Introduction to Game Theory

ECON 212

Joel Blit

LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of ECON 212 during Fall 2021 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

For any questions, send me an email via https://notes.sibeliusp.com/contact.

You can find my notes for other courses on https://notes.sibeliusp.com/.

Sibelius Peng

Contents

1

Introduction

What is game theory? It is study of strategic interactions between agents (players). One player is maximizing its utility subject to other constraints. The outcome for an agent depends not only on his choice of action but also on the action of the other player(s).

What do we mean by game? Unlike common definition of game: a competitive activity played according to rules like chess, monopoly, in this course, we mean strategic interactions between agents in any arena: politics, economic competition.

We are going to analyze these situations by building models (which are necessarily an incomplete abstraction). If we understand the "game", more likely to "win" at it. If we don't like it, we can sometimes change it, or not play it at all.

1.1 Strategic Game

For a complete mathematical definition, please refer to chapter 2 of CO 456.

strategic game

A **strategic game** consists of

- a set of **players**,
- a set of **actions** for each player,
- **preferences** over the set of action profiles for each player.

Action profile is an action for each different player, and there are many action profiles, which lead to different outcomes for the game.

Note that in a strategic game, "time" is absent. In other words, all players choose an action simultaneously. Once an action is chosen it cannot be changed This is different from the extensive games. In extensive games, there is a sequence of actions happening in different time. Then the players can actually see what other players played before.

1.1.1 Prison's Dilemma

A famous example of strategic game is **prison's dilemma** (PD). Two suspects are held in separate cells. The police has enough evidence to convict each of them of a minor offense. If one of them finks they can convict the other of a major offense (and they set free the one who finked). If they both fink, they are both convicted of a major offense, but the punishment is slightly less harsh because they cooperated.

Now let's formalize this model. There are two players (suspects) p_1, p_2. There are two actions for each player: {Quiet, Fink}, or $\{Q, F\}$ for $i = 1, 2$. Thus there are four action profiles, and players have preferences over these four action profiles as follows:

$$u_1(F,Q) > u_1(Q,Q) > u_1(F,F) > u_1(Q,F)$$
$$u_2(Q,F) > u_2(Q,Q) > u_2(F,F) > u_2(F,Q)$$

Here $u_i(ap_1, ap_2)$ denotes the utility or payoff for player i in the action profile (ap_1, ap_2). We can also express this using a payoff table:

		Suspect 2	
		Fink	Quiet
Suspect 1	Fink	b, b	d, a
	Quiet	a, d	c, c

Here we typically have $a > b > c > d$. Take the top left cell as an example: first b is payoff to suspect 1 if they are both quiet, and second b refers to payoff to suspect 2 if they are both quiet. Similarly, second a in the top right cell is payoff to suspect 2 if suspect 1 is quiet and suspect 2 finks. First c in the bottom right cell is payoff to suspect 1 if they both fink.

As we can see here, prisoner's dilemma is quite simple, but it captures something really important. It models a situation where there are gains from cooperation, but each player is better off being uncooperative regardless of what the other does. Many real-life situations can be represented by this type of strategic interaction.

Some selected examples include arms race, two countries are deciding between no nuke or build nuke. It is true that there is no way for these two countries to coordinate, unless we go outside of this game: two countries can get together and say: "we realize that we are in prison's dilemma. The outcome would not be good for either one of us." Then two countries can sign some legally binding agreement, which can increase the cost of building nuclear weapons or diplomatic cost.

Tariff Wars

Another examples include advertising in duopoly (ads/no ads), tariff wars. In tariff wars, there are two countries engaging in free trade. Each country can choose whether to impose a tariff. From https://en.wikipedia.org/wiki/Tariff,

> A tariff is a tax imposed by a government of a country or of a supranational union on imports or exports of goods.

Tariff results in the country paying lower prices for imports and therefore a gain of 8 at the other country's expense. But Tariffs also result in a deadweight loss of 5 for the country that imposes them. From https://www.investopedia.com/terms/d/deadweightloss.asp,

> A deadweight loss is a cost to society created by market inefficiency, which occurs when supply and demand are out of equilibrium. Mainly used in economics, deadweight loss can be applied to any deficiency caused by an inefficient allocation of resources.

Then we can depict the tariff wars with the following payoff table:

		Canada: No Tariff	Canada: Tariff
U.S.	No Tariff	$0, 0$	$-8, 3$
	Tariff	$3, -8$	$-5, -5$

Consider the bottom left cell. Imagine the U.S. imposes tariff on Canada, not vice versa. The U.S. gets a benefit of 8, but also suffers from economic inefficiency of 5, so the U.S. is 3 better off than they were in free trade. Canada doesn't have inefficiency cost, but 8 dollars are taken from the U.S., thus the payoff is 8 dollars lower than they were in free trade.

Now consider the bottom right cell. Both countries are taking 8 from the other. Then net-net, the result is 0 for both. And they both have -5 inefficiency cost.

Carbon Emissions

There are two countries both of which are currently emitting lots of carbon. Each country can choose whether to continue to emit at current levels or whether to curb their emissions. Assume the cost of curbing emissions is 6, and benefit of a country curbing emissions is 5 (to each country in the world). This can be described in the following payoff table:

		Canada: Curb Emissions	Canada: Emit
U.S.	Curb Emissions	$4, 4$	$-1, 5$
	Emit	$5, -1$	$0, 0$

Location of Production and IP Theft

Over the last 10 years, lots of high-tech U.S. companies have moved to China. They found that local competitors learned the technologies quickly from these companies, then they are going to have lower costs. The explanation lies in the game theory.

Assume there is an industry with two U.S. firms (duopoly) that are splitting the world market equally (earning $10M each). Each firm can choose whether to continue producing in the U.S. or shift production to China. Because producing in China lowers the costs, firm can drop prices and drive the other firm out of business. But when a firm produces in China their product will be imitated and they will face new Chinese entrants in their industry (which decreases profit by $5M).

		Firm 2: U.S.	Firm 2: China
Firm 1	U.S.	10M, 10M	0M, 15M
	China	15M, 0M	5M, 5M

Joint Project

There are two students Alice and Bob working on a joint project. Each student can choose whether to put effort into their part of the project. If neither student works, they get 50%. If one works, they get 70%. If both work, they get 90%. But note that effort is costly. Each student is indifferent between working and getting a grade X, and shirking and getting $X - 30$.

		Bob	
		Work	Shirk
Alice	Work	$60, 60$	$40, 70$
	Shirk	$70, 40$	$50, 50$

To break this scenario, professor could observe how much effort each student has put in. Another way is that we can turn it into a repeated game. Then the outcome of this game will affect the games in the future.

1.1.2 Battle of the Sexes (BoS)

Early before the invention of the internet, Barb and Sam would like to get together in the evening. The "what's on" section of the paper says that there are two events going on in Waterloo that evening: Ballet and Soccer. Barb really likes watching Ballet, and Sam really likes watching Soccer. But they both really like each other and would most of all like to meet up. The problem is that they forgot to coordinate on the venue (and cell phones were not yet invented).

		Sam	
		Ballet	Soccer
Barb	Ballet	$2, 1$	$0, 0$
	Soccer	$0, 0$	$1, 2$

Observer that BoS is a coordination game. Players want to cooperate, so they both pick what's best for both of them. Once they cooperate, they are not incentive to do something else. This is quite different from the prison's dilemma where each player will deviate from the cooperation.

BoS can also model a number of different real-life situations. For example, the Prime Minister and the Finance Minister trying to decide which position to take on an issue. Another example would be that two firms trying to agree on an interface between their respective products.

1.1.3 Matching Pennies

Each of two players has a coin. They simultaneously choose whether to show the head or the tail of the coin If they show the same side. then player 2 pays player 1 a dollar. If they show a different side, then player 1 pays player 2 a dollar.

		PII	
		Head	Tail
PI	Head	$1, -1$	$-1, 1$
	Tail	$-1, 1$	$1, -1$

Notice that this is an example of game where one wins and the other loses. In other words, this is a "strictly competitive" game (the interests of the two players are diametrically opposed). A typical example would be penalties kick in football, in which a player is allowed to take a single shot on the goal while it is defended only by the opposing team's goalkeeper (from wiki).

1.1.4 Stag Hunt

A group of hunters is trying to catch a stag. Each hunter can focus on catching the stag, or he can instead catch a hare. If all hunters pursue the stag, then they catch it and share it equally. If any hunter goes for a hare (which he catches and keeps to himself), then the stag escapes All hunters prefer a stag, to a hare, to nothing.

Now let's formalize this model. Players are all the hunters. For each player, the set of actions is {Stag, Hare}. The preferences are: for each hunter the action profile in which they all choose Stag is ranked highest, followed by any profile in which he chooses Hare, followed by any profile in which he chooses Stag and at least one other hunter chooses Hare. If we have two hunters, we can draw a payoff table:

		HII	
		Stag	Hare
HI	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

Stag Hunt is similar to the prison's dilemma except that players would rather cooperate as long as everyone else is cooperating.

Perhaps this is a better model of an arm's race. Country would rather not have arms when the other country doesn't. In other words, we don't care the military superiority over other countries, but we do care other countries not having the stronger militaries. Perhaps it is also a better model of working on a joint project for the similar reason. Student would rather put in effort as long as it is not futile.

Golden balls: Nick & Ibrahim

Consider a television show: https://youtu.be/S0qjK3TWZE8, where its scenario can be summarized in the following payoff table:

		N	
		Split	Steal
I	Split	6.8, 6.8	0, 13.6
	Steal	13.6, 0	0, 0

First, is this a prison's dilemma? Unfortunately, it is very close to a prison's dilemma, but no, because here we don't have strict inequality $u_2(\text{Steal}, \text{Split}) < u_2(\text{Steal}, \text{Steal})$.

What Nick did in this game is that he eliminated the left column of this payoff table. Thus there would be no incentive for Ibrahim to choose Steal, and given that he values being honest, he chose to Split.

2

Nash Equilibrium in Pure Strategies

2.1 Definition

It would be nice to be able to say something about the outcomes of these games. In particular, what actions will be chosen by the players in a given strategic game? Thus we need a solution concept: **Nash Equilibrium**.

Consider a simple game **Odd-Even**: Both players say even/odd at the same time. If both players choose the same option, either even or odd, they both get payoff 1. Otherwise they both get payoff -1. Now if we try to do this experiment in the whole class: a student tries to play this game with colleagues as many people as possible. This student will record his choice of action and payoff of each play. It turns out there is an *evolution* in what people are playing. People have beliefs about what others are going to do. These form *given beliefs*, which are *correct*, and no ones want to deviate. This leads to the definition of Nash Equilibrium.

Nash Equilibrium (NE)

The action profile a^* in a strategic game with ordinal preferences is a **Nash Equilibrium** if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j.

Mathematically, let $N = \{1, \ldots, N\}$ be a set of players. Then a^* in a strategic game is a Nash Equilibrium if for all $i \in N$, for all action a_i of player i, $u_i(a^*) \geq u_i(a_i, a^*_{-i})$.

Note that we typically use a^* to denote the Nash Equilibrium. a_{-i} is the action profile obtained from a by dropping a_i.

2.2 Rationales for NE

Consider a real-life situation where there are two people walking in different directions in a street. They don't want to bump into each other. They can only choose to go left or right. If they choose both left/right, then they won't bump and get payoff 1, otherwise get payoff -1 and bump.

It turns out that people tend to go right in countries in Canada, which is a social norm. This brings the rationale for NE: NE establishes **stable social convention**. If we pair up players at random from a large population, and there are a bunch of one-shot games over time, all players will adopt a behavior that is consistent with a NE. The beliefs will be developed about what other players will choose, then they will choose the best action according to given beliefs. In a long run, there will be an equilibrium, and these beliefs will be correct.

Moreover, a NE is a **self-enforcing agreement**. Thus no one is incentivized to deviate. Note that in prison's dilemma, no cooperative equilibrium is achievable. An agreement to be quiet is not self-enforcing. We are more likely to observe self enforcing outcomes in reality because they can be achieved more realistically.

In practice, NE is not always going to be the outcome in a strategic game. First, not everyone is rational and/or an optimizing agent. Second, the NE says something about the equilibrium, not the path that player actions take on their way to the equilibrium.

2.3 Finding NE

We can determine NE of a (pure) strategic game by ruling out all action profiles that are not an equilibrium according to the definition. Another way is to use best response functions to determine the NE.

Stag Hunt > 2 players

It turns out there are 2 NEs in this game: (Stag, . . ., Stag) and (Hare, . . ., Hare). It's impossible to check every action profiles when the number of players goes to infinity. Here we can show both are NE and no other action profile is a NE.

First consider everyone picks Stag. They are getting the highest payoff, so there is no profitable deviation. Then consider everyone picks Hare. If any hunter deviates, then he gets nothing, which is worse off.

Now we need to show there are no other NE. Consider an action that is not the two above. Then there is at least one person chasing Stag and at least one person chasing Hare. Then the person chasing the Stag is better off deviating to get a Hare. Therefore, this is not an NE.

Guessing $2/3$ of Average

If there are only two players, regardless what the other person does, choosing 1 will always win (or tie instead of outright losing). The action "1" weakly dominates all other actions. Weak domination requires at least one strict inequality between this action profile and others. Detailed definition later.

When there are three players, then we can show $(1, 1, 1)$ is an NE by iterated elimination of weakly dominated strategies (IEWDS). CO 456 discusses this method in details.

2.4 Best Response Functions

We are interested in determining the best actions for a player given the actions of the other players.

best response function

For any player i, the **best response function** associates a set of actions by player i to any list of the other players' actions.

Formally, we let

$$B_i(a_i) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \quad \forall a_i' \in A_i\}$$

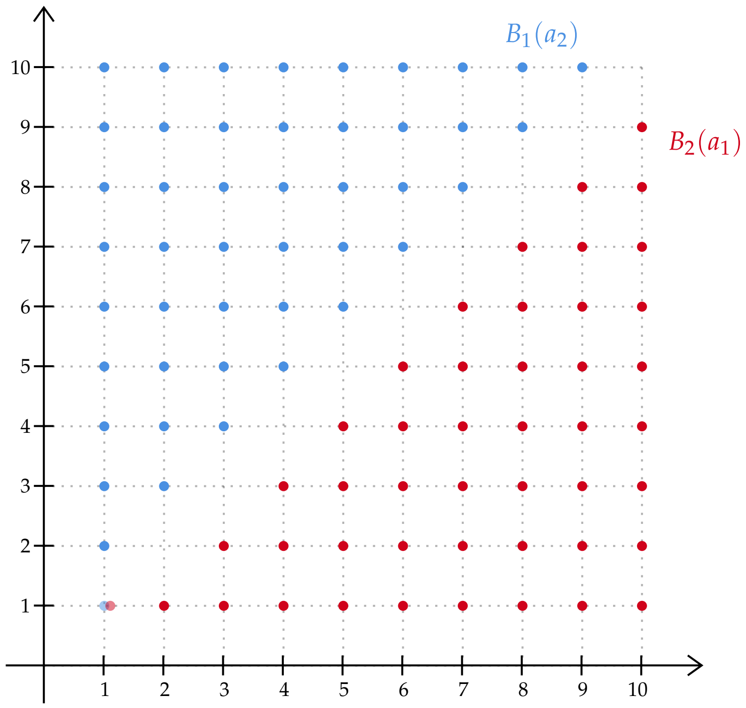
where A_i is the set of possible actions for player i.

A quick trivial note from this definition: each element of $B_i(a_{-i})$ is a best response of player i to a_{-i}. In other words, given other players choose a_{-i}, then player i can do no better than an element of $B_i(a_{-i})$.

Best response function examples skipped, as one can find more examples in section 2.1.1 of CO 456.

As mentioned earlier, one can find NE using best response functions, because sometimes a game has too many action profiles to check feasibly.

Now consider the game of choosing 2/3 average where there are two players, and $A_i = [10]$ for each i. The best responses for both players can be depicted as follows:



We notice that the only point overlapped between two best response functions is Nash Equilibrium. This is not a coincidence and this is how we will use best response functions to find Nash Equilibria. If the best response functions overlap, at that point, each player is playing the best response at the other's play. Then by definition, they don't have a profitable deviation, and this is the definition of Nash Equilibrium.

Recall that an NE is an action profile such that no player can do better by changing his action, given the other players' actions. Therefore, we can define a NE as an action profile in which every player's action is a best response to the other players' actions.

Proposition 36.1

The action profile a^* is a Nash Equilibrium of a strategic game with ordinal preferences if and only if every player's action is a best response to the other players' actions:

$$a_i^* \in B_i(a_{-i}^*) \qquad \forall i \in N. \tag{36.2}$$

Note that this is the same as Lemma 2.1 in CO 456. Because of the difference of nature between these two courses, CO 456 calls it a lemma, while ECON 212 calls it a proposition.

Now consider the special case where every player has a single best response. If each player i has a single best response to each list a_{-i} we can write (36.2) as equations:

$$a_i^* = b_i(a_{-i}^*) \qquad \forall i \in N, \tag{36.3}$$

where $b_i(a_{-i}^*)$ is the single member in $B_i(a_{-i}^*)$. Note that (36.3) is a set of n equations in n unknowns, where $n = |N|$ is the number of players.

Now we can find NE using BRF by first finding BRF for each player, then finding the action profiles that satisfy (36.2) (which reduces to (36.3) if all players have single best responses). Check two examples after Lemma 2.1 in CO 456.

Consider a more complicated example modelling a relationship. Players are two individuals. Each player's set of actions is the set of effort levels that they put into the relationship. We require $a_i \geq 0$ for $i = 1, 2$ and a_i can be any real nonnegative value. Player i's preferences (utilities) are given by:

$$u_i(a_i, a_j) = a_i(c + a_j - a_i) \qquad \text{for } i = 1, 2 \text{ and } c > 0.$$

First note that there is a complementarity between two players: if a_j is bigger, then if we pick bigger a_i, we get a bigger number. Also note there is a diminishing term $-a_i^2$.

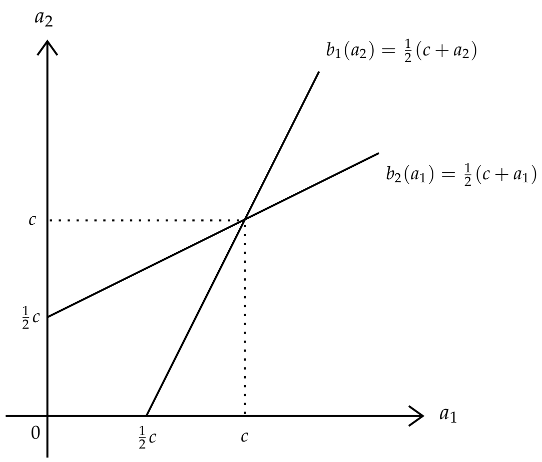
We are going to use BRF to find NE. Player i's best response function to a_j is:

$$B_i\left(a_j\right) = \left\{a_i \geq 0 : a_i\left(c + a_j - a_i\right) \geq a_i'\left(c + a_i - a_i'\right) \text{ for all } a_i' \geq 0\right\}$$

Because the payoff function is quadratic, it will have a single maximum for any a_j:

$$b_i(a_j) = \max_{a_i} a_i(c + a_j - a_i) = \frac{1}{2}(c + a_j)$$

Now let's draw BRFs for $i = 1, 2$:



So (c, c) is NE. In practice, we can plot before solving this algebraically using (36.3).