



# *Partial Differential Equations 2*

AMATH 453



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# Preface

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# Waves and Diffusions

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## 1.1 The wave equation

We already know the wave equation ( $c > 0$ ):

$$u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty,$$

and the general solution is of the form

$$u(x, t) = f(x + ct) + g(x - ct).$$

With initial conditions imposed, we have the IVP

$$u_{tt} - c^2 u_{xx} = 0, \quad \begin{cases} u(x, 0) = \phi(x), \\ u_t(x, 0) = \psi(x). \end{cases}$$

The solution to IVP is then

$$u(x) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds.$$

To interpret the integral, we can let  $\psi(x) = \mu'(x)$ , then the integral becomes

$$\int_{x-ct}^{x+ct} \psi(s) ds = \mu(x + ct) - \mu(x - ct).$$

## 1.2 Conservation laws

Given a wave equation, we multiply by  $u_t$ :

$$\begin{aligned} u_t u_{tt} - c^2 u_t u_{xx} &= 0 \\ \frac{\partial}{\partial t} \left( \frac{1}{2} u_t^2 \right) - c^2 \left[ \frac{\partial}{\partial x} (u_t u_x) - u_{tx} u_x \right] &= 0 \\ \frac{\partial}{\partial t} \left( \frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) - \frac{\partial}{\partial x} (c^2 u_t u_x) &= 0 \end{aligned}$$

Then the conservation law states that

$$\frac{\partial R}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where  $R \in (-\infty, +\infty)$ , and  $F \rightarrow 0$  with  $x \rightarrow \pm\infty$ .

### 1.3 The Diffusion Equation & Maximum principle

The diffusion equation is given by

$$u_t = ku_{xx}, \quad -\infty < x < \infty$$

with diffusion constant  $k > 0$ .

We define

$$\begin{aligned} R &= (a, b) \times (0, \infty) \\ R_T &= (a, b) \times (0, T] \\ \overline{R_T} &= [a, b] \times [0, T] \\ C_T &= \{a \leq x \leq b, t = 0\} \cup \{a, 0 \leq t \leq T\} \cup \{b, 0 \leq t \leq T\} \end{aligned}$$

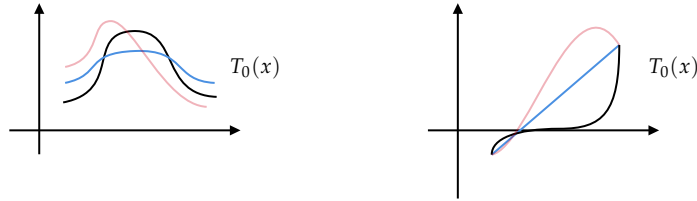
#### Theorem 1.1: Maximum principle

If  $u \in C(\overline{R_T}) \cap C^2(R_T)$  is a solution of the diffusion equation, then  $u(x, t) \leq \max_{C_T} \{u\}$  for all  $(x, t) \in R_T, T > 0$ . Here  $C_T$  is called the parabolic boundary of  $R_T$ .

#### Remark:

1. We can replace  $u_t - ku_{xx} = 0$  with  $u_t - ku_{xx} \leq 0$ .
2. A stronger version of the theorem exists which says that  $u(x, t) < \max_{C_T} \{u\}$  unless  $u$  is constant.
3. Same result applies to the minimum of  $u$  by replacing  $u$  with  $-u$ . However, in this case, (1) doesn't apply. Now we need  $u_t - ku_{xx} \geq 0$ .

Here are some intuitions. Consider a rod lying on  $[a, b]$  with initial non-constant temperature  $T_0(x)$ . Then as time goes, only blue  $T$  is possible, not red  $T$ .



#### Proof:

Let  $M = \max_{C_T} u$ . Note that  $M$  exists since  $u$  is continuous on  $C_T$ , and  $C_T$  is a closed boundary. We need to show that  $u \leq M$  on  $\overline{R_T}$ .

Let

$$v(x, t) = u(x, t) + \epsilon x^2, \quad \epsilon > 0$$

Let  $r = \max\{|a|, |b|\}$ . Then  $v(x, t) \leq M + \epsilon r^2$  on  $C_T$ . Now we prove that  $v \leq M + \epsilon r^2$  on  $R_T$ .

On  $R_T$ , we have

$$u = v - \epsilon x^2 \leq M + \epsilon(r^2 - x^2)$$

Now if we take the derivative,

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon = -2k\epsilon < 0 \quad (*)$$

- (i) Suppose  $v(x, t)$  has a maximum at an interior point  $(x_0, t_0)$ , i.e.,  $(x_0, t_0) \in (a, b) \times (0, T)$ . Then

$v_t(x_0, t_0) = 0$ . Moreover,  $v_{xx}(x_0, t_0) \leq 0$ . Then

$$v_t(x_0, t_0) - kv_{xx}(x_0, t_0) = -kv_{xx}(x_0, t_0) \geq 0$$

contradicting (\*), thus there are no interior max.

(ii) Suppose  $v(x, t)$  has a maximum at an interior point of the upper boundary.  $v_t(x_0, T) \geq 0$ . Then

$$v_t(x_0, T) - kv_{xx}(x_0, T) \geq 0$$

contradicting (\*), thus there are no maximum along the upper boundary.

But  $v$  is continuous on  $\overline{R_T}$ , thus it has a maximum value which we now know must occur on  $C_T$ . Hence  $v \leq M + \epsilon r^2$  on  $\overline{R_T}$ . Letting  $\epsilon \rightarrow 0$ , we have  $u \leq M$  on  $R_T$ .  $\square$

## 1.4 Uniqueness of the Dirichlet Problem

$$\begin{aligned} u_t - ku_{xx} &= f(x, t) & a < x < b, 0 < t < \infty \\ u(x, 0) &= \phi(x) \\ u(a, t) &= g(t) \\ u(b, t) &= h(t) \end{aligned} \tag{1.1}$$

### Theorem 1.2

The solution of (1.1) is unique.

**Proof:**

Suppose there are two solutions  $u_1(x, t)$  and  $u_2(x, t)$ . Let  $w(x, t) = u_1 - u_2$ . Now we calculate

$$\begin{aligned} w_t - kw_{xx} &= (u_{1t} - ku_{1xx}) - (u_{2t} - u_{2xx}) = f - f = 0 \\ w(x, 0) &= u_1(x, 0) - u_2(x, 0) = \phi - \phi = 0 \\ w(a, t) &= w(b, t) = 0 \end{aligned}$$

By maximum principle, we have  $w \leq 0$  on the boundary, and by minimum principle,  $w \geq 0$ , since  $\max_{C_T} \{w\} = \min_{C_T} \{w\} = 0$ . Then we conclude that  $w \equiv 0$ .  $\square$

Now we present a second proof using energy method:

**Proof:**

Given  $w_t - kw_{xx} = 0$ , multiply both sides by  $w$ :

$$0 = ww_t - kw_{xx} = \frac{\partial}{\partial t} \left( \frac{1}{2} w^2 \right) - k \frac{\partial}{\partial x} (ww_x) + kw_x^2$$

If we integrate both sides,

$$\frac{d}{dt} \int_a^b \frac{1}{2} w^2 dx = k \int_a^b (ww_x)_x dx - k \int_a^b w_x^2 dx = kww_x \Big|_a^b - k \int_a^b w_x^2 dx$$

Thus

$$\frac{d}{dt} \int_a^b \frac{1}{2} w^2 dx = -k \int_a^b w_x^2 dx$$

Then

$$\int_a^b \frac{1}{2} w^2 dx = 0 \quad \text{for all the time}$$

Then  $w \equiv 0$  on  $a \leq x \leq b, 0 \leq t \leq T$ .  $\square$

Now let's examine stability. Consider

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(a, t) &= u(b, t) = 0 \end{aligned}$$

and let  $u_j(x, t)$  be the solution for  $u(x, 0) = \phi_j(x)$  for  $j = 1, 2$ .

Let  $w = u_1 - u_2$ . Proceeding as before (energy method) we have

$$\int_a^b (u_1 - u_2)^2 dx \leq \int_a^b (\phi_1 - \phi_2)^2 dx$$

This tells us  $\|u_1 - u_2\|_2 \rightarrow 0$  as  $\|\phi_1 - \phi_2\|_2 \rightarrow 0$ . This is called **stability in the square integrable sense**.

Alternatively, by maximum principle,

$$\max |u_1 - u_2| \leq \max |\phi_1 - \phi_2|$$

using maximum & minimum principle, i.e.,

$$\begin{aligned} \max\{u_1 - u_2\} &\leq \max\{\phi_1 - \phi_2\} \\ \min\{u_1 - u_2\} &\geq \min\{\phi_1 - \phi_2\} \end{aligned}$$

This is called **stability in the uniform sense**.

## 1.5 Diffusion on the Whole Line

Consider the initial value problem

$$u_t - ku_{xx} = 0 \quad \text{on } -\infty < x < \infty, \quad 0 < t < \infty \quad (1.2)$$

$$u(x, 0) = \phi(x) \quad (1.3)$$

If  $s(x, t)$  is a solution of (1.2), then so is

$$u(x, t) = \int_{-\infty}^{\infty} s(x - y, t) g(y) dy \quad (1.4)$$

for any function  $g(y)$ . We can find  $u_t, u_x, u_{xx}$  and take it into (1.2):

$$u_t - ku_{xx} = \int_{-\infty}^{\infty} [s_t(x - y, t) - ks_{xx}(x - y, t)] g(y) dy = 0$$

So we now find a solution of (1.2) with the property that  $s(x, 0) = \delta(x)$ , i.e., solve

$$\begin{aligned} s_t - ks_{xx} &= 0 \\ s(x, 0) &= \delta(x) \end{aligned}$$

To do this, consider the problem:

$$\begin{aligned} v_t - kv_{xx} &= 0 \\ v(x, 0) &= v_0 H(x) \end{aligned} \quad (1.5)$$

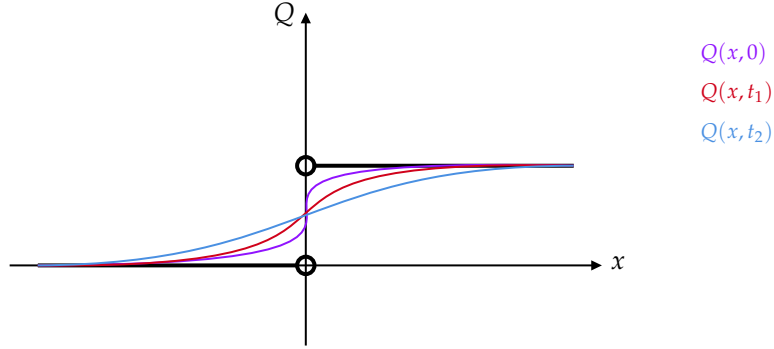
$H = \text{Heaviside function}$

$v_0$  carries the dimension of  $v$ , thus  $H(x)$  is dimensionless.

**Similarity solution of (1.5)**

Let  $Q = \frac{v}{v_0}$  which is dimensionless, then the original problem gets transformed to

$$\begin{aligned} Q_t &= kQ_{xx} \\ Q(x, 0) &= H(x) \end{aligned}$$



The solution can only be a function of  $x, t$  and  $k$ :  $Q = F(x, t, k)$ . Then we can apply dimensionless analysis. This means  $Q$  can only depend on dimensionless combinations of  $x, t$  and  $k$ . We have

$$\begin{aligned} [x] &= L \\ [t] &= T \\ [k] &= \frac{L^2}{T} \end{aligned}$$

Then

$$[x^a t^b k^c] = L^a T^b \frac{L^{2c}}{T^c} \implies b = c, 2c = -a$$

This tells us

$$Q = f(\theta) \quad \text{where } \theta = \frac{x}{\sqrt{kt}}$$

By chain rule, we have

$$\begin{aligned} Q_t &= f'(\theta) \cdot \theta_t = -\frac{1}{2} \frac{\theta}{t} f'(\theta) \\ Q_x &= f'(\theta) \cdot \theta_x = \frac{1}{\sqrt{kt}} f'(\theta) \\ Q_{xx} &= \frac{1}{kt} f''(\theta) \end{aligned}$$

Then

$$\begin{aligned} Q_t - kQ_{xx} &= -\frac{\theta}{2t} f' - \frac{k}{kt} f'' = 0 \\ f''(\theta) &= -\frac{1}{2} \theta f'(\theta) \\ f'(\theta) &= A e^{-\frac{\theta^2}{4}} \\ f(\theta) &= A \int_{-\infty}^{\theta} e^{-s^2/4} ds + C \end{aligned}$$

As  $x \rightarrow +\infty, \theta \rightarrow +\infty$ , and  $Q(x, t) = f(\theta) \rightarrow 1$ . Then  $\lim_{\theta \rightarrow +\infty} f(\theta) = 1$ .

As  $x \rightarrow -\infty, \theta \rightarrow -\infty$  and  $Q(x, t) = f(\theta) \rightarrow 0$ ,  $\lim_{\theta \rightarrow -\infty} f(\theta) = 0$ .

Therefore,  $C$  must be 0, and  $A \int_{-\infty}^{\infty} e^{-s^2/4} ds = 1$ . Using the change of variable  $\eta = \frac{s}{2}$ :

$$\int_{-\infty}^{\theta} e^{-s^2/4} ds = 2 \int_{-\infty}^{\theta/2} e^{-\eta^2} d\eta = 2 \int_{-\infty}^{x/\sqrt{4kt}} e^{-\eta^2} d\eta$$



So if we take  $\theta = \frac{x}{\sqrt{4kt}}$  at the beginning, we get  $\tilde{A} = 2A$  and

$$\tilde{A} \int_{-\infty}^{\infty} e^{-s^2} ds = 1 \implies \tilde{A} = \frac{1}{\sqrt{\pi}}$$

Thus we get

$$Q = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-s^2} ds$$

Note that for  $x > 0$ , as  $t \rightarrow 0^+$ ,  $\frac{x}{\sqrt{4kt}} \rightarrow +\infty$  and  $Q(x, t) \rightarrow \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 1$ .

And for  $x < 0$  as  $t \rightarrow 0^+$ ,  $Q \rightarrow 0$ . The reason for the name “similarity solution” is because the curve is being stretched over time.

$s(x, t)$  has many names: source function (not a great name), Green’s function, fundamental solution, propagator of the diffusion equation, diffusion kernel...

Consider a diffusion equation with initial condition

$$\begin{aligned} u_t + ku_{xx} &= 0 \\ u(x, 0) &= \delta(x) \end{aligned}$$

The solution is Gaussian

$$u = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

For any  $t > 0$ ,  $u$  is non-zero. It gets instantaneously non-zero everywhere.

## Reflections and Sources

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### 2.1 Diffusion on the Half-Line

We will start with diffusion on the half line Dirichlet problem.

$$\begin{aligned} v_t - kv_{xx} &= 0 & 0 < x < \infty, 0 < t < \infty \\ v(x, 0) &= \phi(x) \\ v(0, t) &= 0 \quad \text{for } t > 0 \end{aligned}$$

Let

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}$$

and solve

$$\begin{aligned} u_t + ku_{xx} &= 0 & \text{on } -\infty < x < \infty \\ u(x, 0) &= \phi_{odd}(x) \end{aligned}$$

Then  $v(x, t)$  is restriction of  $u$  to  $x > 0$ . From an earlier result

$$u(x, t) = \int_{-\infty}^{\infty} s(x - y, t) \phi_{odd}(y) \, dy$$

where

$$s(x, t) = \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi kt}}$$

**Claim** From the property of  $s$  and  $\phi_{odd}$ , we can show that  $u(x, t)$  is an odd function of  $x$ . Thus  $u(0, t) = 0$ .

Now we see that

$$\begin{aligned} u(x, t) &= \int_{-\infty}^0 s(x - y, t) [-\phi(-y)] \, dy + \int_0^{\infty} s(x - y, t) \phi(y) \, dy \\ &= \int_{-\infty}^0 s(x + y, t) \phi(y) \, dy + \int_0^{\infty} s(x - y, t) \phi(y) \, dy & \text{let } y = -y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[ e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y) \, dy \end{aligned} \tag{2.1}$$

Example:

$$\begin{aligned} v_t - kv_{xx} &= 0 & 0 < x < \infty \\ v(x, 0) &= 1 & x > 0 \\ v(0, t) &= 0 \end{aligned}$$

Then  $\phi_{odd} = -1 + 2H(x)$ .

Recall the solution of

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x, 0) &= H(x) \end{aligned} \tag{2.2}$$

is

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4k\pi}} e^{-s^2} ds$$

Let  $u(x, t) = -1 + 2q(x, t)$ . Then  $q(x)$  is the solution to (2.2). Hence we have

$$u = -1 + \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4k\pi t}} e^{-s^2} ds = \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Another way to solve is to use (2.1).

Consider Neumann Boundary condition ( $0 < x < \infty$ ):

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x, 0) &= \phi(x) \\ u_x(0, t) &= 0 \end{aligned}$$

We can let

$$\phi_{even} = \begin{cases} \phi(x) & x > 0 \\ \phi(-x) & x < 0 \end{cases}$$

and solve

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x, 0) &= \phi_{even} \end{aligned}$$

With some algebra, we get

$$u = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[ e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y) dy$$

## 2.2 Reflections of Waves

Dirichlet Problem on the half line

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 & 0 < x < \infty \\ v(x, 0) &= \phi(x) \\ v_t(x, 0) &= \psi(x) \\ v(0, t) &= 0 \end{aligned}$$

The idea is  $u(-x, t) = -u(x, t)$ , then  $u = 0$  at  $x = 0$ . So consider an odd reflection about  $x = 0$ :

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases} \quad \psi_{odd} = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \end{cases}$$

We know that the solution of  $(-\infty < x < \infty)$

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(x, 0) &= \phi_{odd}(x) \\ u_t(x, 0) &= \psi_{odd}(x) \end{aligned}$$

is

$$u(x, t) = \frac{1}{2} [\phi_{odd}(x + ct) + \phi_{odd}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{odd}(y) \, dy$$

Note that  $(t > 0)$

$$u(0, t) = \frac{1}{2} [\phi_{odd}(ct) + \phi_{odd}(-ct)] + \frac{1}{2c} \int_{-ct}^{ct} \psi_{odd}(y) \, dy = 0$$

which satisfies the initial condition.

3 cases of the solution

(a)  $x > c|t|$ , then  $x + ct > 0, x - ct > 0$ , then the solution  $(t > 0)$  becomes

$$u(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) \, dy$$

(b) Consider  $0 < x < ct, t > 0$ , we have  $x - ct < 0, x + ct > 0$ . Then

$$\begin{aligned} \phi_{odd}(x - ct) &= -\phi(-x + ct) \\ \phi_{odd}(x + ct) &= \phi(x + ct) \end{aligned}$$

and

$$\begin{aligned} \int_{x-ct}^{x+ct} \psi_{odd}(y) \, dy &= \int_{x-ct}^0 [-\psi(-y)] \, dy + \int_0^{x+ct} \psi(y) \, dy \\ &= -\int_0^{-x+ct} \psi(y) \, dy + \int_0^{x+ct} \psi(y) \, dy \\ &= \int_{-(x-ct)}^{x+ct} \psi(y) \, dy \end{aligned}$$

Therefore

$$u = \frac{1}{2} [\phi(x + ct) - \phi(-(x - ct))] + \frac{1}{2c} \int_{-(x-ct)}^{x+ct} \psi(y) \, dy$$