Rings and Fields

PMATH 334

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LATEXed by Sibelius Peng

Preface

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References:

- Dummit, Foote: *Abstract algebra*.
- http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf
- https://notes.sibeliusp.com/pmath347

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1

Introduction & Motivation

1.1 Fermat's Last Theorem

Fermat's Last Theorem

The equation $x^m + y^m = z^m$ has no non-trivial solutions in integers for $m \geq 3$.

For example, $(1, 0, 1)$, $(-1, 0, 1)$ for m even, are trivial solutions.

In 1897, Gabriel Lamé announced that he has a proof. First he assumed that m is a prime. He writes

$$z^p = x^p + y^p = (x+y)(x+\zeta_p y)(x+\zeta_p^2 y)\cdots(x+\zeta_p^{p-1} y)$$

where $\zeta_p = \cos(\frac{2\pi}{p}) + i\sin(\frac{2\pi}{p})$. Consider the ring

$$\mathbb{Z}[\zeta_p] = \{a_1 + a_2\zeta_p + a_3\zeta_p^2 + \cdots + a_{p-2}\zeta^{p-2} : a_i \in \mathbb{Z}\}$$

which is the smallest ring containing $\mathbb{Z}$ and ζ_p.

Then the next step is to show that $(x+\zeta_p^j y)$'s are coprime in $\mathbb{Z}[\zeta_p]$. Let q_i's be primes.

$$\prod_i q_i^{p\alpha_i} = z^p = (x+y)(x+\zeta_p y)\cdots(x+\zeta_p^{p-1} y)$$

If $(x+\zeta_p^j y)$'s are coprime in $\mathbb{Z}[\zeta_p]$, then $(x+\zeta_p^j y) = (\cdots)^p$ is of p-th power $(*)$. But this is wrong if the factorization is non-unique. However, we have $\mathbb{Z}[\zeta_p]$ can be a unique factorization domain (UFD). This means $(*)$ works. Kummer salvages the argument for approximately (conjecturally) 60% of prime exponents. And these primes are called regular primes.

1.2 Straightedge and compass construction

We are given a length 1 straightedge ruler, and a compass. With these, we can

- connect two points with a straightedge,
- draw a circle, centered at A, and going through B,
- draw intersections of two line segments, circle & line, two circles.

What lengths are constructible? where length means distance between two points. We can do $+, -, \times, \div, \sqrt{}$. Then we can do field extensions:

$$\mathbb{Q} \to \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \cdots$$

Is trisection of an angle doable? No, not possible.

Possible to double the cube, square the circle of the same area?

What regular m-gons are constructible? This is equivalent to the question: is $\cos(\frac{2\pi}{m}) + i\sin(\frac{2\pi}{m})$ constructible?

These can be answered via field extensions.

Other applications including coding theory.

2

An introduction to Rings

2.1 Definitions and basic properties

ring

A ring is a set with two binary operations $+, \times$, such that

1. $(R, +)$ is an abelian group.
   - $+$ is commutative and associative.
   - $\exists 0 \in \mathbb{R}$, $0 + a = a + 0 = a$ for all $a \in R$.
   - $\forall a \in \mathbb{R}$, $\exists (-a) \in R$, $a + (-a) = (-a) + a = 0$.
2. $\times$ is associative $(a \times b) \times c = a \times (b \times c)$.
3. distributive laws hold: $(a + b) \times c = (a \times c) + (b \times c)$.

The ring is called commutative if $\times$ is commutative. The ring is said to have an identity if $\exists 1 \in R$, $1 \times a = a \times 1 = a$, for all $a \in R$, and this does not require the existence of inverse.

For simplicity, we write

$$ab := a \times b, \qquad b - a = b + (-a)$$

Example:

$\mathbb{Z}$ is a commutative ring with identity.

Trivial rings: Let $(R, +)$ be an abelian group. We define $a \times b = 0$ for all $a, b \in R$. The result is a commutative ring with "trivial structure".

$R = \{0\}$ is a zero ring. $0 = 1$ in this case, and it is the only such ring. It leads to assumption $0 \neq 1$, saying $R \neq \{0\}$.

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commutative rings with identity.

$\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ with $+, \times \mod m$ is a ring with identity, and commutative.

The real quaternions: $\{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$. Addition is "component-wise". And the multiplication follows

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

And this is non-commutative ring, with identity 1.

Let X be a set, A be a ring. Consider the set $F = \{f : X \to A\}$. Define

$$(f+g)(x) = f(x) + g(x), \qquad (f \times g)(x) = f(x) \times g(x)$$

F commutative & having identity is inherited from the ring A.

$M_m(\mathbb{Z})$ is the ring of square $m \times m$ matrices with coefficients in $\mathbb{Z}$. It is non-commutative ring with identity.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have compact support, if $\exists a, b \in \mathbb{R}$, $f(x) = 0$ for $x \notin [a, b]$. $R = \{f : \mathbb{R} \to \mathbb{R} : f \text{ has compact support}\}$ is a commutative ring, without identity.

Proposition 2.1

Let R be a ring. Then

1. $0a = a0$ for all $a \in R$.
2. $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.
3. $(-a)(-b) = ab$ for all $a, b \in R$.
4. If R has an identity 1, then it is unique, and $(-a) = (-1)a$.

Proof:
We see that

$$0a = (0+0)a = 0a + 0a$$
$$0a - 0a = (0a + 0a) - 0a = 0a + (0a - 0a)$$
$$0a = 0$$

We also see that

$$(-a)b + ab = ((-a) + a)b = 0b = 0$$

□

We would like to be able to cancel with respect to x: $ab = ac$ then $b = c$. However, this is not true in general.

Example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

However,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2.2 Zero divisor and integral domain

zero divisor

A nonzero element $a \in R$ is called a zero divisor, if there exists $b \in R$ and $b \neq 0$, such that $ab = 0$ or $ba = 0$.

integral domain

A commutative ring with identity, $1 \neq 0$, is called an integral domain, if it contains no zero divisor.

Proposition 2.2

Let R be a ring. Assume that $a, b, c \in R$, and a is not a zero divisor. If $ab = ac$, then either $a = 0$ or $b = c$ (i.e., we can multiplicatively cancel).

Proof:
Observe that

$$ab = ac$$

$$ab - ac = 0$$

$$a(b - c) = 0$$

As a is not zero divisor, then either $a = 0$ or $b - c = 0$. $\square$

If zero divisors exist, then cancellation does not hold:

$$ab = 0 = a \cdot 0 \not\Rightarrow b = 0$$

Remark:
In integral domains, $ab = 0 \implies a = 0$ or $b = 0$.

2.3 Field

division ring

A ring with identity 1, $1 \neq 0$, is called a division ring, if every nonzero element has a multiplicative inverse, i.e., for all $a \in R, a \neq 0$, there exists $b \in R$, such that $ab = ba = 1$.

Consider an example $ab = 1$ existing and $ba = 1$ not existing.

Example:
Real sequences $(x_1, x_2, \ldots)$. Ring of operators on the sequences, $\times$ is composition. Take

$$D : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$$
$$S : (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots)$$

Then

$$D(S(x_1, x_2, \ldots)) = Id(x_1, x_2, \ldots)$$

but $S \circ D \neq Id$.

field

A commutative division ring is called a field.

Example:
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. Quaternions are "only" a division ring because non-commutative. $\mathbb{Z}_p$ is a field for p prime.

Proposition 2.3

Any finite integral domain is a field.

$\mathbb{Z}$ is an integral domain, but far from a field.

Proof:
Check Corollary 10.13 of PMATH 347. □

2.4 Subring

subring

Let R be a ring. A nonzero subset $S \subseteq R$ is called a subring of R, if it is a ring with the operations from $(R, +, \times)$ restricted to S.

That means: $S \neq \varnothing$. $x + (-y) \in S, \forall x, y \in S$. $xy \in S, \forall x, y \in S$.

Example:

$\mathbb{Z}_2 \subseteq \mathbb{Z}$, but $\mathbb{Z}_2$ is not a subring of $\mathbb{Z}$.

$2\mathbb{Z} = \{2 \cdot z : z \in \mathbb{Z}\}$ (ring has no identity) is a subring of $\mathbb{Z}$ (ring has identity).

Ring of matrices $M_2(\mathbb{R})$ (1 is identity matrix) has a subring $S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \right\}$ and $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is the identity in S.

2.5 Unit

unit

Assume that R is a ring with an identity $1 \neq 0$. A $a \in R$ is called a unit, if there exists $b \in R$ such that $ab = ba = 1$. Set of units of R is denoted by $R^\times$.

Example:

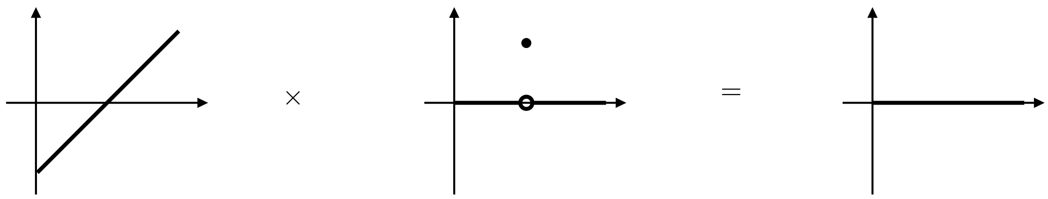
$\mathbb{Z}^\times = \{\pm 1\}$

$\mathbb{Z}_m^\times = \{a \in \mathbb{Z}_m : \gcd(a, m) = 1\}$, $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus \{0\}$ for p prime.

Consider ring R of $[0,1] \to \mathbb{R}$, where $(f \times g)(x) = f(x) \cdot g(x)$, $1_R = 1(x)$. Units are the functions such that $f(x) \neq 0$ for $\forall x \in [0,1]$. Then $f(x)^{-1} = \frac{1}{f(x)}$. All non-units are zero divisors. If $g(y) = 0$, then $h(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ gives $(g \times h) = 0(x) = 0_R$.

Ring of all continuous functions $[0,1] \to \mathbb{R}$ is a subring of the previous ring. Units as before, because $1/f$ exists and is continuous.

Consider $f(x) = x - 1/2$.



3

Ring Homomorphisms

ring homomorphism

Let R, S be rings.

1. A ring homomorphism is $\phi : R \to S$, such that
   (a) $\phi(a+b) = \phi(a) + \phi(b)$, for all $a, b \in R$.
   (b) $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in R$.
2. The kernel of ϕ, $\ker \phi = \{a \in R : \phi(a) = 0_S\}$.
3. A bijective homomorphism is called isomorphism.

Remark:
Isomorphism means "same ring", denote $R \cong S$.

Example:
$\{0, 1\} = \mathbb{Z}_2 = R$, $S = \{a, b\}$ with $a + a = a, a + b = b, \ldots$ Then $R \cong S$.

Example:
$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$ with cancellation $\frac{a}{b} = \frac{ca}{cb}$

Can we say $\mathbb{Z} \subseteq \mathbb{Q}$? not in the purest sense. $\mathbb{Z}$ corresponds to $\{\frac{a}{1} : a \in \mathbb{Z}\}$.

$\mathbb{Q}$ contains an isomorphic copy of $\mathbb{Z}$. $S \subseteq \mathbb{Q}$ such that $S \cong \mathbb{Z}$.

Example:
$\phi : \mathbb{Z} \to \mathbb{Z}_2$. $\phi(2k) = 0$, $\phi(2k+1) = 1$. Then

$$\begin{aligned}
\ker \phi &= 2\mathbb{Z} \\
\phi^{-1}(0) &= 2\mathbb{Z} = \ker \phi \\
\phi^{-1}(1) &= 1 + 2\mathbb{Z} \\
&= 1 + \ker \phi \\
&= 3 + \ker \phi
\end{aligned}$$

Example:

$\phi : \mathbb{Z}[x] \to \mathbb{Z} : p(x) \mapsto p(0)$. Then

$$\begin{aligned}\ker\phi = \phi^{-1}(0) &= \{a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + 0 : a_i \in \mathbb{Z}\} \\ &= x\mathbb{Z}[x] = \{x \cdot p(x) : p(x) \in \mathbb{Z}[x]\}\end{aligned}$$

and

$$\phi^{-1}(a) = x\mathbb{Z}[x] + ax^0 = \ker\phi + ax^0$$

Example:

$\phi : \mathbb{Z}[x] \to \mathbb{Z}_2 : p(x) \mapsto p(0) \bmod 2$. Then

$$\ker\phi = \phi^{-1} = x\mathbb{Z}[x] + 2\mathbb{Z}$$

$$\phi^{-1}(1) = 1 + \ker\phi$$

Example:

$\phi : \mathbb{Z} \to \mathbb{R} : a \mapsto a$, then $\ker\phi = \{0_{\mathbb{R}}\}$.

Proposition 3.1

Let R, S be rings, $\phi : R \to S$ be homomorphism.

1. The image of ϕ, ($\mathrm{Im}(\phi)$, or $\phi(R)$) is a subring of S.
2. $\ker\phi$ is a subring of R. Moreover, $\forall r \in R, \forall \alpha \in \ker\phi$, $r\alpha \in \ker\phi, \alpha \in \ker\phi$. (That is $\ker\phi$ is closed under multiplication by the elements from R)

Proof:

1. If $a, b \in \phi(R)$, then

$$a - b = \phi(x_a) - \phi(x_b) = \phi(x_a - x_b) = \phi(x_{a-b}) \in \phi(R)$$

2. $\phi(r\alpha) = \phi(r) \cdot \phi(\alpha) = \phi(r) \cdot 0 = 0$

$\square$

Can we get a ring structure on $a + \ker\phi$? There is a factor ring $R/\ker\phi$. For example, $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$.

3.1 Ideals & Quotient rings

ideal

Let R be a ring, let $I \subseteq R$ be a subring, let $r \in R$.

1. I is called a left ideal, if $rI \subseteq I$ where $rI = \{ri : i \in I\}$.
2. I is called a right ideal, if $Ir \subseteq I$.
3. I is an ideal, if it is left & right ideal (two sided ideal).

Ideal Test

Check K is an ideal of R:

- $k - j \in K$ for all $j, k \in K$; and
- $rk, kr \in K$ for all $k \in K, r \in R$.

It is a quick generalization of previous definition. Reference: Laurent W. Marcoux's 334 notes.

additive quotient

Let $I \subseteq R$ be an ideal. The additive quotient is defined as $R/I = \{a + I : a \in R\}$.

Example:

$\mathbb{Z}/3\mathbb{Z} = \Big\{\{\ldots, -6, 3, 0, 3, 6, \ldots\}, \{\ldots, -5, -2, 1, 4, \ldots\}, \{\ldots, -4, -1, 2, 5, 8, \ldots\}\Big\}$. Additive group.

Let $I = 3\mathbb{Z}$. Then $a + I$ are called (additive) cosets.

Proposition 3.2

Let R be a ring, I an ideal of R, then R/I is a ring with the operations

$$(a + I) +_{R/I} (b + I) =: (a +_R b) + I$$

$$(a + I) \times_{R/I} (b + I) = (a \times_R b) + I$$

The ring properties R/I follow from R being a ring.

quotient ring

R/I is called the quotient ring of R by I.

Remark:

If I is not an ideal, then the definition of the operations on R/I is not well defined.

Example:

Let R be commutative ring with identity $1 \neq 0$, $m \geq 2$. Let $M_m(R)$ be ring of square matrices with coefficients in R.

Denote

$$L_j(R) = \{A \in M_m(R) \mid A_{ik} = 0, \forall i \in [n], k \in [m] \setminus \{j\}\}$$

which means only the j-th column can have non-zero entries. Then $L_j(R)$ is a left ideal in $M_m(R)$. This can be verified by the matrix multiplication. $L_j(R)$ is not a right ideal, i.e., $L_j(R) \cdot M \notin L_j(R)$ for some $M \in M_m(R)$.

Analogously, a right ideal can be obtained by taking

$$T_i(R) = \Big\{A \in M_m(R) \,\Big|\, A_{kj} = 0, \forall k \in [n] \setminus \{i\}, j \in [m]\Big\}$$

Example:

Let $R = \mathbb{Z}[x]$ and $I = x^2\mathbb{Z}[x]$.

Then $R/I = \{a + bx + p(x) : a, b \in \mathbb{Z}, p(x) \in I\}$.

For $a \in R/I$, $\bar{a}$ denotes $a + I$.

3.2 Isomorphism theorems

Lemma 3.3

Let I be an ideal in R, then $a + I = b + I$ ($\bar{a} = \bar{b}$) if and only if $b - a \in I$. Namely, every member of the coset can be the representative.

Theorem 3.4: First isomorphism theorem

If $\phi : R \to S$ is a ring homomorphism, then $\ker\phi$ is an ideal in R, Im ϕ is a subring of S, and $R/\ker\phi \cong \text{Im } \phi$.

Proof:
Theorem 4.2 of http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf

Consider $\tau : R/\ker\phi \to \phi(R) : r + \ker\phi \mapsto \phi(r)$. □

Example:
$\mathbb{Z}[x]/2\mathbb{Z}[x] \cong \mathbb{Z}_2[x]$. We can define $\phi : p(x) \mapsto p(x) \mod 2$.

Theorem 3.5

For any ideal $I \subseteq R$, the map $R \to R/I$ defined by $\pi : r \mapsto r + I$ is a surjective ring homomorphism with kernel I. It is called the natural projection of R onto R/I. Thus every ideal is a kernel of some homomorphism.

Proof:
Prove surjectivity is as before in first iso theorem. The prove homomorphism, both $\times$ and $+$. Now prove $\ker\phi$.

- Let $i \in I$, then $\pi(i) = i + I = I = 0_{R/I}$.
- Let $a \in R/I$, then $\pi(a) = a + I$, but $a \notin I$. Thus by lemma, $a + I \neq I = 0 + I$. □

Theorem 3.6: Second isomorphism theorem

Let A be a subring of R, B an ideal of R. Then $A + B = \{a + b : a \in A, b \in B\}$ is a subring of R. $A \cap B$ is an ideal of R and $(A + B)/B \cong A/A \cap B$.

Proof:
Consider the map $\phi : A \to (A + B)/B : a \mapsto a + B$. Then apply first isomorphism theorem.

Or check Theorem 4.3 of http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf. □

Remark:
$(A + B)/B = \{a + b + B : a \in A, b \in B\} = \{a + B : a \in A\} \stackrel{?}{=} A/B$

This reduction can't happen because B is not necessarily an ideal of A.

Example:
Let $R = \mathbb{Z}$, then $a\mathbb{Z} + b\mathbb{Z} = \gcd(a,b) \cdot \mathbb{Z}$. $a\mathbb{Z} \cap b\mathbb{Z} = \operatorname{lcm}(a,b) \cdot \mathbb{Z}$. Then by second iso thm

$$\frac{\gcd(a,b)\mathbb{Z}}{b\mathbb{Z}} \cong \frac{a\mathbb{Z}}{\operatorname{lcm}(a,b)\mathbb{Z}}$$

Lemma 3.7

If $m \mid n$, then $n\mathbb{Z}$ is an ideal of $m\mathbb{Z}$, and $|m\mathbb{Z}/n\mathbb{Z}| = \frac{n}{m}$.

The coset representative in $(m\mathbb{Z}/n\mathbb{Z})$ are $\{0, m, 2m, \ldots, (\frac{n}{m} - 1)m\}$. Applying to $A + B/B \cong A/A \cap B$, we have

$$\frac{b}{\gcd(a,b)} = \frac{\operatorname{lcm}(a,b)}{a} \implies ab = \operatorname{lcm}(a,b) \cdot \gcd(a,b)$$

Theorem 3.8: Third isomorphism theorem

Let $I \subseteq J$ be ideals in R. Then J/I is an ideal in R/I and $(R/I)/(J/I) \cong R/J$.

Proof:
Define $\phi : R/I \to R/J : a + I \mapsto a + J$. Then show that $\ker \phi = J/I$ and then use first isomorphism theorem.

Or check Theorem 4.4 of `http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf` □

Example:
$(\mathbb{Z}/6\mathbb{Z})/(3\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$.

Theorem 3.9: Fourth isomorphism theorem/correspodence theorem

Let R be ring, I ideal in R. The correspondence $A \leftrightarrow A/I$ is an inclusion preserving bijection between the set of subrings (A) of R, $I \subseteq A \subseteq R$, and the set of subrings of R/I. Furthermore, A/I is an ideal in R/I if and only if A is an ideal in R $(I \subseteq A)$.

Proof:
No first isomorphism theorem. Expand and verify the definitions. □

The interesting part is: subring of R/I gives subring of R.

4

More on Ideals

Let $A \subseteq R$ with identity.

(A)

1. $(A) =$ the smallest ideal containing A (in R)
2. Let

$$RA = \left\{ \sum r_i a_i : r_i \in R, a_i \in A \right\}$$
$$AR = \left\{ \sum a_i r_i : r_i \in R, a_i \in A \right\}$$
$$RAR = \left\{ \sum r_i a_i r_i' : r_i, r_i' \in R, a_i \in A \right\}$$

   where these are all finite sums.
3. If $A = \{a\}$, then $(A) =: (a)$ is called a principal ideal.
4. If an ideal $I = (A)$ for A finite, we call I finitely generated.

Remark:

$$(A) = \bigcap_{\substack{I \text{ ideal of } R \\ A \subseteq I}} I$$

The intersection is indeed an ideal.

$(A) \subseteq \cap I$ because (A) is the smallest. $\cap I \subseteq (A)$ because it contains $I = (A)$.

Note that $\cup I_\alpha$ is not an ideal in general.

What is (A)?

Assume R is commutative. Then (A) contains $a \in R$, and also $ra, r \in R, a \in R$, and their sums. This is precisely the definition of RA. Thus $RA \subseteq (A)$.

Note that $1 \in R$. Then $A \subseteq RA$, and RA is an ideal itself. By minimality, $(A) \subseteq RA$.

To conclude, $(A) = RA = AR = RAR$ in the commutative case.

In particular, the principal ideal $(A) = a \cdot R = \{ar : r \in R\}$, because let $A = \{a\}$, we have

$$AR = \left\{\sum ar_i : r_i \in R\right\} = \left\{a\left(\sum r_i\right) : r_i \in R\right\}$$

works in commutative rings.

Warning In non-commutative rings, we have $(A) = RAR$, so

$$(a) = RaR \neq \{r_i a r_i' : r_i, r_i' \in R\}$$

Example:
$R = \mathbb{Z}$, the principal ideal (m) is $m\mathbb{Z}$.

Example:
Let $R = \{f : [0,1] \to \mathbb{R}\}$. Then $I = \{f \in R : f(1/2) = 0\}$ is an ideal. And $I = (g)$ where

$$g(x) = \begin{cases} 0 & \text{if } x = 1/2 \\ 1 & \text{otherwise} \end{cases}$$

For $h \in I$, $h = g \cdot h \in (g)$. Note that g is an identity element of I, but not of R.

Example:
$C = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ is a subring of R. $I = \{f \in C : f(1/2) = 0\}$ is again an ideal. BUT! I is not a principal ideal, I is not even finitely generated (not easily proven).

Note that I here is different from last example, where the instructor made a mistake at first.

Example:
Let $R = \mathbb{Q}[x]$. Consider subring $S = x\mathbb{Q}[x] + \mathbb{Z}$. An ideal $I = x\mathbb{Q}[x]$.

1. $I = (x)$ in R
2. I is an ideal in S where I is not finitely generated

If I is finitely generated in S, then there exists $p_1, \dots, p_k \in I$

$$I = (p_1, \dots, p_k) = \left\{\sum_{i=1}^{k} p_i(x) q_i(x) : q_i \in S\right\}$$

As p_i are in ideal $I = x\mathbb{Q}[x]$, p_i don't have constant term. However, this is not possible. Take an element $\frac{a}{b}x \in I$, then

$$\frac{a}{b}x = \sum_{i=1}^{k} p_i(x) q_i(x)$$

As p_i's are fixed, one need to find proper q_i's to make this equation hold. Now consider b to be a prime such that b does not divide the product of denominators of p_i's, then it's impossible to find any q_i's to make this equation holds. Therefore I is not a finite generated ideal in S.

Proposition 4.1

Let I be an ideal in R with identity $1 \neq 0$.

1. $I = R$ if and only if I contains a unit.
2. Let R be commutative. Then R is a field if and only if the only ideals in R are 0 and R.

Proof:

Statement 1

($\Rightarrow$) Because $1 \in R = I$, and 1 is a unit.

($\Leftarrow$) Let $u \in I$ be a unit. Then $u \cdot u^{-1} = 1 \in I$. Let $r \in I$, as $1 \in I$, then $1 \cdot r \in I$, hence $I = R$.

Statement 2

($\Rightarrow$) Let $0 \neq I \subseteq R$ be an ideal. Then it contains a unit. Then by (1), $I = R$.

($\Leftarrow$) Take arbitrary $0 \neq r \in R$. The ring (r) can't be zero ideal, hence $(r) = R$. Thus $1 \in (r)$. That means there exists $s \in R$, such that $1 = r \cdot s$. Then $s = r^{-1}$. Hence r is a unit. □

Corollary 4.2

A nonzero homomorphism from a field to a ring is an injection.

Proof:
Let ϕ be such a homomorphism. $\ker \phi$ is an ideal of the field. This implies $\ker \phi = 0$ (injective homomorphism) or R, the whole field. And the second possibility tells us ϕ is a zero map, which is eliminated by the assumption. □

4.1 Maximal ideals

maximal ideal

An ideal M is an arbitrary ring R is called a maximal ideal if $M \neq R$ and there is no proper $(\neq R)$ ideal I, $M \subsetneq I \subsetneq R$.

Alternatively, ideal I of a ring R is maximal if the only ideals containing I are I and R.

Theorem 4.3

Assume that R ring is commutative. The ideal M is maximal if and only if R/M is a field.

Proof:
By 4th iso thm, or correspondence theorem, R/M is a field $\Leftrightarrow$ ideals of R/M are zero ideals and $R/M \Leftrightarrow$ only ideals of R containing M are M and $R \Leftrightarrow M$ is maximal. □

Example:
$p\mathbb{Z}$ is maximal ideal for any p prime.

Theorem 4.4

$p\mathbb{Z}$ is maximal if and only if $\mathbb{Z}/p\mathbb{Z}$ is a field.

Example:
$(2, x)$ in $\mathbb{Z}[x]$ is maximal. $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}_2$ because $(2, x)$ is a kernel of $\phi : p(x) \mapsto p(0) \bmod 2$.

Example:
Let $R = \{f : [0, 1] \to \mathbb{R}\}$ and $M_c = \{f \in R : f(c) = 0\}$. Consider $\phi : R \to \mathbb{R} : f \mapsto f(c)$. Then $\ker \phi = M_c$. As $\mathbb{R} = \phi(R)$, then $R/M_c \cong \mathbb{R}$ is a field. Hence M_c maximal.

4.2 Maximal ideals and Zorn's Lemma

Consult Section 10.3 of PMATH 347 if needed.

Is every ideal (proper) contained in some maximal ideal? No. Consider $\mathbb{Q}$ with standard $+$ and $a \times b = 0_+$ for all $a, b \in \mathbb{Q}$. We have ideals

$$\left\{\frac{a}{2} : a \in \mathbb{Z}\right\} \subseteq \left\{\frac{a}{4} : a \in \mathbb{Z}\right\} \subseteq \cdots \subseteq \left\{\frac{a}{2^k} : a \in \mathbb{Z}\right\} \subseteq \cdots$$

These ideals are not contained in a maximal ideal. This happens because there's no identity.

Theorem 4.5

In a ring with an identity, every proper is contained in some maximal ideal.

Wrong idea Given I, then $I \subseteq \bigcup_{\substack{I \subseteq A \\ A \neq R}} A$. But this is not an ideal. For example, $\mathbb{Z}_6 \subseteq \mathbb{Z}_2 \cup \mathbb{Z}_3$ is not an ideal.

Right idea $I \subseteq \bigcup_{A \in C} A$ for C being a "chain"

$$I \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq \cdots$$

partial order

A partial order on a set S is a relation on X such that

1. $a \leq a$ for all $a \in S$,
2. If $a \leq b$ and $b \leq a$ then $a = b$ for all $a, b \in S$,
3. If $a \leq b$ and $b \leq c$, then $a \leq c$ for all $a, b, c \in S$.

So set inclusion $\subseteq$ is a partial order.

The ordering does not have to be "linear": sth $\leq$ sth $\leq$ sth $\leq \cdots$. For sets, we can have

$$\begin{tikzcd}
 & \{a,b\} \arrow[r, phantom, "\subseteq"] & \{a,b,d\} \arrow[dr, phantom, sloped, "\subseteq"] & \\
\{a\} \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & & \{a,b,c,d,e\} \\
 & \{a,c\} \arrow[r, phantom, "\subseteq"] & \{a,c,e\} \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

A chain C in a partially ordered set $(S, \leq)$ is a subset such that for all $x, y \in C$, $x \leq y$ or $y \leq x$ (i.e., all elements are comparable).

Zorn's Lemma

Let $(S, \leq)$ be a partially ordered set with the property that each chain has an upper bound in S. Then S contains a maximal element.

Theorem 4.6

Let R be a ring with 1. Then every proper ideal I is contained in some maximal ideal.

Proof:
Let $F = \{J : J \text{ is a } \textbf{proper} \text{ ideal of } R, M \subseteq J\}$. Notice $(F, \subseteq)$ is a poset (partially ordered set). Recall some notations/definitions:

- Chain: subset $G \subseteq F$, s.t. $\forall x, y \in G$, $x \subseteq y$ or $y \subseteq x$ (comparable)
- Upper bound of $G \in F$, $m \in F$, s.t. $\forall g \in G$, $g \subseteq m$.
- Maximal in F: $m \in F$, s.t. $\forall a \in F$, $(m \leq a) \implies (a = m)$.

Let $C \subseteq F$ be a chain. Put $M := \bigcup_{A \in C} A$. M is an ideal because

1. nonempty: $A \in C$, $I \subseteq A$, then $I \in M$.
2. Let $a \in A, b \in B$, and $A, B \in C$. WLOG, assume $A \subseteq B$. Then $a, b \in B$, then $a - b \in B$, then $a - b \in M$.
3. $\forall r \in R, a \in M$, we have $a \in A \in C$, then $ra \in A$, $ra \in M$.

We claim that M is an upper bound of C in F. If $M = R$, then $1 \in A \in C$. But then by proposition, $A = R$. Contradiction.

Then apply Zorn lemma.

Or check proposition 10.8 of PMATH 347. $\square$

5

Polynomial Rings & Rings of Fractions

5.1 How to make new rings from old rings?

I don't want to put this section to the previous chapter. So here it is.

Direct products

Let $(R_i, +_i, \times_i)$ be rings. $R_1 \times R_2$ is a ring with

$$(r_1, r_2) \oplus (s_1, s_2) = (r_1 +_1 s_1, r_2 +_2 s_2)$$
$$(r_1, r_2) \otimes (s_1, s_2) = (r_1 \times_1 s_1, r_2 \times_2 s_2)$$

Then this applies to $\prod_i R_i$ (works for at most countable R_i's).

Direct sum

For finitely many R_i's, it is just direct product. For infinitely many R_i's

$$\bigoplus_{i \in I} R_i = \left\{ (r_1, r_2, r_3, \ldots) : r_i \in R_i, \text{ only finitely many } r_1 \neq 0 \right\}$$

5.2 Basic Definitions and Examples

Let R be a commutative ring with identity. A polynomial with coefficients in R with undeterminate/-variable x is a **formal** expression

$$p(x) = a^n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with $a_i \in R, \forall i \in 0, \ldots, n$. If $a_m \neq 0$, then $\deg p = n$. If $a_n = 1$, we call $p(x)$ monic.

$R[x] = \{a^n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : n \in \mathbb{N}, a_i \in R\}$ with operations

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i$$

$$\left(\sum_{i=0}^{n} a_i x^i \right) \times \left(\sum_{i=0}^{m} b_i x^i \right) = \sum_{k=0}^{n+m} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k$$

Observe that R appears in $R[x]$ as constant polynomials. $R[x]$ is commutative ring with identity.

Proposition 5.1

Let R be an integral domain, let $p, q \in R[x]$ be nonzero elements. Then

1. $\deg pq = \deg p + \deg q$
2. the units of $R[x]$ are precisely the units of R.
3. $R[x]$ is an integral domain.

Proof:
$p(x)q(x) = \underbrace{a_n b_m}_{\neq 0} x^{n+m} + \cdots$

Let $p(x) \in R[x]$ be invertible, then there exists q such that $pq = 1$. By (1), $\deg p = 0$. Thus $\deg q = 0$. p, q are constant polynomials.

$pq = 0$, then $\deg p + \deg q = 0$. Then $\deg p = \deg q = 0$. Then they are all constant polynomials. As R is integral domain, we have $p = q = 0$. □

Formal power series

Ring of all power series $R[[x]] = \left\{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\right\}$ with the same operations defined as polynomial rings.

1. $R[[x]]$ is a commutative ring with identity.
2. Units of $R[[x]]$ are $\sum_{i=0}^{\infty} a_i x^i$ with a_0 unit in R.

Laurent series

$$R((x)) = \left\{\sum_{i=N}^{\infty} a_i x^i : a_i \in R, N \in \mathbb{Z}\right\}$$

5.3 Rings of fractions

Construct $\mathbb{Q}$ from $R = \mathbb{Z}$. Define
$$\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$$

$\frac{p}{q}$ is a "formal" fraction. ($p \cdot q^{-1}$ does not work). However, $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}$ are distinct formal fractions. We want to have them be in equivalent classes.

We define $\frac{a}{b} \sim \frac{c}{d}$ iff $ad = bc$ (use only the ring operations). Then define $\mathbb{Q}$ be the equivalence classes of $\sim$. For that, we need to show that $\sim$ is equivalence: reflexive, symmetric, transitive.

We define addition as
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
is well-defined on equivalence classes. We can obtain $+$ on the equivalence classes through definition of $+$.

We define multiplication as
$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$
is well-defined on equivalence classes.

Then we obtain $\mathbb{Q}$. Note that the well-definednesses need a proof. See Section 11.1 of PMATH 347.

$\frac{2}{1}, \frac{1}{2} \in \mathbb{Q}$, then $\frac{2}{1} \cdot \frac{1}{2} = \frac{2}{2} \sim \frac{1}{1}$ is an identity. Thus 2 is invertible in $\mathbb{Q}$. Every integer is a unit in $\mathbb{Q}$.

If R have zero divisors, $ab = 0$ and $a, b \neq 0$. Then if a invertible: $1 = a^{-1} \cdot a$, then $b = a^{-1}(a \cdot b) = 0$. Contradiction. Thus **zero divisors do not have inverses in any ring**. Now consider

$$a = \frac{a}{1} = \frac{ab}{b} = \frac{0}{b} = 0$$

contradiction to $a \neq 0$. Thus we will avoid zero divisors.

Theorem 5.2

Let R be a commutative ring. Let D be any subset of R closed under multiplication and not containing zero divisors and 0. Then there exists a commutative ring Q with identity such that Q contains R as a subring and every element of D is a unit of Q. Moreover,

1. every element of Q is of the form $\frac{r}{d}$ for some $r \in R, d \in D$. If $D = R \setminus \{0\}$, then Q is a field.
2. The ring Q is the smallest ring containing R in which all elements of D are units.

Here we formalize the definition of "smallest": Let S be any commutative ring with identity and let $\phi : R \to S$ be any injective homomorphism such that $\phi(d)$ is a unit of S for each $d \in D$. Then there is an injective homomorphism $\Phi : Q \to S$ such that $\Phi_R = \phi$. In other words, any ring containing an isomorphic copy of R in which elements of D become units must contain Q.

$$\begin{tikzcd} R \arrow[r, hook, "id"] \arrow[dr, "\phi \text{ injective}"'] & Q \arrow[d, "\Phi \text{ injective}"] \\ & S \end{tikzcd}$$

Thus R"$\subseteq$"S.

Proof:

Almost the same as the proof of Theorem 11.3 of pmath347. Below are some main points.

$F := \{(r,s) : r \in R, d \in D\}$. Then $\sim$ is an equivalence relation: $(r,s) \sim (g,h)$ iff $rh = sg$. Then denote by $\frac{r}{d}$ the equivalence class of (r,d). As above, we define $+$ and $\times$.

Let $Q/\sim$ be the set of equivalence classes of $\sim$. We verify it is a ring.

Q contains an isomorphic image of R: consider a homomorphism $\sigma : R \to Q, r \mapsto \frac{rd}{d}$ for any $d \in D$ (does not depend on choice of d). We need to prove injectivity here.

Every $d \in D$ (i.e., $\sigma(d)$) is invertible in Q.

Now let's prove (1) and (2). (1) is trivial. Now prove (2). We claim that there exists $\psi : Q \to S$ injective such that $\psi|_R = \phi$. Note that $\phi(d)$ invertible for all $d \in D$, thus we can define $\psi(\frac{r}{d}) = \phi(r)\phi(d)^{-1}$ for all $r \in R$, $d \in D$. ψ is well defined. ψ is a homomorphism because ϕ is. Not hard to see ψ is injective. Finally, we see that $\psi|_R = \phi$. $\square$

Example:

$R = \mathbb{Z}$, then $Q = \mathbb{Q}$.

If R is a field, then $Q = R$.

$R = 2\mathbb{Z}$ is a ring without identity, then $Q = \mathbb{Q}$. $1_\mathbb{Q} = \frac{2}{2}$ for example.

$R := R[x]$, then Q is a ring of $\frac{p(x)}{q(x)}$, $q(x) \neq 0$. This is rational functions. If we start with $\mathbb{Z}[x]$, then $Q = \{\frac{p(x)}{q(x)} : q(x) \neq 0\}$. If we start with $\mathbb{Q}[x]$, then its Q is the same.

$R := R[[x]]$, then $Q = R((x))$.

6

Chinese Remainder Theorem

comaximal

The ideals $A, B \subseteq R$ are said to be comaximal if $A + B = R$.

m, n coprime iff $\exists a, b \in \mathbb{Z}, an + bm = 1$.

$A + B$

$A + B := \{a + b : a \in A, b \in B\}$.

Example:
$5\mathbb{Z}, 3\mathbb{Z} \subseteq Z$. As $10 + (-9) \in 5\mathbb{Z} + 3\mathbb{Z}$, hence $5\mathbb{Z}, 3\mathbb{Z}$ are comaximal.

AB

$AB := \{\sum_{\text{finite sums}} a_i b_i : a_i \in A, b_i \in B\}$. Similarly we have $A_1 \cdots A_k := \{\sum a_{i1} \cdots a_{ik} : a_{ij} \in A_j\}$.

Theorem 6.1

Let R be a commutative ring with an identity. Let $I_1, I_2, \ldots, I_k$ be ideals in R, such that I_n, I_m are comaximal for $n \neq m$. Then

$$^R/_{I_1 I_2 \cdots I_k} = {}^R/_{I_1 \cap I_2 \cap \cdots \cap I_k} \cong R/I_1 \times R/I_2 \times \cdots \times R/I_k$$

In particular, $I_1 I_2 \cdots I_k = I_1 \cap I_2 \cap \cdots \cap I_k$.

Proof:
By induction. The proof here is the same as Theorem 11.24 of pmath 347. □

Remark:
Consider the units of $R/I_1 \cdots I_k$ and $R/I_1 \times \cdots \times R/I_k$. The units are the same (under isomorphism). That means that

$$(R/I_1 \cdots I_k)^\times \cong (R/I_1)^\times (R/I_2)^\times \times \cdots \times (R/I_k)^\times$$

Because units in the product of rings are units in each component.

> An element of a product ring is a unit iff each component is a unit in its respective ring.

Then apply this remark to integers: $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

$$(\mathbb{Z}/m\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z})^\times \times \cdots (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})^\times$$

Euler's totient function: $\varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$. Thus from the relation above, we have

$$\varphi(m) = \varphi(p_1^{\alpha_1}) \cdots \varphi(p_k^{\alpha_k})$$

which means $\varphi(\cdot)$ is multiplicative arithmetic function.

7

Domains

7.1 Euclidean Domains

norm

A norm on a ring R is a function $N : \mathbb{R} \to \mathbb{Z}^+ \cup \{0\}$, s.t. $N(0) = 0$.

Euclidean domain

An integral domain (identity, commutative, no zero divisors) for which there exists a Norm, such that: $\forall a, b \in R$, $b \neq 0$, there exists $q, r \in R$ s.t. $a = qb + r$ with $N(r) < N(b)$ or $r = 0$. This is called a Euclidean domain.

Example:

$R = \mathbb{Z}$, $N(x) = |x|$. Then $a = qb + r$ follows from division with remainder. We don't have to keep r positive/negative.

Example:

Fields with $N(x) = 0$. We have $a = (ab^{-1})b = 0$

Example:

F a field. Then $F[x]$ is a Euclidean domain with $N(p(x)) = \deg(p(x))$. Then we can have polynomial long division.

Example:

Consider Gaussian integers.

$$Z[i] = \{a + bi : a, b \in \mathbb{Z}\}$$

is ED with $N(a + bi) = a^2 + b^2 = (a + bi)(a - bi)$.

Theorem 7.1

Every ideal in a Euclidean domain is principal.

Proof:
Let $I \subseteq R$ an ideal. Take a nonzero element d in I of the smallest norm. Let $x \in I$, then $x = qd + r$ where $N(r) = 0$ or $N(r) < N(d)$. But $N(r) < N(d)$ is not possible. So $N(r) = 0$. Since $r = x - qd \in I$, then we must have $r = 0$. Then $x = qd$. This holds for any $x \in I$. Thus $I = (d)$. □

Remark:
Every ideal is principal: principal ideal domain (PID). We have ED $\subseteq$ PID, not other way around.

7.2 GCD & Bézout domains

greatest common divisor

Let R be commutative.

1. We say that $b \mid a$ (b divides a), if there exists $x \in R$, $a = bx$.
2. $d \in R$ is called a $\gcd(a, b)$ if
   $*$) $d \mid a, d \mid b$
   $\triangle$) if $d' \mid a$ and $d' \mid b$, then $d' \mid d$.

We can rephrase two conditions:

$*$) $(a, b) \subseteq (d) \subseteq R$

$\triangle$) If $(a, b) \subseteq (d')$, then $(a, b) \subseteq (d) \subseteq (d')$.

Bézout domain

Bézout domain is a form of a Prüfer domain. It is an integral domain in which the sum of two principal ideals is again a principal ideal.

Proposition 7.2

In Bezout domains (every (a, b) is principal), $(a, b) = (d)$ where $d = \gcd(a, b)$.

Proof:
Assume $(a, b) = (\alpha)$. We know that $(a, b) = (\alpha) \subseteq (d)$ because (d) is the smallest ideal containing (a, b). Then by definition of gcd, we conclude that $(\alpha) = (d)$. □

Bezout domain is not necessary for existence of gcd.

Example:
$R = \mathbb{Z}[x]$, what is $\gcd(2, x)$? $(2, x)$ is not principal. It is a maximal ideal, because $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}_2$.

We see that $(2, x) \subseteq (1)$. Because $(2, x)$ is maximal, there are no ideals in between. Hence $\gcd(2, x^2) = 1$.

Theorem 7.3

Let R be an integral domain (commutative ring with identity), then $(d) = (d')$ if and only if $d = d'u$ for a unit $u \in R$.

Example:

In $\mathbb{Z}[i]$, units are $\{\pm 1, \pm i\}$, then $(2) = (-2i)$.

Proof:

We know that $d \in (d')$ and $d' \in (d)$. Thus we can find $x, y \in R$ such that $d = d'x$ and $d' = dy$. Hence $d(1 - xy) = 0$. If $d = 0$, then it's a trivial ring. If $d \neq 0$, then $xy = 1$. $\square$

Corollary 7.4

If $\gcd(a, b) = d$, then all gcd's are ud, for u a unit.

7.3 Euclidean Algorithm

It unfolds as follows

$$\begin{aligned} a &= q_0 b + r_0, \qquad N(r_0) < N(b) \\ b &= q_1 r_0 + r_1 \\ r_0 &= q_2 r_1 + r_2 \\ &\vdots \\ r_{m-2} &= q_m r_{m-1} + r_m \\ r_{m-1} &= q_{m+1} r_m + 0 \end{aligned}$$

Theorem 7.5

Let R be a Euclidean domain, $a, b \neq 0$, $a, b \in R$.

1. The last nonzero remainder, r_m, in Euclidean algorithm is $\gcd(a, b)$.

2. Moreover, $r_m = ax + by$ for $x, y \in R$. And x, y can be obtained from Euclidean algorithm.

Proof:

By going backwards in Euclidean algorithm, we obtain inductively that $r_m \mid r_{m-1}, r_{m-2}, \ldots, r_1, r_0$, $r_m \mid a, b$. This shows that $(a, b) \subseteq (r_m)$, which means r_m is a common divisor. It remains to show that $(r_m) \subseteq (a, b)$. We see that

$$\begin{aligned} r_0 &= a - q_0 b \in (a, b) \\ r_1 &= b - q_1 \underset{\in (a,b)}{\underset{\uparrow}{r_0}} \in (a, b) \\ &\vdots \\ r_m &= r_{m-2} - q_m r_{m-1} \in (a, b) \end{aligned}$$

Thus $(r_m) \subseteq (a, b)$.

Therefore $(r_m) = (a, b)$. $\square$

7.4 Principal Ideal Domain

Example:

$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right] = \left\{a + b\frac{1+\sqrt{-19}}{2} : a, b \in \mathbb{Z}\right\}$ is PID, but not Euclidean domain. To prove it is not ED, we follow the definition: $\forall a, b \in R$, $a = qb + r$, $N(r) < N(b)$ or $b = 0$. Take b a non-unit, non-zero with minimal norm. Then every x can be written as $x = qb + r$, $r = 0$ or r is a unit. If b is defined above, and we know units are $\{\pm 1\}$, then $x = qb \pm 1$ or $x = qb + 0$.

Take $2 = qb + r$, $r \in \{0, \pm 1\}$. This gives us three possibilities: $b \mid 2, b \mid 1, b \mid 3$.

For the rest, check https://math.stackexchange.com/a/23872 or page 282 of Dummit & Foote.

Principal Ideal Domain

An integral domain in which every ideal is principal is called a Principal Ideal Domain (PID).

Example:

$\mathbb{Z}$, $F[x]$ for F a field. $\mathbb{Z}[x]$ is not PID.

Proposition 7.6

Let R be a PID, $a, b \neq 0$, $a, b \in R$. Then if $(d) = (a, b)$, then

1. $d = \gcd(a, b)$.
2. $d = ax + by$ for $x, y \in R$.
3. d is unique up to a multiplication by a unit in R.

prime ideal

An ideal $I \subsetneq R$ is called a prime ideal if $ab \in I \implies a \in I$ or $b \in I$.

Example:

$6\mathbb{Z}$ is not prime ideal as $2 \times 3 \in 6\mathbb{Z}$ and $2, 3 \notin 6\mathbb{Z}$.

$7\mathbb{Z}$ is prime ideal.

Remark:

A prime p satisfies $p \mid ab \implies p \mid a$ or $p \mid b$.

Proposition 7.7

Every maximal ideal is prime.

Proof:

Maximal $\Leftrightarrow R/I$ field $\Rightarrow R/I$ integral domain $\Leftrightarrow I$ prime. $\square$

Theorem 7.8

Every nonzero prime ideal in PID is a maximal ideal.

Proof:
Suppose there exists a maximal ideal (m) where $m \in R$ such that a prime ideal $(p) \subseteq (m) \subseteq R$. Then $p = rm$. Then $rm \in (p)$. As (p) is prime ideal, thus either $r \in (p)$ or $m \in (p)$.

If $m \in (p)$, then $(m) \subseteq (p)$, then $(m) = (p)$.

If $r \in (p)$, then $r = sp$ for some $s \in R$. Sub it back, $p = rm = spm$. Then $p(1 - sm) = 0$. As $p \neq 0$, then $sm = 1$, thus s, m are units. Thus $(m) = R$. $\square$

Corollary 7.9

$\mathbb{Q}[x]/(p(x))$ for $p(x)$ irreducible (thus (p) is primal). $\mathbb{Q}[x]/(p) \cong \mathbb{Q}(\alpha)$, where α is a root of p.

Corollary 7.10

If $F[x]$ is a PID (ED), then F is a field.

Proof:

(x) is an ideal. We know that $F \cong F[x]/(x)$ is an integral domain. We also know that F is integral domain iff (x) is a prime ideal. As $F[x]$ is PID, then (x) is also maximal. Thus we conclude that $F[x]/(x) \cong F$ is a field.
♡

$\square$

Remark:
In ED, $\forall a, b \neq 0, a = qb - r, N(r) < N(b)$ or $b \mid a$.

The norm above generalizes to **Dedekind-Hasse norm**: $N(0) = 0, N(a) > 0$ if $a \neq 0$. Such that $\forall a, b \in R$, $a, b \neq 0$, $\exists s, t \in R : 0 < N(sa - tb) < N(b)$ or $b \mid a$.

Proposition 7.11

R is PID iff R has a Dedekind-Hasse norm.

Corollary 7.12

$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is PID.

7.5 Unique Factorization Domain

irreducible/prime

Let R be an integral domain.

1. Let $r \in R$, $r \neq 0$, r not a unit. We say that r is irreducible, if $r = ab \Rightarrow a$ or b is a unit of R.
2. $p \in R$, non-unit is called a prime, if $p \mid ab \Rightarrow p \mid a$ or $p \mid b$.

2′. (alternatively) p is prime if (p) is a prime ideal.

3. $a, b \in R$ are associated ($a \sim b$) if $a = ub$ for u a unit.

Proposition 7.13

A prime is irreducible.

Proof:

Let p be prime and $p = a \cdot b$. Then $p \mid ab$, then $p \mid a$ or $p \mid b$. WLOG, assume $p \mid a$. Then $a = px$. Hence $p = pxb$. This implies $xb = 1$, then x, b are units.
♡

□

Example:

$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. We found that

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two factorizations into irreducibles, and

$$2 \nmid (1 + \sqrt{-5}) \qquad \text{and} \qquad 2 \nmid (1 - \sqrt{-5})$$

Note that $N(a + b\sqrt{-5}) = a^2 + 5b^2 \in \mathbb{Z}$. Then we observe

$$4 = N(2) = N(\alpha\beta) = N(\alpha)N(\beta)$$

Even better, we have

$$(6) = P_2^2 P_3 P_3',$$

where $P_2 = (2, 1 + \sqrt{-5})$, $P_3 = (3, 2 + \sqrt{-5})$, $P_3' = (3, 2 - \sqrt{-5})$ are all prime ideals. In particular,

$$\begin{aligned} (2) &= P_2^2 \\ (3) &= P_3 P_3' \\ (1 + \sqrt{-5}) &= P_2 P_3 \\ (1 - \sqrt{-5}) &= P_2 P_3' \end{aligned}$$

Theorem 7.14

In PID, primes are precisely irreducibles. In other words, irreducible in PID is prime.

Proof:

Let r be an irreducible. We want to show if (r) is prime ideal. Let $(r) \subseteq M = (m)$ for some ideal M. Then $r = mx$. Because r is irreducible, then either m or x is a unit. If m is a unit, then $M = R$. If x is a unit, then $r \sim m$, then $(r) = (m)$. This proves (r) is maximal. As this is PID, then (r) is prime ideal. Hence r is prime.

♡

□

Unique Factorization Domain

An integral domain R is called a UFD if every non-zero non-unit $r \in R$ satisfies

1. $p_1 p_2 \cdots p_k$ where p_i's are irreducibles of R.
2. if $r = q_1 q_2 \cdots q_m$, with q_i's irreducibles, then $m = k$, and there exists a permutation π of $\{1, 2, \ldots, k\}$, such that $p_i \sim q_{\pi(i)}$.

Example:

A field is a UFD.

$\mathbb{Z}[x]$ is a UFD (if R is UFD, then $R[x]$ is UFD)

PID is UFD.

$\mathbb{Z}[\sqrt{-5}]$ is NOT a UFD.

Proposition 7.15

In UFD, every irreducible is a prime.

Proof:

Let p be irreducible, let $p \mid ab$. I.e., $ab = px$. Then we can write

$$(a_1 \cdots a_n)(b_1 \cdots b_m) = p(x_1 \cdots x_{m+n-1})$$

where a_i, b_i, p, x_i are irreducibles. By UFD property, WLOG assume $p \sim a_i$, then $pu = a_i$ for u a unit, i.e., $p \mid a_i$. Then

$$p(ua_1 \cdots a_{i-1}a_{i+1} \cdots a_m) = a$$

Hence $p \mid a$. □

Proposition 7.16

Let $a, b \neq 0$ in UFD. If

$$a = up_1^{e_1} p_2^{e_2} \cdots p_n^{e_n} \tag{7.1}$$
$$b = vp_1^{f_1} p_2^{f_2} \cdots p_n^{f_n} \tag{7.2}$$

with u, v units, $e_i, f_i \geq 0$ integers, p_i primes. Then

$$d = p_1^{\min\{e_1, f_2\}} \cdots p_n^{\min\{e_n, f_n\}}$$

is a $\gcd(a, b)$.

Proof:

Obviously, $d \mid a$, $d \mid b$. In d, $p_i^{\min\{e_i, f_i\}+1}$ if for some i, then it is not a divisor for both a, b. Thus the exponents have to be $\leq \min\{e_i, f_i\}$. If all $\leq$ are $=$, then we obtain d. If not all $\leq$ are strict, then we get something that divides d. □

Example:

$\mathbb{Z}[i]$ is UFD, but $\mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}$ is not UFD.

$$4 = 2 \cdot 2 = (-2i)(2i)$$

but $i \notin \mathbb{Z}[2i]$, so $2 \nsim (2i)$ or $(-2i)$.

Also $2i$ is not a prime, because $(2i)$ is not a prime ideal:

$$\mathbb{Z}[2i]/(2i) \cong \mathbb{Z}/4\mathbb{Z}$$

in which $2 \times 2 = 0$, which is not an integral domain. This isomorphism is obtained by

$$\phi(a + 2bi) = a \mod 4$$

Theorem 7.17

Every PID is UFD.

Proof:

Two steps:

1. Every nonzero non-unit element is a finite product of irreducibles.
2. Uniqueness.

Let $r \neq 0$, non-unit. Either r is irreducible, or $r = r_1 \cdot r_2$, r_1, r_2 non-units. Then r is irreducible or $r = r_1 r_2$ (r_1, r_2 are non-units). Then either r_1 is irreducible or $r_1 = r_{11} r_{12}$; r_2 is irreducible or $r_2 = r_{21} r_{22}$. We continue this process, iteratively factor r. We want to show the factorization is finite.

Assume factorization does not end. Then we obtain an infinite chain C:

$$(r) \subseteq (r_1) \subseteq (r_{11}) \subseteq (r_{112}) \subseteq \cdots \subseteq R$$

which corresponds to an infinite chain of factorization. Then $(m) = \bigcup_{(r_\alpha) \in C} (r_\alpha)$ is an ideal in PID. Since $m \in (r_\alpha)$, for some r_α in chain, then $(r_\alpha) = (m)$. Then

$$(r) \subseteq (r_1) \subseteq \cdots \subseteq (r_\alpha) = (m) \subseteq (m) \subseteq \cdots \subseteq R$$

The chain stabilizes (Noether Domain). Contradicts infinite factorization.

Let $r = p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$ for p_i, q_i irreducibles. WLOG, assume $q_1 \mid p_1$ then $p_1 = uq_1$ for u a unit. Then $p_1 \sim q_1$. Then

$$uq_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m \implies (up_2) \cdots p_n = q_2 \cdots q_m$$

Finish by induction on $\min\{m, n\}$. □

Corollary 7.18

If R is a PID, then there exists a Dedekind-Hasse norm on R.

Proof:

Define the norm: $N(0) = 0$ and $N(p_1 p_2 \cdots p_k) = 2^k$ where $p_1 p_2 \cdots p_k$ is unique factorization to irreducibles and p_i's do not need to be distinct. We observe that $N(ab) = N(a)N(b)$, and $N(a) > 0$ iff $a \neq 0$.

Let $a, b \in R$, then $(a, b) = (r)$ for some $r \in R$, and we know $r = \gcd(a, b)$. Then there exist $s, t \in R$ such that $sa - tb = r$. Taking norms $N(sa - tb) = N(r)$. Here r has less factors than a, b, thus $N(r) < N(b)$ because $r \mid b$. □

Remark:

gcd in UFD always exist, but consider an example. In $\mathbb{Z}[x]$, $\gcd(2, x) = 1$, but $(2, x) \subsetneq (1) = \mathbb{Z}[x]$. And there don't exist α, β s.t. $1 = \alpha \cdot 2 + \beta \cdot x$, namely gcd is not a combination of a, b in this case.

$$\begin{array}{ccccccccc} \text{Field} & \subsetneq & \text{ED} & \subsetneq & \text{PID} & \subsetneq & \text{UFD} & \subsetneq & \text{Integral domain} \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \mathbb{Z} & & \mathbb{Z}[1+\sqrt{-19}/2] & & \mathbb{Z}[x] & & \mathbb{Z}[\sqrt{-5}] & \end{array}$$

8

Polynomial Rings

Previously: If $R[x]$ is PID (or ED), then R is a field.

Remember: If R is ID, then $R[x]$ is ID.

$R[x_1, x_2, \cdots, x_n]$

For commutative ring R with identity, $x_1, \ldots, x_n$ commuting variables, we have

$$R[x_1, x_2, \cdots, x_n] = (R[x_1, x_2, \ldots, x_{m-1}])[x_m]$$

Proposition 8.1

Let I be an ideal in commutative ring R, with identity. Then $(R/I)[x] \cong R[x]/(I)$, where $(I) = I[x]$ is in $R[x]$. Moreover, if I is a prime ideal in R, then $(I) = I[x]$ is a prime ideal in $R[x]$.

Example:

$(\mathbb{Z}_5)[x] \cong \mathbb{Z}[x]/5\mathbb{Z}[x]$

Proof:

Consider a homomorphism $\phi : R[x] \to (R/I)[x]$ where ϕ is a coefficient reduction mod I. To check ϕ is a homomorphism, we want to check if $\phi(pq) = \phi(p)\phi(q)$. At x^k of $p(x)q(x)$, after ϕ applied to $\sum_{i=0}^{k} p_i q_{k-i}$, we get $(\sum p_i q_{k-i}) \mod I = \sum_{i=0}^{k} (p_i \mod I)(q_{k-i} \mod I)$. We observe that $\ker \phi = I[x]$. We also see that $\phi(R[x]) = (R/I)[x]$, which is just operations.

I prime ideal, then R/I ID, then $(R/I)[x]$ ID, then $R[x]/I[x]$ ID, thus $I[x]$ is prime ideal. $\square$

8.1 Polynomial rings over fields

Recall norm on $R[x]$: $N(p(x)) = \deg(p)$.

Theorem 8.2

Let F be a field, then $F[x]$ is a ED. Namely, if $a(x), b(x) \in F[x]$, then there exists unique $q, r \in F[x]$ such that $a(x) = b(x)q(x) + r(x)$ with $\deg(r) < \deg(b)$ or $r = 0$. (if $F \subseteq E$, then $F[x] \subseteq E[x]$), where E is a ED.

Proof:
By induction for existence.

1. If $\deg(a) < \deg(b)$, then $r = a, q = 0$.
2. If $\deg(a) \geq \deg(b)$, we can write

$$a(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$
$$b(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

where $m \leq n$.

Then the polynomial $\tilde{a}(x) = a(x) - \frac{a_n}{b_m}x^{n-m}b(x)$ and $\deg(\tilde{a}) < \deg(b)$. Then there exists $\tilde{q}, \tilde{r}$ such that $\tilde{a}(x) = \tilde{q}(x)b(x) + \tilde{r}(x)$ with $\deg(b) > \deg(\tilde{r})$. Sub in $a(x)$, we get

$$a(x) = \left(\tilde{q}(x) + \frac{a_n}{b_m}x^{n-m}\right)b(x) + \tilde{r}(x)$$

Note that $\frac{a_n}{b_m} = a_n b_m^{-1}$ is well-defined because F is a field.

As for the uniqueness, assume that $a = qb + r = q'b + r'$. Subtracting these two, we have

$$0 = b(x)(q(x) - q'(x)) + (r(x) - r'(x))$$

where $\deg(r - r') < \deg(b)$. Then

$$b(x)(q(x) - q'(x)) = (r(x) - r'(x)) = 0$$

Because integral domain,

$$q(x) - q'(x) = r(x) - r'(x) = 0$$

Thus $q(x) = q'(x)$ and $r(x) = r'(x)$. □

Corollary 8.3

If F is a field, then $F[x]$ is a UFD and a PID.

Example:
$\mathbb{Z}[x]$ is not a PID because $(2, x)$ is not principal.

$\mathbb{Q}[x]$ is a PID as $\mathbb{Q}$ is a field, then $(2, x) = (1) = \mathbb{Q}[x]$.

$\mathbb{Z}[x]/p\mathbb{Z}[x] \cong (\mathbb{Z}/p\mathbb{Z})[x] \cong \mathbb{Z}_p[x]$. What happens to $(2, x)$ in $(\mathbb{Z}/p\mathbb{Z})[x]$. If $p = 2$, then $(2, x) = (x)$ in $\mathbb{Z}_p[x]$. If $p > 2$, then 2 is invertible, then $(2, x) = (1)$ in $\mathbb{Z}_p[x]$.

8.2 Polynomial rings that are UFDs

Proposition 8.4: Gauss' Lemma

Let R be a UFD with a field of fraction F, and let $p(x) \in R[x]$. If $p(x)$ is irreducible in $R[x]$, then $p(x)$ is also irreducible in $F[x]$. (i.e., if $p(x)$ is reducible in $F[x]$, it is also reducible in $R[x]$)

More precisely, if $p(x) = a(x)b(x)$ in $F[x]$, then $\exists r, s \in F$ such that $p(x) = \underbrace{(ra(x))}_{\in R[x]}\underbrace{(sb(x))}_{\in R[x]}$, and nothing more, with respect to $a(x), b(x)$.