Partial Differential Equations 2

AMATH 453

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Preface

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Waves and Diffusions

1.1 The wave equation

We already know the wave equation ($c > 0$):

$$u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < \infty,$$

and the general solution is of the form

$$u(x,t) = f(x+ct) + g(x-ct).$$

With initial conditions imposed, we have the IVP

$$u_{tt} - c^2 u_{xx} = 0, \qquad \begin{cases} u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x). \end{cases}$$

The solution to IVP is then

$$u(x) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

To interpret the integral, we can let $\psi(x) = \mu'(x)$, then the integral becomes

$$\int_{x-ct}^{x+ct} \psi(s)\,ds = \mu(x+ct) - \mu(x-ct).$$

1.2 Conservation laws

Given a wave equation, we multiply by u_t:

$$\begin{aligned} u_t u_{tt} - c^2 u_t u_{xx} &= 0 \\ \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2\right) - c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{tx}u_x\right] &= 0 \\ \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) - \frac{\partial}{\partial x}\left(c^2 u_t u_x\right) &= 0 \end{aligned}$$

Then the conservation law states that

$$\frac{\partial R}{\partial t} + \frac{\partial F}{\partial x} = 0\,,$$

where $R \in (-\infty, +\infty)$, and $F \to 0$ with $x \to \pm\infty$.

1.3 The Diffusion Equation & Maximum principle

The diffusion equation is given by

$$u_t = ku_{xx}, \qquad -\infty < x < \infty$$

with diffusion constant $k > 0$.

We define

$$\begin{aligned} R &= (a,b) \times (0,\infty) \\ R_T &= (a,b) \times (0,T] \\ \overline{R_T} &= [a,b] \times [0,T] \\ C_T &= \{a \le x \le b, t = 0\} \cup \{a, 0 \le t \le T\} \cup \{b, 0 \le t \le T\} \end{aligned}$$

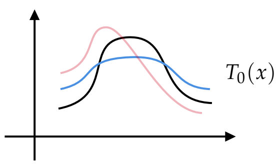
Theorem 1.1: Maximum principle

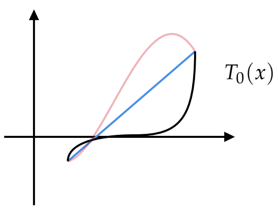
If $u \in C(\overline{R_T}) \cap C^2(R_T)$ is a solution of the diffusion equation, then $u(x,t) \le \max_{C_T}\{u\}$ for all $(x,t) \in R_T, T > 0$. Here C_T is called the parabolic boundary of R_T.

Remark:

1. We can replace $u_t - ku_{xx} = 0$ with $u_t - ku_{xx} \le 0$.
2. A stronger version of the theorem exists which says that $u(x,t) < \max_{C_T}\{u\}$ unless u is constant.
3. Same result applies to the minimum of u by replacing u with $-u$. However, in this case, (1) doesn't apply. Now we need $u_t - ku_{xx} \ge 0$.

Here are some intuitions. Consider a rod lying on $[a,b]$ with initial non-constant temperature $T_0(x)$. Then as time goes, only blue T is possible, not red T.





Proof:
Let $M = \max_{C_T} u$. Note that M exists since u is continuous on C_T, and C_T is a closed boundary. We need to show that $u \le M$ on $\overline{R_T}$.

Let

$$v(x,t) = u(x,t) + \epsilon x^2, \quad \epsilon > 0$$

Let $r = \max\{|a|, |b|\}$. Then $v(x,t) \le M + \epsilon r^2$ on C_T. Now we prove that $v \le M + \epsilon r^2$ on R_T.

On R_T, we have

$$u = v - \epsilon x^2 \le M + \epsilon(r^2 - x^2)$$

Now if we take the derivative,

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon = -2k\epsilon < 0 \tag{*}$$

(i) Suppose $v(x,t)$ has a maximum at an interior point (x_0, t_0), i.e., $(x_0, t_0) \in (a,b) \times (0,T)$. Then

$v_t(x_0, t_0) = 0$. Moreover, $v_{xx}(x_0, t_0) \leq 0$. Then

$$v_t(x_0, t_0) - kv_{xx}(x_0, t_0) = -kv_{xx}(x_0, t_0) \geq 0$$

contradicting (*), thus there are no interior max.

(ii) Suppose $v(x, t)$ has a maximum at an interior point of the upper boundary. $v_t(x_0, T) \geq 0$. Then

$$v_t(x_0, t_0) - kv_{xx}(x_0, t_0) \geq 0$$

contradicting (*), thus there are no maximum along the upper boundary.

But v is continuous on $\overline{R_T}$, thus it has a maximum value which we now know must occur on C_T. Hence $v \leq M + \epsilon r^2$ on $\overline{R_T}$. Letting $\epsilon \to 0$, we have $u \leq M$ on R_T. □