Matroid Theory

CO 446

Jim Geelen

LATEXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 446 during Spring 2021 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

This course is an introduction to matroid theory for graph theorists. Tree, cycle, vertex connectivity, minors, planar duality extend to matroids.

We will generalize

- Hall's Theorem (matching in bipartite graphs),
- Menger's Theorem (disjoint paths),
- Tutte's Wheel's Theorem (3-connectivity),
- Jaeger's Theorem (flows),
- Kuratowski's Theorem (planar graphs).

We also prove Tutte's Theorem (matching). We also find analogues of Ramsey's Theorem, Turan's Theorem, Erdős-Stone Theorem (maybe).

For any questions, send me an email via https://notes.sibeliusp.com/contact.

You can find my notes for other courses on https://notes.sibeliusp.com/.

Sibelius Peng

Contents

1

Matroid

What is a matroid?

matroid

A **matroid** is a pair $(E, \mathcal{I})$ consisting of a finite set E, called the **ground set**, and a collection $\mathcal{I}$ of subsets of E, called **independent sets**, satisfying

(I1) the empty set is independent,

(I2) subsets of independent sets are independent, and

(I3) for each $X \subseteq E$, all maximal independent subsets of X have the same size, denoted $r_M(X)$ or $r(X)$; this is called the **rank** of X.

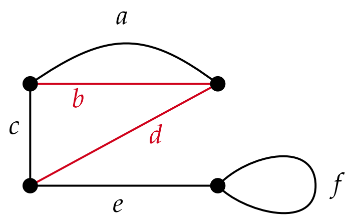
We are using the following notations. For a matroid $M = (E, \mathcal{I})$ we write:

- $E(M)$ for E,
- $\mathcal{I}(M)$ for $\mathcal{I}$,
- $|M|$ for $|E(M)|$, and
- $r(M)$ for $r(E(M))$.

1.1 Examples

1.1.1 Cycle-matroid of graphs

G



$\{b, d\} \in \mathcal{F}$

Let $G = (V, E)$ be a graph. Define $M(G) := (E, \mathcal{F})$ where $\mathcal{F}$ is the collection of all edge-sets that induce a forest in G.

Then we can check $M(G)$ is a matroid:

(I1) Clearly, empty set is acyclic, then a forest.

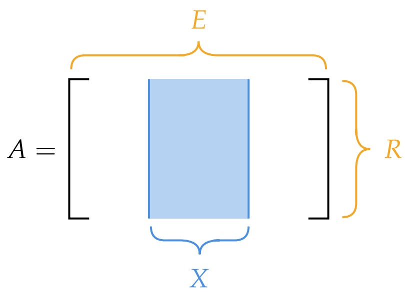
(I2) If we throw away edges from a forest, it is still a forest.

(I3) If we build a forest in a greedy way in a connected graph, we end up with a spanning tree, which is of the same size.

What is $r_M(X)$? $r_M(X) = |V| - \#$ of components of $G[V, X]$, which denotes the subgraph containing all the vertices and the edges in X.

We call $M(G)$ the **cycle-matroid** of G. A matroid is **graphic** if it is the cycle-matroid of some graph.

1.1.2 The column-matroid of a matrix



$A \in \mathbb{F}^{R \times E}$ where $\mathbb{F}$ is a field and R and E are finite sets. The **column-matroid** of A is $M(A) := (E, \mathcal{I})$ where $\mathcal{I}$ is the collection of all sets that index a set of linearly independent columns.

$M(A)$ is a matroid:

(I1) Trivial.

(I2) Trivial.

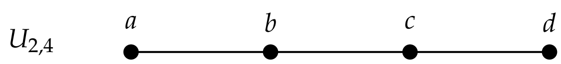
(I3) From linear algebra.

Remark:

The rank of a set $X \subseteq E$ is the rank of the submatrix $A[R, X]$.

- We call $M(A)$ the **column-matroid** of A.
- A matroid is **$\mathbb{F}$-representable** if it is the column-matroid over a matrix over the field $\mathbb{F}$.
- We abbreviate $GF(2)$-representable to **binary**.
- A matroid is **representable** if it is $\mathbb{F}$-representable over some field $\mathbb{F}$.

1.1.3 The 4-point line



$$E(U_{2,4}) = \{a, b, c, d\}, \qquad \mathcal{I}(U_{2,4}) = \{\text{all sets of size at most 2}\}$$

Claim $U_{2,4}$ is not binary.

Proof:

There are only three distinct non-zero vectors in $GF(2)^2$. $\square$

Index

C

F

G

I

M

R