Coding Theory

CO 331

Alfred Menezes

LaTeXed by Sibelius Peng

Preface

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Sibelius Peng

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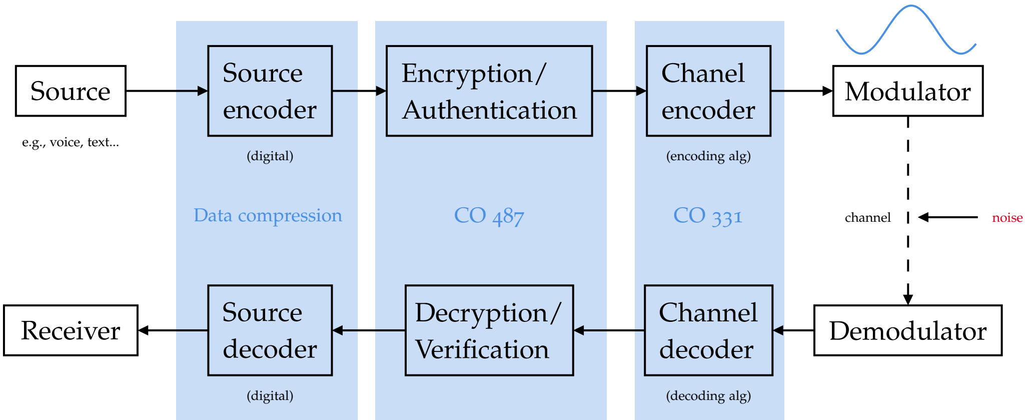
Introduction

Coding theory is about clever ways of adding redundancy to messages to allow (efficient) error detection and error correction.

Here is our communication model:



Example: Parity Code

Encoding algorithm Add a 0 bit to the (binary) msg m if the number of 1's in m is even; else add a 1 bit.

Decoding algorithm If the number of 1's in a received msg r is even, then accept r; else declare that an error has occurred.

Example: Replication Code

<table>
<tr><th>Source msgs</th><th>Codeword</th><th># err/codeword (always) detected</th><th># err/codeword (always) corrected *</th><th>Information rate</th></tr>
<tr><td>0</td><td>0</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">1</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>0</td><td>00</td><td rowspan="2">1</td><td rowspan="2">0</td><td rowspan="2">$\frac{1}{2}$</td></tr>
<tr><td>1</td><td>11</td></tr>
<tr><td>0</td><td>000</td><td rowspan="2">2</td><td rowspan="2">1</td><td rowspan="2">$\frac{1}{3}$</td></tr>
<tr><td>1</td><td>111</td></tr>
<tr><td>0</td><td>0000</td><td rowspan="2">3</td><td rowspan="2">1</td><td rowspan="2">$\frac{1}{4}$</td></tr>
<tr><td>1</td><td>1111</td></tr>
<tr><td>0</td><td>00000</td><td rowspan="2">4</td><td rowspan="2">2</td><td rowspan="2">$\frac{1}{5}$</td></tr>
<tr><td>1</td><td>11111</td></tr>
</table>

$\xrightarrow{\text{encoding algorithm}}$

*: using "nearest neighbour decoding"

Goal of Coding Theory

Design codes so that:

1. High information rate
2. High error-correcting capability
3. Efficient encoding & decoding algorithms

Course Overview

This course deals with *algebraic methods* for designing good (block) codes. The focus is on error correction (not on error detection). These codes are used in wireless communications, space probes, CD/DVD players, storage, QR codes, etc.

Some modern stuff are not covered: Turbo codes, LDPC codes, Raptor codes, . . . Their math theories are not so elegant as algebraic codes.

The big picture

Coding theory in its broadest sense deals with techniques for the *efficient*, *secure* and *reliable* transmission of data over communication channels that may be subject to *non-malicious errors* (noise) and *adversarial intrusion*. The latter includes passive intrusion (eavesdropping) and active intrusion (injection/deletion/modification).

1

Fundamentals

1.1 Basic Definitions and Concepts

alphabet

An **alphabet** A is a finite set of $q \geq 2$ symbols.

word

A **word** is a finite sequence of symbols from A (also: vector, tuple).

length

The **length** of a word is the number of symbols it has.

code

A **code** C over A is a set of words (of size ≥ 2).

codeword

A **codeword** is a word in the code C.

block code

A **block code** is a code in which all codewords have the same length.

A **block code of length** n **containing** M **codewords over** A is a subset $C \subseteq A^n$ with $|C| = M$. C is called an $[n, M]$-code over A.

Example:

$A = \{0, 1\}$. $C = \{00000, 11100, 00111, 10101\}$ is a $[5, 4]$-code over $\{0, 1\}$.

Messages		Codewords
00	$\to$	00000
10	$\to$	11100
01	$\to$	00111
11	$\to$	10101

Encoding of messages (1-1 map)

Assumptions about the communications channel

(1) The channel only transmits symbols from A ("hard decision decoding").

(2) No symbols are deleted, added, interchanged or transposed during transmission.

(3) The channel is a q-symmetric channel:

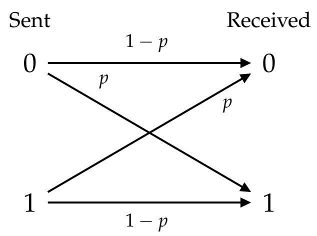
Let $A = \{a_1, \ldots, a_q\}$. Let $X_i =$ the i^{th} symbol sent. Let $Y_i =$ the i^{th} symbol received. Then for all $i \geq 1$, and all $i \leq j, k \leq q$,

$$\Pr(Y_i = a_j | X_i = a_k) = \begin{cases} 1 - p, & \text{if } j = k \\ \frac{p}{q-1}, & \text{if } j \neq k. \end{cases}$$

p is called the **symbol error probability** of the channel $(0 \leq p \leq 1)$.

Binary Symmetric Channel (BSC)

A 2-symmetric channel is called a binary symmetric channel.



For a BSC:

1. If $p = 0$, the channel is *perfect*.
2. If $p = 1/2$, the channel is *useless*.
3. If $1/2 < p \leq 1$, then flipping all received bits converts the channel to a BSC with $0 \leq p < 1/2$.
4. Henceforth, we will assume that $0 < p < 1/2$ for a BSC.

Exercise:

For a q-symmetric channel, show that one can take $0 < p < \frac{q-1}{q}$ WLOG.

One can first consider the case $q = 3$.

information rate

The **information rate** (or rate) R of an $[n, M]$-code C over A is $R = \frac{\log_q M}{n}$.

If C encodes messages that are k-tuples over A (so $M = |A^k| = q^k$), then $R = \frac{k}{n}$.

Note:
$0 \le R \le 1$. Ideally, R should be close to 1.

Example:
The rate of the binary code $C = \{00000, 11100, 00111, 10101\}$ is $R = \frac{2}{5}$.

Hamming distance

The **Hamming distance** (or distance) between two n-tuples over A is the number of coordinate positions in which they differ.

The Hamming distance (or distance) of an $[n, M]$-code C is $d(C) = \min\{d(x, y) : x, y \in C, x \neq y\}$.

Example:
The distance of $C = \{00000, 11100, 00111, 10101\}$ is $d(C) = 2$.

Theorem 1.1: properties of Hamming distance

For all $x, y, z \in A^n$,

1. $d(x, y) \ge 0$, with $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) + d(y, z) \ge d(x, y)$ ($\triangle$ inequality).

1.2 Decoding Strategy

Example:
Let $C = \{00000, 11100, 00111, 10101\}$. C is a $[5, 4]$-code over $\{0, 1\}$ (a binary code).

Error Detection If C is used for error detection only, the strategy is the following: A received word $r \in A^n$ is accepted if and only if $r \in C$.

Error Correction Let C be an $[n, M]$-code over A with distance d. Suppose $c \in C$ is transmitted, and $r \in A^n$ is received. The (channel) decoder must decide one of the following:

(i) No errors have occurred; *accept* r.

(ii) Errors have occurred; *correct*[1] (*decode*) r to a codeword $c \in C$?

(iii) Errors have occurred; *no correction is possible*.

Nearest Neighbour Decoding

(i) Incomplete Maximum Likelihood Decoding (IMLD):

If there is a unique codeword $c \in C$ such that $d(r, c)$ is minimum, then correct r to c. If no such c exists, then report that errors have occurred, but correction is not possible (ask for retransmission, or disregard information).

(ii) Complete Maximum Likelihood Decoding (CMLD):

[1] Error correction does not guarantee that the channel decoder always makes the correct decision. For example, $00000 \xrightarrow{\text{transmit}} 11100$ which is accepted.

Same as IMLD, except that if there are two or more $c \in C$ for which $d(r, c)$ is minimum, correct r to an arbitrary one of these.

Is IMLD a reasonable strategy?

Theorem 1.2

IMLD chooses the codeword c for which the conditional probability

$$P(r|c) = P(r \text{ is received}|c \text{ is sent})$$

is largest.

Proof:
Suppose $c_1, c_2 \in C$ with $d(c_1, r) = d_1$ and $d(c_2, r) = d_2$. Suppose $d_1 > d_2$.

Now

$$P(r|c_1) = (1-p)^{n-d_1}\left(\frac{p}{q-1}\right)^{d_1}$$

and

$$P(r|c_2) = (1-p)^{n-d_2}\left(\frac{p}{q-1}\right)^{d_2}$$

So,

$$\frac{P(r|c_1)}{P(r|c_2)} = (1-p)^{d_2-d_1}\left(\frac{p}{q-1}\right)^{d_1-d_2} = \left(\frac{p}{(1-p)(q-1)}\right)^{d_1-d_2}$$

Recall

$$p < \frac{q-1}{q} \implies pq < q-1 \implies 0 < q - pq - 1$$

$$\implies p < p + q - pq - 1 \implies p < (1-p)(q-1) \implies \frac{p}{(1-p)(q-1)} < 1$$

Hence

$$\frac{P(r|c_1)}{P(r|c_2)} < 1$$

and so

$$P(r|c_1) < P(r|c_2)$$

and the result follows. □

Minimum Error Probability Decoding (MED)

An *ideal strategy* would be to correct r to a codeword $c \in C$ for which $P(c|r) = P(r \text{ is received}|c \text{ is sent})$ is largest. This is MED.

Example: (IMLD/CMLD) is not the same as MED

Consider $C = \{\underset{c_1}{\underset{\uparrow}{000}}, \underset{c_2}{\underset{\uparrow}{111}}\}$. Suppose $P(c_1) = 0.1$ and $P(c_2) = 0.9$. Suppose $p = \frac{1}{4}$ (for a BSC).

Suppose $r = 100$ is the received word. Then

$$P(c_1|r) = \frac{P(r|c_1)\cdot P(c_1)}{P(r)} = \frac{p(1-p)^2 \times 0.1}{P(r)} = \frac{9}{640}\cdot\frac{1}{P(r)}$$
$$P(c_2|r) = \frac{P(r|c_2)\cdot P(c_2)}{P(r)} = \frac{(1-p)p^2 \times 0.9}{P(r)} = \frac{27}{640}\cdot\frac{1}{P(r)}$$

So, MED decodes r to c_2. But IMLD decodes r to c_1.

IMLD vs. MED

- IMLD maximizes $P(r|c)$. MED maximizes $P(c|r)$.
- (i) MED has the drawback that the decoding algorithm depends on the probability distribution of source messages.

  (ii) If all source messages are equally likely, then CMLD and MED are equivalent:

$$P(r|c_i) = P(c_i|r) \cdot P(c_i)/P(r) = P(c_i|r) \cdot \underbrace{\left[\frac{1}{M \cdot P(r)}\right]}_{\substack{\text{does not} \\ \text{depend on } c_i}}$$

  (iii) In practice IMLD (or CMLD) is used.
- In this course, we will use IMLD/CMLD.

1.3 Error Correcting & Detecting Capabilities of a Code

Detection Only

Strategy: If r is received, then accept r if and only if $r \in C$.

e-error detecting code

A code C is an **e-error detecting code** if the decoder always makes the correct decision if e or fewer errors per codeword are introduced by the channel.

Example:

Consider $C = \{000, 111\}$.

C is a 2-error detecting code.

C is not a 3-error detecting code.

Theorem 1.3

A code C of distance d is a $(d-1)$-error detecting code (but is not a d-error detecting code).

Proof:

Suppose $c \in C$ is sent.

If no errors occur, then c is received (and is accepted).

Suppose that # of errors is ≥ 1 and $\leq d-1$; let r be the received word. Then $1 \leq d(r,c) \leq d-1$, so $r \notin C$. Thus r is rejected. This proves that it is $(d-1)$-error detecting code.

Since $d(C) = d$, there exist $c_1, c_2 \in C$ with $d(c_1, c_2) = d$. If c_1 is sent and c_2 is received, then c_2 is accepted; the d errors go undetected. □

Correction

Strategy: IMLD/CMLD

e-error correcting code

A code C is an **e-error correcting code** if the decoder always makes the correct decision if e or fewer errors per codeword are introduced by the channel.

Example:

Consider $C = \{000, 111\}$.

C is a 1-error correcting code.

C is not a 2-error correcting code.

Theorem 1.4

A code C of distance d is an e-error correcting code, where $e = \lfloor \frac{d-1}{2} \rfloor$.

Proof:

Suppose that $c \in C$ is sent, at most $\frac{d-1}{2}$ errors are introduced, and r is received. Then $d(r, c) \leq \frac{d-1}{2}$.

On the other hand, if c_1 is any other codeword, then

$$\begin{aligned} d(r, c_1) &\geq d(c, c_1) - d(r, c) && \triangle \text{ ineq} \\ &\geq d - \frac{d-1}{2} && \text{since } d(C) = d \\ &= \frac{d+1}{2} \\ &> \frac{d-1}{2} \geq d(r, c) \end{aligned}$$

Hence c is the unique codeword at minimum distance from r, so the decoder correctly concludes that c was sent. □

Exercise:

Suppose $d(C) = d$, and let $e = \lfloor \frac{d-1}{2} \rfloor$. Show that C is *not* an $(e+1)$-error correcting code.

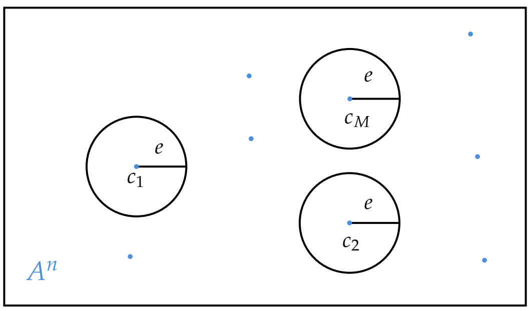
A natural question to ask is: given A, n, M, d, does there exist an $[n, M]$-code C over A of distance $\geq d$. This can be phrased as an equivalent sphere packing problem:

Sphere packing

Can we place M spheres of radius $e = \lfloor \frac{d-1}{2} \rfloor$ in A^n so that no two spheres overlap?

$C = \{c_1, \ldots, c_M\}$, $e = \lfloor \frac{d-1}{2} \rfloor$, S_c = sphere of radius e centered at c = all words within distance e of c.

We proved: if $c_1, c_2 \in C, c_1 \neq c_2$, then $S_{c_1} \cap S_{c_2} = \varnothing$.



Let $n = 128, q = 2, M = 2^{64}$. Does there exists a binary $[n, M]$-code with $d \geq 22$? If so, can encoding and decoding be done efficiently?

We'll view $\{0,1\}^{128}$ as a vector space of dimension 128 over $\mathbb{Z}_2$. We'll choose C to be a 64-dimensional subspace of this vector space. We will construct such a code at the end of the course. The main tools used will be linear algebra (over finite fields) and abstract algebra (rings and fields).

2

Introduction to Finite Fields

2.1 Definitions

ring

A **(commutative) ring** $(R, +, \cdot)$ consists of a set R and two operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$, such that

1. $a + (b + c) = (a + b) + c \quad \forall a, b, c \in R.$
2. $a + b = b + a, \quad \forall a, b \in R.$
3. $\exists 0 \in R$ such that $a + 0 = a$, $\forall a \in R$.
4. $\forall a \in R$, $\exists - a \in R$ such that $a + (-a) = 0$.
5. $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall a, b, c \in R.$
6. $a \cdot b = b \cdot a, \qquad \forall a, b \in R.$
7. $\exists 1 \in R, 1 \neq 0$, such that $a \cdot 1 = a, \quad \forall a \in R.$
8. $a \cdot (b + c) = a \cdot b + b \cdot c, \qquad \forall a, b, c \in R.$

Notation We will denote $(R, +, \cdot)$ by R.

Example:
$\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}$ are commutative rings.

field

A **field** $(F, +, \cdot)$ is a commutative ring with the additional property:

9. $\forall a \in F, a \neq 0, \exists a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.

Example:
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. $\mathbb{Z}$ is *not* a field.

infinite/finite field

A field $(F, +, \cdot)$ is a **finite field** if F is a finite set; otherwise it is an **infinite field**. If F is a finite field , its **order** is $|F|$.

Example:

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are infinite fields.

For which integers $n \geq 2$ does there *exist* a finite field of order n? How does one *construct* such a field, i.e., what are the field elements, and how are the field operations performed?

The Integers Modulo n

Let $n \geq 2$. Recall that $\mathbb{Z}_n$ consists of the set of equivalence classes of integers modulo n, $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$, with addition and multiplication: $[a] + [b] = [a+b]$, $[a] \cdot [b] = [a \cdot b]$.

More simply, we write $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, and perform addition and multiplication modulo n.

Example:

$\mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$. In $\mathbb{Z}_9$, $3 + 7 = 1$ and $3 \cdot 7 = 3$.

More precisely, $3 + 7 \equiv 1 \pmod 9$ and $3 \cdot 7 \equiv 3 \pmod 9$.

$\mathbb{Z}_n$ is a commutative ring (i.e., axioms 1-8 in the definition are satisfied).

When is $\mathbb{Z}_n$ is a field?

Theorem 2.1

$\mathbb{Z}_n$ is a field if and only if n is prime.

Proof:

$\Leftarrow$ Suppose n is prime. Let $a \in \mathbb{Z}_n$, $a \neq 0$ (so $1 \leq a \leq n-1$). Since n is prime, $\gcd(a, n) = 1$. Hence $\exists s, t \in \mathbb{Z}$ such that $as + nt = 1$. Reducing both sides modulo n gives $as \equiv 1 \pmod n$. Hence $a^{-1} = s$. Thus $\mathbb{Z}_n$ is a field.

$\Rightarrow$ Suppose n is composite, say $n = ab$ where $2 \leq a, b \leq n-1$.

Now, if a^{-1} exists, say $ac \equiv 1 \pmod n$, then $abc \equiv b \pmod n$, so $nc \equiv b \pmod n$. Thus $b \equiv 0 \pmod n$, so $n \mid b$ which is absurd since $2 \leq b \leq n-1$. Thus $\mathbb{Z}_n$ is not a field.

$\square$

We have established the existence of finite fields of order n, for each prime n. What about finite fields of order n, where n is composite? In particular, is there a field of order 4? Order 6?

characteristic

Let F be a field. The **characteristic** of F, denoted $\text{char}(F)$, is the smallest positive integer m such that $\underbrace{1 + \cdots + 1}_{m} = 0$. If no such m exists, then $\text{char}(F) = 0$.

Example:

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ have characteristic 0. $\mathbb{Z}_p$ (p prime) has characteristic p.

Theorem 2.2

If $\mathrm{char}(F) = 0$, then F is an infinite field.

Proof:
The elements $1, 1+1, 1+1+1, \ldots$ are distinct, because if $\underbrace{1+\cdots+1}_{a} = \underbrace{1+\cdots+1}_{b}$ where $a < b$, then $(\underbrace{1+\cdots+1}_{b}) - (\underbrace{1+\cdots+1}_{a}) = 1+\cdots+1_{b-a} = 0$, contradicting $\mathrm{char}(F) = 0$. $\square$

Theorem 2.3

Let F be a field with $\mathrm{char}(F) = m \neq 0$. Then m is prime.

Proof:
Suppose m is composite, say $m = ab$ where $2 \leq a, b \leq m-1$. Let $s = \underbrace{1+\cdots+1}_{a}$ and $t = \underbrace{1+\cdots+1}_{b}$; note that $s, t \neq 0$. Then $s \cdot t = (\underbrace{1+\cdots+1}_{a}) \cdot (\underbrace{1+\cdots+1}_{b}) = \underbrace{1+\cdots+1}_{ab=m} = 0$. Thus

$$s \cdot t \cdot t^{-1} = s \cdot 1 = s = 0,$$

a contradiction. We then conclude that m is prime. $\square$

Let F be a finite field of characteristic p. Consider the subset of elements of F:

$$E = \{0, 1, 1+1, 1+1, \ldots, \underbrace{1+\cdots+1}_{p-1}\}.$$

The elements of E are distinct. One can verify that E is a field, using the same operations as F. E is a **subfield** of F. If we identify the elements of E with the elements of $\mathbb{Z}_p$ in the natural way, then E is essentially the same field as $\mathbb{Z}_p$. We have proven:

Theorem 2.4

Let F be a finite field of char p. Then $\mathbb{Z}_p$ is a subfield of F.

Finite fields as vector spaces

Let F be a finite field of characteristic p. Identify:

$$\begin{array}{rcl} \text{vectors} & \leftrightarrow & \text{elements of } F \\ \text{scalars} & \leftrightarrow & \text{elements of } \mathbb{Z}_p \\ \text{vector addition} & \leftrightarrow & \text{addition of } F \\ \text{scalar multiplication} & \leftrightarrow & \text{multiplication of } F \end{array}$$

Then F is a vector space over $\mathbb{Z}_p$ (i.e., the axioms of what it means to be a vector space are satisfied).

2.2 Finite fields: Non-existence

Theorem 2.5

Let F be a finite field of characteristic p. Then the order of F is p^n, for some positive integer n.

Proof:

Let the dimension of F as a vector space over $\mathbb{Z}_p$ be n. Let $\alpha_1, \ldots, \alpha_n$ be a basis for F over $\mathbb{Z}_p$. Then each element $\beta \in F$ can be written uniquely in the form $\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$, where $c_i \in \mathbb{Z}_p$. Thus $F = \left\{ \sum_{i=1}^{n} c_i\alpha_i : c_i \in \mathbb{Z}_p \right\}$, so $|F| = p^n$. $\square$

For example, there do not exist finite fields of order $6, 10, 12, 14, 15, \ldots$

Do finite fields of orders $4, 8, 9, 16, 25, 27, \ldots$ exist?

2.3 Existence of finite fields

Polynomial rings Let F be a field. $F[x]$ denotes the set of all polynomials in x with coefficients from F. Addition and multiplication of polynomials in $F[x]$ is done in the usual way, with coefficient arithmetic done in F.

Example:

In $\mathbb{Z}_5[x]$,

$$\begin{aligned}(3x^4 + 2x^3 + x + 4) + (x^5 + 2x^4 + x^2 + 2x + 3) &= x^5 + 2x^3 + x^2 + 3x + 2 \\ (3x^2 + 4x + 1) \cdot (2x^2 + x + 2) &= (x^4 + x^3 + 2x^2 + 4x + 2).\end{aligned}$$

Note that $F[x]$ is an infinite commutative ring.

Construction of finite fields: main idea

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\text{fix } n \geq 2", "a \equiv b \pmod{n}"'] & \mathbb{Z}_n \arrow[r, "n \text{ prime}"] & \mathbb{Z}_n \end{tikzcd}$$

infinite commutative ring | finite commutative ring | finite field of order n

$$\begin{tikzcd} \mathbb{Z}_p[x] \arrow[r, "{\text{fix } f \in \mathbb{Z}_p[x],\ \deg(f) = n \geq 1}", "g \equiv h \pmod{f}"'] & \mathbb{Z}_p[x]/(f) \arrow[r, "f \text{ irreducible}"] & \mathbb{Z}_p[x]/(f) \end{tikzcd}$$

infinite commutative ring | finite commutative ring | finite field of order p^n

Polynomial division

Let $f, g \in F[x]$, with $g \neq 0$. Then there exist unique polynomials $r, s \in F[x]$ such that $f = \underset{\substack{\uparrow \\ \text{quotient}}}{s}\, g + \underset{\substack{\uparrow \\ \text{remainder}}}{r}$, $\deg(r) < \deg(g)$. By convention, $\deg(0) = -\infty$.

Example:

Consider $f = 3x^4 + 2x^2 + x + 1, g = 2x^2 + 3x + 4 \in \mathbb{Z}_5[x]$.

$$\begin{array}{r|lllll} & 4x^2 & +3 & & & \\ \hline 2x^2+3x+4 & 3x^4 & +2x^3 & +2x^2 & +x & +1 \\ & 3x^4 & +2x^3 & +x^2 & & \\ \hline & & & x^2 & +x & +1 \\ & & & x^2 & +4x & +2 \\ \hline & & & & 2x & +4 \end{array}$$

So, $f = (4x^2 + 3)g + (2x + 4)$.

The ring $F[x]/(f)$

$g \equiv h \pmod f$

Let $f \in F[x]$ with $\deg(f) \geq 1$. Let $g, h \in F[x]$. Then g is congruent to h modulo f, written $g \equiv h \pmod f$, if $g - h = \ell f$ for some $\ell \in F[x]$ (equivalently, $f \mid (g - h)$, or g, h leave the same remainder upon division by f).

The relation $\equiv \pmod f$ is an equivalence relation, and partitions $F[x]$ into equivalence classes:

$$[g] = \{h \in F[x] : g \equiv g \pmod f\}.$$

Addition & multiplication: $[g] + [h] = [g + h], [g] \cdot [h] = [g \cdot h]$.

$F[x]/(f)$

The set of equivalence classes is denoted $F[x]/(f)$.

Theorem 2.6

$F[x]/(f)$ is a commutative ring.

Suppose now that $\deg(f) = n$. Let $g \in F[x]$. Then we can write $g = sf + r$, where $s, r \in F[x]$, and $\deg(r) < n$. Thus $g \equiv r \pmod f$, so $[g] = [r]$.

If $r_1, r_2 \in F[x], r_1 \neq r_2, \deg(r_1), \deg(r_2) < n$, then $f \nmid (r_1 - r_2)$, so $r_1 \not\equiv r_2 \pmod f$. Thus $[r_1] \neq [r_2]$.

Thus the polynomials in $F[x]$ of degree $< n$ are a complete set of representatives of the equivalence classes of $F[x]/(f)$.

Now, let $F = \mathbb{Z}_p$. Then $\mathbb{Z}_p[x]/(f) = \{[r] : r \in \mathbb{Z}_p[x], \deg(r) < n\}$. Thus $|\mathbb{Z}_p[x]/(f)| = p^n$, so $\mathbb{Z}_p[x]/(f)$ is a commutative ring of order p^n.

When is $F[x]/(f)$ a field?

irreducible over F

Let $f \in F[x]$, with $\deg(f) \geq 1$. Then **irreducible over** F if f cannot be written as $f = g \cdot h$, $g, h \in F[x]$, $\deg(g) \geq 1, \deg(h) \geq 1$.

Example:

$x^2 + 1$ is irreducible over $\mathbb{R}$, since it has no roots in $\mathbb{R}$.

$x^2 + 1$ is reducible over $\mathbb{C}$, since $x^2 + 1 = (x + i)(x - i)$.

$x^2 + 1$ is reducible over $\mathbb{Z}_2$, since $x^2 + 1 = (x + 1)(x + 1)$.

$x^2 + 1$ is irreducible over $\mathbb{Z}_3$, since it has no roots in $\mathbb{Z}_3$.

Theorem 2.7

$F[x]/(f)$ is a field if and only if f is irreducible over F.

Proof:
Analogous to the proof of the theorem: $\mathbb{Z}_n$ is a field if and only if n is prime. □

Now let's construct finite fields.

Theorem 2.8

Let $f \in \mathbb{Z}_p[x]$ be an irreducible polynomial of degree $n \geq 1$. Then $\mathbb{Z}_p[x]/(f)$ is a finite field of order p^n and characteristic p. The elements are the polynomials in $\mathbb{Z}_p[x]$ of degree $< n$.

Example: finite field of order $4 = 2^2$

Here $p = 2$ and $n = 2$. Let $f(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$. Then $f(0) = 1$, $f(1) = 1$, so f has no roots in $\mathbb{Z}_2$. Thus f is irreducible over $\mathbb{Z}_2$.

So, $F = \mathbb{Z}_2[x]/(x^2 + x + 1)$ is a finite field of order $2^2 = 4$. The elements are $\{0, 1, x, x + 1\}$ where $[\,]$ is omitted.

Example of addition: $x + (x + 1) = 1$

Example of multiplication: $x \cdot (x + 1) = x^2 + x = 1$

Example: finite field of order $2^3 = 8$

Here, $p = 2$ and $n = 3$. We need an irreducible polynomial. We need an irreducible polynomial of degree 3 over $\mathbb{Z}_2$.

Candidates: $x^3, x^3 + 1, x^3 + x, x^3 + x + 1, x^3 + x^2, x^3 + x^2 + 1, x^3 + x^2 + x, x^3 + x^2 + x + 1$

- Try $f(x) = x^3 + x + 1$.

  Since $f(0) = f(1) = 1$, f has no roots in $\mathbb{Z}_2$, and thus no linear factors in $\mathbb{Z}_2[x]$. Thus f is irreducible over $\mathbb{Z}_2$, and $F_1 = \mathbb{Z}_2[x]/(x^3 + x + 1)$ is a finite field of order $2^3 = 8$.

  The elements of F_1 are $\{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$.

  Example of addition: $(x^2 + x) + (x^2 + x + 1) = 1$.

  Example of multiplication: $(x^2 + x) \cdot (x^2 + x + 1) = x^4 + x = x^2$.

  Example of inversion: $x^{-1} = x^2 + 1$, since $x \cdot (x^2 + 1) = 1$.

- $x^3 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$, so $F_2 = \mathbb{Z}_2[x]/(x^3 + x^2 + 1)$ is a finite field of order 8. The elements of F_2 are $\{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$.

Note that F_1 and F_2 are not the same field. For example, in F_1, $x \cdot x^2 = x + 1$, whereas in F_2, $x \cdot x^2 = x^2 + 1$. However, F_1 and F_2 are isomorphic (essentially the same). Formally, there is a bijection $\phi : F_1 \to F_2$ such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ $\forall a, b \in F_1$.

Existence and uniqueness of finite fields

Let p be prime and $n \geq 1$. Then there exists an irreducible polynomial of degree n over $\mathbb{Z}_p$.

Theorem 2.9

There exists a finite field of order q if and only if $q = p^n$ for some prime p and $n \geq 1$.

Actually, any two finite fields of the same order are isomorphic.

We will denote *the* finite field of order q by $\mathrm{GF}(q)$ "the Galois Field of order q".

In the previous example, we saw two ways of representing the finite field $\mathrm{GF}(2^3)$.

2.4 Properties of finite fields

Theorem 2.10: Frosh's dream

Let F be a finite field of characteristic p, and let $\alpha, \beta \in F$. Then $(\alpha+\beta)^{p^m} = \alpha^{p^m} + \beta^{p^m}$ $\forall m \geq 1$.

Proof ($m = 1$):
By the Binomial Theorem,

$$(\alpha+\beta)^p = \binom{p}{0}\alpha^p + \sum_{i=1}^{p-1}\binom{p}{i}\alpha^i\beta^{p-i} + \binom{p}{p}\beta^p.$$

Now for $1 \leq i \leq p-1$,

$$\binom{p}{i} = \frac{p(p-1)(p-2)\cdots(p-i+1)}{1\cdot 2\cdot 3\cdots i} \equiv 0 \pmod{p},$$

since p divides the numerator but not the denominator, and since $\binom{p}{i}$ is an integer. Thus

$$\binom{p}{i}\alpha^i\beta^{p-i} = \underbrace{\alpha^i\beta^{p-i} + \cdots + \alpha^i\beta^{p-i}}_{\binom{p}{i}} = (1+\cdots+1)\alpha^i\beta^{p-i} = 0.$$

Hence $(\alpha+\beta)^p = \alpha^p + \beta^p$. The statement for $m \geq 1$ can be proven by induction. □

The multiplicative group $\mathrm{GF}(q)^*$

multiplicative group of $\mathrm{GF}(q)$

The **multiplicative group of** $\mathrm{GF}(q)$ is $\mathrm{GF}(q)^* = \mathrm{GF}(q) \setminus \{0\}$.

Theorem 2.11

Let $\alpha \in \mathrm{GF}(q)^*$. Then $\alpha^{q-1} = 1$.

Note that if $\mathrm{GF}(q) = \mathbb{Z}_p$, this is Fermat's Little Theorem.

Proof:
Let the (distinct) elements of $\mathrm{GF}(q)^*$ be $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$. Consider the (nonzero) elements $\alpha\alpha_1, \alpha\alpha_2, \ldots, \alpha\alpha_{q-1}$. These elements are distinct because if $\alpha\alpha_i = \alpha\alpha_j$ for some $i \neq j$, then $\alpha^{-1}(\alpha\alpha_i) = \alpha^{-1}(\alpha\alpha_j)$, so $\alpha_i = \alpha_j$, a contradiction. Hence $\{\alpha\alpha_1, \alpha\alpha_2, \ldots, \alpha\alpha_{q-1}\} = \{\alpha_1, \alpha_2, \ldots, \alpha_{q-1}\}$, so

$$(\alpha\alpha_1)(\alpha\alpha_2)\cdots(\alpha\alpha_{q-1}) = \alpha_1\alpha_2\cdots\alpha_{q-1}.$$

Cancelling gives $\alpha^{q-1} = 1$. □

Corollary 2.12

Let $\alpha \in \mathrm{GF}(q)$. Then $\alpha^q = \alpha$.

Order of finite elements

order of α

Let $\alpha \in \mathrm{GF}(q)^*$. The **order of** α, denoted $\mathrm{ord}(\alpha)$, is the smallest positive integer t such that $\alpha^t = 1$.

Theorem 2.13

Let $\alpha \in \mathrm{GF}(q)^*$, $\mathrm{ord}(\alpha) = t$. Then $\alpha^s = 1$ if and only if $t \mid s$.

Proof:
Let $s \in \mathbb{Z}$. Then long division of s by t yields

$$s = \ell t + r, \quad \text{where } 0 \leq r < t.$$

Now, $\alpha^s = \alpha^{\ell t + r} = (\alpha^t)^\ell \cdot \alpha^r = \alpha^r$. Hence $\alpha^s = 1 \iff \alpha^r = 1 \iff r = 0 \iff t \mid s$. □

Corollary 2.14

Let $\alpha \in \mathrm{GF}(q)^*$. Then $\mathrm{ord}(\alpha) \mid (q-1)$.

Example:
There is only one element in $\mathrm{GF}(q)$ of order 1, namely the element 1.

Example:
Consider $\mathrm{GF}(2^3) = \mathbb{Z}_2[x]/(x^3 + x + 1)$. The order of $\alpha = x^2 + 1$ is 7.

Example:
Consider $\mathrm{GF}(2^4) = \mathbb{Z}_2[x]/(x^4 + x + 1)$.

$f(x) = x^4 + x + 1$ has no roots in $\mathbb{Z}_2$, thus no linear factors. Also, $f(x)$ has no irreducible quadratic factors, since $(x^2 + x + 1) \nmid f(x)$. Note here $x^2, x^2 + 1, x^2 + x$ are reducible quadratic polynomials. Thus f is irreducible over $\mathbb{Z}_2$.

Find $\mathrm{ord}(x)$ in $\mathrm{GF}(2^4)$.

Solution: We have $x^1 = x, x^2 = x^2, x^3 = x^3, x^4 = x + 1, x^5 = x^2 + x \neq 1$. Thus $\mathrm{ord}(x) \neq 1, 3, 5$. Since $\mathrm{ord}(x) \mid 15$, we must have $\mathrm{ord}(x) = 15$.

Let $\alpha \in \mathrm{GF}(q)^*$ with $\mathrm{ord}(\alpha) = t$. Then the elements $\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{t-1}$ are distinct. In particular, if

$\mathrm{ord}(\alpha) = q-1$, then $\mathrm{GF}(q)^* = \{\alpha^0, \alpha^1, \alpha^2, \dots, \alpha^{q-2}\}$.

generator

A **generator** of $\mathrm{GF}(q)^*$ is an element of order $q-1$.

Example:

$\alpha = x$ is a generator of $\mathrm{GF}(2^4) = \mathbb{Z}_2[x]/(x^4+x+1)$ since $\mathrm{ord}(x) = 15$. Let's verify the above fact:

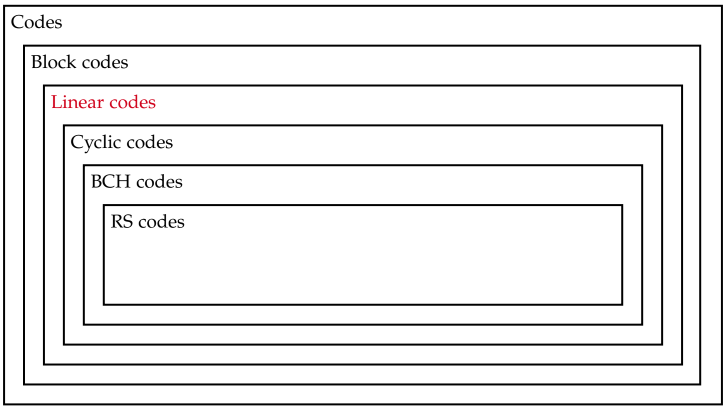
$$
\begin{array}{llll}
x^0 = 1, & x^1 = x, & x^2 = x^2, & x^3 = x^3, \\
x^4 = x+1, & x^5 + x^2 + x, & x^6 = x^3 + x^2, & x^7 = x^3 + x + 1, \\
x^8 = x^2 + 1, & x^9 = x^3 + x, & x^{10} = x^2 + x + 1, & x^{11} = x^3 + x^2 + x, \\
x^{12} = x^3 + x^2 + x + 1, & x^{13} = x^3 + x^2 + x + 1, & x^{14} = x^3 + 1, & x^{15} = 1.
\end{array}
$$

Theorem 2.15

Every finite field $\mathrm{GF}(q)$ has a generator.

3

Linear codes



3.1 Definition

Let $F = \mathrm{GF}(q)$. Let $V_n(F) = \underbrace{F \times \cdots \times F}_{n}$. $V_n(F)$ is an n-dimensional space over F. $|V_n(F)| = q^n$.

linear (n,k)-code over F

A **linear (n,k)-code over F** is a k-dimensional subspace of $V_n(F)$.

Recall a *subspace* S of a vector space V over F is non-empty subset $S \subseteq V$ such that:

$$\text{(i) } a, b \in S \implies a + b \in S, \text{ (ii) } a \in S, \lambda \in F \implies \lambda a \in S.$$

If S is a subspace of V, then S is itself a vector space over F; also $0 \in S$. A *basis* of S is a linearly independent, spanning subset of S. All bases of S have the same cardinality, called the *dimension* of S.

3.2 Properties of Linear Codes

Let C be an (n,k)-code over F, and let $v_1, v_2, \ldots, v_k$ be an ordered basis for C.

1. Number of codewords

The elements of C are precisely

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k, \quad c_i \in F.$$

Thus, $|C| = M = q^k$.

2. Rate

The rate of C is $R = \frac{\log_q M}{n} = \frac{\log_q q^k}{n} = \frac{k}{n}$.

3. Weight

Hamming weight

The **Hamming weight** $w(v)$ of a vector $v \in V_n(F)$ is the number of nonzero coordinates in v. The **Hamming weight** of a linear code C is $w(C) = \min\{w(c) : c \in C, c \neq 0\}$.

Theorem 3.1

If C is a linear code, then $w(c) = d(C)$.

Proof:
We have

$$\begin{aligned} d(C) &= \min\{d(x,y) : x,y \in C, x \neq y\} \\ &= \min\{w(x-y) : x,y \in C, x \neq y\} && \text{since } d(x,y) = w(x-y) \\ &= \min\{w(c) : c \in C, c \neq 0\} && \text{since } C \text{ is linear }, x-y \in C \\ &= w(C). \end{aligned}$$

4. Encoding

Since there are q^k codewords, there are also q^k source messages. We shall assume that source messages are the elements of F^k. Then a convenient and natural bijection (i.e., *encoding rule*) between F^k and C is defined by:

$$\boxed{m = (m_1, m_2, \ldots, m_k) \mapsto c = m_1v_1 + m_2v_2 + \cdots + m_kv_k.}$$

Note that different ordered bases for C yield different encoding rules.

5. Generator matrix

A convenient way to represent C.

generator matrix

A **generator matrix** G for an (n,k)-code C is a $k \times n$ matrix whose rows form a basis for C:

$$G = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}_{k \times n}.$$

Note that the encoding rule is $c = mG$.

Example: linear code

Consider the $(5,3)$-binary code:

$$C = \langle \underset{c_1}{10011}, \underset{c_2}{01001}, \underset{c_3}{00110} \rangle$$

and c_1, c_2, c_3 are linearly independent over $\mathrm{GF}(2)$.

A generator matrix for C is $G = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]_{3\times 5}$

The encoding rule (with respect to the ordered basis $\{c_1, c_2, c_3\}$) is $c = mG$.

$$\begin{array}{ccc@{\qquad}ccc} 000 & \to & 00000 & 100 & \to & 10011 \\ 001 & \to & 00110 & 101 & \to & 10101 \\ 010 & \to & 01001 & 110 & \to & 11010 \\ 011 & \to & 01111 & 111 & \to & 11100 \end{array}$$

Other properties: $M = |C| = 2^3 = 8$, $R = 3/5$, $d(C) = w(C) = 2$.

Standard form GM

standard form generator matrix

Let C be an (n,k)-code over F. A GM G for C of the form $G = [I_k | A]_{k\times n}$ is said to be in **standard form**.

systematic code

If C has a GM in standard form, then C is a **systematic code**.

Example: systematic/non-systematic code

$C = \langle 100011, 001001, 000110 \rangle$ is a *non-systematic* $(6,3)$-binary code.

But $C' = \langle 10011, 001001, 010010 \rangle$ is *systematic*. A GM for C' is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

equivalent codes

Two codes C, C' over F are **equivalent** if C' can be obtained from C by choosing a permutation of the coordinate positions $\{1, 2, \ldots, n\}$, and then consistently rearranging every codeword of C according to this permutation.

Below are some facts of equivalent codes:

1. If C is linear, and C' is equivalent to C, then C' is linear.
2. Equivalent codes have the same length, dimension, distance.
3. Every linear code is equivalent to a systematic code.

3.3 The Dual Code

inner product

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in V_n(F)$. The **inner product** of x and y is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \in F$$

For all $x, y, z \in V_n(F)$ and $\lambda \in F$:

1. $x \cdot y = y \cdot x$.
2. $x \cdot (y + z) = x \cdot y + x \cdot z$.
3. $(\lambda x) \cdot y = \lambda(x \cdot y)$
4. $x \cdot x$ does *not* imply that $x = 0$.

Example:
Consider $x = 111100 \in V_6(\mathbb{Z}_2)$. Then $x \cdot x = 0$, but $x \neq 0$. More generally, if $x \in V_n(\mathbb{Z}_2)$, then $x \cdot x = 0$ if and only if $w(x)$ is even.

orthogonal vectors

Two vectors x, y are **orthogonal** if $x \cdot y = 0$.

dual code

Let C be an (n, k)-code over F. The **dual code** or **orthogonal code** of C is

$$C^\perp = \{x \in V_n(F) : x \cdot y = 0, \quad \forall y \in C\}.$$

Theorem 3.2

If C is an (n, k)-code over F, then $C^\perp$ is an $(n, n-k)$-code over F.

Proof:
Let G be a GM for C, and let the rows of G be $v_1, v_2, \ldots, v_k$.

Claim Let $x \in V_n(F)$. Then $x \in C^\perp$ if and only if $v_1 \cdot x = v_2 \cdot x = \cdots = v_k \cdot x = 0$.

Let's prove the claim.

($\Rightarrow$) is clear since $v_1, v_2, \ldots, v_k \in C$.

($\Leftarrow$) Suppose $v \in C$. Then we can write $v = \lambda_1 v_1 + \cdots + \lambda_k v_k$, where $\lambda_i \in F$. Then

$$v \cdots x = (\lambda_1 v_1 + \cdots + \lambda_k v_k) \cdot x = \lambda_1(v_1 \cdot x) + \cdots + \lambda_k(v_k \cdot x) = 0.$$

Thus, $C^\perp = \{x \in V_n(F) : Gx^T = 0\}$ = null space of G. Since G has rank k, $C^\perp$ is a subspace of $V_n(F)$ of dimension $n - k$. □

3.4 Parity-Check Matrix

Let G be a GM for a linear code C. Then $C^\perp$ = null space of G.

Theorem 3.3

If C is a linear code, then $(C^\perp)^\perp = C$.

Proof:
Let C be an (n, k)-code. Then $C^\perp$ is an $(n, n-k)$-code.

Furthermore, $(C^\perp)^\perp$ is an (n, k)-code, and $C \subseteq (C^\perp)^\perp$.

Since $\dim(C) = \dim\big((C^\perp)^\perp\big)$, it follows that $C = (C^\perp)^\perp$. □

parity-check matrix

If C is a linear code, then a generator matrix H for $C^\perp$ is called a **parity-check matrix** (PCM) for C.

Note:
H is an $(n-k) \times n$ matrix.

C has many PCMs.

Constructing a GM for $C^\perp$

Theorem 3.4

Let C be an (n, k)-code with GM $G = [I_k | A]$. Then $H = [-A^T | I_{n-k}]$ is a GM for $C^\perp$.

Note that A is $k \times (n-k)$ matrix.

Proof:
Since $\operatorname{rank}(H) = n - k$, H is a GM for an $(n, n-k)$-code $\overline{C}$. Also,

$$GH^T = [I_k | A] \left[\begin{array}{c} -A \\ \hline I_{n-k} \end{array}\right] = -A + A = 0.$$

Thus $\overline{C} \subseteq C^\perp$. Since $\dim(\overline{C}) = \dim(C^\perp)$, we have $\overline{C} = C^\perp$. Hence H is a GM for $C^\perp$. □

Example:

Consider the $(5, 2)$-code C over $\mathbb{Z}_3$ with GM $G = \begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$. Find a PCM for C.

Solution Find a GM for C in standard form:

$$G \xrightarrow{R_1 \leftarrow 2R_1} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right].$$

So,

$$H = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

is a PCM for C.

We have

$$C = \left\{ \begin{array}{lll} 00000, & 20210, & 10120, \\ 11001, & 22002, & 01211, \\ 02122, & 21121, & 12212 \end{array} \right\}$$

Thus $d(C) = w(C) = 3, R = 2/5$.

Notes on PCMs

Let C be an (n,k)-code over F with GM G.

1. An $(n-k) \times n$ matrix H over F is a PCM for C iff $GH^T = 0$ and $\operatorname{rank}(H) = n - k$.
2. G is a PCM for $C^\perp$ (since $(C^\perp)^\perp = C$).
3. $C = \operatorname{null}(H)$.
4. Let H be a PCM for C, and let $x \in V_n(F)$, then $x \in C$ iff $Hx^T = 0$.

V3c skipped for now. V3f

3.5 Syndrome Decoding

Let C be an (n,k)-code over $F = \mathrm{GF}(q)$ with PCM H.

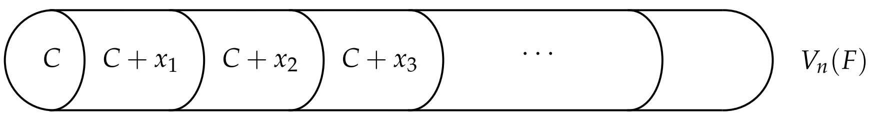
$x \equiv y \pmod{C}$

Let $x, y \in V_n(F)$. We write $x \equiv y \pmod{C}$ if $x - y \in C$.

Facts:

1. $\equiv \pmod{C}$ is an *equivalence relation*.
2. The set of equivalences classes partitions $V_n(F)$.
3. The equivalence class containing $x \in V_n(F)$ is called a **coset** of C. THis class is

$$C + x = \{y \in V_n(F) : y \equiv \pmod{C}\} = \{c + x : c \in C\}.$$



Example: Cosets

Consider a $(5,2)$-binary code C with GM

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Find all cosets of C.

Solution The cosets of C are:

$$\begin{aligned} C = C + 00000 &= \{00000, 10111, 01110, 11001\} = C + 10111 = C + 01110 + C + 11001 \\ C + 10000 &= \{10000, 00111, 11110, 01001\} = C + 00111 = C + 11110 = C + 01001 \\ C + 01000 &= \{01000, 11111, 00110, 10001\} \\ C + 00100 &= \{00100, 10011, 01010, 11101\} \\ C + 00010 &= \{00010, 10101, 01100, 11011\} \\ C + 00001 &= \{00001, 10110, 01111, 11000\} \\ C + 10100 &= \{10100, 00011, 11010, 01101\} \\ C + 10010 &= \{10010, 00101, 11100, 01011\} \end{aligned}$$

Facts on cosets:

4

Golay codes

5

Cyclic codes

cyclic space

A subspace S of $V_n(F)$ is **cyclic** if $(a_0, a_1, \ldots, a_{n-1}) \in S \implies (a_{n-1}, a_0, a_1, \ldots, a_{n-2}) \in S$.

cyclic code

A **cyclic code** is a cyclic subspace of $V_n(F)$.

5.1 The Polynomial Ring $R = F[x]/(x^n - 1)$

Let $R = F[x]/(x^n - 1)$, where $F = \mathrm{GF}(q)$. Then R is a commutative ring (but not a field, since $x^n - 1$ is reducible over F).

We have the following bijection between $V_n(F)$ and R:

$$a = (a_0, a_1, a_2, \ldots, a_{n-1}) \longleftrightarrow a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}.$$

Vector addition and scalar multiplication preserved.

$a \cdot b$

Let $a, b \in V_n(F)$. Then $a \cdot b = c \in V_n(F)$, where $c \leftrightarrow c(x) = a(x) \cdot b(x) \bmod (x^n - 1)$.

ideal

Let R be a (finite) commutative ring. A non-empty subset I of R is an **ideal** of R if

(i) $a, b \in I \implies a + b \in I$;

(ii) $a \in I, b \in R \implies a \cdot b \in I$.

Theorem 5.1: algebraic characterization of cyclic subspaces of $V_n(F)$

Let S be a non-empty subset of $V_n(F)$. Let I be the associated polynomials in $R = F[x]/(x^n - 1)$. Then S is a cyclic subspace of $V_n(F)$ if and only if I is an ideal of R.

5.2 Ideals of $R = F[x]/(x^n - 1)$

ideal generated by g

Let R be a (commutative) ring, and let $g \in R$. Let $\langle g \rangle = \{g \cdot r : r \in R\}$. Then $\langle g \rangle$ is an ideal of R, called the **ideal generated by** g.

principal ideal

An ideal I of R is said to be **principal** if $I = \langle g \rangle$ for some $g \in I$.

principal ideal ring

A ring R is a **principal ideal ring** if every ideal of R is principal.

Theorem 5.2

$R = F[x]/(x^n - 1)$ is a principal ideal ring.

generator polynomial

Let I be an ideal of $R = F[x]/(x^n - 1)$.

- If $I = \{0\}$, then $x^n - 1$ is the **generator polynomial** of I.
- If $I \neq \{0\}$, then *the* monic polynomial of smallest degree in I is called the **generator polynomial** of I.

Theorem 5.3

Let I be a nonzero ideal of $R = F[x]/(x^n - 1)$.

1) There is a unique monic polynomial $g(x)$ of smallest degree in I; $I = \langle g \rangle$.
2) $g(x) \mid (x^n - 1)$ in $F[x]$.

Theorem 5.4

Let $h(x)$ be a monic divisor of $x^n - 1$ in $F[x]$. Then $h(x)$ is *the* generator polynomial of $\langle h(x) \rangle$.

Corollary 5.5

There is a 1-1 correspondence between ideals of R and monic divisors of $x^n - 1$, and thus also a 1-1 correspondence between cyclic subspaces of $V_n(F)$ and monic divisors of $x^n - 1$.

5.3 Dimension of a Cyclic Code

Theorem 5.6

Let $g(x)$ be a monic divisor of $x^n - 1$ over F, where $F = \mathrm{GF}(q)$. Suppose $\deg(g) = n - k$. Then the cyclic subspace S of $V_n(F)$ generated by $g(x)$ has dimension k.

5.4 GM of a Cyclic Code

Theorem 5.7

Let $g(x)$ be the generator polynomial of an (n,k)-cyclic code C over F (so $g(x)$ is a monic divisor of $x^n - 1$ over F of degree $n - k$). Then a (non-standard) GM for C is

$$G = \begin{bmatrix} g(x) \\ xg(x) \\ x^2 g(x) \\ \vdots \\ x^{k-1} g(x) \end{bmatrix}_{k \times n} .$$

Encoding Source messages are the polynomials in $F[x]$ of degree $< k$. If $m(x) = m_0 + m_1 x + \cdots + m_{k-1} x^{k-1}$, then encoding of m with respect to G is

$$c = \begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \end{bmatrix} G = m_0 g(x) + m_1 x g(x) + \cdots + m_{k-1} x^{k-1} g(x),$$

so $c(x) = m(x)g(x)$. Note that no reduction by $x^n - 1$ is needed.

5.5 The Dual Code of a Cyclic Code

Let C be an (n,k)-cyclic code over F with generator polynomial $g(x)$. Let

$$g(x) = \underset{\substack{\uparrow \\ \neq 0}}{g_0} + g_1 x + \cdots + \underset{\substack{\uparrow \\ = 1}}{g_{n-k}} x^{n-k} + \underbrace{g_{n-k+1} x^{n-k+1} + \cdots + g_{n-1} x^{n-1}}_{0} .$$

parity-check polynomial

The **parity-check polynomial** is $h(x) = (x^n - 1)/g(x)$.

Let

$$h(x) = \underset{\substack{\uparrow \\ \neq 0}}{h_0} + h_1 x + \cdots + \underset{\substack{\uparrow \\ = 1}}{h_k} x^k + \underbrace{h_{k+1} x^{k+1} + \cdots + h_{n-1} x^{n-1}}_{0}$$

Also define $h_j = h_{j \bmod n}$ for all $j \in \mathbb{Z}$.

Observe that $g = (g_0, \ldots, g_{n-1})$ is orthogonal to the vector $\overline{h} = (h_{n-1}, \ldots, h_0)$ and all its cyclic shifts.

6

BCH codes

6.1 Subfields and Extension fields

For any prime power q, $\mathrm{GF}(q)$ is a subfield of $\mathrm{GF}(q^m)$ and we can view $\mathrm{GF}(q^m)$ as an m-dimensional vector space over $\mathrm{GF}(q)$.

Example:

$\mathrm{GF}(2^{16})$ is a 16-dimensional vector space over $\mathrm{GF}(2)$.
$\mathrm{GF}(2^{16})$ is a 8-dimensional vector space over $\mathrm{GF}(2^2)$.
$\mathrm{GF}(2^{16})$ is a 4-dimensional vector space over $\mathrm{GF}(2^4)$.
$\mathrm{GF}(2^{16})$ is a 2-dimensional vector space over $\mathrm{GF}(2^8)$.
$\mathrm{GF}(2^{16})$ is a 1-dimensional vector space over $\mathrm{GF}(2^{16})$.

6.2 Minimal Polynomials

We call $\mathrm{GF}(q^m)$ the **extension field**, and $\mathrm{GF}(q)$ the subfield.

minimal polynomial of α over $\mathrm{GF}(q)$

Let $\alpha \in \mathrm{GF}(q^m)$. The **minimal polynomial of α over** $\mathrm{GF}(q)$, denoted $m_\alpha(y)$ is the monic polynomial of smallest degree in $\mathrm{GF}(q)[y]$ that α has a root.

Theorem 6.1

Let $\alpha \in \mathrm{GF}(q^m)$.

1. The minimal polynomial $m_\alpha(y)$ of α over $\mathrm{GF}(q)$ is unique.
2. $m_\alpha(y)$ is irreducible over $\mathrm{GF}(q)$.
3. $\deg(m_\alpha) \leq m$.
4. If $f \in \mathrm{GF}(q)[y]$, then $f(\alpha) = 0 \iff m_\alpha(y) \mid f(y)$.

6.3 Computing Minimal Polynomials

Theorem 6.2

Let $\alpha \in \mathrm{GF}(q^m)$. Then $\alpha \in \mathrm{GF}(q)$ if and only if $\alpha^q = \alpha$.

set of conjugates of α w.r.t. GF(q)

Let $\alpha \in \mathrm{GF}(q^m)$. Let t be the smallest positive integer such that $\alpha^{q^t} = \alpha$ (note: $t \leq m$). Then the **set of conjugates of α w.r.t.** GF(q) is $C(\alpha) = \{\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{t-1}}\}$.

The t elements in $C(\alpha)$ are distinct.

Theorem 6.3

Let $\alpha \in \mathrm{GF}(q^m)$. Then the minimal polynomial of α over GF(q) is

$$m(y) = \prod_{\beta \in C(\alpha)} (y - \beta)\,.$$

6.4 Factoring $x^n - 1$ over GF(q)

Preliminaries Let p be the characteristic of GF(q). If $\gcd(n, q) \neq 1$, then write $n = \overline{n} p^\ell$, where $\ell \geq 1$ and $\gcd(\overline{n}, p) = 1$. Then $x^n - 1 = (x^{\overline{n}} - 1)^{p^\ell}$. So, WLOG, we shall assume that $\gcd(n, q) = 1$.

Let m be the smallest integer such that $q^m \equiv 1 \pmod{n}$, i.e., $n \mid (q^m - 1)$. Note that such an m exists.

Let α be a generator of $\mathrm{GF}(q^m)^*$. Let $\beta = \alpha^{(q^m-1)/n}$; note that $\beta \in \mathrm{GF}(q^m)$.

Also note, $\mathrm{ord}(\beta) = n$, and $1, \beta, \beta^2, \ldots, \beta^{n-1}$ are distinct. Furthermore, $(\beta^i)^n = (\beta^n)^i = 1$ for each $i \in [0, n-1]$. Hence $1, \beta, \beta^2, \ldots, \beta^{n-1}$ are roots of $x^n - 1$; and there aren't any other roots. So,

$$x^n - 1 = (x-1)(x-\beta)(x-\beta^2)\cdots(x-\beta^{n-1})$$

is the complete factorization of $x^n - 1$ over GF(q^m).

However, we seek the factorization of $x^n - 1$ over GF(q).

Consider β^i, where $0 \leq i \leq n-1$. Since β^i is a root of $x^n - 1$, we have $m_{\beta^i}(x) \mid (x^n - 1)$. Also, the roots of $m_{\beta^i}(x)$ are $C(\beta^i) = \{\beta^i, \beta^{iq}, \beta^{iq^2}, \ldots, \beta^{iq^{t-1}}\}$, where t is the smallest positive integer such that $iq^t \equiv i \pmod{n}$.

cyclotomic coset of q (mod n) containing i

Suppose $\gcd(n, q) = 1$, and let $0 \leq i \leq n-1$. The **cyclotomic coset of** q (mod n) **containing** i is

$$C_i = \{i, \quad iq \bmod n, \quad iq^2 \bmod n, \quad \ldots, \quad iq^{t-1} \bmod n\},$$

where t is the smallest positive integer such that $iq^t \equiv i \pmod{n}$. Also $C = \{C_i : 0 \leq i \leq n-1\}$ is the **set of cyclotomic cosets of** q mod n.

Theorem 6.4

Suppose $\gcd(n, q) = 1$.

- The number of monic irreducible factors of $x^n - 1$ over $\mathrm{GF}(q)$ is equal to the number of (distinct) cyclotomic cosets of q mod n.
- The number of monic irreducible factors of degree d is equal to the number of (distinct) cyclotomic cosets of q mod n of size d.

Theorem 6.5

Suppose $\gcd(n, q) = 1$. Let m be the smallest positive integer such that $q^m \pmod n$, and let $\beta \in \mathrm{GF}(q^m)$ be an element of order n. Then the monic irreducible factor of $x^n - 1$ over $\mathrm{GF}(q)$ are $\{m_{\beta^i}(x) : 0 \le i \le n-1\}$, where

$$m_{\beta^i}(x) = \prod_{j \in C_i} (x - \beta^j) .$$

If $j \in C_i$, then $m_{\beta^j}(x) = m_{\beta^i}(x)$.

6.5 BCH Codes: Definition

BCH codes are cyclic codes that are constructed in such a way that a (useful) lower bound on their distance is known.

BCH code

A **BCH code** C over $\mathrm{GF}(q)$ of block length n and **designed distance** δ is a cyclic code generated by $g(x) = \mathrm{lcm}\,\{m_{\beta^i}(x) : a \le i \le a + \delta - 2\}$, for some integer a.

6.6 BCH Bound

Vanderdmonde matrix

A **Vanderdmonde matrix** over a field F is a matrix of the form

$$A(x_1, x_2, \ldots, x_t) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_t \\ x_1^2 & x_2^2 & \cdots & x_t^2 \\ \vdots & \vdots & \vdots & \\ x_1^{t-1} & x_2^{t-1} & \cdots & x_t^{t-1} \end{bmatrix}_{t \times t} ,$$

where $x_1, x_2, \ldots, x_t \in F$.

Theorem 6.6

$\det(A(x_1, \ldots, x_t)) \neq 0$ if and only if $x_1, \ldots, x_t$ are distinct.

Corollary 6.7

A Vanderdmonde matrix $A(x_1, \ldots, x_t)$ is non-singular if and only if $x_1, x_2, \ldots, x_t$ are distinct.

Theorem 6.8: BCH bound

Let C be an (n, k)-BCH code over GF(q) with designed distance δ. Then $d(C) \geq \delta$.

6.7 BCH Decoding

Later, because not covered in A4.

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