



Rings and Fields

PMATH 334



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Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of PMATH 334 during Winter 2022 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

For the table of contents, I am most likely following Dummit, Foote: *Abstract algebra*.

For any questions, send me an email via <https://notes.sibeliusp.com/contact>.

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Contents

Preface	1
1 Introduction & Motivation	3
1.1 Fermat's Last Theorem	3
1.2 Straightedge and compass construction	3
2 An introduction to Rings	5
2.1 Definitions and basic properties	5
2.2 Zero divisor and integral domain	6
2.3 Field	7
2.4 Subring	8
2.5 Unit	8
3 Ring Homomorphisms	10
3.1 Quotient rings	11

Introduction & Motivation

1.1 Fermat's Last Theorem

Fermat's Last Theorem

The equation $x^m + y^m = z^m$ has no non-trivial solutions in integers for $m \geq 3$.

For example, $(1, 0, 1)$, $(-1, 0, 1)$ for m even, are trivial solutions.

In 1897, Gabriel Lamé announced that he has a proof. First he assumed that m is a prime. He writes

$$z^p = x^p + y^p = (x + y)(x + \zeta_p y)(x + \zeta_p^2 y) \cdots (x + \zeta_p^{p-1} y)$$

where $\zeta_p = \cos(\frac{2\pi}{p}) + i \sin(\frac{2\pi}{p})$. Consider the ring

$$\mathbb{Z}[\zeta_p] = \{a_1 + a_2 \zeta_p + a_3 \zeta_p^2 + \cdots + a_{p-2} \zeta_p^{p-2} : a_i \in \mathbb{Z}\}$$

which is the smallest ring containing \mathbb{Z} and ζ_p .

Then the next step is to show that $(x + \zeta_p^j y)$'s are coprime in $\mathbb{Z}[\zeta_p]$. Let q_i 's be primes.

$$\prod_i q_i^{p\alpha_i} = z^p = (x + y)(x + \zeta_p y) \cdots (x + \zeta_p^{p-1} y)$$

If $(x + \zeta_p^j y)$'s are coprime in $\mathbb{Z}[\zeta_p]$, then $(x + \zeta_p^j y) = (\cdots)^p$ is of p -th power (*). But this is wrong if the factorization is non-unique. However, we have $\mathbb{Z}[\zeta_p]$ can be a unique factorization domain (UFD). This means (*) works. Kummer salvages the argument for approximately (conjecturally) 60% of prime exponents. And these primes are called **regular primes**.

1.2 Straightedge and compass construction

We are given a length 1 straightedge ruler, and a compass. With these, we can

- connect two points with a straightedge,
- draw a circle, centered at A , and going through B ,
- draw intersections of two line segments, circle & line, two circles.

What lengths are constructible? where length means distance between two points. We can do $+$, $-$, \times , \div , $\sqrt{}$. Then we can do field extensions:

$$\mathbb{Q} \rightarrow \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3}) \rightarrow \dots$$

Is trisection of an angle doable? No, **not possible**.

Possible to **double the cube**, **square the circle** of the same area?

What regular m -gons are constructible? This is equivalent to the question: is $\cos(\frac{2\pi}{m}) + i \sin(\frac{2\pi}{m})$ constructible?

These can be answered via field extensions.

Other applications including **coding theory**.

An introduction to Rings

2.1 Definitions and basic properties

ring

A ring is a set with two binary operations $+$, \times , such that

1. $(R, +)$ is an abelian group.
 - $+$ is commutative and associative.
 - $\exists 0 \in R, 0 + a = a + 0 = a$ for all $a \in R$.
 - $\forall a \in R, \exists (-a) \in R, a + (-a) = (-a) + a = 0$.
2. \times is associative $(a \times b) \times c = a \times (b \times c)$.
3. distributive laws hold: $(a + b) \times c = (a \times c) + (b \times c)$.

The ring is called commutative if \times is commutative. The ring is said to have an identity if $\exists 1 \in R, 1 \times a = a \times 1 = a$, for all $a \in R$, and this does not require the existence of inverse.

For simplicity, we write

$$ab := a \times b, \quad b - a = b + (-a)$$

Example:

\mathbb{Z} is a commutative ring with identity.

Trivial rings: Let $(R, +)$ be an abelian group. We define $a \times b = 0$ for all $a, b \in R$. The result is a commutative ring with “trivial structure”.

$R = \{0\}$ is a zero ring. $0 = 1$ in this case, and it is the only such ring. It leads to assumption $0 \neq 1$, saying $R \neq \{0\}$.

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commutative rings with identity.

$\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ with $+, \times \bmod m$ is a ring with identity, and commutative.

The real quaternions: $\{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$. Addition is “component-wise”. And the multiplication follows

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

And this is non-commutative ring, with identity 1.

Let X be a set, A be a ring. Consider the set $F = \{f : X \rightarrow A\}$. Define

$$(f + g)(x) = f(x) + g(x), \quad (f \times g)(x) = f(x) \times g(x)$$

F commutative & having identity is inherited from the ring A .

$M_m(\mathbb{Z})$ is the ring of square $m \times m$ matrices with coefficients in \mathbb{Z} . It is non-commutative ring with identity.

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to have compact support, if $\exists a, b \in \mathbb{R}$, $f(x) = 0$ for $x \notin [a, b]$.
 $R = \{f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ has compact support}\}$ is a commutative ring, without identity.

Proposition 2.1

Let R be a ring. Then

1. $0a = a0$ for all $a \in R$.
2. $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.
3. $(-a)(-b) = ab$ for all $a, b \in R$.
4. If R has an identity 1, then it is unique, and $(-a) = (-1)a$.

Proof:

We see that

$$\begin{aligned} 0a &= (0 + 0)a = 0a + 0a \\ 0a - 0a &= (0a + 0a) - 0a = 0a + (0a - 0a) \\ 0a &= 0 \end{aligned}$$

We also see that

$$(-a)b + ab = ((-a) + a)b = 0b = 0$$

□

We would like to be able to cancel with respect to x : $ab = ac$ then $b = c$. However, this is not true in general.

Example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

However,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2.2 Zero divisor and integral domain

zero divisor

A nonzero element $a \in R$ is called a zero divisor, if there exists $b \in R$ and $b \neq 0$, such that $ab = 0$ or $ba = 0$.

integral domain

A commutative ring with identity, $1 \neq 0$, is called an integral domain, if it contains no zero divisor.

Proposition 2.2

Let R be a ring. Assume that $a, b, c \in R$, and a is not a zero divisor. If $ab = ac$, then either $a = 0$ or $b = c$ (i.e., we can multiplicatively cancel).

Proof:

Observe that

$$ab = ac$$

$$ab - ac = 0$$

$$a(b - c) = 0$$

As a is not zero divisor, then either $a = 0$ or $b - c = 0$. □

If zero divisors exist, then cancellation does not hold:

$$ab = 0 = a \cdot 0 \not\Rightarrow b = 0$$

Remark:

In integral domains, $ab = 0 \implies a = 0$ or $b = 0$.

2.3 Field**division ring**

A ring with identity 1 , $1 \neq 0$, is called a division ring, if every nonzero element has a multiplicative inverse, i.e., for all $a \in R, a \neq 0$, there exists $b \in R$, such that $ab = ba = 1$.

Consider an example $ab = 1$ existing and $ba = 1$ not existing.

Example:

Real sequences (x_1, x_2, \dots) . Ring of operators on the sequences, \times is composition. Take

$$D : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$$

$$S : (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$$

Then

$$D(S(x_1, x_2, \dots)) = Id(x_1, x_2, \dots)$$

but $S \circ D \neq Id$.

field

A commutative division ring is called a field.

Example:

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. Quaternions are “only” a division ring because non-commutative. \mathbb{Z}_p is a field for p prime.

Proposition 2.3

Any finite integral domain is a field.

\mathbb{Z} is an integral domain, but far from a field.

Proof:

Check Corollary 10.13 of PMATH 347. □

2.4 Subring

subring

Let R be a ring. A nonzero subset $S \subseteq R$ is called a subring of R , if it is a ring with the operations from $(R, +, \times)$ restricted to S .

That means: $S \neq \emptyset$. $x + (-y) \in S, \forall x, y \in S$. $xy \in S, \forall x, y \in S$.

Example:

$\mathbb{Z}_2 \subseteq \mathbb{Z}$, but \mathbb{Z}_2 is not a subring of \mathbb{Z} .

$2\mathbb{Z} = \{2 \cdot z : z \in \mathbb{Z}\}$ (ring has no identity) is a subring of \mathbb{Z} (ring has identity).

Ring of matrices $M_2(\mathbb{R})$ (1 is identity matrix) has a subring $S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \right\}$ and

$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is the identity in S .

2.5 Unit

unit

Assume that R is a ring with an identity $1 \neq 0$. A $a \in R$ is called a unit, if there exists $b \in R$ such that $ab = ba = 1$. Set of units of R is denoted by R^\times .

Example:

$$\mathbb{Z}^\times = \{\pm 1\}$$

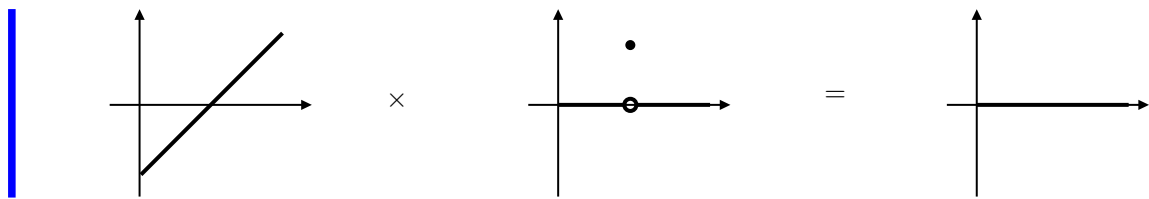
$$\mathbb{Z}_m^\times = \{a \in \mathbb{Z}_m : \gcd(a, m) = 1\}, \mathbb{Z}_p^\times = \mathbb{Z}_p \setminus \{0\} \text{ for } p \text{ prime.}$$

Consider ring R of $[0, 1] \rightarrow \mathbb{R}$, where $(f \times g)(x) = f(x) \cdot g(x)$, $1_R = 1(x)$. Units are the functions such that $f(x) \neq 0$ for $\forall x \in [0, 1]$. Then $f(x)^{-1} = \frac{1}{f(x)}$. All non-units are zero divisors. If $g(y) = 0$,

$$\text{then } h(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \text{ gives } (g \times h) = 0(x) = 0_R.$$

Ring of all continuous functions $[0, 1] \rightarrow \mathbb{R}$ is a subring of the previous ring. Units as before, because $1/f$ exists and is continuous.

Consider $f(x) = x - 1/2$.



Ring Homomorphisms

ring homomorphism

Let R, S be rings.

1. A ring homomorphism is $\phi : R \rightarrow S$, such that
 - (a) $\phi(a + b) = \phi(a) + \phi(b)$, for all $a, b \in R$.
 - (b) $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in R$.
2. The kernel of ϕ , $\ker \phi = \{a \in R : \phi(a) = 0_S\}$.
3. A bijective homomorphism is called isomorphism.

Remark:

Isomorphism means “same ring”, denote $R \cong S$.

Example:

$\{0, 1\} = \mathbb{Z}_2 = R, S = \{a, b\}$ with $a + a = a, a + b = b, \dots$. Then $R \cong S$.

Example:

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$ with cancellation $\frac{a}{b} = \frac{ca}{cb}$

Can we say $\mathbb{Z} \subseteq \mathbb{Q}$? not in the purest sense. \mathbb{Z} corresponds to $\{\frac{a}{1} : a \in \mathbb{Z}\}$.

\mathbb{Q} contains an isomorphic copy of \mathbb{Z} . $S \subseteq \mathbb{Q}$ such that $S \cong \mathbb{Z}$.

Example:

$\phi : \mathbb{Z} \rightarrow \mathbb{Z}_2$. $\phi(2k) = 0, \phi(2k + 1) = 1$. Then

$$\begin{aligned}
 \ker \phi &= 2\mathbb{Z} \\
 \phi^{-1}(0) &= 2\mathbb{Z} = \ker \phi \\
 \phi^{-1}(1) &= 1 + 2\mathbb{Z} \\
 &= 1 + \ker \phi \\
 &= 3 + \ker \phi
 \end{aligned}$$

Example: $\phi : \mathbb{Z}[x] \rightarrow \mathbb{Z} : p(x) \mapsto p(0)$. Then

$$\begin{aligned}\ker \phi &= \phi^{-1}(0) = \{a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + 0 : a_i \in \mathbb{Z}\} \\ &= x\mathbb{Z}[x] = \{x \cdot p(x) : p(x) \in \mathbb{Z}[x]\}\end{aligned}$$

and

$$\phi^{-1}(a) = x\mathbb{Z}[x] + ax^0 = \ker \phi + ax^0$$

Example: $\phi : \mathbb{Z}[x] \rightarrow \mathbb{Z}_2 : p(x) \mapsto p(0) \bmod 2$. Then

$$\ker \phi = \phi^{-1}(0) = x\mathbb{Z}[x] + 2\mathbb{Z}$$

$$\phi^{-1}(1) = 1 + \ker \phi$$

Example: $\phi : \mathbb{Z} \rightarrow \mathbb{R} : a \mapsto a$, then $\ker \phi = \{0_{\mathbb{R}}\}$.**Proposition 3.1**Let R, S be rings, $\phi : R \rightarrow S$ be homomorphism.

1. The image of ϕ , $\text{Im}(\phi)$, or $\phi(R)$ is a subring of S .
2. $\ker \phi$ is a subring of R . Moreover, $\forall r \in R, \forall \alpha \in \ker \phi, r\alpha \in \ker \phi, \alpha \in \ker \phi$. (That is $\ker \phi$ is closed under x by the elements from R)

Proof:

1. If $a, b \in \phi(R)$, then

$$a - b = \phi(x_a) - \phi(x_b) = \phi(x_a - x_b) = \phi(x_{a-b}) \in \phi(R)$$

2. $\phi(r\alpha) = \phi(r) \cdot \phi(\alpha) = \phi(r) \cdot 0 = 0$

□

Can we get a ring structure on $a + \ker \phi$? There is a factor ring $R / \ker \phi$. For example, $\mathbb{Z} / 2\mathbb{Z} \cong \mathbb{Z}_2$.

3.1 Quotient rings

idealLet R be a ring, let $I \subseteq R$ be a subring, let $r \in R$.

1. I is called a left ideal, if $rI \subseteq I$ where $rI = \{ri : i \in I\}$.
2. I is called a right ideal, if $Ir \subseteq I$.
3. I is an ideal, if it is left & right ideal (two sided ideal).

additive quotientLet $I \subseteq R$ be an ideal. The additive quotient is defined as $R/I = \{a + I : a \in R\}$.

Example:

$\mathbb{Z}/3\mathbb{Z} = \{\{\dots, -6, 3, 0, 3, 6, \dots\}, \{\dots, -5, -2, 1, 4, \dots\}, \{\dots, -4, -1, 2, 5, 8, \dots\}\}$. Additive group.

Let $I = 3\mathbb{Z}$. Then $a + I$ are called (additive) cosets.

Proposition 3.2

Let R be a ring, I an ideal of R , then R/I is a ring with the operations

$$(a + I) +_{R/I} (b + I) =: (a +_R b) + I$$

$$(a + I) \times_{R/I} (b + I) = (a \times_R b) + I$$

The ring properties R/I follow from R being a ring.

quotient ring

R/I is called the quotient ring of R by I .

Remark:

If I is not an ideal, then the definition of the operations on R/I is not well defined.

Example:

Let R be commutative ring with identity $1 \neq 0$, $m \geq 2$. Let $M_m(R)$ be ring of square matrices with coefficients in R .

Denote

$$L_j(R) = \{A \in M_m(R) \mid A_{ik} = 0, \forall i \in [n], k \in [m] \setminus \{j\}\}$$

which means only the j -th column can have non-zero entries. Then $L_j(R)$ is a left ideal in $M_m(R)$. This can be verified by the matrix multiplication. $L_j(R)$ is not a right ideal, i.e., $L_j(R) \cdot M \not\subseteq L_j(R)$ for some $M \in M_m(R)$.

Analogously, a right ideal can be obtained by taking

$$T_i(R) = \left\{ A \in M_m(R) \mid A_{kj} = 0, \forall k \in [n] \setminus \{i\}, j \in [m] \right\}$$

Example:

Let $R = \mathbb{Z}[x]$ and $I = x^2\mathbb{Z}[x]$.

Then $R/I = \{a + bx + p(x) : a, b \in \mathbb{Z}, p(x) \in I\}$.

For $a \in R/I$, \bar{a} denotes $a + I$.

Theorem 3.3: First isomorphism theorem

If $\phi : R \rightarrow S$ is a ring homomorphism, then $\ker \phi$ is an ideal in R , $\text{Im } \phi$ is a subring of S , and $R/\ker \phi \cong \text{Im } \phi$.

Proof:

Theorem 4.2 of <http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf>

□

Theorem 3.4

For any ideal $I \subseteq R$, the map $R \rightarrow R/I$ defined by $r \mapsto r + I$ is a surjective ring homomorphism with kernel I . It is called the natural projection of R onto R/I . Thus every ideal is a kernel of some homomorphism.

Index

A

additive quotient 11

D

division ring 7

F

field 7

I

ideal 11

integral domain 7

Q

quotient ring 12

R

ring 5

ring homomorphism 10

S

subring 8

U

unit 8

Z

zero divisor 6