Deterministic OR Models

CO 370

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Preface

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What is operations research (OR)? There's no standard definitions for it. One particular definition: use of mathematical models to make complex decisions for real life problems. The origin is British military in WW2. OR is actually everywhere today. Key milestone: Simplex algorithm (1947).

Recall optimization problem is of the form:

$$\begin{aligned} \max \quad & f(x) \\ \text{s.t.} \quad & \text{a set of constraints} \end{aligned}$$

There are some applications: mail delivery, machine scheduling, inventory problem, network design, facility location, class scheduling, portfolio optimization, surgery planning, sensor location.

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LP formulations

1.1 Production problem

Products $J = \{1, \ldots, n\}$
Resources $I = \{1, \ldots, m\}$

Data:

- $\forall j \in J : c_j =$ value of unit of product j
- $\forall i \in I : b_i =$ number of units of resource i available
- $\forall i \in I, \forall j \in J : a_{ij} =$ number of units of resource i going to product j

Goal: maximize values of product made subject to available resources

Var: $x_j =$ number of units of product j produced

Then problem is

$$\begin{array}{rll} \max & \sum_{j \in J} c_j x_j & \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \leq b_i & (i \in I) \\ & x_j \geq 0 & (j \in J) \end{array}$$

Now let's generalize this problem to have more than one period.

Products $J = \{1, \ldots, n\}$
Resources $I = \{1, \ldots, m\}$
Periods $K = \{1, \ldots, p\}$

Then we have **data**

- $\forall j \in J, k \in K$: $c_{jk} =$ unit value of product j in period k
- $\forall i \in I, k \in K$: $w_{ik} =$ unit price for resource i in period k
- $\forall i \in I, j \in J$: $a_{ij} =$ number of units of resource i going into product j

and the **goal**: decide how much of each resource to buy & how much of each product to make during each period, to maximize total profit. Unused resources are available at next time period.

Var:

- p_{ik} = number of units of resource i, purchased at start of period k
- x_{jk} = number of units of product j made in period k
- z_{ik} = number of units of resource i at the end of period k

$$\text{Profit} = \sum_{k\in K}\left[\sum_{j\in J} c_{jk}x_{jk} - \sum_{i\in I} w_{ik}p_{ik}\right] \tag{1.1}$$

Then we keep track of resources: for $i \in I$, $k \in K$

$$z_{ik} = z_{i(k-1)} + p_{ik} - \sum_{j\in J} a_{ij}x_{jk} \tag{1.2}$$

and we define for $i \in I$,

$$z_{i0} = 0 \tag{1.3}$$

Thus the optimization problem is

$$\begin{aligned} \max \quad & (1.1) \\ \text{s.t.} \quad & (1.2), (1.3) \\ & p, x, z \geq 0 \end{aligned} \tag{P}$$

Remark:
If (P) has a feasible solution of value that is bigger than 0, then (P) is unbounded. So we are missing some assumptions, maybe? For example, b_{ik} = amount of resource i that can be bound during period k. Then we can add constraints: $p_{ik} \leq b_{ik}$.

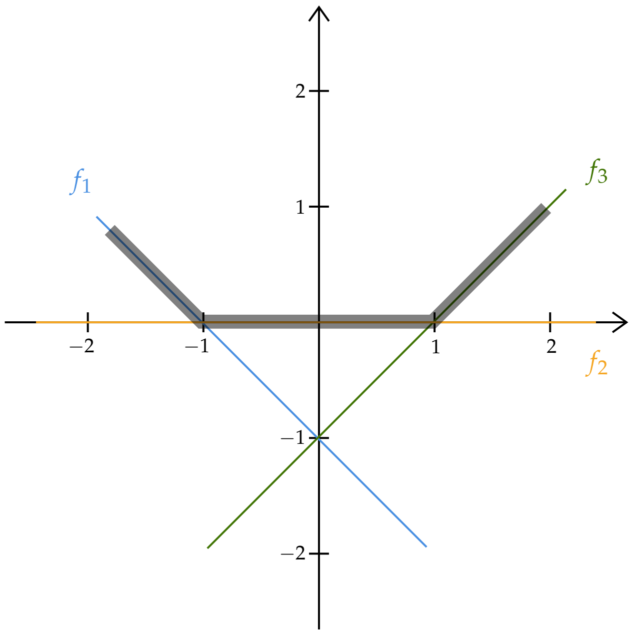
1.2 Minimax

Consider the problem of the form

$$\begin{aligned} &\min_x \max\{f_1(x), \ldots, f_k(x)\} := g(x) \\ &\text{s.t.} \ \cdots \end{aligned}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$.

Example:
$f_1(x) = -x - 1, f_2(x) = 0, f_3(x) = x - 1$. Then $\max\{f_1(x), f_2(x), f_3(x)\}$ is as follows



A motivation

- $\forall i \in [k]$, $f_i(x)$ = completion time for task i.
- Project consists of task $1, \ldots, k$.
- $g(x)$ = completion time of entire project

Note that minimax is not an optimization problem as we defined it. We can revise it as follows

$$\begin{array}{rl} \min & y \\ \text{s.t.} & y \geq f_1(x) \\ & y \geq f_2(x) \\ & \vdots \\ & y \geq f_k(x) \\ & \cdots \end{array}$$

An application minimize a piece-wise linear convex function using linear programming.

2 Flows

digraph

A directed graph (digraph) is a pair (V, E) where

- V is a set of vertices,
- E is a set of ordered pairs of vertices called arcs.

Notation Let $q \in V$, then

$$\delta^+(q) = \{e \in E \mid e \text{ leaves } q\}$$
$$\delta^-(q) = \{e \in E \mid e \text{ arrives at } q\}$$

2.1 Max st-flow model

Given

1. digraph $G = (V, E)$,
2. two vertices $s, t \in V$, and $s \neq t$,
3. $\forall e \in E$, arc e has capacity $u_e \geq 0$.

Now we construct an LP.

For every arc e, we will have a variable x_e, and x_e will be called the **flow** on arc e.

Notation Let $q \in V$:

$$f_x(q) := \sum_{e \in \delta^+(q)} x_e - \sum_{e \in \delta^-(q)} x_e$$

The maximization problem is then

$$\begin{array}{rll} \max & f_x(s) & \\ \text{s.t.} & f_x(q) = 0 & (q \in V, q \neq s, q \neq t) \\ & 0 \leq x_e \leq u_e & (e \in E) \end{array} \tag{P}$$

A feasible solution to (P) is a flow. An optimal solution to (P) is a maximum flow. The value of a flow x is $f_x(S)$.

Remark:
(P) is always feasible. It is not unbounded, so there always exists a maximum flow.

Application computer network. Suppose we have

- computers s and t ($s \neq t$)
- capacity u_e (gb/s) for every link e

The goal is to computer number gb that can be sent from s to t across network. Then x_e is the amount of information across e. $f_x(q) = 0$ means no information lost.

Magic property

If u is integer, then there exists an optimal solution to (P) that is integer.

Remark:
We need the condition "u is integer" so that the property is still true. Also, an optimal solution to (P) is not necessarily integers.

Generalize max st-flows

We can add lower bounds to arcs: $\ell_e \ \forall e \in E$

$$\begin{array}{rll} \max & f_x(s) & \\ \text{s.t.} & f_x(q) = 0 & (q \in V, q \neq s, q \neq t) \\ & \ell_e \leq x_e \leq u_e & (e \in E) \end{array} \tag{Q}$$

Magic property - revised

If ℓ, u is integer, and there exists an optimal solution to (Q), then there exists an optimal solution to (Q) that is integer.

Example: Consistent rounding
The goal is to round all entries to nearest up/down integer, so that row sums & column sums still hold.

Any feasible solutions give consistent rounding.

2.2 Min cost flow model

Given

- Digraph $G = (V, E)$
- Capacities $u_e \geq 0$ ($e \in E$)
- Costs c_e ($e \in E$)
- Supply/demands b_q ($q \in V$)

Then the model is

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ \text{s.t.} & f_x(q) = b_q & (q \in V) \\ & 0 \le x_e \le u_e & (e \in E) \end{array} \tag{P}$$

Similarly, feasible solution to (P) is a flow. An optimal solution to (P) is a min cost flow.

Magic property - min cost flows

Suppose u, b are integer, then if there exists a min cost flow, then there exists a min cost flow that is integer.

Similarly, we can add a lower bound:

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ \text{s.t.} & f_x(q) = b_q & (q \in V) \\ & \ell_e \le x_e \le u_e & (e \in E) \end{array} \tag{P}$$

Magic property - min cost flows (revised)

Suppose u, b, ℓ are integer, then if there exists a min cost flow, then there exists a min cost flow that is integer.

What is a necessary condition for b_q so that there exists a flow? $\sum_{q \in V} b_q = 0$.

Example: Staffing problem

hours	# of employees required
1-2	2
2-3	3
3-4	3
4-5	2

and we have cost of hiring a single employee between

hours	cost
1-5	6
1-4	4
3-5	5
2-4	3

The goal is to minimize cost of hiring employees while meeting staff needs.

3

IP formulations

3.1 IP tricks

Imaging we are forcing variable to take some prescribed set of values: $x \in \{5, 9, 13, 36\}$. Then we can introduce variables $z_1, z_2, z_3, z_4 \in \{0, 1\}$ so that we have two constraints:

$$\begin{aligned} 1 &= z_1 + z_2 + z_3 + z_4 \\ x &= 5z_1 + 9z_2 + 13z_3 + 36z_4 \end{aligned}$$

3.2 Modeling piece-wise linear function

Let $f : \mathbb{R} \to \mathbb{R}$ be piecewise linear. Given $a_1, \ldots, a_k$ and $f_1, \ldots, f_k$ such that $f(a_i) = f_i$. The goal is to write IP constraints with variables x, y such that $y = f(x)$, $x \in \underset{\substack{\uparrow \\ \mathrm{dom}(f)}}{[a_1, a_k]}$.

To generalize,

1. $\lambda_1, \ldots, \lambda_k \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$
2. $x = \sum_{i=1}^{k} \lambda_i a_i$
3. $y = \sum_{i=1}^{k} \lambda_i f_i$
4. $z_0, \ldots, z_k \in \{0, 1\}$, $z_0 = z_k = 0$
5. $\sum_{i=0}^{k} z_i = 1$
6. $\forall p \in [k]$, $\lambda_p \leq z_{p-1} + z_p$

By (4) and (5), we may assume that $z_p = 1$ and $z_j = 0\ \forall j \neq p$.

Claim λ_p, λ_{p+1} are the only non-zero λ variables.

Proof:
Pick $j \neq p, j \neq p+1$. As

$$0 \leq \lambda_j \leq z_{j-1} + z_j = 0 + 0 = 0$$

□

With this claim, we can simplify (2) and (3).

3.3 Modeling union of polyhedra

Let

$$P_1 = \{x \mid A^1 x \leq b^1\}$$
$$P_2 = \{x \mid A^2 x \leq b^2\}$$

Goal: write condition: $x \in P_1 \cup P_2$ as part of IP.

Hypothesis: If $x \in P_1 \cup P_2$, then $0 \leq x \leq U$ for some U.

Constraints:

1. $y_1, y_2 \in \{0, 1\}$
2. $y_1 + y_2 = 1$
3. $x = x^1 + x^2$
4. $A^i x^i \leq y_i b^i$, $i = 1, 2$
5. $0 \leq x^i \leq y_i U$, $i = 1, 2$

To show that $x \in P_1 \cup P_2 \iff \exists x^1, x^2, y_1, y_2$ such that all 5 conditions hold.

Proof:
First let's assume (1) - (5) hold. We may assume $y_1 = 1, y_2 = 0$. Then (5) tells us $x^2 = 0$. (3) tells us $x = x^1$. (4) implies

$$A^1 x^1 \leq y_1 b^1 \implies A^1 x \leq b^1 \implies x \in P_1 \subseteq P_1 \cup P_2$$

Now suppose $x \in P_1 \cup P_2$. We may assume $x \in P_1$. Then we set $y_1 = 1, y_2 = 0, x^1 = x, x^2 = 0$. Then we can verify that all 5 conditions hold. □

3.4 Perfect formulations

basis

Let A be a matrix with column indices $\{1, \ldots, n\}$. Then $B \subseteq \{1, \ldots, n\}$ is a basis if

1. A_B square,
2. A_B non-singular.

Remark:
A has a basis $\iff$ rows of A are independent.

basic solution

Let A be a matrix with column indices $1, \ldots, n$. Consider

$$Ax = b \tag{$*$}$$

Pick B as basis of A. Then x is a basic solution of $(*)$ if

1. $Ax = b$.
2. $x_j = 0$ if $j \notin B$.

x is a basic solution for $(*)$ if x is a basic solution for some basis B.

standard equality form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and rows of A are independent.

The correctness of simplex algorithm implies the following theorem:

Theorem 3.1

If an LP is SEF has an optimal solution, then it has an optimal solution that is basic.

Let A be matrix, b vector with same number of entries as rows of A. Then $A \leftarrow_j b$ denotes matrix obtained from A by replacing column j by b.

Theorem 3.2: Cramer's rule

Let M be a non-singular matrix and consider $Mx = b$.

$$\bar{x}_j = \frac{\det(M \leftarrow_j b)}{\det(M)} \quad \forall j$$

then $M\bar{x} = b$.

Proposition 3.3

Let M be square matrix with $\det(M) = \pm 1$, and M, b are integer. Then there exists a unique solution to $Mx = b$ is integer.

Proof:
Directly from Cramer's rule. □

Totally unimodular matrix

A matrix A is totally unimodular if every square submatrix N of A, $\det(N) \in \{0, +1, -1\}$.