Matroid Theory

CO 446

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LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 446 during Spring 2021 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

This course is an introduction to matroid theory for graph theorists. Tree, cycle, vertex connectivity, minors, planar duality extend to matroids.

We will generalize

- Hall's Theorem (matching in bipartite graphs),
- Menger's Theorem (disjoint paths),
- Tutte's Wheel's Theorem (3-connectivity),
- Jaeger's Theorem (flows),
- Kuratowski's Theorem (planar graphs).

We also prove Tutte's Theorem (matching). We also find analogues of Ramsey's Theorem, Turan's Theorem, Erdős-Stone Theorem (maybe).

For any questions, send me an email via `https://notes.sibeliusp.com/contact`.

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Sibelius Peng

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Matroid

What is a matroid?

matroid

A **matroid** is a pair $(E, \mathcal{I})$ consisting of a finite set E, called the **ground set**, and a collection $\mathcal{I}$ of subsets of E, called **independent sets**, satisfying

(I1) the empty set is independent,

(I2) subsets of independent sets are independent, and

(I3) for each $X \subseteq E$, all maximal independent subsets of X have the same size, denoted $r_M(X)$ or $r(X)$; this is called the **rank** of X.

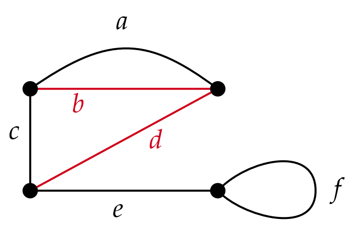
We are using the following notations. For a matroid $M = (E, \mathcal{I})$ we write:

- $E(M)$ for E,
- $\mathcal{I}(M)$ for $\mathcal{I}$,
- $|M|$ for $|E(M)|$, and
- $r(M)$ for $r(E(M))$.

1.1 Examples

1.1.1 Cycle-matroid of graphs

G



$\{b, d\} \in \mathcal{F}$

Let $G = (V, E)$ be a graph. Define $M(G) := (E, \mathcal{F})$ where $\mathcal{F}$ is the collection of all edge-sets that induce a forest in G.

Then we can check $M(G)$ is a matroid:

(I1) Clearly, empty set is acyclic, then a forest.

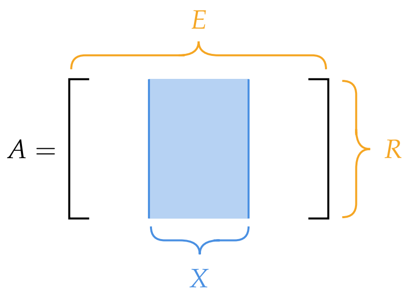
(I2) If we throw away edges from a forest, it is still a forest.

(I3) If we build a forest in a greedy way in a connected graph, we end up with a spanning tree, which is of the same size.

What is $r_M(X)$? $r_M(X) = |V| - \#$ of components of $G[V, X]$, which denotes the subgraph containing all the vertices and the edges in X.

We call $M(G)$ the **cycle-matroid** of G. A matroid is **graphic** if it is the cycle-matroid of some graph.

1.1.2 The column-matroid of a matrix



$A \in \mathbb{F}^{R \times E}$ where $\mathbb{F}$ is a field and R and E are finite sets. The **column-matroid** of A is $M(A) := (E, \mathcal{I})$ where $\mathcal{I}$ is the collection of all sets that index a set of linearly independent columns.

$M(A)$ is a matroid:

(I1) Trivial.

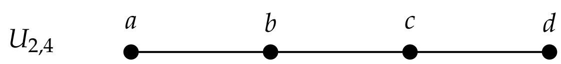
(I2) Trivial.

(I3) From linear algebra.

Remark:
The rank of a set $X \subseteq E$ is the rank of the submatrix $A[R, X]$.

- We call $M(A)$ the **column-matroid** of A.
- A matroid is **$\mathbb{F}$-representable** if it is the column-matroid over a matrix over the field $\mathbb{F}$.
- We abbreviate $GF(2)$-representable to **binary**.
- A matroid is **representable** if it is $\mathbb{F}$-representable over some field $\mathbb{F}$.

1.1.3 The 4-point line



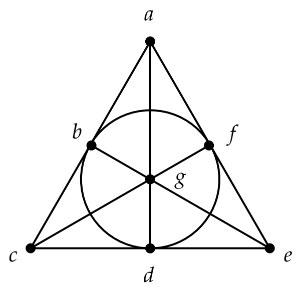
$$E(U_{2,4}) = \{a, b, c, d\}, \qquad \mathcal{I}(U_{2,4}) = \{\text{all sets of size at most 2}\}$$

Claim $U_{2,4}$ is not binary.

Proof:
There are only three distinct non-zero vectors in $GF(2)^2$. $\square$

1.1.4 The Fano matroid

F_7



$$E(F_7) = \{a, \ldots, g\}, \quad \mathcal{I}(F_7) = \left\{ \begin{array}{l} \text{all sets of size at most 3 except for} \\ \text{the seven triples depicted by lines} \end{array} \right\}$$

The binary representation:

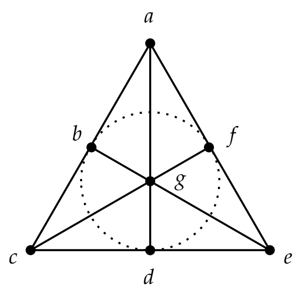
$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Claim The Fano is $\mathbb{F}$-representable $\Leftrightarrow$ $\mathbb{F}$ has characteristic 2.

Proof:
Do the calculations. First let a, c, e be basis. □

1.1.5 The non-Fano matroid

F_7^-



Exercise: non-Fano matroid
The non-Fano matroid is $\mathbb{F}$-representable if and only if $\mathbb{F}$ has characteristic different from 2.

All all matroids representable?

No, $F_7 \oplus F_7^-$ is not representable.

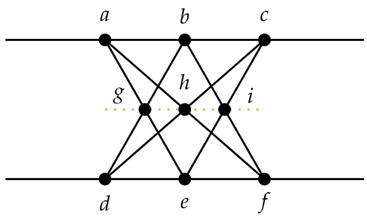
Direct sum

Let M and N be matroids with $E(M) \cap E(N) = \varnothing$. Define

$$M \oplus N := \big(E(M) \cup E(N), \{I \cup J : I \in \mathcal{I}(M), J \in \mathcal{I}(N)\}\big).$$

Note that $M \oplus N$ is a matroid; this is the direct sum of M and N.

1.1.6 The non-Pappus matroid



Exercise: Pappus 340 AD

The non-Pappus matroid is not representable.

Almost all matroids are non-representable.

Theorem (Nelson 2018)

The fraction of n-element matroids that are representable tends to zero as n tends to infinity.

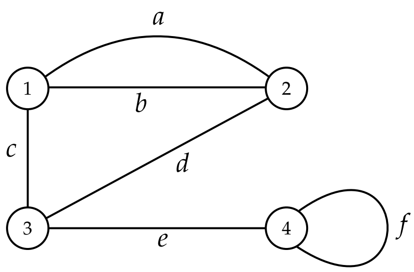
1.2 Graphic Matroids

Theorem 1.1

Graphic matroids are binary.

Consider the following graph G.

G



Consider the incidence matrix of G:

$$
\begin{array}{c c}
 & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}
\end{array}
$$

Note that the entry $4f$ can be viewed as 0 if we are in $GF(2)$.

For $v \in GF(2)^V$, consider $v^T A$.

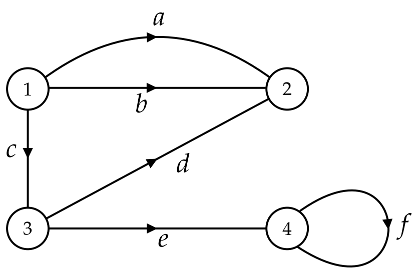
$$\begin{array}{c|cccccc|c} v & a & b & c & d & e & f & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ \hline v^T A & 0 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

Proof:
Observe that if $v^T A = 0$, then v is constant on each component. Thus the dimension of the left null space of A is the number of components of G. Therefore, $\text{rank}(A) = |V| -$ # components of G. Thus $M(G) = M(A)$. □

Theorem 1.2

Graphic matroids are representable over all fields.

G



Instead of incidence matrix, we consider the signed-incidence matrix:

$$\begin{array}{c|cccccc|} & a & b & c & d & e & f \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}$$

To construct a signed-incidence matrix, we first put an orientation on each edge, then construct the matrix with ± 1. The orientation is arbitrary. If we swap the orientation of edge a over, we are just swapping 1 and -1 in the first column, corresponding to scaling this column by -1, which doesn't change the matroid. Every field has $0, \pm 1$, so we can do this in every field. Entry $4f$ is achieved by $0 = 1 - 1$.

Exercise:
If A is the signed incidence matrix of G, then $M(A) = M(G)$ over any field.

1.3 Alternative definitions of matroids

We defined matroid by their independent sets. However, we can recover $\mathcal{I}(M)$ from any the following:

- circuits of M (a **circuit** is a minimal dependent set)
- bases of M (a **basis** is a maximal independent set)
- rank function, r_M, of M

- flats of M (a **rank-k flat** is a maximal set of rank-k)
- hyperplanes of M (a **hyperplane** is a flat of rank $r(M) - 1$)

Each of these give rise to a "cryptomorphically equivalent" definition of a matroid.

1.3.1 Circuits

A **circuit** is a minimal dependent set. In the cycle-matroid of a graph, a circuit is the edge set of a cycle.

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