Applied Cryptography

CO 487

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LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 487 during Winter 2021 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

Notations:

- $k \in_R K$ means that k is chosen uniformly and independently at random from K.
- $a \| b$ and a, b denote the concatenation between strings a and b.

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Sibelius Peng

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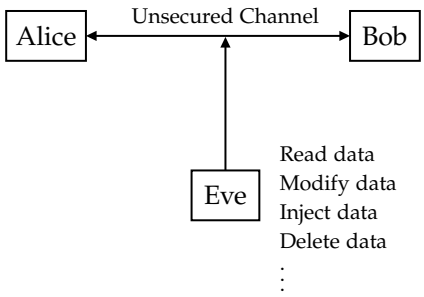
Introduction

Cryptography is about securing communications in the presence of *malicious* adversaries.



Note that even if we call him "Eve", he can do more than eavesdropping... The adversary is malicious, powerful and unpredictable.

Fundamental Goals of Cryptography

1. **Confidentiality**: Keeping data secret from all but those authorized to see it.
2. **Data integrity**: Ensuring data has not been altered by unauthorized means.
3. **Data origin authentication**: Corroborating the source of data.
4. **Non-repudiation**: Preventing an entity from denying previous commitments or actions.

Some examples of unsecured channel:

- Secure Browsing: The Internet
- Online Shopping: The Internet
- Automatic Software Upgrades: The Internet
- Cell Phone Service: Wireless
- Wi-Fi: Wireless
- Bluetooth: Wireless
- Messaging: Wired/Wireless

Communicating Parties

Alice and Bob are two communicating devices.

Alice	Bob	Communication channel
person	person	telephone cable
person	person	cellular network
person	web site	internet
iPhone	wireless	router wireless
iPhone	headphones	wireless
iPhone	service provider	cellular network
your car's brakes	another car	wireless
smart card	bank machine	financial network
smart meter	energy provider	wireless
military commander	satellite	space

1.1 Secure Web Transactions

Transport Layer Security (TLS): The cryptographic protocol used by web browsers for secure web transactions for secure access to amazon, gmail, hotmail, facebook etc.

TLS is used to assure an individual user (called a client) of the authenticity of the web site (called the server) he or she is visiting, and to establish a secure communications channel for the remainder of the session.

Symmetric-key cryptography: The client and server a priori share some secret information k, called a key.

They can subsequently engage in secure communications by encrypting their messages with AES and authenticating the resulting ciphertexts with HMAC.

How do they establish the shared secret key k?

Public-key cryptography: Communicating parties a priori share some authenticated (but non-secret) information.

To establish a secret key, the client selects the secret session key k, and encrypts it with the server's RSA public key. Then only the server can decrypt the resulting ciphertext with its RSA private key to recover k.

How does the client obtain an authentic copy of the server's RSA public key?

Signature scheme: The server's RSA public key is signed by a Certifying Authority using the RSA signature scheme.

The client can verify the signature using the Certifying Authority's RSA public verification key which is embedded in its browser. In this way, the client obtains an authentic copy of the server's RSA public key.

1.2 The TLS Protocol

1. When a client first visits a secured web page, the server transmits its certificate to the client.
    - The certificate contains the server's identifying information (e.g., web site name and URL) and RSA public key, and the RSA signature of a certifying authority.

- The certifying authority (e.g., Verisign) is trusted to carefully verify the server's identity before issuing the certificate.

2. Upon receipt of the certificate, the client verifies the signature using the certifying authority's public key, which is embedded in the browser. A successful verification confirms the authenticity of the server and of its RSA public key.
3. The client selects a random session key k, encrypts it with the server's RSA public key, and transmits the resulting ciphertext to the server.
4. The server decrypts the ciphertext to obtain the session key, which is then used with symmetric-key schemes to encrypt (e.g. with AES) and authenticate (e.g. with HMAC) all sensitive data exchanged for the remainder of the session.
5. The establishment of a secure link is indicated by a closed padlock in the browser. Clicking on this icon reveals the server's certificate and information about the certifying authority.

TLS is one of the most successful security technologies ever deployed. But is TLS really secure?

There are many potential security vulnerabilities:

1. The crypto is weak (e.g., AES, HMAC, RSA).
2. Quantum attacks on the underlying cryptography.
3. Weak random number generation.
4. Issuance of fraudulent certificates
   - In 2001, Verisign erroneously issued two Class 3 code-signing certificates to a person masquerading as a Microsoft representative.
   - Mistake due to human error.
5. Software bugs (both inadvertent and malicious).
6. Phishing attacks.
7. TLS only protects data during transit. It does not protect your data when it is collected at the server.

   Many servers store large amounts of credit card data and other personal information.
8. The National Security Agency (NSA)

1.3 Cryptography in Context

Cybersecurity is comprised of the concepts, technical measures, and administrative measures used to protect networks, computers, programs and data from deliberate or inadvertent unauthorized access, disclosure, manipulation, loss or use. Also known as information security. Cybersecurity includes the study of computer security, network security and software security.

Note that Cryptography $\neq$ Cybersecurity.

- Cryptography provides some mathematical tools that can assist with the provision of cybersecurity services. It is a small, albeit an indispensable, part of a complete security solution.
- Security is a chain
  - Weak links become targets; one flaw is all it takes.

- Cryptography is usually not the weakest link. However, when the crypto fails the damage can be catastrophic.

2

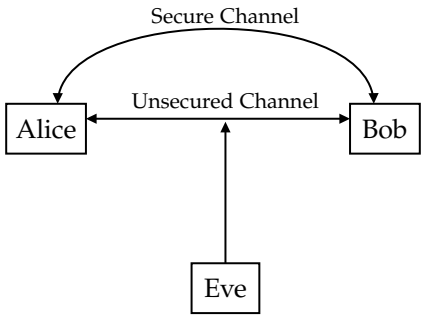
Symmetric-Key Cryptography

2.1 Basic concepts

symmetric-key encryption scheme

A **symmetric-key encryption scheme (SKES)** consists of:

- M - the plaintext space,
- C - the ciphertext space,
- K - the key space,
- a family of encryption functions: $E_k : M \to C$, $\forall k \in K$,
- a family of decryption functions: $D_k : M \to C$, $\forall k \in K$, such that $D_k(E_k(m)) = m$ for all $m \in M, k \in K$.



1. Alice and Bob agree on a secret key $k \in K$ by communicating over the secure channel.
2. Alice computes $c = E_k(m)$ and sends the ciphertext c to Bob over the unsecured channel.
3. Bob retrieves the plaintext by computing $m = D_k(c)$.

Some examples:

- WWII: Enigma Machine, Alan Turing,

  https://en.wikipedia.org/wiki/Cryptanalysis_of_the_Enigma

- WWII: Lorenz Machine, Bill Tutte, Colossus

http://tinyurl.com/COBillTutte, http://billtuttememorial.org.uk/

2.1.1 The Simple Substitution Cipher

- M = all English msgs.
- C = all encrypted msgs.
- K = all permutations of the English alphabet.
- $E_k(m)$: Apply permutation k to m, one letter at a time.
- $D_k(c)$: Apply inverse permutations k^{-1} to c, one letter at a time.

Example:

$$k = \begin{array}{cccccccccccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p & q & r & s & t & u & v & w & x & y & z \\ D & N & X & E & S & K & O & J & T & A & F & P & Y & I & Q & U & B & R & Z & G & V & C & H & M & W & L \end{array}$$

Encryption: $m = the\ big\ dog$, $c = E_k(the\ big\ dog) = GJS\ NTO\ EQO$.

Decryption: $c = GJS\ NTO\ EQO$, $m = E_k^{-1}(GJS\ NTO\ EQO) = the\ big\ dog$.

2.1.2 Security of SKES

Is the simple substitution cipher a secure SKES? but wait, what does it mean for a SKES to be secure? We need a **security definition**.

1. What are the computational powers of the adversary?
2. How does the adversary interact with the two communicating parties?
3. What is the adversary's goal?

Security model: Defines the computational abilities of the adversary, and how she interacts with the communicating parties.

Basic assumption: The adversary knows everything about the SKES, except the particular key k chosen by Alice and Bob. (Avoid security by obscurity!!) Security should only depend on the secrecy of the secret keying material, not on the secrecy of the algorithms that describe the cryptographic scheme.

Let's now consider the *computational power* of the adversary.

- **Information-theoretic security**: Eve has infinite computational resources.
- **Complexity-theoretic security**: Eve is a 'polynomial-time Turing machine'.
- **Computational security**: Eve has 36,768 Intel E5-2683 V4 cores running at 2.1 GHz at her disposal. See: Graham in the basement of MC

  We say: Eve is "*computationally bounded*".

In this course, we will be concerned with computational security.

Next consider how the adversary can *interact* with Alice and Bob.

- Passive attacks:
  - **Ciphertext-only attack**: The adversary knows some ciphertext (that was generated by Alice or Bob).
  - **Known-plaintext attack**: The adversary also knows some plaintext and the corresponding

ciphertext. (stronger because more info than before)[1]

- Active attacks: (even stronger)
  - **Chosen-plaintext attack**: The adversary can also *choose* some plaintext and obtains the corresponding ciphertext. This is at least as strong as known-plaintext attack because the attacker gets to choose the plain text for which it is given the corresponding ciphertext.
- Other attacks (not considered in this course):
  - Clandestine attacks: bribery, blackmail, etc.
  - Side-channel attacks: monitor the encryption and decryption equipment (timing attacks, power analysis attacks, electromagnetic-radiation analysis, etc.)

Finally, we will consider the *adversary's goal* (from strongest to weakest).

1. Recover the secret key.
2. Systematically recover plaintext from ciphertext (without necessarily learning the secret key).
3. Learn some partial information about the plaintext from the ciphertext (other than its length).

If the adversary can achieve 1 or 2, the SKES is said to be **totally insecure** (or **totally broken**).

If the adversary cannot learn any partial information about the plaintext from the ciphertext (except possibly its length), the SKES is said to be **semantically secure**. So the ciphertext hides all the meaning about the corresponding plaintext.

security of SKES

A symmetric-key encryption scheme is said to be **secure** if it is semantically secure against chosen-plaintext attack by a computationally bounded adversary.

Note that the definitions captures the computational abilities of the adversary, namely some concrete upper bound on its computational resources. It captures how the adversary interacts with Alice and Bob, namely by mounting a chosen-plaintext attack. And finally, it captures the goal of the adversary, namely breaking semantic security. Note also that in the definition the attacker's capabilities are *as strong as possible*, namely a chosen-plaintext attack and its goal is *as weak as possible* namely, breaking semantic security

From the definition, we can also deduce what it means to break an encryption scheme.

1. The adversary is given a challenge ciphertext c (generated by Alice or Bob using their secret key k).
2. During its computation, the adversary can select plaintexts and obtains (from Alice or Bob) the corresponding ciphertexts.
3. After a feasible amount of computation, the adversary obtains some information about the plaintext m corresponding to the challenge ciphertext c (other than the length of m).

If the attacker can win this game specified in these three steps, then we'll say that the encryption scheme is in secure.

[1] In practice, Eve might get such plaintext because in many applications data is first formatted by prepending to it some header information whose contents might be predictable, for example email headers.

Desirable Properties of a SKES

1. Efficient algorithms should be known for computing E_k and D_k (i.e., for encryption and decryption).
2. The secret key should be small (but large enough to render exhaustive key search infeasible).
3. The scheme should be secure.
4. The scheme should be secure even against the designer of the system.

The last point is to ensure that the designer hasn't somehow inserted a backdoor into the encryption scheme whereby the adversary can learn a secret keying material by looking at ciphertext, while this method would be hard to find by other people. This fourth property can sometimes be very difficult to obtain in practice.

Now we can see whether the simple substitution cipher is secure: *Totally insecure against a chosen-plaintext attack*. This is because the adversary can simply give the plaintext message that's comprised of the English alphabet from a to z to Alice for encryption. Alice returns to the adversary the ciphertext, which it turns out is the secret key itself. And so by simply giving Alice a short plaintext and getting back the corresponding ciphertext the adversary has learnt Alice's secret key.

What about security under a ciphertext-only attack? Is exhaustive key search possible?

In this attack, we are given sufficient amounts of ciphertext c, decrypt c using each possible key until c decrypts to a plaintext message which "makes sense" (normal English). In principle, 30 characters of ciphertext are sufficient on average to yield a unique plaintext that is a sensible English message. In practice, a few hundred characters are needed.

If we tried exhaustive search, the number of keys to try is $26! \approx 2^{88}$, which is very large in the following sense: If the adversary uses 10^6 computers, each capable of trying 10^9 keys per second, then exhaustive key search takes about 10^4 years. So, exhaustive key search is infeasible.

In this course, a cryptographic task that requires

- 2^{40} operations is considered very easy.
- 2^{56} operations is considered easy.
- 2^{64} operations is considered feasible.
- 2^{80} operations is considered barely feasible.
- 2^{128} operations is considered infeasible.

The Bitcoin network is presently performing hash operations at the rate of $2^{66.4}$ per second (or $2^{91.3}$ per year).

The Landauer limit from thermodynamics suggests that exhaustively trying 2^{128} symmetric keys would require $\gg$ 3000 gigawatts of power for one year (which is $\gg$ 100% of the world's energy production). http://en.wikipedia.org/wiki/Brute-force_attack

security level of a cryptographic scheme

A cryptographic scheme is said to have a **security level** of ℓ bits if the fasted known attack on the scheme takes approximately 2^ℓ operations.

As of the year 2021, a security level of 128 bits is desirable in practice.

Let's return to our question of whether the simple substitution cipher is secure against a ciphertext-only attack. In fact, simple frequency analysis of ciphertext letters can be used to recover the key. One would simply find the most frequently occurring ciphertext letter; this would most likely correspond to the English letter e which is the most frequently occurring letter in the English alphabet. Hence, the simple substitution cipher is *totally insecure* even against a ciphertext-only attack.

2.1.3 Polyalphabetic Ciphers

Basic idea: Use several permutations, so a plaintext letter is encrypted to one of several possible ciphertext letters.

Example: Vigenère cipher

Secret key is an English word having no repeated letters. E.g., $k = CRYPTO$.

Example of encryption:

	m	=	t	h	i	s	i	s	a	m	e	s	s	a	g	e
+	k	=	C	R	Y	P	T	O	C	R	Y	P	T	O	C	R
	c	=	V	Y	G	H	B	G	C	D	C	H	L	O	I	V

Here, $A = 0, B = 1, \ldots, Z = 25$; addition of letters is mod 26.

Decryption is subtraction modulo 26: $m = c - k$.

Frequency distribution of ciphertext letters is flatter (than for a simple substitution cipher).

The Vigenère cipher is totally insecure against a chosen-plaintext attack. Not unexpectedly, the Vigenère cipher is also totally insecure against a ciphertext-only attack.

2.2 The One-Time Pad

The one-time pad is a symmetric-key encryption scheme invented by Vernam in 1917 for the telegraph system. The key is a random[2] string of letters. Example of encryption:

	m	=	t	h	i	s	i	s	a	m	e	s	s	a	g	e
+	k	=	Z	F	K	W	O	G	P	S	M	F	J	D	L	G
	c	=	S	M	S	P	W	Y	P	F	Q	X	C	D	R	K

Note that the key is as long as the plaintext. One major disadvantage of the one-time pad is that the secret key is of the same length as the plaintext. This is especially inconvenient if Alice and Bob are exchanging long messages over a period of time, and so it might be tempting in practice to reuse the secret key.

The key should *not* be re-used: If $c_1 = m_1 + k$ and $c_2 = m_2 + k$, then $c_1 - c_2 = m_1 - m_2$. Thus $c_1 - c_2$ depends only on the plaintext (and not on the key) and hence can leak information about the plaintext. In particular, if m_1 is known, then m_2 can be easily computed.

In the one-time pad, we took the plaintext letters to be the english alphabet. From now on in the course, unless otherwise stated, messages and keys will be assumed to be binary/*bit strings*. This is a reasonable convention since computers process binary data.

[2] independently and uniformly at random

$\oplus$

$\oplus$ is bitwise exclusive-or (XOR) i.e., bitwise addition modulo 2. For example:

$$1011001010 \oplus 1001001001 = 0010000011.$$

Note that $x \oplus x = 0$ and $x \oplus y = y \oplus x$. Hence if $x = y \oplus z$, then $x \oplus y = z$.

So, for the one-time pad:

Encryption: $c = m \oplus k$.

Decryption: $m = c \oplus k$.

https://cryptosmith.com/2008/05/31/stream-reuse/ is an example which reuses the same key...

Security of One-Time Pad

Perfect secrecy: The one-time pad is semantically secure against ciphertext-only attack by an adversary with infinite computational resources. This can be proven formally using concepts from information theory [Shannon 1949].

The bad news with perfect secrecy is that Shannon (1949) proved that if plaintexts are m-bit strings, then any symmetric-key encryption scheme with perfect secrecy must have $|K| \geq 2^m$. So, perfect secrecy (and the one-time pad) is fairly useless in practice. However, all is not lost. One can use one-time pads to inspire the construction of so-called *stream ciphers*.

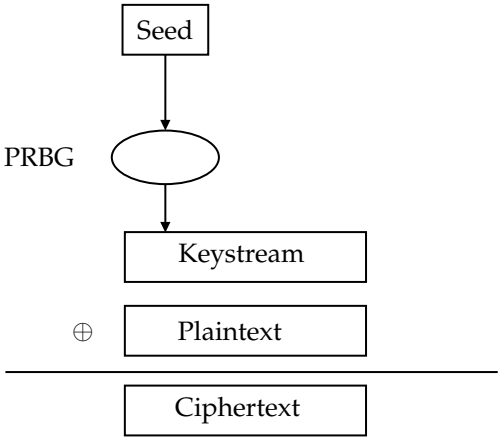
2.3 Stream Ciphers

Basic idea: Instead of using a random key in the one-time pad, use a "pseudorandom" key.

A **pseudorandom bit generator** (PRBG) is a deterministic algorithm that takes as input a (relatively small random) seed, and outputs a longer "pseudorandom" sequence called the keystream.

Using a PRBG for encryption (a stream cipher):

The seed is the secret key shared by Alice and Bob.



The XOR portion of the stream cipher emulates a one-time pad. Note though that we no longer have perfect secrecy because the keystream is no longer purely random but pseudo-random. Thus security depends on the quality of the PRBG.

There are two *security requirements* for the pseudo-random bit generator.

First, The keystream should be "indistinguishable" from a random sequence (the **indistinguishability requirement**).

If an adversary knows a portion c_1 of ciphertext and the corresponding plaintext m_1, then she can easily find the corresponding portion $k_1 = c_1 \oplus m_1$ of the keystream. Thus, given portions of the keystream, it should be infeasible to learn any information about the rest of the keystream (the **unpredictability requirement**).

Aside: Don't use UNIX random number generators (`rand` and `srand`) for cryptography!

- $X_0 = \text{seed}$, $X_{i+1} = aX_i + b \mod n$, $i \geq 0$.
- a, b and n are fixed and public constants.

And so the UNIX routines `rand` and `srand` do not meet the unpredictability requirement.

One difficulty with using stream ciphers in practice is that we still have the requirement as we did with the one-time pad that keystream should never be reused.

Then we will see two examples of stream ciphers: Rc4 and ChaCha20.

2.4 The RC4 Stream Cipher

It was designed by Ron Rivest in 1987. He is the "R" in RSA. Until recently, was widely used in commercial products including Adobe Acrobat, Windows password encryption, Oracle secure SQL, TLS, etc.

- *Pros*: Extremely simple; extremely fast; variable key length. No catastrophic weakness has been found since 1987.
- *Cons*: Design criteria are proprietary; not much public scrutiny until the year 2001.

In the past 10 years, many weaknesses have been found in RC4 and so its use has rapidly declined in practice.

RC4 has two components: (i) a *key scheduling algorithm*, and (ii) a *keystream generator*.

In the following, $K[i], \overline{K}[i]$ and $S[i]$ are 8-bit integers (bytes).

Algorithm 1: RC4 Key Scheduling Algorithm

Input: Secret key $K[0], K[1], \ldots, K[d-1]$. (Keylength is $8d$ bits.)
Output: 256-long array: $S[0], S[1], \ldots, S[255]$.

for $i = 0, \ldots, 255$ **do**
    $S[i] \leftarrow i$
    $\overline{K}[i] \leftarrow K[i \mod d]$
$j \leftarrow 0$
for $i = 0, \ldots, 255$ **do**
    $j \leftarrow (\overline{K}[i] + S[i] + j) \mod 256$
    Swap$(S[i], S[j])$

The idea of key scheduling is to begin with the identity permutation capital S of the integers 0 through 255, and then apply 256 random swaps to this permutation with the hope that the resulting permutation is a random looking permutation. S is a "random-looking" permutation of $\{0, 1, 2, \ldots, 255\}$ that is generated from the secret key.

In keystream generation, the array S from key scheduling is used to generate keystream of the desired

length. Alice would generate keystream of a certain length, add it to plaintext to get ciphertext. And Bob would generate the same keystream and add it to ciphertext to get plaintext.

Algorithm 2: RC4 Keystream Generator

Input: 256-long byte array: $S[0], S[1], \ldots, S[255]$ produced by the RC4 Key Scheduling Algorithm
Output: Keystream.

```
1 i ← 0; j ← 0
2 while keystream bytes are required do
3     i ← (i + 1) mod 256
4     j ← (S[i] + j) mod 256
5     Swap(S[i], S[j])
6     t ← (S[i] + S[j]) mod 256
7     Output(S[t])
```

ENCRYPTION: The keystream bytes are stored with the plaintext bytes to produce ciphertext bytes.

2.4.1 Case Study: Wired Equivalent Privacy

Wireless Security

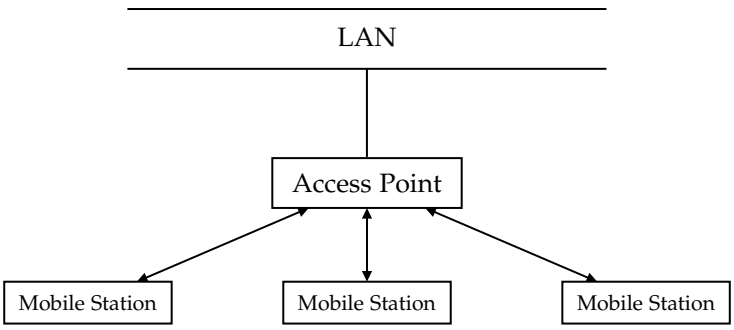
Wireless networks have become prevalent. Popular standards for wireless networks: IEEE 802.11 (longer range, higher speeds, commonly used for wireless LANs) and Bluetooth (short range, low speed).

New security concerns:

- More attack opportunities (no need for physical access).
- Attack from a distance ($>$ 1 km with good antennae).
- No physical evidence of attack.

IEEE 802.11 Security

IEEE 802.11 standard for wireless LAN communications includes a protocol called **Wired Equivalent Privacy** (WEP). This standard was ratified in September 1999. This was a very exciting time because this was the first time that wi-fi was available to the consumer market. Multiple amendments: 802.11a (1999), 802.11b (1999), 802.11g (2003), 802.11i (2004), 802.11n (2009)... WEP's goal is (only) to protect link-level data during wireless transmission between mobile stations and access points. A mobile station might be your cell phone or your laptop, and an access point is a wi-fi router somewhere in the building.



WEP had three main *security goals*.

1. *Confidentiality*: Prevent casual eavesdropping. To achieve this, RC4 is used for encryption.
2. *Data Integrity*: Prevent tampering with transmitted messages. To achieve this, an 'integrity checksum' is used for WEP.
3. *Access Control*: Protect access to a wireless network infrastructure. This was achieved by discarding all packets that are not properly encrypted using WEP.

Description of WEP Protocol

Mobile stations share a secret key k with access point. Here k is either 40 bits or 104 bits in length. The IEEE standard does not specify how the key is to be distributed. But in practice, one shared key per LAN is common; this key is manually injected into each access point and mobile station; the key is not changed very frequently.

Plaintext messages are divided into packets of some fixed length (e.g., 1500 bytes). These packets were encrypted using RC4. Since a mobile station or access point might encrypt many packets in a given period of time, WEP had to be careful not to reuse RC4 keystreams.

Thus WEP uses a per-packet 24-bit *initialization vector* (IV) v to process each packet. WEP does not specify how the IVs are managed. In practice: either a random IV is generated for each packet; or the IV is set to 0 and incremented by 1 for each use.

To send a packet m, an entity (e.g., Alice) does the following:

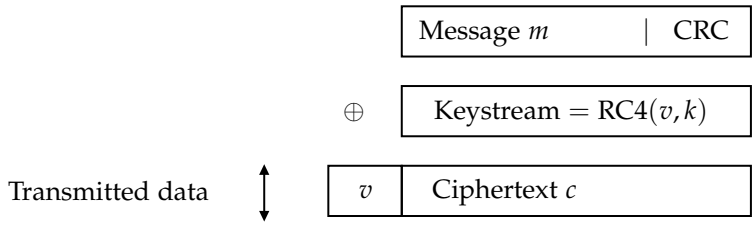
1. Select a 24-bit IV v.
2. Compute a 32-bit checksum: $S = \text{CRC}(m)$.

   802.11 specifies that a CRC-32 checksum be used. CRC-32 is linear. That is, for any two messages m_1 and m_2 of the same bitlength,

   $$\text{CRC}(m_1 \oplus m_2) = \text{CRC}(m_1) + \text{CRC}(m_2).$$

   (The details of CRC-32 are not important to us)
3. Compute $c = (m \parallel S) \oplus \text{RC4}(v \parallel k)$.
   - $\parallel$ (and also a comma) denotes concatenation
   - $(v \parallel k)$ is the key used the RC4 stream cipher.
4. Send (v, c) over the wireless channel.

Here's a diagram of the encryption process.



The receiver of (v, c) does the following:

1. Compute $(m \parallel S) = c \oplus \text{RC4}(v \parallel k)$.
2. Compute $S' = \text{CRC}(m)$; reject the packet if $S' \neq S$.

Are confidentiality, data integrity, and access control achieved? Intuitively the answer is yes. No (Borisov, Goldberg & Wagner; 2001),

Problem 1: IV Collision

Suppose that two packets (v, c) and (v, c') use the same IV v. Let m, m' be the corresponding plaintexts. Then $c \oplus c' = (m \parallel S) \oplus (m' \parallel S')$. Thus, the eavesdropper can compute $m \oplus m'$. If m is known, the, m' is immediately available. If m is not known, then one may be able to use the expected distribution of m and m' to discover information about them. (Some contents of network traffic is predictable.)

Since there are only 2^{24} choices for the IV, collisions are guaranteed after enough time - a few days on a busy network (5 Mbps). If IVs are randomly selected, then one can expect a collision after about 2^{12} ($= \sqrt{2^{24}}$) packets. This is due to the birthday paradox.

Birthday paradox

Suppose that an urn contains n numbered balls. Suppose that balls are drawn from the urn, one at a time, with replacement. The expected number of draws before a ball is selected for a second time (called a *collision*) is approximately $\sqrt{\pi n/2} \approx \sqrt{n}$.

Collisions are more likely if keys k are long-lived and the same key is used for multiple mobile stations in a network.

Conclusion: WEP does not provide a high degree of confidentiality.

Problem 2: Checksum is Linear

CRC-32 is used to check integrity. This is fine for random errors, but not for deliberate ones.

It is easy to make controlled changes to (encrypted) packets: Suppose (v, c) is an encrypted packet. Let $c = \text{RC4}(c \parallel k) \oplus (m \parallel S)$, where k, m, S are unknown to the adversary.

The adversary chooses a bit string Δ of the same length as m, where the ones in Δ correspond to the bits of m that she wishes to change. Let $m' = m \oplus \Delta$, where Δ is a bit string. And so in this way the attacker can make controlled changes to the unknown plaintext m.

Let $c' = c \oplus (\Delta \parallel \text{CRC}(\Delta))$. Then (v, c') is a valid encrypted packet for m'.

$$\begin{aligned} c' &= c \oplus (\Delta \parallel \text{CRC}(\Delta)) \\ &= \text{RC4}(v \parallel k) \oplus (m \parallel S) \oplus (\Delta \parallel \text{CRC}(\Delta)) \\ &= \text{RC4}(v \parallel k) \oplus (m \oplus \Delta \parallel \text{CRC}(m) \oplus \text{CRC}(\Delta)) \\ &= \text{RC4}(c \parallel k) \oplus (m' \parallel \text{CRC}(m \oplus \Delta)) \\ &= \text{RC4}(v \parallel k) \oplus (m' \parallel \text{CRC}(m')) \end{aligned}$$

Hence the receiver will accept c' as a validly encrypted WEP packet, and will accept m' as the plaintext. In this way, the adversary has successfully made a controlled change to the plaintext m without actually knowing the plaintext.

Conclusion: WEP does not provide data integrity.

Problem 3: Integrity Function is Unkeyed

Suppose that an attacker learns the plaintext m corresponding to a single encrypted packet (v, c). Then, the attacker can compute the RC4 keystream $\text{RC4}(v \parallel k) = c \oplus (m \parallel \text{CRC}(m))$. Henceforth, the attacker can compute a valid encrypted packet for *any* plaintext m' of her choice: (v, c'), where $c' = \text{RC4}(v \parallel k) = c \oplus (m' \parallel \text{CRC}(m'))$. Note that this is a validly encrypted ciphertext for the plaintext m'.

Conclusion: WEP does not provide access control.

Optional Reading: Sections 1, 2, 3, 4.1, 4.2, 6 of "Intercepting mobile communications: The insecurity of 802.11", by N. Borisov, I. Goldberg and D. Wagner.

2.4.2 Fluhrer-Mantin-Shamir Attack

Shortly after these attacks on WEP were disclosed, a *more devastating attack* was discovered. This was due to Fluhrer, Mantin & Shamir, 2001. Shamir is the "S" in RSA.

The attack makes the following three assumptions:

1. The same 104-bit key k is used for a long period of time. [Most products do this.]
2. The IV is incremented for each packet, or a random IV is selected for each packet. [Most products do this.]
3. The first plaintext byte of each packet (i.e. the first byte of each m) is known to the attacker. [Most wireless protocols prepend the plaintext with some header bytes which are non-secret.]

In the attack, a passive adversary who can collect about 5,000,000 encrypted packets can very easily recover k (and thus totally break the system). [Details not covered in this course.]

The attack can be easily mounted in practice: Can buy a $100 wireless card and hack drivers to capture (encrypted) packets. On a busy wireless network (5Mbps), 5 million packets can be captured in a few hours, and then k can be immediately computed.

Implementation details: A. Stubbefield, J. Ionnidis, A. Rubin, "Using the Fluhrer, Mantin and Shamir attack to break WEP", AT&T Technical Report, August 2001.

If you want to mount the WEP attack today, all you have to do is find a wireless network that still uses WEP and then go to one of these two websites: Aircrack-ng, WEPCrack. The latest enhancement of the Fleur-Mantin-Shamir attack is aircrack ptw. This can break WEP in under 60 seconds (only about 40,000 packets are needed).

WEP was blamed for the 2007 theft of 45 million credit-card numbers from T.J. Maxx. (T.J. Maxx is an American department store chain). A subsequent class action lawsuit settled for $40,900,000.

See: `http://tinyurl.com/WEP-TJMaxx`

IEEE 802.11 Update

`http://en.wikipedia.org/wiki/Wi-Fi_Protected_Access`

WPA2 (Wi-Fi Protected Access) Implements the new IEEE 802.11i standard (2004). Uses the AES block cipher instead of RC4. Deployed ubiquitously in Wi-fi networks. However, deployments of WEP are still out there, check `https://wigle.net/stats`. KRACK attack disclosed in October 2017.

WPA3 Adopted in January 2018. This contains further improvements to WPA2, including counter-measures to the CRACK attack. Dragonblood attack disclosed in April 2019.

This cycle of finding attacks and then fixing them with counter-measures continues today.

The Fluhrer-Mantin-Shamir attack exploits known biases in the first few bytes of the keystream. The attack can be defeated by discarding the first few bytes (e.g., 768 bytes) of the keystream. [Details omitted]

As of 2013, approximately 50% of all TLS traffic was secured using RC4. 2013-2021: Because of several

new weaknesses discovered, RC4 has been deprecated in applications such as TLS. (As of October 2019, $\approx$ 11.6% of websites have RC4 enabled, and only $\approx$ 1.1% actually use it.)

Conclusions: Don't use RC4. Instead use ChaCha20 or AES-CTR (more on these later).

> Lessons Learned
>
> 1. Details matter: RC4 was considered to be a secure stream cipher. However, it was improperly used in WEP and the result was a highly insecure communications protocol. Do not assume that "obvious" ways of using cryptographic functions are secure.
> 2. Attacks only get better; they never get worse.
>   (a) Moore's law (1965): computers get twice as fast every two years.
>   (b) Known attacks are constantly being tweaked and improved.
>   (c) New attacks are constantly being invented.
> 3. Designing security is hard:
>   - Designing cryptographic protocols is complicated and difficult.
>   - Clearly state your security objectives.
>   - Hire cryptography experts to design your security. Programming or engineering experts are not good enough.
>   - Make your protocols available for public scrutiny.
> 4. There is a big demand in industry for "security engineers": People who have a deep understanding of applied cryptography and have excellent programming skills.

2.5 ChaCha20 Stream Cipher

It was designed by Dan Bernstein in 2008.

The ChaCha20 stream cipher is conceptually simple, word-oriented, and uses only simple arithmetic operations (integer addition modulo 2^{32}, xor, and left rotations[3]). As a result, it is extremely fast in software, and does not require any special hardware. To date, no security weaknesses have been found. ChaCha20 is widely deployed in practice, including in TLS.

Let's first describe ChaCha20's *initial state*. We will use the following notation:

- 256-bit key: $k = (k_1, k_2, \ldots, k_8)$
- 96-bit nonce: $n = (n_1, n_2, n_3)$
- 128-bit constant: $f = (f_1, f_2, f_3, f_4)$
- 32-bit counter: c

A hexadecimal digit is a 4-bit number.

The words are f_1, f_2, f_3, f_4, written as hexadecimal strings of length 8. For example, $f_1 =$ `0x61707865`. Each of the eight hexadecimal digits is between `0` and `9` or `a` to `f`. These represent the decimal numbers between 0 and 15, or a binary string of length 4. And so this hexadecimal string of length 8 can be viewed as a binary string of length 32.

A nonce (or IV) is a non-repeating quantity (either a counter, or a randomly-generated string).

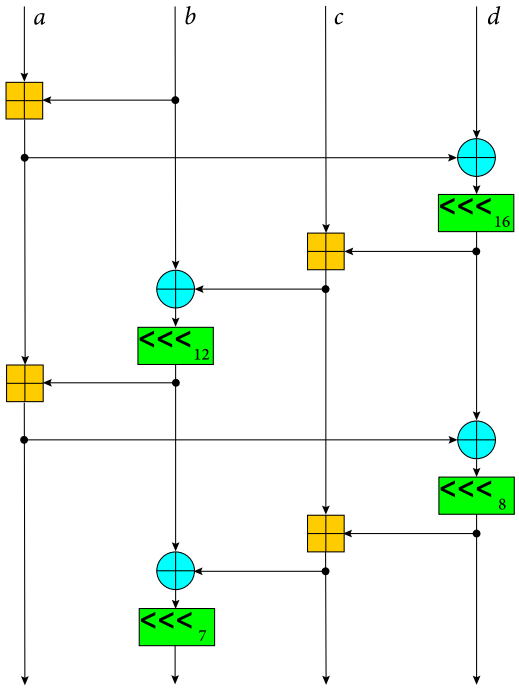
[3] These three arithmetic operations are included in the instruction sets of most modern processors.

The initial state of ChaCha20 can be represented by this 4×4 array of 32-bit words:

$$\begin{array}{|c|c|c|c|} \hline f_1 & f_2 & f_3 & f_4 \\ \hline k_1 & k_2 & k_3 & k_4 \\ \hline k_5 & k_6 & k_7 & k_8 \\ \hline c & n_1 & n_2 & n_3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline S_1 & S_2 & S_3 & S_4 \\ \hline S_5 & S_6 & S_7 & S_8 \\ \hline S_9 & S_{10} & S_{11} & S_{12} \\ \hline S_{13} & S_{14} & S_{15} & S_{16} \\ \hline \end{array}$$

2.5.1 ChaCha20 Quarter Round Function

It's called a quarter round function since it operates on a quarter of the state, which is a binary string of length 128 bits.



picture from wiki: https://commons.wikimedia.org/wiki/File:ChaCha_Cipher_Quarter_Round_Function.svg.

The purpose of the quarter round function is to mix up the bits of a, b, c, and d in a complicated nonlinear fashion. This is done with three arithmetic operations defined on 32-bit words, $\oplus$: XOR; and $\boxplus$: integer addition modulo 2^{32}; and $\lll t$: left-rotate by t bit positions.

QR: Quarter Round Function

Input Four 32-bit words a, b, c, d

$$\begin{array}{lll} a \leftarrow a \boxplus b, & d \leftarrow d \oplus a, & d \leftarrow d \lll 16 \\ c \leftarrow c \boxplus d, & b \leftarrow b \oplus c, & b \leftarrow b \lll 12 \\ a \leftarrow a \boxplus b, & d \leftarrow d \oplus a, & d \leftarrow d \lll 8 \\ c \leftarrow c \boxplus d, & b \leftarrow b \oplus c, & b \leftarrow b \lll 7 \end{array}$$

Output a, b, c, d

Finally, here is a description of ChaCha20's *keystream generator*.

Algorithm 3: ChaCha20 Keystream Generator

1 Select a nonce n and initialize the counter c.
2 **while** *keystream bytes are required* **do**
3 Create the initial state S.
4 Make a copy S' of S.
5 Update S by repeating the following 10 times:

$$\begin{array}{ll} QR(S_1, S_5, S_9, S_{13}) & QR(S_2, S_6, S_{10}, S_{14}) \\ QR(S_3, S_7, S_{11}, S_{15}) & QR(S_4, S_8, S_{12}, S_{16}) \\ QR(S_1, S_6, S_{11}, S_{16}) & QR(S_2, S_7, S_{12}, S_{13}) \\ QR(S_3, S_8, S_9, S_{14}) & QR(S_4, S_5, S_{10}, S_{15}) \end{array}$$

6 Output $S \oplus S'$ (64 keystream bytes).
7 Increment the counter.

ENCRYPTION: The keystream bytes are xored with the plaintext bytes to produce ciphertext bytes. The nonce is included in the ciphertext.

When Bob receives the ciphertext and the nonce, he regenerates a keystream using the keystream generation algorithm just as Alice did, and then he xors the keystream bytes with the ciphtext bytes to produce the plaintext bytes.

We can now see why ChaCha20 is very efficient. 64 key stream bytes can be generated using only 960 computer instructions.

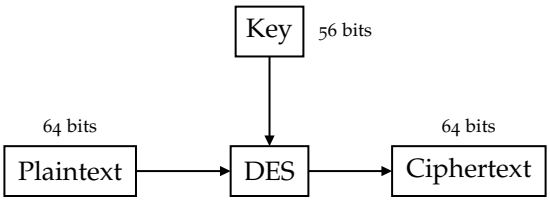
Unlike RC4, ChaCha20 incorporates a nonce and a counter into the encryption algorithm and thus the problems we saw with using RC4 in WEP are automatically avoided, even though Alice and Bob might use the same secret key for a long period of time. Since they use a 96-bit nonce for each plaintext that they encrypt and they use a counter for each 64 byte portion of the keystream for encrypting a particular plaintext, with very high probability ChaCha20 keystream is never reused by Alice and Bob.

2.6 Block Ciphers

A *block cipher* is a SKES that breaks up the plaintext into blocks of a fixed length (e.g. 128 bits), and encrypts the blocks one at a time.

In contrast, a stream cipher encrypts the plaintext one character (usually a bit) at a time.

Historically important example of a block cipher: The Data Encryption Standard (DES)



Key length: 56 bits; Size of key space 2^{56}; Block length: 64 bits.

2.6.1 Brief History of Block Ciphers

Late 1960's: Feistel network and LUCIFER designed at IBM.

1972: NBS (now NIST: National Institute of Standards and Technology) solicits proposals for encryption algorithms for the protection of computer data.

1973-1974: IBM develops DES.

1975: NSA (National Security Agency) "fixes" DES. Reduces the key length from 64 bits to 56 bits. (NSA tried for 48 bits; compromised at 56 bits.)

To understand NSA's motivation for doing this, let's take a quick look at the NSA organization and its mission.

NSA

The National Security Agency: Founded in 1952. Budget is Classified (estimated: $10.8 billion/year), as well as its number of employees (30,000–40,000).

NSA has two main divisions: *Signals Intelligence* (SIGINT) which produces foreign intelligence information and *Information Assurance* (IA) which protects all classified and sensitive information that is stored or sent through US government equipment.

The NSA is very influential in setting US government export policy for cryptographic products (especially encryption). Canadian counterpart: Communications Security Establishment (CSE).

Most countries around the world have organizations analogous to the NSA. Here is the logo of Ministry of State Security (China):

Picture from wiki: https://commons.wikimedia.org/wiki/File:Ministry_of_State_Security_of_the_People%27s_Republic_of_China.svg

> NSA Shenanigans
>
> Edward Snowden. CIA/NSA contractor who disclosed in 2013 up to 200,000 NSA classified documents to the press. Currently in Russia under temporary asylum. Documents revealed the enormous capabilities of NSA and its partners (including CSE and GCHQ):
>
> - "Groundbreaking cryptanalytic capabilities"
> - Collects vast amounts of internet communications and automatically analyzes the data
> - Directly attacks routers, switches, firewalls, etc.

> - Installs malware on individual computers
> - Breaks into individual computers
>
> Bruce Schneier, a well-known security expert, "The NSA is subverting the Internet and turning it into a massive surveillance tool."

Block Cipher history continued...

1977: DES adopted as US Federal Information Processing Standard (FIPS 46).

1981: DES adopted as a US banking standard (ANSI X3.92).

1997: NIST begins the AES (Advanced Encryption Standard) competition.

1999: 5 finalists for AES announced.

2001: Rijndael adopted for AES (FIPS 197). AES has three key lengths: 128, 192 and 256 bits.

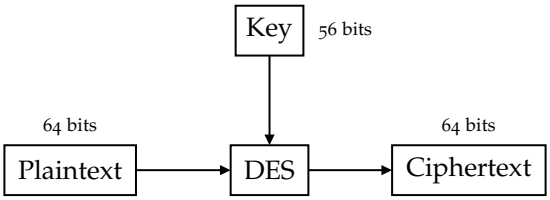
2021: No significant weaknesses found with AES (as yet). AES uptake is rapidly increasing. However, (Triple-)DES is still deployed.

2.6.2 Some Desirable Properties of Block Ciphers

Design principles described by Claude Shannon in 1949:

- Security:
  - *Diffusion*: each ciphertext bit should depend on all plaintext bits.
  - *Confusion*: the relationship between key and ciphertext bits should be complicated.
  - *Key length*: should be small, but large enough to preclude exhaustive key search.
- Efficiency:
  - Simplicity (easier to implement and analyze).
  - High encryption and decryption rate.
  - Suitability for hardware or software.

2.6.3 The Data Encryption Standard (DES)



Key length: 56 bits; Size of key space 2^{56}; Block length: 64 bits.

The design principles of DES were classified by the NSA in the mid-1970s and they remain classified today, making analysis difficult. Nonetheless, hundreds of research papers have been written on the security of DES since the mid-1970s. (Triple-)DES is still deployed is practice, although its use is rapidly decreasing.

DES Problem 1: Small Key Size

Exhaustive search on key space takes 256 steps and can be easily parallelized.

In the mid-1990s, the RSA security company set some DES challenges. In the challenge, you were given three known plaintext-ciphertext pairs generated with a secret key. The plaintext were the ASCII messages "the unknown message is", each block being eight ASCII characters or 64 bits. You were also given a fourth ciphertext and finding the corresponding plaintext was the challenge.

The unkn	own mess	age is:	????????

The DES challenge was first solved in June of 1997. This was done in three months by Internet search. In 1999, it was broken in 56 hours by DeepCrack machine (1800 chips; $250,000). In 2006, broken in 153 hours by COPACOBANA machine ($10,000). In 2012, broken in 11.5 hours by crack.sh ($200). These experiments show us that the cost and time to perform exhaustive key search decrease over time as computers get faster and cheaper.

The RSA security company also set some 64-bit challenges using the RC5 block cipher which has a key length of 64 bits. In July 2002, broken in 1757 days. Participation by 331,252 individuals.

DES Problem 2: Small Block Size

If plaintext blocks are distributed "uniformly at random", then the expected number of ciphertext blocks observed before a collision occurs is $\approx 2^{32}$ (by the birthday paradox). Hence the ciphertext reveals some information about the plaintext.

Small block length is also damaging to some authentication applications (more on this later).

To summarize, the only (substantial) weaknesses known in DES are the obvious ones: small key length and small block length.

Question How can one construct a more secure block cipher from DES? (i.e. without changing the internals of DES.)

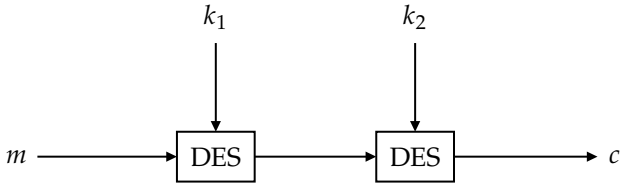
One approach is to use **multiple encryption**: Re-encrypt the ciphertext one or more times using independent keys. Hope that this increases the effective key length.

Note that multiple encryption does not always result in increased security. For example, if E_π denotes the encryption function for the simple substitution cipher with key π, then is $E_{\pi_2} \circ E_{\pi_1}$ any more secure than E_π? No, since $E_{\pi_2} \circ E_{\pi_1} = E_{\pi_2 \circ \pi_1}$.

2.6.4 Double-DES

Key is $k = (k_1, k_2)$, $k_1, k_2 \in_R \{0,1\}^{56}$. Here $k \in_R K$ means that k is chosen uniformly and independently at random from K.

Encryption: $c = E_{k_2}(E_{k_1}(m))$. (E = DES encryption, E^{-1} = DES decryption)



Decryption: $m = E_{k_1}^{-1}(E_{k_2}^{-1}(c))$.

The Double-DES key length is $\ell = 112$, so exhaustive key search takes 2^{112} steps (infeasible). Note also that the block length of Double DES is the same as that of DES, namely 64 bits.

Main idea $c = E_{k_2}(E_{k_1}(m))$ if and only if $E_{k_2}^{-1}(c) = E_{k_1}(m)$. Suppose now that (k_1, k_2) is Alice and Bob's secret key. Suppose also that we are given three plaintext-ciphertext pairs $(m_1, c_1), (m_2, c_2) and (m_3, c_3)$ that were generated by Alice or Bob using their secret key (k_1, k_2). In the meet-in-the-middle attack, we'll search for keys (h_1, h_2) which also encrypt $m_i \to c_i$ for $i = 1, 2, 3$. We do this by searching all possible candidate keys (h_1, h_2). If we have found such key (h_1, h_2), we'll conclude that (h_1, h_2) equals (k_1, k_2) with high probability.

Algorithm 4: Meet-In-The-Middle Attack on Double-DES

Input: 3 known PT/CT pairs $(m_1, c_1), (m_2, c_2), (m_3, c_3)$
Output: The secret key (k_1, k_2)

foreach $h_2 \in \{0,1\}^{56}$ **do**
  Compute $E_{h_2}^{-1}(c_1)$, and store $[E_{h_2}^{-1}(c_1), h_2]$ in a table sorted by first component.
foreach $h_1 \in \{0,1\}^{56}$ **do**
  Compute $E_{h_1}(m_1)$.
  Search for $E_{h_1}(m_1)$ in the table.
  // We say that $E_{h_1}(m_1)$ matches table entry $[E_{h_2}^{-1}(c_1), h_2]$ if $E_{h_2}^{-1}(c_1) = E_{h_1}(m_1)$.
  foreach *match* $[E_{h_2}^{-1}(c_1), h_2]$ *in the table* **do**
    if $E_{h_2}(E_{h_1}(m_2)) = c_2$ **then**
      if $E_{h_2}(E_{h_1}(m_3)) = c_3$ **then**
        Output (h_1, h_2) and STOP.

We need to justify the *correctness* and analyze the *running time* of the attack.

Correctness

We would like to find the number of known plaintext/ciphertext pairs needed for unique key determination.

Let E be a block cipher with key space $K = \{0,1\}^\ell$, and plaintext and ciphertext space $\{0,1\}^L$.

Let $k' \in K$ be the secret key chosen by Alice and Bob, and let $(m_i, c_i), 1 \le i \le t$, be known plaintext/-ciphertext pairs, where the plaintext m_i are distinct. (Note that $c_i = E_{k'}(m_i)$ for all $1 \le i \le t$)

Question Then how large should t be to ensure (with probability very close to 1) that there is only one key $k \in K$ such that $E_k(m_i) = c_i$ for all $1 \le i \le t$?

Answer Select t so that $FK \approx 0$ where FK is so-called "false keys".

For each $k \in K$, then encryption function $E_k : \{0,1\}^L \to \{0,1\}^L$ is a permutation. This is because the encryption function E_k is an invertible function.

We make the *heuristic assumption* that for each $k \in K, E_k$ is a random function (i.e., a randomly selected function). This assumption is certainly false since E_k is *not* random, and because a random function is almost certainly not a permutation. Nonetheless, it turns out that the assumption is good enough for our analysis.

Now, fix $k \in K, k \neq k'$. The probability that $E_k(m_i) = c_i$ for all $1 \le i \le t$ is

$$\underbrace{\frac{1}{2^L} \cdot \frac{1}{2^L} \cdots \frac{1}{2^L}}_{t} = \frac{1}{2^{Lt}}.$$

Here the first probability, $\frac{1}{2^L}$, is the probability that E_k encrypts m_1 to c_1. This is because there are 2^L possible ciphertext blocks. Similarly for $i = 2, \ldots, t$. Then we multiply these probabilities together.

Thus the expected number of false keys $k \in K$ (not including k') for which $E_k(m_i) = c_i$ for all $1 \leq i \leq t$ is

$$FK = \frac{2^\ell - 1}{2^{Lt}}.$$

This expected number is the probability that a single key is a false key times the number of possible false keys, which is the total number of keys $2^\ell - 1$ because we aren't considering k' as a false key. And so the expected number of false keys is given by this expression.

Now let's return to the correctness of the meet-in-the-middle attack. Let E be the DES encryption function, so Double-DES encryption is $c = E_{k_2}(E_{k_1}(m))$.

1. If $\ell = 112, L = 64, t = 3$, then $FK \approx 1/2^{80} \approx 0$. Thus if a Double-DES key (h_1, h_2) is found for which $E_{h_2}(E_{h_1}(m_i)) = c_i$ for $i = 1, 2, 3$, then with very high probability we have $(h_1, h_2) = (k_1, k_2)$.
2. If $\ell = 112, L = 64, t = 1$, then $FK \approx 2^{48}$. Thus the expected number of Double-DES keys (h_1, h_2) for which $E_{h_2}(E_{h_1}(m_1)) = c_1$ is $\approx 2^{48}$.
3. If $\ell = 112, L = 64, t = 2$, then $FK \approx 1/2^{16}$. Thus the expected number of Double-DES keys (h_1, h_2) for which $E_{h_2}(E_{h_1}(m_1)) = c_1$ and $E_{h_2}(E_{h_1}(m_2)) = c_2$ is $\approx 1/2^{16}$.

This justifies correctness of the meet-in-the-middle attack on Double DES.

Runtime Analysis

In the analysis, we'll take one operation to be either a single DES encryption operation E, or a single DES decryption operation E^{-1}.

Now let's take a closer look at the algorithm. The running time of line 2 is 2^{56} operations because we do one single decryption for each key h_2 in the single DES key space.[4] Similarly, the total running time of line 4 is 2^56 operations. Line 7 is executed every time there is a match in line 6.

The number of matches equals the number of keys h_1 for which this equation holds, in other words (h_1, h_2) encrypts m_1 to c_1. We need to count the number of keys (h_1, h_2) that encrypt m_1 to c_1. That is given by the number of double DES false keys when $t = 1$. By the formula for false keys, we see that this is roughly 2^{48}, so the attack will execute line 7 a total of 2^{48} times. Thus the cost of line 7 is 2×2^{48} as this line involves 2 DES operations.

Finally, the attack will execute line 8 for each false key (h_1, h_2) that encrypts $m_1 \to c_1, m_2 \to c_2$. By the formula for false keys, the expected number of double DES keys when $t = 2$ is roughly $1/2^{16}$, and so we expect that line 8 is executed $1/2^{16}$ times for false keys (h_1, h_2), and once for the actual secret key (k_1, k_2). Therefore, the expected running time of line 8 is $2 \times \left(1 + \frac{1}{2^{16}}\right)$.

To summarize, the number of DES operations is $\approx 2^{56} + 2^{56} + 2 \cdot 2^{48} \approx 2^{57}$. (We are not counting the time to do the sorting and searching)

Note that this is a lot less than the expected running time of exhaustive key search for double DES, which is roughly 2^{112}. Note also that this is a feasible amount of computation and so we might conclude that the meet-in-the-middle attack on Double DES demonstrates that Double DES offers no more security than single DES.

Space requirements

In line 2, the table entry has length $64 + 56$ bits, and we have 2^{56} table entries. Therefore, the space of the table in line 2 is $2^{56}(64 + 56)$ bits $\approx 1,080,863$ Tbytes.

[4] More precisely, the running time here means this particular line would do one operation 2^{56} times: this line has been executed 2^{56} times, and this line consists of 1 operation.

To put this in context, we consider storage units conversion:

- 10^3 bytes $= 1$ Kbyte (kilo) $\approx 2^{10}$ bytes
- 10^3 Kbytes $= 1$ Mbyte (mega) $\approx 2^{20}$ bytes
- 10^3 Mbytes $= 1$ Gbyte (giga) $\approx 2^{30}$ bytes
- 10^3 Gbytes $= 1$ Tbyte (tera) $\approx 2^{40}$ bytes
- 10^3 Tbytes $= 1$ Pbyte (peta) $\approx 2^{50}$ bytes
- 10^3 Pbytes $= 1$ Ebyte (exa) $\approx 2^{60}$ bytes

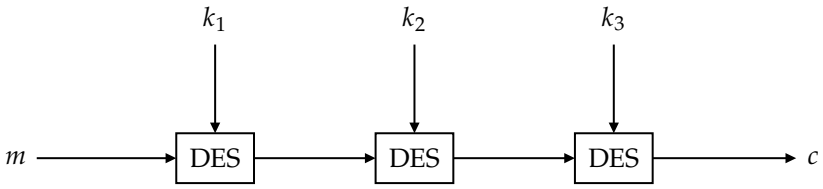
Accessing such a large storage would slow down the attack. Whether the meet-in-the-middle attack on Double DES is indeed cost effective, and in particular is much better than exhaustive key search on single DES? To address this concern, consider time-memory tradeoff. The attack can be modified to decrease the storage requirements at the expense of time: for $s \in [1, 55]$, we have time 2^{56+s} steps and memory 2^{56-s} units.

Conclusions: Double-DES has the same effective key length as DES. Double-DES is not much more secure than DES.

2.6.5 Triple-DES

Triple-DES. Key is $k = (k_1, k_2, k_3)$, $k_1, k_2, k_3 \in_R \{0,1\}^{56}$.

Encryption: $c = E_{k_3}(E_{k_2}(E_{k_1}(m)))$ where $E =$ DES encryption, $E^{-1} =$ DES decryption



Decryption: $m = E_{k_1}^{-1}(E_{k_2}^{-1}(E_{k_3}^{-1}(c)))$.

Key length of Triple-DES is $\ell = 168$, so exhaustive key search takes 2^{168} steps (infeasible)

One can verify meet-in-the-middle attack takes $\approx 2^{112}$ steps. So, the effective key length of Triple-DES against exhaustive key search is ≈ 112 bits, still infeasible. However, no proof that Triple-DES is more secure than DES.

Note that block length is unchanged.

Triple-DES is still deployed in practice, especially by some financial institutions, where its use can be expected to continue until 2030.

2.6.6 The Advanced Encryption Standard (AES)

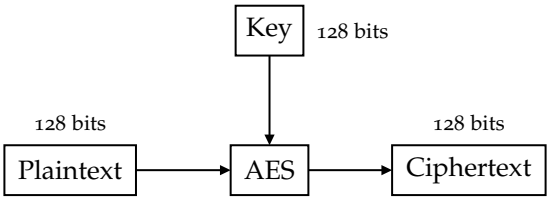
In this section, we will discuss AES, the block cipher that is the most widely deployed symmetric-key encryption scheme.

http://www.nist.gov/aes

In 1997, NIST began the competition for selecting the Advanced Encryption Standard or AES to replace the aging Data Encryption Standard.

Four main requirements:

- Key lengths: 128, 192 or 256 bits.
- Block length: 128 bits.
- Efficient on both hardware and software platforms.
- Availability on a worldwide, non-exclusive, royalty-free basis.



In this lecture, we'll describe AES encryption for the case where the secret key as length 128 bits.

The AES Process: 1998: 15 submissions in Round 1. 1999: 5 finalists selected by NIST: MARS, RC6, Rijndael, Serpent, Twofish. 1999: NSA performed a hardware efficiency comparison. **2000: Rijndael was selected.** 2001: The AES standard is officially adopted (FIPS 197). Rijndael is an example of a substitution-permutation network. 2021: No attacks have been found on AES that are (significantly) faster than exhaustive key search.

Substitution-Permutation Networks

A **substitution-permutation network** (SPN) is an iterated block cipher where a round consists of a substitution operation followed by a permutation operation.

The components of an SPN cipher are the following:

- n: the block length.
- ℓ: the key length.
- h: the number of rounds.
- A fixed invertible function $S : \{0,1\}^b \to \{0,1\}^b$, where b is a divisor of n.
- A fixed permutation P on $\{1,2,\ldots,n\}$.
- A key scheduling algorithm that determines subkeys $k_1, k_2, \ldots, k_h, k_{h+1}$ from a key k.

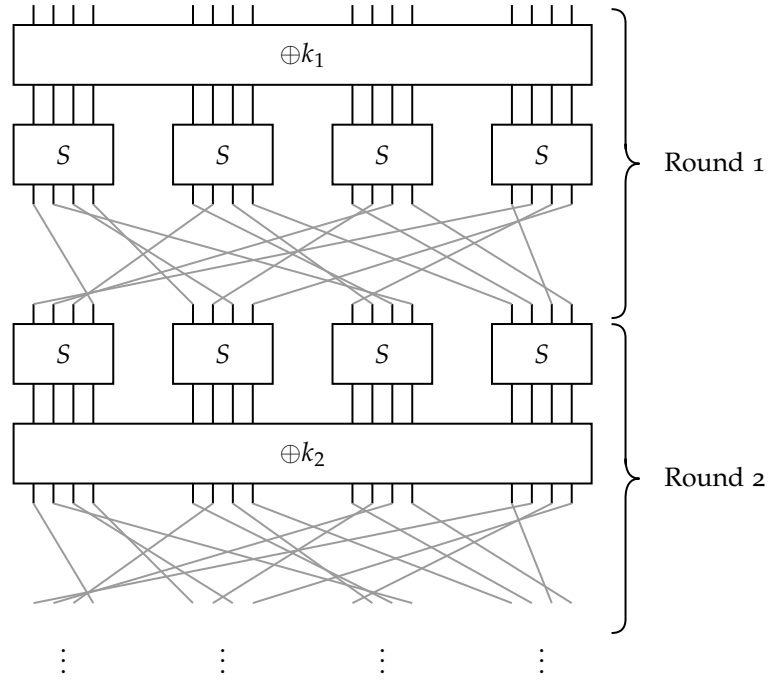
Note that n, ℓ, h, S, P and the key scheduling algorithm are public. The only secret in AES is the key k that is selected.

Here is a description of encryption:

Algorithm 5: Encryption of Substitution-Permutation Networks

$A \leftarrow$ plaintext
for $i = 1 \ldots h$ **do**
  $A \leftarrow A \oplus k_i$ // XOR
  $A \leftarrow S(A)$ // Substitution
  $A \leftarrow P(A)$ // Permutation
$A \leftarrow A \oplus k_{h+1}$
ciphertext $\leftarrow A$

The substitution provides confusion. The permutation provides diffusion. Decryption is just the reverse of encryption.



AES is an SPN, where the permutation operation is replaced by two invertible linear transformations. All operations are byte oriented (e.g., $b = 8$ so the S-box maps 8-bits to 8-bits). This allows AES to be efficiently implemented on software platforms. The block length of AES is $n = 128$ bits. Each subkey is 128 bits. AES accepts three different key lengths. The number of rounds h depends on the key length:

cipher	key length ℓ	h
AES-128	128	10
AES-192	192	12
AES-256	256	14

AES Round Operations

Each round updates a variable called State which consists of a 4×4 array of bytes (note: $4 \cdot 4 \cdot 8 = 128$, the block length). State is initialized with the plaintext:

$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$
$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$
$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
$a_{3,0}$	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$

$\leftarrow$ plaintext

After h rounds are completed, a final subkey is XOR-ed with State, the result being the ciphertext.

The AES round function uses four invertible operations:

1. `AddRoundKey` (key mixing)
2. `SubBytes` (S-box)
3. `ShiftRows` (permutation)
4. `MixColumns` (linear transformation).

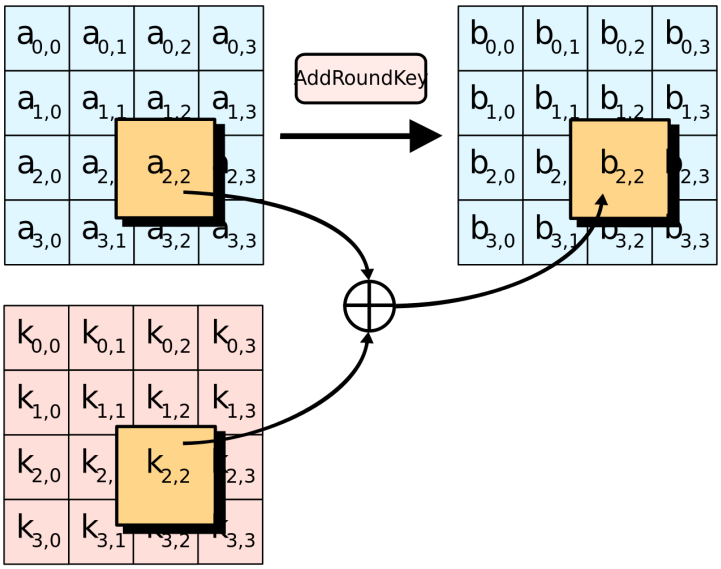
The pictures below are from wiki:

- https://commons.wikimedia.org/wiki/File:AES-AddRoundKey.svg

- https://commons.wikimedia.org/wiki/File:AES-SubBytes.svg
- https://commons.wikimedia.org/wiki/File:AES-ShiftRows.svg
- https://commons.wikimedia.org/wiki/File:AES-MixColumns.svg

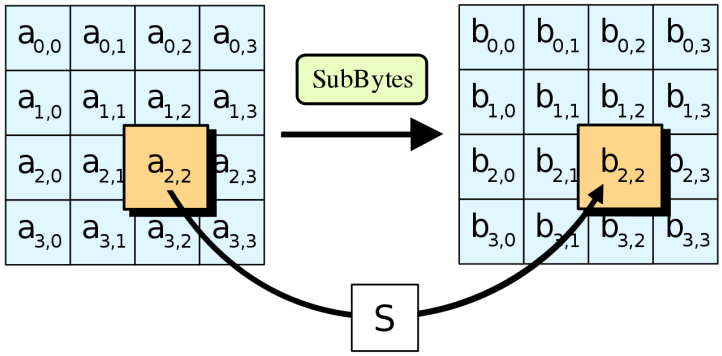
Add Round Key

Bitwise-XOR each byte of State with the corresponding byte of the subkey.



Substitute Bytes

Take each byte in State and replace it with the output of the S-box.



$S : \{0,1\}^8 \to \{0,1\}^8$ is a fixed, public, invertible function. The S-box is a non-linear function.

The Finite Field $GF(2^8)$

The elements of the finite field $GF(2^8)$ are the polynomials of degree at most 7 in $\mathbb{Z}_2[y]$, with addition and multiplication performed modulo the irreducible polynomial $f(y) = y^8 + y^4 + y^3 + y + 1$.

Interpret an 8-bit string $a = a_7a_6a_5a_4a_3a_2a_1a_0$ as coefficients of the polynomial $a(y) = a_7y^7 + a_6y^6 + a_5y^5 + \cdots + a_1y + a_0$ and vice versa.

Example:

Let $a = 11101100 = \texttt{ec}$ and $b = 00111011 = \texttt{3b}$, so $a(y) = y^7 + y^6 + y^5 + y^3 + y^2$ and similarly $b(y) = y^5 + y^4 + y^3 + y + 1$.

1. Addition: $a(y) + b(y) = y^7 + y^6 + y^t + y^2 + y + 1$, so $\texttt{ec} + \texttt{3b} = \texttt{d7}$.
2. Multiplication: $a(y) \cdot b(y) = y^{12} + y^{10} + y^8 + y^4 + y^2$, which leaves a remainder of $r(y) = y^7 + y^6 + y^3$ upon division by $f(y)$. Hence $a \cdot b = 11001000$ in $GF(2^8)$, or $\texttt{ec} \cdot \texttt{3b} = \texttt{c8}$.
3. Inversion: $\texttt{ec}^{-1} = 5f$ since $\texttt{ec} \cdot \texttt{5d} = \texttt{01}$.

S-box Definition

Let $p \in \{0,1\}^8$, and consider p as an element of $GF(2^8)$.

1. Let $q = p^{-1}$ if $p \neq 0$, and $q = p$ if $p = 0$.
2. Define $q = (q_7 q_6 q_5 q_4 q_3 q_2 q_1 q_0)$.
3. Compute

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \mod 2$$

Then $S(p) = r = (r_7 r_6 r_5 r_4 r_3 r_2 r_1 r_0)$.

So S maps the byte p to the byte r. This is a nonlinear function since inversion in the field $GF(2^8)$ is a non-linear function.

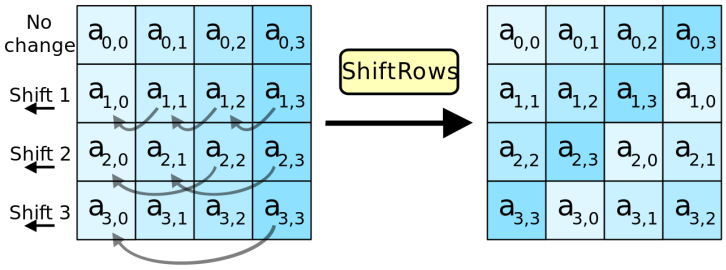
Instead of the algebraic description of the S-box, I can define the S function by this table.

	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
0	63	7c	77	7b	f2	6b	6f	c5	30	01	67	2b	fe	d7	ab	76
1	ca	82	c9	7d	fa	59	47	f0	ad	d4	a2	af	9c	a4	72	c0
2	b7	fd	93	26	36	3f	f7	cc	34	a5	e5	f1	71	d8	31	15
3	04	c7	23	c3	18	96	05	9a	07	12	80	e2	eb	27	b2	75
4	09	83	2c	1a	1b	6e	5a	a0	52	3b	d6	b3	29	e3	2f	84
5	53	d1	00	ed	20	fc	b1	5b	6a	cb	be	39	4a	4c	58	cf
6	d0	ef	aa	fb	43	4d	33	85	45	f9	02	7f	50	3c	9f	a8
7	51	a3	40	8f	92	9d	38	f5	bc	b6	da	21	10	ff	f3	d2
8	cd	0c	13	ec	5f	97	44	17	c4	a7	7e	3d	64	5d	19	73
9	60	81	4f	dc	22	2a	90	88	46	ee	b8	14	de	5e	0b	db
a	e0	32	3a	0a	49	06	24	5c	c2	d3	ac	62	91	95	e4	79
b	e7	c8	37	6d	8d	d5	4e	a9	6c	56	f4	ea	65	7a	ae	08
c	ba	78	25	2e	1c	a6	b4	c6	e8	dd	74	1f	4b	bd	8b	8a
d	70	3e	b5	66	48	03	f6	0e	61	35	57	b9	86	c1	1d	9e
e	e1	f8	98	11	69	d9	8e	94	9b	1e	87	e9	ce	55	28	df
f	8c	a1	89	0d	bf	e6	42	68	41	99	2d	0f	b0	54	bb	16

For example, $S(\texttt{a8}) = \texttt{c2}$.

Shift Rows

Permute the bytes of State by applying a *cyclic shift* to each row.



Mix Columns

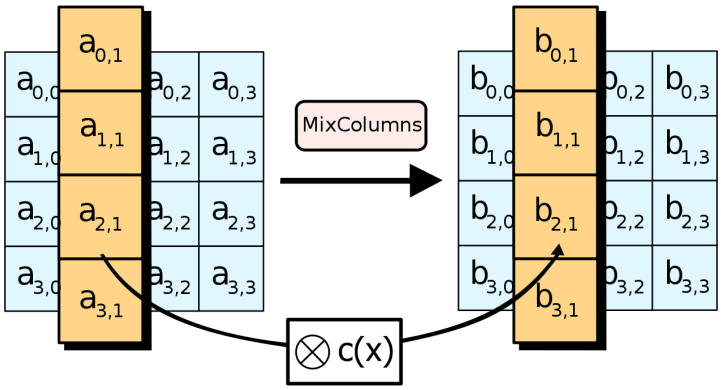
The purpose of `MixColumns` is to mix up the bits in the four bytes of each column of the State.

1. Read column i of `State` as a polynomial:

$$(a_{0,i}, a_{1,i}, a_{2,i}, a_{3,i}) = a_{0,i} + a_{1,i}x + a_{2,i}x^2 + a_{3,i}x^3$$

   (interpret the coefficients as elements of the finite field $GF(2^8)$).

2. Multiply this polynomial with the constant polynomial $c(x) = \texttt{03} \cdot x^3 + \texttt{01} \cdot x^2 + \texttt{01} \cdot x + \texttt{02}$ and reduce modulo $x^4 - 1$. This gives a new polynomial: $b_{0,i} + b_{1,i}x + b_{2,i}x^2 + b_{3,i}x^3$



The $\otimes c(x)$ Operation

Let $a(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, where each $a_i \in GF(2^8)$.

Let $c(x) = \texttt{02} + \texttt{01}x + \texttt{01}x^2 + \texttt{03}x^3$, where $\texttt{01}, \texttt{02}, \texttt{03}$ are elements in $GF(2^8)$ (in hexadecimal).

To compute $a(x) \otimes c(x)$:

- Compute $d(x) = a(x) \times c(x)$ (polynomial multiplication, where coefficient arithmetic is in $GF(2^8)$).
- Divide by $x^4 - 1$ to find the remainder polynomial $r(x)$ (equivalently, replace x^4 by 1, x^5 by x, and x^6 by x^2).
- Then $a(x) \otimes c(x) = r(x)$.

Let $a(x) = \texttt{d0f112bb} = \texttt{d0} + \texttt{f1}x + \texttt{12}x^2 + \texttt{bb}x^3$. $a(x) \otimes c(x) = \texttt{1a} + \texttt{a4}x + \texttt{d3}x^2 + \texttt{e5}x^3 = \texttt{1aa4d3e5}$.

AES Encryption

From the key k derive $h+1$ subkeys $k_0, k_1, \ldots, k_h$. Below is a description of encryption:

Algorithm 6: AES Encryption

```
State ← plaintext
State ← State ⊕ k_0
for i = 1...h − 1 do
    State ← SubBytes(State)
    State ← ShiftRows(State)
    State ← MixColumns(State)
    State ← State ⊕ k_i
State ← SubBytes(State)
State ← ShiftRows(State)
State ← State ⊕ k_h
ciphertext ← State
```

Note that in the final round, `MixColumns` is not applied. This helps make decryption more similar to encryption, and thus is useful when implementing AES. [Details omitted]

AES Decryption

From the key k derive $h+1$ subkeys $k_0, k_1, \ldots, k_h$. Below is a description of decryption:

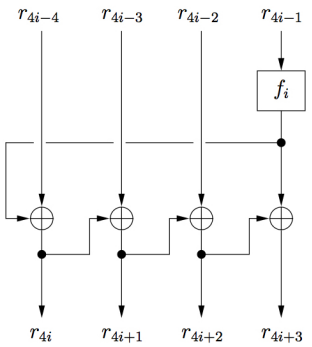
Algorithm 7: AES Decryption

```
State ← ciphertext
State ← State ⊕ k_h
State ← InvShiftRows(State)
State ← InvSubBytes(State)
for i = h − 1...1 do
    State ← State ⊕ k_i
    State ← InvMixColumns(State)
    State ← InvShiftRows(State)
    State ← InvSubBytes(State)
State ← State ⊕ k_0
plaintext ← State
```

`InvMixColumns` is multiplication by $d(x) = \texttt{0e} + \texttt{09}x + \texttt{0d}x^2 + \texttt{0b}x^3$ modulo $x^4 - 1$.

To complete the description of AES encryption, let's see *AES key schedule* for 128-bit keys. For 128-bit keys, AES has 10 rounds, so we need 11 subkeys.

- The first subkey $k_0 = (r_0, r_1, r_2, r_3)$ which is the actual AES key.
- The second subkey is $k_1 = (r_4, r_5, r_6, r_7)$.
- The third subkey is $k_2 = (r_8, r_9, r_9, r_{10})$.
- ...
- The 11^{th} subkey is $k_{10} = (r_{40}, r_{41}, r_{42}, r_{43})$.



Finally, here is a description of the *key schedule functions* f_i. The functions $f_i : \{0,1\}^{32} \to \{0,1\}^{32}$ are defined as follows:

1. The input is divided into 4 bytes: (a, b, c, d).
2. Left-rotate the bytes: (b, c, d, a).
3. Apply the AES S-box to each byte: $(S(b), S(c), S(d), S(a))$.
4. XOR the leftmost byte with the constant ℓ_i, and output the result: $(S(b) \oplus \ell_i, S(c), S(d), S(a))$.

The constants ℓ_i:

i	ℓ_i	i	ℓ_i	i	ℓ_i	i	ℓ_i	i	ℓ_i
1	0x01	2	0x02	3	0x04	4	0x08	5	0x10
6	0x20	7	0x40	8	0x80	9	0x1b	10	0x36

The AES encryption scheme is superior to Triple DES in many ways:

1. AES has three key lengths 128, 192 and 256 bits, whereas Triple DES has only one key length, namely 168 bits. Although its effective key length is 112 bits.
2. AES has block length 128 bits whereas the block length of Triple DES is only 64 bits.
3. AES is byte-oriented and so has very fast software implementations. On the other hand, Triple DES is bit-oriented and is significantly slower in software.

2.6.7 Block Cipher Modes of Operation

From https://en.wikipedia.org/wiki/Block_cipher_mode_of_operation:

> In cryptography, a block cipher mode of operation is an algorithm that uses a block cipher to provide information security such as confidentiality or authenticity. A block cipher by itself is only suitable for the secure cryptographic transformation (encryption or decryption) of one fixed-length group of bits called a block. A mode of operation describes how to repeatedly apply a cipher's single-block operation to securely transform amounts of data larger than a block.

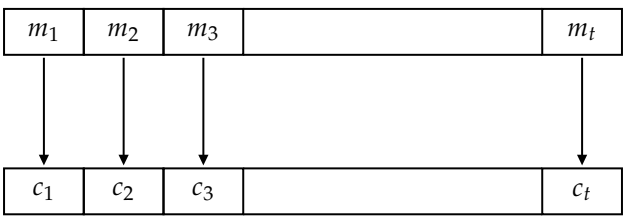
In practice, one might need to encrypt a large quantity of data. Plaintext message is $m = m_1, m_2, \ldots, m_t$, where each m_i is an L-bit block.

Question How should we use a block cipher $E_k : \{0,1\}^L \to \{0,1\}^L$ to encrypt m?

One way to encrypt the multi-block message m is to use electronic codebook or ECB mode.

Electronic Codebook (ECB) Mode

Encrypt blocks independently, one at a time: $c = c_1, c_2, \ldots, c_t$, where $c_i = E_k(m_i)$.



Decryption: $m_i = E_k^{-1}(c_i)$, $i = 1, 2, \ldots, t$.

Drawback: Identical plaintexts result in identical ciphertexts (under the same key), and thus ECB

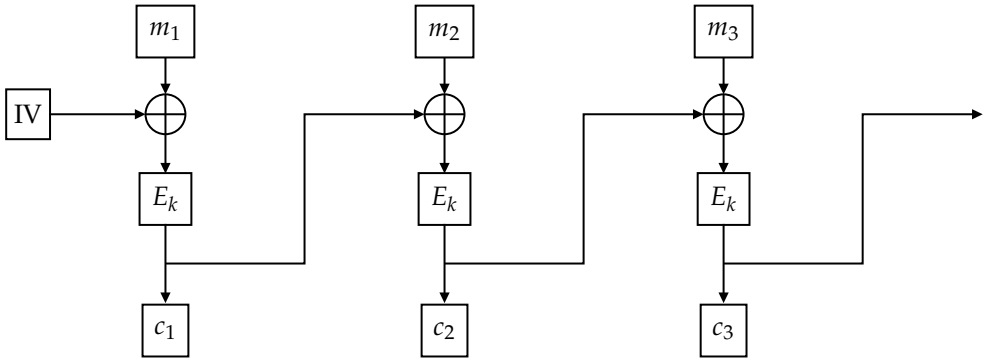
encryption is not (semantically) secure against chosen-plaintext attacks.

Suppose that the adversary has a challenge ciphertext c of length t blocks. The adversary would like to learn something about the plaintext m that corresponds to c. The adversary can select a plaintext m' of block length t, and obtain from Alice the encryption c' of m'. Now if c' equals c, then the adversary can conclude that m equals m'. On the other hand, if c' is not equal to c, then the adversary can conclude that m is not equal to m'. In either case, the adversary has learned something about the unknown plaintext m, and thus has broken semantic security.

The main problem with ECB mode is that the encryption process is *deterministic*. And so, for security one must include randomization into the encryption process. One way to do that is using *Cipher Block Chaining* or CBC mode.

Cipher Block Chaining (CBC) Mode

Encryption: Select $c_0 \in_R \{0,1\}^L$ (c_0 is a random non-secret IV). Then compute $c_i = E_k(m_i \oplus c_{i-1})$, $i = 1, 2, \ldots, t$.



Ciphertext is $(c_0, c_1, c_2, \ldots, c_t)$.

Decryption: $m_i = E_k^{-1}(c_i) \oplus c_{i-1}, \quad i = 1, 2, \ldots, t$.

Note that Alice selects a new random IV c_0 for every plaintext m that she encrypts. Identical plaintexts with different IVs result in different ciphertexts and for this reason CBC encryption is (semantically) secure against chosen-plaintext attacks (for a well chosen block cipher E).

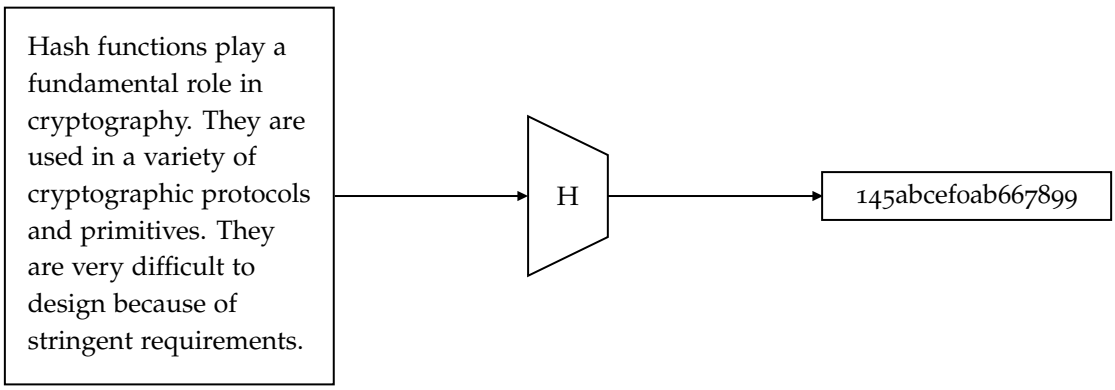
3

Hash Functions

3.1 Introduction

Hash functions play a fundamental role in cryptography. They are used in a variety of cryptographic primitives and protocols. They are very difficult to design because of very stringent security and performance requirements. Over the years, many hash functions have been broken. The main candidates available today are: SHA-1; SHA-2 family: SHA-224, *SHA-256*, SHA-384, SHA-512

What is a Hash Function? Informally speaking, a hash function is a fixed public function that takes as input messages of any length and outputs a hash value of a fixed length. In some sense the output serves as a fingerprint of the much longer input.



See these websites

- http://www.xorbin.com/tools/md5-hash-calculator
- http://www.xorbin.com/tools/sha1-hash-calculator
- http://www.xorbin.com/tools/sha256-hash-calculator

for calculators for the hash functions MD5, SHA-1 and SHA-256.

hash function

A **hash function** is a mapping H such that:

1. H maps binary messages of arbitrary lengths $\leq L$ to outputs of a fixed length n:

$$H : \{0,1\}^{\leq L} \to \{0,1\}^n.$$

(L is large, e.g., $L = 2^{64}$; n is small, e.g., $n = 256$)

2. $H(x)$ can be efficiently computed for all $x \in \{0,1\}^{\leq L}$.

H is called an n-bit hash function. $H(x)$ is called the hash or message digest of x.

Note:
The description of a hash function is public and fixed. There are no secret keys.

For simplicity, we will usually write $\{0,1\}^*$ instead of $\{0,1\}^{\leq L}$.

Toy Hash Function

$H : \{0,1\}^{\leq 4} \to \{0,1\}^2$

x	$H(x)$	x	$H(x)$	x	$H(x)$	x	$H(x)$
0	00	1	01				
00	11	01	01	10	01	11	00
000	00	001	10	010	11	011	11
100	11	101	01	110	01	111	10
0000	00	0001	11	0010	11	0011	00
0100	01	0101	10	0110	10	0111	01
1000	11	1001	01	1010	00	1011	01
1100	10	1101	00	1110	00	1111	11

$(00, 1000)$ is a *collision* because these two messages have the same hash value, namely 11.

1001 is a *preimage* of 01 since the hash of 1001 is 01.

10 is a *second preimage* of 1011 since the hash of 1011 is 01 and hash of 10 is also 01.

Example: SHA-256

SHA-256: $\{0,1\}^* \to \{0,1\}^{256}$

SHA-256("Hello there") =
`0x4e47826698bb4630fb4451010062fadbf85d61427cbdfaed7ad0f23f239bed89`

SHA-256("Hello There") =
`0xabf5dacd019d2229174f1daa9e62852554ab1b955fe6ae6bbbb214bab611f6f5`

SHA-256("Welcome to CO 487") =
`0x819ba4b1e568e516738b48d15b568952d4a35ea73f801c873907d3ae1f5546fb`

SHA-256("Welcome to CO 687") =
`0x404fb0ee527b8f9f01c337e915e8beb6e03983cfd9544296b8cf0e09c9d8753d`

So the idea behind hash functions is that for each message, the hash function outputs a hash value that appears random and has no apparent resemblance to the original message.

3.1.1 Hash Functions from Block Ciphers

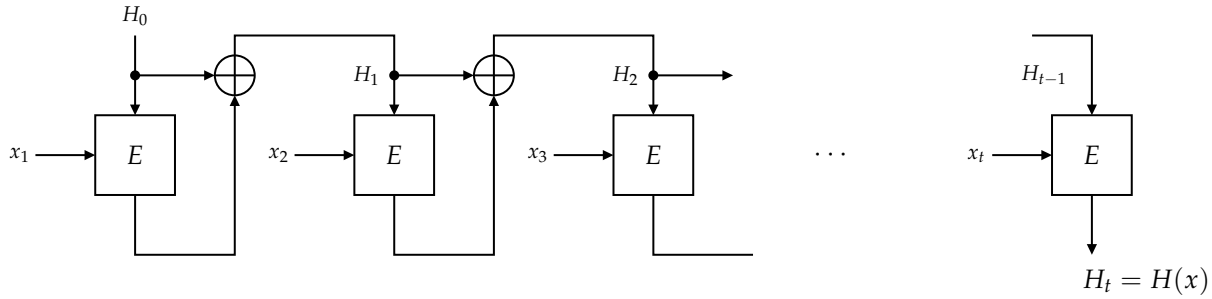
Davies-Meyer hash function

Let E_k be an m-bit block cipher with n-bit key k. Let IV be a fixed m-bit initializing value.

To compute $H(x)$, do:

Algorithm 8: Davies-Meyer hash function

1 Break up $x \parallel 1$ into n-bit blocks: $\overline{x} = x_1, x_2, \ldots, x_t$, padding out the last block with 0 bits if necessary.
2 $H_0 := IV$
3 **for** $i = 1, \ldots, t$ **do**
4 Compute $H_i = E_{x_i}(H_{i-1}) \oplus H_{i-1}$
5 $H(x) := H_t$



Note that hashing x does not use any secret keying information. The blocks of the message x are used as the keys in the hashing process, and so anyone who knows the message x can compute its hash.

Hash functions are the Swiss Army knife of cryptography. Hash functions are used for all kinds of applications they were not designed for, and for this reason it's sometimes difficult to assess security of these applications. One reason for this widespread use of hash functions is *speed*.

3.1.2 Desirable security properties for hash functions

Preimage Resistance (PR)

$H : \{0,1\}^* \to \{0,1\}^n$.

preimage resistance hash function

Given a hash value $y \in_R \{0,1\}^n$, it is computationally infeasible to find (with non-negligible probability of success) *any* $x \in \{0,1\}^*$ such that $H(x) = y$.

Here x is called *a* preimage of y.

Note that $x \in_R S$ means that x is chosen independently and uniformly at random from S.

Preimage resistance is required if hash functions are used for *password protection* on a multi-user computer system: Server stores (userid, H(password)) in a password file. Thus, if an attacker gets a copy of the password file, she does not learn any passwords. She only learns the hashes of the passwords.

2nd Preimage Resistance (2PR)

$H : \{0,1\}^* \to \{0,1\}^n$.

second-preimage resistance hash function

Given an input $x \in_R \{0,1\}^*$, it is computationally infeasible to find (with non-negligible probability of success) a second input $x' \in \{0,1\}^*$, $x' \neq x$, such that $H(x') = H(x)$.

2PR is needed if hash functions are used as *Modification Detection Codes* (MDCs): To ensure that a message m is not modified by unauthorized means, one computes $H(m)$ and protects $H(m)$ from unauthorized modification. Protecting $H(m)$ might be easier than protecting m itself because $H(m)$ is in general much smaller than m.

Modification detection codes can be used for *virus protection*. A user Alice wishes to download a piece of software x from a website that might be untrusted. Suppose also that Alice has an authentic copy of the hash of x, which might have been obtained from the print copy of a trade magazine. So Alice will download x from the website, compute its hash, and compare the hash value with the hash from the trade magazine. If the two hash values are equal, then Alice installs the software on her computer. An adversary who wants Alice to install a piece of malware needs to format the malware x' so that the hash of x' equals the hash of x. In other words the adversary needs to find a second preimage for the message x, and so for this application we need H to be second-preimage resistant.

Collision Resistance (CR)

$H : \{0,1\}^* \to \{0,1\}^n$.

collision-resistant hash function

It is computationally infeasible to find (with non-negligible probability of success) two distinct inputs $x, x' \in \{0,1\}^*$ such that $H(x) = H(x')$.

The pair (x, x') is called a *collision* for H.

Note that H has many collisions since the domain is much larger in general than its codomain, and so by the pigeonhole principle there are many pairs of messages with the same hash value. So the issue isn't whether collisions exist, but whether one can find a collision *efficiently*.

Collision resistance is required when hash functions are used to compute *message digests or digital signature schemes*: For reasons of efficiency, instead of signing a (long) message, the (much shorter) message digest is signed. This action requires preimage-resistance, 2nd preimage resistance, and collision resistance. (More on this later)

To see why collision resistance is required: Suppose that the legitimate signer Alice can find two messages x_1 and x_2, with $x_1 \neq x_2$ and $H(x_1) = H(x_2)$. Alice can sign x_1 and later claim to have signed x_2. In this way, Alice might be able to repudiate her signature on x_1.

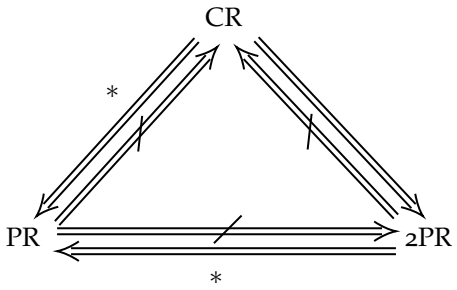
Later on in the course, we'll see many *other applications* of hash functions:

1. Message Authentication Codes (HMAC). [More on this later]
2. Pseudorandom bit generation: Distilling random bits s from several "pseudorandom" sources $x_1, \ldots, x_t$. Output $s = H(x_1, \ldots, x_t)$.
3. Key derivation function (KDF): Deriving a cryptographic key from a shared secret. [More later]

4. Proof-of-work in cryptocurrencies (Bitcoin): [More on this later]
5. Quantum-safe signature schemes: [More on this later]

3.1.3 Relationships between PR, 2PR, CR

Let $H : \{0,1\}^* \to \{0,1\}^n$ be a hash function.



$*$ for somewhat uniform hash functions

1. CR implies 2PR

Proof:
Suppose that H is not 2PR.

Select $x \in_R \{0,1\}^*$. Since H is not 2PR, we can efficiently find $x' \in \{0,1\}^*$, $x' \neq x$, with the same hash value $H(x') = H(x)$. Thus we have efficiently found a collision (x, x') for H.

Hence H is not CR. □

Note that this proof establishes the contrapositive statement.

2. 2PR does not guarantee CR

Proof:
Suppose that $H : \{0,1\}^* \to \{0,1\}^n$ is 2PR.

Consider $\overline{H} : \{0,1\}^* \to \{0,1\}^n$ defined by:

$$\overline{H}(x) = \begin{cases} H(0), & \text{if } x = 1, \\ H(x), & \text{if } x \neq 1. \end{cases}$$

Thus we have $\overline{H}(1) = H(0) = \overline{H}(0)$.

Suppose that $\overline{H}$ is not 2PR. So, given $x \in_R \{0,1\}^*$, we can efficiently find $x' \in \{0,1\}^*$, $x' \neq x$, with $\overline{H}(x') = \overline{H}(x)$. With probability essentially 1, we can assume that $x \neq 0, 1$. Hence $\overline{H}(x) = H(x)$.

Now, if $x' \neq 1$, then $\overline{H}(x') = H(x') = H(x)$; while if $x' = 1$, then $\overline{H}(x') = \overline{H}(1) = H(0) = H(x)$. In either case, we have found a second preimage for x with respect to H. This contradicts the assumption that H is second preimage resistant. So, $\overline{H}$ must be 2PR.

Also, $\overline{H}$ is not CR, since $(0,1)$ is a collision for $\overline{H}$.

Hence $\overline{H}$ is second preimage resistant but not collision resistant. □

3. CR does not guarantee PR

Proof:
Suppose that $H : \{0,1\}^* \to \{0,1\}^n$ is CR.

Consider $\overline{H} : \{0,1\}^* \to \{0,1\}^{n+1}$ defined by:

$$\overline{H}(x) = \begin{cases} 1 \parallel x, & \text{if } x \in \{0,1\}^n, \\ 0 \parallel H(x), & \text{if } x \notin \{0,1\}^n. \end{cases}$$

Then we claim that $\overline{H}$ is CR.

First note that hash values that begin with the 1 have unique preimages, and hence there does not exist a collision for H-bar where the colliding messages have hash values that begin with a 1.

Suppose next that we can officially find a collision for $\overline{H}$ where the colliding messages have hash values that begin with a 0. But then this collision for $\overline{H}$ is also a collision for H that we have officially found. This contradicts the assumption that H is collision resistant, and so we conclude that $\overline{H}$ is also collision resistant.

And, $\overline{H}$ is not PR since preimages can be efficiently found for at least half of all $y \in \{0,1\}^{n+1}$. □

However, if H is "*somewhat uniform*" (i.e., all hash values have roughly the same number of preimages), then *CR does imply PR*.

Proof:
Suppose H us a somewhat uniform hash function that is not PR. Select $x \in_R \{0,1\}^*$ and compute $y = H(x)$. Since H is somewhat uniform, y is also a randomly selected hash value.

Since H is not PR, we can efficiently find a preimage x' of y. Since H is somewhat uniform, we expect that y has many preimages, and thus $x' \neq x$ with very high probability. Thus, (x, x') is a collision for H that we have efficiently found, so H is not CR. □

Note that we shall henceforth assume that hash functions are somewhat uniform.

4. PR does not guarantee 2PR

Proof:
Suppose $H : \{0,1\}^* \to \{0,1\}^n$ is PR.

Define $\overline{H} : \{0,1\}^* \to \{0,1\}^n$ by

$$\overline{H}(x_1, x_2, \ldots, x_t) = H(0, x_2, \ldots, x_t).$$

Then we first claim $\overline{H}$ is PR.

Suppose that $\overline{H}$ is not preimage resistant. Then, given a random hash value y, we can efficiently find a message x such that $\overline{H}(x)$ equals y. But then $H(x') = y$ where x' is the same as x except that its first bit is now zero. And so we can efficiently find preimages for H. This contradicts the assumption that H is preimage resistant, and so we conclude that $\overline{H}$ is also preimage resistant.

Then let's prove $\overline{H}$ is not 2PR.

Suppose that we're given a random message x. Then we can easily find a second preimage for x with respect to $\overline{H}$, namely by flipping the first bit of x to get the message x'. And this shows that $\overline{H}$ is not second preimage resistant.

So $\overline{H}$ is a hash function that is preimage resistant but is not second preimage resistant. □

5. 2PR implies PR (for somewhat uniform H)

Proof:
Exercise. □

Note that CR is the hardest property to satisfy because if a hash function is CR then it is also PR and also 2PR. On the other hand, PR does not imply CR and 2PR does not imply CR.

3.2 Generic Attacks

generic attack

A **generic attack** on a hash function $H : \{0,1\}^* \to \{0,1\}^n$ does not exploit any properties the specific hash function may have.

In the analysis of a generic attack, we view H as a *random selected function* in the sense that for each $x \in \{0,1\}^*$, we assume that the value $y = H(x)$ was chosen by selecting $y \in_R \{0,1\}^n$.

From a security point of view, a random function is an *ideal* hash function. However, random functions are not suitable for practical applications because they cannot be compactly stored.

3.2.1 Find Preimages

Here's a *generic attack for finding preimages* for an n-bit hash function $H : \{0,1\}^* \to \{0,1\}^n$.

Given $y \in_R \{0,1\}^n$, repeatedly select arbitrary $x \in \{0,1\}^*$ until $H(x) = y$.

For each message x, the probability that $H(x)$ equals the specific y is $1/2^n$ because the total number of hash values is 2^n and we're assuming that H is a randomly selected function. Hence the *expected number of steps* is $\approx 2^n$. (Here, a 'step' is a hash function evaluation.)

This attack is infeasible if $n \geq 128$.

Note:
It has been proven that this generic attack for finding preimages is optimal, i.e., no faster *generic* attack exists.

Of course, for a specific hash function, there might exist a preimage finding algorithm that is faster than the generic attack.

3.2.2 Find Collisions

Select arbitrary $x \in \{0,1\}^*$ and store $(H(x), x)$ in a table sorted by first entry. Continue until a collision is found.

Since there are 2^n possible hash values, by the birthday paradox we conclude that *the expected number of steps* is $\sqrt{\pi 2^n/2} \approx \sqrt{2^n}$. (Here, a "step" is a hash function evaluation.)

This attack is infeasible if $n \geq 256$.

It has been proven that this generic attack for finding collisions is optimal in terms of the number of hash function evaluations.

A major draw back of this algorithm is the *expected space required*: $\sqrt{\pi 2^n/2} \approx \sqrt{2^n}$.

If $n = 128$, then the expected running time is 2^{64} steps (feasible). However, the expected space required is 7×10^8 Tbytes (infeasible).

3.2.3 VW Parallel Collision Search

VW: Paul van Oorschot, Michael Wiener (1993)

Expected number of steps is $\sqrt{2^n}$. However, expected space required is negligible.

It is easy to parallelize: m-fold speedup with m processors.

The VW attack finds a random pair of n-bit messages with the same hash value. It might appear that such a collision is useless in practice. However, the VW attack can easily be modified to find "meaningful" collisions. See the details in Parallel Collision Search with Cryptanalytic Applications.

Conclusion: If collision resistance is desired, then use an n-bit hash function with $n \geq 256$

Parallel collision search (VW Method)

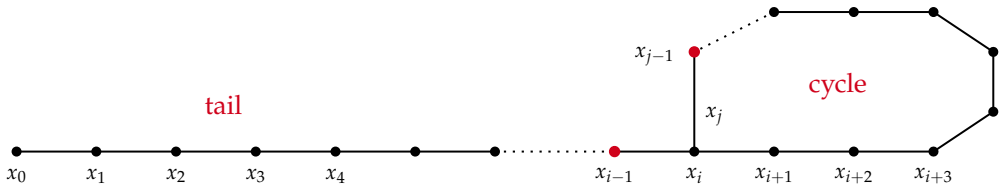
Problem: Find a collision for $H : \{0,1\}^* \to \{0,1\}^n$.

Assumption: H is a random function.

Notation: Let $N = 2^n$.

Define a sequence $\{x_i\}_{i \geq 0}$ by

$$x_0 \in_R \{0,1\}^n, \quad x_i = H(x_{i-1}) \text{ for } i \geq 1.$$



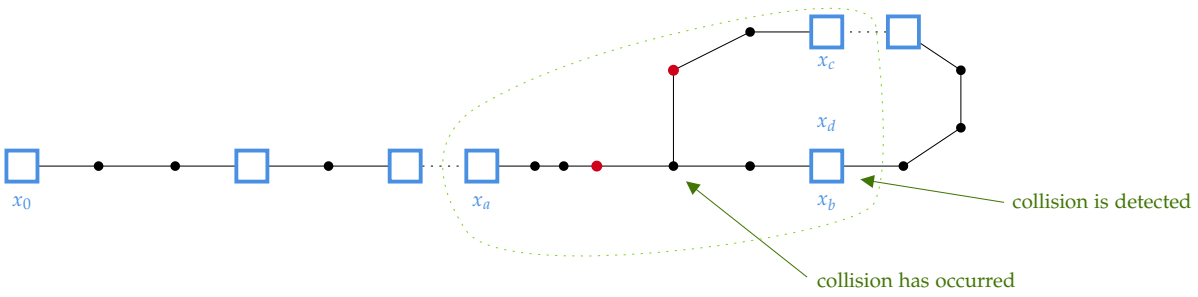
This sequence cannot be distinct forever, because the terms are drawn from a finite set, and so the sequence must eventually collide with itself. Let j be the smallest index for which $x_j = x_i$ for some $i < j$; such a j must exist. Then $x_{j+\ell} = x_{i+\ell}$ for all $\ell \geq 1$.

By the birthday paradox, $E[j] \approx \sqrt{\frac{\pi N}{2}} \approx \sqrt{N}$. In fact, $E[i] \approx \frac{1}{2}\sqrt{N}$ where i is the length of the tail of the sequence. And $E[j-i] \approx \frac{1}{2}\sqrt{N}$ where $j-i$ is the length of the cycle.

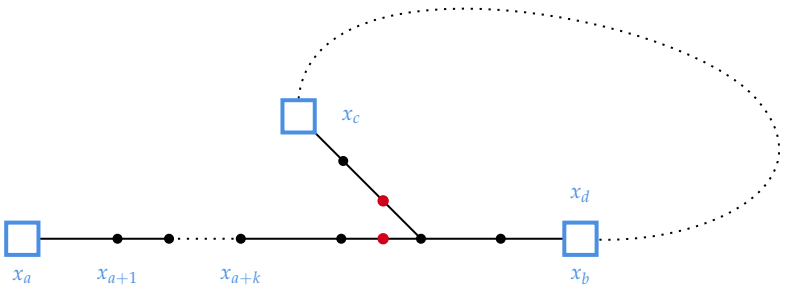
Now, $i \neq 0$ with overwhelming probability; in that event, (x_{i-1}, x_{j-1}) (red points) is a collision for H.

One way to find this collision is to compute the terms $x_0, x_1, x_2, \ldots$ and store them in a table. However the table would be too large, namely of size roughly $\sqrt{N}$. How to find (x_{i-1}, x_{j-1}) without using much storage? VW method's idea is to store only *distinguished points*.

Select an easily testable distinguishing property for elements of $\{0,1\}^n$ (e.g., leading 32 bits are all 0). Let θ be the proportion of elements of $\{0,1\}^n$ that are distinguished. In a nutshell, VW algorithm is to compute $x_0, x_1, x_2, \ldots$ and only store the points in the sequence that are distinguished.



We can expand the green portion of the diagram as follows:



Algorithm 9: VW collision finding

```
   // Stage 1 (detecting a collision)
1  Select x_0 ∈_R {0,1}^n
2  Store (x_0, 0, −) in a sorted table
3  LP ← x_0 // LP = last point stored in the table
4  for d = 1, 2, 3, 4, ... do
5      Compute x_d = H(x_{d−1})
6      if x_d is distinguished then
7          if x_d is already in table then
               // Say x_d = x_b where b < d
8              Go to Stage 2.
9          else
10             Store (x_d, d, LP) in a table.
11             LP ← x_d
   // Stage 2 (finding a collision)
12 ℓ_1 ← b − d,   ℓ_2 ← d − c
   // WLOG suppose ℓ_1 ≥ ℓ_2
13 k ← ℓ_1 − ℓ_2
14 Compute x_{a+1}, x_{a+2}, ..., x_{a+k}.
15 for m = 1, 2, 3, ... do
16     Compute (x_{a+k+m}, x_{c+m})
17 until x_{a+k+m} = x_{c+m}
18 The collision is (x_{a+k+m−1}, x_{c+m−1})
```

Let's analyze the VW attack.

- Stage 1: Expected # of H-evaluations is

$$\underbrace{\sqrt{\frac{\pi N}{2}}}_{\text{collision occurs}} + \underbrace{\frac{1}{\theta}}_{\text{collision is detected}} \approx \sqrt{N} + \frac{1}{\theta}$$

- Stage 2: Expected # of H-evaluations is $\leq \frac{3}{\theta}$ (see optional readings)

Overall expected running time: $\boxed{\sqrt{N} + \frac{4}{\theta}}$

Storage: $\approx \theta\sqrt{N}(3n)$ bits (each table entry has bitlength $3n$)

Let's revisit the example when $n = 128$. Take $\theta = \frac{1}{2^{32}}$. Then the expected time of VW collision search

is 2^{64} H-evaluations (feasible), and the expected storage is 241 Gbytes (free).

Parallelizing VW collision search

Suppose we had m processors available to us. Run a copy of VW on each of the m processors. Each processor would report distinguished points to a central server , and the central server would check for repeated distinguish points.

Note that each processor begins with its own random starting point, and so each processor computes a different sequence. Each sequence can now collide either with itself or with another sequence as shown here. When two sequences from different processors collide then we have a collision of n-bit strings. This collision is detected by the central server because this distinguished point will be computed twice, once by this processor and once by this processor. When a collision is detected then the collision can be found as before using Stage 2 of the VW attack.

Expected time $\approx \sqrt{N}/m + 4/\theta$. Expected storage $\approx \theta\sqrt{N}(3n)$ bits

Note that we have factor-m speedup. No communication between processors. Processors only occasionally communicate with central server. So VW parallelizes very well.

3.3 Iterated Hash Functions (Merkle Meta-Method)

Iterated hash functions has two components:

- Fixed initializing value $IV \in \{0,1\}^n$,
- Compression function $f : \{0,1\}^{n+r} \to \{0,1\}^n$ (efficiently computable).

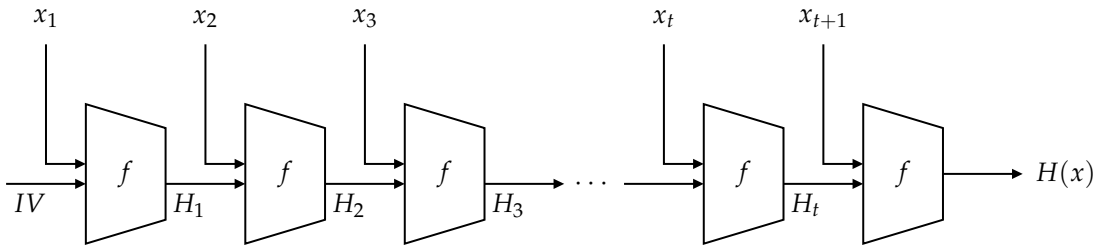
To compute $H(x)$ where x has bitlength $b < 2^r$ do:

Algorithm 10: Merkle's hash function

1 Break up x into r-bit blocks: $\overline{x} = x_1, x_2, \ldots, x_t$, padding out the last block with 0 bits if necessary.
2 Define x_{t+1}, the *length-block*, to hold the right-justified binary representation of b.
3 $H_0 := IV$
4 **for** $i = 1, 2, \ldots, t+1$ **do**
5 Compute $H_i = f(H_{i-1}, x_i)$
6 $H(x) := H_{t+1}$

The H_i's are called *chaining variables*.

Here is a diagram that illustrates Merkle's has function construction.



3.3.1 Collision Resistance of Iterated Hash Functions

Merkle's Theorem

If the compression function f is collision resistant, then the hash function H is also collision resistant.

Merkle's theorem reduces the problem of designing collision-resistant hash functions to that of designing collision-resistant compression functions.

Proof:

Suppose H is not CR. Then we can *efficiently* find $x, x' \in \{0,1\}^*$, $x \neq x'$, with $H(x) = H(x')$.

Let

$$\begin{aligned} \overline{x} &= x_1, x_2, \ldots, x_t, \quad b = \texttt{bitlength}(x), \quad x_{t+1} = \text{length block} \\ \overline{x}' &= x'_1, x'_2, \ldots, x'_t, \quad b' = \texttt{bitlength}(x'), \quad x'_{t'+1} = \text{length block.} \end{aligned}$$

We efficiently compute:

$$\begin{aligned} H_0 &= IV \\ H_1 &= f(H_0, x_1) \\ H_2 &= f(H_1, x_2) \\ &\vdots \\ H_{t-1} &= f(H_{t-2}, x_{t-1}) \\ H_t &= f(H_{t-1}, x_t) \\ H_{t+1} &= f(H_t, x_{t+1}) \end{aligned} \qquad \begin{aligned} H_0 &= IV \\ H'_1 &= f(H_0, x'_1) \\ H'_2 &= f(H'_1, x'_2) \\ &\vdots \\ H'_{t'-1} &= f(H'_{t'-2}, x'_{t'-1}) \\ H'_{t'} &= f(H_{t'-1}, x'_{t'}) \\ H_{t'+1} &= f(H'_{t'}, x'_{t'+1}) \end{aligned}$$

Since $H(x) = H(x')$, we have $H_{t+1} = H'_{t'+1}$.

Now if $b \neq b'$, then $x_{t+1} \neq x'_{t'+1}$. Thus $(H_t, x_{t+1}), (H'_{t'}, x'_{t'+1})$ is a collision for f that we have efficiently found. Hence f is not collision resistant.

Suppose $b = b'$. Then $t = t'$ and $x_{t+1} = x'_{t'+1}$.

Let i be the largest index, $i \in [0, t]$, for which $(H_i, x_{i+1}) \neq (H'_i, x'_{i+1})$. Such i must exist since $x \neq x'$. Then

$$H_{i+1} = f(H_i, x_{i+1}) = f(H'_i, x'_{i+1}) = H'_{i+1},$$

so $(H_i, x_{i+1}), (H'_i, x'_{i+1})$ is a collision for f that we have efficiently found. Hence f is not CR. □

3.3.2 Provable Security

A major theme of cryptographic research is to formulate precise security definitions and assumptions, and then *prove* that a cryptographic protocol is secure. A *proof of security* is certainly desirable since it rules out the possibility of attacks being discovered in the future. However, it isn't always easy to assess the practical security assurances (if any) that a security proof provides.

- The assumptions might be unrealistic, or false, or circular.
- The security proof might be fallacious.
- The security model might not account for certain kinds of realistic attacks.

- The security proof might be asymptotic.
- The security proof might have a large tightness gap.

Optional reading: http://anotherlook.ca

3.4 MDx-Family of Hash Functions

MDx is a family of iterated hash functions.

MD4

MD4 was proposed by Ron Rivest in 1990. MD4 has 128-bit outputs. (VW finds an MD4 collision in time 2^{64}) Wang et al. (2004) found collisions for MD4 by hand. So this showed that MD4 was completely insecure from the point of view of collision resistance. Leurent (2008) discovered an algorithm for finding MD4 preimages in 2^{102} steps.

MD5

MD5 is a strengthened version of MD4. Designed by Ron Rivest in 1991. MD5 has 128-bit outputs.

Wang and Yu (2004) found MD5 collisions in 2^{39} steps. MD5 collisions can now be found in 2^{24} steps (in a few seconds on a laptop computer). Sasaki & Aoki (2009) discovered a method for finding preimages for MD5 in $2^{123.4}$ steps.

Therefore, MD5 should not be used if collision resistance is required, but is probably okay as a preimage-resistant hash function.

MD5 is still used today. The reason is that MD5 was widely deployed in the 1990s and the 2000s and it's very expensive and cumbersome to replace all these implementations. For example, in 2006, MD5 is implemented more than 850 times in Microsoft Windows source code. In 2014, Microsoft issued a patch that restricts the use of certificates with MD5 in Windows: http:tinyurl.com/MicrosoftMD5. The reason for the patch was the discovery of the Flame malware.

Flame Malware

See https://en.wikipedia.org/wiki/Flame_(malware)

Discovered in May 2012. Highly sophisticated espionage tool. Targeted computers in Iran and the Middle East. Suspected to originate from the US and/or Israeli government; US government has denied all responsibility. Contains a forged Microsoft certificate for Windows code signing. Forged certificate used a new, "zero-day MD5 chosen-prefix" collision attack. Microsoft no longer allows the use of MD5 for code signing.

3.5 SHA

3.5.1 SHA-1

Secure Hash Algorithm (SHA) was designed by NSA and published by NIST in 1993 (FIPS 180). 160-bit iterated hash function, based on MD4. Slightly modified to SHA-1 (FIPS 180-1) in 1994 in order to fix an (undisclosed) security weakness. Wang et al. (2005) found collisions for SHA in 239 steps. Wang et al. (2005) discovered a collision-finding algorithm for SHA-1 that takes 263 steps. The first SHA-1 collision was found on February 23, 2017. No preimage or 2nd preimage attacks (that are faster than the generic attacks) are known for SHA-1.

Microsoft's SHA-1 Plan

See http://tinyurl.com/MicrosoftSHA1. As of May 9, 2017: TLS server-authentication certificates that use SHA-1 will be considered invalid. However, Microsoft still permits the use of SHA-1 in Code signature file hashes, Code signing certificates, Timestamp signature hashes, Timestamping certificates, etc.

3.5.2 SHA-2 Family

In 2001, NSA proposed variable output-length versions of SHA-1.

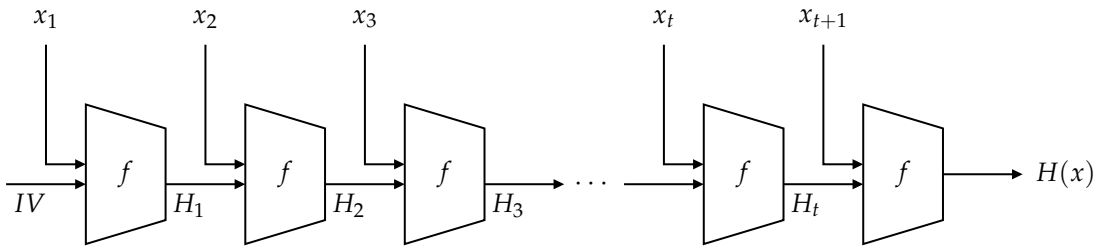
Output lengths are 224 bits (SHA-224 and SHA-512/224), 256 bits (SHA-256 and SHA-512/256), 384 bits (SHA-384) and 512 bits (SHA-512).

2020: No weaknesses in any of these hash functions have been found.

Note: The security levels of these hash functions against collision-finding attacks is the same as the security levels of Triple-DES, AES-128, AES-192 and AES-256 against exhaustive key search attacks. For example, the VW attack on SHA-224 can find a collision in 2^{112} operations, whereas the meet-in-the-middle attack on Triple-DES finds a secret key in 2^{112} steps.

The SHA-2 hash functions are standardized in FIPS 180-2.

3.5.3 Description of SHA-256



SHA-256 is an iterated hash function (Merkle meta-method).

$n = 256, r = 512$.

Compression function is $f : \{0,1\}^{256+512} \to \{0,1\}^{256}$.

Input: bitstring x of arbitrary bitlength $b \geq 0$.

Output: 256-bit hash value $H(x)$ of x.

SHA-256 Notation

A, B, C, D, E, F, G, H	32-bit words.
$+$	addition modulo 2^{32}.
$\overline{A}$	bitwise complement.
$A \gg s$	shift A right through s positions.
$A \hookrightarrow s$	rotate A right through s positions.
AB	bitwise AND.
$A \oplus B$	bitwise exclusive-OR.
$f(A, B, C)$	$AB \oplus \overline{A}C$.
$g(A, B, C)$	$AB \oplus AC \oplus BC$.
$r_1(A)$	$(A \hookrightarrow 2) \oplus (A \hookrightarrow 13) \oplus (A \hookrightarrow 22)$.
$r_2(A)$	$(A \hookrightarrow 6) \oplus (A \hookrightarrow 11) \oplus (A \hookrightarrow 25)$.
$r_3(A)$	$(A \hookrightarrow 7) \oplus (A \hookrightarrow 18) \oplus (A \gg 3)$
$r_4(A)$	$(A \hookrightarrow 17) \oplus (A \hookrightarrow 19) \oplus (A \gg 10)$.

SHA-256 Constants

32-bit initial chaining values (IVs):

$$h_1 = \texttt{0x6a09e667}, \quad h_2 = \texttt{0xbb67ae85}, \quad h_3 = \texttt{0x3c6ef372}, \quad h_4 = \texttt{0xa54ff53a},$$
$$h_5 = \texttt{0x510e527f}, \quad h_6 = \texttt{0x9b05688c}, \quad h_7 = \texttt{0x1f83d9ab}, \quad h_8 = \texttt{0x5be0cd19}.$$

These words were obtained by taking the first 32 bits of the fractional parts of the square roots of the first 8 prime numbers.

Per-round integer additive constants:

$$y_0 = \texttt{0x428a2f98}, \quad y_1 = \texttt{0x71374491}, \quad y_2 = \texttt{0xb5c0fbcf}, \quad y_3 = \texttt{0xe9b5dba5},$$
$$\ldots \quad \ldots \quad y_{62} = \texttt{0xbef9a3f7}, \quad y_{63} = \texttt{0xc67178f2}.$$

These words were obtained by taking the first 32 bits of the fractional parts of the cube roots of the first 64 prime numbers.

Algorithm 11: SHA-256 Preprocessing

Pad x (with 1 followed by as few 0's as possible) so that its bitlength is 64 less than a multiple of 512.
Append a 64-bit representation of $b \mod 2^{64}$.
The formatted input is $x_0, x_1, \ldots, x_{16m-1}$, where each x_i is a 32-bit word.
Initialize chaining variables: $(H_1, H_2, \ldots, H_6, H_7, H_8) \leftarrow (h_1, h_2, \ldots, h_6, h_7, h_8)$.
foreach $i = 0 \ldots m-1$ **do**
  Copy the i-th block of sixteen 32-bit words into temporary storage: $X_j \leftarrow x_{16i+j}$, $0 \le j \le 15$.
  `// Expand the 16-word block into a 64-word block:`
  for $j = 16 \ldots 63$ **do**
    $X_j \leftarrow r_4(X_{j-2}) + X_{j-7} + r_3(X_{j-15}) + X_{j-16}$.
  `// Initialize working variables`
  $(A, B, \ldots, F, G, H) \leftarrow (H_1, H_2, \ldots, H_6, H_7, H_8)$.
  for $j = 0 \ldots 63$ **do**
    $T_1 \leftarrow H + r_2(E) + f(E, F, G) + y_j + X_j$
    $T_2 \leftarrow r_1(A) + g(A, B, C)$
    $H \leftarrow G, \quad G \leftarrow F, \quad F \leftarrow E, \quad E \leftarrow D + T_1$
    $D \leftarrow C, \quad C \leftarrow B, \quad B \leftarrow A, \quad A \leftarrow T_1 + T_2$.
  Update chaining values: $(H_1, H_2, \ldots, H_7, H_8) \leftarrow (H_1 + A, H_2 + B, \ldots, H_7 + G, H_8 + H)$.

Output: SHA-256$(x) = H_1 \parallel H_2 \parallel H_3 \parallel H_4 \parallel H_5 \parallel H_5 \parallel H_6 \parallel H_7 \parallel H_8$.

Performance: Speed benchmarks (2017) for software implementations on an Intel Core i9 2.9 GHz 6-core Coffee Lake (8950HK) using OpenSSL 1.1.1d. SHA-256 is quite fast in software. It's not as fast as MD5 or SHA-1, but that's probably just as well since MD5 and SHA-1 are considered to be insecure as far as collision resistance is concerned.

3.5.4 SHA-3

The SHA-2 design is similar to SHA-1, and thus there are lingering concerns that the SHA-1 weaknesses could eventually extend to SHA-2.

SHA-3: NIST hash function competition. 64 candidates submitted by Oct 31 2008 deadline.
2012: Keecak was selected as the winner. Keecak uses the "sponge construction" and *not* the Merkle iterated hash design. SHA-3 is being used in practice, but is not as widely deployed as SHA-2.

3.5.5 NIST's Policy on Hash Functions

August 5, 2015. See: http://csrc.nist.gov/groups/ST/hash/policy.html

Should stop using SHA-1 for digital signatures and other applications that require collision resistance. May still use SHA-1 for HMAC, KDFs, and random number generators. May use SHA-2 for all applications that employ secure hash algorithms. SHA-3 may also be used, but this is not required.

4

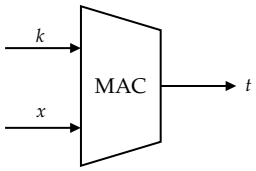
Message authentication code schemes

4.1 Introduction

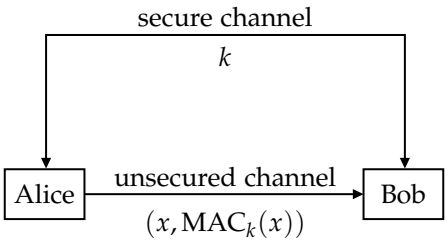
message authentication code

A **message authentication code** scheme is a family of functions $\text{MAC}_k : \{0,1\}^* \to \{0,1\}^n$ parameterized by an ℓ-bit key k, where each function MAC_k can be efficiently computed.

$t = \text{MAC}_k(x)$ is called the *MAC* or *tag* of x with key k.



MAC schemes are used for providing (symmetric-key) data integrity and data origin authentication.



To provide data integrity and data origin authentication:

1. Alice and Bob establish a secret key $k \in \{0,1\}^\ell$.
2. Alice computes tag $t = \text{MAC}_k(x)$ and sends (x,t) to Bob.
3. Bob verifies that $t = \text{MAC}_k(x)$.

Note:
No confidentiality since the message x is sent in the clear. If confidentiality is required, then one has to properly combine an encryption scheme with a MAC scheme.

Non-repudiation. This is because when Bob receives a tag message from Alice and the tag is valid,

then Bob is convinced that the message came from Alice. However, Bob cannot convince a third party that the message came from Alice because Bob could have generated the message and its tag himself. This is the consequence of Alice and Bob sharing the same secret key.

An adversary who captures the message and tag can *replay* this tag message to Bob at a future point in time and Bob will accept the tagged message. To avoid replay, add a timestamp or sequence number.

4.1.1 Security Definition

Let K be the secret key shared by Alice and Bob.

The adversary does not know k, but is allowed to obtain (from Alice or Bob) tags for messages of her choosing. The adversary's goal is to obtain the tag of *any* new message, i.e., a message whose tag she did not already obtain from Alice or Bob.

secure MAC scheme

A MAC scheme is **secure** if given some tags $\text{MAC}_k(x_i)$ for x_i's of one's own choosing, it is computationally infeasible to compute (with non-negligible probability of success) a message-tag pair $(x, \text{MAC}_k(x))$ for any new message x.

More concisely, a MAC scheme is secure if it is existentially unforgeable against chosen-message attack.

Note: A secure MAC scheme can be used to provide *data integrity* and *data origin authentication*.

An *ideal* MAC scheme has the following property:

For each key $k \in \{0,1\}^\ell$, the function $\text{MAC}_k : \{0,1\}^* \to \{0,1\}^n$ is a random function.

Ideal MAC schemes are useless in practice. However, when analyzing a generic attack on a MAC scheme, it is reasonable to assume that the MAC scheme is ideal.

4.2 Generic Attacks on MAC schemes

Guessing the MAC of a message $x \in \{0,1\}^*$:

Select $y \in_R \{0,1\}^n$ and guess that $\text{MAC}_k(x) = y$.

Assuming that MAC_k is random function, the probability of success is $1/2^n$. Note that the adversary cannot check her guess directly because the adversary does not know the key k. MAC guessing is infeasible if $n \geq 128$.

Exhaustive search on the key space:

Given r known message-tag paris $(x_1, t_1), \ldots, (x_r, t_r)$, one can check whether a guess k of the key is correct by verifying that $\text{MAC}_k(x_i) = t_i$, for $i = 1, 2, \ldots, r$.

Assume that MAC_k's are random functions, the expected number of keys for which the tags verify is

$$1 + FK = 1 + (2^\ell - 1)/2^{nr}$$

For example, if $\ell = 128$, $n = 128$, $r = 2$, then $FK \approx 1/2^{128}$.

Expected number of steps $\approx 2^\ell$. Exhaustive search is infeasible if $\ell \geq 128$.

4.3 MACs Based on Block Ciphers

CBC-MAC

Let E be an n-bit block cipher with key space $\{0,1\}^\ell$.

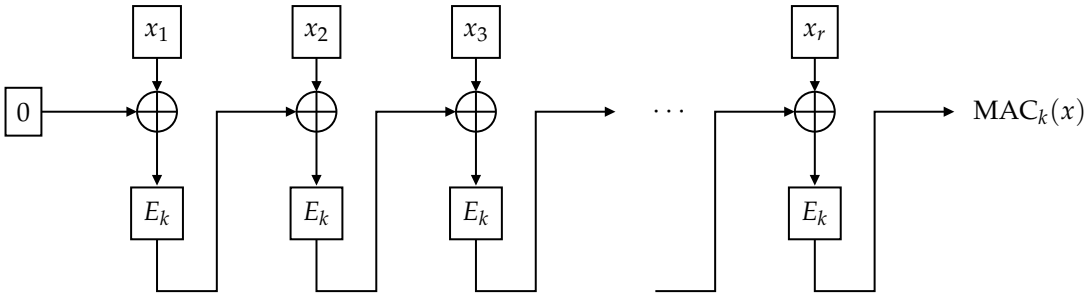
Assumption: Suppose that plaintext messages all have lengths that are multiplies of n, if not then we would add some padding to the message.

To compute $\text{MAC}_k(x)$:

1. Divide x into n-bit blocks $x_1, x_2, \ldots, x_r$.
2. Compute $H_1 = E_k(x_1)$.
3. For $2 \leq i \leq r$, compute $H_i = E_k(H_{i-1} \oplus x_i)$.
4. Then $\text{MAC}_k(x) = H_r$.

Algorithm 12: CBC-MAC

Divide x into n-bit blocks $x_1, x_2, \ldots, x_r$.
Compute $H_1 = E_k(x_1)$.
for $i = 2, \ldots, r$ **do**
    Compute $H_i = E_k(H_{i-1} \oplus x_i)$.
$\text{MAC}_k(x) := H_r$



4.3.1 Security of CBC-MAC

CBC MAC comes with a rigorous security analysis from 1994 [Bellare, Kilian & Rogaway 1994]. Here's an informal statement of a Theorem:

Informal statement of a Theorem

Suppose that E is an "ideal" encryption scheme. (That is, for each $k \in \{0,1\}^\ell$, $E_k : \{0,1\}^n \to \{0,1\}^n$ is a 'random' permutation.) Then CBC-MAC *with fixed-length inputs* is a secure MAC scheme.

CBC-MAC (as described above without additional measures) is not secure if *variable length* messages are allowed. Here is a *chosen-message attack* on CBC-MAC:

1. Select an arbitrary n-bit block x_1.
2. Obtain the tag t_1 of the one-block message x_1 (so $t_1 = E_k(x_1)$).
3. Obtain the tag t_2 of the one-block message t_1 (so $t_2 = E_k(t_1)$).
4. Then t_2 is the tag of the 2-block message $(x_1, 0)$ since $t_2 = E_k(0 \oplus E_k(x_1)) = E_k(E_k(x_1)) = E_k(t_1)$.

This attack might not be very realistic, but nonetheless it illustrates that CBC MAC does not meet our very strong notion of security, namely existential unforgeability against chosen message attack.

4.3.2 Encrypted CBC-MAC (EMAC)

One countermeasure for variable-length messages is **Encrypted CBC-MAC**:

Encrypt the last block under a second key s: $\text{EMAC}_{k,s}(x) = E_s(H_r)$, where $H_r = \text{CBC-MAC}_k(x)$.

EMAC has a rigorous security analysis from the year 2000. [Petrank & Rackoff 2000]:

Informal statement of a Theorem

Suppose that E is an "ideal" encryption scheme. Then EMAC is a secure MAC scheme (for inputs of any length).

4.4 MACs Based on Hash Functions

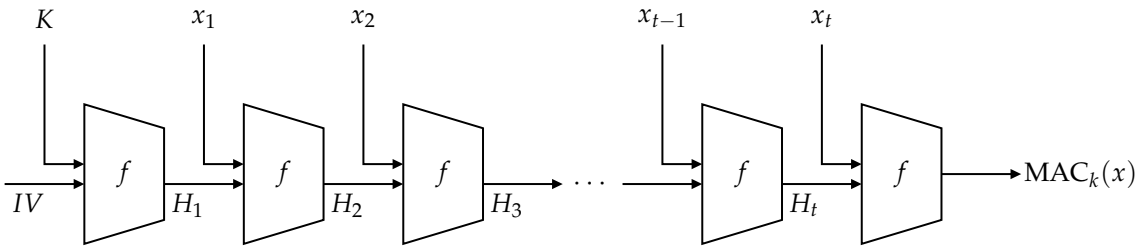
Hash functions were not originally designed for message authentication; in particular they are not "keyed" primitives.

Question How to use them to construct secure MACs?

Let H be an iterated n-bit hash function (without the length-block). Let $n + r$ be the input blocklength of the compression function $f : \{0,1\}^{n+r} \to \{0,1\}^n$. For example, for SHA-256, $n = 256, r = 512$. Let $k \in \{0,1\}^n$. Let K denote k padded with $(r - n)$ 0's. So K has bitlength r. Thus $K = \underbrace{\boxed{\leftarrow 0 \rightarrow}}_{r-n}\underbrace{\boxed{\leftarrow k \rightarrow}}_{n}$.

4.4.1 Secret Prefix Method

MAC definition: $\text{MAC}_k(x) = H(K, x)$.



This is *insecure*. Here is a *length extension attack*: suppose $(x, \text{MAC}_k(x))$ is known. Suppose the bitlength of x is a multiple of r. Then $\text{MAC}_k(x \parallel y)$ can be computed for any y (without knowledge of k).

Also insecure if a length block is postpended to $K \parallel x$ prior to application of H.

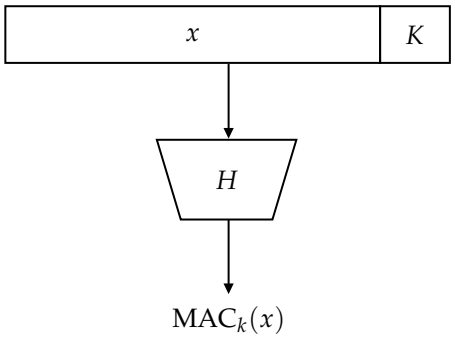
4.4.2 Secret Suffix Method

MAC definition: $\text{MAC}_k(x) = H(x, K)$.

The attack on the secret prefix method does not work here.

Suppose that a collision (x_1, x_2) can be found for H (i.e., $H(x_1) = H(x_2)$). We assume that x_1 and x_2 both have bitlengths that are multiples of r.

Thus $H(x_1, K) = H(x_2, K)$, and so $\text{MAC}_k(x_1) = \text{MAC}_k(x_2)$. Then the MAC for x_1 can be requested, giving the MAC for x_2. Hence if H is not *CR*, then the secret suffix method MAC is *insecure*.

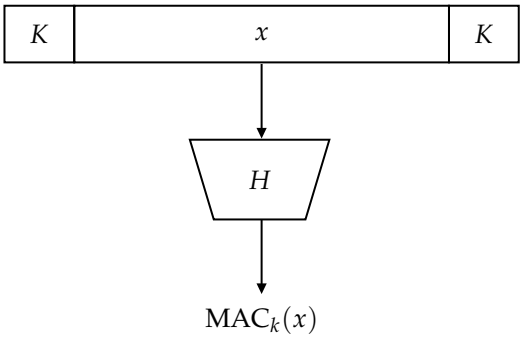


4.4.3 Envelope Method

MAC definition: $\text{MAC}_k(x) = H(K, x, K)$.

The MAC key is used both at the start and end of the MAC computation. Thus attacks on the secret prefix method and the secret suffix method do not work here because the key block is appended to the left and to the right of the message x.

The envelope method appears to be secure (i.e., no serious attacks have been found).



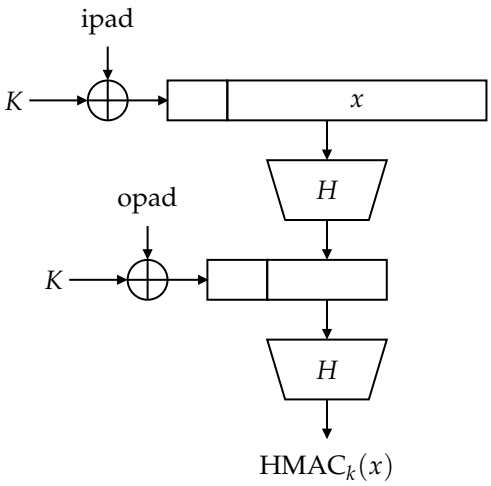
4.4.4 HMAC

MAC definition: $\text{HMAC}_k(x) = H(K \oplus \text{opad}, H(K \oplus \text{ipad}, x))$.

The most commonly used MAC scheme today is HMAC. It is "Hash-based" MAC and was designed by Bellare, Canetti & Krawczyk (1996).

Define two r-bit strings (in hexadecimal notation): ipad = `0x36`, opad = `0x5C`; each repeated $r/8$ times. These bytes `0x36` and `0x5C` are arbitrarily chosen. The important thing is that they be fixed and different from each other.

The main part of the HMAC algorithm is the secret prefix method, which is highly insecure. However since the key block is appended to the left of the resulting hash value and then this two-block message is hashed once more, the length extension attack is prevented.



HMAC comes with a rigorous security analysis.

Informal statement of a Theorem

Suppose that the compression function f used in H is a secure MAC with fixed length messages and a secret IV as the key. Then HMAC is a secure MAC algorithm.

In practice, one should use HMAC with the SHA-256 hash function. HMAC is specified in IETF[1] RFC 2104 and FIPS 198. HMAC is used in IPsec (Internet Protocol Security) and TLS.

Key Derivation Function

HMAC is commonly used as a *key derivation function* (KDF).

Suppose that Alice has a secret key k, and wishes to derive several *session keys* (e.g., to encrypt data in different communication sessions).

Alice computes $sk_1 = \text{HMAC}_k(1), sk_2 = \text{HMAC}_k(2), sk_3 = \text{HMAC}_k(3), \ldots$

Rationale: Without knowledge of k, an adversary is unable to learn anything about any particular session key sk_j , even though it may have learnt some other session keys.

This works because HMAC is existentially unforgeable against chosen-message attack.

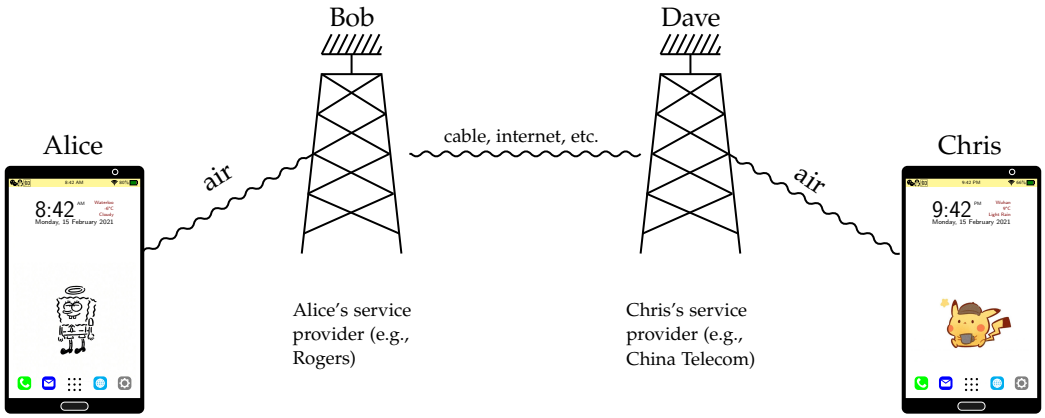
4.5 Case study: GSM

Global standards for mobile communications:

- 2G, 2.5G: GSM (Global System for Mobile Communication)
- 3G: UMTS (Universal Mobile Telecommunications System)
- 4G: LTE (Long Term Evolution)
- 5G: in the process of gradually moving on to 5G...

We will sketch the basic security mechanisms in GSM. GSM security is notable since it uses only symmetric-key primitives. UMTS and LTE security improves upon GSM security in several ways, but will not be discussed here.

Here is the basic GSM setup[2].



The objective of GSM security is to secure the communications between Alice and Bob, and between Dave and Chris, i.e., between a cell phone and its nearest base station. GSM does not provide security

[1] Internet Engineering Task Force
[2] phone drawings are from https://tex.stackexchange.com/a/479893

for the communications between base stations – this communications could be secured using a variety of means depending on the service provider, location, country, and so on. And it's even possible that no security is used at all.

4.5.1 GSM Security

Cryptographic ingredients:

- Enc: A symmetric-key encryption scheme.
- MAC: A symmetric-key MAC scheme.
- KDF: A key derivation function.

How do Alice and Bob agree upon a shared secret key? The GSM solution is to use *SIM cards*. A SIM card manufacturer randomly selects a secret key k, and installs it in a SIM card. A copy of k is given to the cell phone service provider. When a user purchases cell phone service, she gets the SIM card which she installs in her phone. Note that A different key k is chosen for each user.

The security objectives of GSM are

1. *Entity authentication*: Cell phone service provider needs to be assured that entities accessing its service are legitimate subscribers.
2. *Confidentiality*: Users need the assurance that their cell phone communications are private.

Here is the basic GSM security protocol. Alice: cell phone user, Bob: cell phone service provider.

Algorithm 13: Basic GSM security protocol

1 Alice sends an authentication request to Bob.
2 Bob selects a challenge $r \in_R \{0,1\}^{128}$, and sends r to Alice.
3 Alice's SIM card uses k to compute the response $t = \text{MAC}_k(r)$. Alice sends t to Bob.
4 Bob retrieves Alice's key k from its database, and verifies that $t = \text{MAC}_k(r)$.
5 Alice and Bob compute an encryption key $K_E = \text{KDF}_k(r)$, and thereafter use the encryption algorithm Enc_{K_E} to encrypt and decrypt messages for each other for the remainder of the session.

One drawback with using only symmetric-key crypto is that the SIM card manufacturer and the cell phone service providers have to securely maintain a large database of SIM keys k.

In 2015, the Snowden leaks revealed that NSA and GCHQ had stolen SIM keys from Gemalto, which manufactures about 2 billion SIM cards each year. See http://tinyurl.com/NSASIM

5

Authentic Encryption

A symmetric-key encryption scheme E provides confidentiality, e.g., $E = \text{AES}$, but not authentication. On the other hand, a MAC scheme provides authentication (data origin authentication and data integrity), e.g., MAC = HMAC, but not confidentiality. What if confidentiality and authentication are *both* required?

First Method: Encrypt-and-MAC

- Alice sends $(c, t) = (E_{k_1}(m), \text{MAC}_{k_2}(m))$ to Bob, where m is the plaintext and k_1, k_2 are secret keys she shares with Bob.
- Bob decrypts c to obtain $m = E_{k_1}^{-1}(c)$ and then verifies that $t = \text{MAC}_{k_2}(m)$.

Intuitively, this provides authenticated encryption because the ciphertext hides the plaintext and the tag authenticates the plaintext. However, this generic method might have some security vulnerabilities: it isn't clear that the tag also hides the plaintext completely. Indeed, it's possible that the tag of a message does leak some bits about the plaintext. This is because MAC schemes were not designed for confidentiality, but rather for authentication.

Second Method: Encrypt-then-MAC

- Alice sends $(c, t) = (E_{k_1}(m), \text{MAC}_{k_2}(E_{k_1}(m)))$ to Bob, where m is the plaintext and k_1, k_2 are secret keys she shares with Bob.
- Bob first verifies that $t = \text{MAC}_{k_2}(c)$ and then decrypts c to obtain $m = E_{k_1}^{-1}(c)$.

This method has been deemed to be secure, provided of course that the encryption scheme E and the MAC scheme employed are secure.

Special-Purpose AE Schemes

Many specialized authenticated encryption schemes have been developed, the most popular of these being **Galois/Counter Mode (GCM)**.

These modes can be faster than generic Encrypt-then-MAC, and also allow for the authentication (but not encryption) of "header" data. This is a useful feature because in many communications protocols the data to be transmitted is preceded by some header data which is specified by the communications protocol. This header data should not be encrypted, but it's usually desirable to protect its authenticity.

5.1 AES-GCM

This is an authenticated encryption scheme proposed by David McGrew and John Viega in 2004. It was adopted as a NIST standard in 2007. AES-GCM eses the CTR mode of encryption and a custom-designed MAC scheme.

5.1.1 CTR: CounTeR Mode of Encryption

Let $k \in_R \{0,1\}^{128}$ be the secret key.

Let $M = (M_1, M_2, \ldots, M_u)$ be a plaintext message, where each M_i is a 128-bit block, $u \leq 2^{32} - 2$.

To encrypt M, Alice does the following:

Algorithm 14: CTR: Encryption

Select $IV \in_R \{0,1\}^{96}$.
$J_0 := IV \parallel 0^{31} \parallel 1$.
for $i = 1 \ldots u$ **do**
    $J_i \leftarrow J_{i-1} + 1$ `// increment the counter`
    Compute $C_i = \text{AES}_k(J_i) \oplus M_i$.
Send $(IV, C_1, C_2, \ldots, C_u)$ to Bob.

So just like ChaCha20, the counter mode of encryption uses an IV and a counter. This prevents against the accidental reuse of keystream.

To decrypt, Bob does the following:

Algorithm 15: CTR: Decryption

$J_0 := IV \parallel 0^{31} \parallel 1$.
for $i = 1 \ldots u$ **do**
    $J_i \leftarrow J_{i-1} + 1$ `// increment the counter`
    Compute $M_i = \text{AES}_k(J_i) \oplus C_i$.

Note:

1. CTR mode of encryption can be viewed as a stream cipher.
2. As was the case with CBC encryption, identical plaintexts with different IVs result in different ciphertexts.
3. It is critical that the IV should not be repeated; this can be difficult to achieve in practice.
4. Unlike CBC encryption, CTR encryption is parallelizable.
5. Note that AES^{-1} is not used.

Multiplying Blocks

Let $a = a_0 a_1 a_2 \ldots a_{127}$ be a 128-bit block.

We associate the binary polynomial $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{127} x^{127} \in \mathbb{Z}_2[x]$ with a.

Let $f(x) = 1 + x + x^2 + x^7 + x^{128}$.

If a and b are 128-bit blocks then define $c = a \bullet b$ to be the block corresponding to the polynomial $c(x) = a(x) \cdot b(x) \mod f(x)$ in $\mathbb{Z}_2[x]$. In other words, that is, $c(x)$ is the remainder upon dividing

$a(x) \cdot b(x)$ by $f(x)$, where coefficient arithmetic is performed modulo 2. This is multiplication in the Galois Field $GF(2^{128})$.

> Example: 6 bit-blocks, $GF(2^6)$
>
> Let $f(x) = 1 + x + x^6$. Let $a = 001101$ and $b = 100111$.
>
> Then $a(x) = x^2 + x^3 + x^5$, $b(x) = 1 + x^3 + x^4 + x^5$. Thus, $c(x) = x^2 + x^5$ and $c = a \bullet b = 001001$.

Now let's describe AES-GCM authenticated encryption.

Input:

- Data to be authenticated (but not encrypted) $A = (A_1, A_2, \ldots, A_v)$. This might be some header data.
- Data to be encrypted and authenticated: $M = (M_1, M_2, \ldots, M_u)$.
- Secret key $k \in_R \{0,1\}^{128}$.

Output: (IV, A, C, t), where

- IV is a 96-bit initialization vector.
- $A = (A_1, A_2, \ldots, A_v)$ is authenticated data.
- $C = (C_1, C_2, \ldots, C_u)$ is the encrypted/authenticated data.
- t is a 128-bit authentication tag.

5.1.2 AES-GCM Encryption, Decryption/Authentication Procedure

Alice does the following:

Algorithm 16: AES-GCM encryption/authentication

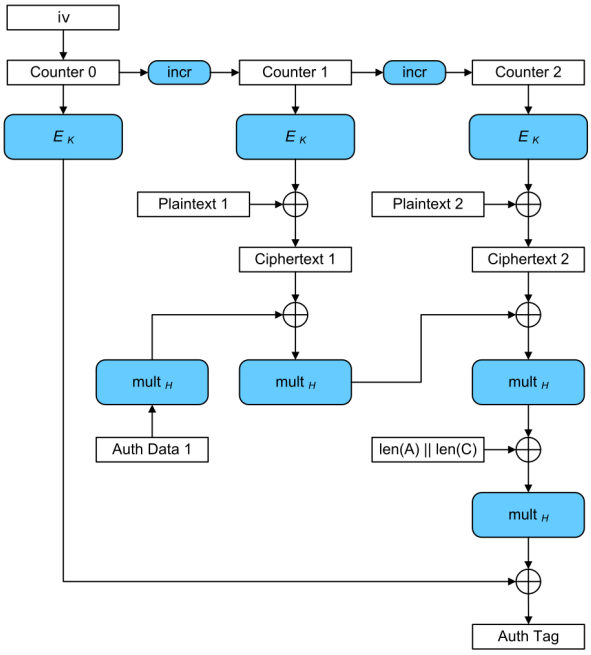
1. $L := L_A \parallel L_M$, where L_A, L_M are the bitlengths of A, M expressed as 64-bit integers. (L is the length block.)
2. Select $IV \in_R \{0,1\}^{96}$ and let $J_0 = IV \parallel 0^{31} \parallel 1$.

   `// Encryption`
3. **for** $i = 1 \ldots u$ **do**
4. $\quad J_i \leftarrow J_{i-1} + 1$
5. $\quad C_i \leftarrow \text{AES}_k(J_i) \oplus M_i$

   `// Authentication`
6. $T := 0^{128}$
7. $H := \text{AES}_k(0^{128})$.
8. **for** $i = 1 \ldots v$ **do**
9. $\quad T \leftarrow (T \oplus A_i) \bullet H$.
10. **for** $i = 1 \ldots u$ **do**
11. $\quad T \leftarrow (T \oplus C_i) \bullet H$.
12. $T \leftarrow (T \oplus L) \bullet H$.
13. $t := \text{AES}_k(J_0) \oplus T$.
14. **Output**: (IV, A, C, t).

> **Note:**
> A secret key should be used to encrypt at most 2^{32} messages in order to minimize the possibility of keystream reusage.

Here is a depiction of AES-GCM when the authentication data A is one-block long ($v = 1$) and the plaintext data M is two-blocks long ($u = 2$). Picture from https://commons.wikimedia.org/wiki/File:GCM-Galois_Counter_Mode_with_IV.svg



Upon receiving (IV, A, C, t), Bob does the following:

Algorithm 17: AES-GCM decryption/authentication

$L := L_A \parallel L_C$, where L_A, L_C are the bitlengths of A, C expressed as 64-bit integers.
// Authentication
$T := 0^{128}$
$H := \text{AES}_k(0^{128})$.
for $i = 1 \dots v$ **do**
  $T \leftarrow (T + \oplus A_i) \bullet H$.
for $i = 1 \dots u$ **do**
  $T \leftarrow (T \oplus C_i) \bullet H$.
$T \leftarrow (T \oplus L) \bullet H$.
$t' := \text{AES}_k(J_0) \oplus T$.
if $t' = t$ **then**
  Proceed to decryption;
else
  Reject.
// Decryption
$J_0 := IV \parallel 0^{31} \parallel 1$.
for $i = 1 \dots u$ **do**
  $J_i \leftarrow J_{i-1} + 1$
  $M_i \leftarrow \text{AES}_k(J_i) \oplus C_i$
Accept and output (A, M).

5.1.3 Insights into Authentication Mechanism

The purpose of GCM authentication is to mix up the bits of the secret key k, the bits of the message A, the bits of the ciphertext C, and the bits of the length block L, and the IV which is contained in the counter block J_0, to produce the authentication tag t. This is done by these four lines of codes.

- $T \leftarrow 0^{128}$, $H \leftarrow \text{AES}_k(0^{128})$
- For i from 1 to v do: $T \leftarrow (T + A_i)H$

$$\begin{aligned} T &= ((((0 + A_1)H + A_2)H + A_3)H + \cdots + A_v)H \\ &= A_1H^v + A_2H^{v-1} + A_3H^{v-2} + \cdots + A_{v-1}H^2 + A_vH \end{aligned}$$

- For i from 1 to u do: $T \leftarrow (T + C_i)H$.
- $T \leftarrow (T + L)H$

$$T = A_1H^{u+v+1} + A_2H^{u+v} + \cdots + A_vH^{u+2} + C_1H^{u+1} + \cdots + C_uH^2 + LH = f_{A,M}(H),$$

where

$$f_{A,M}(x) = A_1x^{u+v+1} + \cdots + A_vx^{u+2} + C_1x^{u+1} + \cdots + C_ux^2 + Lx \in GF(2^{128})[x]$$

- Hence, $t = \text{AES}_k(J_0) \oplus f_{A,M}(H)$.

To justify the security of the AES-GCM authentication mechanism, we'll consider the case where AES-GCM is used for authentication only.

Consider AES-GCM with no M, so $u = 0, L_M = 0$. The message to be authenticated is $A = (A_1, A_2, \ldots, A_v)$, where $v \leq \ell$. The tag is (IV, t), where $IV \in_R \{0,1\}^{96}$ and $t = \text{AES}_k(J_0) + f_A(H)$. Again, $J_0 = IV \parallel 0^{31} \parallel 1, H = \text{AES}_k(0)$.

Attack goal: this is the one in the security definition for MAC schemes. Eve has message-tag pairs (for messages of her choosing): (A^j, IV^j, t^j), $1 \leq j \leq r$. Her goal is to produce a message-tag forgery (A^*, IV^*, t^*), where $A^* \notin \{A^1, A^2, \ldots, A^r\}$.

We can assume that no two IV's in this list are the same. This is because the IV is produced by the MACing oracle are 96-bit strings and were randomly generated. We can also assume that Eve does not know k or H.

Security Argument (Informal)

Now, suppose that Eve outputs a forgery (A^*, IV^*, t^*). The security argument is divided into two cases.

1. If $IV^* \notin \{IV^1, \ldots, IV^r\}$, then $J_0^* \notin \{J_0^1, \ldots, J_0^r\}$, and so Eve doesn't know $\text{AES}_k(J_0^*)$, which serve as a one-time pad for $f_{A^*}(H)$. Thus, the probability that Eve can output a valid tag (i.e., $t^* = \text{AES}_k(J_0^*) + f_{A^*}(H)$) is only $1/2^{128}$.
2. Suppose that $IV^* = IV^j$, for some $1 \leq j \leq r$, so $J_0^* = J_0^j$. Then

$$t^* - t^j = \text{AES}_k(J_0^*) + f_{A^*}(H) - \text{AES}_k(J_0^j) - f_{A^j}(H) = f_{A^*}(H) - f_{A^j}(H)$$

So, Eve has produced A^*, A^j and α such that $\alpha = f_{A^*}(H) - f_{A^j}(H)$, without knowledge of H. But this can only be done with negligible probability, as the following lemma shows.

Lemma

For all distinct $A, B \in \left(\{0,1\}^{128}\right)^{\leq \ell}$ and $\alpha \in \{0,1\}^{128}$,

$$\Pr[f_A(H) - f_B(H) = \alpha] \leq (\ell + 1)/2^{128}$$

(which is negligible), where the probability is assessed over random choices of $H \in \{0,1\}^{128}$.

Proof:

Let $A, B \in \left(\{0,1\}^{128}\right)^{\leq \ell}$ with $A \neq B$, and let $\alpha \in \{0,1\}^{128}$. Suppose $A \in \{0,1\}^{128r}$ and $B \in \{0,1\}^{128w}$. Then

$$\begin{aligned} f_A(H) - f_B(H) - \alpha = (A_1 H^{v+1} + A_2 H^v + \cdots + A_v H^2 + L_A H) \\ - (B_1 H^{w+1} + B_2 H^w + \cdots + B_w H^2 + L_B H) - \alpha, \end{aligned}$$

which is a polynomial in H of degree $\leq \max(v, w) + 1 \leq \ell + 1$.

Since a nonzero polynomial of degree $\leq \ell + 1$ can have at most $\ell + 1$ roots, there are at most $\ell + 1$ $H \in \{0,1\}^{128}$ satisfying $f_A(H) - f_B(H) - \alpha = 0$. □

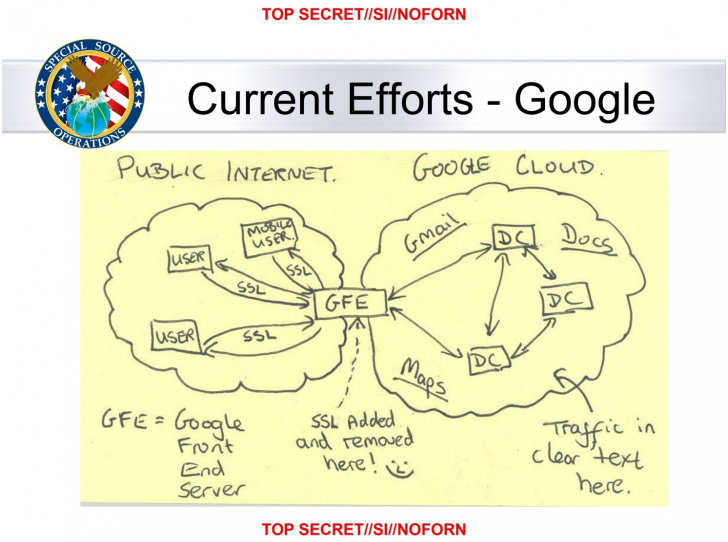
5.1.4 Some Features of AES-GCM

1. Performs authentication and encryption.
2. Supports authentication only (by using empty M).
3. Very fast implementations on Intel and AMD processors because of special AES-NI and PCLMULQDQ instructions for the AES and • operations.
4. Encryption and decryption can be parallelized.
5. AES-GCM can be used in streaming mode[1].
6. Security is justified by a security proof: Original McGrew-Viega security proof (2004) was wrong. The proof was fixed in 2012 by Iwata-Ohashi-Minematsu.

AES-GCM is widely used today.

[1]Streaming mode means that both the encryption and the authentication operations can be performed on the blocks of m as they are received, one at a time. This is useful in streaming applications.

5.2 Google Encryption



This is from NSA classified documents that were leaked by Snowden in 2013. On the left we see a bunch of users on the internet accessing Google's services such as gmail and search, and they do this by communicating with a Google front-end server GFE. The communications between the computers and Google's front-end servers are protected using TLS, also known as SSL. The data transmitted by the users are decrypted by the Google front-end servers, and then transmitted through Google's private network to its data centers situated around the world. Communications between the front-end servers and data centers and between data centers is through these private links. the slide had two comments, one noting that SSL is added and removed here, and the other is that traffic in the clear text is transmitted in the cloud.

GCHQ/NSA MUSCULAR program

Disclosed by Edward Snowden on October 30, 2013. Surveillance program conducted by GCHQ (British spy agency) in partnership with NSA. An unnamed telecommunications operator provided GCHQ with secret access to its fibre optic cables that transported data between Google and Yahoo! data centres. Millions of records collected each day, so that they couldn't possibly store all of it, so they had to process and discard a lot of it on the fly. Google was not encrypting user data when it transported it between data centers. In November 2013, Google and Yahoo! announced they were encrypting all traffic between their data centres.

5.2.1 Google Data

Google has 21 data centres around the world. A data centre (DC) contains tens of thousands of servers. It has lots of physical security (cameras, biometric identification, metal detectors, vehicle barriers, etc.) Communication between these servers and the outside world is all done via Google Front End (GFE) servers. Servers within a data centre communicate via a LAN (Local Area Network). Servers in different data centres communicate via a WLAN (Wide Local Area Network).

Broadly speaking, there are three kinds of data to protect:

1. Data communicated between individual users (browsers) and Google (GFEs).
    - Data is encrypted and authenticated using TLS.
    - TLS uses symmetric-key enc. (e.g., AES), symmetric-key authentication (e.g., HMAC); au-

thenticated enc. (e.g., AES-GCM); key establishment (e.g., RSA public-key enc., ECDH); public key certificates (RSA signatures).

- TLS is used by all web servers and browsers (not Google specific).

2. Data communicated between Google servers (perhaps in different data centres).

   Data is secured using Google's version of TLS (Application Layer Transport Security, ALTS). ALTS handles roughly 10^{10} remote procedure calls (RPCs) per second.

3. Data stored at data centres.

In the remainder of this section, we'll just discuss Google's methods for storing data at its data centers.

5.2.2 Key Management Service (KMS)

All data stored within Google data centres is encrypted with AES256-GCM (GCM = Galois Counter Mode). AES128-CTR + HMAC-SHA256 (Encrypt-then-MAC) is used in some legacy applications. However, Google is in the process of moving the encryption for these legacy applications to AES-256 GCM.

What secret keys does Google use to encrypt all this data? At one extreme, Google could use one secret key to encrypt all the data in its backend. However, this is clearly risky when the adversary gets his hands on this secret key. Google's solution is at the other extreme: a unique session key to encrypt each piece of data. This raised a lot of practical questions, which are addressed by Google's KMS.

The KMS manages the many AES secret keys that are used to encrypt/decrypt data by the many storage services within data centres. So the KMS has some very stringent requirements. Some of the KMS requirements:

- Availability: $>$ 99.9995% requests are served.
- Latency: 99% of requests are served in $<$ 10 ms.
- Scalability: Handle $\approx 10^7$ requests/second.
- Security: Effortless and foolproof key rotation (3 months).
- Efficiency: To minimize number of machines needed.

On January 24, 2014, a KMS configuration file was truncated by error. As a result, the KMS did not know the secret keys used to decrypt stored data. Gmail, Calendar, Docs, etc. crashed for 25+ minutes. "This got a lot of attention within Google." Subsequently, Google made many changes to its KMS. $\gg$ 99.9999% of KMS requests are served. 99.9% of requests are served in $< 200\mu$s.

5.2.3 Google's Key Hierarchy

1. Storage systems (millions of processes)

   Encrypts data with DEKs (Data Encryption Keys).

2. KMS (tens of thousands)

   Encrypts DEKs with KEKs (Key Encryption Keys).

3. Root KMS (hundreds)

   Encrypts KEKs with KMS Master Keys.

4. Root KMS Master Key Distributor (hundreds)

Encrypts KMS Master Keys with the Root KMS Master Key.

5. Physical safes (two)

   The Root KMS Master Key is backed up on hardware devices.

1. Storage Systems

Suppose that a storage system wishes to encrypt some data item m. The storage system does the following.

1. Break up m into chunks, $m_1, \ldots, m_\ell$.

   Each chunk can be up to several Gigabytes in size.

2. Generate ℓ Data Encryption Keys (DEKs), $k_1, \ldots, k_\ell$.

   Multiple sources of entropy are sampled (e.g., Intel's RDRAND instruction; inter-packet arrival times; measurement of disk seeks) The samples are then combined and hashed using a key derivation function (KDF) to produce a 256-bit secret key.

3. Encrypt with AES256-GCM: $c_1 = \text{AES}_{k_1}(m_1), \ldots, c_\ell = \text{AES}_{k_\ell}(m_\ell)$,

4. Send the DEKs $k_1, \ldots, k_\ell$ to a KMS.

   This transmission is protected with ALTS, Google's internal version of TLS.

5. Receive the wrapped (encrypted) keys $w_1, \ldots, w_\ell$ from the KMS.

   The KMS encrypts the DEKs with its Key Encryption Keys (KEKs).

6. Store $(c_1, w_1), \ldots, (c_\ell, w_\ell)$.

   These encrypted chunks are replicated and distributed across Google's storage systems.

When an application needs a piece of data, the stored system needs to decrypt the corresponding chunk, (c_j, w_j):

1. The storage system sends the wrapped key w_j to the KMS.
2. The KMS decrypts w_j using the appropriate KEK, and sends the DEK k_j to the storage system.
3. The storage system decrypts c_j using k_j .

Note:

1. Each data chunk has a unique identifier.
2. The KMS maintains an Access Control List (ACL) to ensure that a data chunk can only be decrypted by the authorized storage system.
3. Note that each chunk of data is encrypted using a different DEK. This ensures that if a DEK is compromised, then only one chunk of data is potentially compromised.
4. The storage system does not store the DEKs $k_1, \ldots, k_\ell$ but rather the wrapped data encryption keys.
5. If a chunk of data m_j is updated, it is re-encrypted with a new DEK k'_j rather than using the old DEK k_j. So each data encryption key is only used once.

2. KMS

Key Management Services (KMSs) generate the AES256-GCM Key Encryption Keys (KEKs), and maintain the Access Control Lists (one ACL list for each KEK). The KMS encrypts/decrypts DEKs using the KEKs, in accordance with the ACL. The KEKs never leave the KMS. The KMS also maintains an audit trail of when a KEK was used. KEKs are rotated (i.e., changed). The standard rotation frequency is once every 90 days.

3,4. Root KMS

The Root KMS wraps KEKs with AES256-GCM KMS Master Keys. There are about a dozen KMS Master Keys. The Root KMSs are run on dedicated secured machines in Google's data centres.

The KMS Master Keys are wrapped with the AES256-GCM Root KMS Master Key. The Root KMS Master Key is stored in RAM on the Root KMS machines. The Root KMS Master Key Distributor ensures that all Root KMSs always have the same version of the Root KMS Master Key.

5. Physical Safes

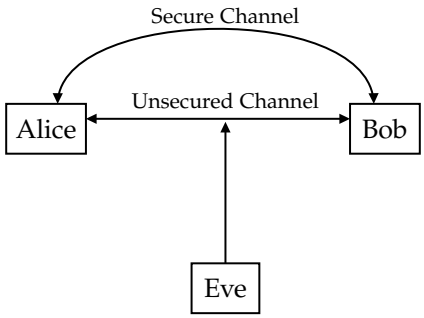
The Root KMS Master Key is backed up on two hardware devices stored in physical safes in highly secured areas in two physically separated Google locations. Fewer than 20 Google employees have access to these safes. The backups will be used if Google ever has to do a complete reboot.

6

Introduction to Public-Key Cryptography

6.1 Drawbacks with Symmetric-Key Cryptography

Symmetric-key cryptography: Communicating parties a priori share some secret keying information.

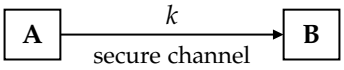


The shared secret keys can then be used to achieve confidentiality (e.g., using AES), or authentication (e.g., using HMAC), or both (e.g., using AES-GCM).

6.1.1 Key Establishment Problem

How do Alice and Bob establish the secret key k?

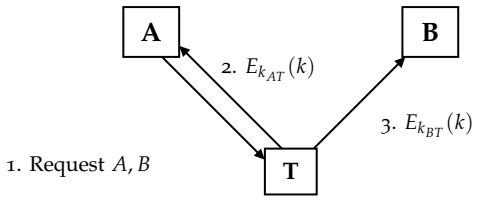
Method 1 Point-to-point key distribution. (Alice selects the key and sends it to Bob over a secure channel)



The secure channel could be: A trusted courier, a face-to-face meeting; installation of an authentication key in a SIM card. This is generally not practical for large-scale applications.

Method 2 Use a Trusted Third Party (TTP) T.

Each user A shares a secret key k_{AT} with T for a symmetric-key encryption scheme E. To establish this key, A must visit T once. T serves as a **key distribution centre** (KDC):



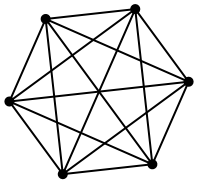
1. A sends T a request for a key to share with B.
2. T selects a session key k, and encrypts it for A using k_{AT}.
3. T encrypts k for B using k_{BT}.

Drawbacks of using a KDC:

1. The TTP must be unconditionally trusted.
2. The TTP is an attractive target.
3. The TTP must be on-line. Potential bottleneck. Critical reliability point.

6.1.2 Key Management Problem

In a network of n users, each user has to share a different key with every other user.



Each user thus has to store $n-1$ different secret keys. The total number of secret keys is $\binom{n}{2} \approx n^2/2$.

6.1.3 Non-Repudiation is Difficult to Achieve

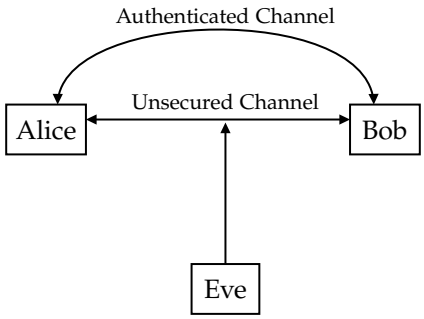
Recall non-repudiation is preventing an entity from denying previous actions or commitments. Denying being the source of a message.

Strictly speaking, symmetric-key techniques cannot be used to achieve non-repudiation. This is because Alice and Bob share a secret key, so Bob can do with the secret key whatever Alice can do with that secret key. If Alice uses the secret key to send Bob a message, perhaps say using HMAC, Bob knows that the message came from Alice but Bob could not prove to a judge that the message came from Alice because Bob could have generated the message and its authentication tag himself.

6.2 Public-Key Cryptography

public-key cryptography

Communicating parties a priori share some *authenticated* (but non-secret) information.



Invented by Ralph Merkle, Whitfield Diffie, Martin Hellman in 1975.

Excerpts from Merkle's CS 244 project proposal (Computer Security, UC Berkeley, Fall 1974)

> "Secure communications are made possible because of knowledge, known to both people, which is not known to anyone else. Usually, both people know this knowledge because they were able to hold a private conversation with each other before they began to send encrypted messages over an unsecure channel."
>
> "It might seem intuitively obvious that if two people have never had opportunity to prearrange an encryption method, then they will be unable to communicate securely over an insecure channel. While this might seem intuitively obvious, I believe it is false. I believe that it is possible for two people to communicate securely without having made any prior arrangements that are not completely public."

6.2.1 Merkle Puzzles

This method is meant to be proof-of-concept. It's not meant to be practical. The goal for Alice and Bob is to establish a secret session key by communicating over an authenticated (but non-secret) channel.

1. Alice creates N puzzles P_i, $1 \leq i \leq N$ (e.g., $N = 10^9$). Each puzzle takes t hours to solve (e.g., $t = 5$). The solution to P_i reveals a 128-bit session key sk_i and a randomly-selected 128-bit serial number n_i (which Alice selected and stored).
2. Alice sends $P_1, P_2, \ldots, P_N$ to Bob.
3. Bob selects j at random from $[1, N]$ and solves puzzle P_j to obtain sk_j and n_j.
4. Bob sends n_j to Alice.
5. The secret session key is sk_j.

An eavesdropper has to solve 500,000,000 puzzles on average to determine the puzzle index j (and thus sk_j).

Example:

$P_i = \text{AES-CBC}_{k_i}(sk_i, n_i, n_i)$, where $k_i = (r_i \parallel 0^{88})$ and r_i is a randomly selected 40-bit string.

P_i can be solved in 2^{40} steps by exhaustive key search.

Key Pair Generation for Public-Key Cryptography

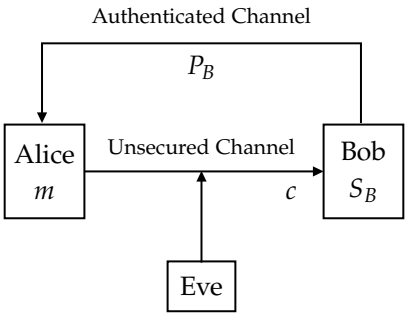
Each entity A does the following:

1. Generate a key pair (P_A, S_A).
2. S_A is A's secret key.
3. P_A is A's public key.

Security requirement: It should be infeasible for an adversary to recover S_A from P_A.

For example, $S_A = (p, q)$ where p and q are randomly-selected prime numbers; $P_A = p \cdot q$.

6.2.2 Public-Key Encryption



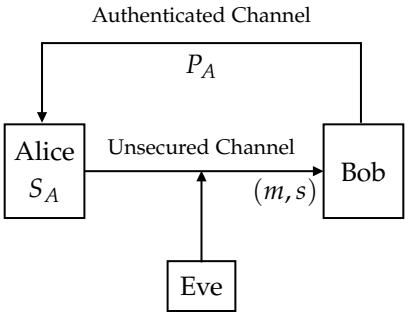
To encrypt a secret message m for Bob, Alice does:

1. Obtain an authenticated copy of Bob's public key P_B.
2. Compute $c = E(P_B, m)$; E is the encryption function.
3. Send c to Bob.

To decrypt c, Bob does:

1. Compute $m = D(S_B, c)$; D is the decryption function.

6.2.3 Digital Signatures



To sign a message m, Alice does:

1. Compute $s = \text{Sign}(S_A, m)$.
2. Send m and s to Bob.

To verify Alice's signature s on m, Bob does:

1. Obtain an authenticated copy of Alice's public key P_A.
2. Accept if Verify(P_A, m, s) = "Accept".

Suppose that Alice generates a signed message (m, s). Then *anyone* who has an authentic copy of Alice's public key P_A can verify the authenticity of the signed message. This authentication property cannot be achieved with a symmetric-key MAC scheme. Digital signatures are widely used to sign software updates which are then broadcast to computers around the world.

6.2.4 Hybrid Schemes

Advantages of public-key cryptography:

- No requirement for a secured channel.
- Each user has only 1 key pair, which simplifies key management.
- A signed message can be verified by anyone.
- Facilitates the provision of non-repudiation services (with digital signatures).

Disadvantages of public-key cryptography:

- Public-key schemes are slower than their symmetric-key counterparts.

Therefore, in practice, symmetric-key and public-key schemes are used together. Here is an example:

To encrypt a secret signed message m, Alice does:

Algorithm 18: Hybrid scheme encryption

1 $s \leftarrow \text{Sign}(S_A, m)$
2 Select a secret key k for a symmetric-key encryption scheme such as AES.
3 Obtain an authentic copy of Bob's public key P_B.
4 Send $c_1 = E(P_B, k)$ and $c_2 = \text{AES}_k(m, s)$.

To recover m and verify its authenticity, Bob does:

Algorithm 19: Hybrid scheme decryption

1 Decrypt $c_1 : k = D(S_B, c_1)$
2 Decrypt c_2 using k to obtain (m, s).
3 Obtain an authentic copy of Alice's public key P_A.
4 Check that Verify(P_A, m, s) = "Accept".

6.3 Algorithmic Number Theory

Fundamental Theorem of Arithmetic

Every integer $n \geq 2$ has a unique prime factorization (up to ordering of factors).

Note that this is an example of unique factorization domain. Check PMATH 347 for more details.

We then have some interesting questions:

- Given an integer $n \geq 2$, how do we find its prime factorization efficiently?
- How do we efficiently verify an alleged prime factorization of an integer $n \geq 2$?

- Given an integer $n \geq 2$, how do we efficiently decide whether n is prime or composite?

6.3.1 Complexity Theory Review

It's a review because the reader is assumed to be familiar with CS 341 and CS 466.

algorithm

An **algorithm** is a "well-defined computational procedure" (e.g., a Turing machine) that takes a variable input and eventually halts with some output.

For an integer factorization algorithm, the input is a positive integer n, and the output is the prime factorization of n.

efficiency

The **efficiency** of an algorithm is measured by the scarce resources it consumes (e.g. time, space, number of processors, number of chosen plaintext-ciphertext pairs).

input size

The **input size** is the number of bits required to write down the input using a reasonable encoding.

Example:

The size of a positive integer n is $\lfloor \log_2 n \rfloor + 1$ bits.

running time

The **running time** of an algorithm is an upper bound as a function of the input size, of the worst case number of basic steps the algorithm takes over all inputs of a fixed size.

polynomial-time

An algorithm is a **polynomial-time** (efficient) algorithm if its (expected) running time is $O(k^c)$, where c is a fixed positive integer, and k is the input size.

6.3.2 Basic Integer Operations

Input: Two k-bit positive integers a and b

Input size: $O(k)$ bits.

Operation		Running time of naive alg (in bit ops)
Addition:	$a + b$	$O(k)$
Subtraction:	$a - b$	$O(k)$
Multiplication:	$a \cdot b$	$O(k^2)$
Division:	$a = qb + r$	$O(k^2)$
GCD:	$\gcd(a, b)$	$O(k^2)$

GCD: Can be efficiently computed using the Euclidean Algorithm.

6.3.3 Basic Modular Operations

Input: A k-bit integer n, and integers $a, b, m \in [0, n-1]$.

Input size: $O(k)$ bits.

Operation		Running time of naive alg (in bit ops)
Addition:	$a+b \mod n$	$O(k)$
Subtraction:	$a-b \mod n$	$O(k)$
Multiplication:	$a \cdot b \mod n$	$O(k^2)$
Inversion:	$a^{-1} \mod n$	$O(k^2)$
Exponentiation:	$a^m \mod n$	$O(k^3)$

Note:
$x = a^{-1} \mod n$ means $ax \equiv 1 \mod n$ and $1 \leq x \leq n-1$. $a^{-1} \mod n$ exists iff $\gcd(a, n) = 1$.
$a^{-1} \mod n$ can be efficiently computed using the Extended Euclidean Algorithm.

Modular Exponentiation

Now let's take a closer look at modular exponentiation.

Input: A k-bit integer n, and integers $a, m \in [0, n-1]$.

Output: $a^m \mod n$.

We first have two naive algorithm:

Algorithm 20: Modular exponentiation naive algorithm 1

```
1 Compute d = a^m
2 return d mod n
```

Note the bitlength of d is (approximately) $\log_2 d = \log_2 a^m = m \cdot \log_2 a = O(2^k k)$ since $m \approx 2^k$. Hence the algorithm is not polytime.

Algorithm 21: Modular exponentiation naive algorithm 2

```
1 A ← a
2 for i ← 2 ... m do
3 |  A ← A × a mod n
4 return A
```

Let the binary representation of m be $m = \sum_{i=0}^{k-1} m_i 2^i$ where $m_i \in \{0, 1\}$. Then

$$a^m \equiv a^{\sum_{i=0}^{k-1} m_i 2^i} \equiv \prod_{i=0}^{k-1} a^{m_i 2^i} \equiv \prod_{\substack{0 \leq i \leq k-1 \\ m_i = 1}} a^{2^i} \pmod{n}$$

This suggests the following repeated square-and-multiply algorithm for computing $a^m \mod n$:

Algorithm 22: Modular exponentiation: repeated square-and-multiply

```
1  Write m in binary: m = Σ_{i=0}^{k-1} m_i 2^i.
2  if m_0 = 1 then
3  |  B ← a
4  else
5  |  B ← 1
6  A ← a for i ← 1 ... k − 1 do
7  |  A ← A^2 mod n
8  |  if m_i = 1 then
9  |  |  B ← B × A mod n
10 return B
```

Analysis: At most k modular squarings and k modular multiplications, so worst-case running time is $O(k^3)$ bit operations (polytime).

7

RSA

7.1 Basic RSA

Ron Rivest, Adi Shamir, Len Adleman

7.1.1 RSA Encryption

Each entity Alice does the following:

Algorithm 23: RSA Key Generation

1. Randomly select two large, distinct primes p, q of the same bitlength. `// This can be done efficiently.`
2. Compute $n = pq$ and $\phi = \phi(n) = (p-1)(q-1)$.
3. Select arbitrary large e, $1 < e < \phi$, with $\gcd(e, \phi) = 1$.
4. Compute the integer d, $1 < d < \phi$, with $ed \equiv 1 \mod \phi$.
5. Alice's public key is (n, e); her private key is d.

Note that n is called the *RSA modulus*; e is the *encryption exponent*; d is the *decryption exponent*.

To encrypt a message for Alice, Bob does:

Algorithm 24: RSA Encryption

1. Obtain an authentic copy of Alice's public key (n, e).
2. Represent the message as an integer $m \in [0, n-1]$.
3. Compute the ciphertext $c = m^e \mod n$.
4. Send c to Alice.

To decrypt c, Alice does

Algorithm 25: RSA Decryption

1. Compute $m = c^d \mod n$.

Example: Toy example on RSA Key Generation/Encryption

Alice does the following for key generation:

1. Select primes $p = 23, q = 37$.

2. Compute $n = pq = 851$ and $\phi(n) = (p-1)(q-1) = 792$.
3. Select $e = 631$ satisfying $\gcd(631, 792) = 1$.
4. Solves $631d \equiv 1 \mod 792$ to get $d \equiv -305 \equiv 487 \mod 792$, and selects $d = 487$.
5. Alice's public key is $(n = 851, e = 631)$; her private key is $d = 487$.

To encrypt a plaintext $m = 13$ for Alice, Bob does:

1. Obtains Alice's public key $(n = 851, e = 631)$.
2. Computes $c = 13^{631} \mod 851$ using the repeated-square-and-multiply algorithm:
   (a) Write $e = 631$ in binary: $e = 2^9 + 2^6 + 2^5 + 2^4 + 2^2 + 2^1 + 2^0$.
   (b) Compute successive squarings of $m = 13 \mod n$:

$$\begin{aligned} 13 &\equiv 13 \pmod{851} & 13^2 &\equiv 169 \pmod{851} \\ 13^2 &\equiv 478 \pmod{851} & 13^{2^3} &\equiv 416 \pmod{851} \\ 13^{2^4} &\equiv 303 \pmod{851} & 13^{2^5} &\equiv 752 \pmod{851} \\ 13^{2^6} &\equiv 440 \pmod{851} & 13^{2^7} &\equiv 423 \pmod{851} \\ 13^{2^8} &\equiv 219 \pmod{851} & 13^{2^9} &\equiv 305 \pmod{851}. \end{aligned}$$

   (c) Multiply together the squares 13^{2^i} for which the ith bit (where o i 9) of the binary representation of 631 is 1:

$$\begin{aligned} 13^{631} &= 13^{2^9+2^6+2^5+2^4+2^2+2^1+2^0} \\ &= 13^{2^9} \cdot 13^{2^6} \cdot 13^{2^5} \cdot 13^{2^4} \cdot 13^{2^2} \cdot 13^{2^1} \cdot 13^{2^0} \\ &\equiv 305 \cdot 440 \cdot 752 \cdot 303 \cdot 478 \cdot 169 \cdot 13 \pmod{851} \\ &\equiv 616 \pmod{851}. \end{aligned}$$

3. Bob sends the ciphertext $c = 616$ to Alice.

To decrypt $c = 616$, Alice uses her private key d = 487 as follows:

1. Compute $m = 616^{487} \mod 851$ to get $m = 13$.

Theorem (RSA works)

For all integers m, $m^{ed} \equiv m \mod n$.

Proof:
Since $ed \equiv 1 \mod \phi$, we can write $ed = 1 + k(p-1)(q-1)$ for some $k \in \mathbb{Z}$. Since $ed > 1$ and $p, q \geq 2$, we have $k \geq 1$. We will prove that $m^{ed} \equiv m \mod p$.

Suppose first that $p \mid m$. Then $m \equiv 0 \mod p$, so $m^{ed} \equiv 0 \mod p$. Hence $m^{ed} \equiv m \mod p$.

Suppose now that $p \nmid m$. Then $\gcd(p, m) = 1$, so $m^{p-1} \equiv 1 \mod p$ by Fermat's Little Theorem. Rasing both sides to the power $k(q-1)$ and then multiply by m gives $m^{k(p-1)(q-1)+1} \equiv m \mod p$. Hence $m^{ed} = m \mod p$.

Similarly, $m^{ed} \equiv m \mod q$. Since p, q are distinct primes, $\gcd(p, q) = 1$ and hence $m^{ed} \equiv m \mod pq$. □

7.1.2 Basic RSA Signature Scheme

Key Generation Same as for RSA encryption: Algorithm 23.

To sign $m \in \{0,1\}^*$, Alice does:

Algorithm 26: RSA Signature Generation

1 Compute $M = H(m)$, where H is a hash function.
2 Compute the signature $s = M^d \mod n$. // so $s^e \equiv M^{ed} \equiv M \mod n$
3 Alice's signed message is (m, s). // the signed message is (m, s)

To verify (m, s), Bob does:

Algorithm 27: RSA Signature Verification

1 Obtain an authentic copy of Alice's Public key (n, e).
2 Compute $M = H(m)$
3 Compute $M' = s^e \mod n$.
4 Accept (m, s) iff $M = M'$.

7.2 Case Study: QQ Browser

Background: The Snowden documents suggested that the NSA (and collaborators) were exploiting vulnerabilities in the UC Browser (a browser for mobile devices that is popular in China) to track users.

This prompted Knockel, Senft and Deibert (UToronto) in 2016 to study the security of browsers used in China.

They studied QQ Browser, a free web browser for Android, Windows, Mac, iOS, developed by Tencent.

QQ Browser is used by hundreds of millions of cell phone users in China.

7.2.1 Version 1

When a user launches the QQ Browser (the client) on Android, the browser makes a series of WUP requests to QQ Browser's server (the server).

Via a WUP request, the browser sends personal user data to the server. This data includes: QQ username, WiFi MAC address of client, MAC addresses of all nearby WiFi access points, URL of each page visited by the browser... (UC Browser and Baidu Browser also collect similar data.)

Of course, the personal user data needs to be protected as it is transmitted over the internet.

QQ Browser WUP Encryption

1. To encrypt a WUP request m, the client does the following:
   (a) Randomly generate a 128-bit AES session key k:

```
int i = 10000000 + new Random().nextInt(89999999);
int j = 10000000 + new Random().nextInt(89999999);
return (String.valueOf(i) + String.valueOf(j)).getBytes();
```

   (b) Encrypt k with the server's RSA public key (n, e): $c_1 = k^e \mod n$.

   Here $e = 65537$ and $n = 24540641757374088471004774586996502346 3$ (128 bits).

(c) Encrypt m: $c_2 = \text{AES-ECB}_k(m)$.

(d) Send (c_1, c_2) to the server.

2. The server does the following:

(a) Decrypt c_1 using its RSA private key d: $k = c_1^d \mod n$.

(b) Decrypt c_2: $m = \text{AES-ECB}_k^{-1}(c_2)$.

(c) Encrypt the WUP response m': $c' = \text{TEA-CBC}_{k'}(m')$, where $k' =$ `sDf434ol*123+-KD` (in ASCII). Send c'.

3. The client decrypts $m' = \text{TEA-CBC}_{k'}^{-1}(c')$ using the hard-coded 128-bit k'.

WUPS!

1. i and j are each randomly selected integers in the interval $[10000000, 99999998]$, so the total number of keys k is $\approx 2^{26.4} \times 2^{26.4} \approx 2^{52.8}$ (not 2^{128}).

2. The server's RSA public key is easily factored: $n = 14119218591450688427 \times 17381019776996486069$.

3. The response uses the fixed key k' that is hard-coded in all QQ browsers!

4. Don't use ECB mode!

Summary: The user data is protected using extremely weak crypto, and thus is very vulnerable to passive eavesdropping.

7.2.2 Version 2

1. To encrypt a WUP request m, the client does the following:

(a) Randomly generate a 128-bit AES session key k.

(b) Encrypt k with the server's RSA public key (n, e): $c_1 = k^e mod n$. Here, $e = 65537$ and n is a 1024-bit RSA modulus.

(c) Encrypt m: $c_2 = \text{AES-ECB}_k(m)$.

(d) Send (c_1, c_2) to the server.

2. The server does the following:

(a) Decrypt c_1 using its RSA private key d: $K = c_1^d \mod n$, and let k be the 128 least significant bits of K.

(b) Decrypt c_2: $m = \text{AES-ECB}_k^{-1}(c_2)$.

(c) If m is a properly formatted WUP request, then encrypt the WUP response m': $c' = \text{AES-ECB}_k(m')$ and send c'. If m is not properly formatted, then don't respond.

3. The client decrypts $m' = \text{AES-ECB}_k^{-1}(c')$.

In step 2(a), the integer K is represented as a 1024-bit number, of which the least significant 128-bits are taken to be k. However, the QQ server software did not check that the remaining $1024 - 128 = 896$ bits of K are all 0 (as they should be). This flaw can be exploited using a *restricted chosen-ciphertext attack*:

(a) The adversary intercepts a ciphertext $c = (c_1, c_2)$.

(b) She then sends to the QQ server a carefully-chosen modification $\hat{c} = (\hat{c}_1, \hat{c}_2)$ of c.

(c) Depending on whether the server responds or not, the adversary learns 1 bit of the secret key k.

(d) Steps (b) and (c) are repeated to obtain all the 128 bits of k.

The attack requires 128 interactions with the QQ server and very little computation. (Note that the RSA private key d is not computed.)

7.3 Security of RSA Encryption

Security of RSA Key Generation If an adversary can factor n, then she can compute d from (n, e). It has been proven that any efficient method for computing d from (n, e) is equivalent to factoring n.

Security of Basic RSA Encryption A basic notion of security is that is should be computationally infeasible to compute m from c. This is known as the RSA problem:

RSA Problem (RSAP)

Given an RSA public key (n, e) and $c = m^e \mod n$, where $m \in [0, n-1]$, compute m.

The only effective method known for solving RSAP is to factor n (and then compute d and m). Henceforth, we shall assume that RSAP is *intractable*.

Dictionary Attack Suppose that the plaintext is chosen from a relatively small (and known) set M of messages. Then, given a target ciphertext c, the adversary can encrypt each message in M until c is obtained.

Countermeasure: Append a randomly selected 128-bit string (called a *salt*) to m priory to encryption. $\boxed{salt|m}$ Note that m is now encrypted to one of 2^{128} possible ciphertexts.

Chosen-Ciphertext Attack Suppose the adversary E has a target ciphertext c intended for Alice. Suppose also that E can induce Alice to decrypt *any* ciphertext for her, *except for c itself* (we say that Eve has a *decryption oracle*). Then E can decrypt c as follows:

1. Select arbitrary $x \in [2, n-1]$ with $\gcd(x, n) = 1$.
2. Compute $\hat{c} = c \cdot x^e \mod n$, where (n, e) is Alice's public key.
   Note that $\hat{c} \neq c$, unless $\gcd(c, n) \neq 1$.
3. Obtain the decryption $\hat{m}$ of $\hat{c}$ from the decryption oracle.
   Note that $\hat{m} \equiv \hat{c}^d \equiv (cx^e)^d \equiv c^d x^{ed} \equiv m \cdot x \mod n$
4. Compute $m = \hat{m} \cdot x^{-1} \mod n$.

Countermeasure: Add some prescribed *formatting* to m prior to encryption. After decrypting the ciphertext c, if the plaintext is not properly formatted, then c is rejected (so the decryption oracle does not return a plaintext).

So, RSA encryption should incorporate **salting** and **formatting**.

secure public-key encryption scheme

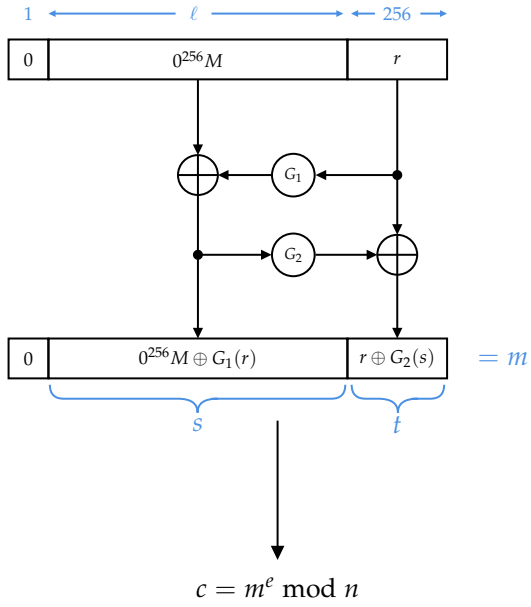
A public-key encryption scheme is **secure** if it is semantically secure against chose-ciphertext attack by a computationally bounded adversary.

To *break* a public-key encryption scheme, E should do:

1. E is given a *challenge ciphertext* c (and the public key (n, e)).
2. E has a decryption oracle, to which she can present any ciphertexts for decryption except for c itself.
3. After a feasible amount of computation, E should learn *something* about the plaintext m that corresponds to c (other than its length).

RSA Optimal Asymmetric Encryption Padding (OAEP)

Encryption:



- k = bitlength of n (e.g., 3072)
- $\ell = k - 256 - 1$
- $r \in_R \{0,1\}^{256}$ (salt)
- M plaintext ($\ell - 256$ bits)
- $G_1 : \{0,1\}^{256} \to \{0,1\}^{\ell}$
- $G_2 : \{0,1\}^{\ell} \to \{0,1\}^{256}$

G_1, G_2: *masking functions* built from $H = \text{SHA256}$.

e.g., $G_1(r) = H(0,r) \parallel H(1,r) \parallel H(2,r) \parallel \cdots$.

Decryption: To decrypt c, do the following:

1. Compute $m = c^d \mod n$.
2. Parse m: $\underbrace{\boxed{0}}_{1}\underbrace{\boxed{\leftarrow s \rightarrow}}_{\ell}\underbrace{\boxed{\leftarrow t \rightarrow}}_{256}$
3. Compute $r = G_2(s) \oplus t$
4. Compute $G_1(r) \oplus s = \underbrace{\boxed{\leftarrow a \rightarrow}}_{256}\underbrace{\boxed{\leftarrow b \rightarrow}}_{256}$
5. If $a = 0^{256}$, then output $M = b$; else reject c.

Theorem

Suppose that RSAP is intractable. Suppose that G_1, G_2 are random functions. Then RSA-OAEP is a secure public-key encryption scheme.

7.4 Integer Factorization

7.4.1 Review from complexity theory

Big-O notation, little-o notation.

polynomial-time algorithm

One whose worst-case running time function is of the form $O(n^c)$, where n is the input size and c is a constant.

exponential-time algorithm

One whose worst-case running time function is not of the form $O(n^c)$.

In this course, **fully** exponential-time functions are of the form 2^{cn}, where c is a constant, e.g., $O(2^{n/2})$.

subexponential-time algorithm

One whose worst-case running time function is of the form $2^{o(n)}$, and not of the form $O(n^c)$ for any constant c, e.g., $O(2^{\sqrt{n}})$.

Roughly speaking, "polynomial-time = efficient", "fully exponential-time = terribly inefficient", "subexponential-time = inefficient, but not terribly so".

Example: Trial Division

Consider the following algorithm (trial division) for factoring RSA-moduli n. Then trial divide n by the primes $2, 3, 5, 7, \ldots, \lfloor\sqrt{n}\rfloor$, If any of these, say ℓ, divides n, then stop and output the factor ℓ of n.

The running time of this method is at most $\sqrt{n}$ trial divisions, which is $O(\sqrt{n})$. Is this a polynomial-time algorithm for factoring RSA moduli?

Let A be an algorithm whose inputs are elements of the integers modulo n, $\mathbb{Z}_n$, or an integer n (so the input size is $O(n \log n)$). If the expected running time of A is of the form

$$L_n[\alpha, c] = O\left(\exp\left((c + o(1))\,(\log_e n)^{\alpha}\,(\log_e \log_e n)^{1-\alpha}\right)\right),$$

where c is a positive constant, and α is a constant satisfying $0 < \alpha < 1$, then A is a *subexponential-time* algorithm. Note that if $\alpha = 0$, $L_n[0, c] = O\left((\log n)^{c+o(1)}\right)$, which is *polytime*. If $\alpha = 1$, $L_n[1, c] = O\left(n^{c+o(1)}\right)$, which is *fully exponential time*.

7.4.2 Special-Purpose Factoring Algorithms

Examples: Trial division, Pollard's p-1 algorithm, Pollard's ρ algorithm, elliptic curve factoring algorithm, special number field sieve.

These are only efficient if the number n being factored has a *special form* (e.g., n has a prime factor p such that $p - 1$ has only small factors; or n has a prime factor p that is relatively small).

To maximize resistance to these factoring attacks on RSA moduli, one should select the RSA primes p and q *at random* and *of the same bitlength*.

7.4.3 General-Purpose Factoring Algorithms

These are factoring algorithms whose running times do not depend of any properties of the number being factored. There have been two major developments in the history of factoring:

1. (1982) Quadratic sieve factoring algorithm (QS): Running time: $L_n[\frac{1}{2}, 1]$.

2. (1990) Number field sieve factoring algorithm (NFS): Running time: $L_n[\frac{1}{3}, 1.923]$.

7.4.4 History of Factoring

Year	Number	Bits	Method	Notes
1903	$2^{67}-1$	67	Naive	F. Cole (3 years of Sundays). 0.02 secs in Maple (2020)
1988	$\approx 10^{100}$	332	QS	Distributed computation by 100's of computers; communication by email
1994	RSA-129	425	QS	1600 computers around the world; 8 months
1999	RSA-155	512	NFS	300 workstations + Cray; 5 months
2003	RSA-174	576	NFS	
2005	RSA-200	663	NFS	(55 years on a single workstation)
2009	RSA-768	768	NFS	2000 core years
2019	RSA-240	795	NFS	900 core years
2020	RSA-250	829	NFS	2700 core years

The RSA Factoring Challenge: https://en.wikipedia.org/wiki/RSA_Factoring_Challenge

The largest 'hard' number factored to date is **RSA-250** (250 decimal digits, 829 bits), which was factored on Feb 28 2020:

$$214032465024074496126442307283933356300861471514475501779775492088141802344714013664334551909580467961099285187247091458768739626192155736304745477052080511905649310668769159001975940569345745223058932597669747168173806936489469987157849497593749793 7$$
$$= 641352894770715802787901901705773890848250147429434472081168596320245323446302386235987526683477087376619255856946397988533367 \times 333720275949781565562260106053551142279407603447675546667845209870238417292100370802574486732968818775657189862580369320627 11$$

RSA-1024 factoring challenge (1024 bits, 309 decimal digits):

$$135066410865995223349603216278805969938881475605667027524485143851526510604859533833940287150571909441798207282164471551373680419703964191743046496589274256239341020864383202110372958725762385059643110564073501508187510676594629205563685329475213500852879416377328533906109750543439998111505059772368092756 3$$

Equivalent Security Levels

Security in bits	Block cipher	Hash function	RSA $\log_2 n$
80	SKIPJACK	(SHA-1)	1024
112	Triple-DES	SHA-224	2048
128	AES Small	SHA-256	3072
192	AES Medium	SHA-384	7680
256	AES Large	SHA-512	15360

Recall that a cryptographic scheme is said to have a security level of ℓ bits if the fastest known attack on the scheme takes approximately 2^ℓ operations.

Summary

Factoring is believed to be a hard problem. However, we have no proof or theoretical evidence that factoring is indeed hard. However, factoring is known to be easy on a quantum computer (Shor's algorithm). The largest number factored with Shor's algorithm is the number 21. The big open question is whether large-scale quantum computers can ever be built. 512-bit RSA is considered insecure today. 1024-bit RSA is considered risky today (but still deployed). Applications are moving to 2048-bit and 3072-bit RSA.

7.5 RSA Signature Scheme

Recall Algorithm 23 (RSA key generation), Algorithm 26 (RSA signature generation) and Algorithm 27 (RSA signature verification).

We require that RSAP be intractable, since otherwise E could forge A's signature as follows:

1. Select arbitrary m.
2. Compute $M = H(m)$.
3. Solve $s^e \equiv M \pmod{n}$ for s.
4. Then s is A's signature on m.

If H is not *preimage resistant*, and the range of H is $[0, n-1]$, E can forge signatures as follows:

1. Select $s_R \in [0, n-1]$.
2. Compute $M = s^e \mod n$.
3. Find m such that $H(m) = M$.
4. Then s is A's signature on m.

If H is not *2nd preimage resistant*, E could forge signatures as follows:

1. Suppose that (m, s) is a valid signed message.
2. Find an m', $m \neq m'$, such that $H(m) = H(m')$.
3. Then (m', s) is a valid signed message.

If H is not *collision resistant*, E could forge signatures as follows:

1. Select m_1, m_2 such that $H(m_1) = H(m_2)$, where m_1 and m_2 are two distinct messages.
2. Induce A to sign m_1: $s = H(m_1)^d \mod n$.
3. Then s is also A's signature on m_2.

7.5.1 Attack Model

Goals of the adversary:

1. *Total break*: E recovers A's private key, or a method for systematically forging A's signatures (i.e., E can compute A's signature for arbitrary messages).
2. *Existential forgery*: E forges A's signature for a single message; E may not have any control over the content or structure of this message.

Types of attacks E can launch:

1. *Key-only attack*: The only information E has is A's public key.
2. *Known-message attack*: E knows some message/signature pairs.
3. *Chosen-message attack*: E has access to a signing oracle which it can use to obtain A's signatures on some messages of its choosing.

7.5.2 Security of a Signature Scheme

security of a signature scheme

A signature scheme is said to be **secure** if it is existentially unforgeable by a computationally bounded adversary who launches a chosen-message attack.

Note: The adversary has access to a signing oracle. Its goal is to compute a single valid message/signature pair for any message that was not previously given to the signing oracle.

Is the basic RSA signature scheme secure? NO, if H is SHA-256; [Details not covered in this course.] YES if H is a 'full domain' hash function.

Full Domain Hash RSA (RSA-FDH)

Same as the basic RSA signature scheme, except that the hash function is $H : \{0,1\}^* \to [0, n-1]$. In practice, one could use: $H(m) = \text{SHA-256}(1, m) \parallel \text{SHA-256}(2, m) \parallel \cdots \text{SHA-256}(t, m)$.

Theorem (Bellare & Rogaway, 1996)

If RSAP is intractable and H is a random function, then RSA-FDH is a secure signature scheme.

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