Introduction to Real Analysis

PMATH 333

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Preface

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Sibelius Peng

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Axioms

Lecture 1 Axioms on the real number system

- $\mathbb{R}$: the set of real numbers
- $\mathbb{Z}$: the set of integers
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{N}$: the set of positive integers

The axioms fall into 3 groups

Group I (Addition and multiplication)

Any two real numbers x, y have a sum $x + y$ and a product $x \cdot y$, which are also real numbers. In addition, $+$ and $\cdot$ have the following properties:

(A1) $x + y = y + x$.

(A2) $(x + y) + z = x + (y + z)$.

(A3) There exists a real number, denoted 0, such that $x + 0 = x$ for all $x \in \mathbb{R}$.

(A4) For all $x \in \mathbb{R}$, there exists a real number, denoted $-x$, such that $x + (-x) = 0$.

(M1) $x \cdot y = y \cdot x$.

(M2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(M3) There exists a real number distinct from 0, denoted 1, such that $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

(M4) For all $x \in \mathbb{R} \setminus \{0\}$, there exists a real number, denoted x^{-1}, such that $x \cdot x^{-1} = 1$.

(D) $x \cdot (y + z) = x \cdot y + x \cdot z$.

Group II (Order)

There is a relation $<$ between real numbers such that

(O1) Given real numbers x, y, exactly one of the following holds: $x < y$ or $x = y4$ or $y < x$.

(O2) If $x < y$ and $y < z$, then $x < z$.

(O3) If $x < y$, then $x + z < y + z$ for all $z \in \mathbb{R}$.

(O4) If $x < y$, then $x \cdot z < y \cdot z$ for all $0 < z$.

Group III (Completeness)

Note that the following definitions will be defined later.

(C) Any non-empty subset of $\mathbb{R}$ which is bounded from above has a least upper bound.

Example 1.1:
$x \cdot 0 = 0$ for all $x \in \mathbb{R}$.

Example 1.2:
$x \cdot y = 0$ if and only if $x = 0$ or $y = 0$.

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Topology of $\mathbb{R}^n$

Lecture 2 The n-dimensional Euclidean space

Definition 2.1:

1. $\mathbb{R}^n = \{\vec{x} = (x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$. Given $\vec{x} \in \mathbb{R}^n$
2. For $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, define
$$\begin{aligned} \vec{x} + \vec{y} &= (x_1 + y_1, \ldots, x_n + y_n) \\ \alpha\vec{x} &= (\alpha x_1, \ldots, \alpha x_n) \end{aligned}$$
3. For $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the inner product
$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$$

Lemma 2.2

The following properties of the inner product are easy to check.

1. $(\alpha x + \beta y) \cdot z = (\alpha x \cdot z) + \beta(y \cdot z)$
2. $x \cdot y = y \cdot x$
3. $x \cdot x \geq 0$, with equality holding if and only if $x = \vec{0}$.

Definition 2.3: Euclidean norm

Given $x \in \mathbb{R}^n$, define the **Euclidean norm** of x by $\|x\| := (x \cdot x)^{1/2}$.

Remark 2.4:
Existence of the square root can be traced back to the completeness axiom.

Lemma 2.5

For all $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, we have

1. $\|\alpha x\| = |\alpha| \|x\|$.
2. $\|\alpha\| \geq 0$ with equality if and only if $x = 0$.

Proposition 2.6: Cauchy-Schwarz inequality

For all $x, y \in \mathbb{R}^n$, $|x \cdot y| \leq \|x\| \|y\|$.

Proposition 2.7: Triangle inequality

For all $x, y \in \mathbb{R}^n$,

1. $\|x + y\| \leq \|x\| + \|y\|$.
2. $|\|x\| - \|y\|| \leq \|x - y\|$.

Definition 2.8: norm

A function $\rho : \mathbb{R}^n \to [0, \infty)$ is called a norm if

1. $\rho(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\rho(x) = 0$ if and only if $x = 0$.
2. $\rho(\alpha x) = |\alpha| \rho(x)$ for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.
3. $\rho(x + y) \leq \rho(x) + \rho(y)$.

Lecture 3 Another proof of Cauchy-Schwarz

The proof last time generalizes Hölder's inequality:

Hölder's inequality

Let (S, Σ, μ) be a measure space and let $p, q \in [1, \infty)$ with $1/p + 1/q = 1$. Then for all measurable real- or complex-valued functions f and g on S,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Lecture 4 Open sets and closed sets

Notation 4.1

Open and closed ball For $x_0 \in \mathbb{R}^n$, $r > 0$, define

1. $B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$
2. $\overline{B_r(x_0)} = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$

Definition 4.2: open and closed subset

1. A subset E of $\mathbb{R}^n$ is said to be open if for all $x_0 \in E$, there exists $\delta > 0$ such that $B_\delta(x_0) \subseteq E$.
2. A subset E of $\mathbb{R}^n$ is said to be closed if $\mathbb{R}^n \setminus E$ is open.

Example 4.3:

1. $\mathbb{R}^n, \varnothing$ both open. Hence both closed as $\mathbb{R}^n \setminus \varnothing = \mathbb{R}^n$ and $\mathbb{R}^n \setminus \mathbb{R}^n = \varnothing$.
2. For all $a \in \mathbb{R}^n$, $\{a\}$ is closed.
3. $B_r(x_0)$ is open and not closed. Note that "not closed" is not a consequence of openness.
4. $\overline{B_r(x_0)}$ is closed and not open.
5. $E = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n < 1\}$ is open.

Remark 4.4:
Not open $\not\Rightarrow$ closed, closed $\not\Rightarrow$ not open.

1. $\mathbb{R}^n$ and $\varnothing$ are clopen.
2. $E = (a, b]$ for $a < b$. E is not open and not closed.

Lecture 5 New open sets from old

Proposition 5.1

1. The union of an arbitrary collection of open sets in $\mathbb{R}^n$ is open.
2. The intersection of finitely many open sets in $\mathbb{R}^n$ is open.

Corollary 5.2

1. The intersection of an arbitrary collection of closed sets is closed.
2. The union of finitely many closed sets is closed.

Remark 5.3:
Finiteness is necessary in previous propositions. For example, $\bigcup_{a \in B_\delta(0)} \{a\} = B_\delta(0)$ is an infinite collection of closed sets, and it is not closed. $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ by completeness axiom. It is infinite collection of open sets, and it is not open.

Lecture 6 Interior and closure (I)

Warm-up $[a, \infty)$ closed and not open.

Definition 6.1: interior, closure, boundary

Let $E \subseteq \mathbb{R}^n$.

1. x belongs to the interior of E, denoted E°, if $\exists \delta > 0$ such that $B_\delta(x) \subseteq E$.
2. x belongs to the closure of E, denoted $\overline{E}$, if $\forall \delta > 0$, $B_\delta(x) \cap E \neq \varnothing$.
3. x belongs to the boundary of E, denoted ∂E, if $x \in \overline{E} \setminus E^\circ$. Equivalently,

$$\partial E = \{x \in \mathbb{R}^n \mid \forall \delta > 0, B_\delta(x) \cap E = \varnothing \text{ and } B_\delta(x) \setminus E \neq \varnothing\}$$

Remark 6.2:
$E^\circ \subseteq E \subseteq \overline{E}$. Each inclusion can be proper.

Proposition 6.3

Let $E \subseteq \mathbb{R}^n$.

1. $E^\circ = \cup\{A \subseteq E \mid A \text{ is open}\}$
2. E° is open.
3. E is open if and only if $E = E^\circ$.

Lecture 7 Interior and closure (II)

Proposition 7.4

Let $E \subseteq \mathbb{R}^n$.

1. $\overline{E} = \cap\{A \subseteq \mathbb{R}^n \mid A \supseteq E \text{ and } A \text{ is closed}\}$
2. $\overline{E}$ is closed.
3. E is closed if and only if $E = \overline{E}$.

Remark 7.5:

1. (3) gives an alternative way to prove closedness.
2. $\partial E = \overline{E} \cap (\mathbb{R}^n \setminus E^\circ)$ is closed. Intersection of closed sets is closed.

Lecture 8 Examples (I)

Example 8.1:
$\{x_0\}$ is closed, for some $x_0 \in \mathbb{R}^n$.

$\overline{\{x_0\}} = \{x_0\}$, $\{x_0\}^\circ = \varnothing$, $\partial\{x_0\} = \{x_0\}$.

Example 8.2:
$E = (a, b]$ for $a < b$.

$E^\circ = (a, b)$, $\overline{E} = [a, b]$, $\partial E = \{a, b\}$.

Example 8.3:
$E = \mathbb{Z} \subseteq \mathbb{R}$.

$\mathbb{Z}$ is closed, $\mathbb{Z}^\circ = \varnothing$, $\partial\mathbb{Z} = \mathbb{Z}$.

Example 8.4:
$E = \mathbb{Q} \subseteq \mathbb{R}$.

$\mathbb{Q}^\circ = \varnothing$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\partial\mathbb{Q} = \mathbb{R}$.

Lecture 9 Examples (II)

Example 9.1:
$E = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n \leq 1\}$

E is closed, $E^\circ = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n < 1\}$, $\partial E = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 1\}$.

Example 9.2:
$E = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n < 1\}$

E is open, $\overline{E} = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n \leq 1\}$ and $\partial E = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 1\}$.

Example 9.3:
$E = B_r(x_0)$

E is open. We can prove that closure of $B_r(x_0)$ is $\overline{B_r(x_0)}$. $\partial B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$.

Lecture 10 Relative openness and closedness

Definition 10.1: open/closed relative to A

Let $E \subseteq A \subseteq \mathbb{R}^n$.

1. E is open relative to A, or open in A, if $\forall x \in E$, $\exists\delta > 0$ such that $B_\delta(x) \cap A \subseteq E$.
2. E is closed relative to A, or closed in A, if $A \setminus E$ is open relative to A.

Remark 10.2:

1. Openness and closedness defined in lecture 4 is strictly speaking openness and closedness relative to $\mathbb{R}^n$.

2. Unless otherwise stated, "E is open" and "E is closed" (with specifying relative to which A) means relative to $\mathbb{R}^n$.

Proposition 10.3

Let $E \subseteq A \subseteq \mathbb{R}^n$. Then E is open relative to A iff $E = A \cap G$ for some G open relative to $\mathbb{R}^n$.

Proposition 10.4

Let $E \subseteq A \subseteq \mathbb{R}^n$. Then E is closed relative to A iff $E = A \cap F$ for some F closed relative to $\mathbb{R}^n$.

Example 10.5:

1. $A = \{x \in \mathbb{R}. \mid x_n \geq 0\}$; $E = \{x \in \mathbb{R}^n \mid \|x\| < 1, x_n \geq 0\}$.

   $E = B_1(0) \cap A$, thus E is open relative to A, but E not open relative to $\mathbb{R}^n$.

2. $A = [0,1) \cup (1,2]$; $E = [0,1)$.

   $E = (-1,1) \cap A$, then E is open relative to A.

   $A \setminus E = (1,3) \cap A$, then $A \setminus E$ open relative to A, so E is closed relative to A.

   But E is neither open nor closed relative to $\mathbb{R}$.

3. $A = \mathbb{Z}$; $E = \{0\}$.

   $E = \{0\} \cap \mathbb{Z}$, then E is closed relative to $\mathbb{Z}$.

   $E = (-\frac{1}{2}, \frac{1}{2}) \cap \mathbb{Z}$, then E is open relative to $\mathbb{Z}$.

   But E is closed and not open relative to $\mathbb{R}$.

Lecture 11 Connected sets

Definition 11.1: disconnected

Let $A \subseteq \mathbb{R}^n$. We say that A is disconnected if there exists subset E, F of A such that

(i) E, F both non-empty;

(ii) $E \cap F = \varnothing$, $E \cup F = A$;

(iii) E, F both open relative to A.

Equivalently, A is disconnected if there exists a subset E of A such that

(i') $E \neq \varnothing$, $E \neq A$.

(ii') E both open and closed relative to A.

Example 11.2:

$a, b \in \mathbb{R}^n$, $a \neq b$. Then $A = \{a, b\}$ is disconnected.

$A = [0,1) \cup (1,2]$ is disconnected.

$A = \{x \in \mathbb{R}^n \mid \|x\| \neq 1\}$ is disconnected.

$A = \mathbb{Z}$ is disconnected.

Definition 11.3: connected

Let $A \subseteq \mathbb{R}^n$. We say that A is connected if A is not disconnected. That is if $E = \varnothing$ or $F = \varnothing$ whenever $E, F \subseteq A$ satisfy $E \cap F = \varnothing$, $E \cup F = A$ and E, F both open relative to A.

Example 11.4:

$\{x_0\}$ connected.

Intervals $[a, b], \ldots, (-\infty, b], \mathbb{R}$ are connected.

Convex sets in $\mathbb{R}^n$ are connected.

Definition 11.5: convex

$A \subseteq \mathbb{R}^n$ is said to be convex if for all $x, y \in A$ and $t \in [0, 1]$, we have $tx + (1-t)y \in A$.

Lecture 12 New connected sets from old

Lemma 12.1

Let $B \subseteq \mathbb{R}^n$. If $E \subseteq B$ is open relative to B, then $E \cap A$ is open relative to A for all $A \subseteq B$.

Proposition 12.2

Let $A \subseteq \mathbb{R}^n$ be a connected set. Then $\overline{A}$ is connected.

Proposition 12.3

If $A_1, A_2 \subseteq \mathbb{R}^n$ are connected and $A_1 \cap A_2 \neq \varnothing$, then $A := A_1 \cup A_2$ is connected.

Remark 12.4:

Generalizations of proposition 3: Let $\{A_i\}_{i \in I}$ be an arbitrary collection of connected subsets of $\mathbb{R}^n$ and assume that $A_i \cap A_j$ for all $i, j \in I$. Prove that $\cup_{i \in I} A_i$ is connected.

Lecture 13 Convex sets (I)

Definition 13.1: convex

$A \subseteq \mathbb{R}^n$ is said to be convex if for all $x, y \in A$ and $t \in [0, 1]$, we have $tx + (1-t)y \in A$.

Example 13.2:

$B_r(x_0)$ convex.

$a \in \mathbb{R}^n \setminus \{0\}, c \in \mathbb{R}$. Then $E = \{x \in \mathbb{R}^n \mid x \cdot a < c\}$ is convex.

$E = \mathbb{R}^n \setminus \{0\}$ not convex.

Proposition 13.3

Intersection of an arbitrary collection of convex sets is convex.

Proposition 13.4

Let $E \subseteq \mathbb{R}^n$ be convex. Then $\overline{E}$ and E° is convex.

Lecture 14 Convex sets (II)

Proposition 14.1

If $A \subseteq \mathbb{R}^n$ is convex, then A is connected.

Proof assumes connectedness of intervals.

Example 14.2:

$E = \mathbb{R}^2 \setminus \{(x_1, 0) \mid x_1 \geq 0\}$ is connected, but not convex.

$E = \mathbb{R}^2 \setminus \{0\}$ is connected, but not convex.

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The completeness of $\mathbb{R}$

Lecture 15 Least upper bounds

Definition 15.1: upper bound

Let $E \subseteq \mathbb{R}$.

1. We say that $a \in \mathbb{R}$ is an upper bound of E if $x \leq a$ for all $x \in E$.
2. E is said to be bounded above if it has an upper bound.

Definition 15.2: least upper bound

Let $E \subseteq \mathbb{R}$. We say that $a \in \mathbb{R}$ is a least upper bound of E if

1. a is an upper bound of E.
2. $a \leq b$ for all upper bound b of E. (Equivalently, if $b < a$ then b is not an upper bound of E.)

Lemma 15.3

Let $E \subseteq \mathbb{R}$. E can only have at most one least upper bound.

By lemma 3, if E has a least upper bound, it is actually "the" least upper bound, and we denote it by $\sup E$, supremum of E.

> Example 15.4:
> $E = \{a_1, \ldots, a_k\}$ is a finite subset of $\mathbb{R}$. $\sup E = \max_{1 \leq i \leq k} a_i$.
>
> $\sup[0, 1] = \sup(0, 1) = 1$

Proposition 15.5

Let $E \subseteq \mathbb{R}$ and suppose $\sup E$ exists. Then $\forall \delta > 0$, $\exists x \in E$ such that $\sup E - \delta < x \leq \sup E$. In particular, $\sup E \in \overline{E}$.

Proposition 15.6

Let $E \subseteq \mathbb{Z}$ and suppose $\sup E$ exists, then $\sup E \in E$ and $\sup E \in \mathbb{Z}$.

Lecture 16 The completeness axiom

The completeness axiom

Let $E \subseteq \mathbb{R}$ be non-empty and bounded from above. Then E has a least upper bound.

Then completeness axiom + Lemma 15.3 imply: If $E \subseteq \mathbb{R}$ non-empty and bounded above, then $\sup E$ exists.

Lemma 16.1

1. Let $\varnothing \neq A \subseteq B \subseteq \mathbb{R}$. Suppose B is bounded above. Then so is A. $\sup A \leq \sup B$.
2. Let $A, B \subseteq \mathbb{R}$ be non-empty and bounded above. Then so is $A + b := \{a + b \mid a \in A, b \in B\}$. Moreover, $\sup(A + B) = \sup A + \sup B$.

Lecture 17 Some consequences of completeness (I)

Proposition 17.1: Archimedean property

Given $a, b \in \mathbb{R}$, with $a > 0$ and $b \geq 0$, there exists $n \in \mathbb{N}$ such that $(n-1)a \leq b < na$.

Corollary 17.2

1. Let $E = \left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\}$. Then $\sup E = 1$.
2. Let $V_k = \left(-\frac{1}{k}, \frac{1}{k}\right)$ for all $k \in \mathbb{N}$. Then $\bigcap_{k=1}^{\infty} V_k = \{0\}$.

Lecture 18 Some consequences of completeness (II)

Proposition 18.1: Density of the rationals

Given $x, y \in \mathbb{R}$ with $x < y$, $\exists r \in \mathbb{Q}$ such that $x < r < y$.

Corollary 18.2

1. $\overline{\mathbb{Q}} = \mathbb{R}$.
2. $\overline{\mathbb{Q}^n} = \mathbb{R}^n$.

Remark 18.3:
Density of rationals $\iff$ $\overline{\mathbb{Q}} = \mathbb{R}$.

Lecture 19 Some consequences of completeness (III)

Proposition 19.1: Existence if the square root

Given $x > 0$, there exists a unique $y > 0$ such that $y^2 = x$, and we denote this y by $\sqrt{x}$ or $x^{\frac{1}{2}}$.

Remark 19.2:
The above proof can be adapted to prove the existence of the n-th root. We will say more about exponential functions later.

Remark 19.3:

1. $\sqrt{2} \neq \mathbb{Q}$.
2. Then from (1) one can prove that $E = \{a \in \mathbb{Q} \mid a > 0, a^2 < 2\}$, we have $E \neq \varnothing$ and bounded above, but there exists no $r \in \mathbb{Q}$ such that

   (a) $r \geq x$ for all $x \in E$,

   (b) $rr \leq s$ for all upper bound $s \in \mathbb{Q}$ of E.

   In fact, if such an r existed, it would have to satisfy $r^2 = 2$.

   Hence $\mathbb{Q}$ is not complete. There are non-empty subsets of $\mathbb{Q}$ which are bounded from above but have no least upper bound in $\mathbb{Q}$.

Lecture 20 Connected of intervals

Proposition 20.1

Intervals are connected.

Lecture 21 Decimal expansions

Proposition 21.1

For all $x \in [0,1)$, there exists a unique function $a : \mathbb{N} \in \{0,\ldots,9\}$ such that, writing a_n for $a(n)$ for all $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} \frac{a_i}{10^i} \leq x < \left(\sum_{i=1}^{n} \frac{a_i}{10^i}\right) + \frac{1}{10^n} \qquad \forall n \in \mathbb{N}.$$

Remark 21.2:
The base 10 can be replaced with any $\ell \in \mathbb{N}$, $\ell \geq 2$.

Lecture 22 Greatest lower bounds

Definition 22.1: lower bound

Let $E \subseteq \mathbb{R}$.

1. We say that $a \in \mathbb{R}$ is a lower bound of E if $a \leq x$ for all $x \in E$.
2. E is said to be bounded from below if it has a lower bound.

Definition 22.2: greatest lower bound

Let $E \subseteq \mathbb{R}$. We say that $a \in \mathbb{R}$ is a greatest lower bound of E if

1. a is a lower bound of E.
2. $b \leq a$ for all lower bound b of E. Equivalently, if $b > a$ then b is not a lower bound of E.

Lemma 22.3

Given $E \subseteq \mathbb{R}$, define $-E = \{-x \mid x \in E\}$. Then

1. a is a lower bound of E iff $-a$ is an upper bound of $-E$.
2. a is a greatest lower bound of E iff $-a$ is a least upper bound of $-E$.

Remark 22.4:
A subset $E \subseteq \mathbb{R}$ can have at most one lower bound. We denote it by $\inf E$, the infimum of E.

If $E \subseteq \mathbb{R}$ is non-empty and bounded from below, then it has a greatest lower bound. Furthermore, in this case, $-E$ is non-empty and bounded from above, and $\inf E = -\sup(-E)$.

Example 22.5:
$\inf\{\frac{1}{n} \mid n \in \mathbb{N}\} = 0$

$\inf\{r^n \mid n \in \mathbb{N}\} = 0$ where $0 < r < 1$

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Sequences in $\mathbb{R}$ and $\mathbb{R}^n$

Lecture 23 Sequences and limits (I)

Definition 23.1: convergence seqeunce

Let (a_n) be a sequence in $\mathbb{R}^d$.

1. We say that (a_n) converges or is convergent if for some $x \in \mathbb{R}^d$ we have $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|a_n - x\| < \epsilon$ for all $n \geq N$.

   In this case x is called the limit of (a_n), denoted $\lim_{n\to\infty} a_n$, and (a_n) is said to converge to x as $n \to \infty$.

2. (a_n) is said to diverge if it does not converge.

Remark 23.2:
Since $\|a_n - x\| = |\|a_n - x\| - 0|$, we have that $a_n \to x$ as $n \to \infty$ iff $\|a_n - x\| \to 0$ as $n \to \infty$.

Lemma 23.3

$a_n \to x$ as $n \to \infty$ iff $\forall$ open set $U \subseteq \mathbb{R}^d$ containing x, $\exists N \in \mathbb{N}$ such that $a_n \in U$ for all $n \geq N$.

Proposition 23.4: Uniqueness of limit

Let (a_n) be a sequence in $\mathbb{R}^d$. Then it has at most one limit.

Definition 23.5: bounded sequence & Cauchy sequence

Let (a_n) be a sequence in $\mathbb{R}^d$.

1. (a_n) is said to be bounded if $\exists R > 0$ such that $\|a_n\| \leq R$ for all $n \in \mathbb{N}$, in other words, $a_n \in \overline{B_R(0)}$ for all $n \in \mathbb{N}$.
2. (a_n) is said to be a Cauchy sequence if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|a_n - a_m\| < \epsilon$ for all $n, m \geq N$.

Proposition 23.6

Let (a_n) be a sequence in $\mathbb{R}^d$ and suppose (a_n) converges. Then (a_n) is bounded and (a_n) is Cauchy.

Lecture 24 Sequences and limits (II)

Proposition 24.1

Let (a_n) be a sequence in $\mathbb{R}^d$. Write $a_n = (a_{n1}, \ldots, a_{nd})$ for all n. Then $a_n \to x$ iff $a_{ni} \to x_i$ as $n \to \infty$ for all $i \in [d]$.

Lemma 24.2

Let (a_n) be a convergent sequence in $\mathbb{R}^d$ with $\|a_n\| \leq R$ for all $n \in \mathbb{N}$. Then writing x for $\lim_{n\to\infty} a_n$, we have $\|x\| \leq R$.

Remark 24.3:
Given $\varnothing \neq E \subseteq \mathbb{R}^d$, if (a_n) is a sequence in E converging to x, then $x \in \overline{E}$.

Conversely, $\forall x \in \overline{E}$ (by considering $B_{1/n}(x)$) there exists a sequence in E converging to x.

Proposition 24.4

$a_n \to x$ and $b_n \to y$ as $n \to \infty$ in $\mathbb{R}^d$. Then

1. $a_n + b_n \to x + y$.
2. $\forall \alpha \in \mathbb{R}$, $\alpha a_n \to \alpha x$.
3. $a_n \cdot b_n \to x \cdot y$.

Lecture 25 Some examples

Const sequence is convergent.

Example 25.1:

$a_n = (-1)^n$. We can prove that (a_n) diverges by prove that it is not Cauchy. It is bounded, however.

Example 25.2:

$a_n = \frac{1}{n^k}$ for k some natural number. $a_n \to 0$ as $n \to \infty$.

$a_n = r^n$, $r \in (0, 1)$. $a_n \to 0$ as $n \to \infty$.

Lemma 25.3

Suppose for some $x \in \mathbb{R}^d$ we have $\|a_n - x\| \leq t_n \ \forall n \in \mathbb{N}$, where (t_n) is a sequence in $[0, \infty)$ converging to 0. Then $a_n \to x$.

Example 25.4:

Given $x \in [0, 1)$, let $a : \mathbb{N} \in \{0, \ldots, 9\}$ such that, writing a_n for $a(n)$ for all $n \in \mathbb{N}$, namely that

$$\sum_{i=1}^{n} \frac{a_i}{10^i} \leq x < \left(\sum_{i=1}^{n} \frac{a_i}{10^i} \right) + \frac{1}{10^n} \qquad \forall n \in \mathbb{N}.$$

Define $q_n = \sum_{i=1}^{n} \frac{a_i}{10^i}$. (q_n) is Cauchy.

In fact, $|q_n - x| \leq \frac{1}{10^n}$. Then $q_n \to x$ as $n \to \infty$.

Lecture 26 Monotone sequences

Definition 26.1: increasing, decreasing and monotone

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$.

1. (a_n) increasing (strictly increasing, resp.) if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ (if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$ resp.)
2. (a_n) decreasing (strictly decreasing, resp.) if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ (if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$ resp.)
3. (a_n) is said to be monotone if it is increasing or decreasing.

Definition 26.2: bounded sequence

Let (a_n) be a sequence in $\mathbb{R}$.

1. (a_n) bounded from above if $\{a_n \mid n \in \mathbb{N}\}$ bounded from above.
2. similar for bounded from below.

Example 26.3:

1. $a_n = (-1)^n$, not monotone. Bounded from above and below.
2. $a_n = \frac{1}{n^k}$, $k \in \mathbb{N}$. Strictly decreasing. Bounded from above and below.
3. $a_n = r^n$, $r \in (0, 1)$. Strictly decreasing. Bounded from above and below.
4. (q_n) in lec 25. Bounded above and below. Increasing.

Proposition 26.4

1. (a_n) in $\mathbb{R}$, increasing and bounded from above. Then $\exists x \in \mathbb{R}$ such that $a_n \leq x \ \forall x \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$.
2. (a_n) in $\mathbb{R}$, decreasing and bounded from below. Then $\exists x \in \mathbb{R}$ such that $a_n \geq x \ \forall x \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$.

Remark 26.5:
Consider the following statements.

(C) Every non-empty $E \subseteq \mathbb{R}$ and bounded above has a least upper bound.

(M) (a_n) in $\mathbb{R}$, increasing and bounded above. Then (a_n) converges and $a_m \leq \lim_{n\to\infty} a_n \ \forall m \in \mathbb{N}$.

We have assumed (C) as an axiom and deduced (M) as a theorem. We can also do the opposite.

Lecture 27 Cauchy sequences in $\mathbb{R}^d$

Lemma 27.1

Cauchy sequence are bounded.

Lemma 27.2

Let (a_n) be a bounded sequence in $\mathbb{R}$.

1. For $m \in \mathbb{N}$, $\inf\{a_n \mid n \geq m\}$ exists.
2. Letting $b_m = \inf\{a_n \mid n \geq m\}$, then $(b_m)_{m\in\mathbb{N}}$ is increasing and bounded above.

Proposition 27.3

Let (a_n) be a Cauchy sequence in $\mathbb{R}^d$. Then (a_n) converges.

Remark 27.4:
Proposition 3 is useful when proving a sequence converges but we don't have a good idea what the limit might be.

Assuming Archimedean property & Convergence of Cauchy sequence in $\mathbb{R}$ as axioms, we then can deduce a theorem that every non-empty subset of $\mathbb{R}$ which is bounded from above has a least upper bound.

Lecture 28 Nested sequence of closed sets in $\mathbb{R}^d$

Definition 28.1: nested sequence

1. A sequence $E_1, E_2, \ldots, E_n, \ldots$ of subsets of $\mathbb{R}^d$ is said to be nested if $E_{n+1} \subseteq E_n \ \forall n \in \mathbb{N}$.
2. A nested sequence $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots$ of subsets of $\mathbb{R}^d$ is said to have **diameters going to zero** if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ and $a \in \mathbb{R}^d$ such that $E_n \subseteq B_\epsilon(a) \ \forall n \geq N$.

Remark 28.2:

1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$. Define $E_m = \{x_n \mid n \geq m\}$. Then $(E_m)_{m\in\mathbb{N}}$ is a nested sequence of subsets of $\mathbb{R}^d$.
2. If $(E_m)_{m\in\mathbb{N}}$ is a nested sequence of subsets of $\mathbb{R}^d$, then so is $(\overline{E_m})_{m\in\mathbb{N}}$

   We can use A4 Q3:

   > Let $\varnothing \neq E \subseteq \mathbb{R}^d$. Then $x \in \overline{E}$ iff there exists a sequence in E converging to x.

   to prove that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
3. If $(E_m)_{m\in\mathbb{N}}$ and $(F_m)_{m\in\mathbb{N}}$ are nested sequence of subsets of $\mathbb{R}^d$, with (F_m) having diameters going to zero, and with $E_m \subseteq F_m \ \forall m \in \mathbb{N}$, then (E_m) has diameters going to zero.

Proposition 28.3

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$. For all $m \in \mathbb{N}$, define $E_m = \{x_n \mid n \geq m\}$. By remark 2, $(\overline{E}_m)$ is a nested sequence of subsets of $\mathbb{R}^d$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence iff the nested sequence $(\overline{E_m})_{m\in\mathbb{N}}$ has diameters going to zero.

Proposition 28.4

Let $(F_n)_{n\in\mathbb{N}}$ be a nested sequence of non-empty closed subsets of $\mathbb{R}^d$, and assume that $(F_n)_{n\in\mathbb{N}}$ has diameters going to zero. Then $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one element.

Lecture 29 Subsequences

Definition 29.1: subsequence

Let $x : \mathbb{N} \to \mathbb{R}^d$ be a sequence in $\mathbb{R}^d$. A subsequence of x is a sequence in $\mathbb{R}^d$ of the form $x \circ f : \mathbb{N} \to \mathbb{R}^d$ where f is a strictly increasing function from $\mathbb{N} \to \mathbb{N}$. (That is, $f(k+1) > f(k)$ $\forall k \in \mathbb{N}$).

Remark 29.2:
Given a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, by induction on k we get $f(k) \geq k \ \forall k \in \mathbb{N}$.

Example 29.3:

1. $a_n = (-1)^n$. $(a_{2k})_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$, here $f(k) = 2k$.
2. $(a_n)_{n\in\mathbb{N}}$ is any sequence in $\mathbb{R}^d$. $m \in \mathbb{N}$ given. Then $(a_{m+k})_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$,

here $f(k) = m + k$.

3. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ converging to 0. Then there exists a subsequence $(a_{f(m)})_{m\in\mathbb{N}}$ such that $\|a_{f(m)}\| < \frac{1}{m}$ for all $m \in \mathbb{N}$.

Lemma 29.4

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ and suppose $a_n \to x$ as $n \to \infty$. Then every subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to x.

Proposition 29.5

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$, so that $b_m := \inf\{a_n \mid n \geq m\}$ exists $\forall m \in \mathbb{N}$ and that $(b_m)_{m\in\mathbb{N}}$ converges by lec 27 and lec 26. Then there exists a subsequence of $(a_n)_{n\in\mathbb{N}}$ converging to $\lim_{m\to\infty} b_m$.

Corollary 29.6: Bolzano-Weierstrass theorem in $\mathbb{R}$

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Remark 29.7:
We take for granted the well-ordering principle: Every non-empty subset of $\mathbb{N}$ has a smallest element.

Lecture 30 The Bolzano-Weierstrass theorem

Definition 30.1: $a_n \in E$ for infinitely many $n \in \mathbb{N}$

Given $E \subseteq \mathbb{R}^d$ and a sequence (a_n) in $\mathbb{R}^d$, we say that $a_n \in E$ for infinitely many $n \in \mathbb{N}$ if $\forall N \in \mathbb{N}, \exists n \geq N$ such that $a_n \in E$.

Definition 30.2: d-cube

1. A closed d-cube is a subset C of $\mathbb{R}^d$ of the form

$$C = [a_1, b_1] \times \cdots \times [a_d, b_d],$$

where $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$ $\forall i \in [d]$.

2. Given a closed d-cube $C = [a_1, b_1] \times \cdots [a_d, b_d] \subseteq \mathbb{R}^d$, let

$$J_{i,0} = \left[a_i, \frac{a_i + b_i}{2}\right], \quad J_{i,1} = \left[\frac{a_i + b_i}{2}, b_i\right]$$

and let

$$C'_{k_1\cdots k_d} = J_{1,k_1} \times \cdots J_{d,k_d} \qquad (k_1, \ldots, k_d \in \{0, 1\})$$

Write $\mathcal{L}_C$ for $\{C_{k_1\cdots k_d} \mid k_1, \ldots, k_d \in \{0, 1\}\}$.

Remark 30.3:

1. Closed d-cubes are closed.
2. If $a_n \in C$ for infinitely many n, then $\exists C' \in \mathcal{L}_C$ such that $a_n \in C'$ for infinitely many n.

Lemma 30.4

Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed d-cubes such that $C_{n+1} \in \mathcal{L}_{C_n}$ for all $n \in \mathbb{N}$. Then $(C_n)_{n\in\mathbb{N}}$ is a nested sequence with diameters going to zero.

Proposition 30.5: Bolzano-Weierstrass theorem

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}^d$. Then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Lecture 31 Some applications of the Bolzano-Weierstrass theorem

Definition 31.1: boudned subset

A subset E of $\mathbb{R}^d$ is said to be bounded if $\exists R > 0$ such that $E \subseteq \overline{B_R(0)}$.

Proposition 31.2

Let $E \subseteq \mathbb{R}^d$ be non-empty, closed and bounded. Then $\exists x_0 \in E$ such that $\|x\| \leq \|x_0\|$ $\forall x \in E$.

Proposition 31.3

Let $\rho : \mathbb{R}^d \to [0, \infty)$ be a norm on $\mathbb{R}^d$. $\exists \alpha > 0$ such that $\rho(x) \geq \alpha\|x\|$ $\forall x \in \mathbb{R}^d$.

Remark 31.4:

Combining proposition 3 with A4 Q5a:

$\rho(x) \leq \max\{\rho(e_1), \ldots, \rho(e_d)\}\sqrt{d}\|x\|$ for all $x \in \mathbb{R}^d$.

we infer that given a norm ρ, $\exists \alpha, C > 0$ such that

$$\alpha\|x\| \leq \rho(x) \leq C\|x\| \qquad \forall x \in \mathbb{R}^d$$

In particular,

1. $\|a_n - x\| \to 0$ iff $\rho(a_n - x) \to 0$, so any norm on $\mathbb{R}^d$ defines the same notion of convergence as the Euclidean norm.
2. $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq E$ iff $\exists r > 0$ such that $\{y \in \mathbb{R}^d \mid \rho(y - x) < r\} \subseteq E$. So any norm on $\mathbb{R}^d$ defines the same notion of openness as the Euclidean norm.

Lecture 32 Equivalent formulations of completeness (I)

Consider the following statements:

(C) If $E \subseteq \mathbb{R}$ is non-empty and bounded above, then E has a least upper bound.

(M) If (a_n) is a sequence in $\mathbb{R}$ which is increasing and bounded above, then $\exists x \in \mathbb{R}$ such that $a_n \leq x$ $\forall n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$.

(S) If (a_n) is a Cauchy sequence in $\mathbb{R}$, then (a_n) converges.

(A) $\forall a, b \in \mathbb{R}$ with $a > 0$, $b \geq 0$, $\exists n \in \mathbb{N}$ such that $na > b$.

In this course, we assume (C) as an axiom, and we have seen (M) and (S) + (A) follow as theorems.

Lemma 32.1: doesn't use any of (C), (M), (S) and (A)

E non-empty subset of $\mathbb{R}$ and bounded from above. Suppose E contains no upper bound of itself. Then there exist sequences $(a_n), (b_n)$ such that for all $n \in \mathbb{N}$,

(i) b_n is an upper bound of E while a_n isn't.

(ii) $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$.

(iii) $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.

Lemma 32.2

Assume (A). Let $E, (a_n), (b_n)$ be as in Lemma 1. If $c \in \bigcap_{m=1}^{\infty} [a_m, b_m]$ then c is a least upper bound of E.

Lecture 33 Equivalent formulations of completeness (II)

Lemma 33.1

(A) is a consequence of either (C) or (M).

Proposition 33.2

1. Assuming (M) as an axiom in place of (C), then (C) follows as a theorem.
2. Assuming (S) + (A) as an axiom in place of (C), then (C) follows as a theorem.

5

Countability

Lecture 34 Countable and uncountable sets

Definition 34.1: countable, at most countable, uncountable

A set E is said to be

1. countable, if there exists a bijection $f : \mathbb{N} \to E$.
2. at most countable, if E is either finite or countable.
3. uncountable, if E is neither finite nor countable.

Proposition 34.2

Any infinite subset E of $\mathbb{N}$ is countable.

Corollary 34.3

Let E be an infinite set.

1. If F is countable and and if there exists an injection $h : E \to F$, then E is countable.
2. If F is countable and if there exists a surjection $h : F \to E$, then E is countable.

Lecture 35 Some examples

Example 35.1:
$\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}$ are countable.

Lemma 35.2

Let $E_1, \dots, E_k$ be countable. Then $E_1 \times \cdots E_k$ is countable.

Example 35.3:

$\mathbb{Q}, \mathbb{Q}_+ = \mathbb{Q} \cap (0, \infty), \mathbb{Q}^d, U := \{B_r(x) \mid r \in \mathbb{Q}_+, x \in \mathbb{Q}^d\}$ are countable.

Let U be an arbitrary collection of mutually disjoint, non-empty open subsets of $\mathbb{R}^d$. Then U is at most countable.

Lecture 36 Cantor's diagonal argument

Proposition 36.1

Let $(E_m)_{m \in \mathbb{N}}$ be a sequence of non-empty, at most countable sets. Then $E := \bigcup_{m=1}^{\infty} E_m$ is at most countable.

Corollary 36.2

If $E_1, \dots, E_k$ are non-empty, at most countable. Then $\bigcup_{j=1}^{k} E_j$ is at most countable.

Example 36.3:

Let $E = \{x : \mathbb{N} \to \{0, \dots, 9\} \mid \exists N \in \mathbb{N} \text{ such that } x_k = 9 \quad \forall k > N\}$. Then E is countable.

Let $A = \{\text{all sequences } x : \mathbb{N} \to \{0, \dots, 9\}\}$. Then A is uncountable.

Let $B = A \setminus E$. Then B is uncountable.

Lecture 37 Uncountability of $\mathbb{R}$

Proposition 37.1: Complements of Proposition 21.1

Let $B = \{a : \mathbb{N} \to \{0, \dots, 9\} \mid \forall N \in \mathbb{N}, \exists n > N \text{ such that } a_n \neq 9\}$. Given $a \in B$, there exists a unique $x \in [0, 1)$ such that

$$\sum_{i=1}^{n} \frac{a_i}{10^i} \leq x < \sum_{i=1}^{n} \frac{a_i}{10^i} + \frac{1}{10^n} \quad \forall n \in \mathbb{N}.$$

Proposition 37.2

$[0, 1)$ is uncountable.

Corollary 37.3

$\mathbb{R}$ is uncountable.

6

Compactness

Lecture 38 Open coverings and compactness

Remark 38.1:

Let $E \subseteq \mathbb{R}^d$ be closed and bounded. By Bolzano-Weierstrass theorem, and A4 Q3:

> Let $\varnothing \neq E \subseteq \mathbb{R}^d$. Then $x \in \overline{E}$ iff there exists a sequence in E converging to x.

any sequence $(x_n)_{n\in\mathbb{N}}$ in E has a convergence subsequence whose limit lies in E. This is important for existence of minimizers/maximizers.

We want an equivalent formulations of above remark purely in terms of open sets.

Definition 38.2: open covering, subcovering, compact

Let $E \subseteq \mathbb{R}^d$.

1. An open covering of E is a collection U of open subsets of $\mathbb{R}^d$ such that $E \subseteq \bigcup_{V\in U} V$. We call this "$U$ covers E".
2. Given an open covering U of E, a subcovering of U is a subcollection $U' \subseteq U$ such that $E \subseteq \bigcup_{V\in U'} V$.
3. E is said to be compact if every open covering of E has finite subcovering, that is, if for all open covering U of E, $\exists N \in \mathbb{N}$ and $V_1, \ldots, V_N \in U$ such that $E \subseteq V_1 \cup \cdots \cup V_N$.

Example 38.3:

$E = \{x_1, \ldots, x_N\}$ is a finite subset of $\mathbb{R}^d$. Then E is compact.

$E = \{0\} \cup \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$ is compact.

$E = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$ is not compact.

Lecture 39 Some properties (I)

Proposition 39.1

Let $E_1, \dots, E_k \in \mathbb{R}^d$ be compact. Then $E := \bigcup_{j=1}^k E_j$ is compact.

Proposition 39.2

Let $F \subseteq E \subseteq \mathbb{R}^d$ with F closed and E compact. Then F is compact.

Proposition 39.3

Let $E \subseteq \mathbb{R}^d$ be compact. Then E is closed and bounded.

Lecture 40 Some properties (II)

Lemma 40.1

Let $K \subseteq \mathbb{R}^d$ be compact. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ and $x_1, \dots, x_N \in K$ such that $K \subseteq \bigcup_{j=1}^N B_\epsilon(x_j)$.

Remark 40.2:

Given a sequence (x_n) in $\mathbb{R}^d$ and $E \subseteq \mathbb{R}^d$, recall that we say "$x_n \in E$ for infinitely many $n \in \mathbb{N}$" if $\forall N \in \mathbb{N}$, $\exists n \geq N$ such that $x_n \in E$.

If $x_n \in E$ for infinitely many $n \in \mathbb{N}$ and if $\exists N \in \mathbb{N}$ and $A_1, \dots, A_n \subseteq \mathbb{R}^d$ such that $E \subseteq A_1 \cup \dots \cup A_N$, then $\exists j \in [N]$ such that $x_n \in A_j$ for infinitely many $n \in \mathbb{N}$.

Proposition 40.3

Let $K \subseteq \mathbb{R}^d$ be compact and suppose (x_n) is a sequence in K. Then (x_n) has a convergent subsequence with limit lying in K.

Lecture 41 Countable subcoverings

Proposition 41.1

Given $E \subseteq \mathbb{R}^d$ an an open covering U of E, there exists at most countable subcollection U' of U such that $E \subseteq \bigcup_{V \in U'} V$.

Lecture 42 Heine-Borel theorem

Lemma 42.1

Let $(F_n)_{n\in\mathbb{N}}$ be a nested sequence of non-empty closed subsets of $\mathbb{R}^d$, with F_1 bounded. Then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

Proposition 42.2: Heine-Borel Theorem

Let $E \subseteq \mathbb{R}^d$ be closed and bounded. Then E is compact.

Lecture 43 Equivalent formulations of compactness

Definition 43.1: sequentially compact

Let $E \subseteq \mathbb{R}^d$. Then E is said to be sequentially compact if every sequence in E has a convergent subsequence, with limit lying in E.

Proposition 43.2

Let E be a subset of $\mathbb{R}^d$. Then the following are equivalent:

(i) E is closed and bounded.

(ii) E is compact.

(iii) E is sequentially compact.

Remark 43.3:

For a subset E of $\mathbb{R}^d$, we have proved

$$\begin{tikzcd}
\text{(ii) } E \text{ compact} \arrow[dr, "(1)"'] & & \text{(iii) } E \text{ sequentially compact} \arrow[dl, "(4)"] \\
& \text{(i) } E \text{ closed and bounded} \arrow[ul, bend right, "(2)"'] \arrow[ur, bend left, "(3)"] &
\end{tikzcd}$$

- In particular (ii) $\Leftrightarrow$ (iii).
- (2) (3) are special to $\mathbb{R}^d$. (1), (4) hold in more generality.
- (ii) $\Leftrightarrow$ (iii) hold in more generality as will, but general version has a much harder proof.

Lecture 44 Accumulation points

Definition 44.1: accumulation point, isolated point

1. For $A \subseteq \mathbb{R}^d$, we say that $x_0 \in \mathbb{R}^d$ is an accumulation point of A if $x_0 \in \overline{A \setminus \{x_0\}}$, that is, if $B_\delta(x_0) \cap (A \setminus \{x_0\}) \neq \varnothing \; \forall \delta > 0$. The set of accumulation points of A is denoted by A'.
2. $x_0 \in \mathbb{R}^d$ is said to be an isolated point of A if $\exists \delta > 0$, $B_\delta(x_0) \cap A = \{x_0\}$.

Remark 44.2:
An accumulation point of A need not lie in A, while an isolated point of A lies in A from the definition.

If x_0 is an isolated point of A, then there exists $\delta > 0$ such that $B_\delta(x_0) \cap A = \{x_0\}$, then we have $B_\delta(x_0) \cap (A \setminus \{x_0\}) = \varnothing \implies x_0 \notin A'$.

Lemma 44.3

For all $A \subseteq \mathbb{R}^d$, we have $\overline{A} = A' \cup \left\{ x \in \mathbb{R}^d \,\middle|\, x \text{ is an isolated point of } A \right\}$. Moreover, the two sets on the RHS are disjoint.

Example 44.4:
Let $A \subseteq \mathbb{R}^d$. Then $A^\circ \subseteq A'$.

Let $V \subseteq \mathbb{R}^d$ be an open set, then $\partial V \subseteq V'$.

7

Continuous functions

Lecture 45 Limit of functions (I)

Definition 45.1: limit of a function

Let $A \subseteq \mathbb{R}^m$. Let $f : A \to \mathbb{R}^n$ be a function. Let $x_0 \in A'$.

1. Given $y \in \mathbb{R}^n$, we write $f(x) \to y$ as $x \to x_0, x \in A$ [a] if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|f(x) - y\| < \epsilon$ whenever $x \in B_\delta(x_0) \cap (A \setminus \{x_0\})$.
2. f is said to have a limit as $x \to x_0$, $x \in A$ if $\exists y \in \mathbb{R}^n$ such that $f(x) \to y$ as $x \to x_0, x \in A$.

[a] $f(x)$ tends to y as x tends to x_0 through points in A

Proposition 45.2

Let $A \subseteq \mathbb{R}^m$ and suppose $f : A \to \mathbb{R}^n$ is a function. Take $x_0 \in A'$. Then given $y \in \mathbb{R}^n$, the following are equivalent:

1. $f(x) \to y$ as $x \to x_0$, $x \in A$.
2. $(f(x_k))_{k\in\mathbb{N}}$ converges to y whenever $(x_k)_{k\in\mathbb{N}}$ is a sequence in $A \setminus \{x_0\}$ converging to x_0.

Corollary 45.3

$A \subseteq \mathbb{R}^m$, $f : A \to \mathbb{R}^n$ a function, $x_0 \in A'$. Then f has at most one limit as $x \to x_0$ through points in A.

Remark 45.4:

If $f : A \to \mathbb{R}^n$ has a limit as $x \to x_0, x \in A$, it must be "the" limit, denote by $\lim_{\substack{x\to x_0 \\ x\in A}} f(x)$.

When $A = \mathbb{R}^m$, we drop "$x \in A$" .

Corollary 45.5

$A \subseteq \mathbb{R}^m$, $f : A \to \mathbb{R}^n$ and $x_0 \in A'$ as above. $\forall x \in A$, write $f(x) = (f_1(x), \dots, f_n(x))$. Then given $y \in \mathbb{R}^n$, $f(x) \to y$ as $x \to x_0, x \in A$ if and only if $f_i(x) \to y_i$ as $x \to x_0, x \in A$, $\forall i \in [n]$.

Lecture 46 Limit of functions (II)

Remark 46.1:

$A \subseteq \mathbb{R}^m$, $f : A \to \mathbb{R}^n$ and $x_0 \in A'$. Suppose $B \subseteq A$ is such that $x_0 \in B'$. Consider $f|_B : B \to \mathbb{R}^n : x \mapsto f(x)$. Given $y \in \mathbb{R}^n$, if $f(x) \to y$ as $x \to x_0$, $x \in A$, then $(f|_B)(x) \to y$ as $x \to x_0, x \in B$. Some times we denote restrictions of f still by "f", by abuse of notation.

The converse is not true.

Remark 46.2:

Let f, g be functions from $A \subseteq \mathbb{R}^m$ to $\mathbb{R}^n$. Suppose $x_0 \in A'$ and that $\exists r > 0$ such that $f(x) = g(x)$ $\forall x \in B_r(x_0) \cap (A \setminus \{x_0\})$. Given $y \in \mathbb{R}^n$, if $f(x) \to y$ as $x \to x_0, x \in A$, then $g(x) \to y$ as $x \to x_0$, $x \in A$.

Lecture 47 Some examples of limits

Example 47.1:

$(\mathbb{R}^m)' = \mathbb{R}^m$.

ρ be any norm. $\lim_{x \to x_0} \rho(x) = \rho(x_0)$.

Example 47.2:

Take $a > 0$. For $x \in \mathbb{Q}$, write $x = \frac{p}{q}$ where $p \in \mathbb{Z}, q \in \mathbb{N}$ and define $a^x = (a^{1/q})^p$. We take for granted that

1. (well-defined) $(a^{1/n})^m = (a^{1/q})^p$ if $\frac{m}{n} = \frac{p}{q}$ $(m, p \in \mathbb{Z},\ n, q \in \mathbb{N})$
2. $a^x a^y = a^{x+y}$ for all $x, y \in \mathbb{Q}$.
3. $(ab)^x = a^x b^x$ $\forall a, b > 0$, $x \in \mathbb{Q}$.

Then $\forall x_0 \in \mathbb{Q}$, $\lim_{\substack{x \to x_0 \\ x \in \mathbb{Q}}} a^x = a^{x_0}$.

Lecture 48 Continuity

Definition 48.1: continuous

Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$ be a function.

1. f is said to be continuous at $x_0 \in A$ relative to A if either
   (i) x_0 is an isolated point of A, or
   (ii) $x_0 \in A'$ and $\lim_{x \to x_0, x \in A} f(x) = f(x_0)$.
2. Given $B \subseteq A$, f is said to be continuous on B relative to A if f is continuous at x relative to A for all $x \in B$.

Remark 48.2:
We sometimes drop "relative to A" if $A = \mathbb{R}^m$.

Given $B \subseteq A$ and $x_0 \in B$, we sometimes simply write f for $f|_B$ in the statement "$f|_B$ is continuous at x_0 relative to B".

Lemma 48.3

Let $A \subseteq \mathbb{R}^m$. $f : A \to \mathbb{R}^n$ is continuous at $x_0 \in A$ relative to A iff $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|f(x) - f(x_0)\| < \epsilon \; \forall x \in B_\delta(x_0) \cap A$.

Proposition 48.4

Let $f, g : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be two functions. Suppose $x_0 \in A$ and both f, g are continuous at x_0 relative to A. Then

1. so are $f + g, \alpha f, f \cdot g$.
2. If in addition $n = 1$ and $g(x_0) \neq 0$, then $\exists r > 0$ such that $g(x) \neq 0 \; \forall x \in B_r(x_0) \cap A$ and $\frac{1}{g}$ is continuous at x_0 relative to $B_r(x_0) \cap A$.

Lecture 49 Some examples I

Example 49.1:

1. $f : \mathbb{R}^m \to \mathbb{R} : x \mapsto x_1^{k_1} \cdots x_m^{k_m}$ is continuous on $\mathbb{R}^m$.
2. Any norm is continuous on $\mathbb{R}^m$.
3. Fix $a > 0$. $f : \mathbb{Q} \to \mathbb{R} : x \mapsto a^x$ is continuous on $\mathbb{Q}$ relative to $\mathbb{Q}$.
4. Polynomials in x are continuous on $\mathbb{R}$.
5. $p, q : \mathbb{R} \to \mathbb{R}$ are two polynomials. Then
   (a) $V := \{x \in \mathbb{R} \mid q(x) \neq 0\}$ is open in $\mathbb{R}$.
   (b) $\frac{p}{q}$ is continuous on V relative to V.

Example 49.2:

Given $k \in \mathbb{N} \setminus \{1\}$, define $f : [0, \infty) \to [0, \infty)$ by $f(x) = \begin{cases} x^{1/k} & x > 0, \\ 0 & x = 0. \end{cases}$ Then f is continuous on $[0, \infty)$ relative to $[0, \infty)$.

Proposition 49.3

Suppose $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ and $g : B \subseteq \mathbb{R}^n \to \mathbb{R}^\ell$, and that $f(A) \subseteq B$ (so that $g \circ f$ makes sense). Given $x_0 \in A$, if f is continuous at x_0 relative to A and g is continuous at $f(x_0)$ relative to B, then $g \circ f$ is continuous at x_0 relative to A.

Lecture 50 Some examples (II)

Example 50.1:

For $f_1, \ldots, f_N : A \subseteq \mathbb{R}^n \to \mathbb{R}$, define $A \to \mathbb{R}$ by $h(x) = \max\{f_1(x), \ldots, f_N(x)\}$. Given $x_0 \in A$. If $f_1, \ldots, f_N$ are continuous at x_0 relative to A, then so is h.

Let $E_+ = \{x \in \mathbb{R}^m \mid x_m \geq 0\}$, $E_- = \{x \in \mathbb{R}^m x_m \leq 0\}$. Suppose $f : E_+ \to \mathbb{R}^n$ is continuous on E_+ relative to E_+ and $g : E_- \to \mathbb{R}^n$ is continuous on E_- relative to E_- with $f(x) = g(x)\ \forall x \in E_+ \cap E_-$. Then $h = \begin{cases} f(x) & x_m \geq 0 \\ g(x) & x_m \leq 0 \end{cases}$ is continuous on $\mathbb{R}^m$ relative to $\mathbb{R}^m$.

Lecture 51 Some examples (III)

Example 51.1: Discontinuity

$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is discontinuous at $x_0\ \forall x_0 \in \mathbb{R}$.

$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & x \in \mathbb{Q}, x = \frac{p}{q} \text{ lowest terms } (p \in \mathbb{Z}, q \in \mathbb{N}) \end{cases}$ is continuous at x_0 if $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and discontinuous at x_0 if $x_0 \in \mathbb{Q}$.

$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ is continuous at $(x_0, y_0) \neq (0, 0)$ and discontinuous at $(0, 0)$.

Lecture 52 Continuity and openness

Definition 52.1: neighborhood

1. Given $x \in \mathbb{R}^m$, a neighborhood of x is a subset $V \subseteq \mathbb{R}^m$ such that $x \in V$ and V is open.
2. Given $A \subseteq \mathbb{R}^m$ and $x \in A$, a neighborhood of x relative to A is a subset $V \subseteq A$ such that $x \in V$ and V is open relative to A.

Notation 52.2

Let $A \subseteq \mathbb{R}^m$. $f : A \to \mathbb{R}^n$ a function.

1. For $E \subseteq \mathbb{R}^n$, let $f^{-1}(E) = \{x \in A \mid f(x) \in E\}$.
2. For $B \subseteq A$, let $f(B) = \{y \in \mathbb{R}^n \mid y = f(x) \text{ for some } x \in B\}$.

Proposition 52.3

Let $f : A \to \mathbb{R}^n$ be a function, where $A \subseteq \mathbb{R}^m$. Given $x_0 \in A$, f is continuous at x_0 relative to A iff for all neighborhood W of $f(x_0)$, $f^{-1}(W)$ contains a neighborhood of x_0 relative to A.

Corollary 52.4

$f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ a function. Then f is continuous on A relative to A iff $\forall W \subseteq \mathbb{R}^n$ open, $f^{-1}(W)$ is open relative to A.

Remark 52.5:
It is NOT true that if $f : A \to \mathbb{R}^n$ is continuous on A relative to A then $f(V)$ is open whenever V is open relative to A.

For example, consider $f(x) = 0\ \forall x \in \mathbb{R}^m$. Then $f(V) = \{0\}$ for all non-empty open $V \subseteq \mathbb{R}^m$. However, $\{0\}$ is not open in $\mathbb{R}^n$.

Lecture 53 Continuous functions on connected sets

Lemma 53.1

Let $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$ be a function. Let $V, W \subseteq \mathbb{R}^m$.

1. $f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$.
2. $f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W)$.
3. $f^{-1}(V) = f^{-1}(V \cap f(A))$.
4. $f(f^{-1}(V)) = V \cap f(A)$.

Proposition 53.2

Let $A \subseteq \mathbb{R}^m$ be connected and let $f : A \to \mathbb{R}^n$ be continuous on A relative to A. Then $f(A)$ is connected.

Corollary 53.3: Intermediate value theorem

Let $A \subseteq \mathbb{R}^m$ be connected and let $f : A \to \mathbb{R}$ be continuous on A relative to A. Given $x_0, x_1 \in A$ with $f(x_0) \leq f(x_1)$, then $\forall c \in [f(x_0), f(x_1)]$, $\exists x_* \in A$ such that $f(x_*) = c$.

Lecture 54 Continuous functions on compact sets (I)

Proposition 54.1

Let $\varnothing \neq A \subseteq \mathbb{R}^m$ be compact and let $f : A \to \mathbb{R}^n$ be a function which is continuous on A relative to A. Then $f(A)$ is compact.

Corollary 54.2

Let $\varnothing \neq A \subseteq \mathbb{R}^m$ be compact and let $f : A \to \mathbb{R}$ be continuous on A relative to A. Then

1. f is bounded on A. That is $\exists R > 0$ such that $|f(x)| \leq R \ \forall x \in A$.
2. $\sup_{x\in A} f(x), \inf_{x\in A} f(x)$ both exist. Moreover, $\exists x^*, x_* \in A$ such that $f(x^*) = \sup_{x\in A} f(x)$ and $f(x_*) = \inf_{x\in A} f(x)$.

Remark 54.3:
Compactness assumption on A is necessary in corollary 2. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. Since $(0, \infty)$ not closed, thus not compact. f is continuous on $(0, \infty)$, but unbounded.

Lecture 55 Continuous functions on compact sets (II)

Definition 55.1: uniformly continuous

Suppose $A \subseteq \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$ be a function. Given $B \subseteq A$, f is said to be uniformly continuous on B if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ whenever $x, y \in B$ and $\|x - y\| < \delta$.

Remark 55.2:
If f is uniformly continuous on A, then f is continuous on A relative to A. Converse is false in general.

Example 55.3:
Norm function is uniformly continuous on $\mathbb{R}^m$.

$f(x) = x^2$ is uniformly continuous on $[-K, K]$ $\forall K > 0$, but not uniformly continuous on $\mathbb{R}$.

Proposition 55.4

Let $\varnothing \neq A \subseteq \mathbb{R}^m$ be compact and let $f : A \to \mathbb{R}^n$ be continuous on A relative to A. Then f is uniformly continuous on A.

Lecture 56 More on uniform continuity (I)

Lemma 56.1

Suppose $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is uniformly continuous on A. Given $x_0 \in \overline{A} \setminus A$. $\forall k \in \mathbb{N}$, define $E_k = f(B_{1/k}(x_0) \cap (A \setminus \{x_0\}))$. Then $(\overline{E_k})_{k\in\mathbb{N}}$ is a nested sequence of non-empty closed sets with diameters going to zero.

Lemma 56.2

Suppose $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is uniformly continuous on A. Then $\forall x_0 \in \overline{A} \setminus A$, $\lim_{x\to x_0, x\in A} f(x)$ exists.

As noted above, $x_0 \in A'$, so statement makes sense.

Proposition 56.3

Let $A \subseteq \mathbb{R}^m$ and suppose $f : A \to \mathbb{R}^n$ is uniformly continuous on A. Then there exists unique $F : \overline{A} \to \mathbb{R}^n$ such that $F(x) = f(x)$ $\forall x \in A$. This is called "F extends f". And F is continuous on $\overline{A}$ relative to $\overline{A}$.

Lecture 57 More on uniform continuity (II)

Continue the proof of last proposition.

Lecture 58 More on uniform continuity (III)

Proposition 58.1

Let $a > 0$ and define $f : \mathbb{Q} \to \mathbb{R}$ be $f(x) = a^x$. Then $\forall L > 0$, f is uniform continuous on $(-L \cap L) \cap \mathbb{Q}$.

Proposition 58.2

Let $a > 0$. Then there exists a unique $F : \mathbb{R} \to \mathbb{R}$ such that F is continuous on $\mathbb{R}$ and $F(x) = a^x$ $\forall x \in \mathbb{Q}$.

Remark 58.3:
We still denote $F(x)$ by a^x.

Remark 58.4:
Let $a, b > 0$, then for all $x, y \in \mathbb{R}$:

1. $a^x a^y = a^{x+y}$,
2. $(a^x)^y = a^{xy}$,

3. $a^x b^x = (ab)^x$.

These can be extended from $x, y \in \mathbb{Q}$ to $x, y \in \mathbb{R}$ by continuity.

8

Sequences of functions

Lecture 59 Pointwise and uniform convergence

Definition 59.1: pointwise and uniform convergence

Let $A \subseteq \mathbb{R}^m$ and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions $A \to \mathbb{R}^n$. Given $f : A \to \mathbb{R}^n$, and $B \subseteq A$

1. $(f_k)_{k\in\mathbb{N}}$ is said to converge pointwise to f on B if $\forall x \in B$, $f_k(x) \to f(x)$ as $k \to \infty$.
2. $(f_k)_{k\in\mathbb{N}}$ is said to converge uniformly to f on B if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\|f_k(x) - f(x)\| < \epsilon \quad \forall k \geq N \text{ and } x \in B.$$

Remark 59.2:
Uniform convergence on B implies pointwise convergence on B.

Example 59.3:

$f_k : [0,1] \to \mathbb{R} : x \mapsto x^k$. Define $f(x) = \begin{cases} 0 & 0 \leq x < 1, \\ 1 & x = 1, \end{cases}$ we see that $f_k \to f$ pointwise on $[0,1]$. However, (f_k) does NOT converge uniformly to f on $[0,1]$.

Suppose $a \in (0,1)$. For all $k \in \mathbb{N}$, define $f_k : \mathbb{R} \to \mathbb{R} : x \mapsto \sum_{j=0}^{k} x^j$. Then $(f_k)_{k\in\mathbb{N}}$ converges uniformly on $[-a,a]$.

Lecture 60 Uniform convergence and continuity

Proposition 60.1

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions from $A \subseteq \mathbb{R}^m$ to $\mathbb{R}^n$. Suppose f_k is continuous on A relative to A $\forall k \in \mathbb{N}$ and that $(f_k)_{k\in\mathbb{N}}$ converges uniformly on A to $f : A \to \mathbb{R}^n$. Then f is continuous on A relative to A.

Remark 60.2:
Example 59.3 shows that uniform convergence is necessary in proposition 1, and pointwise convergence is not enough.

9

Integration

Lecture 61 Partitions (I)

Definition 61.1: partition, refinement, regular partition

Let $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a closed n-cube.

1. $v(S) := (b_1 - a_1) \cdots (b_n - a_n)$. Below assume $v(S) > 0$.
2. A partition of S is a finite collection $\mathcal{P}$ of closed n-cubes such that $v(P) > 0$ $\forall P \in \mathcal{P}$, $S = \bigcup_{P \in \mathcal{P}} P$, and $P^\circ \cap (\tilde{P})^\circ = \varnothing$ whenever $P, \tilde{P} \in \mathcal{P}$ with $P \neq \tilde{P}$.
3. Given two partitions $\mathcal{P}, \mathcal{P}'$ of S, we say that $\mathcal{P}'$ is a refinement of $\mathcal{P}$ ("$\mathcal{P}' \leq \mathcal{P}$") if $\forall P \in \mathcal{P}'$, $\exists R \in \mathcal{P}$ such that $P \subseteq R$.
4. A partition $\mathcal{P}$ of S is said to be regular if $\exists$ partitions $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of $[a_1, b_1], \ldots, [a_n, b_n]$ respectively, such that $\mathcal{P} = \{I_1 \times \cdots \times I_n \mid I_1 \in \mathcal{P}_1, \ldots, I_n \in \mathcal{P}_n\}$.

Remark 61.2:
If $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$, then $S^\circ = (a_1, b_1) \times \cdots (a_n, b_n)$. In particular, $v(S) > 0$ iff $S^\circ \neq \varnothing$.

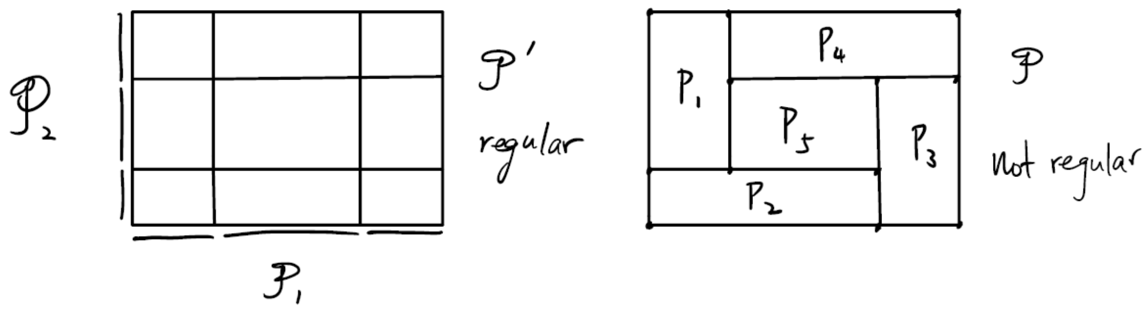
Suppose $\mathcal{P}, \mathcal{P}'$ are partitions of S such that $\mathcal{P}' \leq \mathcal{P}$. Then $\mathcal{P}' = \bigcup_{R \in \mathcal{P}} \{\mathcal{P} \in \mathcal{P}' \mid P \subseteq R\}$ and this is a disjoint union.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be partitions of $[a_1, b_1], \ldots, [a_n, b_n]$ respectively, and define

$$\mathcal{P} = \{I_1 \times \cdots \times I_n \mid I_1 \in \mathcal{P}_1, \ldots, I_n \in \mathcal{P}_n\}.$$

Then $\mathcal{P}$ is indeed a partition of $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$.

Example 61.3:



Lecture 62 Partitions (II)

Lemma 62.1

Suppose $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is closed n-cube with $v(S) > 0$. Then every partition of S has a regular refinement.

Remark 62.2:
The above proof yields the following more general statement.

Let S be closed n-cube with $v(S) > 0$. Let $\mathcal{R}$ be finite collection of closed n-cubes such that $v(R) > 0$ $\forall R \in \mathcal{R}$ and $R \subseteq S$ $\forall R \in \mathcal{R}$. Then there exists a regular partition $\mathcal{P}$ of S such that $\forall P \in \mathcal{P}$ and $R \in \mathcal{R}$, either $P \subseteq R$ or $P^\circ \cap R^\circ = \varnothing$.

Lecture 63 Partitions (III)

Corollary 63.1

Let S be closed n-cube with $v(S) > 0$.

1. Let $\mathcal{P}, \mathcal{P}'$ be partitions of S. Then there exists regular partition $\mathcal{P}''$ of S such that $\mathcal{P}'' \leq \mathcal{P}'$ and $\mathcal{P}'' \leq \mathcal{P}$.
2. Let R be a closed n-cube with $v(R) > 0$ and $R \subseteq S$ and suppose $\mathcal{P}$ is a partition of S. Then there exists regular refinement $\mathcal{P}'$ of $\mathcal{P}$ such that $\forall P \in \mathcal{P}'$, either $P \subseteq R$ or $P^\circ \cap R^\circ = \varnothing$.

Proposition 63.2

Let $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a closed n-cube with $v(S) > 0$, and let $\mathcal{P}$ be a partition of S. Then $v(S) = \sum_{P \in \mathcal{P}} v(P)$.

Lecture 64 Integrability (I)

Definition 64.1: $U(f,\mathcal{P}), L(f,\mathcal{P})$

Let S be closed n-cube with $v(S) > 0$, $f : S \to \mathbb{R}$ a bounded function. Given partition $\mathcal{P}$ of S, define

$$U(f,\mathcal{P}) = \sum_{P\in\mathcal{P}} \left(\sup_{x\in P} f(x)\right) v(P)$$
$$L(f,\mathcal{P}) = \sum_{P\in\mathcal{P}} \left(\inf_{x\in P} f(x)\right) v(P)$$

Lemma 64.2

Define S, f as above.

1. If $\mathcal{P}', \mathcal{P}$ are partitions of S such that $\mathcal{P}' \leq \mathcal{P}$, then $L(f,\mathcal{P}) \leq L(f,\mathcal{P}')$, $U(f,\mathcal{P}') \leq U(f,\mathcal{P})$.
2. For any two partitions $\mathcal{P}, \mathcal{R}$ of S, $L(f,\mathcal{P}) \leq U(f,\mathcal{R})$.

Definition 64.3: $\overline{\int}_S f, \underline{\int}_S f$

Let f, S as in definition 1. Define

$$\overline{\int}_S f = \inf\{U(f,\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } S\},$$
$$\underline{\int}_S f = \sup\{L(f,\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } S\}.$$

Remark 64.4:
Since $\{S\}$ is a partition of S, $\{U(f,\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } S\}$ and $\{L(f,\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } S\}$ are both non-empty. Moreover, $\overline{\int}_S f$ and $\underline{\int}_S f$ are well-defined. And $\overline{\int}_S f \geq \underline{\int}_S f$.

Definition 64.5: integrable

f is said to be integrable on S if $\underline{\int}_S f = \overline{\int}_S f$, in which case the common value is denoted $\int_S f$.

Lecture 65 Integrability (II)

Proposition 65.1

Let S be closed n-cube with $v(S) > 0$. Suppose $c \in \mathbb{R}$ and define $f : S \to \mathbb{R}$ bt $f(x) = c\ \forall x \in S$. Then f is integrable on S and $\int_S f = c \cdot v(S)$.

Proposition 65.2

Let S be closed n-cube with $v(S) > 0$, $f : S \to \mathbb{R}$ be bounded. Then the following are equivalent:

1. f is integrable on S.
2. $\forall \epsilon > 0$, $\exists$ partition $\mathcal{P}$ of S such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.

Proposition 65.3

Let $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be closed n-cube with $v(S) > 0$. $f : S \to \mathbb{R}$ continuous on S relative to S. Then f is bounded and integrable on S.

Lecture 66 New integrable functions from old ones (I)

Proposition 66.1

Let S be closed n-cube with $v(S) > 0$. Suppose $f, g : S \to \mathbb{R}$ are bounded and integrable on S.

1. $\forall c \in \mathbb{R}$, cf is integrable on S and $\int_S cf = c \int_S f$.
2. $f + g$ is integrable on S and $\int_S f + g = \int_S + \int_S g$.
3. $|f|$ is integrable on S and $|\int_S f| \leq \int_S |f|$.

Lecture 67 New integrable functions from old ones (II)

Proposition 67.1

Let S be closed n-cube with $v(S) > 0$. $f : S \to \mathbb{R}$ bounded and integrable on S.

1. Let $R \subseteq S$ be a closed n-cube with $v(R) > 0$. Then f is integrable on R.
2. Given a partition $\mathcal{P}$ of S, f is integrable on P $\forall P \in \mathcal{P}$, and $\int_S f = \sum_{P \in \mathcal{P}} \int_P f$.

Lecture 68 Examples

Example 68.1:

$f(x) = \begin{cases} 0 & 0 \leq x < 1, \\ 1 & x = 1 \end{cases}$ is integrable on $[0, 1]$, and $\int_{[0,1]} f = 0$.

Example 68.2:

$f(x) = \begin{cases} 1 & x \in [0, 1] \cap \mathbb{Q}, \\ 0 & x \in [0, 1] \setminus \mathbb{Q} \end{cases}$ is NOT integrable on $[0, 1]$.

Example 68.3:

Define $f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & x \in \mathbb{Q}, x = \frac{p}{q} \text{ lowest terms } (p \in \mathbb{Z}, q \in \mathbb{N}) \end{cases}$ as in lecture 51. Then f is integrable on $[0, 1]$.

Lecture 69 Fubini's theorem (I)

Proposition 69.1: Fubini's theorem

S_1 closed m-cube, S_2 closed n-cube. $v(S_1), v(S_2) > 0$. $f : S_1 \times S_2 \to \mathbb{R}$ bounded. Assume

1. f is integrable on $S_1 \times S_2$.
2. $\forall x \in S_1$, the function $g_x : S_2 \to \mathbb{R}$ given by $g_x(y) = f(x, y)$ is integrable on S_2.

Then the function $G : S_1 \to \mathbb{R}$ given by $G(x) = \int_{S_2} g_x$ is bounded and integrable on S_1 and $\int_{S_1} G = \int_{S_1 \times S_2} f$.

Remark 69.2:

$\displaystyle\int_{S_1} G$ is referred to as an iterated integral since we can write it as $\displaystyle\int_{S_1} \left(\int_{S_2} g_x \right)$.