Rings and Fields

PMATH 334

Tomáš Vávra

LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of PMATH 334 during Winter 2022 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

References:

- Dummit, Foote: *Abstract algebra*.
- http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf
- https://notes.sibeliusp.com/pmath347

The list of theorems is almost following Dummit, Foote. Moreover, the proofs might be slightly different that what are in class.

For any questions, send me an email via https://notes.sibeliusp.com/contact.

You can find my notes for other courses on https://notes.sibeliusp.com/.

Sibelius Peng

Contents

1

Introduction & Motivation

1.1 Fermat's Last Theorem

Fermat's Last Theorem

The equation $x^m + y^m = z^m$ has no non-trivial solutions in integers for $m \geq 3$.

For example, $(1, 0, 1)$, $(-1, 0, 1)$ for m even, are trivial solutions.

In 1897, Gabriel Lamé announced that he has a proof. First he assumed that m is a prime. He writes

$$z^p = x^p + y^p = (x+y)(x+\zeta_p y)(x+\zeta_p^2 y)\cdots(x+\zeta_p^{p-1} y)$$

where $\zeta_p = \cos(\frac{2\pi}{p}) + i\sin(\frac{2\pi}{p})$. Consider the ring

$$\mathbb{Z}[\zeta_p] = \{a_1 + a_2\zeta_p + a_3\zeta_p^2 + \cdots + a_{p-2}\zeta^{p-2} : a_i \in \mathbb{Z}\}$$

which is the smallest ring containing $\mathbb{Z}$ and ζ_p.

Then the next step is to show that $(x + \zeta_p^j y)$'s are coprime in $\mathbb{Z}[\zeta_p]$. Let q_i's be primes.

$$\prod_i q_i^{p\alpha_i} = z^p = (x+y)(x+\zeta_p y)\cdots(x+\zeta_p^{p-1} y)$$

If $(x + \zeta_p^j y)$'s are coprime in $\mathbb{Z}[\zeta_p]$, then $(x + \zeta_p^j y) = (\cdots)^p$ is of p-th power $(*)$. But this is wrong if the factorization is non-unique. However, we have $\mathbb{Z}[\zeta_p]$ can be a unique factorization domain (UFD). This means $(*)$ works. Kummer salvages the argument for approximately (conjecturally) 60% of prime exponents. And these primes are called regular primes.

1.2 Straightedge and compass construction

We are given a length 1 straightedge ruler, and a compass. With these, we can

- connect two points with a straightedge,
- draw a circle, centered at A, and going through B,
- draw intersections of two line segments, circle & line, two circles.

What lengths are constructible? where length means distance between two points. We can do $+, -, \times, \div, \sqrt{}$. Then we can do field extensions:

$$\mathbb{Q} \to \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \cdots$$

Is trisection of an angle doable? No, not possible.

Possible to double the cube, square the circle of the same area?

What regular m-gons are constructible? This is equivalent to the question: is $\cos(\frac{2\pi}{m}) + i\sin(\frac{2\pi}{m})$ constructible?

These can be answered via field extensions.

Other applications including coding theory.

2

An introduction to Rings

2.1 Definitions and basic properties

ring

A ring is a set with two binary operations $+, \times$, such that

1. $(R, +)$ is an abelian group.
   - $+$ is commutative and associative.
   - $\exists 0 \in \mathbb{R}$, $0 + a = a + 0 = a$ for all $a \in R$.
   - $\forall a \in \mathbb{R}$, $\exists(-a) \in R$, $a + (-a) = (-a) + a = 0$.
2. $\times$ is associative $(a \times b) \times c = a \times (b \times c)$.
3. distributive laws hold: $(a + b) \times c = (a \times c) + (b \times c)$.

The ring is called commutative if $\times$ is commutative. The ring is said to have an identity if $\exists 1 \in R$, $1 \times a = a \times 1 = a$, for all $a \in R$, and this does not require the existence of inverse.

For simplicity, we write

$$ab := a \times b, \qquad b - a = b + (-a)$$

Example:

$\mathbb{Z}$ is a commutative ring with identity.

Trivial rings: Let $(R, +)$ be an abelian group. We define $a \times b = 0$ for all $a, b \in R$. The result is a commutative ring with "trivial structure".

$R = \{0\}$ is a zero ring. $0 = 1$ in this case, and it is the only such ring. It leads to assumption $0 \neq 1$, saying $R \neq \{0\}$.

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commutative rings with identity.

$\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ with $+, \times \mod m$ is a ring with identity, and commutative.

The real quaternions: $\{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$. Addition is "component-wise". And the multiplication follows

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

And this is non-commutative ring, with identity 1.

Let X be a set, A be a ring. Consider the set $F = \{f : X \to A\}$. Define

$$(f+g)(x) = f(x) + g(x), \qquad (f \times g)(x) = f(x) \times g(x)$$

F commutative & having identity is inherited from the ring A.

$M_m(\mathbb{Z})$ is the ring of square $m \times m$ matrices with coefficients in $\mathbb{Z}$. It is non-commutative ring with identity.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have compact support, if $\exists a, b \in \mathbb{R}$, $f(x) = 0$ for $x \notin [a, b]$. $R = \{f : \mathbb{R} \to \mathbb{R} : f \text{ has compact support}\}$ is a commutative ring, without identity.

Proposition 2.1

Let R be a ring. Then

1. $0a = a0$ for all $a \in R$.
2. $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.
3. $(-a)(-b) = ab$ for all $a, b \in R$.
4. If R has an identity 1, then it is unique, and $(-a) = (-1)a$.

Proof:
We see that

$$0a = (0+0)a = 0a + 0a$$
$$0a - 0a = (0a + 0a) - 0a = 0a + (0a - 0a)$$
$$0a = 0$$

We also see that

$$(-a)b + ab = ((-a) + a)b = 0b = 0$$

□

We would like to be able to cancel with respect to x: $ab = ac$ then $b = c$. However, this is not true in general.

Example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

However,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2.2 Zero divisor and integral domain

zero divisor

A nonzero element $a \in R$ is called a zero divisor, if there exists $b \in R$ and $b \neq 0$, such that $ab = 0$ or $ba = 0$.

integral domain

A commutative ring with identity, $1 \neq 0$, is called an integral domain, if it contains no zero divisor.

Proposition 2.2

Let R be a ring. Assume that $a, b, c \in R$, and a is not a zero divisor. If $ab = ac$, then either $a = 0$ or $b = c$ (i.e., we can multiplicatively cancel).

Proof:
Observe that

$$ab = ac$$

$$ab - ac = 0$$

$$a(b - c) = 0$$

As a is not zero divisor, then either $a = 0$ or $b - c = 0$. □

If zero divisors exist, then cancellation does not hold:

$$ab = 0 = a \cdot 0 \not\Rightarrow b = 0$$

Remark:
In integral domains, $ab = 0 \implies a = 0$ or $b = 0$.

2.3 Field

division ring

A ring with identity 1, $1 \neq 0$, is called a division ring, if every nonzero element has a multiplicative inverse, i.e., for all $a \in R, a \neq 0$, there exists $b \in R$, such that $ab = ba = 1$.

Consider an example $ab = 1$ existing and $ba = 1$ not existing.

Example:
Real sequences $(x_1, x_2, \ldots)$. Ring of operators on the sequences, $\times$ is composition. Take

$$D : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$$
$$S : (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots)$$

Then

$$D(S(x_1, x_2, \ldots)) = Id(x_1, x_2, \ldots)$$

but $S \circ D \neq Id$.

field

A commutative division ring is called a field.

Example:
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. Quaternions are "only" a division ring because non-commutative. $\mathbb{Z}_p$ is a field for p prime.

Proposition 2.3

Any finite integral domain is a field.

$\mathbb{Z}$ is an integral domain, but far from a field.

Proof:
Check Corollary 10.13 of PMATH 347. □

2.4 Subring

subring

Let R be a ring. A nonzero subset $S \subseteq R$ is called a subring of R, if it is a ring with the operations from $(R, +, \times)$ restricted to S.

That means: $S \neq \varnothing$. $x + (-y) \in S, \forall x, y \in S$. $xy \in S, \forall x, y \in S$.

Example:
$\mathbb{Z}_2 \subseteq \mathbb{Z}$, but $\mathbb{Z}_2$ is not a subring of $\mathbb{Z}$.

$2\mathbb{Z} = \{2 \cdot z : z \in \mathbb{Z}\}$ (ring has no identity) is a subring of $\mathbb{Z}$ (ring has identity).

Ring of matrices $M_2(\mathbb{R})$ (1 is identity matrix) has a subring $S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \right\}$ and $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is the identity in S.

2.5 Unit

unit

Assume that R is a ring with an identity $1 \neq 0$. A $a \in R$ is called a unit, if there exists $b \in R$ such that $ab = ba = 1$. Set of units of R is denoted by $R^\times$.

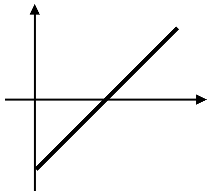
Example:
$\mathbb{Z}^\times = \{\pm 1\}$

$\mathbb{Z}_m^\times = \{a \in \mathbb{Z}_m : \gcd(a, m) = 1\}$, $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus \{0\}$ for p prime.

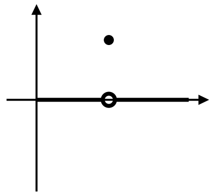
Consider ring R of $[0, 1] \to \mathbb{R}$, where $(f \times g)(x) = f(x) \cdot g(x)$, $1_R = 1(x)$. Units are the functions such that $f(x) \neq 0$ for $\forall x \in [0, 1]$. Then $f(x)^{-1} = \frac{1}{f(x)}$. All non-units are zero divisors. If $g(y) = 0$, then $h(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ gives $(g \times h) = 0(x) = 0_R$.

Ring of all continuous functions $[0, 1] \to \mathbb{R}$ is a subring of the previous ring. Units as before, because $1/f$ exists and is continuous.

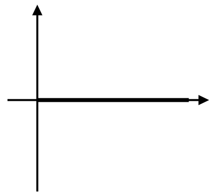
Consider $f(x) = x - 1/2$.



$\times$



$=$



3

Ring Homomorphisms

ring homomorphism

Let R, S be rings.

1. A ring homomorphism is $\phi : R \to S$, such that
   (a) $\phi(a+b) = \phi(a) + \phi(b)$, for all $a, b \in R$.
   (b) $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in R$.
2. The kernel of ϕ, $\ker\phi = \{a \in R : \phi(a) = 0_S\}$.
3. A bijective homomorphism is called isomorphism.

Remark:
Isomorphism means "same ring", denote $R \cong S$.

Example:
$\{0, 1\} = \mathbb{Z}_2 = R$, $S = \{a, b\}$ with $a + a = a, a + b = b, \ldots$ Then $R \cong S$.

Example:
$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$ with cancellation $\frac{a}{b} = \frac{ca}{cb}$

Can we say $\mathbb{Z} \subseteq \mathbb{Q}$? not in the purest sense. $\mathbb{Z}$ corresponds to $\{\frac{a}{1} : a \in \mathbb{Z}\}$.

$\mathbb{Q}$ contains an isomorphic copy of $\mathbb{Z}$. $S \subseteq \mathbb{Q}$ such that $S \cong \mathbb{Z}$.

Example:
$\phi : \mathbb{Z} \to \mathbb{Z}_2$. $\phi(2k) = 0$, $\phi(2k+1) = 1$. Then

$$\begin{aligned}
\ker\phi &= 2\mathbb{Z} \\
\phi^{-1}(0) &= 2\mathbb{Z} = \ker\phi \\
\phi^{-1}(1) &= 1 + 2\mathbb{Z} \\
&= 1 + \ker\phi \\
&= 3 + \ker\phi
\end{aligned}$$

Example:
$\phi : \mathbb{Z}[x] \to \mathbb{Z} : p(x) \mapsto p(0)$. Then

$$\begin{aligned}\ker\phi = \phi^{-1}(0) &= \{a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + 0 : a_i \in \mathbb{Z}\} \\ &= x\mathbb{Z}[x] = \{x \cdot p(x) : p(x) \in \mathbb{Z}[x]\}\end{aligned}$$

and

$$\phi^{-1}(a) = x\mathbb{Z}[x] + ax^0 = \ker\phi + ax^0$$

Example:
$\phi : \mathbb{Z}[x] \to \mathbb{Z}_2 : p(x) \mapsto p(0) \bmod 2$. Then

$$\ker\phi = \phi^{-1} = x\mathbb{Z}[x] + 2\mathbb{Z}$$

$$\phi^{-1}(1) = 1 + \ker\phi$$

Example:
$\phi : \mathbb{Z} \to \mathbb{R} : a \mapsto a$, then $\ker\phi = \{0_{\mathbb{R}}\}$.

Proposition 3.1

Let R, S be rings, $\phi : R \to S$ be homomorphism.

1. The image of ϕ, ($\mathrm{Im}(\phi)$, or $\phi(R)$) is a subring of S.
2. $\ker\phi$ is a subring of R. Moreover, $\forall r \in R, \forall \alpha \in \ker\phi$, $r\alpha \in \ker\phi, \alpha \in \ker\phi$. (That is $\ker\phi$ is closed under multiplication by the elements from R)

Proof:

1. If $a, b \in \phi(R)$, then

$$a - b = \phi(x_a) - \phi(x_b) = \phi(x_a - x_b) = \phi(x_{a-b}) \in \phi(R)$$

2. $\phi(r\alpha) = \phi(r) \cdot \phi(\alpha) = \phi(r) \cdot 0 = 0$

□

Can we get a ring structure on $a + \ker\phi$? There is a factor ring $R/\ker\phi$. For example, $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$.

3.1 Ideals & Quotient rings

ideal

Let R be a ring, let $I \subseteq R$ be a subring, let $r \in R$.

1. I is called a left ideal, if $rI \subseteq I$ where $rI = \{ri : i \in I\}$.
2. I is called a right ideal, if $Ir \subseteq I$.
3. I is an ideal, if it is left & right ideal (two sided ideal).

Ideal Test

Check K is an ideal of R:

- $k - j \in K$ for all $j, k \in K$; and
- $rk, kr \in K$ for all $k \in K, r \in R$.

It is a quick generalization of previous definition. Reference: Laurent W. Marcoux's 334 notes.

additive quotient

Let $I \subseteq R$ be an ideal. The additive quotient is defined as $R/I = \{a + I : a \in R\}$.

Example:

$\mathbb{Z}/3\mathbb{Z} = \Big\{\{\dots, -6, 3, 0, 3, 6, \dots\}, \{\dots, -5, -2, 1, 4, \dots\}, \{\dots, -4, -1, 2, 5, 8, \dots\}\Big\}$. Additive group.

Let $I = 3\mathbb{Z}$. Then $a + I$ are called (additive) cosets.

Proposition 3.2

Let R be a ring, I an ideal of R, then R/I is a ring with the operations

$$(a + I) +_{R/I} (b + I) =: (a +_R b) + I$$

$$(a + I) \times_{R/I} (b + I) = (a \times_R b) + I$$

The ring properties R/I follow from R being a ring.

quotient ring

R/I is called the quotient ring of R by I.

Remark:

If I is not an ideal, then the definition of the operations on R/I is not well defined.

Example:

Let R be commutative ring with identity $1 \neq 0$, $m \geq 2$. Let $M_m(R)$ be ring of square matrices with coefficients in R.

Denote

$$L_j(R) = \{A \in M_m(R) \mid A_{ik} = 0, \forall i \in [n], k \in [m] \setminus \{j\}\}$$

which means only the j-th column can have non-zero entries. Then $L_j(R)$ is a left ideal in $M_m(R)$. This can be verified by the matrix multiplication. $L_j(R)$ is not a right ideal, i.e., $L_j(R) \cdot M \notin L_j(R)$ for some $M \in M_m(R)$.

Analogously, a right ideal can be obtained by taking

$$T_i(R) = \Big\{A \in M_m(R) \,\Big|\, A_{kj} = 0, \forall k \in [n] \setminus \{i\}, j \in [m]\Big\}$$

Example:

Let $R = \mathbb{Z}[x]$ and $I = x^2\mathbb{Z}[x]$.

Then $R/I = \{a + bx + p(x) : a, b \in \mathbb{Z}, p(x) \in I\}$.

For $a \in R/I$, $\bar{a}$ denotes $a + I$.

3.2 Isomorphism theorems

Lemma 3.3

Let I be an ideal in R, then $a + I = b + I$ ($\bar{a} = \bar{b}$) if and only if $b - a \in I$. Namely, every member of the coset can be the representative.

Theorem 3.4: First isomorphism theorem

If $\phi : R \to S$ is a ring homomorphism, then $\ker \phi$ is an ideal in R, Im ϕ is a subring of S, and $R/\ker\phi \cong \text{Im } \phi$.

Proof:
Theorem 4.2 of http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf

Consider $\tau : R/\ker\phi \to \phi(R) : r + \ker\phi \mapsto \phi(r)$. □

Example:
$\mathbb{Z}[x]/2\mathbb{Z}[x] \cong \mathbb{Z}_2[x]$. We can define $\phi : p(x) \mapsto p(x) \mod 2$.

Theorem 3.5

For any ideal $I \subseteq R$, the map $R \to R/I$ defined by $\pi : r \mapsto r + I$ is a surjective ring homomorphism with kernel I. It is called the natural projection of R onto R/I. Thus every ideal is a kernel of some homomorphism.

Proof:
Prove surjectivity is as before in first iso theorem. The prove homomorphism, both $\times$ and $+$. Now prove $\ker\phi$.

- Let $i \in I$, then $\pi(i) = i + I = I = 0_{R/I}$.
- Let $a \in R/I$, then $\pi(a) = a + I$, but $a \notin I$. Thus by lemma, $a + I \neq I = 0 + I$. □

Theorem 3.6: Second isomorphism theorem

Let A be a subring of R, B an ideal of R. Then $A + B = \{a + b : a \in A, b \in B\}$ is a subring of R. $A \cap B$ is an ideal of R and $(A + B)/B \cong A/A \cap B$.

Proof:
Consider the map $\phi : A \to (A + B)/B : a \mapsto a + B$. Then apply first isomorphism theorem.

Or check Theorem 4.3 of http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf. □

Remark:
$(A + B)/B = \{a + b + B : a \in A, b \in B\} = \{a + B : a \in A\} \stackrel{?}{=} A/B$

This reduction can't happen because B is not necessarily an ideal of A.

Example:
Let $R = \mathbb{Z}$, then $a\mathbb{Z} + b\mathbb{Z} = \gcd(a,b) \cdot \mathbb{Z}$. $a\mathbb{Z} \cap b\mathbb{Z} = \operatorname{lcm}(a,b) \cdot \mathbb{Z}$. Then by second iso thm

$$\frac{\gcd(a,b)\mathbb{Z}}{b\mathbb{Z}} \cong \frac{a\mathbb{Z}}{\operatorname{lcm}(a,b)\mathbb{Z}}$$

Lemma 3.7

If $m \mid n$, then $n\mathbb{Z}$ is an ideal of $m\mathbb{Z}$, and $|m\mathbb{Z}/n\mathbb{Z}| = \frac{n}{m}$.

The coset representative in $(m\mathbb{Z}/n\mathbb{Z})$ are $\{0, m, 2m, \ldots, (\frac{n}{m} - 1)m\}$. Applying to $A + B/B \cong A/A \cap B$, we have

$$\frac{b}{\gcd(a,b)} = \frac{\operatorname{lcm}(a,b)}{a} \implies ab = \operatorname{lcm}(a,b) \cdot \gcd(a,b)$$

Theorem 3.8: Third isomorphism theorem

Let $I \subseteq J$ be ideals in R. Then J/I is an ideal in R/I and $(R/I)/(J/I) \cong R/J$.

Proof:
Define $\phi : R/I \to R/J : a + I \mapsto a + J$. Then show that $\ker \phi = J/I$ and then use first isomorphism theorem.

Or check Theorem 4.4 of `http://www.math.uwaterloo.ca/~lwmarcou/notes/pmath334.pdf` □

Example:
$(\mathbb{Z}/6\mathbb{Z})/(3\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$.

Theorem 3.9: Fourth isomorphism theorem/correspodence theorem

Let R be ring, I ideal in R. The correspondence $A \leftrightarrow A/I$ is an inclusion preserving bijection between the set of subrings (A) of R, $I \subseteq A \subseteq R$, and the set of subrings of R/I. Furthermore, A/I is an ideal in R/I if and only if A is an ideal in R $(I \subseteq A)$.

Proof:
No first isomorphism theorem. Expand and verify the definitions. □

The interesting part is: subring of R/I gives subring of R.

4

More on Ideals

Let $A \subseteq R$ with identity.

(A)

1. (A) = the smallest ideal containing A (in R)
2. Let

$$RA = \left\{\sum r_i a_i : r_i \in R, a_i \in A\right\}$$
$$AR = \left\{\sum a_i r_i : r_i \in R, a_i \in A\right\}$$
$$RAR = \left\{\sum r_i a_i r_i' : r_i, r_i' \in R, a_i \in A\right\}$$

   where these are all finite sums.
3. If $A = \{a\}$, then $(A) =: (a)$ is called a principal ideal.
4. If an ideal $I = (A)$ for A finite, we call I finitely generated.

Remark:

$$(A) = \bigcap_{\substack{I \text{ ideal of } R \\ A \subseteq I}} I$$

The intersection is indeed an ideal.

$(A) \subseteq \cap I$ because (A) is the smallest. $\cap I \subseteq (A)$ because it contains $I = (A)$.

Note that $\cup I_\alpha$ is not an ideal in general.

What is (A)?

Assume R is commutative. Then (A) contains $a \in R$, and also $ra, r \in R, a \in R$, and their sums. This is precisely the definition of RA. Thus $RA \subseteq (A)$.

Note that $1 \in R$. Then $A \subseteq RA$, and RA is an ideal itself. By minimality, $(A) \subseteq RA$.

To conclude, $(A) = RA = AR = RAR$ in the commutative case.

In particular, the principal ideal $(A) = a \cdot R = \{ar : r \in R\}$, because let $A = \{a\}$, we have

$$AR = \left\{\sum ar_i : r_i \in R\right\} = \left\{a\left(\sum r_i\right) : r_i \in R\right\}$$

works in commutative rings.

Warning In non-commutative rings, we have $(A) = RAR$, so

$$(a) = RaR \neq \{r_i a r_i' : r_i, r_i' \in R\}$$

Example:
$R = \mathbb{Z}$, the principal ideal (m) is $m\mathbb{Z}$.

Example:
Let $R = \{f : [0,1] \to \mathbb{R}\}$. Then $I = \{f \in R : f(1/2) = 0\}$ is an ideal. And $I = (g)$ where

$$g(x) = \begin{cases} 0 & \text{if } x = 1/2 \\ 1 & \text{otherwise} \end{cases}$$

For $h \in I$, $h = g \cdot h \in (g)$. Note that g is an identity element of I, but not of R.

Example:
$C = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ is a subring of R. $I = \{f \in C : f(1/2) = 0\}$ is again an ideal. BUT! I is not a principal ideal, I is not even finitely generated (not easily proven).

Note that I here is different from last example, where the instructor made a mistake at first.

Example:
Let $R = \mathbb{Q}[x]$. Consider subring $S = x\mathbb{Q}[x] + \mathbb{Z}$. An ideal $I = x\mathbb{Q}[x]$.

1. $I = (x)$ in R
2. I is an ideal in S where I is not finitely generated

If I is finitely generated in S, then there exists $p_1, \ldots, p_k \in I$

$$I = (p_1, \ldots, p_k) = \left\{\sum_{i=1}^{k} p_i(x)q_i(x) : q_i \in S\right\}$$

As p_i are in ideal $I = x\mathbb{Q}[x]$, p_i don't have constant term. However, this is not possible. Take an element $\frac{a}{b}x \in I$, then

$$\frac{a}{b}x = \sum_{i=1}^{k} p_i(x)q_i(x)$$

As p_i's are fixed, one need to find proper q_i's to make this equation hold. Now consider b to be a prime such that b does not divide the product of denominators of p_i's, then it's impossible to find any q_i's to make this equation holds. Therefore I is not a finite generated ideal in S.

Proposition 4.1

Let I be an ideal in R with identity $1 \neq 0$.

1. $I = R$ if and only if I contains a unit.
2. Let R be commutative. Then R is a field if and only if the only ideals in R are 0 and R.

Proof:

Statement 1

($\Rightarrow$) Because $1 \in R = I$, and 1 is a unit.

($\Leftarrow$) Let $u \in I$ be a unit. Then $u \cdot u^{-1} = 1 \in I$. Let $r \in I$, as $1 \in I$, then $1 \cdot r \in I$, hence $I = R$.

Statement 2

($\Rightarrow$) Let $0 \neq I \subseteq R$ be an ideal. Then it contains a unit. Then by (1), $I = R$.

($\Leftarrow$) Take arbitrary $0 \neq r \in R$. The ring (r) can't be zero ideal, hence $(r) = R$. Thus $1 \in (r)$. That means there exists $s \in R$, such that $1 = r \cdot s$. Then $s = r^{-1}$. Hence r is a unit. □

Corollary 4.2

A nonzero homomorphism from a field to a ring is an injection.

Proof:
Let ϕ be such a homomorphism. $\ker \phi$ is an ideal of the field. This implies $\ker \phi = 0$ (injective homomorphism) or R, the whole field. And the second possibility tells us ϕ is a zero map, which is eliminated by the assumption. □

4.1 Maximal ideals

maximal ideal

An ideal M is an arbitrary ring R is called a maximal ideal if $M \neq R$ and there is no proper $(\neq R)$ ideal I, $M \subseteq I \subseteq R$.

Alternatively, ideal I of a ring R is maximal if the only ideals containing I are I and R.

Theorem 4.3

Assume that R ring is commutative. The ideal M is maximal if and only if R/M is a field.

Proof:
By 4th iso thm, or correspondence theorem, R/M is a field $\Leftrightarrow$ ideals of R/M are zero ideals and R/M $\Leftrightarrow$ only ideals of R containing M are M and R $\Leftrightarrow$ M is maximal. □

Example:
$p\mathbb{Z}$ is maximal ideal for any p prime.

Theorem 4.4

$p\mathbb{Z}$ is maximal if and only if $\mathbb{Z}/p\mathbb{Z}$ is a field.

Example:
$(2, x)$ in $\mathbb{Z}[x]$ is maximal. $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}_2$ because $(2, x)$ is a kernel of $\phi : p(x) \mapsto p(0) \bmod 2$.

Example:
Let $R = \{f : [0, 1] \to \mathbb{R}\}$ and $M_c = \{f \in R : f(c) = 0\}$. Consider $\phi : R \to \mathbb{R} : f \mapsto f(c)$. Then $\ker \phi = M_c$. As $\mathbb{R} = \phi(R)$, then $R/M_c \cong \mathbb{R}$ is a field. Hence M_c maximal.

4.2 Maximal ideals and Zorn's Lemma

Consult Section 10.3 of PMATH 347 if needed.

Is every ideal (proper) contained in some maximal ideal? No. Consider $\mathbb{Q}$ with standard $+$ and $a \times b = 0_+$ for all $a, b \in \mathbb{Q}$. We have ideals

$$\left\{\frac{a}{2} : a \in \mathbb{Z}\right\} \subseteq \left\{\frac{a}{4} : a \in \mathbb{Z}\right\} \subseteq \cdots \subseteq \left\{\frac{a}{2^k} : a \in \mathbb{Z}\right\} \subseteq \cdots$$

These ideals are not contained in a maximal ideal. This happens because there's no identity.

Theorem 4.5

In a ring with an identity, every proper is contained in some maximal ideal.

Wrong idea Given I, then $I \subseteq \bigcup_{\substack{I \subseteq A \\ A \neq R}} A$. But this is not an ideal. For example, $\mathbb{Z}_6 \subseteq \mathbb{Z}_2 \cup \mathbb{Z}_3$ is not an ideal.

Right idea $I \subseteq \bigcup_{A \in C} A$ for C being a "chain"

$$I \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq \cdots$$

partial order

A partial order on a set S is a relation on X such that

1. $a \leq a$ for all $a \in S$,
2. If $a \leq b$ and $b \leq a$ then $a = b$ for all $a, b \in S$,
3. If $a \leq b$ and $b \leq c$, then $a \leq c$ for all $a, b, c \in S$.

So set inclusion $\subseteq$ is a partial order.

The ordering does not have to be "linear": sth $\leq$ sth $\leq$ sth $\leq \cdots$. For sets, we can have

$$\begin{tikzcd}
 & \{a,b\} \arrow[r, phantom, "\subseteq"] & \{a,b,d\} \arrow[dr, phantom, sloped, "\subseteq"] & \\
\{a\} \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & & \{a,b,c,d,e\} \\
 & \{a,c\} \arrow[r, phantom, "\subseteq"] & \{a,c,e\} \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

A chain C in a partially ordered set $(S, \leq)$ is a subset such that for all $x, y \in C$, $x \leq y$ or $y \leq x$ (i.e., all elements are comparable).

Zorn's Lemma

Let $(S, \leq)$ be a partially ordered set with the property that each chain has an upper bound in S. Then S contains a maximal element.

Theorem 4.6

Let R be a ring with 1. Then every proper ideal I is contained in some maximal ideal.

Proof:
Let $F = \{J : J \text{ is a } \textbf{proper} \text{ ideal of } R, M \subseteq J\}$. Notice $(F, \subseteq)$ is a poset (partially ordered set). Recall some notations/definitions:

- Chain: subset $G \subseteq F$, s.t. $\forall x, y \in G$, $x \subseteq y$ or $y \subseteq x$ (comparable)
- Upper bound of $G \in F$, $m \in F$, s.t. $\forall g \in G$, $g \subseteq m$.
- Maximal in F: $m \in F$, s.t. $\forall a \in F$, $(m \leq a) \implies (a = m)$.

Let $C \subseteq F$ be a chain. Put $M := \bigcup_{A \in C} A$. M is an ideal because

1. nonempty: $A \in C$, $I \subseteq A$, then $I \in M$.
2. Let $a \in A, b \in B$, and $A, B \in C$. WLOG, assume $A \subseteq B$. Then $a, b \in B$, then $a - b \in B$, then $a - b \in M$.
3. $\forall r \in R, a \in M$, we have $a \in A \in C$, then $ra \in A$, $ra \in M$.

We claim that M is an upper bound of C in F. If $M = R$, then $1 \in A \in C$. But then by proposition, $A = R$. Contradiction.

Then apply Zorn lemma.

Or check proposition 10.8 of PMATH 347. $\square$

5

Polynomial Rings & Rings of Fractions

5.1 How to make new rings from old rings?

I don't want to put this section to the previous chapter. So here it is.

Direct products

Let $(R_i, +_i, \times_i)$ be rings. $R_1 \times R_2$ is a ring with

$$\begin{aligned}(r_1, r_2) \oplus (s_1, s_2) &= (r_1 +_1 s_1, r_2 +_2 s_2) \\ (r_1, r_2) \otimes (s_1, s_2) &= (r_1 \times_1 s_1, r_2 \times_2 s_2)\end{aligned}$$

Then this applies to $\prod_i R_i$ (works for at most countable R_i's).

Direct sum

For finitely many R_i's, it is just direct product. For infinitely many R_i's

$$\bigoplus_{i \in I} R_i = \left\{(r_1, r_2, r_3, \ldots) : r_i \in R_i, \text{ only finitely many } r_1 \neq 0\right\}$$

5.2 Basic Definitions and Examples

Let R be a commutative ring with identity. A polynomial with coefficients in R with undeterminate/-variable x is a **formal** expression

$$p(x) = a^n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with $a_i \in R, \forall i \in 0, \ldots, n$. If $a_m \neq 0$, then $\deg p = n$. If $a_n = 1$, we call $p(x)$ monic.

$R[x] = \{a^n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : n \in \mathbb{N}, a_i \in R\}$ with operations

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i)\, x^i$$

$$\left(\sum_{i=0}^{n} a_i x^i\right) \times \left(\sum_{i=0}^{m} b_i x^i\right) = \sum_{k=0}^{n+m} \left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k$$

Observe that R appears in $R[x]$ as constant polynomials. $R[x]$ is commutative ring with identity.

Proposition 5.1

Let R be an integral domain, let $p, q \in R[x]$ be nonzero elements. Then

1. $\deg pq = \deg p + \deg q$
2. the units of $R[x]$ are precisely the units of R.
3. $R[x]$ is an integral domain.

Proof:
$p(x)q(x) = \underbrace{a_n b_m}_{\neq 0} x^{n+m} + \cdots$

Let $p(x) \in R[x]$ be invertible, then there exists q such that $pq = 1$. By (1), $\deg p = 0$. Thus $\deg q = 0$. p, q are constant polynomials.

$pq = 0$, then $\deg p + \deg q = 0$. Then $\deg p = \deg q = 0$. Then they are all constant polynomials. As R is integral domain, we have $p = q = 0$. □

Formal power series

Ring of all power series $R[[x]] = \left\{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\right\}$ with the same operations defined as polynomial rings.

1. $R[[x]]$ is a commutative ring with identity.
2. Units of $R[[x]]$ are $\sum_{i=0}^{\infty} a_i x^i$ with a_0 unit in R.

Laurent series

$$R((x)) = \left\{\sum_{i=N}^{\infty} a_i x^i : a_i \in R, N \in \mathbb{Z}\right\}$$

5.3 Rings of fractions

Construct $\mathbb{Q}$ from $R = \mathbb{Z}$. Define

$$\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$$

$\frac{p}{q}$ is a "formal" fraction. ($p \cdot q^{-1}$ does not work). However, $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}$ are distinct formal fractions. We want to have them be in equivalent classes.

We define $\frac{a}{b} \sim \frac{c}{d}$ iff $ad = bc$ (use only the ring operations). Then define $\mathbb{Q}$ be the equivalence classes of $\sim$. For that, we need to show that $\sim$ is equivalence: reflexive, symmetric, transitive.

We define addition as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

is well-defined on equivalence classes. We can obtain $+$ on the equivalence classes through definition of $+$.

We define multiplication as

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

is well-defined on equivalence classes.

Then we obtain $\mathbb{Q}$. Note that the well-definednesses need a proof. See Section 11.1 of PMATH 347.

$\frac{2}{1}, \frac{1}{2} \in \mathbb{Q}$, then $\frac{2}{1} \cdot \frac{1}{2} = \frac{2}{2} \sim \frac{1}{1}$ is an identity. Thus 2 is invertible in $\mathbb{Q}$. Every integer is a unit in $\mathbb{Q}$.

If R have zero divisors, $ab = 0$ and $a, b \neq 0$. Then if a invertible: $1 = a^{-1} \cdot a$, then $b = a^{-1}(a \cdot b) = 0$. Contradiction. Thus **zero divisors do not have inverses in any ring**. Now consider

$$a = \frac{a}{1} = \frac{ab}{b} = \frac{0}{b} = 0$$

contradiction to $a \neq 0$. Thus we will avoid zero divisors.

Theorem 5.2

Let R be a commutative ring. Let D be any subset of R closed under multiplication and not containing zero divisors and 0. Then there exists a commutative ring Q with identity such that Q contains R as a subring and every element of D is a unit of Q. Moreover,

1. every element of Q is of the form $\frac{r}{d}$ for some $r \in R, d \in D$. If $D = R \setminus \{0\}$, then Q is a field.

2. The ring Q is the smallest ring containing R in which all elements of D are units.

Here we formalize the definition of "smallest": Let S be any commutative ring with identity and let $\phi : R \to S$ be any injective homomorphism such that $\phi(d)$ is a unit of S for each $d \in D$. Then there is an injective homomorphism $\Phi : Q \to S$ such that $\Phi_R = \phi$. In other words, any ring containing an isomorphic copy of R in which elements of D become units must contain Q.

$$\begin{tikzcd} R \arrow[r, hook, "id"] \arrow[dr, "\phi \text{ injective}"'] & Q \arrow[d, "\Phi \text{ injective}"] \\ & S \end{tikzcd}$$

Thus R"$\subseteq$"S.

Proof:
Almost the same as the proof of Theorem 11.3 of pmath347. Below are some main points.

$F := \{(r,s) : r \in R, d \in D\}$. Then $\sim$ is an equivalence relation: $(r,s) \sim (g,h)$ iff $rh = sg$. Then denote by $\frac{r}{d}$ the equivalence class of (r,d). As above, we define $+$ and $\times$.

Let $Q/\sim$ be the set of equivalence classes of $\sim$. We verify it is a ring.

Q contains an isomorphic image of R: consider a homomorphism $\sigma : R \to Q, r \mapsto \frac{rd}{d}$ for any $d \in D$ (does not depend on choice of d). We need to prove injectivity here.

Every $d \in D$ (i.e., $\sigma(d)$) is invertible in Q.

Now let's prove (1) and (2). (1) is trivial. Now prove (2). We claim that there exists $\psi : Q \to S$ injective such that $\psi|_R = \phi$. Note that $\phi(d)$ invertible for all $d \in D$, thus we can define $\psi(\frac{r}{d}) = \phi(r)\phi(d)^{-1}$ for all $r \in R$, $d \in D$. ψ is well defined. ψ is a homomorphism because ϕ is. Not hard to see ψ is injective. Finally, we see that $\psi|_R = \phi$. $\square$

Example:
$R = \mathbb{Z}$, then $Q = \mathbb{Q}$.

If R is a field, then $Q = R$.

$R = 2\mathbb{Z}$ is a ring without identity, then $Q = \mathbb{Q}$. $1_{\mathbb{Q}} = \frac{2}{2}$ for example.

$R := R[x]$, then Q is a ring of $\frac{p(x)}{q(x)}$, $q(x) \neq 0$. This is rational functions. If we start with $\mathbb{Z}[x]$, then $Q = \{\frac{p(x)}{q(x)} : q(x) \neq 0\}$. If we start with $\mathbb{Q}[x]$, then its Q is the same.

$R := R[[x]]$, then $Q = R((x))$.

6

Chinese Remainder Theorem

comaximal

The ideals $A, B \subseteq R$ are said to be comaximal if $A + B = R$.

m, n coprime iff $\exists a, b \in \mathbb{Z}, an + bm = 1$.

$A + B$

$A + B := \{a + b : a \in A, b \in B\}$.

Example:
$5\mathbb{Z}, 3\mathbb{Z} \subseteq Z$. As $10 + (-9) \in 5\mathbb{Z} + 3\mathbb{Z}$, hence $5\mathbb{Z}, 3\mathbb{Z}$ are comaximal.

AB

$AB := \{\sum_{\text{finite sums}} a_i b_i : a_i \in A, b_i \in B\}$. Similarly we have $A_1 \cdots A_k := \{\sum a_{i1} \cdots a_{ik} : a_{ij} \in A_j\}$.

Theorem 6.1

Let R be a commutative ring with an identity. Let $I_1, I_2, \ldots, I_k$ be ideals in R, such that I_n, I_m are comaximal for $n \neq m$. Then

$$R/_{I_1 I_2 \cdots I_k} = R/_{I_1 \cap I_2 \cap \cdots \cap I_k} \cong R/I_1 \times R/I_2 \times \cdots \times R/I_k$$

In particular, $I_1 I_2 \cdots I_k = I_1 \cap I_2 \cap \cdots \cap I_k$.

Proof:
By induction. The proof here is the same as Theorem 11.24 of pmath 347. $\square$

Remark:
Consider the units of $R/I_1 \cdots I_k$ and $R/I_1 \times \cdots \times R/I_k$. The units are the same (under isomorphism). That means that

$$(R/I_1 \cdots I_k)^\times \cong (R/I_1)^\times (R/I_2)^\times \times \cdots \times (R/I_k)^\times$$

Because units in the product of rings are units in each component.

An element of a product ring is a unit iff each component is a unit in its respective ring.

Then apply this remark to integers: $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

$$(\mathbb{Z}/m\mathbb{Z})^\times \cong (Z/p_1^{\alpha_1}\mathbb{Z})^\times \times \cdots (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})^\times$$

Euler's totient function: $\varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$. Thus from the relation above, we have

$$\varphi(m) = \varphi(p_1^{\alpha_1}) \cdots \varphi(p_k^{\alpha_k})$$

which means $\varphi(\cdot)$ is multiplicative arithmetic function.

7

Domains

7.1 Euclidean Domains

norm

A norm on a ring R is a function $N : \mathbb{R} \to \mathbb{Z}^+ \cup \{0\}$, s.t. $N(0) = 0$.

Euclidean domain

An integral domain (identity, commutative, no zero divisors) for which there exists a Norm, such that: $\forall a, b \in R$, $b \neq 0$, there exists $q, r \in R$ s.t. $a = qb + r$ with $N(r) < N(b)$ or $r = 0$. This is called a Euclidean domain.

Example:

$R = \mathbb{Z}$, $N(x) = |x|$. Then $a = qb + r$ follows from division with remainder. We don't have to keep r positive/negative.

Example:

Fields with $N(x) = 0$. We have $a = (ab^{-1})b = 0$

Example:

F a field. Then $F[x]$ is a Euclidean domain with $N(p(x)) = \deg(p(x))$. Then we can have polynomial long division.

Example:

Consider Gaussian integers.

$$Z[i] = \{a + bi : a, b \in \mathbb{Z}\}$$

is ED with $N(a + bi) = a^2 + b^2 = (a + bi)(a - bi)$.

Theorem 7.1

Every ideal in a Euclidean domain is principal.

Proof:
Let $I \subseteq R$ an ideal. Take a nonzero element d in I of the smallest norm. Let $x \in I$, then $x = qd + r$ where $N(r) = 0$ or $N(r) < N(d)$. But $N(r) < N(d)$ is not possible. So $N(r) = 0$. Since $r = x - qd \in I$, then we must have $r = 0$. Then $x = qd$. This holds for any $x \in I$. Thus $I = (d)$. □

Remark:
Every ideal is principal: principal ideal domain (PID). We have ED $\subseteq$ PID, not other way around.

7.2 GCD & Bézout domains

greatest common divisor

Let R be commutative.

1. We say that $b \mid a$ (b divides a), if there exists $x \in R$, $a = bx$.
2. $d \in R$ is called a $\gcd(a, b)$ if
   $*$) $d \mid a, d \mid b$
   $\triangle$) if $d' \mid a$ and $d' \mid b$, then $d' \mid d$.

We can rephrase two conditions:

$*$) $(a, b) \subseteq (d) \subseteq R$

$\triangle$) If $(a, b) \subseteq (d')$, then $(a, b) \subseteq (d) \subseteq (d')$.

Bézout domain

Bézout domain is a form of a Prüfer domain. It is an integral domain in which the sum of two principal ideals is again a principal ideal.

Proposition 7.2

In Bezout domains (every (a, b) is principal), $(a, b) = (d)$ where $d = \gcd(a, b)$.

Proof:
Assume $(a, b) = (\alpha)$. We know that $(a, b) = (\alpha) \subseteq (d)$ because (d) is the smallest ideal containing (a, b). Then by definition of gcd, we conclude that $(\alpha) = (d)$. □

Bezout domain is not necessary for existence of gcd.

Example:
$R = \mathbb{Z}[x]$, what is $\gcd(2, x)$? $(2, x)$ is not principal. It is a maximal ideal, because $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}_2$.

We see that $(2, x) \subseteq (1)$. Because $(2, x)$ is maximal, there are no ideals in between. Hence $\gcd(2, x^2) = 1$.

Theorem 7.3

Let R be an integral domain (commutative ring with identity), then $(d) = (d')$ if and only if $d = d'u$ for a unit $u \in R$.

Example:

In $\mathbb{Z}[i]$, units are $\{\pm 1, \pm i\}$, then $(2) = (-2i)$.

Proof:

We know that $d \in (d')$ and $d' \in (d)$. Thus we can find $x, y \in R$ such that $d = d'x$ and $d' = dy$. Hence $d(1 - xy) = 0$. If $d = 0$, then it's a trivial ring. If $d \neq 0$, then $xy = 1$. $\square$

Corollary 7.4

If $\gcd(a, b) = d$, then all gcd's are ud, for u a unit.

7.3 Euclidean Algorithm

It unfolds as follows

$$
\begin{aligned}
a &= q_0 b + r_0, \qquad N(r_0) < N(b) \\
b &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
&\vdots \\
r_{m-2} &= q_m r_{m-1} + r_m \\
r_{m-1} &= q_{m+1} r_m + 0
\end{aligned}
$$

Theorem 7.5

Let R be a Euclidean domain, $a, b \neq 0$, $a, b \in R$.

1. The last nonzero remainder, r_m, in Euclidean algorithm is $\gcd(a, b)$.
2. Moreover, $r_m = ax + by$ for $x, y \in R$. And x, y can be obtained from Euclidean algorithm.

Proof:

By going backwards in Euclidean algorithm, we obtain inductively that $r_m \mid r_{m-1}, r_{m-2}, \ldots, r_1, r_0$, $r_m \mid a, b$. This shows that $(a, b) \subseteq (r_m)$, which means r_m is a common divisor. It remains to show that $(r_m) \subseteq (a, b)$. We see that

$$
\begin{aligned}
r_0 &= a - q_0 b \in (a, b) \\
r_1 &= b - q_1 \underset{\substack{\uparrow \\ \in (a,b)}}{r_0} \in (a, b) \\
&\vdots \\
r_m &= r_{m-2} - q_m r_{m-1} \in (a, b)
\end{aligned}
$$

Thus $(r_m) \subseteq (a, b)$.

Therefore $(r_m) = (a, b)$. $\square$

7.4 Principal Ideal Domain

Example:

$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right] = \left\{a + b\frac{1+\sqrt{-19}}{2} : a, b \in \mathbb{Z}\right\}$ is PID, but not Euclidean domain. To prove it is not ED, we follow the definition: $\forall a, b \in R$, $a = qb + r$, $N(r) < N(b)$ or $b = 0$. Take b a non-unit, non-zero with minimal norm. Then every x can be written as $x = qb + r$, $r = 0$ or r is a unit. If b is defined above, and we know units are $\{\pm 1\}$, then $x = qb \pm 1$ or $x = qb + 0$.

Take $2 = qb + r$, $r \in \{0, \pm 1\}$. This gives us three possibilities: $b \mid 2, b \mid 1, b \mid 3$.

For the rest, check https://math.stackexchange.com/a/23872 or page 282 of Dummit & Foote.

Principal Ideal Domain

An integral domain in which every ideal is principal is called a Principal Ideal Domain (PID).

Example:

$\mathbb{Z}$, $F[x]$ for F a field. $\mathbb{Z}[x]$ is not PID.

Proposition 7.6

Let R be a PID, $a, b \neq 0$, $a, b \in R$. Then if $(d) = (a, b)$, then

1. $d = \gcd(a, b)$.
2. $d = ax + by$ for $x, y \in R$.
3. d is unique up to a multiplication by a unit in R.

prime ideal

An ideal $I \subsetneq R$ is called a prime ideal if $ab \in I \implies a \in I$ or $b \in I$.

Example:

$6\mathbb{Z}$ is not prime ideal as $2 \times 3 \in 6\mathbb{Z}$ and $2, 3 \notin 6\mathbb{Z}$.

$7\mathbb{Z}$ is prime ideal.

Remark:

A prime p satisfies $p \mid ab \implies p \mid a$ or $p \mid b$.

Proposition 7.7

Every maximal ideal is prime.

Proof:

Maximal $\Leftrightarrow R/I$ field $\Rightarrow R/I$ integral domain $\Leftrightarrow I$ prime. $\square$

Theorem 7.8

Every nonzero prime ideal in PID is a maximal ideal.

Proof:
Suppose there exists a maximal ideal (m) where $m \in R$ such that a prime ideal $(p) \subsetneq (m) \subsetneq R$. Then $p = rm$. Then $rm \in (p)$. As (p) is prime ideal, thus either $r \in (p)$ or $m \in (p)$.

If $m \in (p)$, then $(m) \subseteq (p)$, then $(m) = (p)$.

If $r \in (p)$, then $r = sp$ for some $s \in R$. Sub it back, $p = rm = spm$. Then $p(1 - sm) = 0$. As $p \neq 0$, then $sm = 1$, thus s, m are units. Thus $(m) = R$. □

Corollary 7.9

$\mathbb{Q}[x]/(p(x))$ for $p(x)$ irreducible (thus (p) is primal). $\mathbb{Q}[x]/(p) \cong \mathbb{Q}(\alpha)$, where α is a root of p.

Corollary 7.10

If $F[x]$ is a PID (ED), then F is a field.

Proof:

(x) is an ideal. We know that $F \cong F[x]/(x)$ is an integral domain. We also know that F is integral domain iff (x) is a prime ideal. As $F[x]$ is PID, then (x) is also maximal. Thus we conclude that $F[x]/(x) \cong F$ is a field.

♡

□

Remark:
In ED, $\forall a, b \neq 0$, $a = qb - r, N(r) < N(b)$ or $b \mid a$.

The norm above generalizes to **Dedekind-Hasse norm**: $N(0) = 0, N(a) > 0$ if $a \neq 0$. Such that $\forall a, b \in R$, $a, b \neq 0$, $\exists s, t \in R : 0 < N(sa - tb) < N(b)$ or $b \mid a$.

Proposition 7.11

R is PID iff R has a Dedekind-Hasse norm.

Corollary 7.12

$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is PID.

7.5 Unique Factorization Domain

irreducible/prime

Let R be an integral domain.

1. Let $r \in R$, $r \neq 0$, r not a unit. We say that r is irreducible, if $r = ab \Rightarrow a$ or b is a unit of R.
2. $p \in R$, non-unit is called a prime, if $p \mid ab \Rightarrow p \mid a$ or $p \mid b$.

2′. (alternatively) p is prime if (p) is a prime ideal.

3. $a, b \in R$ are associated ($a \sim b$) if $a = ub$ for u a unit.

Proposition 7.13

A prime is irreducible.

Proof:

Let p be prime and $p = a \cdot b$. Then $p \mid ab$, then $p \mid a$ or $p \mid b$. WLOG, assume $p \mid a$. Then $a = px$. Hence $p = pxb$. This implies $xb = 1$, then x, b are units.

♡

□

Example:

$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. We found that

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two factorizations into irreducibles, and

$$2 \nmid (1 + \sqrt{-5}) \qquad \text{and} \qquad 2 \nmid (1 - \sqrt{-5})$$

Note that $N(a + b\sqrt{-5}) = a^2 + 5b^2 \in \mathbb{Z}$. Then we observe

$$4 = N(2) = N(\alpha\beta) = N(\alpha)N(\beta)$$

Even better, we have

$$(6) = P_2^2 P_3 P_3',$$

where $P_2 = (2, 1 + \sqrt{-5})$, $P_3 = (3, 2 + \sqrt{-5})$, $P_3' = (3, 2 - \sqrt{-5})$ are all prime ideals. In particular,

$$\begin{aligned}(2) &= P_2^2 \\ (3) &= P_3 P_3' \\ (1 + \sqrt{-5}) &= P_2 P_3 \\ (1 - \sqrt{-5}) &= P_2 P_3'\end{aligned}$$

Theorem 7.14

In PID, primes are precisely irreducibles. In other words, irreducible in PID is prime.

Proof:

Let r be an irreducible. We want to show if (r) is prime ideal. Let $(r) \subseteq M = (m)$ for some ideal M. Then $r = mx$. Because r is irreducible, then either m or x is a unit. If m is a unit, then $M = R$. If x is a unit, then $r \sim m$, then $(r) = (m)$. This proves (r) is maximal. As this is PID, then (r) is prime ideal. Hence r is prime.

♡

□

Unique Factorization Domain

An integral domain R is called a UFD if every non-zero non-unit $r \in R$ satisfies

1. $p_1 p_2 \cdots p_k$ where p_i's are irreducibles of R.
2. if $r = q_1 q_2 \cdots q_m$, with q_i's irreducibles, then $m = k$, and there exists a permutation π of $\{1, 2, \ldots, k\}$, such that $p_i \sim q_{\pi(i)}$.

Example:

A field is a UFD.

$\mathbb{Z}[x]$ is a UFD (if R is UFD, then $R[x]$ is UFD)

PID is UFD.

$\mathbb{Z}[\sqrt{-5}]$ is NOT a UFD.

Proposition 7.15

In UFD, every irreducible is a prime.

Proof:

Let p be irreducible, let $p \mid ab$. I.e., $ab = px$. Then we can write

$$(a_1 \cdots a_n)(b_1 \cdots b_m) = p(x_1 \cdots x_{m+n-1})$$

where a_i, b_i, p, x_i are irreducibles. By UFD property, WLOG assume $p \sim a_i$, then $pu = a_i$ for u a unit, i.e., $p \mid a_i$. Then

$$p(ua_1 \cdots a_{i-1} a_{i+1} \cdots a_m) = a$$

Hence $p \mid a$. □

Proposition 7.16

Let $a, b \neq 0$ in UFD. If

$$a = up_1^{e_1} p_2^{e_2} \cdots p_n^{e_n} \tag{7.1}$$
$$b = vp_1^{f_1} p_2^{f_2} \cdots p_n^{f_n} \tag{7.2}$$

with u, v units, $e_i, f_i \geq 0$ integers, p_i primes. Then

$$d = p_1^{\min\{e_1, f_2\}} \cdots p_n^{\min\{e_n, f_n\}}$$

is a $\gcd(a, b)$.

Proof:

Obviously, $d \mid a$, $d \mid b$. In d, $p_i^{\min\{e_i, f_i\}+1}$ if for some i, then it is not a divisor for both a, b. Thus the exponents have to be $\leq \min\{e_i, f_i\}$. If all $\leq$ are $=$, then we obtain d. If not all $\leq$ are strict, then we get something that divides d. □

Example:

$\mathbb{Z}[i]$ is UFD, but $\mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}$ is not UFD.

$$4 = 2 \cdot 2 = (-2i)(2i)$$

but $i \notin \mathbb{Z}[2i]$, so $2 \nsim (2i)$ or $(-2i)$.

Also $2i$ is not a prime, because $(2i)$ is not a prime ideal:

$$\mathbb{Z}[2i]/(2i) \cong \mathbb{Z}/4\mathbb{Z}$$

in which $2 \times 2 = 0$, which is not an integral domain. This isomorphism is obtained by

$$\phi(a + 2bi) = a \mod 4$$

Theorem 7.17

Every PID is UFD.

Proof:

Two steps:

1. Every nonzero non-unit element is a finite product of irreducibles.
2. Uniqueness.

Let $r \neq 0$, non-unit. Either r is irreducible, or $r = r_1 \cdot r_2$, r_1, r_2 non-units. Then r is irreducible or $r = r_1 r_2$ (r_1, r_2 are non-units). Then either r_1 is irreducible or $r_1 = r_{11} r_{12}$; r_2 is irreducible or $r_2 = r_{21} r_{22}$. We continue this process, iteratively factor r. We want to show the factorization is finite.

Assume factorization does not end. Then we obtain an infinite chain C:

$$(r) \subsetneq (r_1) \subsetneq (r_{11}) \subsetneq (r_{112}) \subsetneq \cdots \subsetneq R$$

which corresponds to an infinite chain of factorization. Then $(m) = \bigcup_{(r_\alpha) \in C} (r_\alpha)$ is an ideal in PID. Since $m \in (r_\alpha)$, for some r_α in chain, then $(r_\alpha) = (m)$. Then

$$(r) \subsetneq (r_1) \subsetneq \cdots \subsetneq (r_\alpha) = (m) \subsetneq (m) \subsetneq \cdots \subsetneq R$$

The chain stabilizes (Noether Domain). Contradicts infinite factorization.

Let $r = p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$ for p_i, q_i irreducibles. WLOG, assume $q_1 \mid p_1$ then $p_1 = uq_1$ for u a unit. Then $p_1 \sim q_1$. Then

$$uq_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m \implies (up_2) \cdots p_n = q_2 \cdots q_m$$

Finish by induction on $\min\{m, n\}$. $\square$

Corollary 7.18

If R is a PID, then there exists a Dedekind-Hasse norm on R.

Proof:

Define the norm: $N(0) = 0$ and $N(p_1 p_2 \cdots p_k) = 2^k$ where $p_1 p_2 \cdots p_k$ is unique factorization to irreducibles and p_i's do not need to be distinct. We observe that $N(ab) = N(a)N(b)$, and $N(a) > 0$ iff $a \neq 0$.

Let $a, b \in R$, then $(a, b) = (r)$ for some $r \in R$, and we know $r = \gcd(a, b)$. Then there exist $s, t \in R$ such that $sa - tb = r$. Taking norms $N(sa - tb) = N(r)$. Here r has less factors than a, b, thus $N(r) < N(b)$ because $r \mid b$. $\square$

Remark:

gcd in UFD always exist, but consider an example. In $\mathbb{Z}[x]$, $\gcd(2, x) = 1$, but $(2, x) \subsetneq (1) = \mathbb{Z}[x]$. And there don't exist α, β s.t. $1 = \alpha \cdot 2 + \beta \cdot x$, namely gcd is not a combination of a, b in this case.

$$\text{Field} \underset{\substack{\uparrow \\ \mathbb{Z}}}{\subsetneq} \text{ED} \underset{\substack{\uparrow \\ \mathbb{Z}[1+\sqrt{-19}/2]}}{\subsetneq} \text{PID} \underset{\substack{\uparrow \\ \mathbb{Z}[x]}}{\subsetneq} \text{UFD} \underset{\substack{\uparrow \\ \mathbb{Z}[\sqrt{-5}]}}{\subsetneq} \text{Integral domain}$$

8

Polynomial Rings

Previously: If $R[x]$ is PID (or ED), then R is a field.

Remember: If R is ID, then $R[x]$ is ID.

$R[x_1, x_2, \cdots, x_n]$

For commutative ring R with identity, $x_1, \ldots, x_n$ commuting variables, we have

$$R[x_1, x_2, \cdots, x_n] = (R[x_1, x_2, \ldots, x_{m-1}])[x_m]$$

Proposition 8.1

Let I be an ideal in commutative ring R, with identity. Then $(R/I)[x] \cong R[x]/(I)$, where $(I) = I[x]$ is in $R[x]$. Moreover, if I is a prime ideal in R, then $(I) = I[x]$ is a prime ideal in $R[x]$.

Example:

$(\mathbb{Z}_5)[x] \cong \mathbb{Z}[x]/5\mathbb{Z}[x]$

Proof:

Consider a homomorphism $\phi : R[x] \to (R/I)[x]$ where ϕ is a coefficient reduction mod I. To check ϕ is a homomorphism, we want to check if $\phi(pq) = \phi(p)\phi(q)$. At x^k of $p(x)q(x)$, after ϕ applied to $\sum_{i=0}^{k} p_i q_{k-i}$, we get $(\sum p_i q_{k-i}) \mod I = \sum_{i=0}^{k}(p_i \mod I)(q_{k-i} \mod I)$. We observe that $\ker \phi = I[x]$. We also see that $\phi(R[x]) = (R/I)[x]$, which is just operations.

I prime ideal, then R/I ID, then $(R/I)[x]$ ID, then $R[x]/I[x]$ ID, thus $I[x]$ is prime ideal. $\square$

8.1 Polynomial rings over fields

Recall norm on $R[x]$: $N(p(x)) = \deg(p)$.

Theorem 8.2

Let F be a field, then $F[x]$ is a ED. Namely, if $a(x), b(x) \in F[x]$, then there exists unique $q, r \in F[x]$ such that $a(x) = b(x)q(x) + r(x)$ with $\deg(r) < \deg(b)$ or $r = 0$. (if $F \subseteq E$, then $F[x] \subseteq E[x]$), where E is a ED.

Proof:
By induction for existence.

1. If $\deg(a) < \deg(b)$, then $r = a, q = 0$.
2. If $\deg(a) \geq \deg(b)$, we can write

$$\begin{aligned} a(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \\ b(x) &= b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0 \end{aligned}$$

where $m \leq n$.

Then the polynomial $\tilde{a}(x) = a(x) - \frac{a_n}{b_m}x^{n-m}b(x)$ and $\deg(\tilde{a}) < \deg(b)$. Then there exists $\tilde{q}, \tilde{r}$ such that $\tilde{a}(x) = \tilde{q}(x)b(x) + \tilde{r}(x)$ with $\deg(b) > \deg(\tilde{r})$. Sub in $a(x)$, we get

$$a(x) = \left(\tilde{q}(x) + \frac{a_n}{b_m}x^{n-m}\right)b(x) + \tilde{r}(x)$$

Note that $\frac{a_n}{b_m} = a_n b_m^{-1}$ is well-defined because F is a field.

As for the uniqueness, assume that $a = qb + r = q'b + r'$. Subtracting these two, we have

$$0 = b(x)(q(x) - q'(x)) + (r(x) - r'(x))$$

where $\deg(r - r') < \deg(b)$. Then

$$b(x)(q(x) - q'(x)) = (r(x) - r'(x)) = 0$$

Because integral domain,

$$q(x) - q'(x) = r(x) - r'(x) = 0$$

Thus $q(x) = q'(x)$ and $r(x) = r'(x)$. □

Corollary 8.3

If F is a field, then $F[x]$ is a UFD and a PID.

Example:

$\mathbb{Z}[x]$ is not a PID because $(2, x)$ is not principal.

$\mathbb{Q}[x]$ is a PID as $\mathbb{Q}$ is a field, then $(2, x) = (1) = \mathbb{Q}[x]$.

$\mathbb{Z}[x]/p\mathbb{Z}[x] \cong (\mathbb{Z}/p\mathbb{Z})[x] \cong \mathbb{Z}_p[x]$. What happens to $(2, x)$ in $(\mathbb{Z}/p\mathbb{Z})[x]$. If $p = 2$, then $(2, x) = (x)$ in $\mathbb{Z}_p[x]$. If $p > 2$, then 2 is invertible, then $(2, x) = (1)$ in $\mathbb{Z}_p[x]$.

8.2 Polynomial rings that are UFDs

Proposition 8.4: Gauss' Lemma

Let R be a UFD with a field of fraction F, and let $p(x) \in R[x]$. If $p(x)$ is irreducible in $R[x]$, then $p(x)$ is also irreducible in $F[x]$. (i.e., if $p(x)$ is reducible in $F[x]$, it is also reducible in $R[x]$)

More precisely, if $p(x) = a(x)b(x)$ in $F[x]$, then $\exists r, s \in F$ such that $p(x) = \underbrace{(ra(x))}_{\in R[x]}\underbrace{(sb(x))}_{\in R[x]}$, and nothing more, with respect to $a(x), b(x)$.

Proof:
Prove by contrapositive. Let $p(x) = a(x)b(x)$ in $F[x]$. We can multiply the denominator, then

$$dp(x) = A(x)B(x) \qquad \text{in } R[x] \tag{$*$}$$

If d is a unit of R, then $p(x) = d^{-1}A(x)B(x)$ in $R[x]$.

If d is not a unit, then $d = p_1p_2\cdots p_k$ is a unique factorization into irreducible primes. Note that (p_1) is a prime ideal of R, then $(R/(p_1))[x]$ is ID. We then take $(\bmod p_1)$ on both sides of $(*)$, then $0 = \overline{A(x)}\,\overline{B(x)}$, where $\overline{A(x)}, \overline{B(x)} \in (R/(p_1))[x]$. Then $\overline{A(x)} = 0$ or $\overline{B(x)} = 0$ as in ID. I.e., either $A(x)$ and $B(x)$ are in (p_1). I.e., either $A(x)$ and $B(x)$ are multiples of p_1. Then WLOG $dp(x) = p_1\cdots p_kp(x) = (p_1A'(x))B(x)$, then $p_2\cdots p_kp(x) = A'(x)B(x)$ in $R[x]$.

Inductively, we'll have $p(x) = \tilde{A}(x)\tilde{B}(x)$ in $R[x]$. I.e., $p(x)$ is reducible in $R[x]$. By contrapositive, the first part holds.

If we write $\tilde{A}(x) = sa(x)$, $\tilde{B}(x) = tb(x)$, then $p(x) = (sa(x))(tb(x))$. Note 1 exists in an UFD. □

Example:
In $\mathbb{Q}[x]$, $x^2 = 2x \cdot \frac{1}{2}x$. We cannot get a factorization in $\mathbb{Z}[x]$ by integer multiples. We can only have $x^2 = x \cdot x = (-x)(-x)$.

Does reducibility in $R[x]$ implies reducibility in $F[x]$? No. In $\mathbb{R}[x]$, $p(x) = 2 \cdot x$ has two irreducibles. In $\mathbb{Q}[x]$, $p(x) = 2 \cdot x$ only has one irreducible as 2 is a unit. Reducibility needs (at least) two irreducible factors.

Corollary 8.5

Let R be UFD, F be its field of fractions and $p(x) \in R[x]$ Let gcd of the coefficients of $p(x)$ be 1. Then $p(x)$ is irreducible if and only if it is irreducible in $F[x]$.

Proof:
$\Rightarrow$ is from Gauss' lemma. For $\Leftarrow$, suppose $p(x)$ is reducible in $R[x]$. Let $p(x) = a(x)b(x)$, then neither of $a(x), b(x)$ are constants, otherwise gcd of the coefficients of $p(x)$ would be this constant. So neither $a(x)$ and $b(x)$ is a unit. Then $p(x)$ is reducible in $F[x]$. □

Theorem 8.6

R is a UFD if and only if $R[x]$ is a UFD.

Proof:
Suppose $R[x]$ is a UFD, then constant polynomials has a unique factorization.

Now supose R is a UFD. Assume $p(x) \in R[x]$, we want to factorize $p(x)$ into irreducibles. Let $p(x) = dp'(x)$, where $d \in R$ and gcd of coefficients of $p'(x)$ is 1.

Since $d \in R$, which is a UFD, then d has a unique factorization. It remains to show that $p(x)$ can be factored uniquely. Factor $p'(x) = g_1(x)g_2(x)\cdots g_r(x)$ in $F[x]$, where $F[x]$ is the field of factions of $R[x]$. By multiplying $c_1, c_2, \ldots, c_r \in F$, we get a factorization in $R[x]$, which is the same trick in the proof of Gauss' lemma. Then

$$p'(x) = (c_1g_1(x))(c_2g_2(x))\cdots(c_rg_r(x)),$$

and $c_ig_i(x)$'s are irreducibles in $F[x]$. We want to show $c_ig_i(x)$'s are irreducibles in $R[x]$. Since $\gcd(\text{coeff. of } p'(x)) = 1$, then $\gcd(c_1, c_2, \ldots, c_r) = 1$, otherwise, we can still factor out a constant from $p'(x)$ and move it to d. This shows that $p(x)$ can be written as a finite product of irreducibles in $R[x]$.

Now suppose in $R[x]$,

$$p(x) = q_1(x) \cdots q_k(x) = q'_1(x) = \cdots q'_r(x)$$

Since $\gcd(coeffs) = 1$, then irreducibility in $R[x]$ implies irreducibility in $F[x]$. This implies $k = r$ and $q_i(x) \sim q'_{\pi(i)}(x)$ in $F[x]$. Then $\exists \frac{a}{b} \in F[x]$ such that $b \cdot q_i(x) = a \cdot q'_{\pi(i)}(x)$. The gcd of LHS coefficients is b and RHS coefficients is a. Thus we must have $a = ub$ for u a unit in R. Then $q_i(x) = \frac{ub}{b} q'_{\pi(i)}(x)$ and $\frac{ub}{b}$ is a unit of R. This proves that $R[x]$ is a UFD. □

Corollary 8.7

$\mathbb{Z}[x]$ is a UFD that is not a PID.

Example:

What about Gauss' lemma for non UFD?

$R = \mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}$ is an ID.

$F = \mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$ is an ID.

$x^2 + 1 = (x + i)(x - i)$ in $F[x]$, but $x^2 + 1$ is irreducible in $R[x]$.

8.3 Irreducibility Criteria

root

$\alpha \in F$ is a root of $p(x)$ if $p(\alpha) = 0$.

Proposition 8.8

Let F be a field and $p(x) \in F[x]$, then $p(x)$ has a factor of degree one if and only if $p(x)$ has a root in F.

Proof:

If $p(x)$ has a factor od degree one, we can assume that the factor is $(x - \alpha)$. Then $p(x) = \underbrace{(1 \cdot x - \alpha)}_{\in F[x]} q(x)$, so $p(\alpha) = 0 \cdot q(\alpha) = 0$.

On the other hand, if $p(\alpha) = 0$, then $p(x) = q(x)(x - \alpha) + r(x)$ where $\deg(r) = 0$ (r constant polynomial) or $r(x) = 0$. As $0 = p(\alpha) = q(\alpha)(\alpha - \alpha) + r(\alpha)$, thus $r(\alpha) = 0$. Hence we conclude $p(x) = q(x)(x - \alpha)$. □

Corollary 8.9

Suppose $p(x) \in F[x]$ is of degree 2 or 3. $p(x)$ is irreducible if and only if $p(x)$ does not of a root in F.

Proof:

If $p(x) = a(x) \cdot b(x)$, nonconstant $a(x), b(x)$, then $\deg(a), \deg(b) < \deg(p)$. □

Example:

In $\mathbb{R}[x]$, $(x^2 + 1)(x^2 + 1)$ is reducible, but does not have a root in $\mathbb{R}$.

Proposition 8.10

Let $p(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$. If $\frac{r}{s} \in \mathbb{Q}$ is a root of p, with $\gcd(r,s) = 1$, then $r \mid a_0$ and $s \mid a_n$.

Proof:
We just plug in, and get

$$0 = a_n \left(\frac{r}{s}\right)^n + \cdots + a_1 \frac{r}{s} + a_0$$

$$0 = a_n r^n + \cdots + a_1 r s^{n-1} + a_0 s^n$$

$$a_n r^n = -s(\cdots\cdots)$$

then $s \mid a_n r^n$, then $s \mid a_n$ from $\gcd(s,r) = 1$.

Analogously for $r \mid a_0$. □

Example:
$p = x^3 - 3x - 1 \in \mathbb{Z}[x]$ is reducible if and only if it has root in $\mathbb{Z}$. A root $r \in \mathbb{Z}$ of p divides -1, then $r = \pm 1$. Then we can check if r is a root.

Example:
Similarly we can check reducibility for $x^2 - p, x^3 - p$ for p prime.

Consider an obvious fact: f reducible in $\mathbb{Z}[x]$, then reducible in $(\mathbb{Z}/m\mathbb{Z})[x]$. Now the following proposition generalizes this fact.

Proposition 8.11

Let I be a proper ideal in an integral domain R. Let $p(x)$ be a nonconstant, monic polynomial in $R[x]$. Then if $\overline{p(x)} \in (R/I)[x]$ cannot be factored into two polynomials of smaller degree, then $p(x)$ is irreducible in $R[x]$.

Proof:
Suppose $p(x) = a(x)b(x) \in R[x]$. Then we know that $a(x)$ and $b(x)$ are monic, and nonconstant. Reducing the coefficients modulo I gives a factorization in $(R/I)[x]$ with nonconstant factors by Proposition 8.1. □

Example:
$x^2 + x + 1$ in $\mathbb{Z}_2[x]$ is irreducible because it has no root in $\mathbb{Z}_2$. Thus irreducible in $\mathbb{Z}[x]$.

$x^2 + 1 = (x+1)(x+1)$ is reducible in $\mathbb{Z}_2[x]$. $x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$, thus $x^2 + 1$ is irreducible in $\mathbb{Z}[x]$.

Example:
$x^4 + 1$ is reducible in $\mathbb{Z}_p[x]$ for any prime p.

$x^4 - 72x^2 + 4$ is reducible in $\mathbb{Z}_n[x]$ for any $m \in \mathbb{N}$.

But they are irreducible in $\mathbb{Z}[x]$.

Theorem 8.12: Eisenstein's criterion

Let P be a prime ideal of an integral domain R. Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x]$ such that $a_{n-1}, \ldots, a_0 \in P$, and $a_0 \notin P^2 = P \cdot P$. Then $p(x)$ is irreducible.

Proof:
Assume we have a factorization $p(x) = a(x)b(x)$ in $R[x]$. Then in $(R/P)[x]$, we reduce the coefficients mod P: $x^n = \overline{a(x)}\,\overline{b(x)}$. Then the constant terms of $\overline{a(x)}$ and $\overline{b(x)}$ are zero, i.e., the constant terms of $a(x)$ and $b(x)$ are elements of P. But then a_0 would be the product of these two would be an element of P^2. Contradiction. $\square$

Example:
$x^4 + 10x + 5$ in $\mathbb{Z}[x]$ is irreducible. Consider prime ideal $P = (5)$.

Example:
$x^n - a \in \mathbb{Z}[x]$ is irreducible for any $a \in \mathbb{Z}$ such that for some prime p with $p \mid a$ and $p^2 \nmid a$.

Example:
For p prime,

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is called p-th cyclotomic polynomial.

If $f(x) = g(x)h(x)$, then $f(x+1) = g(x+1)h(x+1)$. We then can investigate reducibility of $\Phi_p(x+1)$:

$$\Phi_p(x+1) = x^{p-1} + px^{p-2} + \cdots + \frac{p(p-1)}{2}x + p$$

Since all the coefficients except the first are divisible by p by the Binomial Theorem. As before, this shows $\Phi_p(x)$ is irreducible in $\mathbb{Z}[x]$.

9

Field Theory

9.1 Basic Theory of Field Extensions

field extension

Let F be a field. A field K is called an extension of F, if K contains an isomorphic copy of F. We will denote by K/F.

This is not a quotient.

Fact $x^2+1 \in \mathbb{R}[x]$, but no root in $\mathbb{R}$. Make an extension of $\mathbb{R}$ so that $\mathbb{C}$ has a field: $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

Theorem 9.1

Let F be a field, $p(x) \in F[x]$ and irreducible polynomial. Then there exists a field K containing an isomorphic copy of F in which $p(x)$ has a root.

Identifying F with this isomorphic copy shows that there exists an extension of F in which $p(x)$ has a root.

Proof:
Consider the quotient $K = F[x]/(p(x))$. As p is irreducible in PID $f[x]$, (p) is maximal. Hence K is a field. Thus K contains an isomorphic copy of F. Then we have the canonical projection π of $F[x]$ to the quotient $F[x]/(p(x))$ restricted to $F \subseteq F[x]$ gives a homomorphism:

$$\phi = \pi \mid_F \colon F \to K$$

Then we note that it is a zero map because it maps identity 1 of F to the identity 1 of K. Thus ϕ is injective. (or apply Corollary 4.2) Thus $\phi(F) \cong F$ is an isomorphic copy of F contained in K. We identity F with its isomorphic image in K and view F as a subfield of K. Denote $\bar{x} = \pi(x)$ the image of x in the quotient K, then

$$\begin{aligned} p(\bar{x}) &= \overline{p(x)} && \text{(since } \pi \text{ is a homomorphism)} \\ &= p(x) (\operatorname{mod} p(x)) && \text{in } F[x]/(p(x)) \\ &= 0 && \text{in } F[x]/(p(x)) \end{aligned}$$

so that K does indeed contain a root of p. Then K is an extension of F in which p has a root. $\square$

degree/index of a field extension

The degree (or relative degree or index) of a field extension K/F, denoted $[K : F]$, is the dimension of K as a vector space over F (i.e., $[K : F] = \dim_F K$). The extension is said to be finite if $[K : F]$ is finite and is said be infintie otherwise.

Theorem 9.2

Let $p(x) \in F[x]$ be an irreducible polynomial of degree $n \geq 1$ over the field F and let K be the field $F[x]/(p(x))$. Let $\theta = x \bmod (p(x)) \in K$. Then the elements

$$1, \theta, \theta^2, \ldots, \theta^{n-1}$$

are a basis for K over a vector space over F, so the degree of extension is n, i.e., $[K : F] = n$. Hence

$$K = \{a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{n-1}\theta^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\}$$

consists of all polynomials of degree $< n$ in θ.

From linear algebra, $\mathbb{C}$ is a vector space over $\mathbb{R}$. If we multiply a $\mathbb{C}$ by $\mathbb{R}$, it is still $\mathbb{C}$, so it's well defined. Basis in this case is $1, i$. $\dim(\mathbb{C}) = 2$. Thus $[\mathbb{C} : \mathbb{R}] = 2$.

Example:

$\mathbb{R}$ is a vector space over $\mathbb{Q}$. $\mathbb{R}$ are the vectors, $\mathbb{Q}$ are scalars. It is infinite dimensional vector space as it has no finite basis. This is because $\mathbb{R}$ is uncountable.

Proof:

First we want to show $\operatorname{span}\{1, \theta, \ldots, \theta^{n-1}\} = K = F[x]/(p(x))$.

Let $a \in F[x]$, as $F[x]$ is Euclidean domain, we have

$$a(x) = h(x)p(x) + r(x), \qquad \deg r < \deg p = n$$

Then $a \equiv r \bmod p$, which shows that every residue class in $F[x]/(p(x))$ is represented by a polynomial of degree less than n. Hence the images $1, \theta, \ldots, \theta^{n-1}$ in the quotient span the quotient as a vector space over F.

Then we want to show the linear independence of $1, \theta, \ldots, \theta^{n-1}$. Consider the equation

$$a_0 \cdot 1 + a_1\theta + \cdots + a_{n-1}\theta^{n-1} = 0$$

$$a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (p(x)) = 0 + (p(x))$$

in $F[x]/(p(x))$.

$$a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \equiv 0 \bmod (p(x))$$

Namely

$$p(x) \mid a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$$

But $\deg(p) = n$ and $\deg(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) \leq n - 1$, then $a_0 = a_1 = \cdots = a_{n-1} = 0$. $\square$

Addition in K:

$$\sum_{i=0}^{n-1} a_i\theta^i + \sum_{i=0}^{n-1} b_i\theta^i = \sum_{i=0}^{n-1} (a_i + b_i)\theta^i$$

Multiplication is done by

$$a(x)b(x) = h(x)p(x) + r(x)$$

then

$$a(\theta)b(\theta) = r(\theta)$$

where $\deg r \leq n-1$.

Inversion in K: we want to find $a(\theta)\big(a(\theta)\big)^{-1} = 1$. This is equivalent to find $b(x) := \big(a(x)\big)^{-1}$

$$a(x)b(x) + h(x)p(x) = 1$$

which can be done via Extended Euclidean Algorithm.

Example:

Consider $\mathbb{R}[x]/(x^2+1)$. Here $p(x) = x^2+1$. This is equivalent to $\{a + b\theta : a, b \in \mathbb{R}\}$ (pretend that we don't know the complex number).

The addition is

$$(a+b\theta) + (c+d\theta) = (a+c) + (b+d\theta)$$

Multiplication is

$$(a+b\theta)(c+d\theta) = ac + (bd+ac)\theta + bd\theta^2$$

which doesn't fit the form $a + b\theta$. Using the fact that $p(\theta) = 0 = \theta^2 + 1$, then

$$(a+b\theta)(c+d\theta) = (ac-bd) + (bd+ac)\theta$$

Example:

$x^2+1 \in \mathbb{Q}[x]$ has a unit in $\mathbb{Q}[x]/(x^2+1) = \{a + b\theta : a, b \in \mathbb{Q}\}$. $[\mathbb{Q}[x]/(x^2+1) : \mathbb{Q}] = 2$. $1, i$ basis.

Example:

$p(x) = x^2 - 2 \in \mathbb{Q}[x]$, $\theta^2 = 2$.

Then $K = \mathbb{Q}[x]/(x^2-2) = \{a + b\theta : a, b \in \mathbb{Q}\}$. Addition is same as before, multiplication is

$$(a+b\theta)(c+d\theta) = (ac+2bd) + (ad+bc)\theta$$

Example:

$p(x) = x^3 - 2$. Then $\mathbb{Q}[x]/(x^3-2) = \{a_0 + a_1\theta + a_2\theta^2 : a_i \in \mathbb{Q}\}$, where

$$\theta = \sqrt[3]{2} \quad \text{or} \quad \sqrt[3]{2}e^{\frac{2\pi i}{3}} = \sqrt[3]{2}\Big(\frac{-1+i\sqrt{3}}{2}\Big) \quad \text{or} \quad \sqrt[3]{2}e^{\frac{4\pi i}{3}} = \sqrt[3]{2}\Big(\frac{-1-i\sqrt{3}}{2}\Big)$$

Note that if we let $\theta = \sqrt[3]{2}$, then this field it does not contain other two θ's. Similar for other θ's. This brings the idea of splitting fields.

Example:

Let $F = \mathbb{F}_2 = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = GF(2) = \{0, 1\}$ with operations mod 2.

$p(x) = x^2 + x + 1 \in \mathbb{F}_2[x]$ is irreducible, as no roots in $\mathbb{F}_2$. Then can get a degree 2 extension of $\mathbb{F}_2$.

$K = \mathbb{F}_2[x]/(x^2+x+1) \cong \{a + b\theta : a, b \in \{0,1\}\}$ which is a field of four elements. In this field, $\theta^2 = -\theta - 1 = \theta + 1$. The multiplication is defined by

$$\begin{aligned}(a+b\theta)(c+d\theta) &= ac + (ad+bc)\theta + bd\theta^2 \\ &= ac + (ad+bc)\theta + bd(\theta+1) \\ &= (ac+bd) + (ad+bc+bd)\theta\end{aligned}$$

Remark:

It is possible to construct of degree p^n for any $n \geq 1$. All this finite fields are of this form.

Example:

Let $F = k(t)$ be the field of rational functions in the variable t over a field k (for example, $k = \mathbb{Q}$). F is a field of fractions of $k[t]$. Let $p(x) = x^2 - t \in F[x]$, which is irreducible. This is by Eisenstein's criterion, (t) is a primal ideal of $k[t]$.

Then the degree 2 extension is

$$K = F[x]/(x^2 - t) = \{a(t) + b(t)\theta | a, b \in F\}$$

where $\theta^2 = t$.

Every $p(x) \in \mathbb{Q}[x]$ has all roots in $\mathbb{C}$.

field generated by $\alpha, \beta, \ldots$ over F

Let K be a field extension of F, and let $\alpha, \beta, \ldots \in K$. The smallest subfield of K containing $\alpha, \beta, \ldots$ and F is denoted by $F(\alpha, \beta, \ldots)$, which is called the field generated by $\alpha, \beta, \ldots$ over F.

simple extension & primitive element

If we are adjoining only one element α, then $F(\alpha)$ is called a simple extension and α is called a primitive element for the extension.

Theorem 9.3

Let F be a field, $p(x) \in F[x]$ irreducible of degree $n \geq 1$. Suppose that K/F contains a root α of $p(x)$, i.e., $p(\alpha) = 0$. Then $F(\alpha) \cong F[x]/(p(x))$.

Proof:

There is a natural homomorphism

$$\begin{aligned} \varphi : F[x] &\longrightarrow F(\alpha) \subseteq K \\ f(x) &\longmapsto f(\alpha) \end{aligned}$$

Since $p(\alpha) = 0$ by assumption, the $p(x) \in \ker \varphi$. So we obtained an induced homomorphism (also denoted φ):

$$\varphi : F[x]/(p(x)) \longrightarrow F(\alpha)$$

But since $p(x)$ is irreducible, the quotient on the left is a field, as it is not zero map, thus injective. Since this image is then a subfield of $F(\alpha)$ containing F and containing α, by the definition of $F(\alpha)$ the map must be surjective, proving the theorem. □

Corollary 9.4

$F(\alpha) = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\}$ where α is a root of an irreducible polynomial $p(x) \in F[x]$.

Example:

$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} = \{a + b\sqrt{-2} : a, b \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{-2})$ and α is the root of $x^2 - 2$.

$\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(-\sqrt{2})$

$\mathbb{Q}(\sqrt{2})$ has an automorphism (an isomorphism from a mathematical object to itself):

$$\mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2}) : a + b\sqrt{2} \mapsto a - b\sqrt{2}$$

Some facts might be interesting:

- $\mathbb{R}$ has no non-trivial automorphisms.
- $\mathbb{C}$ has identity, $a + bi \mapsto a - bi$, uncountably many "wild" automorphisms

9.2 Algebraic Extensions

algebraic, transcendental

An element α of an extension K/F is called *algebraic* over F, if α is a root of some polynomial in $F[x]$. If α is not algebraic over F, then α is *transcendental* over F. The extension $K/F$is said to be *algebraic* if every element of K is algebraic over F.

Example:

$\sqrt{2}$ is algebraic over $\mathbb{Q}$.

π is transcendental over $\mathbb{Q}(\sqrt{2})$.

e is transcendental over $\mathbb{Q}$.

π is algebraic over $\mathbb{Q}(\pi/2)$, as it is root of $x - \pi$.

Liouville's Constant is Transcendental (1844)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{10^{n!}} &= \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{24}} + \cdots \\ &= 0.11000\,10000\,00000\,00000\,00010\,00\ldots\end{aligned}$$

is transcendental.

Hermite (1873): e is transcendental.

Cantor (1874): almost every complex number is transcendental over $\mathbb{Q}$. (countable many polynomials of $\mathbb{Q}[x]$, uncountably many numbers)

1882: π is transcendental.

Proposition 9.5

Let α be algebraic over F. Then there exists unique monic irreducible $m_{\alpha,F} \in F[x]$ having α as a root.

Moreover, $f(x) \in F[x]$ has α as a root if and only if $m_{\alpha,F}(x) \mid f(x)$ in $F[x]$.

Proof:

Take $g(x)$ as polynomial over F of minimal degree, $g(\alpha) = 0$. We may assume $g(x)$ is monic by multiplying $g(x)$ by a constant. Suppose $g(x)$ were reducible, $g(x) = a(x)b(x)$ with $\deg a, \deg b < \deg g$. As K is a field, $a(\alpha)b(\alpha) = 0$ implies that $a(\alpha) = b(\alpha) = 0$, contradicting the minimality of $\deg g$.

Suppose now $h(x) \in F[x]$ has α as a root, namely $h(\alpha) = 0$. By Euclidean algorithm,

$$h(x) = q(x)g(x) + r(x),$$

then $0 = h(\alpha) = q(\alpha)g(\alpha) + r(\alpha) = 0$, but $\deg r < \deg g$, thus $r(x) = 0$. Consequently, $h(x) = q(x)g(q)$, i.e., $g(x) \mid h(x)$. In particular, if $\deg g = \deg h$, then $g(x) = c \cdot h(x)$. $\square$

minimal polynomial

$m_{\alpha,F}(x)$ (in Proposition 9.5) is called the minimal polynomial of α over F. $\deg \alpha := \deg m_{\alpha,F}(x)$.

Example:

$x^2 - x - 1$ has a root $\approx 1.618 = \varphi$

$\varphi^3 - 2\varphi - 1 = (\varphi + 1)(\varphi^2 - \varphi - 1) = 0$

Proposition 9.6

Let α be algebraic over F. Then $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$. In particular, $[F(\alpha) : F] = \deg \alpha$.

finite extension

An extension K/F, s.t. $[K : F] < \infty$ is called a finite extension of F.

Proposition 9.7

α is algebraic over F if and only if $F(\alpha)/F$ is finite.

More precisely, if α is an element of an extension of degree n, then α is a root of a polynomial of degree at most n, and if α satisfies a polynomial of degree n over F then the degree of $F(\alpha)$ over F is at most n, i.e., $[F(\alpha) : F] \leq n$.

Proof:
Suppose α is algebraic, then $F(\alpha)/F$ is finite as $[F(\alpha) : F] = \deg_F \alpha$.

Conversely, suppose $[K : F] = n$, then $1, \alpha, \alpha^2, \ldots, \alpha^n$ is linear dependent.

$$b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_n\alpha^n = 0$$

has non-trivial solution. Hence α is the root of a nonzero polynomial with coefficients in F (of degree $\leq n$), then α is algebraic of degree $\leq n$. $\square$

Example: Quadratic Extensions over Fields of Characteristic $\neq 2$
Let F be a field, s.t. $1 + 1 \neq 0$. Let $[K : F] = 2$. Let $\alpha \in K \setminus F$, then α satisfies an equation of degree 2, i.e., α is a root of $m_{\alpha,F} = x^2 + bx + c$ for $b, c \in F$. Then $F \subseteq F(\alpha) \subseteq K$, and $[F(\alpha) : F] = [K : F] = 2$, which implies $F(\alpha) = K$.

Roots of $m_{\alpha,F}(x)$ are

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Comments:

1. Derivation of the formula is the same as in $\mathbb{C}$.
2. $\sqrt{b^2 - 4c}$ is derived as a solution to $x^2 - (b^2 - 4c) = 0$.
3. 2 in the denominator is $1 + 1$.

We claim that $F(\alpha) = F(\sqrt{b^2 - 4c})$.

As α is obtained by the field operations from F and $\sqrt{b^2-4c}$. Then $F(\alpha) \subseteq F(\sqrt{b^2-4c})$.

Conversely, as $\sqrt{b^2-4c} = \pm(2\alpha+b) \in F(\alpha)$. Then $F(\sqrt{b^2-4c}) \subseteq F(\alpha)$.

Hence $F(\alpha) = F(\sqrt{b^2-4c})$. And every extension of degree 2 can be written as $F(\sqrt{D})$ for D not a square of F.

Theorem 9.8

Let $F \subseteq K \subseteq L$ be fields. Then
$$[L:F] = [L:K][K:F]$$
i.e. extension degrees are multiplicative, where if one side of the equation is infinite, the other side is also infinite.

Proof:
Let $[L:K]$ and $[K:F]$ be finite, where $[L:K]$ has basis $\beta_1,\dots,\beta_n$ and $[K:F]$ has basis $\alpha_1,\dots,\alpha_k$.

We claim that $[L:F]$ has basis $\alpha_i\beta_j$ for $1 \le i \le k, 1 \le j \le n$.

Let $\gamma \in L$, then $\gamma = b_1\beta_1 + \cdots + b_n\beta_n$ for $b_i \in K$. As $b_i \in K$, we have $b_i = a_{1i}\alpha_1 + \cdots + a_{ki}\alpha_k$ for $a_{\ell j} \in F$. Then
$$\gamma = \sum_{i,j} a_{ij}\alpha_j\beta_i, \quad a_{ij} \in F$$
Note that the indices here are messed up. This proves that $\alpha_i\beta_j$'s span L as a vector space over F.

Now we want to show $\alpha_i\beta_j$ are linearly independent. Consider the equation
$$\sum a_{ij}(\alpha_j\beta_i) = 0, \quad a_{ij} \in F$$
We let $b_i := \sum a_{ij}\alpha_j$, then
$$\sum b_i\beta_i = 0$$
Since β_i are a basis, all b_i must be zero. This gives us
$$0 = b_i = a_{i1}\alpha_1 + \cdots + a_{ik}\alpha_k$$
as α_j are basis, then this implies $a_{ij} = 0$ for all i,j. Hence $\alpha_i\beta_j$ are linearly independent over F, so form a basis for L over F and $[L:F] = nk = [L:K][K:F]$, as claimed.

infinity proof from textbook If $[K:F]$ is infinite, then there are infinitely many elements of K, hence of L, which are linearly independent over F, so that $[L:F]$ is also infinite. Similarly, if $[L:K]$ is infinite, there are infinitely many elements of L linearly independent over K, so certainly linearly independent over F, so again $[L:F]$ is infinite. Finally, if $[L:K]$ and $[K:F]$ are both finite, then the proof above shows $[L:F]$ is finite, so that $[L:F]$ infinite implies at least one of $[L:K]$ and $[K:F]$ is infinite, completing the proof. □

Example:
$\sqrt[6]{2}$ is a root of x^6-2, which is irreducible. Then $[\mathbb{Q}(\sqrt[6]{2}):\mathbb{Q}] = 6$. We know that $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = 2$, and $\sqrt{2} = (\sqrt[6]{2})^3$, then $\sqrt{2} \in \mathbb{Q}(\sqrt[6]{2})$. Then we have
$$\underbrace{[\mathbb{Q}(\sqrt[6]{2}):\mathbb{Q}]}_{6} = \underbrace{[\mathbb{Q}(\sqrt[6]{2}):\mathbb{Q}(\sqrt{2})]}_{3}\underbrace{[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]}_{2}$$
This suggests that $\sqrt[6]{2}$ is a root of $x^3 - \sqrt{2} \in (Q(\sqrt{2}))[x]$, which is irreducible.

Lemma 9.9

$F(\alpha,\beta) = (F(\alpha))(\beta)$.

Proof:
$F(\alpha,\beta)$ contains F,α,β, thus contains $F(\alpha),\beta$.

$(F(\alpha))(\beta) \subseteq F(\alpha,\beta)$ follows from minimality of $F(\alpha)$.

$(F(\alpha))(\beta) \supseteq F(\alpha,\beta)$ follows from $\alpha,\beta,F \in (F(\alpha))(\beta)$ and minimality of $F(\alpha,\beta)$. □

Corollary 9.10

$F(\alpha_1,\alpha_2,\dots,\alpha_k) = (F(\alpha_1,\alpha_2,\dots,\alpha_{k-1}))(\alpha_k)$. And we denote

$$F_0 = F \qquad F_1 = F_0(\alpha_1) \qquad \cdots \qquad F_{k-1} = F_{k-2}(\alpha_{k-1}) \qquad F_k = F_{k-1}(\alpha_k)$$

We see that

$$[K : F] = [F_k : F_{k-1}]\cdots[F_1 : F_0]$$

degree of K/F is a product of degrees of the intermediate extensions.

Example:
We cannot multiply the degree of α's. For example, $\mathbb{Q}(\sqrt[6]{2},\sqrt{2}) = \mathbb{Q}(\sqrt[6]{2})$. We have

$$6 = [\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}] = \underbrace{\mathbb{Q}(\sqrt[6]{2},\sqrt{2}) : \mathbb{Q}(\sqrt[6]{2})}_{1}\underbrace{[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}]}_{6}$$

Example:
$\mathbb{Q}(\sqrt{2},\sqrt{3}) = (\mathbb{Q}(\sqrt{2}))(\sqrt{3})$. Since $\sqrt{3}$ is of degree 2 over $\mathbb{Q}$ of the extension $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}(\sqrt{2})$ is at most 2.

If its degree is 1, then $x^2 - 3$ is reducible over $\mathbb{Q}(\sqrt{2})$, $\Leftrightarrow$ $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$.

Suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$. Then $\sqrt{3} = a + b\sqrt{2}$ for $a,b \in \mathbb{Q}$. Then with some calculation, we see it's impossible. Then

$$[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2} : \mathbb{Q})] = 2 \times 2 = 4$$

Theorem 9.11

The extension K/F is finite ($[K : F] < \infty$) if and only if K is generated over F by a finite number of algebraic elements over F.

Proof:
Assume K is generated over F by a finite number of algebraic elements over F. Then we have $[K : F] = [F_k : F_{k-1}]\cdots[F_1 : F_0]$, where all $[F_{i+1} : F_i] < \infty$. $[F_i(\alpha_{i+1}) : F] < \infty$ because α_{i+1} is algebraic over F_i. This shows $[K : F] < \infty$.

Assume the extension K/F is finite, then there exists a basis of K over F: $\alpha_1,\dots,\alpha_k$, then $K = F(\alpha_1,\dots,\alpha_k)$. □

Theorem 9.12

If α, β are algebraic over F, then $\alpha \pm \beta, \alpha \cdot \beta, \frac{\alpha}{\beta}$ $(\beta \neq 0)$ are algebraic over F. In particular, the collection of all algebra elements over F is a field.

Proof:
They all lie in $F(\alpha, \beta)$, which is a finite extension. Hence they are algebraic by Corollary 13 from textbook:

> If the extension K/F is finite, then it is algebraic.

which is based on Proposition 9.7. $\square$

Corollary 9.13

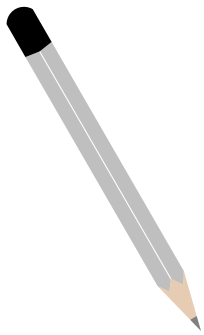
Let K/F be field extension, then all the numbers algebraic over F and belonging to K also form a field (subfield of K).

Example:
Let $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$, it is the set of all algebraic numbers over $\mathbb{Q}$. $\overline{\mathbb{Q}} \cap \mathbb{R}$ is a field as well. These are infinite extensions of $\mathbb{Q}$.

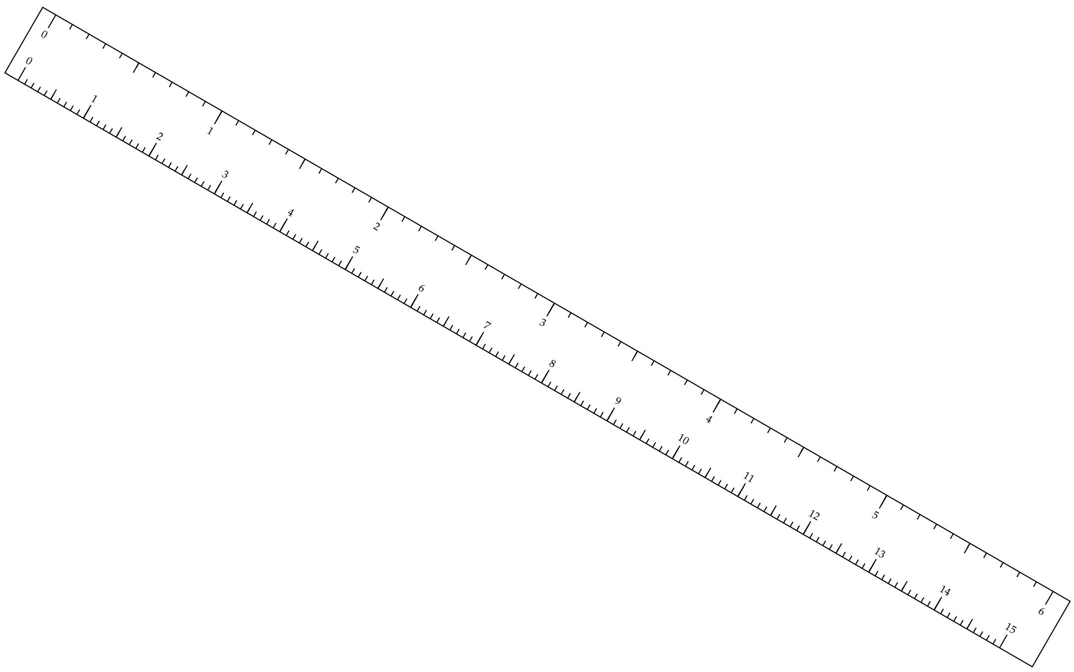
10

Straightedge & Compass Construction

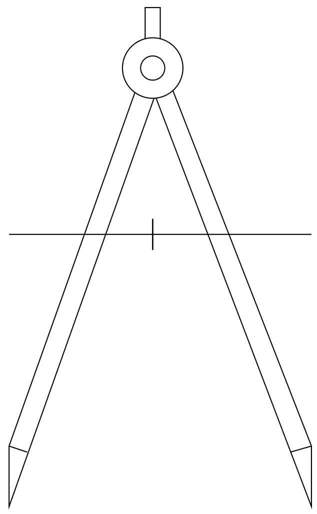
Here is pencil:



Here is ruler (straightedge):



Here is a compass:



Src: https://tex.stackexchange.com/a/147496

constructible number

A constructible number $r \in \mathbb{R}$ is a number where absolute value $|r|$ is a length of constructible line segment.

constructible point

A point of $\mathbb{R}^2$ is called constructible, if it can be constructed by a straightedge and compass with the rules:

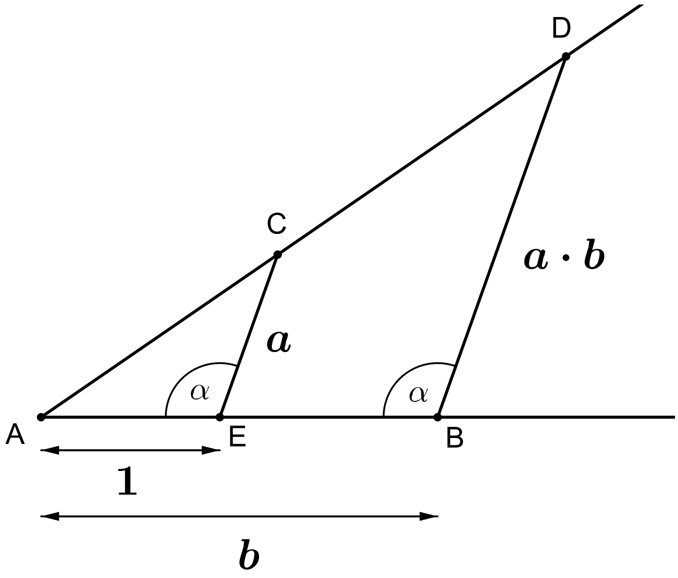
1. We are given unit length, and x, y-axis. So we have $(0, 0)$ and $(1, 0)$.
2. we can construct lines going through two constructible points.
3. We can construct a circle with a center in a constructible point and with radius being a constructible number.
4. We can make intersections of two lines, line + circle, two circles.
5. Finite number of steps.

Proposition 10.1

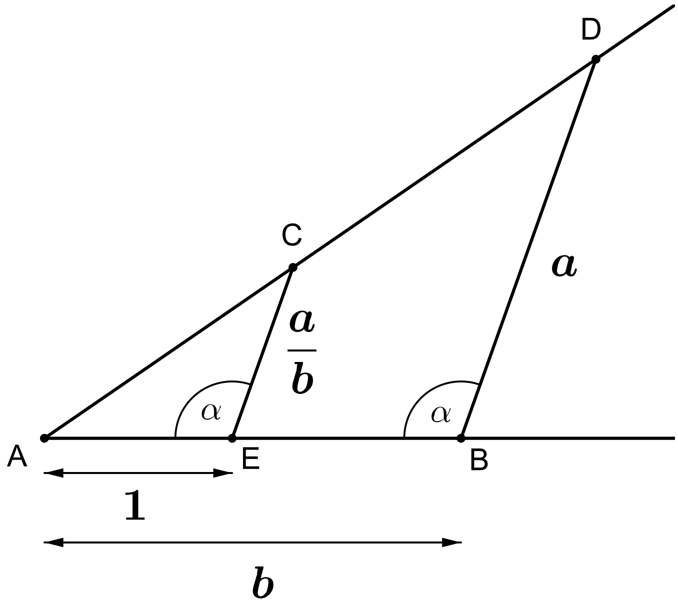
Constructible numbers are a field. If $a \in C$, then $\sqrt{a} \in C$, where C is constructible numbers.

Proof:
$a + b$ is simple.

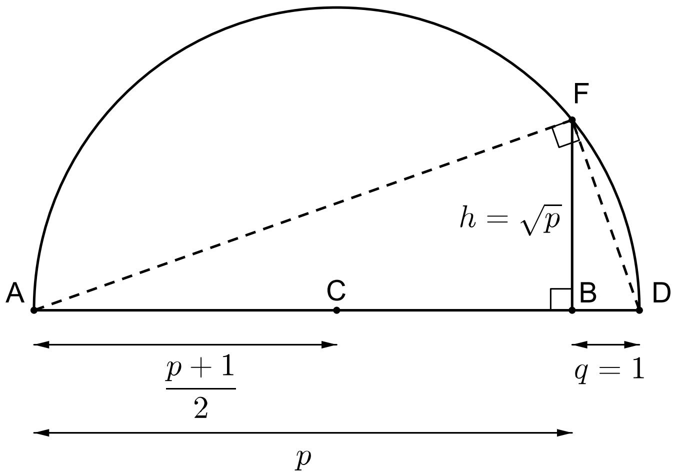
Here is ab based on the intercept theorem.



Here is a/b based on the intercept theorem.



Here is $\sqrt{p}$ based on the geometric mean theorem.



> Img src: https://en.wikipedia.org/wiki/Constructible_number
> Kmhkmh, CC BY 4.0 https://creativecommons.org/licenses/by/4.0, via Wikimedia Commons
>
> □

Thus we can construct the fields $1 \to \mathbb{Q} \to \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3}, \sqrt{2}) \to \mathbb{Q}(\sqrt{3}, \sqrt{2}, \sqrt{\sqrt{3}+\sqrt{2}}) \to \cdots$

What can we construct? Assume that a field F is constructible, i.e., every element is constructible. We can have these new points:

1. Intersection of two lines is in F:

$$a_1x + b_1y + c_1 = 0 \qquad a_2x + b_2y + c_2 = 0$$

   for $a_i, b_i, c_i \in F$.

2. Intersection of circle and line:

$$(x-h)^2 + (y_k)^2 = r^2 \qquad ax + by = c$$

   for $h, k, r, a, b, c \in F$. This is quadratic for x, solution gives a field of degree 2 or 1.

3. Intersection of two circles:

$$(x-a)^2 + (y-b)^2 = r_1^2 \qquad (x-c)^2 + (y-d)^2 = r_2^2$$

   Subtract these two equations, we have a linear equation.

Theorem 10.2

A real number α is constructible if and only if there is a sequence of fields $F_0 = \mathbb{Q}, F_n = \mathbb{Q}(\alpha)$ and $[F_{i+1} : F_i] = 2$. In particular, if α is constructible, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^n$, $n \geq 0$.

Index