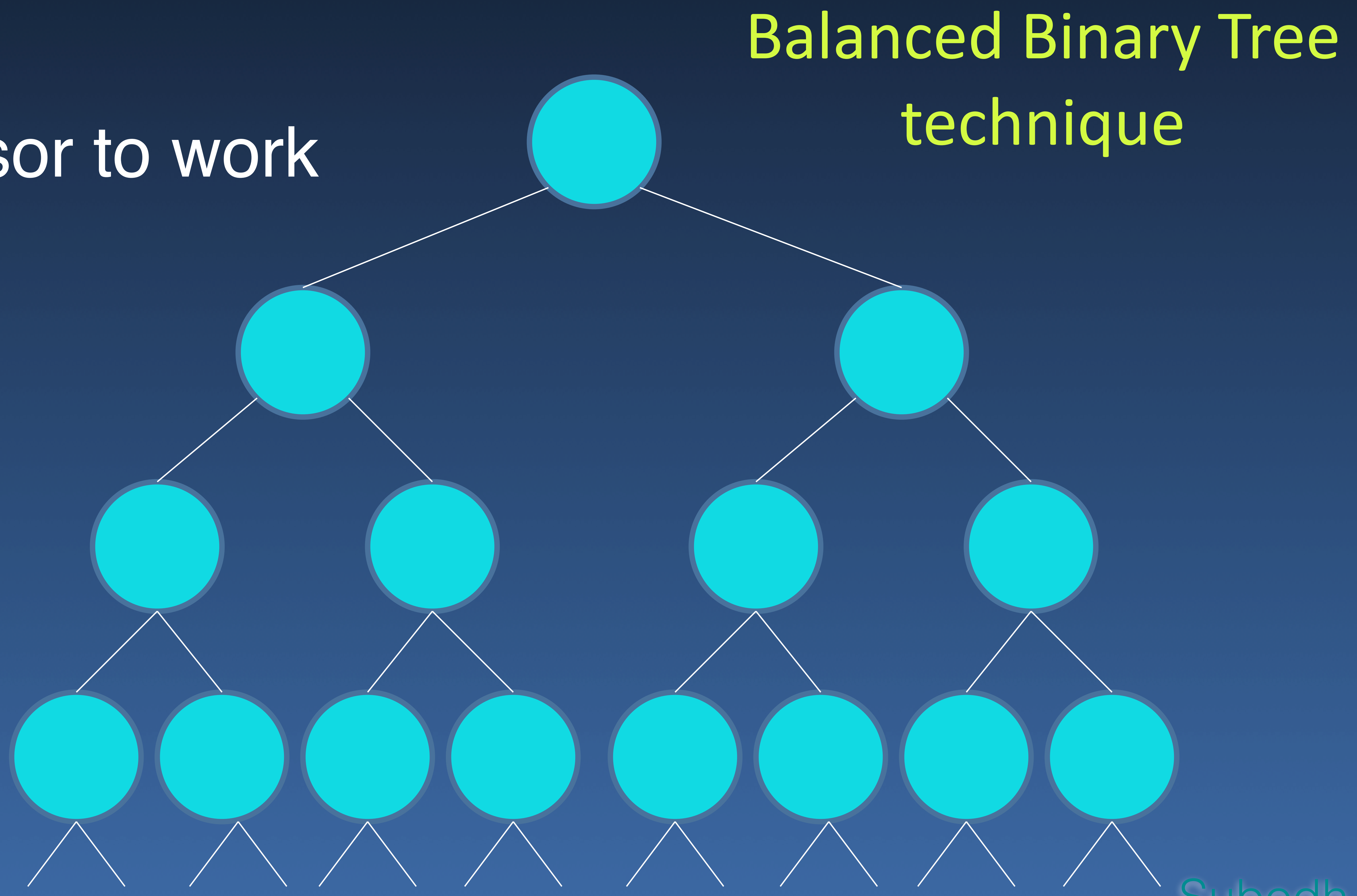
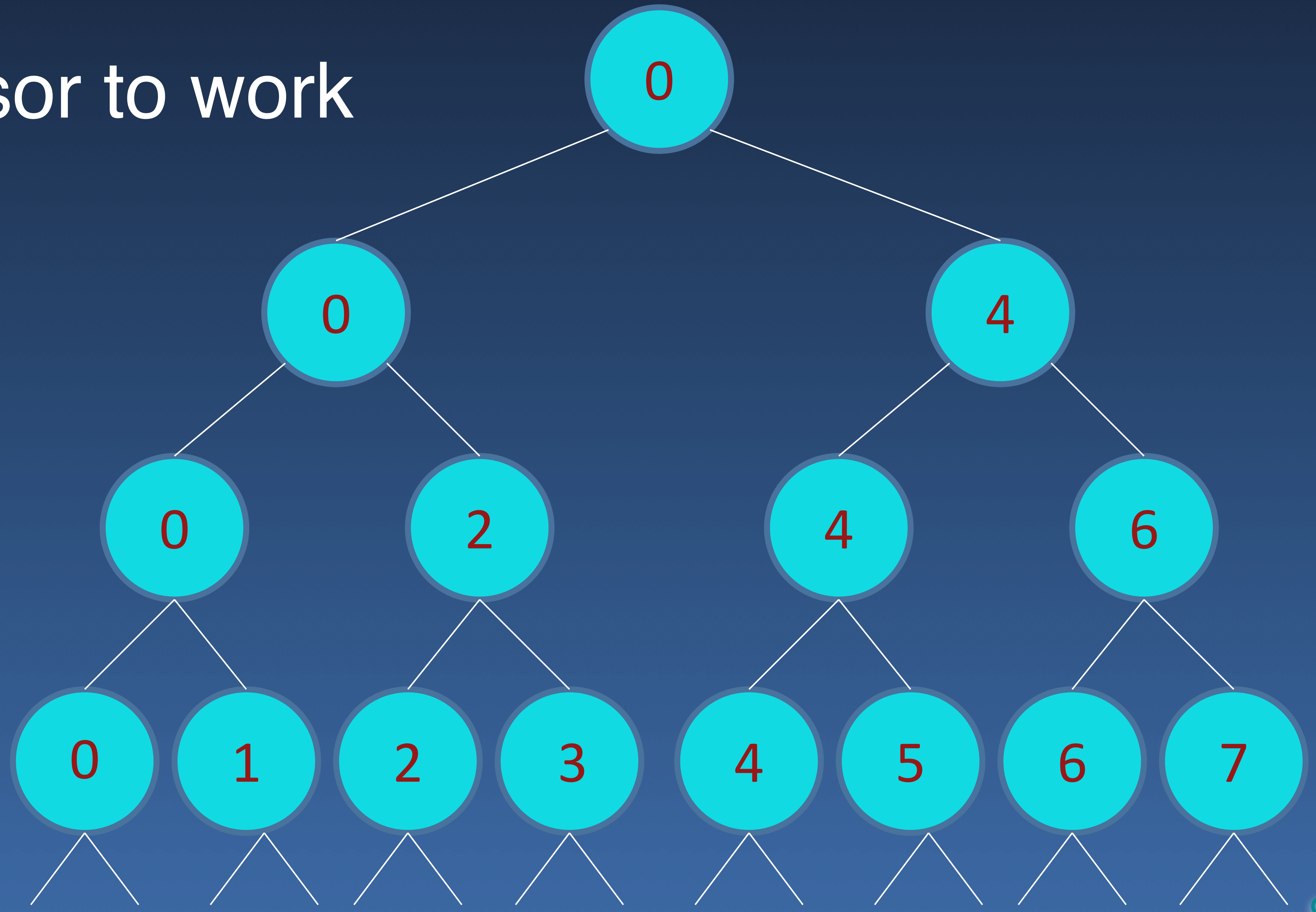
Reduction

- n operands ⇒ log n steps
- Total work = O(n)
- How do you map processor to work



Reduction

- n operands ⇒ log n steps
- Total work = O(n)
- How do you map processor to work
  - ➡ $n/2^i$ processors per step
  - ➡ step i: if $!(id\%2^i)$
    - Read: id, id+1
    - Write id



- n operands ⇒ log n steps
- Total work = O(n)
- How do you map processor to work
  - ➡ Consider $p < n$
  - ➡ Locally reduce at each processor
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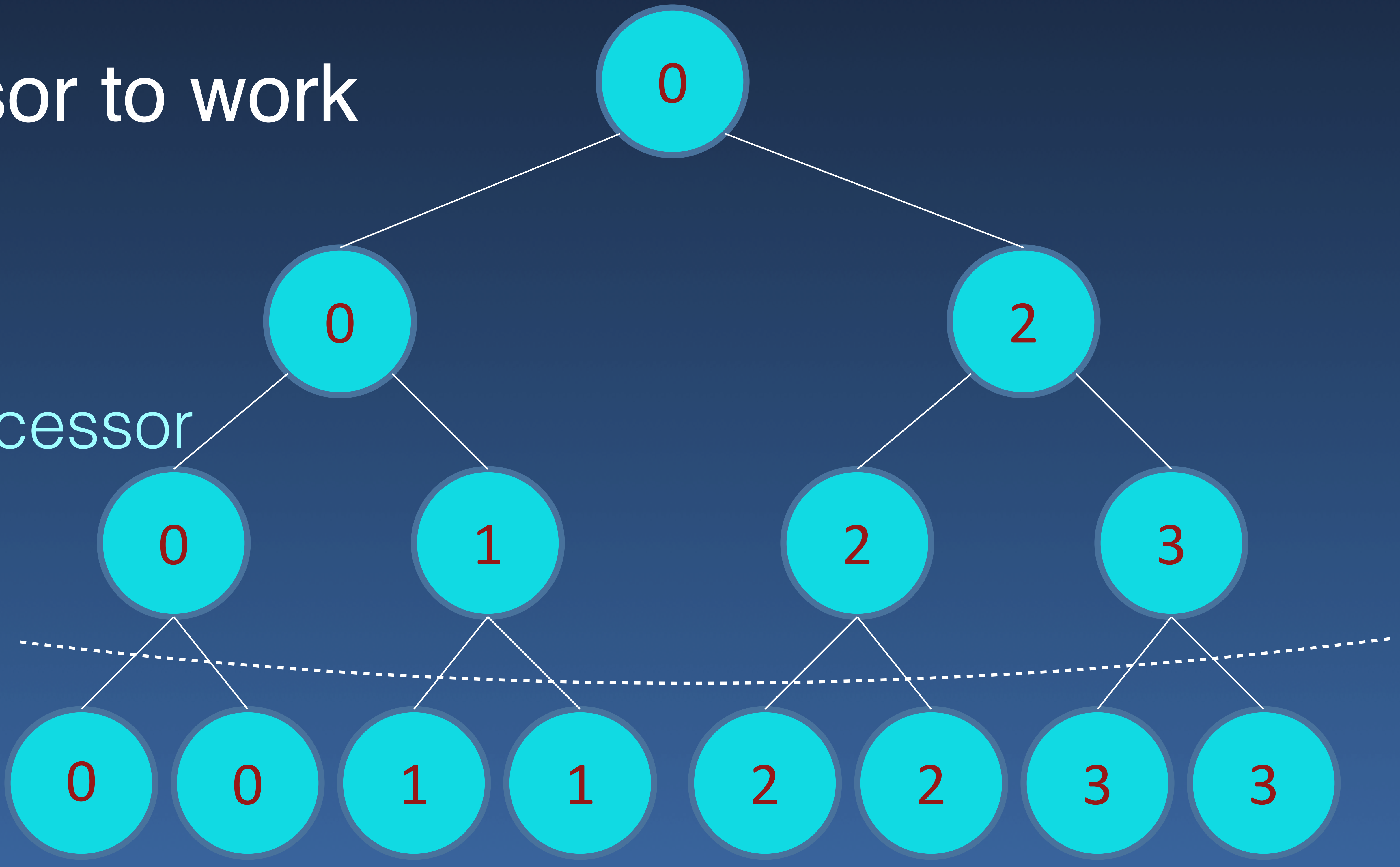

Reduction

- Count the number of operations
  - ➡ Then map to p processors
- Sometime, convenient to start with p
  - ➡ And map operation to processors

- n operands ⇒ log n steps
- Total work = O(n)
- How do you map processor to work
  - ➡ Consider p < n
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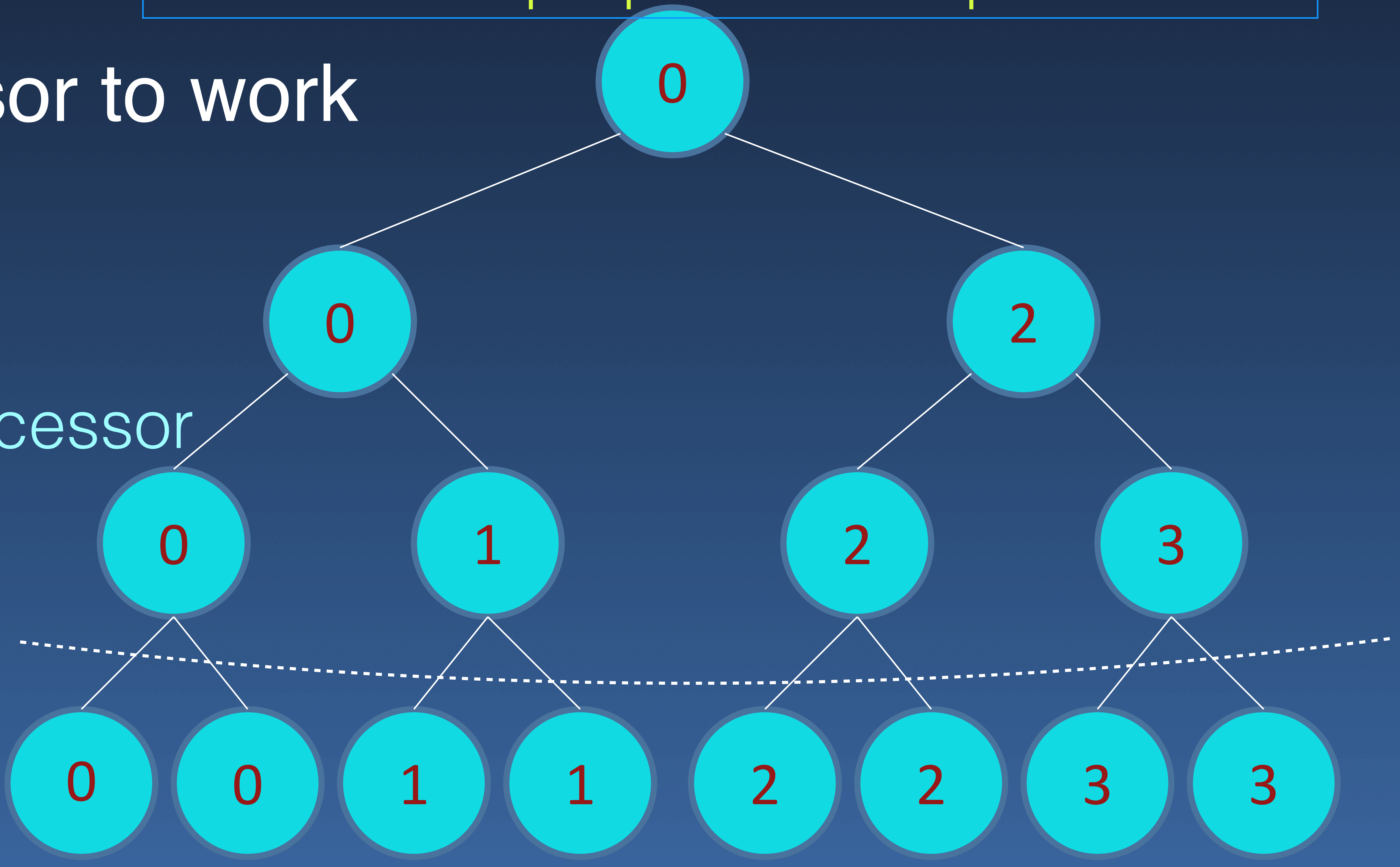

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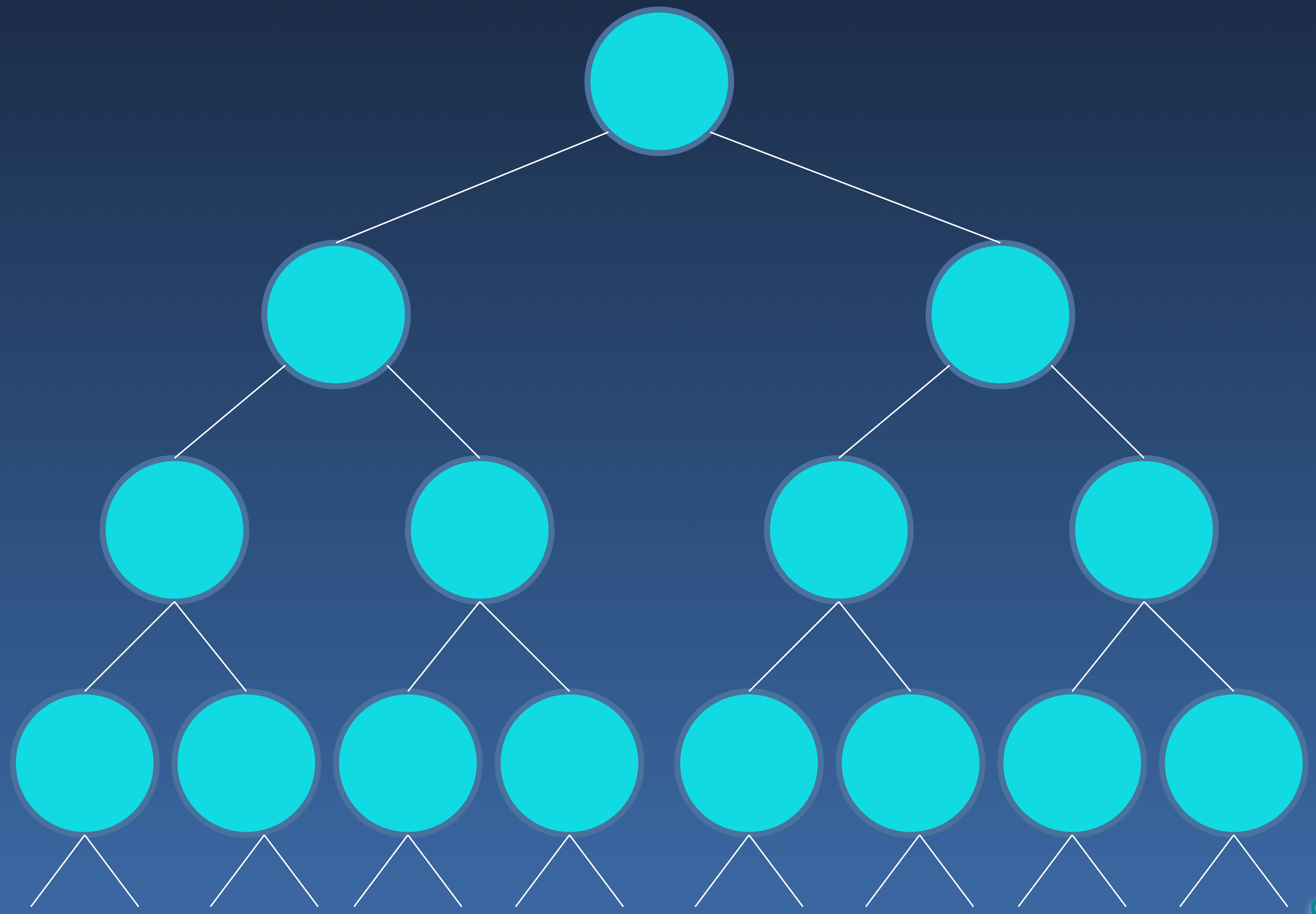
Prefix Sums

Input: x

- P[0] = x[0]
- For i = 1 to n-1
  - ➡ P[i] = P[i-1] + x[i]



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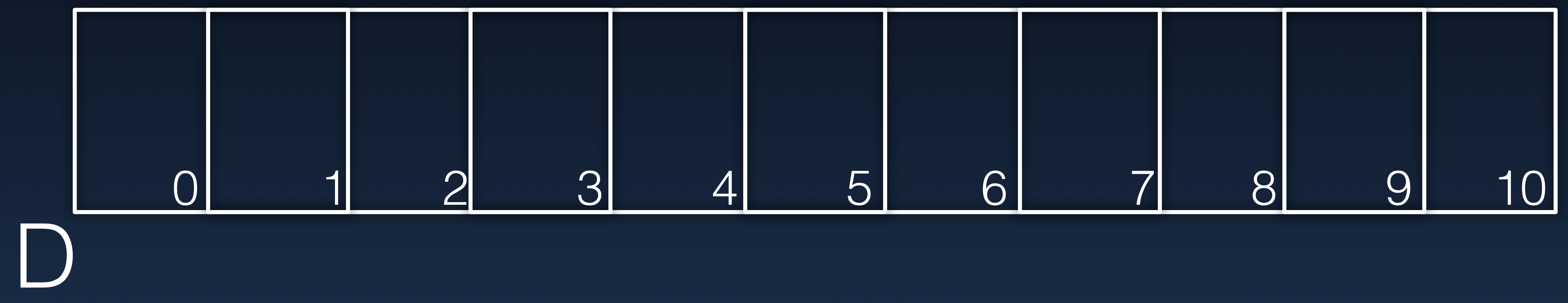
Prefix Sums

```
forall i < n
        R[i] = Map(D[i]);
```



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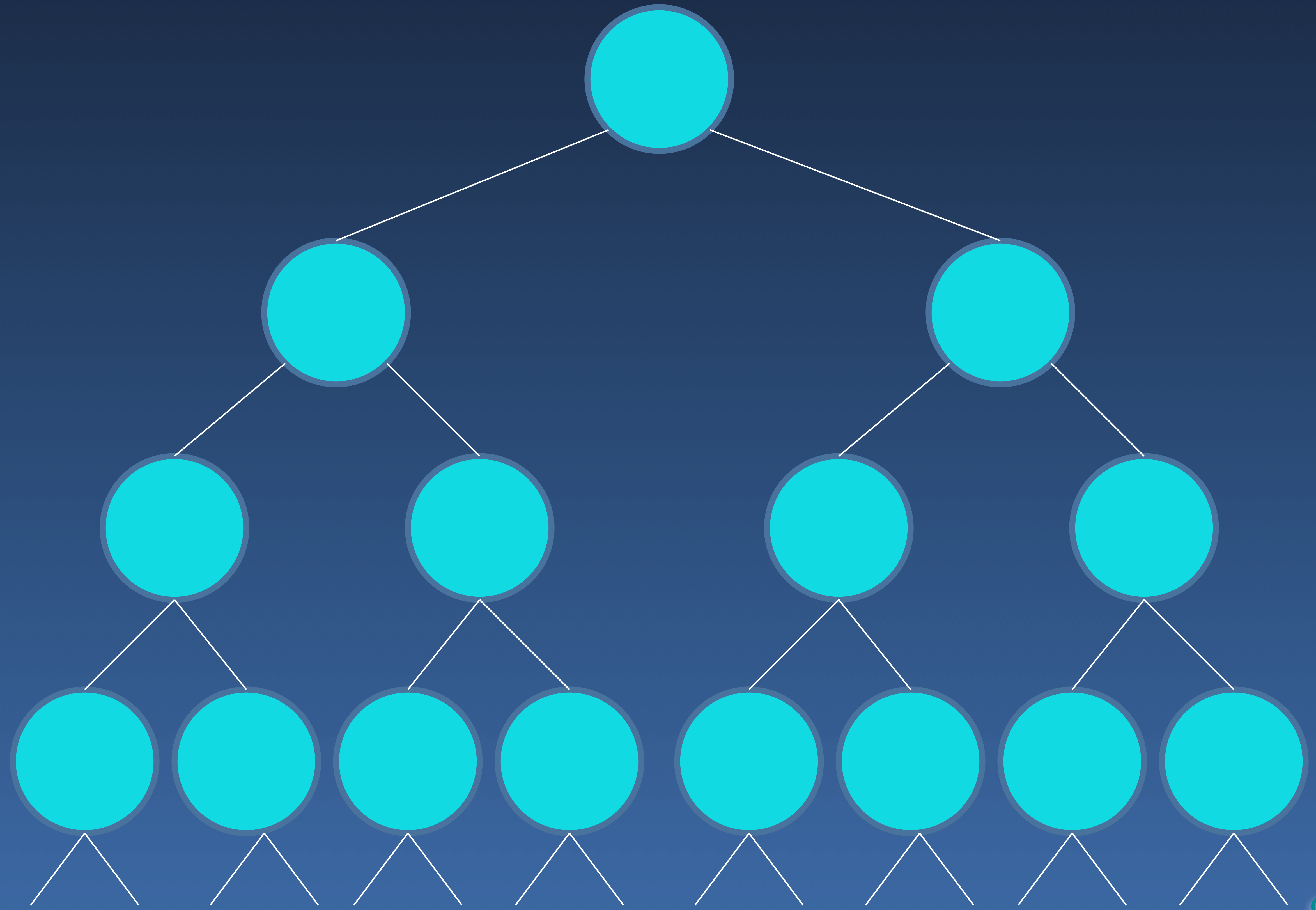

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Prefix Sums

```
forall i < n
   if(filter(D[i]))
      R[i] = Map(D[i]);
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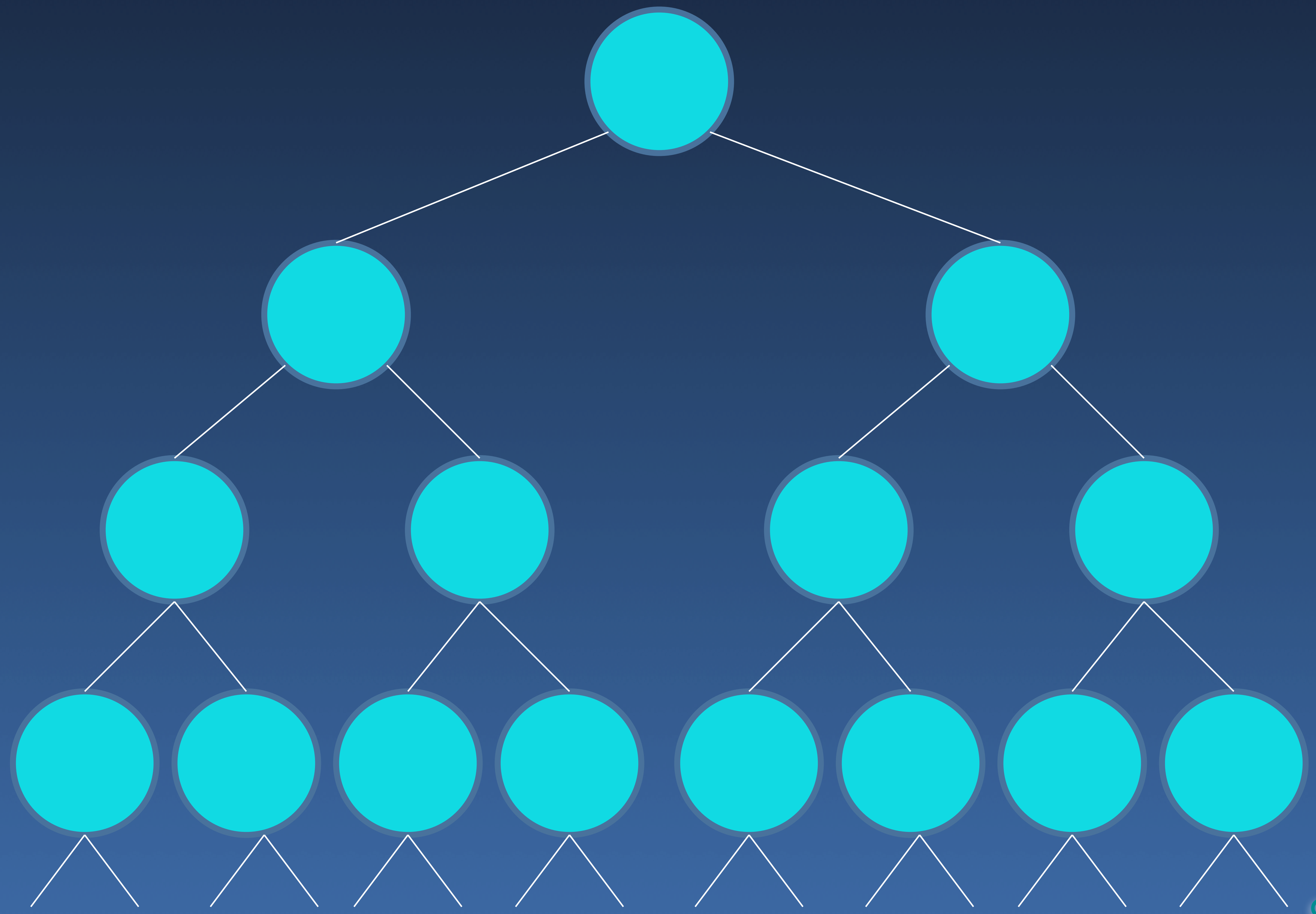
Input: x

- P[0] = x[0]
- For i = 1 to n-1
  - ➡ P[i] = P[i-1] + x[i]

	0	1	2	3	4	5	6	7	8	9	10
filter	0	0	1	1	1	0	1	0	0	0	1

D



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S	0	0	1	2	3	3	4	4	4	4	5
filter	0	0	1	1	1	0	1	0	0	0	1
D	0	1	2	3	4	5	6	7	8	9	10

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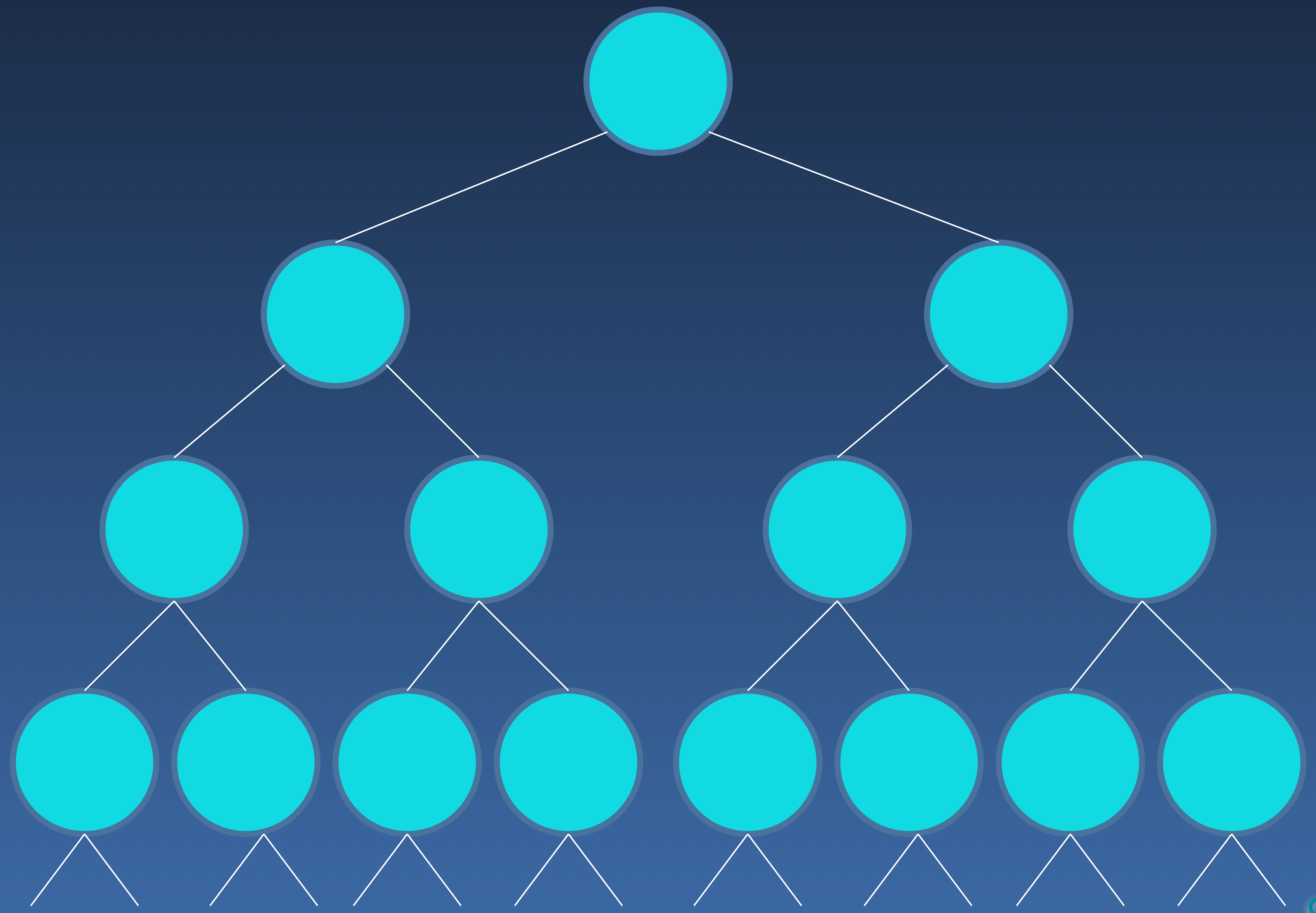

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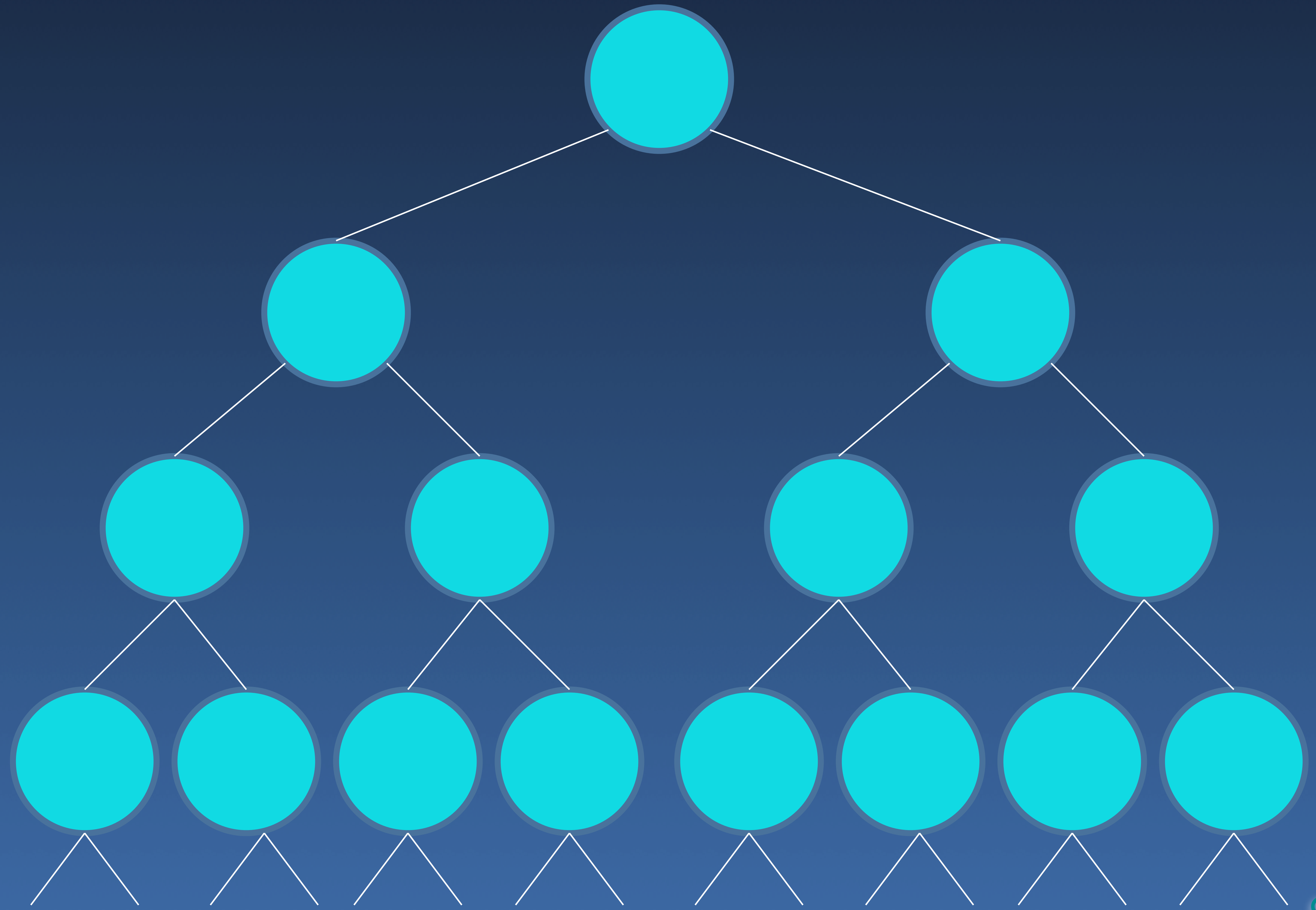

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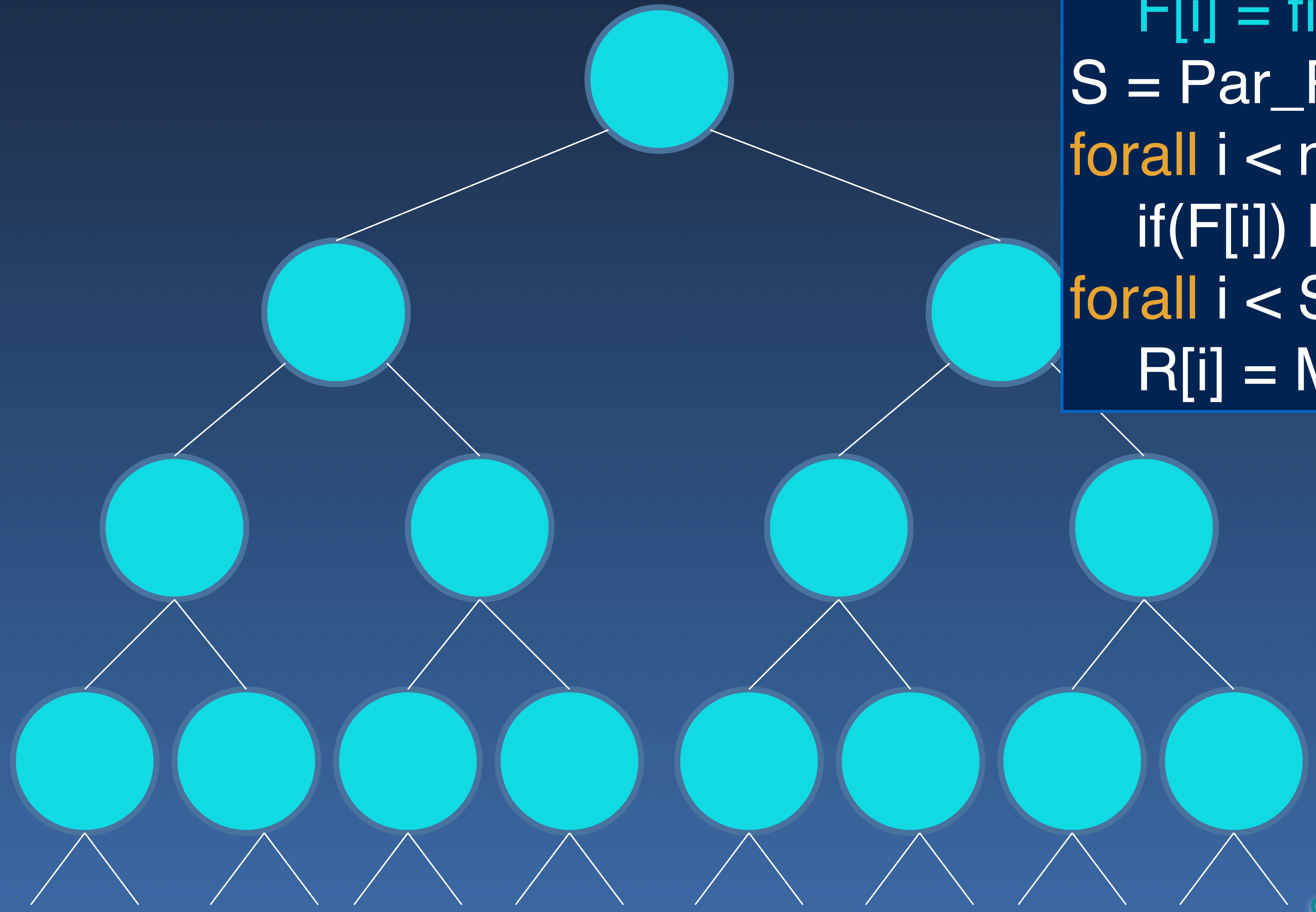

```
forall i < n
    if(filter(D[i]))
        R[i] = Map(D[i]);
```

```
forall i < n
    F[i] = filter(D[i]);
S = Par_PrefixSum(F)
forall i < n
    if(F[i]) P[S[i]-1] = i;
forall i < S[n]
    R[i] = Map(D[P[i]]);
```

Recursive Prefix Sums

- P[0] = x[0]
- For i = 1 to n-1
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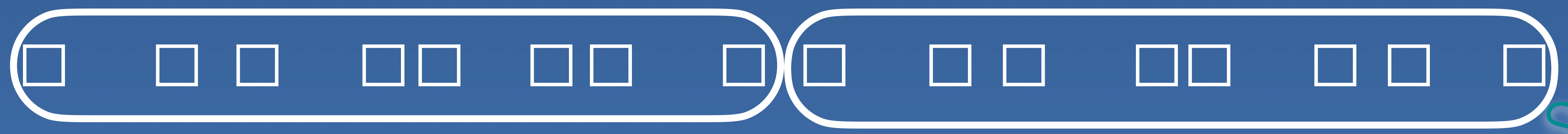
Psum two halves



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Recursive Prefix Sums

- P[0] = x[0]
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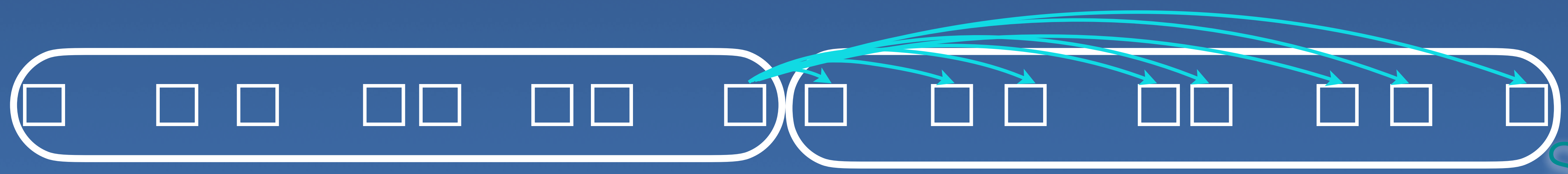
Complete original Psum



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Recursive Prefix Sums

$T(n) = T(n/2) + O(1)$
$W(n) = 2W(n/2)+Kn/2$

$W(n) = O(n \log n)$

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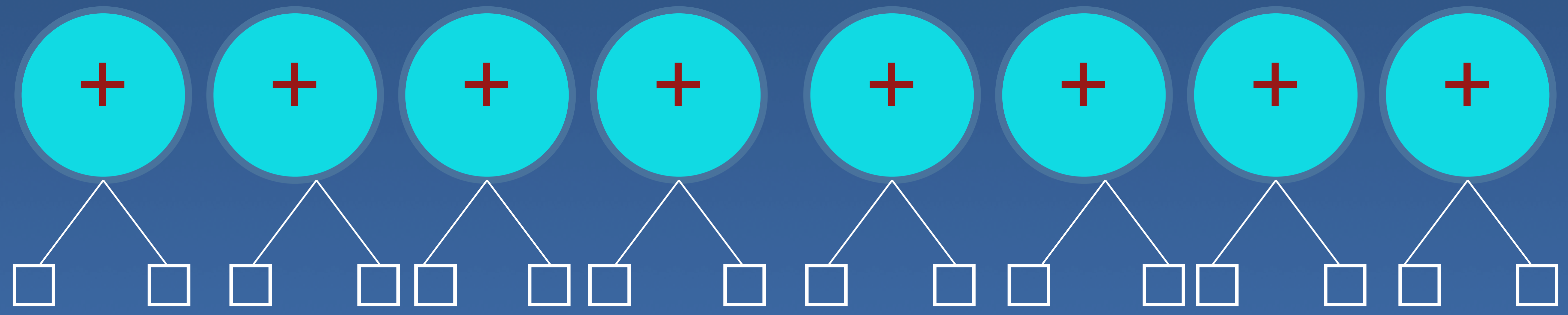
Pair-wise sum



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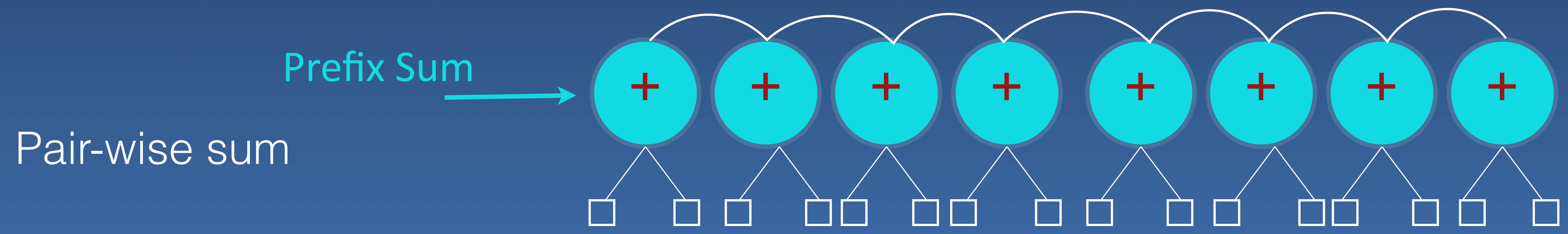

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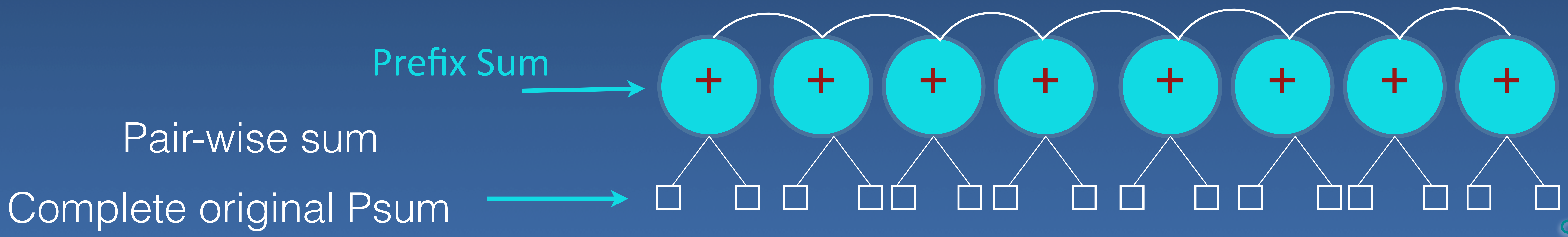

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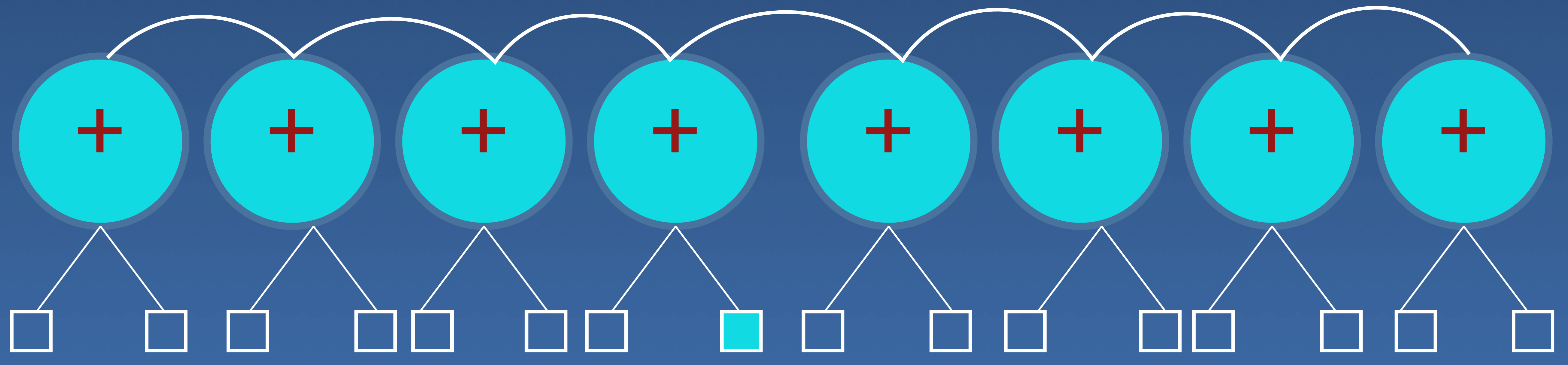
Pair-wise sum

Complete original Psum



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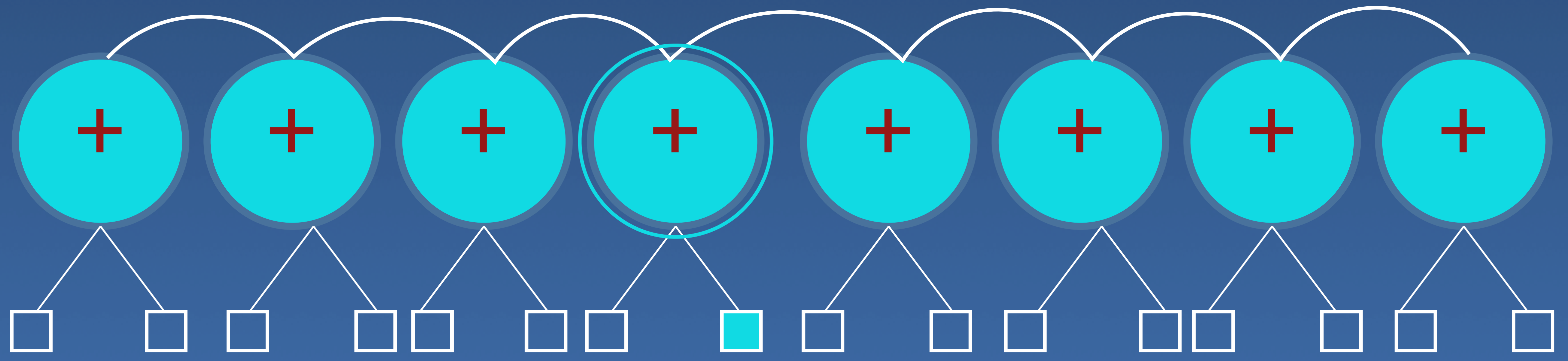

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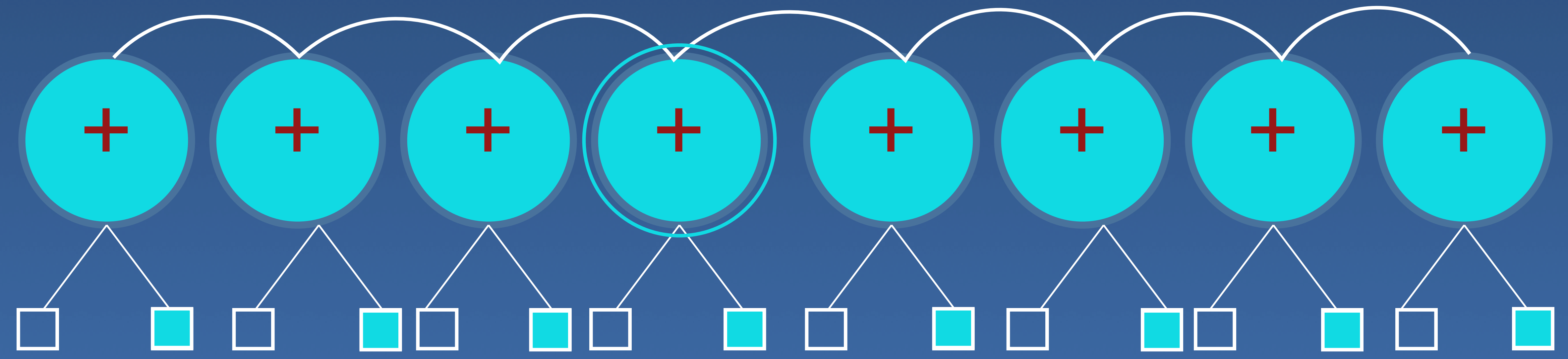

known by recursion

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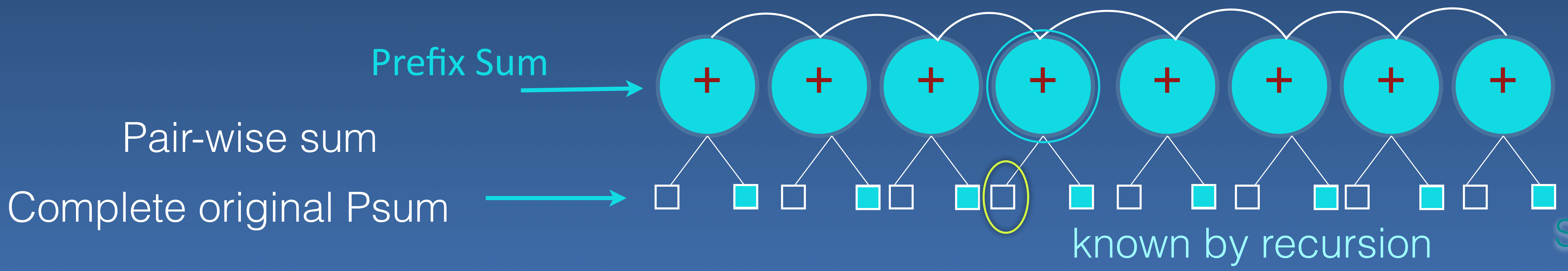

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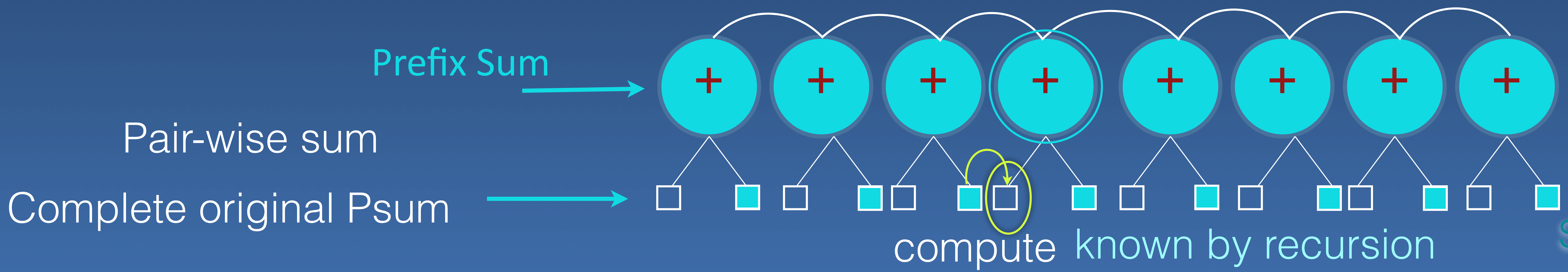

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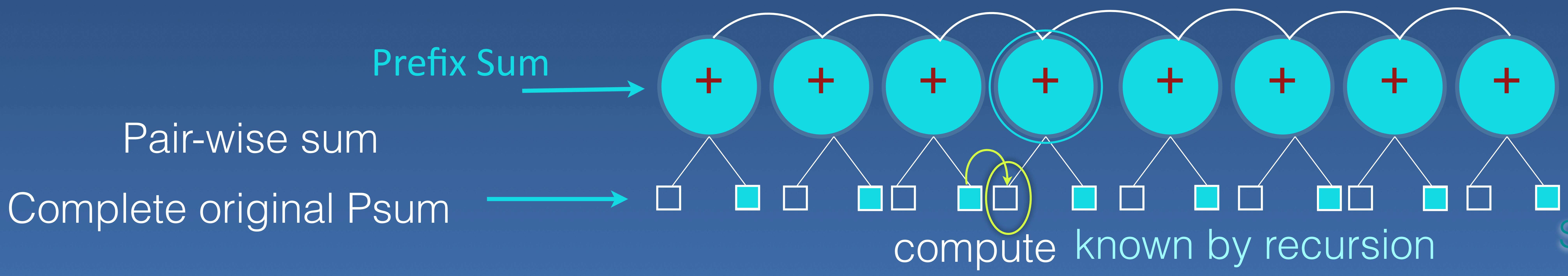

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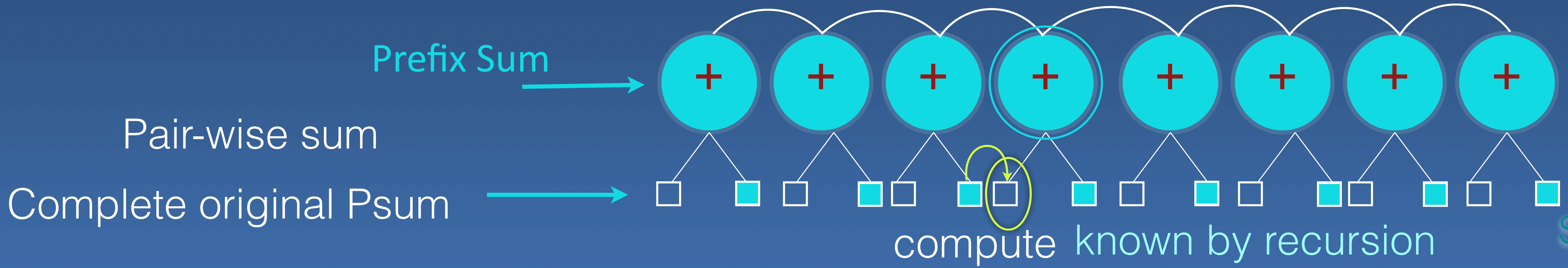

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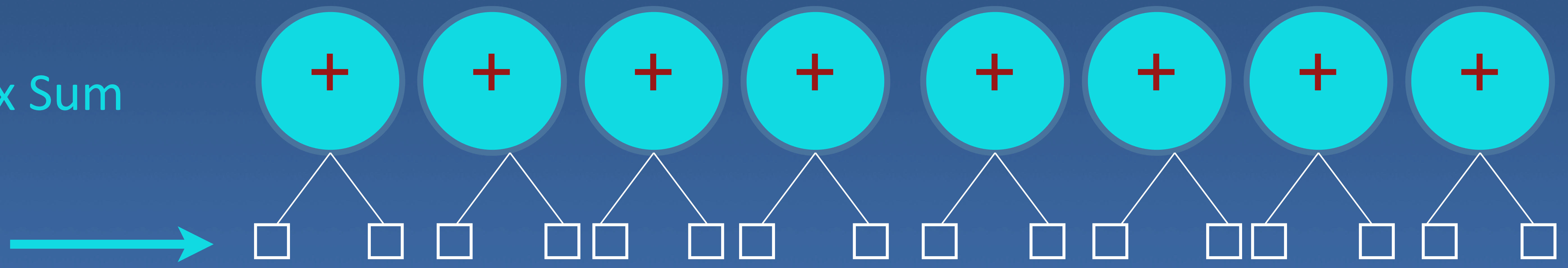
Recursive Prefix Sums

```
prefixSums(P, x, [0:n))
{
    forall i in [0:n/2)
        y[i] =  OP(x[2*i], x[2*i+1])
    prefixSum(z, y, [0:n/2))
    P[0] = x[0]
    forall i in [1:n)
        if(i&1) P[i] = z[i/2]
        else    P[i] = OP(z[i/2-1 ], x[i])
}
```

Or OP^{-1} (z[i/2], x[i]),
if op invertible

Prefix Sum



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Prefix Sum Binary Tree

(Non recursive)

forall i = 0 to n

B[0][i] = A[i]

for h = 1 to log n

forall i in 0:n/2^h

B[h][i] = B[h-1][2i] **OP** B[h-1][2i+1]

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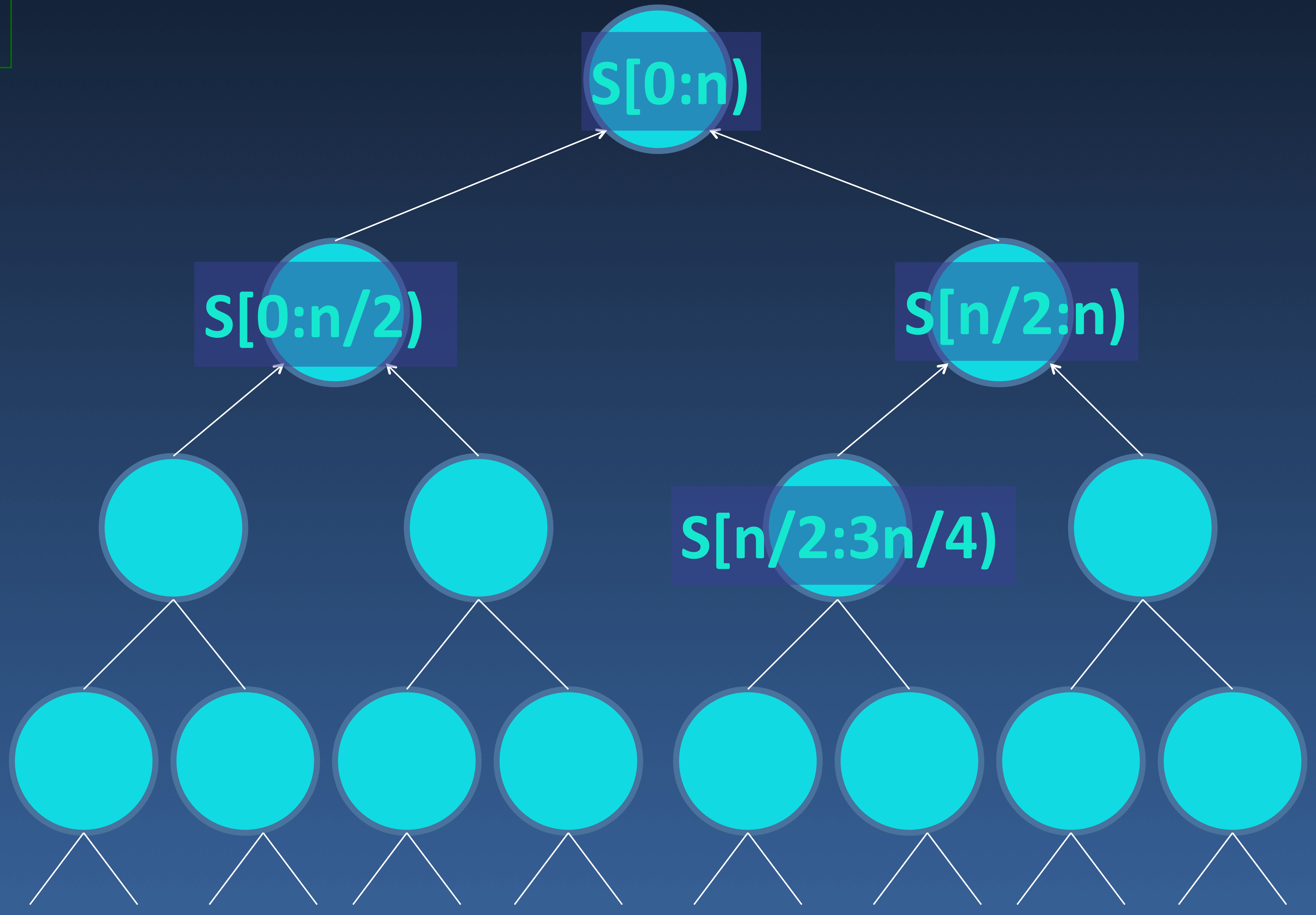

Prefix Sum Binary Tree
(Non recursive)

```
forall i = 0 to n
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for h = 1 to log n
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      B[h][i] = B[h-1][2i] OP B[h-1][2i+1]
for h = log n to 0
   C[h][0] = B[h][0]
   forall i in 1:n/2^h
      Odd i: C[h][i] = C[h+1][i/2]
      Even i: C[h][i] = C[h+1][i/2-1] OP B[h][i]
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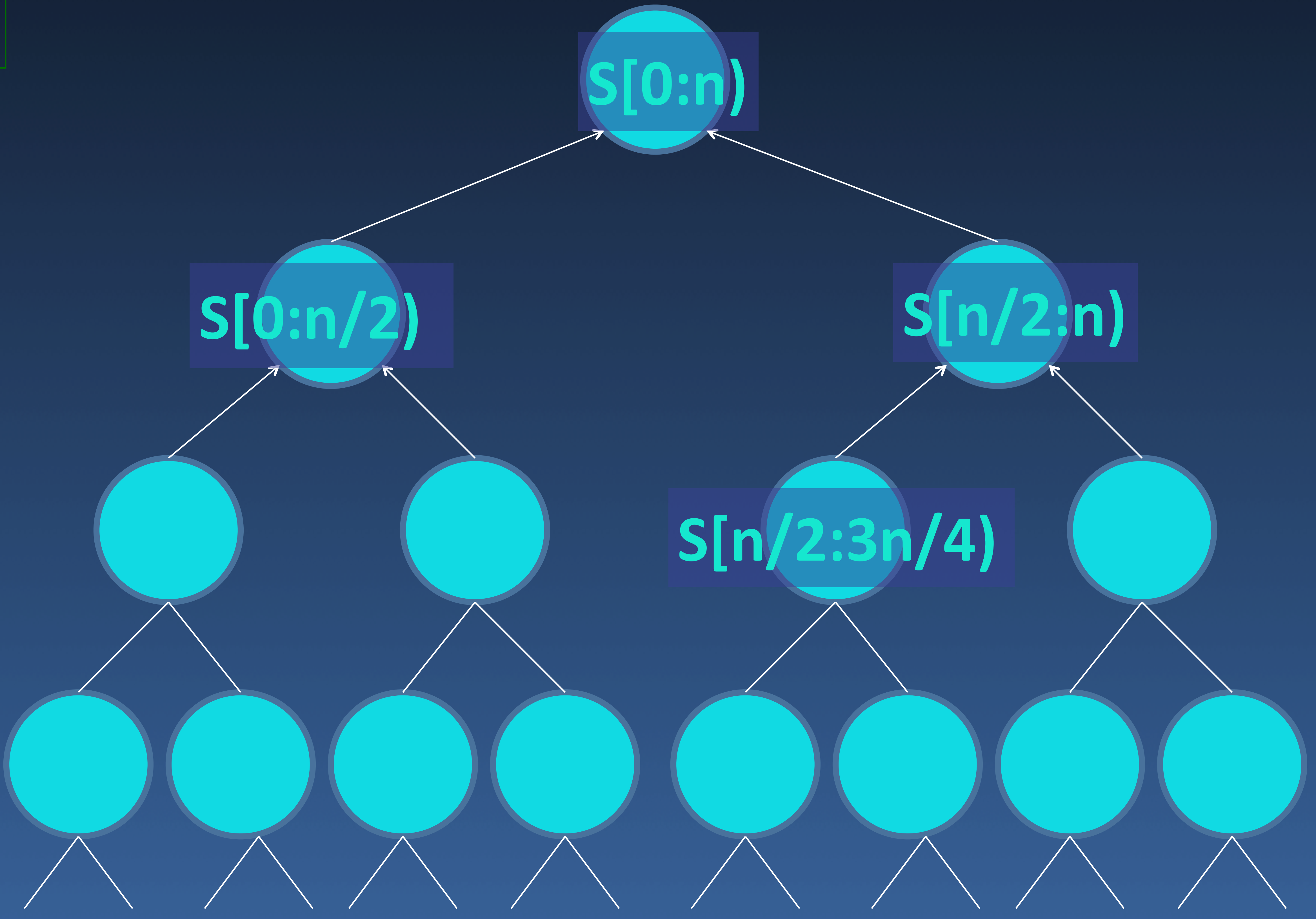

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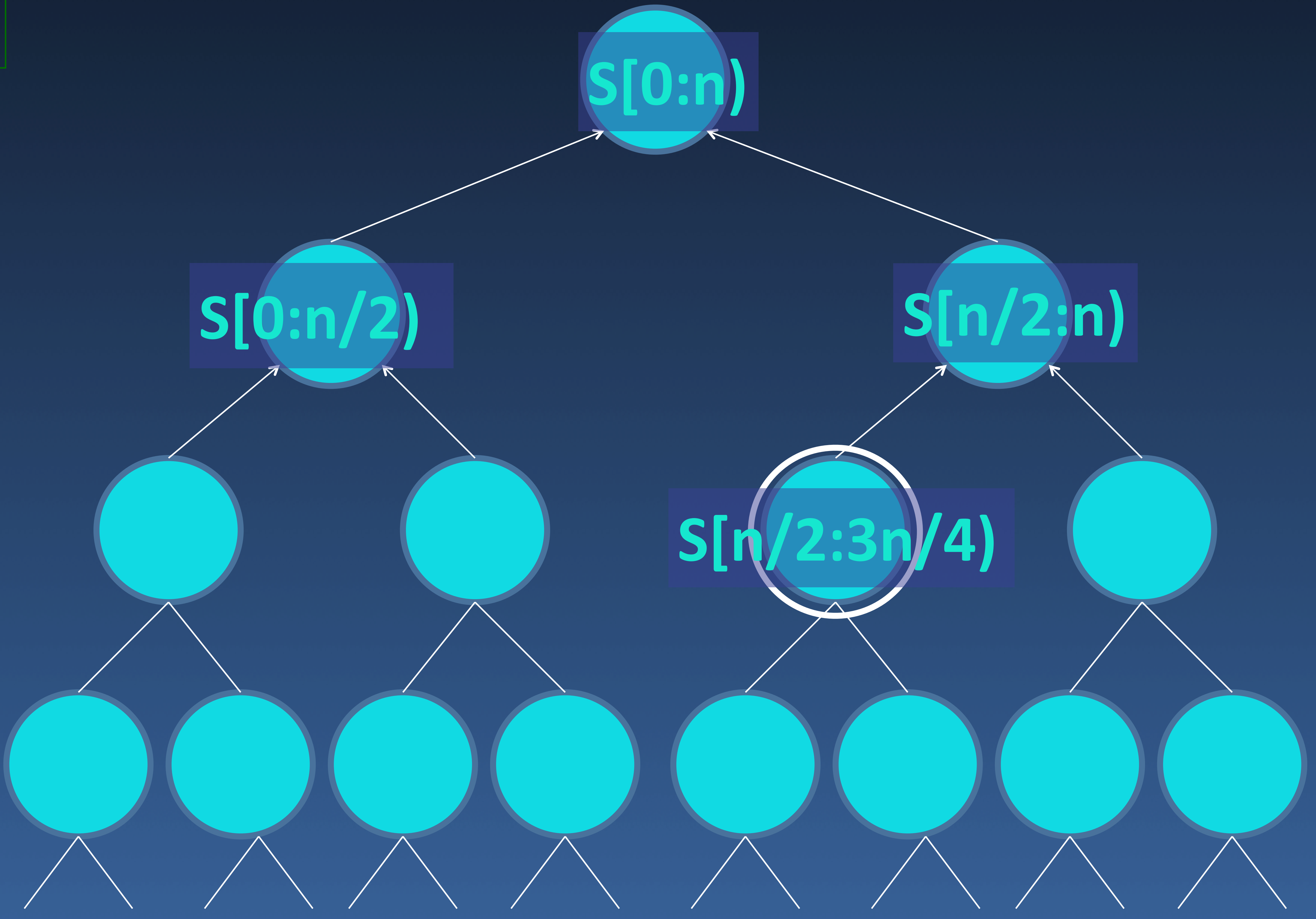

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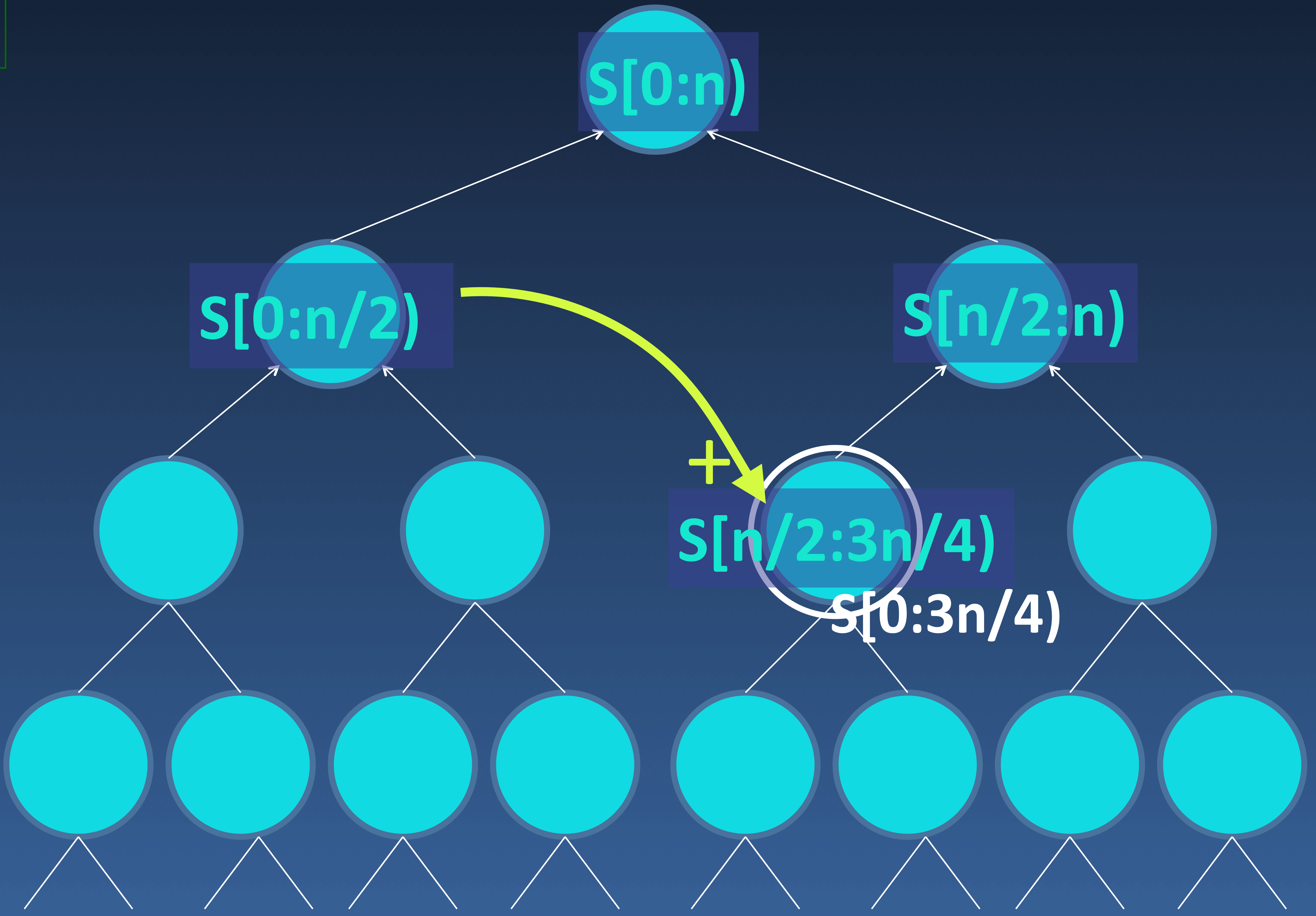

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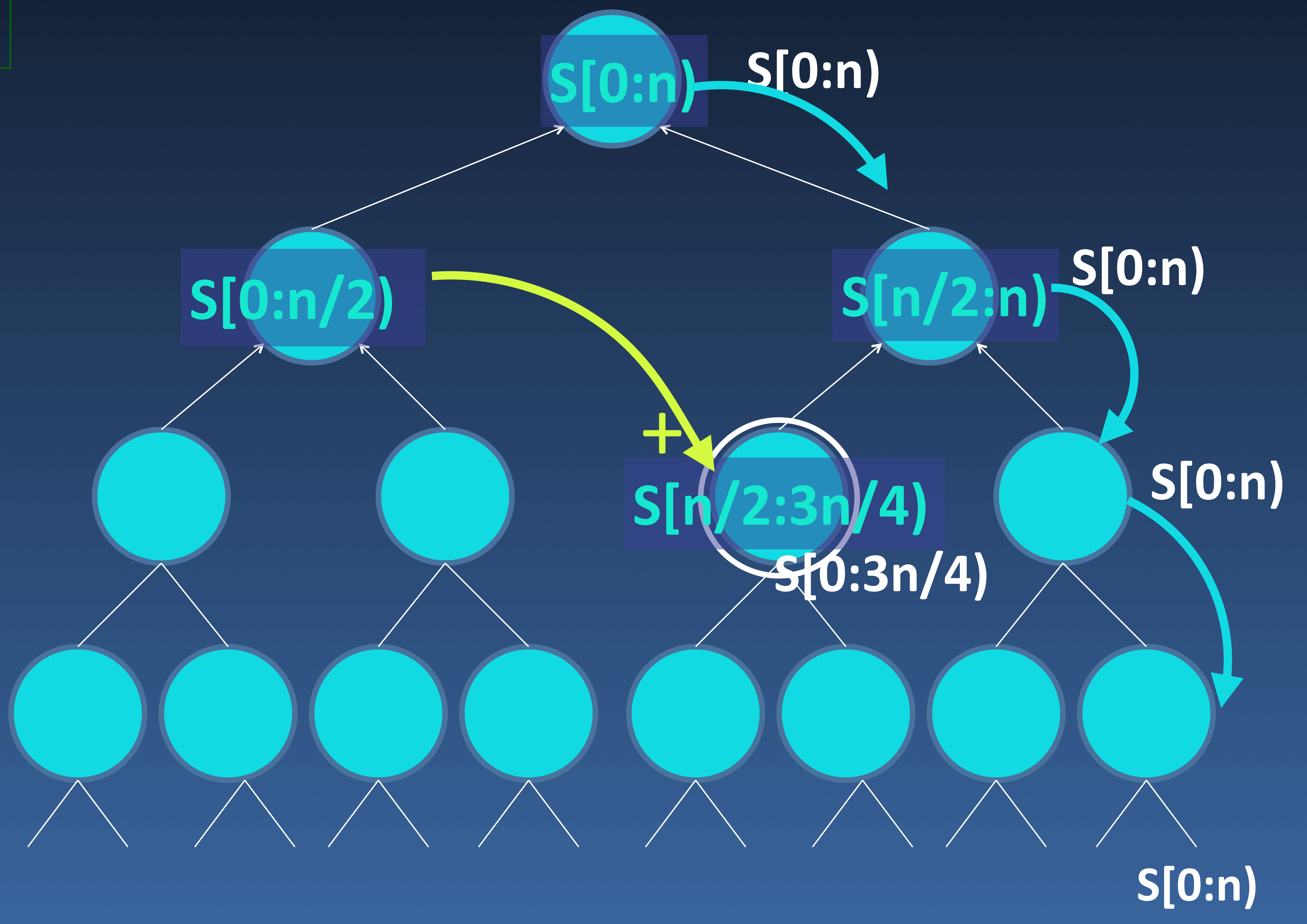

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Balanced Tree Approach

- Build binary tree on the input
- Hierarchically divide into groups
  - ➡ and groups of groups..
- Traverse tree upwards/downwards
- Useful to think of “tree” network topology
  - ➡ Only for algorithm design
  - ➡ Later map sub-trees to processors

Merge Sorted Arrays A,B

```
if(A[i] <= B[j])
   C[k++] = A[i++]
else
   C[k++] = B[j++]
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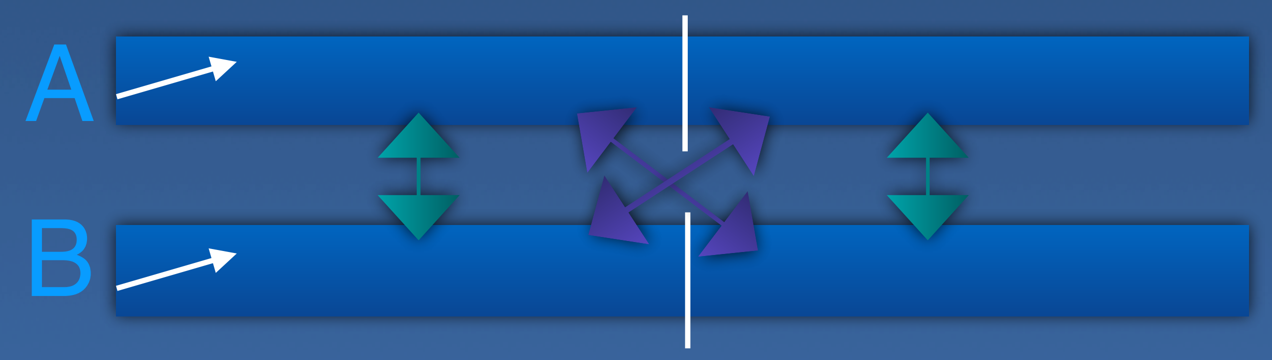

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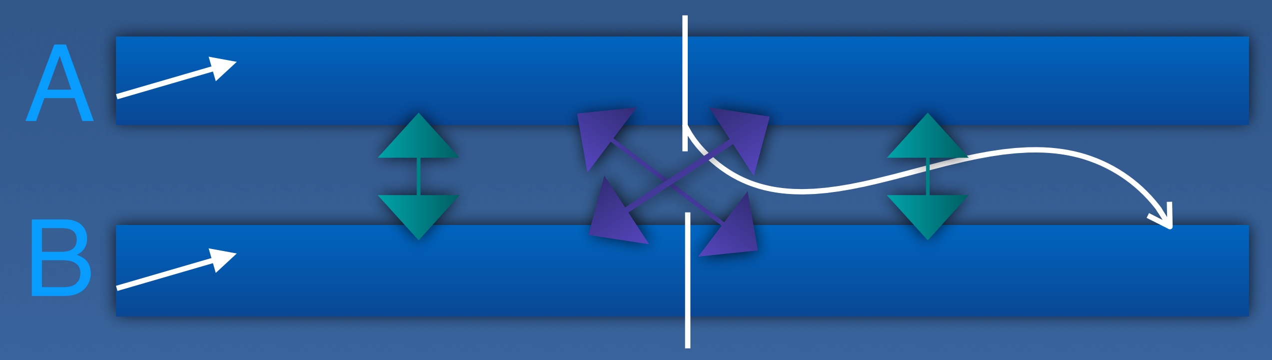

$T(2n) = T(n) + ?$

Subodh Kumar

Merge Sorted Arrays A,B

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if(A[i] <= B[j])
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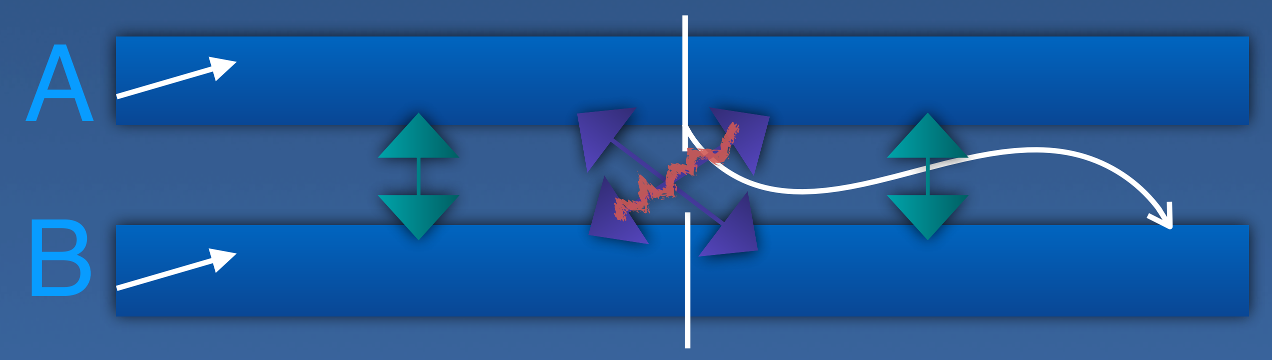

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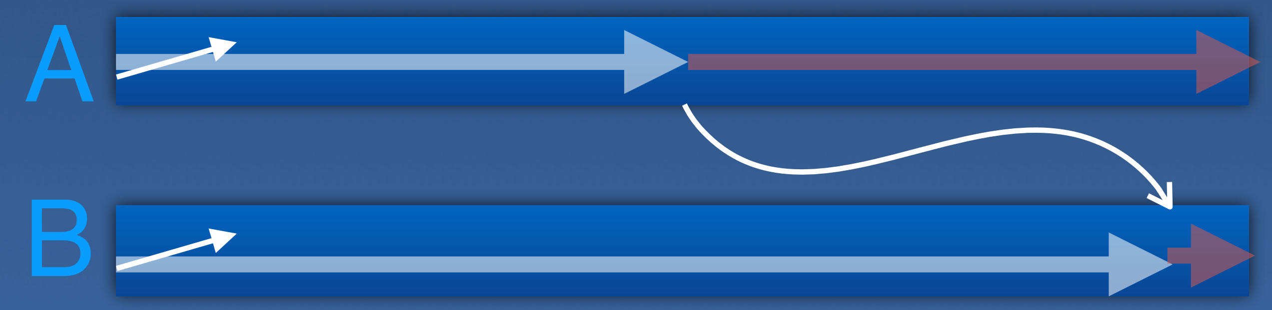

$T(2n) = T(n) + ?$

$T(2n) = T(3n/2) + \log n$

Subodh Kumar

Merge Sorted Arrays A,B

- Determine Rank of each element in A U B
- Rank(x, A U B) = Rank(x, A) + Rank(x, B)
  - ➡ A and B are each sorted; only need to compute the ranks in the other list

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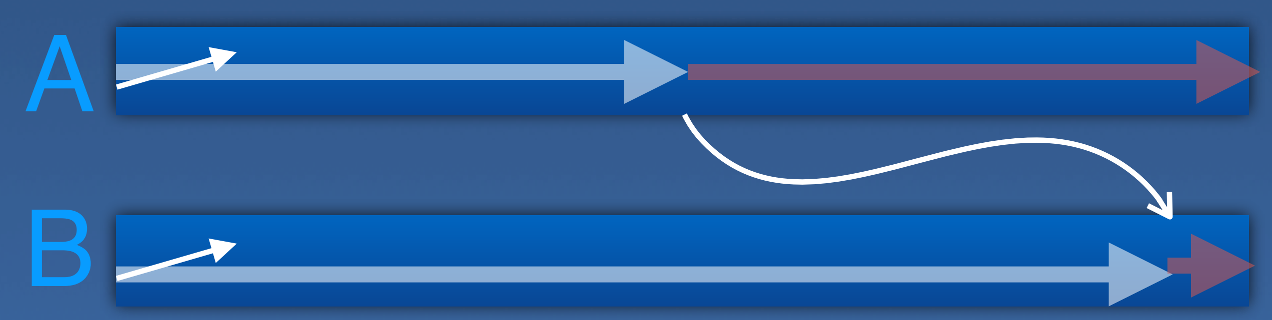

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- Determine Rank of each element in A U B
- Rank(x, A U B) = Rank(x, A) + Rank(x, B)
  - ➡ A and B are each sorted; only need to compute the ranks in the other list
- Find Rank(A[i], B) ∀i and Rank(B[j], A) ∀j
- Find Rank by binary search
  - ➡ O(log n) time
- O(n log n) work

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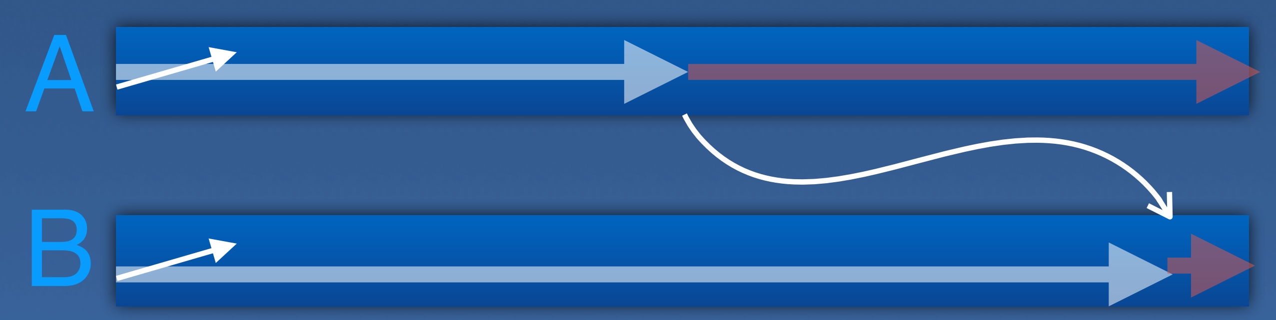

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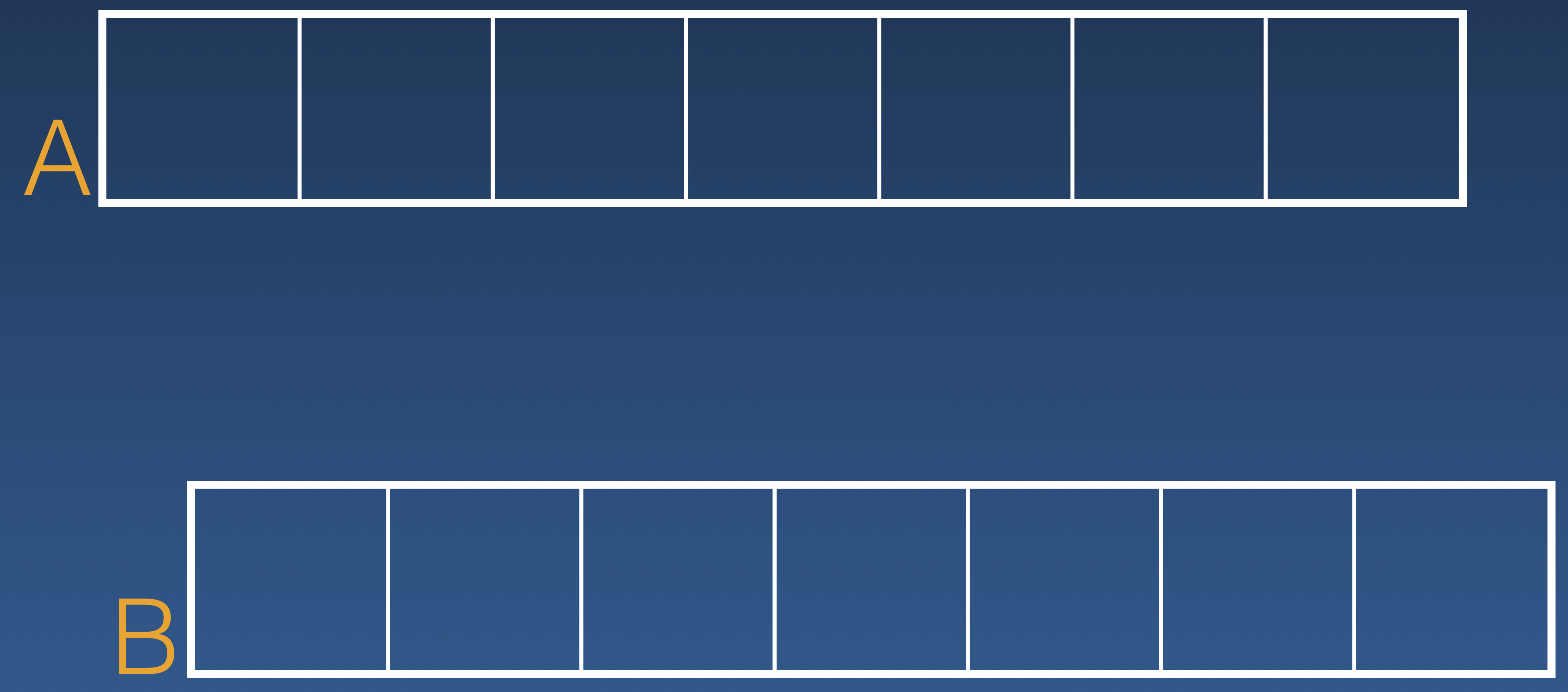
Towards Optimal Merge(A,B)

- Partition A and B into log n sized blocks

Data Partitioning Technique



Towards Optimal Merge(A,B)

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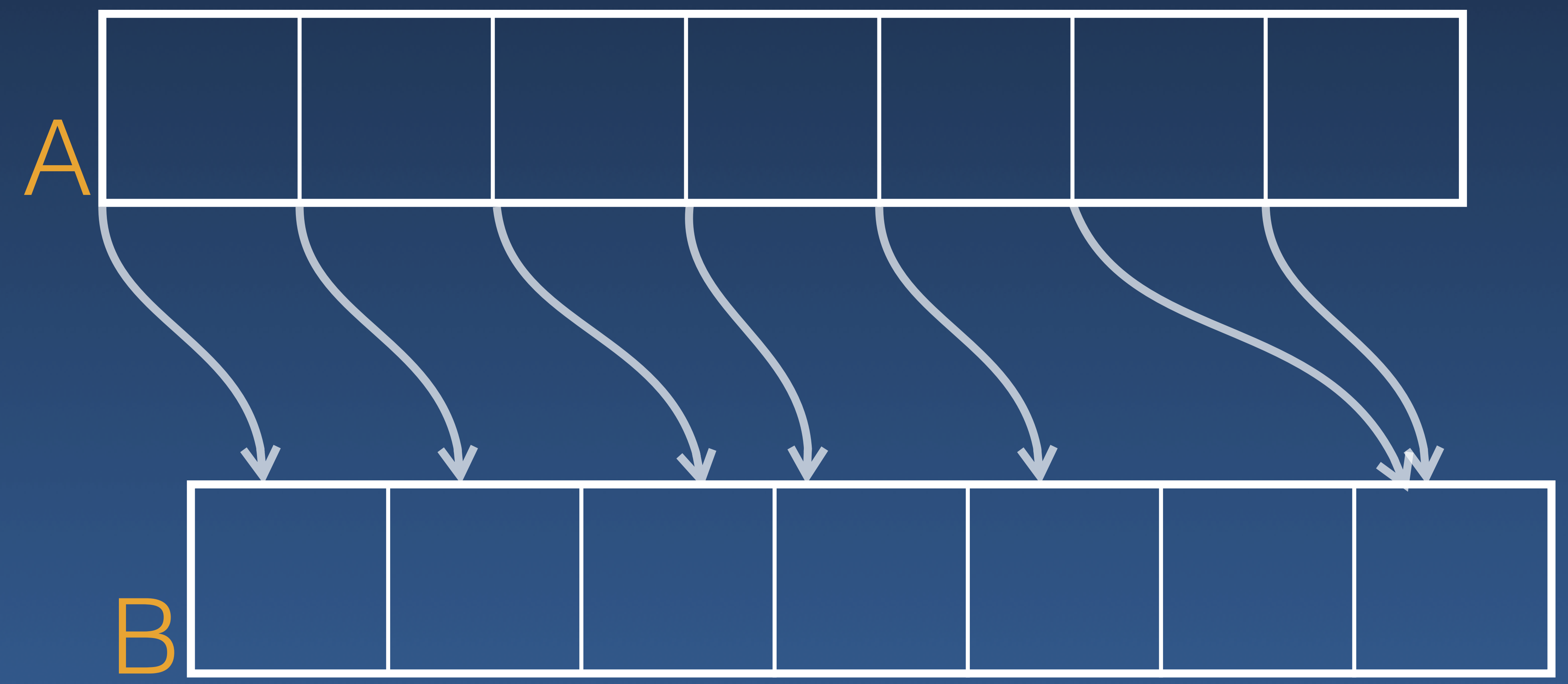

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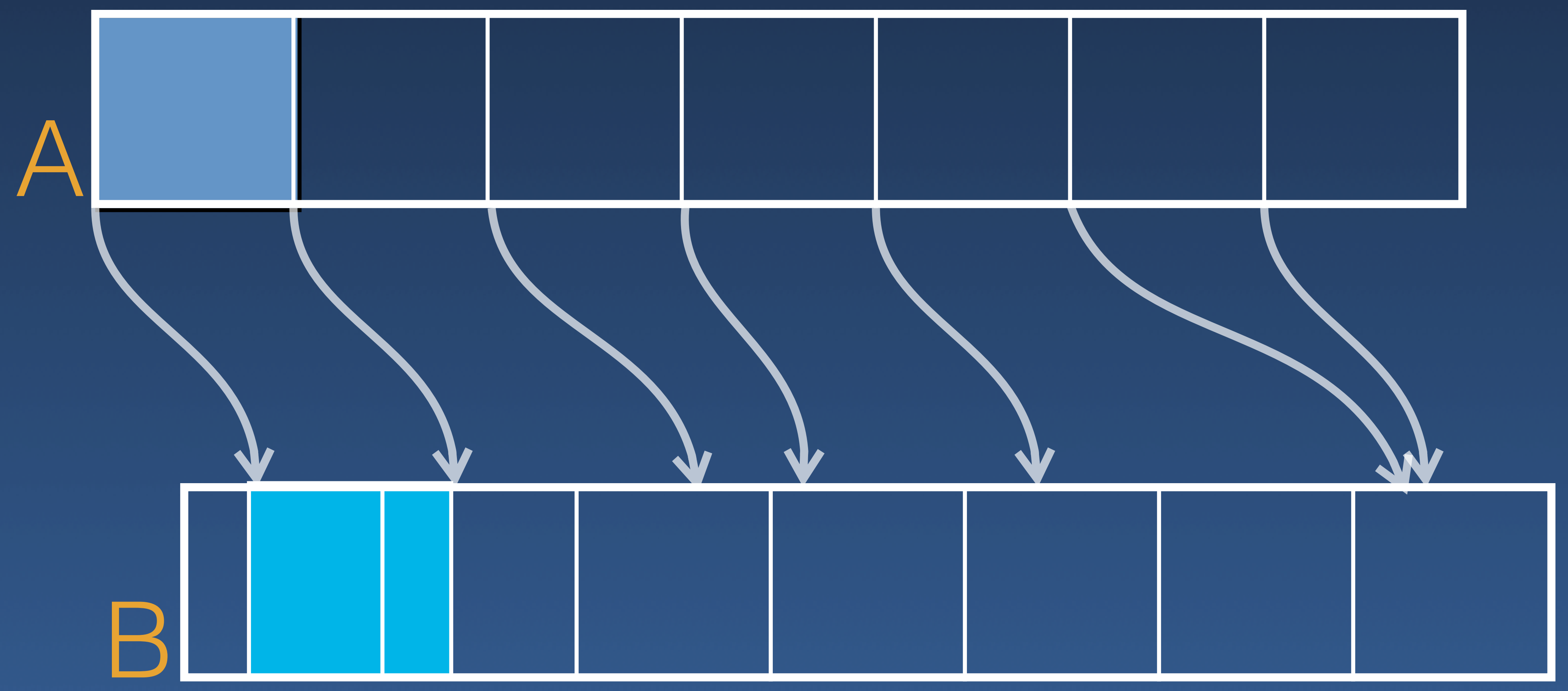

Subodh Kumar

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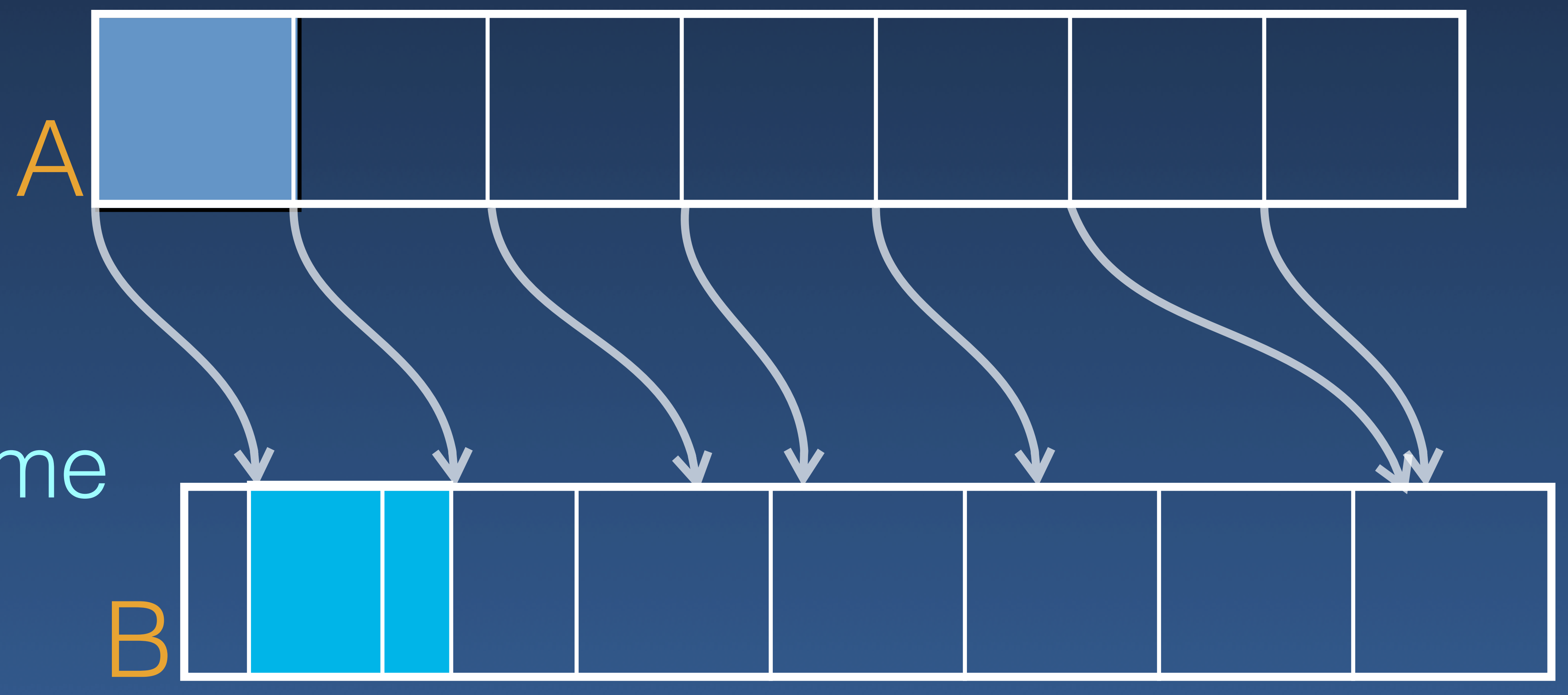

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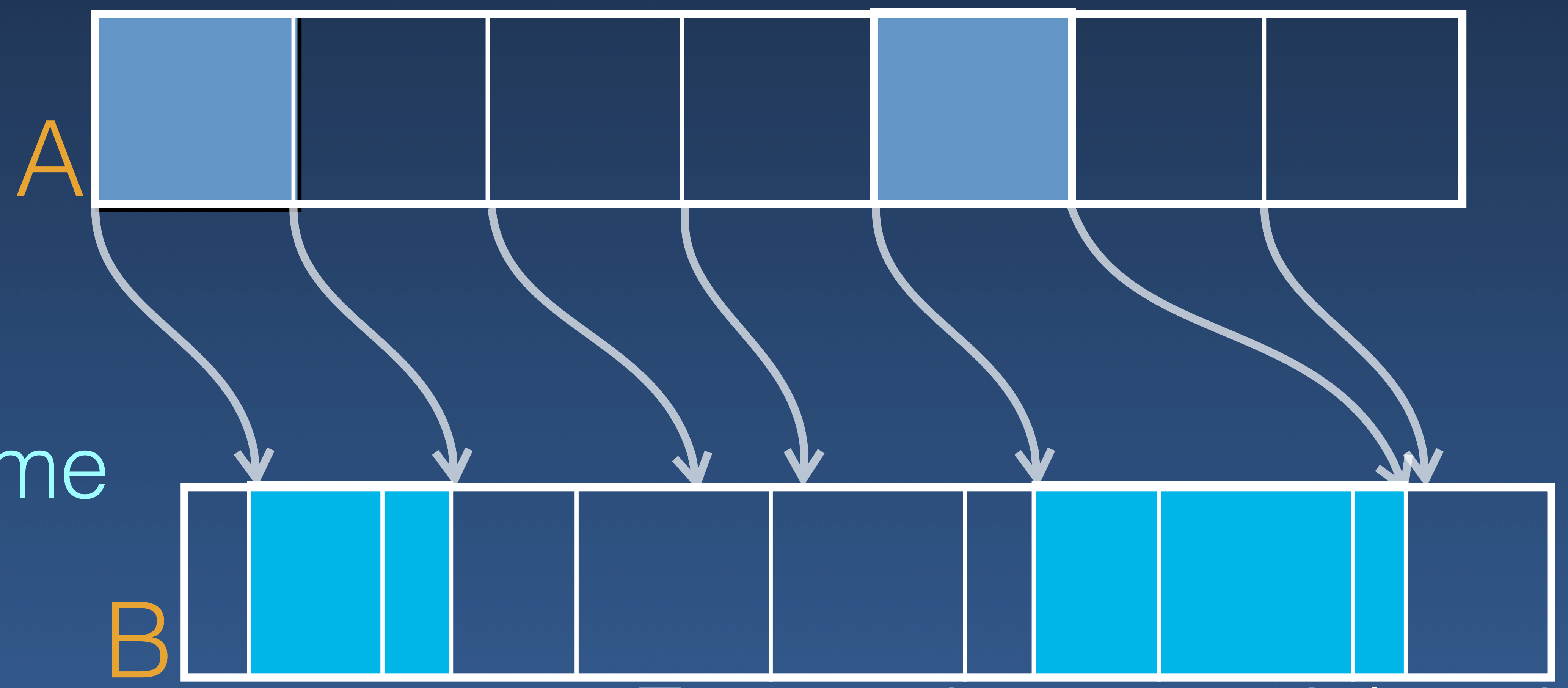

Too much sequential work

Subodh Kumar

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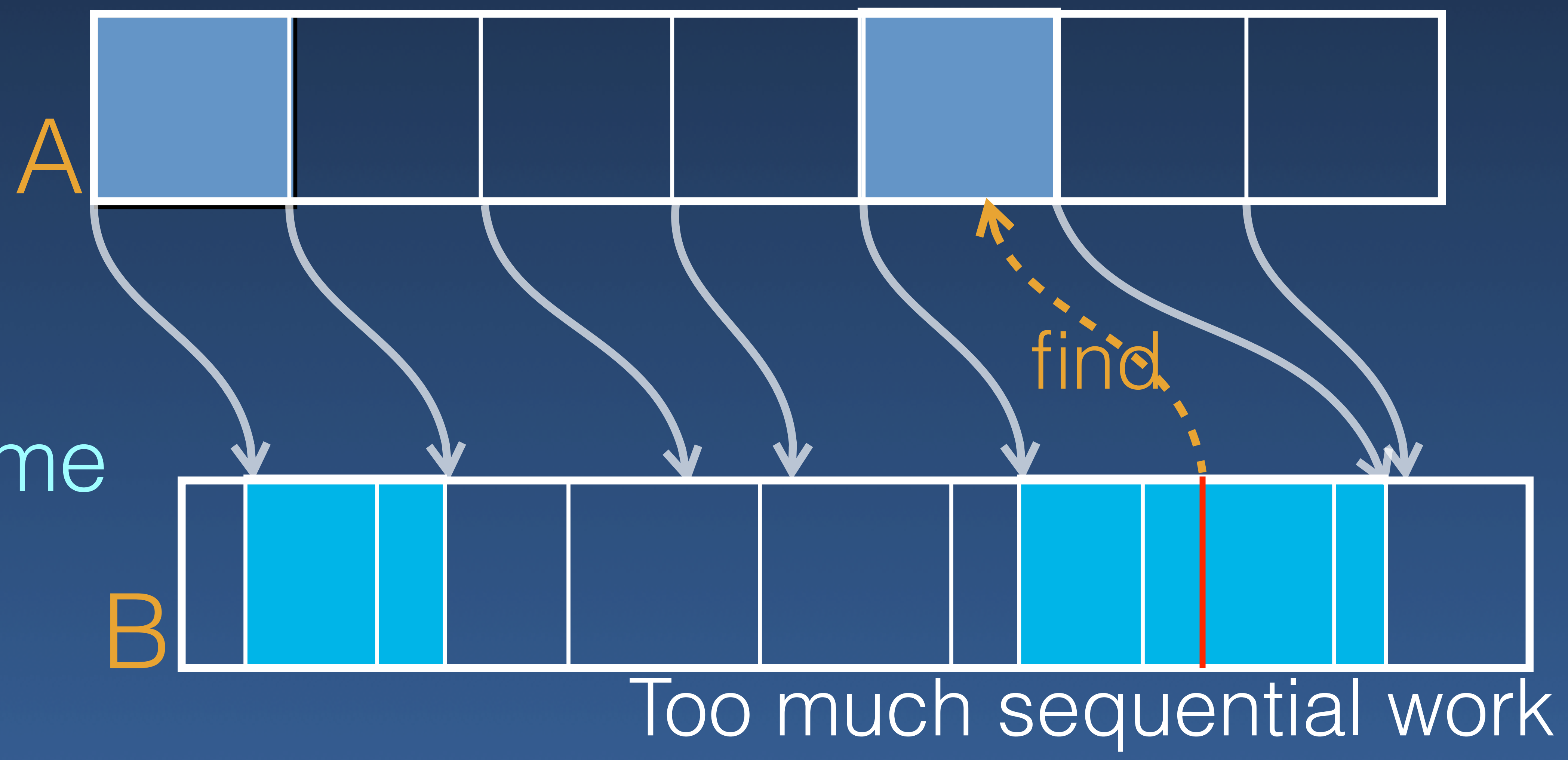

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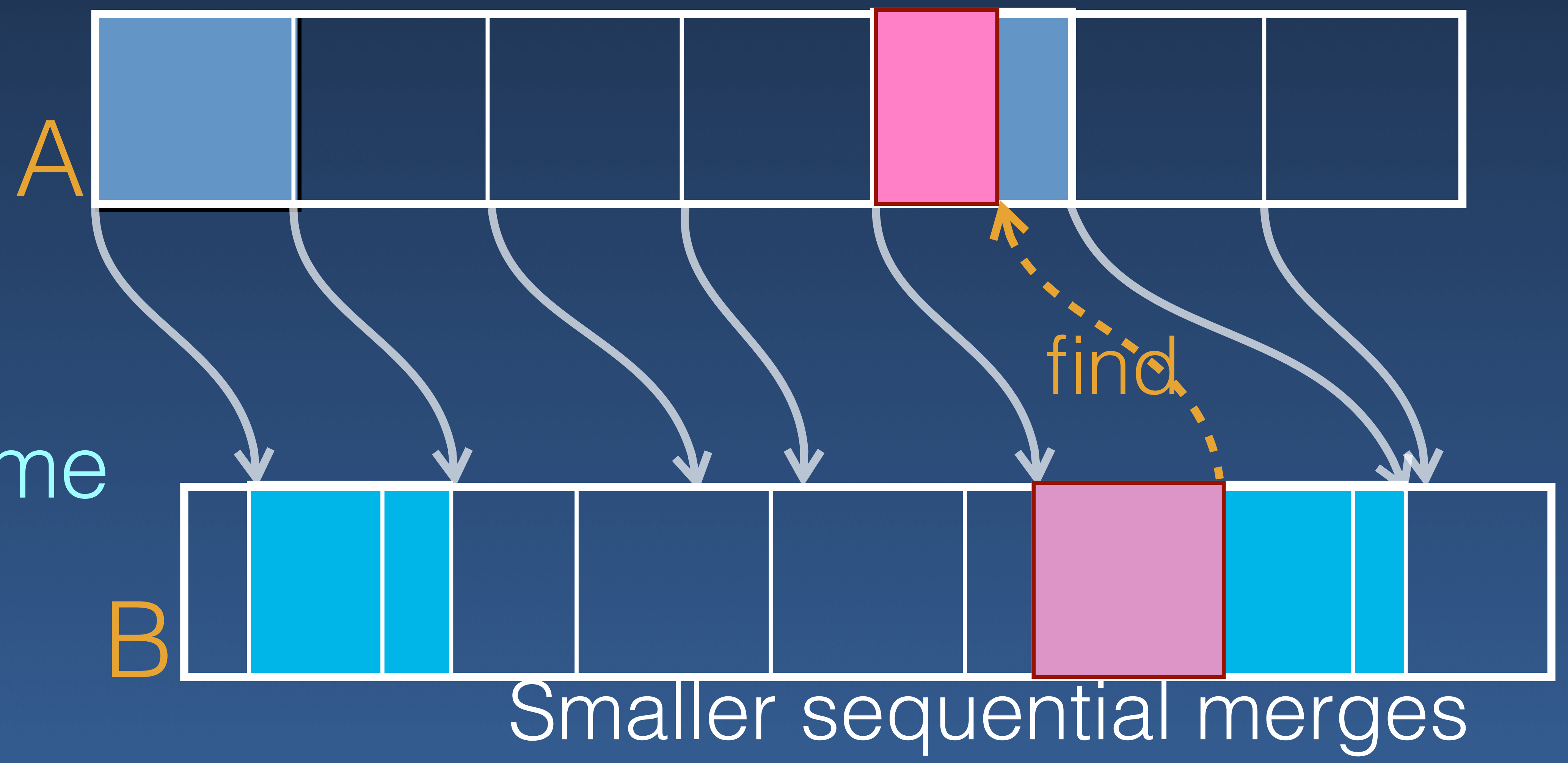

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Total time is O(log n)

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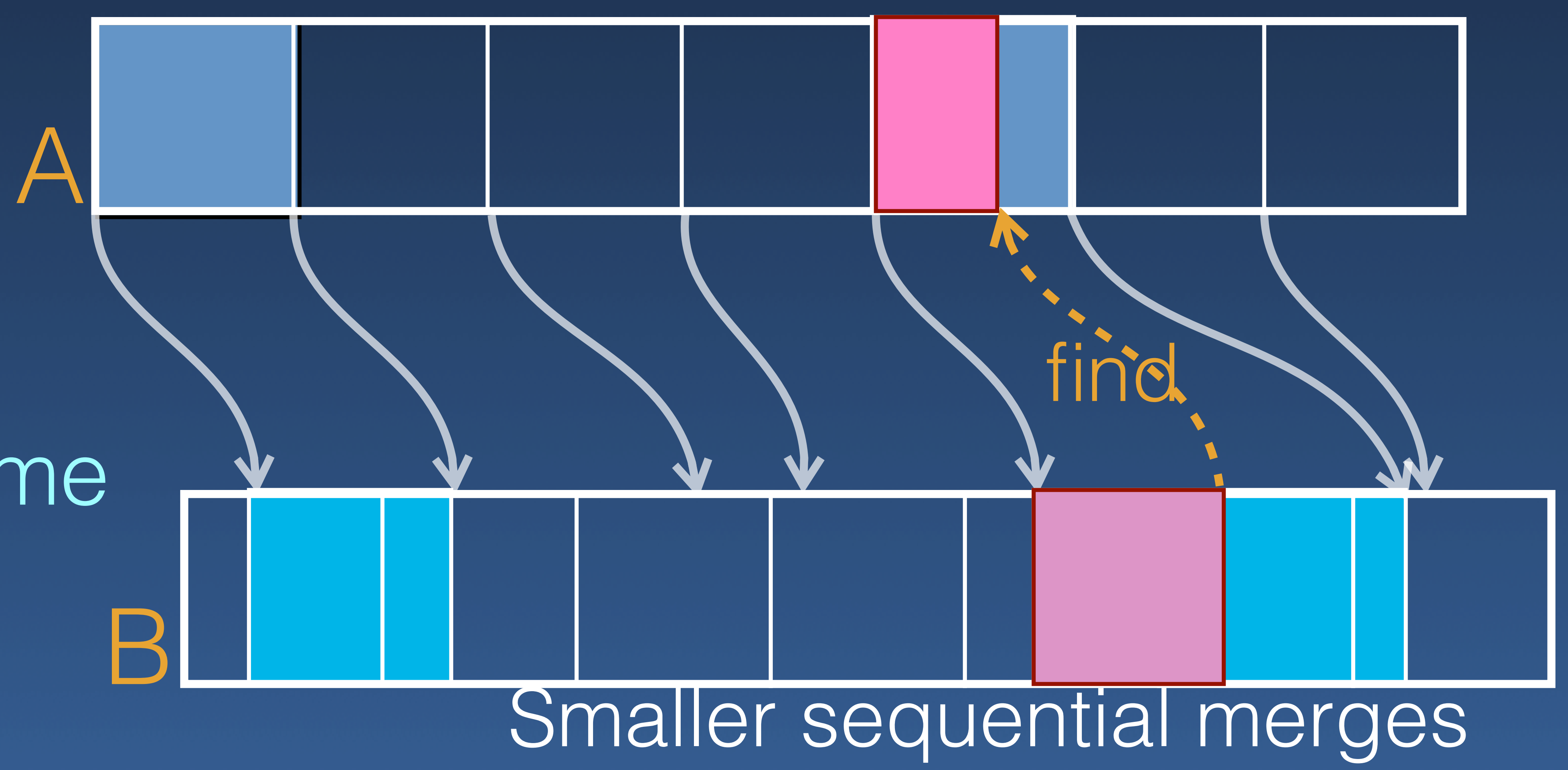

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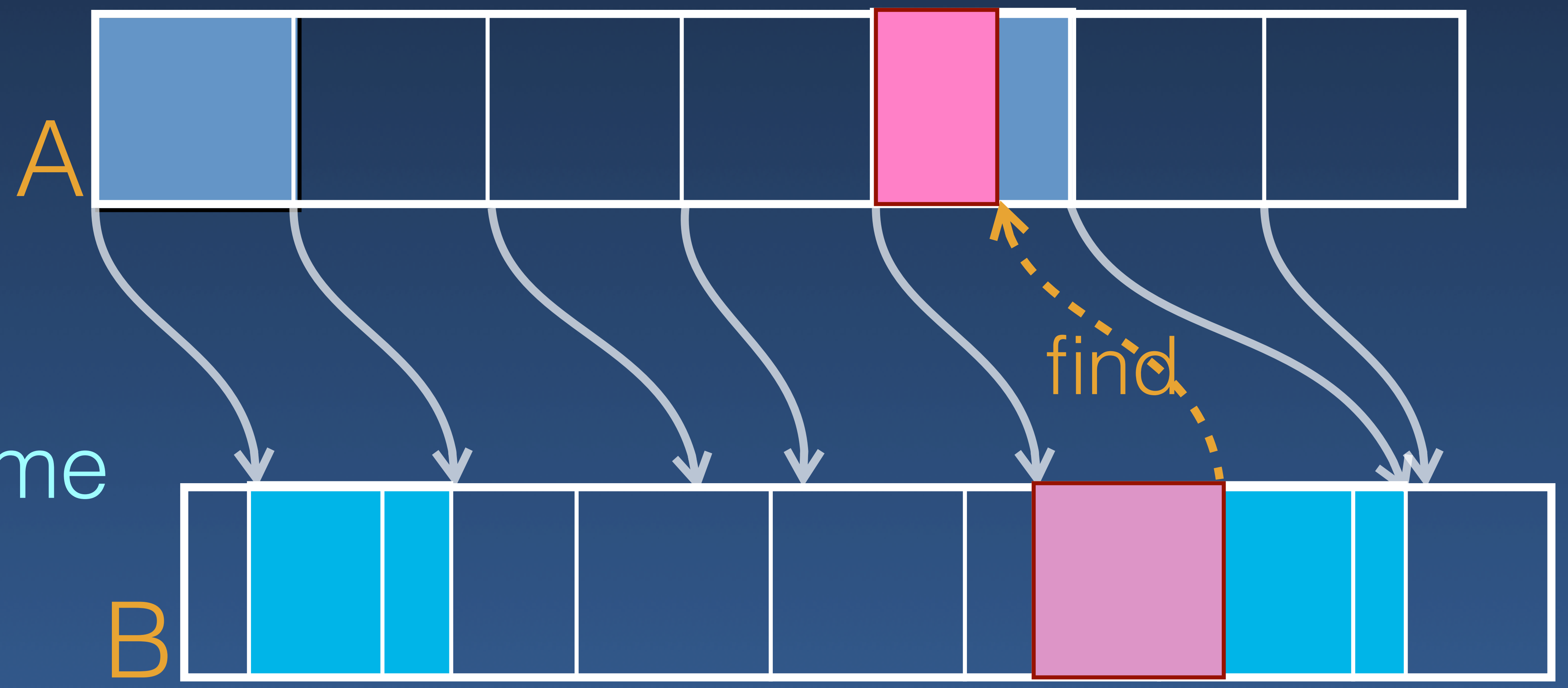

Can we do better?

Subodh Kumar

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limited by

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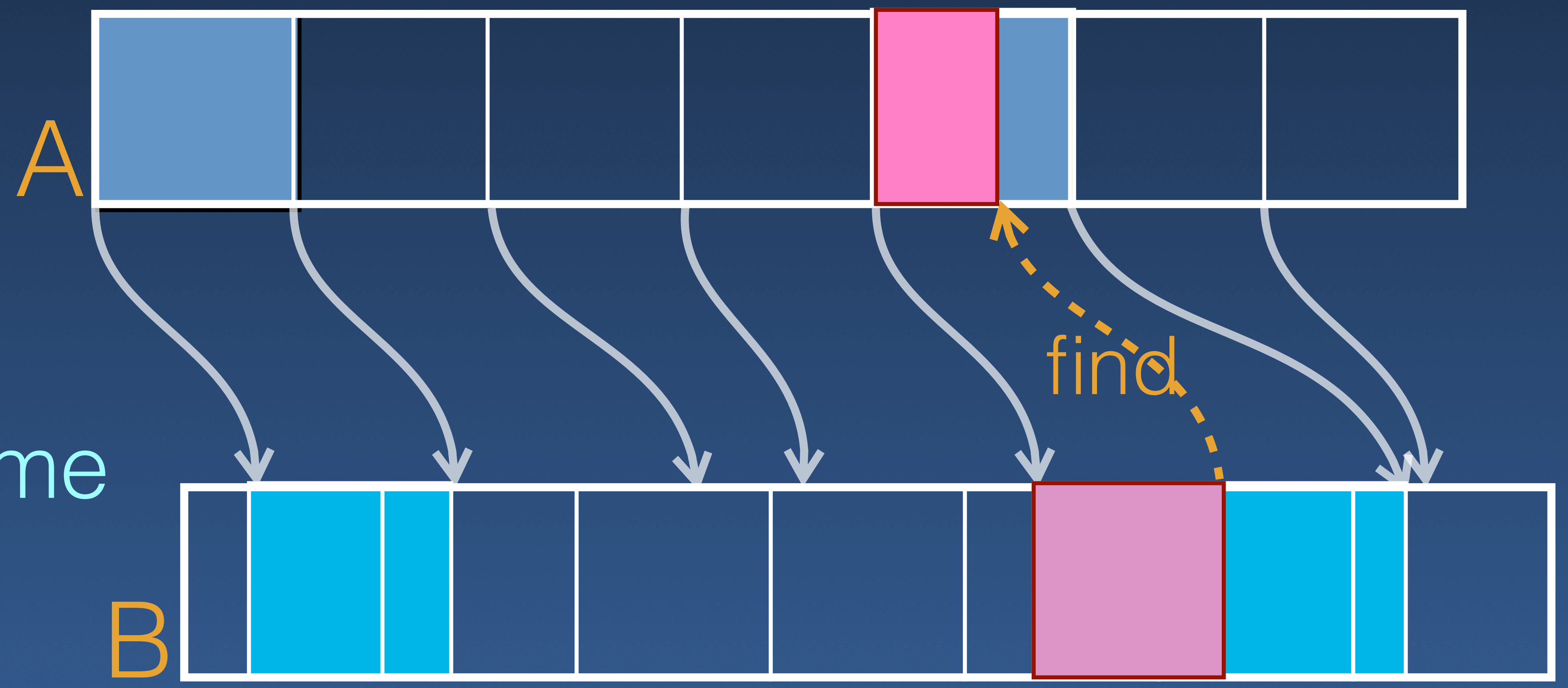

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