Partial Differential Equations 2

AMATH 453

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LaTeXed by *Sibelius Peng*

Preface

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Sibelius Peng

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Waves and Diffusions

1.1 The wave equation

We already know the wave equation ($c > 0$):

$$u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < \infty,$$

and the general solution is of the form

$$u(x,t) = f(x+ct) + g(x-ct).$$

With initial conditions imposed, we have the IVP

$$u_{tt} - c^2 u_{xx} = 0, \qquad \begin{cases} u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x). \end{cases}$$

The solution to IVP is then

$$u(x) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

To interpret the integral, we can let $\psi(x) = \mu'(x)$, then the integral becomes

$$\int_{x-ct}^{x+ct} \psi(s)\, ds = \mu(x+ct) - \mu(x-ct).$$

1.2 Conservation laws

Given a wave equation, we multiply by u_t:

$$\begin{gathered} u_t u_{tt} - c^2 u_t u_{xx} = 0 \\ \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2\right) - c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{tx}u_x\right] = 0 \\ \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right) - \frac{\partial}{\partial x}\left(c^2 u_t u_x\right) = 0 \end{gathered}$$

Then the conservation law states that

$$\frac{\partial R}{\partial t} + \frac{\partial F}{\partial x} = 0\,,$$

where $R \in (-\infty, +\infty)$, and $F \to 0$ with $x \to \pm\infty$.

1.3 The Diffusion Equation & Maximum principle

The diffusion equation is given by

$$u_t = ku_{xx}, \qquad -\infty < x < \infty$$

with diffusion constant $k > 0$.

We define

$$\begin{aligned} R &= (a,b) \times (0,\infty) \\ R_T &= (a,b) \times (0,T] \\ \overline{R_T} &= [a,b] \times [0,T] \\ C_T &= \{a \leq x \leq b, t = 0\} \cup \{a, 0 \leq t \leq T\} \cup \{b, 0 \leq t \leq T\} \end{aligned}$$

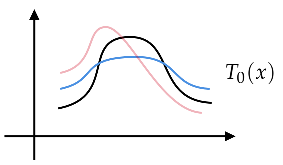
Theorem 1.1: Maximum principle

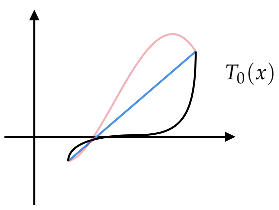
If $u \in C(\overline{R_T}) \cap C^2(R_T)$ is a solution of the diffusion equation, then $u(x,t) \leq \max_{C_T}\{u\}$ for all $(x,t) \in R_T, T > 0$. Here C_T is called the parabolic boundary of R_T.

Remark:

1. We can replace $u_t - ku_{xx} = 0$ with $u_t - ku_{xx} \leq 0$.
2. A stronger version of the theorem exists which says that $u(x,t) < \max_{C_T}\{u\}$ unless u is constant.
3. Same result applies to the minimum of u by replacing u with $-u$. However, in this case, (1) doesn't apply. Now we need $u_t - ku_{xx} \geq 0$.

Here are some intuitions. Consider a rod lying on $[a,b]$ with initial non-constant temperature $T_0(x)$. Then as time goes, only blue T is possible, not red T.





Proof:
Let $M = \max_{C_T} u$. Note that M exists since u is continuous on C_T, and C_T is a closed boundary. We need to show that $u \leq M$ on $\overline{R_T}$.

Let

$$v(x,t) = u(x,t) + \epsilon x^2, \quad \epsilon > 0$$

Let $r = \max\{|a|, |b|\}$. Then $v(x,t) \leq M + \epsilon r^2$ on C_T. Now we prove that $v \leq M + \epsilon r^2$ on R_T.

On R_T, we have

$$u = v - \epsilon x^2 \leq M + \epsilon(r^2 - x^2)$$

Now if we take the derivative,

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon = -2k\epsilon < 0 \tag{*}$$

(i) Suppose $v(x,t)$ has a maximum at an interior point (x_0, t_0), i.e., $(x_0, t_0) \in (a,b) \times (0,T)$. Then

$v_t(x_0, t_0) = 0$. Moreover, $v_{xx}(x_0, t_0) \leq 0$. Then

$$v_t(x_0, t_0) - kv_{xx}(x_0, t_0) = -kv_{xx}(x_0, t_0) \geq 0$$

contradicting (*), thus there are no interior max.

(ii) Suppose $v(x, t)$ has a maximum at an interior point of the upper boundary. $v_t(x_0, T) \geq 0$. Then

$$v_t(x_0, t_0) - kv_{xx}(x_0, t_0) \geq 0$$

contradicting (*), thus there are no maximum along the upper boundary.

But v is continuous on $\overline{R_T}$, thus it has a maximum value which we now know must occur on C_T. Hence $v \leq M + \epsilon r^2$ on $\overline{R_T}$. Letting $\epsilon \to 0$, we have $u \leq M$ on R_T. □

1.4 Uniqueness of the Dirichlet Problem

$$\begin{aligned} u_t - ku_{xx} &= f(x, t) \qquad a < x < b, 0 < t < \infty \\ u(x, 0) &= \phi(x) \\ u(a, t) &= g(t) \\ u(b, t) &= h(t) \end{aligned} \tag{1.1}$$

Theorem 1.2

The solution of (1.1) is unique.

Proof:
Suppose there are two solutions $u_1(x, t)$ and $u_2(x, t)$. Let $w(x, t) = u_1 - u_2$. Now we calculate

$$\begin{aligned} w_t - kw_{xx} &= (u_{1t} - ku_{1xx}) - (u_{2t} - u_{2xx}) = f - f = 0 \\ w(x, 0) &= u_1(x, 0) - u_2(x, 0) = \phi - \phi = 0 \\ w(a, t) &= w(b, t) = 0 \end{aligned}$$

By maximum principle, we have $w \leq 0$ on the boundary, and my minimum principle, $w \geq 0$, since $\max_{C_T}\{w\} = \min_{C_T}\{w\} = 0$. Then we conclude that $w \equiv 0$. □

Now we present a second proof using energy method:

Proof:
Given $w_t - kw_{xx} = 0$, multiply both sides by w:

$$0 = ww_t - kww_{xx} = \frac{\partial}{\partial t}\left(\frac{1}{2}w^2\right) - k\frac{\partial}{\partial x}(ww_x) + kw_x^2$$

If we integrate both sides,

$$\frac{d}{dt}\int_a^b \frac{1}{2}w^2 \,\mathrm{d}x = k\int_a^b (ww_x)_x \,\mathrm{d}x - k\int_a^b w_x^2 \,\mathrm{d}x = kww_x\Big|_a^b - k\int_a^b w_x^2 \,\mathrm{d}x$$

Thus

$$\frac{d}{dt}\int_a^b \frac{1}{2}w^2 \,\mathrm{d}x = -k\int_a^b w_x^2 \,\mathrm{d}x$$

Then

$$\int_a^b \frac{1}{2}w^2 \,\mathrm{d}x = 0 \qquad \text{for all the time}$$

Then $w \equiv 0$ on $a \leq x \leq b, 0 \leq t \leq T$. □

Now let's examine stability. Consider

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(a,t) = u(b,t) &= 0 \end{aligned}$$

and let $u_j(x,t)$ be the solution for $u(x,0) = \phi_j(x)$ for $j = 1, 2$.

Let $w = u_1 - u_2$. Proceeding as before (energy method) we have

$$\int_a^b (u_1 - u_2)^2 \, \mathrm{d}x \le \int_a^b (\phi_1 - \phi_2)^2 \, \mathrm{d}x$$

This tells us $\|u_1 - u_2\|_2 \to 0$ as $\|\phi_1 - \phi_2\|_2 \to 0$. This is called **stability in the square integrable sense**.

Alternatively, by maximum principle,

$$\max |u_1 - u_2| \le \max |\phi_1 - \phi_2|$$

using maximum & minimum principle, i.e.,

$$\begin{aligned} \max\{u_1 - u_2\} &\le \max\{\phi_1 - \phi_2\} \\ \min\{u_1 - u_2\} &\ge \min\{\phi_1 - \phi_2\} \end{aligned}$$

This is called **stability in the uniform sense**.

1.5 Diffusion on the Whole Line

Consider the initial value problem

$$u_t - ku_{xx} = 0 \qquad \text{on } -\infty < x < \infty, \quad 0 < t < \infty \tag{1.2}$$

$$u(x,0) = \phi(x) \tag{1.3}$$

If $s(x,t)$ is a solution of (1.2), then so is

$$u(x,t) = \int_{-\infty}^{\infty} s(x-y,t) g(y) \, \mathrm{d}y \tag{1.4}$$

for any function $g(y)$. We can find u_t, u_x, u_{xx} and take it into (1.2):

$$u_t - ku_{xx} = \int_{-\infty}^{\infty} \Big[s_t(x-y,t) - ks_{xx}(x-y,t) \Big] g(y) \, \mathrm{d}y = 0$$

So we now find a solution of (1.2) with the property that $s(x,0) = \delta(x)$, i.e., solve

$$\begin{aligned} s_t - ks_{xx} &= 0 \\ s(x,0) &= \delta(x) \end{aligned}$$

To do this, consider the problem:

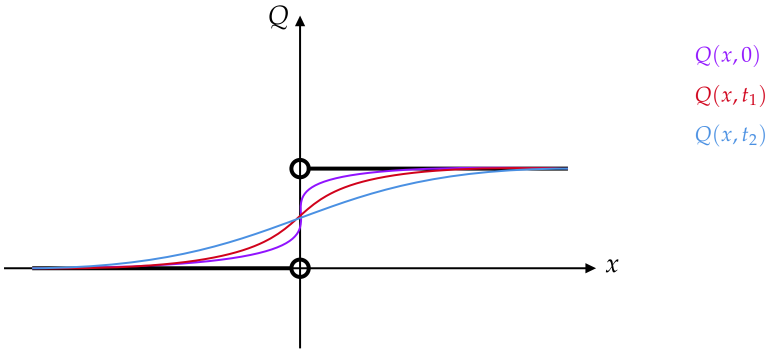
$$\begin{aligned} &v_t - kv_{xx} = 0 \\ &v(x,0) = v_0 H(x) \\ &H = \text{Heaviside function} \end{aligned} \tag{1.5}$$

v_0 carries the dimension of v, thus $H(x)$ is dimensionless.

Similarity solution of (1.5)

Let $Q = \frac{v}{v_0}$ which is dimensionless, then the original problem gets transformed to

$$\begin{aligned} Q_t &= kQ_{xx} \\ Q(x,0) &= H(x) \end{aligned}$$



The solution can only be a function of x, t and k: $Q = F(x,t,k)$. Then we can apply dimensionless analysis. This means Q can only depend on dimensionless combinations of x, t and k. We have

$$\begin{aligned} [x] &= L \\ [t] &= T \\ [k] &= \frac{L^2}{T} \end{aligned}$$

Then

$$[x^a t^b k^c] = L^a T^b \frac{L^{2c}}{T^c} \implies b = c,\ 2c = -a$$

This tells us

$$Q = f(\theta) \qquad \text{where } \theta = \frac{x}{\sqrt{kt}}$$

By chain rule, we have

$$\begin{aligned} Q_t &= f'(\theta) \cdot \theta_t = -\frac{1}{2}\frac{\theta}{t} f'(\theta) \\ Q_x &= f'(\theta) \cdot \theta_x = \frac{1}{\sqrt{kt}} f'(\theta) \\ Q_{xx} &= \frac{1}{kt} f''(\theta) \end{aligned}$$

Then

$$\begin{aligned} Q_t - kQ_{xx} &= -\frac{\theta}{2t} f' - \frac{k}{kt} f'' = 0 \\ f''(\theta) &= -\frac{1}{2}\theta f'(\theta) \\ f'(\theta) &= Ae^{-\frac{\theta^2}{4}} \\ f(\theta) &= A\int_{-\infty}^{\theta} e^{-s^2/4}\, \mathrm{d}s + C \end{aligned}$$

As $x \to +\infty$, $\theta \to +\infty$, and $Q(x,t) = f(\theta) \to 1$. Then $\lim_{\theta\to+\infty} f(\theta) = 1$.
As $x \to -\infty$, $\theta \to -\infty$ and $Q(x,t) = f(\theta) \to 0$, $\lim_{\theta\to-\infty} f(\theta) = 0$.

Therefore, C must be 0, and $A\int_{-\infty}^{\infty} e^{-s^2/4}\, \mathrm{d}s = 1$. Using the change of variable $\eta = \frac{s}{2}$:

$$\int_{-\infty}^{\theta} e^{-s^2/4}\, \mathrm{d}s = 2\int_{-\infty}^{\theta/2} e^{-\eta^2}\, \mathrm{d}\eta = 2\int_{-\infty}^{x/\sqrt{4kt}} e^{-\eta^2}\, \mathrm{d}\eta$$

So if we take $\theta = \dfrac{x}{\sqrt{4kt}}$ at the beginning, we get $\tilde{A} = 2A$ and

$$\tilde{A} \int_{-\infty}^{\infty} e^{-s^2}\, \mathrm{d}s = 1 \implies \tilde{A} = \frac{1}{\sqrt{\pi}}$$

Thus we get

$$Q = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-s^2}\, \mathrm{d}s$$

Note that for $x > 0$, as $t \to 0^+$, $\dfrac{x}{\sqrt{4kt}} \to +\infty$ and $Q(x,t) \to \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-s^2}\, \mathrm{d}s = 1$.

And for $x < 0$ as $t \to 0^+$, $Q \to 0$. The reason for the name "similarity solution" is because the curve is being stretched over time.

$s(x,t)$ has many names: source function (not a great name), Green's function, fundamental solution, propagator of the diffusion equation, diffusion kernel...

Consider a diffusion equation with initial condition

$$\begin{aligned} u_t + k u_{xx} &= 0 \\ u(x,0) &= \delta(x) \end{aligned}$$

The solution is Gaussian

$$u = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$$

For any $t > 0$, u is non-zero. It gets instantaneously non-zero everywhere.

1.6 Reflections and Sources

We will start with diffusion on the half line Dirichlet problem.

$$\begin{aligned} v_t - k v_{xx} &= 0 \qquad 0 < x < \infty,\, 0 < t < \infty \\ v(x,0) &= \phi(x) \\ v(0,t) &= 0 \quad \text{for } t > 0 \end{aligned}$$

Let

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}$$

and solve

$$\begin{aligned} u_t + k u_{xx} &= 0 \qquad \text{on } -\infty < x < \infty \\ u(x,0) &= \phi_{odd}(x) \end{aligned}$$

Then $v(x,t)$ is restriction of u to $x > 0$. From an earlier result

$$u(x,t) = \int_{-\infty}^{\infty} s(x-y,t) \phi_{odd}(y)\, \mathrm{d}y$$

where

$$s(x,t) = \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi k t}}$$

Claim From the property of s and ϕ_{odd}, we can show that $u(x,t)$ is an odd function of x. Thus $u(0,t)=0$.

Now we see that

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{0} s(x-y,t)[-\phi(-y)]\,\mathrm{d}y + \int_{0}^{\infty} s(x-y,t)\phi(y)\,\mathrm{d}y \\ &= \int_{\infty}^{0} s(x+y,t)\phi(y)\,\mathrm{d}y + \int_{0}^{\infty} s(x-y,t)\phi(y)\,\mathrm{d}y \qquad \text{let } y=-y \\ &= \frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty}\left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right]\phi(y)\,\mathrm{d}y \end{aligned} \tag{1.6}$$

Example:

$$\begin{aligned} v_t - kv_{xx} &= 0 \qquad 0<x<\infty \\ v(x,0) &= 1 \qquad x>0 \\ v(0,t) &= 0 \end{aligned}$$

Then $\phi_{odd} = -1 + 2H(x)$.

Recall the solution of

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x,0) &= H(x) \end{aligned} \tag{1.7}$$

is

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4k\pi}} e^{-s^2}\,\mathrm{d}s$$

Let $u(x,t) = -1 + 2q(x,t)$. Then $q(x)$ is the solution to (1.7). Hence we have

$$u = -1 + \frac{2}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4k\pi t}} e^{-s^2}\,\mathrm{d}s = \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Another way to solve is to use (1.6).

Consider Neumann Boundary condition ($0<x<\infty$):

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x,0) &= \phi(x) \\ u_x(0,t) &= 0 \end{aligned}$$

We can let

$$\phi_{even} = \begin{cases} \phi(x) & x>0 \\ \phi(-x) & x<0 \end{cases}$$

and solve

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x,0) &= \phi_{even} \end{aligned}$$

With some algebra, we get

$$u = \frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty}\left[e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}\right]\phi(y)\,\mathrm{d}y$$

Reflections of Waves

Dirichlet Problem on the half line

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 \qquad\qquad 0 < x < \infty \\ v(x,0) &= \phi(x) \\ v_t(x,0) &= \psi(x) \\ v(0,t) &= 0 \end{aligned}$$

The idea is $u(-x,t) = -u(x,t)$, then $u = 0$ at $x = 0$. So consider an odd reflection about $x = 0$:

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases} \qquad\qquad \psi_{odd} = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \end{cases}$$

We know that the solution of $(-\infty < x < \infty)$

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(x,0) &= \phi_{odd}(x) \\ u_t(x,0) &= \psi_{odd}(x) \end{aligned}$$

is

$$u(x,t) = \frac{1}{2}\Big[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\ \mathrm{d}y$$

Note that $(t > 0)$

$$u(0,t) = \frac{1}{2}\Big[\phi_{odd}(ct) + \phi_{odd}(-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\ \mathrm{d}y = 0$$

which satisfies the initial condition.

3 cases of the solution

(a) $x > c|t|$, then $x + ct > 0, x - ct > 0$, then the solution $(t > 0)$ becomes

$$u(x,t) = \frac{1}{2}\Big[\phi(x+ct) + \phi(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\ \mathrm{d}y$$

(b) Consider $0 < x < ct$, $t > 0$, we have $x - ct < 0, x + ct > 0$. Then

$$\begin{aligned} \phi_{odd}(x-ct) &= -\phi(-x+ct) \\ \phi_{odd}(x+ct) &= \phi(x+ct) \end{aligned}$$

and

$$\begin{aligned} \int_{x-ct}^{x+ct} \psi_{odd}(y)\ \mathrm{d}y &= \int_{x-ct}^{0} [-\psi(-y)]\ \mathrm{d}y + \int_0^{x+ct} \psi(y)\ \mathrm{d}y \\ &= -\int_0^{-x+ct} \psi(y)\ \mathrm{d}y + \int_0^{x+ct} \psi(y)\ \mathrm{d}y \\ &= \int_{-(x-ct)}^{x+ct} \psi(y)\ \mathrm{d}y \end{aligned}$$

Therefore

$$u = \frac{1}{2}\Big[\phi(x+ct) - \phi(-(x-ct))\Big] + \frac{1}{2c}\int_{-(x-ct)}^{x+ct} \psi(y)\ \mathrm{d}y$$