



# *Computational Discrete Optimization*

CO 353



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# Preface

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Discrete optimization problems are underlying decisions that have a discrete flavor, e.g., YES/NO or  $\{0,1\}$  decisions.

The focus in this course will be on algorithms, modelling. Broad classes of problems that we will study are network connectivity problems, location problems, general integer programs.

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# Graph Algorithms

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## 1.1 Definitions, Notations & Terminology

A **graph** is a tuple  $(V, E)$ , where  $V$  is set of **nodes/vertices**,  $E$  is set of **edges**, where edges **joins** two nodes.

If  $e$  is an edge that joins nodes  $u, v$ , then we denote this by  $e = uv$ .  $u, v$  are called **ends** of  $e$ .  $e$  is **incident** to nodes  $u, v$ . We are not allowing parallel edges, i.e.,  $e = uv$ , and  $e' = u'v'$  are distinct edges, then  $\{u, v\} \neq \{u', v'\}$ .

An  **$u$ - $v$  path** in  $G = (V, E)$  where  $u, v \in V, u \neq v$ , is a sequence of nodes  $u_1 = u, u_2, \dots, u_k, u_{k+1} = v$ , where  $u_i u_{i+1} \in E \forall i = 1, \dots, k$ . A **cycle** in  $G$  is a sequence of nodes  $u_1, u_2, \dots, u_k, u_{k+1} = u_1$  where  $u_i u_{i+1} \in E \forall i = 1, \dots, k$ , and  $u_i$ 's are distinct. Since there are no parallel edges, we can also identify a path/cycle by its sequence of  $u_i u_{i+1}$  edges. So we will often refer to a path/cycle as a set of edges.

A graph  $G$  is **connected** if it has a  $u - v$  path  $\forall u, v \in V (u \neq v)$ .  $G$  is acyclic if  $G$  does not have a cycle. A **tree** is a connected, acyclic graph.

Let  $G = (V, E)$  be a connected graph, and  $T = (V_T, E_T)$  be a tree. IF  $E_T \subseteq E$  and  $V_T = V$ , then we say that  $T$  is a **spanning tree** of  $G$ .

If  $C$  is a cycle, and  $e \in C$ , then  $C - \{e\}$  still connects all nodes of  $C$ . So if  $G$  is a connected graph, and it contains a cycle  $C$ , and  $e \in C$ , then  $G - \{e\} := (V, E - \{e\})$  is a connected graph. Hence, a spanning tree of  $G$  is a minimal connected subgraph of  $G$ . I.e., if  $T = (V, F)$  where  $F \subseteq E$  is a minimal set such that  $(V, F)$  is connected, then  $T$  is a spanning tree of  $G$ . If  $T = (V, F)$  contains a cycle, then  $F$  is not minimal.

In **directed graph**, each edge has a direction, and goes **from** a node **to** another node.

## 1.2 Shortest paths: Dijkstra's algorithm

**Problem** Given a directed graph  $G = (V, E)$  with edge costs  $\{c_e \geq 0\}$  and a node  $s \in V$ , find the shortest path from  $s$  to all other nodes. The "shortest" path means path with the smallest total edge cost under the  $c_e$  edge costs.

**Notation** For a path  $P$ , let  $c(P) := \sum_{e \in P} c_e$  denote the total cost of  $P$ . Let  $d(u) = \min_{P: P \text{ a } s \rightarrow u \text{ path}} c(P)$ , which is shortest path (SP) distance from  $s$  to  $u$ . If  $u \rightarrow v$  is an edge of  $G$ , we have

$$d(v) \leq d(u) + c_{u,v} \quad (\clubsuit)$$

## Dijkstra's Algorithm

The idea is to maintain a set of explored vertices, and we want to expand this set. Then we can make use of  $(\clubsuit)$  to estimate the shortest path from  $s$  to  $v$ , a vertex to be added to the set. We will maintain a label  $\ell(v)$  for all  $v \notin A$ , which is our current estimate for the  $s \rightarrow v$  shortest path distance.

**Given** Directed graph  $G = (V, E)$ ,  $s \in V$ , edge costs  $\{c_e \geq 0\}$ .

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### Algorithm 1: Dijkstra's Algorithm

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1 Initialize  $A \leftarrow \{s\}$ ,  $d(s) = 0$ ,  $\ell(v) \leftarrow \infty \forall v \notin A$ .
2 while  $A \neq V$  do
3   For all  $v \notin A$  such that  $\exists u \in A$  with edge  $u \rightarrow v$ , update
      
$$\ell(v) = \min \left\{ \ell(v), \min_{u \in A: (u,v) \in E} (d(u) + c_{u,v}) \right\}$$

4   Select  $w \in V - A$  such that  $\ell(w)$  has minimum  $\ell(v)$  value among all  $v \notin A$ .
5   Update  $A \leftarrow A \cup \{w\}$ , set  $d(w) = \ell(w)$ .
```

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#### Remark:

Can obtain actual shortest paths by maintaining along with  $\ell(w)$ , the node  $u \in A$  that determines  $\ell(w)$  (i.e.,  $u \in A$  is s.t.  $\ell(w) = d(u) + c_{u,w}$ ). Call  $u$ , the “parent” of  $w$ , and  $u \rightarrow w$  the parent edge of  $w$ .

The shortest paths obtained via previous point have a special structure: every node  $w \neq s$  has exactly one edge entering it, and there are no cycles, i.e., we have something like “directed” tree. And we denote shortest-path tree: directed tree returned by algorithm.

Also note that  $\ell(v)$  in

$$\ell(v) = \min \left\{ \ell(v), \min_{u \in A: (u,v) \in E} (d(u) + c_{u,v}) \right\}$$

is redundant, since

$$\min_{u \in A: (u,v) \in E} (d(u) + c_{u,v})$$

term only decreases as the set  $A$  only grows.

## Correctness

We may assume that there exists  $s \rightarrow u$  path in  $G \forall u \in V$ . And it's easy to modify Dijkstra's algorithm to detect if this assumption holds, and get shortest path distances from  $s$  to all nodes reachable from  $s$ .

Let  $d^{\text{Alg}}(v)$ :  $d$ -value computed by algorithm. Recall  $d(v)$  is the shortest path distance from  $s$  to  $v$ . The goal then is to show that for all  $v \in V$ ,  $d^{\text{Alg}}(v) = d(v)$ . Clearly this is satisfied when  $v = s$ .

Assume we have correctly computed shortest path distances for all  $u \in A$ ,  $\ell(v)$  is the length of the shortest path  $P$  such that *last edge of  $P$  (which enters  $v$ ) comes from a node in  $A$* .

Why? Consider such a path  $P$ . Let  $u \rightarrow v$  be the last edge of  $P$ . So  $u \in A$ ,  $d^{\text{Alg}}(u) = d(u)$ ,

$$c(P) \geq d(u) + c_{u,v} = d^{\text{Alg}}(u) + c_{u,v} \geq \ell(v)$$

and last inequality is by the definition of  $\ell(v)$ .

### Theorem 1.1

If  $w$  is added to  $A$  in line 5 of the algorithm, then  $d^{\text{Alg}}(w) = d(w)$ . (I.e., we have computed shortest path distance from  $s$  to  $w$ .)

#### Proof:

Assume we have correctly computed shortest path distance  $\forall u \in A$ . Consider an arbitrary  $s \rightarrow w$  path  $P$ . Let  $u$  be the last node on  $P$  that lies in  $A$ . Let  $v$  be the node on  $P$  after  $u$  (so  $v \notin A$ ). Let  $P'$  be the  $s \rightarrow v$  portion of  $P$ . Then

$$c(P) \geq c(P') \geq d(u) + c_{u,v} = d^{\text{Alg}}(u) + c_{u,v} \geq \ell(v) \geq \ell(w)$$

where the last equality is by the definition of  $w$  in the line 4. □

Then following parent edges gives an  $s \rightarrow w$  path of length  $= \ell(w) = d^{\text{Alg}}(w)$ .

## 1.3 Running time and Efficient Algorithms

The goal in this course is to design efficient algorithms. What does efficient mean? The short answer is “reasonable” running time.

**Running time** is number of elementary operations performed by algorithm as a function of input size. **Elementary operations** includes basic arithmetic (e.g., addition), comparisons (is  $x < y$ ?), simple logical constructs (i.e., if-then-else), assignments. **Input size** is the number of *bits* needed to specify the input. Note that number of bits need to specify a number  $x \geq 0$ ,  $x$  integer is roughly  $\log_2 x$ , which is much smaller than  $x$  itself.

For example, the size of an input of the Dijkstra’s algorithm,  $G = (V, E), \{c_e\}_{e \in E}$  is usually taken to be approximately  $|V| + |E| + \sum_{e \in E} \log_2 c_e$ .

**Reasonable running time**, i.e., efficient algorithm means that running time that is **polynomial function** of input size. In order to specify running time & input size in a convenient, compact way, we will use  $O(\cdot)$  notation.

Given two functions:  $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ , we say that  $f(n) = O(g(n))$  if there exist constants  $c > 0$  and  $n_0 \geq 0$  such that  $f(n) \leq c \cdot g(n)$  for all  $n \geq n_0$ .

Here are some examples:

$$\begin{aligned} n &= O(n) \\ 2n + 10 &= O(n) \\ 3n &= O(n^2) \\ \alpha n^c + \beta &= O(n^d) \\ n \log_2 n &= O(n^2) \\ \log_2 n &= O(\log_{10} n) \\ 2^n &= O(3^n) \end{aligned}$$

$f(n) = O(1)$  means  $f(n) \leq c$  for all  $n \geq n_0$ .  $f(n) = O(n^{O(1)})$  is shorthand for  $f(n)$  is bounded by some (fixed) polynomial function of  $n$ :  $f(n) \leq d \cdot n^c$ .

An algorithm with running time  $f(n)$ , where  $n$  is input size, is **efficient** if  $f(n)$  is bounded by a polynomial function of  $n$ , i.e.,  $f(n) = O(n^{O(1)})$ .

Now we can examine the running time of Dijkstra's algorithm (removing unnecessary  $\ell(v)$  in line 3). Let  $m = |E|$ ,  $n = |V|$ . We observe that there are  $n$  iterations of while loop. In each iteration:

1. Computing  $\ell(v)$  takes  $O(d^{\text{in}}(v))$  time where  $d^{\text{in}}(v)$  is the number of edges entering  $v$ .
2. Computing  $\ell(v) \forall v$  takes  $O(m)$  time since  $\sum_{v \in V} d^{\text{in}}(v) = m$ .
3. Line 4 takes  $O(n)$  time.
4. Line 5 takes  $O(1)$  time.

Each iteration takes  $O(m + n)$  time. This is  $O(m)$  if we assume there exists an  $s \rightarrow v$  path  $\forall v \in V$  since then  $m \geq n - 1$ , so  $n = O(m)$ . Then the running time of algorithm is  $O(mn)$  which is a polynomial function of input size.

However, we can have a better implementation. Observe that if  $\{u \in A : (u, v \in E)\}$  does not change across iterations, then  $\ell(v)$  does not change. So instead of recomputing  $\ell(v)$  for all  $v \notin A$ , we do the following:

When we pick  $w \notin A$  to add to  $A$ , we only update  $\ell(v)$  for all  $v \notin A$  such that  $(w, v) \in E$ , and set  $\ell^{\text{new}}(v) = \min(\ell^{\text{old}}(v), d(w) + c_{w,v})$ .

So the steps inside of while loop change as: [Let  $w^*$  be the last node added to  $A$ . Initially  $w^* = s$ .]

- (a) For every edge  $(w^*, v)$ , where  $v \notin A$ , update

$$\ell(v) = \min(\ell(v), d(w^*) + c_{w^*,v})$$

and we call this DecreaseKey operation.

- (b) Find  $w \notin A$  with minimum  $\ell(\cdot)$  value. We call this ExtractMin operation.

- (c) Update  $A \leftarrow A \cup \{w\}$ ,  $d(w) = \ell(w)$ ,  $w^* = w$ .

Across all iterations, we examine each edge  $(u, v)$  at most once in step (a) above (in the iteration when  $w^* = u, v \notin A$ ). So across all iterations,  $\leq m$  DecreaseKey operations,  $\leq n$  ExtractMin operations.

Then we can use a simple array to store  $\ell(\cdot)$  values. Note that DecreaseKey is  $O(1)$ , and ExtractMin operation is  $O(n)$ . Thus the running time =  $O(m + n^2) = O(n^2)$ .

There exist data structures such as priority queue, under which DecreaseKey and ExtractMin take  $O(\log n)$ . Then the running time is then  $O(m \log n)$ .

There exists a data structure called Fibonacci heaps, under which DecreaseKey is  $O(1)$ , and ExtractMin operation is  $O(\log n)$ . Then the running time is  $O(m + n \log n)$ .

## Graph Algorithm cont'd

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### 2.1 Minimum Spanning Trees

**MST Problem** Given a connected, undirected graph  $G = (V, E)$ , edge costs  $\{c_e\}_{e \in E}$ . Find a spanning tree of  $G$  of minimum total edge cost.

We say “ $T$  is a spanning tree” is equivalent to “ $T$  is the edge set of a spanning tree”. We denote the cost of  $T$  by  $c(T) := \sum_{e \in T} c_e$ .

**Note:**

The  $c_e$ 's could be positive, zero, or negative.

If all  $c_e$ 's are  $\geq 0$ , then can equivalently define the MST problem as: find the min-cost connected spanning subgraph of  $G$ . Because there is always an optimal solution that is minimal connected spanning subgraph of  $G$ .

#### Theorem 2.1: Fundamental Theorem about trees

Let  $T = (V, F)$  be a graph, and  $n = |V|$ . The following are equivalent:

- (a)  $T$  is a tree (i.e., connected, acyclic)
- (b)  $T$  is connected, has  $n - 1$  edges.
- (c)  $T$  is acyclic, has  $n - 1$  edges.

**Proof:**

**(a)  $\Rightarrow$  (b)** Pick some  $r \in V$  as root node. Root  $T$  at  $r$ , i.e., draw  $T$  as hanging off of  $r$ . For each  $v \neq r$ , there is a unique edge  $uv$  of  $T$  (incident to  $v$ ) such that  $u$  is closer to  $r$  than  $v$ , and we call  $uv$  the parent edge of  $v$ .

These parent edges cover  $T$ , and number of parent edges  $= n - 1$ , since each  $v \neq r$  has a unique parent edge.

**(b)  $\Rightarrow$  (a)**  $T$  is connected. Let  $T'$  be a spanning tree of  $T$ . So by (a)  $\Rightarrow$  (b), we know that  $T'$  has  $n - 1$  edges. But  $T$  has  $n - 1$  edges. So  $T = T'$ , so  $T$  is a tree.  $\square$



## 2.2 Cut property

**Notation** Let  $v \in V$ .  $\delta(v)$  denotes the set of edges incident to  $v$ . Let  $S \subseteq V$ ,  $\delta(S) := \{uv \in E : u \in S, v \notin S\}$ . In other words,  $\delta(S)$  denotes the “boundary” of  $S$ .

**Now we assume that all edges costs are distinct.** Fix some node  $s \in V$ . Let  $e \in \delta(S)$  have the smallest edge cost among edges in  $\delta(S)$ . Is  $e$  in some MST? In fact,  $e$  is in every MST.

### cut

A cut is any partition  $(A, V - A)$  of the vertex set  $V$ , where  $A \neq \emptyset, A \subsetneq V$ .

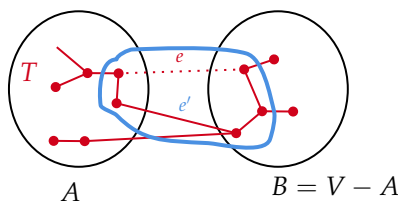
Edges crossing the cut are edges in  $\delta(A)$  ( $= \delta(V - A)$ ). If  $F \subseteq E$ , we say  $F$  crosses the cut to mean  $F \cap \delta(A) \neq \emptyset$ .

### Lemma 2.2: Cut property

Consider any cut  $(A, B)$ , where  $B = V - A$ . If  $e$  is the (unique) min-cost edge across the cut, then  $e$  belongs to every MST.

**Proof (via an exchange argument):**

Suppose  $T$  is an MST such that  $e \notin T$ . We will show that we can find another spanning tree  $T'$  (that contains  $e$ ) such that  $c(T') < c(T)$ , then a contradiction.



$T \cup \{e\}$  contains a cycle  $C$  that contains  $e$ . And this is because  $T \cup \{e\}$  is connected and has  $n$  edges, then it can't be acyclic. Here is a *basic fact*: if a cycle crosses a cut, it crosses the cut at least twice. So  $\exists e' \in C \cap \delta(A)$ ,  $e' \neq e$ . By definition of  $e$ ,  $c_e < c_{e'}$ . And  $e' \in T$ .

Consider  $T' = T \cup \{e\} \setminus \{e'\}$ . We claim that  $T'$  is a spanning tree.  $T'$  is connected since  $e' \in$  cycle in  $T \cup \{e\}$  and  $T'$  has  $n - 1$  edges. Then we have

$$c(T') = c(T) + c_e - c_{e'} < c(T)$$

a contradiction. □

## 2.3 Prim's Algorithm

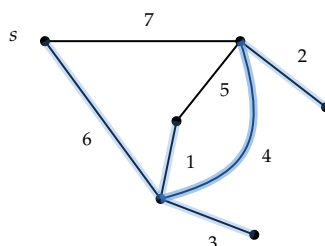
Now we can use cut property as the basis of the greedy algorithm.

### Algorithm 2: Prim's Algorithm

- 1 Pick an arbitrary “seed” node  $s \in V$ .
- 2 Initialize  $A \leftarrow \{s\}$ ,  $T \leftarrow \emptyset$ .
- 3 **while**  $A \neq V$  **do**
- 4     Choose  $e = uv \in \delta(A)$  with smallest cost, where  $u \in A, v \notin A$ .
- 5      $A \leftarrow A \cup \{v\}$ ,  $T \leftarrow T \cup \{e\}$ .
- 6 **return**  $T$

**Example:**

Blue lines are the output of Prim's algorithm.

**Theorem 2.3**

Prim's algorithm correctly computes an MST.

**Proof:**

Let  $T$  be the edge-set returned by Prim's algorithm.

- $T$  is a spanning tree.  $T$  is connected, since every node is connected to  $s$  in  $T$ . Also,  $T$  has  $n - 1$  edges. Thus  $T$  is a spanning tree.
- $T$  is a MST. Every  $e \in T$  belongs to every MST by the cut property, since it is the min-cost edge across some cut. So  $T \subseteq$  every MST. But  $T$  is itself a spanning tree, so  $T$  is MST.

□

**Corollary**

$T$  is the unique MST.

If edge costs are not distinct, then Prim's algorithm still returns an MST; there could be multiple MSTs.

**Implementation & Running Time**

Implementation will be similar to Dijkstra's algorithm. For every unexplored node  $v \notin A$ , maintain a "key"  $a(v) = \min_{e=uv: u \in A} c_e$ . So in each iteration, we choose  $w \notin A$  with smallest  $a(\cdot)$  value similar to Dijkstra, let  $w^*$  be the last node added to  $A$ . In each iteration

- For each edge  $w^*v$ , where  $v \notin A$ , update  $a(v) = \min\{a(v), c_{w^*v}\}$ . **DecKey**
- Find  $w \in V - A$  with smallest  $a(\cdot)$  value. **ExtractMin**
- Set  $A \leftarrow A \cup \{w\}$ , and  $T \leftarrow T \cup \{uw\}$ , where  $u \in A$ , and  $c_{uw} = a(w)$ .

As in Dijkstra's algorithm, across all iterations:

- $n$  **ExtractMin** operations
- $m$  **DecKey** operations

So the running time is

- $O(m + n^2)$  using a simple array to store keys (**DecKey**  $O(1)$ , **ExtractMin**  $O(n)$ )
- $O(m + n \log n)$  using a sophisticated data structure like Fibonacci Heaps (**DecKey**  $O(1)$ , **ExtractMin**  $O(\log n)$ )

## 2.4 Kruskal's algorithm

Kruskal's algorithm finds a MST. In some level, it is more greedy and intuitive than Prim's algorithm. The idea is to keep the edge costs in increasing order, and add edges to the set one by one, as long as no cycle are introduced.

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### Algorithm 3: Kruskal's algorithm

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1 Sort the edges in increasing order of cost.
2 Initialize  $T \leftarrow \emptyset$ .
3 for each edge  $e$  in sorted order do
4   if  $T \cup \{e\}$  does not have a cycle then
5      $T \leftarrow T \cup \{e\}$ 
6 return  $T$ 

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### Theorem 2.4

Kruskal's algorithm returns the unique MST when all edges costs are distinct.

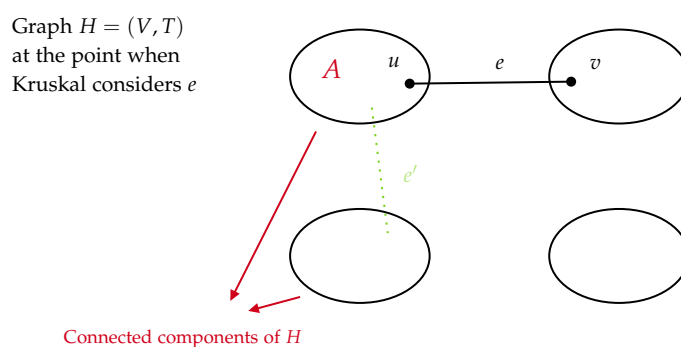
#### Proof:

Let  $T$  be the edge-set returned by Kruskal. Then,  $T$  is acyclic: by construction. Consider a basic fact: a graph is connected  $H = (V_H, E_H)$  if and only if  $\delta_H(A) \neq \emptyset \forall A : \emptyset \neq A \subsetneq V_H$ .

Suppose  $(V, T)$  is not connected. Then from the basic fact,  $\exists A, \emptyset \neq A \subsetneq V$  such that  $\delta(A) \cap T = \emptyset$ . But  $G$  is connected, so  $\exists$  some edge  $e \in \delta(A)$ . Then  $T \cup \{e\}$  is acyclic. So consider the point when Kruskal considers edge  $e$ . Let  $F \subseteq T$  be set of edges Kruskal has added until then. Then  $F \cup \{e\}$  is acyclic, so Kruskal should have added  $e$ . Then  $e \in T$ , a contradiction. So  $T$  is a spanning tree.

Consider any edge  $e = uv \in T$ . Let

$$A = \{w \in V : w \text{ is connected to } u \text{ in } T \text{ at the point when } e \text{ is considered by Kruskal}\}$$



We claim that  $e$  is the min-cost edge in  $\delta(A)$ . Observe that  $e$  is the first edge of  $\delta(A)$  considered by Kruskal. Hence  $e$  is the min-cost edge in  $\delta(A)$ . By claim,  $e \in$  every MST. So  $T \subseteq$  every MST. But  $T$  itself is a spanning tree. So  $T$  is MST.  $\square$

#### Remark:

We can stop Kruskal when  $|T| = |V| - 1$ .

**Running time** Sorting  $m$  edges takes  $O(m \log m)$ . To check if  $e = uv$  can be added, we need to check if  $u, v$  are in different components of  $(V, T)$  at that point. There exist data structures (e.g., Union-Find) for maintaining connected components that allow one to do this  $O(\log n)$  time. So total time for step 3 is  $O(m \log n)$ . Since  $m \leq n^2$ , the total time for the algorithm is  $O(m \log m + m \log n) = O(m \log n)$ .

## 2.5 Application to Clustering

**Clustering** Given a set of objects, and some notion of similarity/dissimilarity between these objects, divide the objects into groups (called clusters) so that

1. Objects in the same group are “similar” to each other.
2. Objects in different groups are “dissimilar” to each other.

**Maximum-Spanning Clustering** Given a set  $V = \{p_1, \dots, p_n\}$  of objects/points, and pairwise distances  $d(p_i, p_j) = d(p_j, p_i) \geq 0 \forall i, j \in [n]$ . The goal is to partition  $V$  into  $k$  clusters  $C_1, \dots, C_k$  ( $C_i \cap C_j = \emptyset \forall i \neq j$ ,  $\bigcup_{i=1}^k C_i = V$ ). So as to maximize the minimum inter-cluster spacing, which is equivalent to minimum distance between a pair of points in different clusters. I.e., maximize

$$\min_{\substack{i, j \in [k] \\ i \neq j}} \min_{\substack{p \in C_i \\ q \in C_j}} d(p, q)$$

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### Algorithm 4: Single-Linkage Clustering

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- 1 Start with every point in a separate cluster.
  - 2 Repeatedly merge the 2 clusters with smallest inter-cluster distance<sup>a</sup>, until we have  $k$  clusters.
- 

<sup>a</sup>Distance between  $C_i, C_j = \min_{p \in C_i, q \in C_j} d(p, q)$

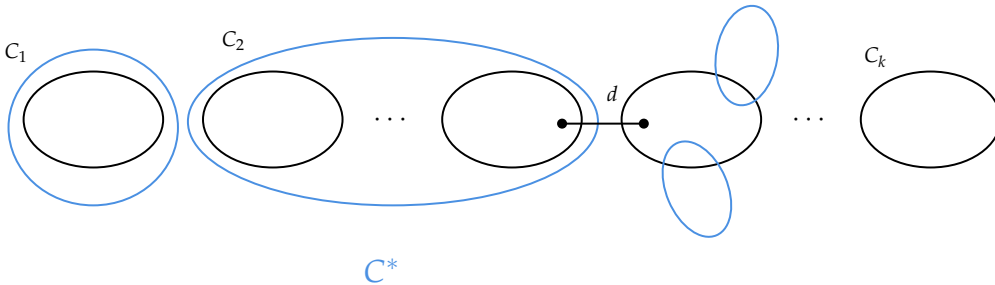
This algorithm is called Single-Linkage Clustering, which is an example of an agglomerative algorithm (i.e., based on merging clusters). Consider a graph  $G$  with  $V$  as vertex set, and an edge between every pair  $p, q \in V$ ,  $p \neq q$ , with cost  $d(p, q)$ .

**Note:**

Single-Linkage Clustering is exactly Kruskal (merging 2 clusters  $C_i, C_j$  due to points  $p \in C_i, q \in C_j$ ) when adding edge  $pq$ . *Except* that we stop when there are  $k$  components. And this is the same as taking an MST and deleting the  $k - 1$  most costly edges, i.e., the  $k - 1$  edges that Kruskal could have added last.

Thus equivalently, Run Kruskal (on complete graph with vertex set  $V$ ,  $d(p, q)$  edge costs) but stop when  $k$  components remain, which is equivalent to take MST and delete  $k - 1$  most costly edges.

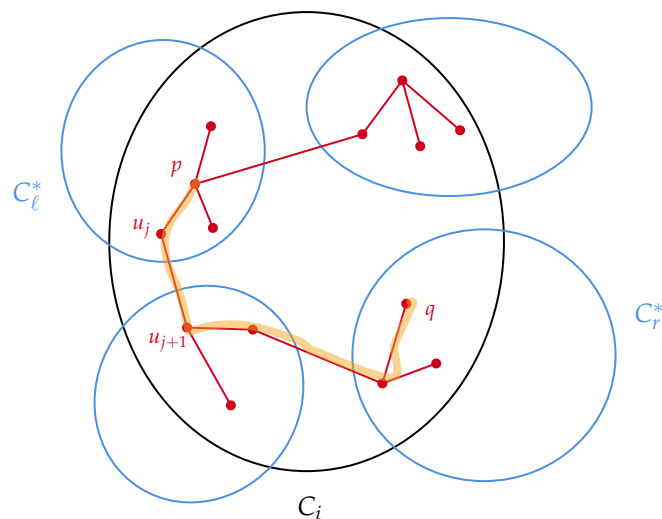
Now let's prove the correctness of Single-Linkage Clustering. Let  $C_1, \dots, C_k$  be clustering produced by MST -  $\{k - 1$  most costly edges $\}$ . Let  $d$  be the spacing of this clustering.



Observe that

$$\begin{aligned} d &= \text{cost of edge Kruskal would have added next} \\ &= (k - 1)\text{th most costly edge of MST} \end{aligned}$$

If  $C = \{C_1, \dots, C_k\}$  is not the optimum, let  $C^* = \{C_1^*, \dots, C_k^*\}$  be the optimum clustering. There might exist the case that  $C_i \subseteq C_j^*$ . As both  $C$  and  $C^*$  have  $k$  partitions, it's not possible to have  $\subseteq$  for all  $k$  partitions. Since  $C \neq C^*$ , there is some cluster  $C_i$  that intersects at least two clusters of  $C^*$ .



Red edges inside  $C_i$  all have cost  $\leq d$  since these are already added by Kruskal.

So there exists points  $p, q$  such that  $p, q \in C_i$ , but  $p, q$  lie in different clusters of  $C^*$ . Suppose  $p \in C_\ell^*$ ,  $q \in C_r^*$ ,  $\ell \neq r$ . Then considering  $p - q$  path in  $C_i$ , there must be two consecutive nodes  $u_j, u_{j+1}$  such that  $u_j \in C_\ell^*, u_{j+1} \notin C_\ell^*$ . But then spacing of  $C^* \leq d(u_j, u_{j+1}) \leq d$  since  $u_j, u_{j+1}$  are in different clusters of  $C^*$ . So this gives a contradiction since we assumed that  $\{C_1, \dots, C_k\}$  is not an optimal clustering, and optimal clustering has spacing strictly larger than  $d$ .

## Graph Algorithms - cont'd

### 3.1 Arborescences

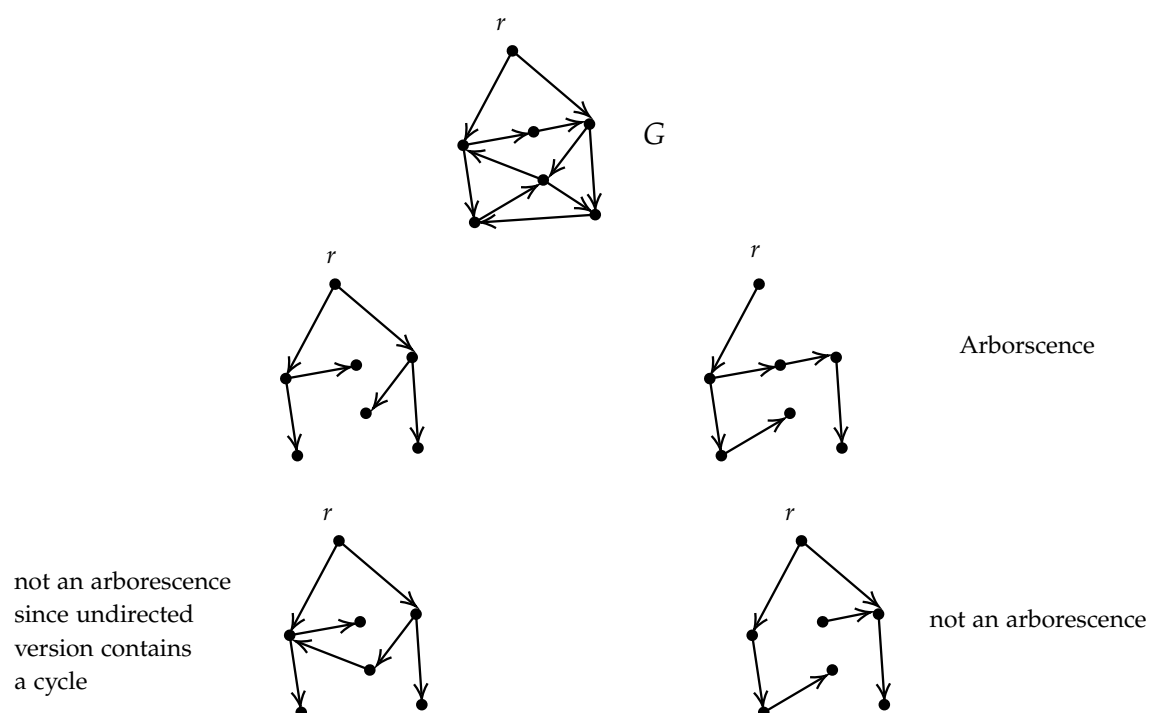
Arborescences is directed spanning tree.  $G = (V, E)$  a directed graph, and let  $r \in V$  be a “root” node.

#### arborescence rooted at $r$

An arborescence rooted at  $r$  (or rooted out of  $r$ ) is a subgraph  $T = (V, F)$  (so  $F \subseteq E$ ) such that

- there is an  $r \rightarrow v$  path in  $T \forall v \in V$
- $T$  is a spanning tree if we ignore the directions of the edges. I.e., “undirected version of  $T$ ” is a spanning tree.

Example:

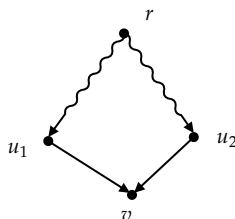


**Lemma 3.1: Useful alternate characterization of arborescence**

$T = (V, F)$  is an arborescence rooted at  $r$  if and only if  $T$  has no (directed) cycles and every  $v \neq r$  has exactly one incoming edge, (and  $r$  has no incoming edges).

**Proof:**

( $\Rightarrow$ ) Suppose  $T$  is arborescence. Since undirected version of  $T$  is a spanning tree,  $T$  has no directed cycles, and there exists  $r \rightarrow v$  path in  $T$  for all  $v \in V$  by the definition of arborescence. The path must be unique, otherwise we might encounter the situation like



then we will have a cycle in the undirected version of  $T$ . Hence every  $v \neq r$  has exactly one incoming edge.

( $\Leftarrow$ ) Suppose  $T$  has no directed cycles, and every node  $v \neq r$  has exactly one incoming edge. For any node  $v \neq r$ , we construct an  $r \rightarrow v$  path as follows: we take  $v$ 's unique incoming edge, say  $u_1 \rightarrow v$ , then  $u_1$ 's unique incoming edge, and so on... Since  $T$  has no cycle, this process must stop because we have reached  $r$ , then we found an  $r \rightarrow v$  path.

This also shows that undirected version of  $T$  is connected. Also, note that  $r$  has no incoming edges in  $T$ : suppose  $T$  had an edge  $v \rightarrow r$ , but then  $r \rightarrow v$  path in  $T$  would create a directed cycle.

So we have shown that (since every  $v \neq r$  has exactly one incoming edge)  $T$  has  $n - 1$  edges. So undirected version of  $T$  is connected, has  $n - 1$  edges, thus is a spanning tree.  $\square$

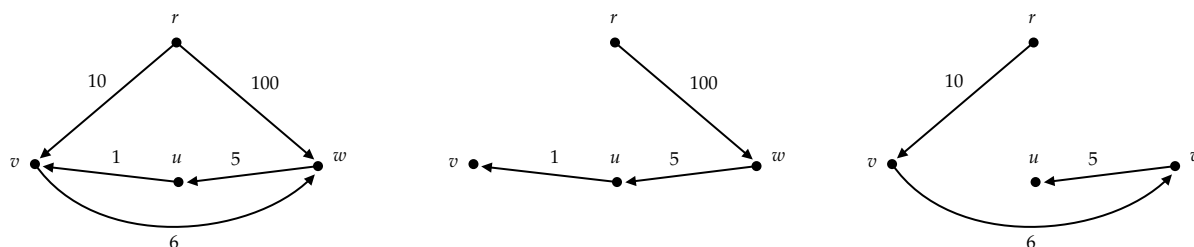
### 3.2 Min Cost Arborescence (MCA) Problem

Given a directed graph  $G = (V, E)$ , and a root node  $r \in V$ , and edge costs  $\{c_e\}_{e \in E}$ , find an arborescence rooted at  $r$  of minimum total edge cost.

**Assume there exists  $r \rightarrow v$  path in  $G$  for all  $v \in V$** , then there exists an arborescence rooted at  $r$ .

There are two greedy strategies (inspired by MST)

**Strategy 1** Pick the cheapest edge entering a node  $v$  (for some fixed node  $v$ ). Does this edge belong to an MCA? Consider the following example for  $v$ . The cheapest edge entering  $v$  is of cost 1. However, if we pick this edge, the only arborescence containing edge  $uv$  is of cost 106, while MCA is of cost 21.



**Strategy 2** Consider some cycle, and delete the most costly edge  $e$  of the cycle. Is this valid, i.e., is there an MCA not containing  $e$ ? In the above example, we take the cycle  $C = u \rightarrow v \rightarrow w \rightarrow u$ . If we

delete the most cost edge  $vw$ , it's impossible to get MCA on the right.

Let us examine greedy strategy (1) again. Pick cheapest edge entering each node  $v \neq r$ . Let  $F^*$  be set of edges picked. Now let's make two observations.

**Observation 1** If  $(V, F^*)$  is an arborescence, then it is an MCA. Every arborescence must pick an incoming edge for every  $v \neq r$ , and  $F^*$  is the cheapest way of picking these edges. If  $(V, F^*)$  is NOT an arborescence, then by Lemma 3.1, we know that  $(V, F^*)$  contains cycle  $Z$ , not containing root  $r$ .

**Observation 2** Suppose for every node  $v \neq r$ , for each edge  $e$  entering  $v$ , we subtract a common amount  $P_v$  from its cost, i.e.,  $\forall v \neq r, \forall$  edges  $e$  entering  $v$ , we set  $c'_e = c_e - P_v$ . Then for any arborescence  $T$ , its  $c$ -cost and  $c'$ -cost differ by a constant; more precisely,  $c(T) - c'(T) = \sum_{v \neq r} P_v$  (simply because  $T$  has exactly one edge entering each  $v \neq r$  by lemma 3.1).

### Corollary

Let  $y_v = \min_{(u,v) \in E} c_{u,v}$ . For all  $v \neq r$ , for all edges  $e$  entering  $v$ , let  $c'_e = c_e - y_v$ <sup>a</sup>. Then  $T$  is an MCA with respect to  $\{c_e\}$  if and only if  $T$  is an MCA with respect to  $\{c'_e\}$  costs.

<sup>a</sup> $c'_e \geq 0$  for all  $e$ .

## 3.3 Edmond's algorithm

Now based on the corollary, imagine we have a cycle  $Z \subseteq F^*$ . And by observation 2, all edges of  $F^*$  (hence  $Z$ ) have  $c'_e = 0$ . Then as long as we can reach one node of the cycle, we can take appropriate edges of the cycle to reach all nodes in the cycle without increasing the  $c'$ -cost. As we don't care about the inside of the cycle, we can contract the cycle and run the algorithm recursively.

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### Algorithm 5: Edmond's Algorithm for MCA

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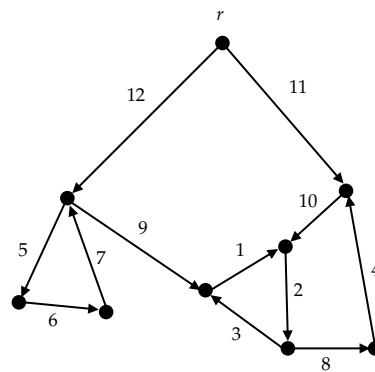
**Input:** Directed graph  $G = (V, E)$ , root  $r \in V$ ,  $\{c_e\}$  edge costs; assume there exists  $r \rightarrow v$  path for all  $v \in V$

- 0 **if**  $G$  has only one node **then return**  $\emptyset$
  - 1 **foreach**  $v \neq r$  **do**
    - $y_v := \min_{(u,v) \in E} c_{u,v}$
    - $c'_e := c_e - y_v$  for all  $e$  entering  $v$
  - 2 For each  $v \neq r$ , choose a 0  $c'$ -cost edge entering  $v$ . Let  $F^*$  be the resulting set of edges.
  - 3 **if**  $F^*$  is an arborescence **then return**  $F^*$
  - 4 Otherwise  $F^*$  contains a directed cycle. Find a cycle  $Z \subseteq F^*$  (not containing  $r$ ). Contract  $Z$  into a single supernode<sup>a</sup> to get a graph  $G' = (V', E')$ .
  - 5 Recursively find an MCA  $(V', F')$  in  $G'$  with respect to  $\{c'_e\}$  edge costs.
  - 6 Extend  $(V', F')$  to an arborescence  $(V, F)$  in  $G$ :
    - Let  $v \in Z$  be the node that has an incoming edge in  $F'$ .
    - Set  $F \leftarrow F' \cup Z \setminus \{\text{edge of } Z \text{ entering } v\}$
  - 7 **return**  $F$
- 

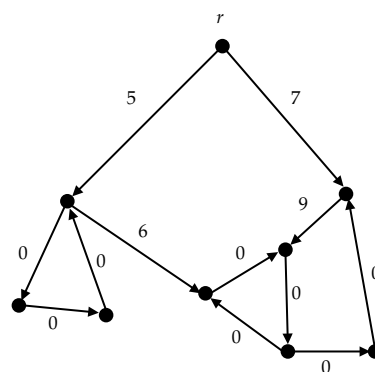
<sup>a</sup>Remove self-loops, but retain parallel edges

Now let's consider an example.

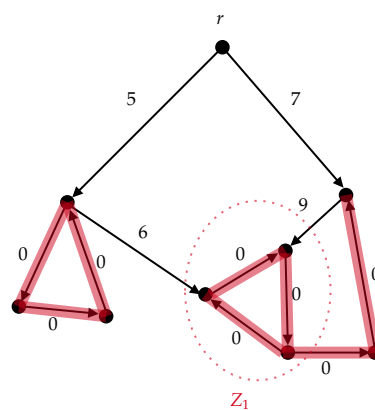




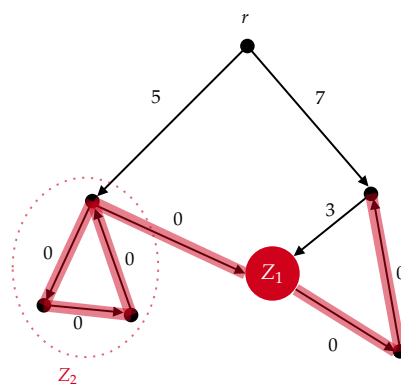
We then change the cost to  $c'_e$ .



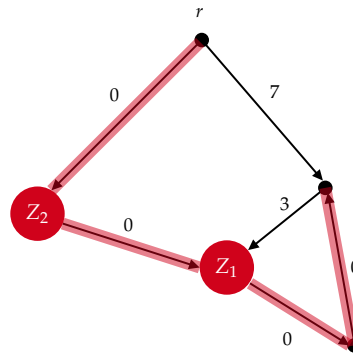
Then we mark the zero costs edges to produce  $F^*$ . And we can see there are cycles in this set of edges.



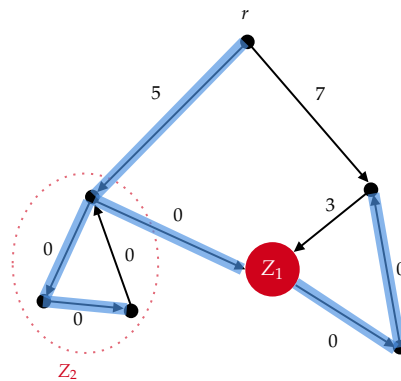
We first pick cycle  $Z_1$  and contract it. And do the cost change again.



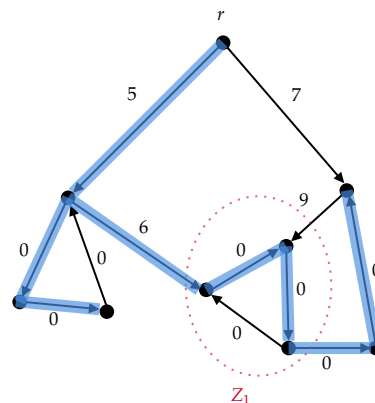
And we found cycle  $Z_2$ , and contract it, and do the cost change.



Now red edges are arborescence. Recursive calls stop. Then we expand it back.



Then expand again.



which is the output of the algorithm.

### Running time

Line 0 takes  $O(1)$ . Line 1-2 are operations on edges, take  $O(m)$ . Line 3 on checking arborescence takes  $O(n)$ : follow incoming edges in  $F^*$  to see if we can reach  $r$ , either  $rv$  path or a cycle. Line 4 takes  $O(|Z|) = O(n)$ . Line 6 takes  $O(|Z|) = O(n)$ .

Algorithm makes  $O(m)$  elementary operations and a recursive call to a smaller graph; at most  $n$  recursive calls. Thus polynomial running time.

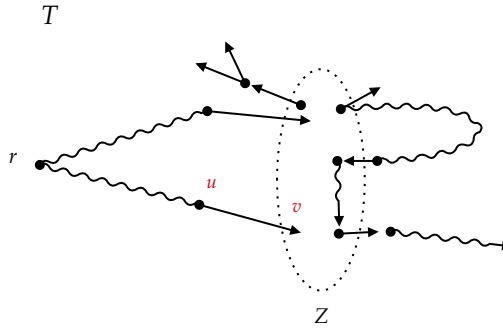
## Correctness

## Lemma 3.2

Suppose we have  $\{d_e\}$  edge costs, and a 0  $d$ -cost cycle  $Z$  such that  $r \notin Z$ . Then there exists an MCA (with respect to  $d$ -costs) that has exactly one edge entering  $Z$ .

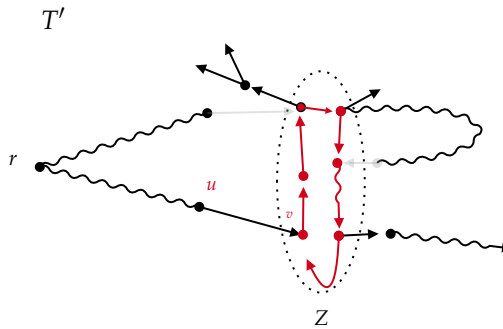
**Proof:**

Let  $T$  be any arborescence, where it might have multiple edges entering  $Z$ . We will show that  $T$  can be modified to an arborescence  $T'$  with the stated property and such that  $d(T') \leq d(T)$  where  $d(T) := \sum_{e \in T} d_e$ .  $T$  might look like the graph below.



Among all edges  $(u', v') \in T$  that enter  $Z$ , let  $(u, v)$  be such that the  $r \rightarrow u$  path in  $T$  has the fewest number of edges. Note that the  $r \rightarrow u$  path in  $T$  has no nodes from  $Z$ . Let

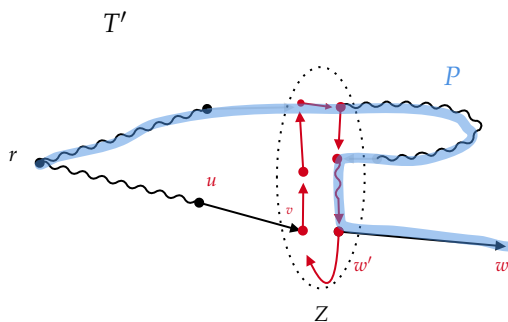
$$T' \leftarrow T - \{(u', v') \in T : v' \in Z, u' \notin Z, v' \neq v\} \cup \{\text{edges of } Z \text{ except for the edge entering } v\}$$



Note that  $d(T') \leq d(T)$  because all edges we are adding in  $Z$  have zero costs. We now have to show  $T'$  is an arborescence.

$T'$  has  $n - 1$  edges since every  $w \neq r$  has exactly one incoming edge. Now it's remained to show that there exists an  $r \rightarrow w$  path  $T' \forall w \neq r$ . Let  $P$  be the  $r \rightarrow w$  path in  $T$ . If  $P$  does not contain any node of  $Z$ , then  $P$  is also an  $r \rightarrow w$  path in  $T'$ .

Now suppose  $P$  contains a node of  $Z$ . Let  $w'$  be the last node of  $P$  in  $Z$ . Then  $T'$  contains:



- $r \rightarrow v$  path in  $T$
- $v \rightarrow w'$  portion of  $Z$
- $w' \rightarrow w$  portion of  $P$

Concatenating these gives an  $r \rightarrow w$  path in  $T'$ .

This also shows that undirected version of  $T'$  is connected, and has  $n - 1$  edges. Thus the undirected version is a spanning tree.  $\square$

**Observation** Any arborescence in  $G$  that enters cycle  $Z$  in step 4 of algorithm exactly once, yields an arborescence in  $G'$  and VICE VERSA, and these two arborescence have the same  $c'$ -cost.

### Theorem 3.3

Edmond's algorithm finds an MCA of  $G$ .

#### Proof:

By induction on  $|V|$ . Base cases where  $|V| = 1$  or  $2$  are clearly true. Suppose inductively, algorithm finds an MCA  $T'$  in  $G'$  with respect to  $c'$ -edge costs.

Let  $T^*$  be an MCA in  $G$  with respect to  $c'$ -edge costs that enters  $Z$  exactly once, which exists by lemma 3.2. Let  $T^{*'}$  be arborescence obtained from  $T^*$  for graph  $G'$ .

We have

$$\begin{aligned} c(T) &= c'(T') && \text{since } c'_e = 0 \quad \forall e \in Z \\ &\leq c'(T^{*'}) && T' \text{ is an MCA of } G' \text{ wrt. } c'\text{-edge costs, induction hypothesis} \\ &= c'(T^*) && \text{edges of } Z \text{ have } 0 \text{ } c'\text{-cost} \end{aligned}$$

So  $T$  is an MCA of  $G$  with respect to  $c'$ -edge costs. Thus  $T$  is an MCA of  $G$  with respect to  $c$ -edge costs.  $\square$

## Introduction to Matroids

### matroid

A matroid is a tuple  $M = (U, \mathcal{I})$ , where  $U$  is a ground set (or universe), and  $\mathcal{I} \subseteq 2^U$  is a collection of subsets of  $U$  satisfying the following properties:

- (a)  $\emptyset \in \mathcal{I}$ .
- (b) If  $A \in \mathcal{I}$ , and  $B \subseteq A$ , then  $B \in \mathcal{I}$ .
- (c) (Exchange property) if  $A, B \in \mathcal{I}$ , with  $|A| < |B|$ , then  $\exists e \in B - A$  such that  $A \cup \{e\} \in \mathcal{I}$ .

Sets in  $\mathcal{I}$  are called **independent sets**. A set not in  $\mathcal{I}$  is called a **dependent set**.

A **maximal independent set**, i.e., a set  $B \in \mathcal{I}$  such that  $B \cup \{e\} \notin \mathcal{I} \forall e \in U - B$ , is called a **basis**.

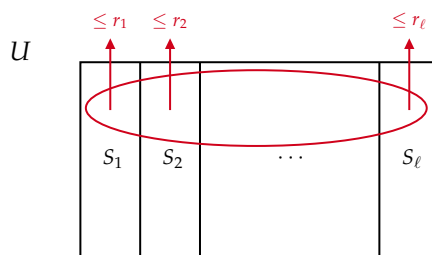
#### Example:

Let  $U$  be any  $n$ -element set, e.g.,  $\{1, \dots, n\}$ . Let  $\mathcal{I} := \{A \subseteq U : |A| \leq k\}$ , where  $k \geq 0$ . Then  $M = (U, \mathcal{I})$  is a matroid (uniform matroid).

- (a)  $\emptyset \in \mathcal{I}$  since  $|\emptyset| = 0 \leq k$
- (b) If  $A \in \mathcal{I}$  and  $B \subseteq A$ , then  $|B| \leq |A| \leq k$ , so  $B \in \mathcal{I}$ .
- (c) Suppose  $A, B \in \mathcal{I}$  with  $|A| < |B| (\leq k)$ . Take any  $e \in B - A$ , and note that  $|A \cup \{e\}| = |A| + 1 \leq |B| \leq k$ , so  $A \cup \{e\} \in \mathcal{I}$ .

#### Example:

Again, let  $U$  be  $n$ -element set. Let  $(S_1, \dots, S_\ell)$  be a partition of  $U$ , and let  $r_1, \dots, r_\ell \geq 0$  be some non-negative integers. Let  $\mathcal{I} := \{A \subseteq U : |A \cap S_i| \leq r_i \quad \forall i = 1, \dots, \ell\}$ .



Then  $M = (U, \mathcal{I})$  is a matroid (partition matroid).

(a), (b) hold trivially. For part (c), suppose  $A, B \in \mathcal{I}$  with  $|A| < |B|$ . Then there exists  $S_i$  such that  $|A \cap S_i| < |B \cap S_i|$  (since  $(S_1, \dots, S_\ell)$  partition  $U$ ). Consider any  $e \in (B \cap S_i) - (A \cap S_i)$ , and  $A' = A \cup \{e\}$ . Then

$$|A' \cap S_j| = \begin{cases} |A \cap S_j| \leq r_j & \text{if } j \neq i \\ |A \cap S_i| + 1 \leq |B \cap S_i| \leq r_i & \text{if } j = i \end{cases}$$

So  $A' \in \mathcal{I}$ .

**Example:**

Let  $U$  be collection of  $n$  vectors in  $\mathbb{R}^d$ . Let

$$\mathcal{I} := \{A \subseteq U : \text{the vectors in } A \text{ are linearly independent}\}$$

Then  $M = (U, \mathcal{I})$  is a matroid.

(a), (b) hold trivially. (c) holds because of basic linear algebra: Suppose  $A, B \in \mathcal{I}$  with  $|A| < |B|$ . If every  $v \in B$  is a linear combinations of vectors in  $A$ , then since  $|B| > |A|$ , the vectors in  $B$  must be linearly dependent. Then there exists  $v \in B$  such that  $v \notin \text{span}(A)$  (where  $\text{span}(A)$  is all vectors that are linear combinations of vectors in  $A$ ). So  $A \cup \{v\}$  consists of linearly independent vectors, i.e.,  $A \cup \{v\} \in \mathcal{I}$ .

**Example:**

Let  $G = (V, E)$  be an undirected graph. Let  $U = E$ ,  $\mathcal{I} := \{A \subseteq E : A \text{ is acyclic}\}$ . Then  $M = (U, \mathcal{I})$  is a matroid (graphic matroid).

(a), (b) hold trivially. Consider (c). Suppose  $A, B \in \mathcal{I}$  with  $|A| < |B|$ . Then  $G_A = (V, A)$  has  $n - |A|$  components where  $n = |V|$ , and  $G_B = (V, B)$  has  $n - |B| < n - |A|$  components.

pic

It cannot be that the vertex-set of every component of  $G_B$  is a subset of the vertex-set of some components of  $G_A$ . I.e., there exists some component of  $G_B$  that intersects at least two components of  $G_A$ .

pic

This means some  $e \in B$  that connects two components of  $G_A$ . Thus  $A \cup \{e\}$  is acyclic, so  $A \cup \{e\} \in \mathcal{I}$ .

## 4.1 Matroid Optimization Problems

### Max-weight independent set (MWIS) problem

Given a matroid  $M = (U, \mathcal{I})$ , and weights  $\{w_e\}_{e \in U}$  (the  $w_e$ 's could be arbitrary), find a max-weight independent set, i.e., find  $A \in \mathcal{I}$  such that

$$w(A) := \sum_{e \in A} w_e = \max_{B \in \mathcal{I}} w(B)$$

### Matroid intersection problem

Given two matroids  $M_1 = (U, \mathcal{I}_1)$  and  $M_2 = (U, \mathcal{I}_2)$  and weights  $\{w_e\}_{e \in U}$ , find a max-weight set that is independent in both matroids (common independent set), i.e., solve

$$\max_{A \in \mathcal{I}_1 \cap \mathcal{I}_2} w(A)$$

Note that max-weight spanning tree with positive edge weights is a special case of MWIS problem, where the matroid is the graphic matroid. (since bases of graphic matroid associated with a connected graph are spanning trees). MST with  $\{c_e\}$  edge costs can be captured by max-weight spanning tree with positive edge weights. By defining  $w_e = M - c_e$ , where  $M > \max_e c_e$  (so that  $w_e > 0$ ).  $w(T) = (n-1)M - c(T)$  for any spanning tree  $T$ .

Now we develop the greedy algorithm for MWIS. Input:  $M = (U, \mathcal{I}), \{w_e\}_{e \in U}$ . We may assume that  $w_e > 0$  for all  $e \in U$ . Because otherwise, we can move to the smaller matroid

$$M' = (U' := \{e \in U : w_e > 0\}, \mathcal{I}' = \{A \subseteq U' : A \in \mathcal{I}\})$$

where we can verify  $M'$  is a matroid. Solving MWIS on  $M'$  will also solve MWIS on  $M$ . Since every independent set of  $M'$  is also independent in  $M$ . If  $A \in \mathcal{I}$ , then  $A \cap U' \in \mathcal{I}'$  and  $w(A \cap U') \geq w(A)$ .

---

**Algorithm 6:** Greedy Algorithm for MWIS

---

**Input:**  $M = (U, \mathcal{I}), \{w_e\}_{e \in U}$

- 1 Sort elements in decreasing order of weight.
  - 2 Initialize  $A \leftarrow \emptyset$
  - 3 Considering elements in sorted order, if  $A \cup \{e\}$  is independent, then set  $A \leftarrow A \cup \{e\}$  where  $e$  is current element being considered.
  - 4 **return**  $A$
- 

Observe that above algorithm run on graphic matroid is equivalent to Kruskal for max-weight spanning tree.

**Claim 4.1**

If  $B, B'$  are two bases of a matroid, then  $|B| = |B'|$ .

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