Coding Theory

CO 331

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Preface

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Sibelius Peng

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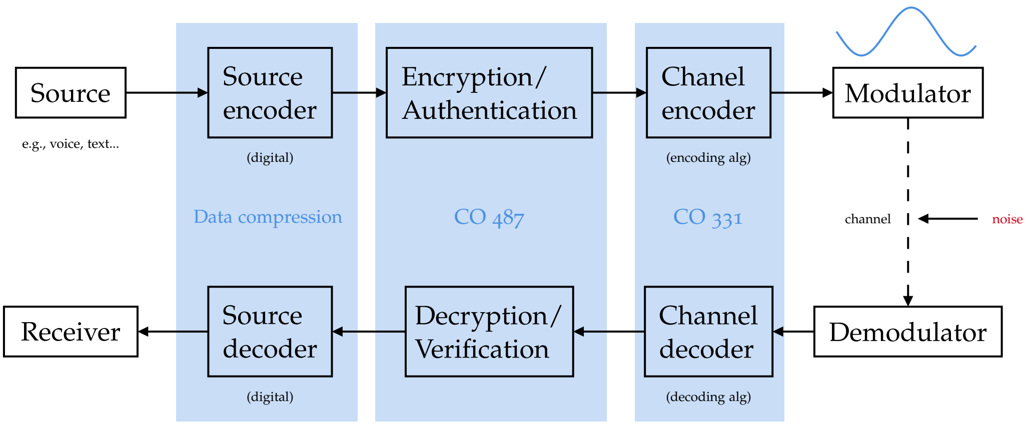
Introduction

Coding theory is about clever ways of adding redundancy to messages to allow (efficient) error detection and error correction.

Here is our communication model:



Example: Parity Code

Encoding algorithm Add a 0 bit to the (binary) msg m if the number of 1's in m is even; else add a 1 bit.

Decoding algorithm If the number of 1's in a received msg r is even, then accept r; else declare that an error has occurred.

Example: Replication Code

Source msgs	Codeword	# err/codeword (always) detected	# err/codeword (always) corrected *	Information rate
0 1	0 1	0	0	1
0 1	00 11	1	0	$\frac{1}{2}$
0 1	000 111	2	1	$\frac{1}{3}$
0 1	0000 1111	3	1	$\frac{1}{4}$
0 1	00000 11111	4	2	$\frac{1}{5}$

$\xrightarrow{\text{encoding algorithm}}$

*: using "nearest neighbour decoding"

Goal of Coding Theory

Design codes so that:

1. High information rate
2. High error-correcting capability
3. Efficient encoding & decoding algorithms

Course Overview

This course deals with *algebraic methods* for designing good (block) codes. The focus is on error correction (not on error detection). These codes are used in wireless communications, space probes, CD/DVD players, storage, QR codes, etc.

Some modern stuff are not covered: Turbo codes, LDPC codes, Raptor codes, . . . Their math theories are not so elegant as algebraic codes.

The big picture

Coding theory in its broadest sense deals with techniques for the *efficient*, *secure* and *reliable* transmission of data over communication channels that may be subject to *non-malicious errors* (noise) and *adversarial intrusion*. The latter includes passive intrusion (eavesdropping) and active intrusion (injection/deletion/modification).

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Fundamentals

1.1 Basic Definitions and Concepts

alphabet

An **alphabet** A is a finite set of $q \geq 2$ symbols.

word

A **word** is a finite sequence of symbols from A (also: vector, tuple).

length

The **length** of a word is the number of symbols it has.

code

A **code** C over A is a set of words (of size ≥ 2).

codeword

A **codeword** is a word in the code C.

block code

A **block code** is a code in which all codewords have the same length.

A **block code of length** n **containing** M **codewords over** A is a subset $C \subseteq A^n$ with $|C| = M$. C is called an $[n, M]$-code over A.

Example:

$A = \{0,1\}$. $C = \{00000, 11100, 00111, 10101\}$ is a $[5,4]$-code over $\{0,1\}$.

Messages		Codewords
00	$\to$	00000
10	$\to$	11100
01	$\to$	00111
11	$\to$	10101

Encoding of messages (1-1 map)

Assumptions about the communications channel

(1) The channel only transmits symbols from A (“hard decision decoding”).

(2) No symbols are deleted, added, interchanged or transposed during transmission.

(3) The channel is a q-symmetric channel:

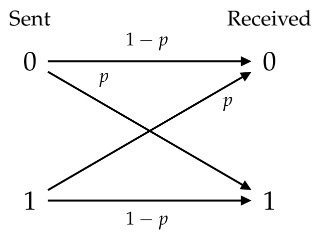
Let $A = \{a_1, \ldots, a_q\}$. Let $X_i =$ the i^{th} symbol sent. Let $Y_i =$ the i^{th} symbol received. Then for all $i \geq 1$, and all $i \leq j, k \leq q$,

$$\Pr(Y_i = a_j | X_i = a_k) = \begin{cases} 1-p, & \text{if } j = k \\ \frac{p}{q-1}, & \text{if } j \neq k. \end{cases}$$

p is called the **symbol error probability** of the channel ($0 \leq p \leq 1$).

Binary Symmetric Channel (BSC)

A 2-symmetric channel is called a binary symmetric channel.



For a BSC:

1. If $p = 0$, the channel is *perfect*.
2. If $p = 1/2$, the channel is *useless*.
3. If $1/2 < p \leq 1$, then flipping all received bits converts the channel to a BSC with $0 \leq p < 1/2$.
4. Henceforth, we will assume that $0 < p < 1/2$ for a BSC.

Exercise:

For a q-symmetric channel, show that one can take $0 < p < \frac{q-1}{q}$ WLOG.

One can first consider the case $q = 3$.

information rate

The **information rate** (or rate) R of an $[n, M]$-code C over A is $R = \frac{\log_q M}{n}$.

If C encodes messages that are k-tuples over A (so $M = |A^k| = q^k$), then $R = \frac{k}{n}$.

Note:
$0 \leq R \leq 1$. Ideally, R should be close to 1.

Example:

The rate of the binary code $C = \{00000, 11100, 00111, 10101\}$ is $R = \frac{2}{5}$.

Hamming distance

The **Hamming distance** (or distance) between two n-tuples over A is the number of coordinate positions in which they differ.

The Hamming distance (or distance) of an $[n, M]$-code C is $d(C) = \min\{d(x, y) : x, y \in C, x \neq y\}$.

Example:

The distance of $C = \{00000, 11100, 00111, 10101\}$ is $d(C) = 2$.

Theorem 1.1: properties of Hamming distance

For all $x, y, z \in A^n$,

1. $d(x, y) \geq 0$, with $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) + d(y, z) \geq d(x, y)$ ($\triangle$ inequality).

1.2 Decoding Strategy

Example:

Let $C = \{00000, 11100, 00111, 10101\}$. C is a $[5, 4]$-code over $\{0, 1\}$ (a binary code).

Error Detection If C is used for error detection only, the strategy is the following: A received word $r \in A^n$ is accepted if and only if $r \in C$.

Error Correction Let C be an $[n, M]$-code over A with distance d. Suppose $c \in C$ is transmitted, and $r \in A^n$ is received. The (channel) decoder must decide one of the following:

(i) No errors have occurred; *accept* r.

(ii) Errors have occurred; *correct*[1] (*decode*) r to a codeword $c \in C$?

(iii) Errors have occurred; *no correction is possible*.

Nearest Neighbour Decoding

(i) Incomplete Maximum Likelihood Decoding (IMLD):

If there is a unique codeword $c \in C$ such that $d(r, c)$ is minimum, then correct r to c. If no such c exists, then report that errors have occurred, but correction is not possible (ask for retransmission, or disregard information).

(ii) Complete Maximum Likelihood Decoding (CMLD):

Same as IMLD, except that if there are two or more $c \in C$ for which $d(r, c)$ is minimum, correct r to an arbitrary one of these.

[1] Error correction does not guarantee that the channel decoder always makes the correct decision. For example, $00000 \xrightarrow{\text{transmit}}$ 11100 which is accepted.

Is IMLD a reasonable strategy?

Theorem 1.2

IMLD chooses the codeword c for which the conditional probability

$$P(r|c) = P(r \text{ is received}|c \text{ is sent})$$

is largest.

Proof:
Suppose $c_1, c_2 \in C$ with $d(c_1, r) = d_1$ and $d(c_2, r) = d_2$. Suppose $d_1 > d_2$.

Now

$$P(r|c_1) = (1-p)^{n-d_1}\left(\frac{p}{q-1}\right)^{d_1}$$

and

$$P(r|c_2) = (1-p)^{n-d_2}\left(\frac{p}{q-1}\right)^{d_2}$$

So,

$$\frac{P(r|c_1)}{P(r|c_2)} = (1-p)^{d_2-d_1}\left(\frac{p}{q-1}\right)^{d_1-d_2} = \left(\frac{p}{(1-p)(q-1)}\right)^{d_1-d_2}$$

Recall

$$p < \frac{q-1}{q} \implies pq < q-1 \implies 0 < q-pq-1$$

$$\implies p < p+q-pq-1 \implies p < (1-p)(q-1) \implies \frac{p}{(1-p)(q-1)} < 1$$

Hence

$$\frac{P(r|c_1)}{P(r|c_2)} < 1$$

and so

$$P(r|c_1) < P(r|c_2)$$

and the result follows. □

Minimum Error Probability Decoding (MED)

An *ideal strategy* would be to correct r to a codeword $c \in C$ for which $P(c|r) = P(r \text{ is received}|c \text{ is sent})$ is largest. This is MED.

Example: (IMLD/CMLD) is not the same as MED

Consider $C = \{\underset{\substack{\uparrow \\ c_1}}{000}, \underset{\substack{\uparrow \\ c_2}}{111}\}$. Suppose $P(c_1) = 0.1$ and $P(c_2) = 0.9$. Suppose $p = \frac{1}{4}$ (for a BSC).

Suppose $r = 100$ is the received word. Then

$$P(c_1|r) = \frac{P(r|c_1)\cdot P(c_1)}{P(r)} = \frac{p(1-p)^2 \times 0.1}{P(r)} = \frac{9}{640}\cdot\frac{1}{P(r)}$$

$$P(c_2|r) = \frac{P(r|c_2)\cdot P(c_2)}{P(r)} = \frac{(1-p)p^2 \times 0.9}{P(r)} = \frac{27}{640}\cdot\frac{1}{P(r)}$$

So, MED decodes r to c_2. But IMLD decodes r to c_1.

IMLD vs. MED

- IMLD maximizes $P(r|c)$. MED maximizes $P(c|r)$.

- (i) MED has the drawback that the decoding algorithm depends on the probability distribution of source messages.

  (ii) If all source messages are equally likely, then CMLD and MED are equivalent:

$$P(r|c_i) = P(c_i|r) \cdot P(c_i)/P(r) = P(c_i|r) \cdot \underbrace{\left[\frac{1}{M \cdot P(r)}\right]}_{\substack{\text{does not} \\ \text{depend on } c_i}}$$

  (iii) In practice IMLD (or CMLD) is used.

- In this course, we will use IMLD/CMLD.

1.3 Error Correcting & Detecting Capabilities of a Code

Detection Only

Strategy: If r is received, then accept r if and only if $r \in C$.

e-error detecting code

A code C is an ***e*-error detecting code** if the decoder always makes the correct decision if e or fewer errors per codeword are introduced by the channel.

Example:

Consider $C = \{000, 111\}$.

C is a 2-error detecting code.

C is not a 3-error detecting code.

Theorem 1.3

A code C of distance d is a $(d-1)$-error detecting code (but is not a d-error detecting code).

Proof:

Suppose $c \in C$ is sent.

If no errors occur, then c is received (and is accepted).

Suppose that # of errors is ≥ 1 and $\leq d-1$; let r be the received word. Then $1 \leq d(r,c) \leq d-1$, so $r \notin C$. Thus r is rejected. This proves that it is $(d-1)$-error detecting code.

Since $d(C) = d$, there exist $c_1, c_2 \in C$ with $d(c_1, c_2) = d$. If c_1 is sent and c_2 is received, then c_2 is accepted; the d errors go undetected. □

Correction

Strategy: IMLD/CMLD

e-error correcting code

A code C is an ***e*-error correcting code** if the decoder always makes the correct decision if e or fewer errors per codeword are introduced by the channel.

Example:

Consider $C = \{000, 111\}$.

C is a 1-error correcting code.

C is not a 2-error correcting code.

Theorem 1.4

A code C of distance d is an e-error correcting code, where $e = \lfloor \frac{d-1}{2} \rfloor$.

Proof:

Suppose that $c \in C$ is sent, at most $\frac{d-1}{2}$ errors are introduced, and r is received. Then $d(r,c) \leq \frac{d-1}{2}$.

On the other hand, if c_1 is any other codeword, then

$$\begin{aligned}
d(r, c_1) &\geq d(c, c_1) - d(r, c) && \triangle \text{ ineq} \\
&\geq d - \frac{d-1}{2} && \text{since } d(C) = d \\
&= \frac{d+1}{2} \\
&> \frac{d-1}{2} \geq d(r, c)
\end{aligned}$$

Hence c is the unique codeword at minimum distance from r, so the decoder correctly concludes that c was sent. $\square$

Exercise:

Suppose $d(C) = d$, and let $e = \lfloor \frac{d-1}{2} \rfloor$. Show that C is *not* an $(e+1)$-error correcting code.

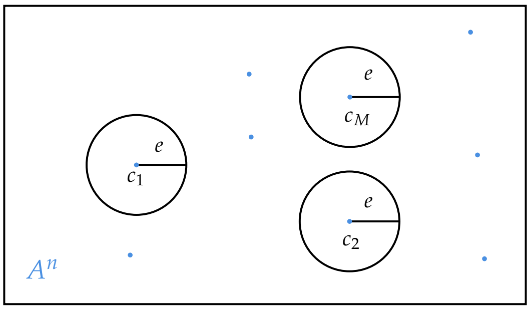
A natural question to ask is: given A, n, M, d, does there exist an $[n, M]$-code C over A of distance $\geq d$. This can be phrased as an equivalent sphere packing problem:

Sphere packing

Can we place M spheres of radius $e = \lfloor \frac{d-1}{2} \rfloor$ in A^n so that no two spheres overlap?

$C = \{c_1, \ldots, c_M\}$, $e = \lfloor \frac{d-1}{2} \rfloor$, S_c = sphere of radius e centered at c = all words within distance e of c.

We proved: if $c_1, c_2 \in C, c_1 \neq c_2$, then $S_{c_1} \cap S_{c_2} = \varnothing$.



Let $n = 128, q = 2, M = 2^{64}$. Does there exists a binary $[n, M]$-code with $d \geq 22$? If so, can encoding and decoding be done efficiently?

We'll view $\{0,1\}^{128}$ as a vector space of dimension 128 over $\mathbb{Z}_2$. We'll choose C to be a 64-dimensional subspace of this vector space. We will construct such a code at the end of the course. The main tools used will be linear algebra (over finite fields) and abstract algebra (rings and fields).

2

Introduction to Finite Fields

2.1 Definitions

ring

A **(commutative) ring** $(R, +, \cdot)$ consists of a set R and two operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$, such that

1. $a + (b + c) = (a + b) + c \quad \forall a, b, c \in R$.
2. $a + b = b + a, \quad \forall a, b \in R$.
3. $\exists 0 \in R$ such that $a + 0 = a$, $\forall a \in R$.
4. $\forall a \in R$, $\exists - a \in R$ such that $a + (-a) = 0$.
5. $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall a, b, c \in R$.
6. $a \cdot b = b \cdot a, \qquad \forall a, b \in R$.
7. $\exists 1 \in R, 1 \neq 0$, such that $a \cdot 1 = a, \quad \forall a \in R$.
8. $a \cdot (b + c) = a \cdot b + b \cdot c, \qquad \forall a, b, c \in R$.

Notation We will denote $(R, +, \cdot)$ by R.

Example:
$\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}$ are commutative rings.

field

A **field** $(F, +, \cdot)$ is a commutative ring with the additional property:

9. $\forall a \in F, a \neq 0$, $\exists a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.

Example:
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. $\mathbb{Z}$ is *not* a field.

infinite/finite field

A field $(F, +, \cdot)$ is a **finite field** if F is a finite set; otherwise it is an **infinite field**. If F is a finite field , its **order** is $|F|$.

Example:

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are infinite fields.

For which integers $n \geq 2$ does there *exist* a finite field of order n? How does one *construct* such a field, i.e., what are the field elements, and how are the field operations performed?

The Integers Modulo n

Let $n \geq 2$. Recall that $\mathbb{Z}_n$ consists of the set of equivalence classes of integers modulo n, $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$, with addition and multiplication: $[a] + [b] = [a+b]$, $[a] \cdot [b] = [a \cdot b]$.

More simply, we write $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, and perform addition and multiplication modulo n.

Example:

$\mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$. In $\mathbb{Z}_9$, $3 + 7 = 1$ and $3 \cdot 7 = 3$.

More precisely, $3 + 7 \equiv 1 \pmod 9$ and $3 \cdot 7 \equiv 3 \pmod 9$.

$\mathbb{Z}_n$ is a commutative ring (i.e., axioms 1-8 in the definition are satisfied).

When is $\mathbb{Z}_n$ is a field?

Theorem 2.1

$\mathbb{Z}_n$ is a field if and only if n is prime.

Proof:

$\Leftarrow$ Suppose n is prime. Let $a \in \mathbb{Z}_n$, $a \neq 0$ (so $1 \leq a \leq n-1$). Since n is prime, $\gcd(a, n) = 1$. Hence $\exists s, t \in \mathbb{Z}$ such that $as + nt = 1$. Reducing both sides modulo n gives $as \equiv 1 \pmod n$. Hence $a^{-1} = s$. Thus $\mathbb{Z}_n$ is a field.

$\Rightarrow$ Suppose n is composite, say $n = ab$ where $2 \leq a, b \leq n-1$.

Now, if a^{-1} exists, say $ac \equiv 1 \pmod n$, then $abc \equiv b \pmod n$, so $nc \equiv b \pmod n$. Thus $b \equiv 0 \pmod n$, so $n \mid b$ which is absurd since $2 \leq b \leq n-1$. Thus $\mathbb{Z}_n$ is not a field.

$\square$

We have established the existence of finite fields of order n, for each prime n. What about finite fields of order n, where n is composite? In particular, is there a field of order 4? Order 6?

characteristic

Let F be a field. The **characteristic** of F, denoted $\text{char}(F)$, is the smallest positive integer m such that $\underbrace{1 + \cdots + 1}_{m} = 0$. If no such m exists, then $\text{char}(F) = 0$.

Example:

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ have characteristic 0. $\mathbb{Z}_p$ (p prime) has characteristic p.

Theorem 2.2

If $\text{char}(F) = 0$, then F is an infinite field.

Proof:
The elements $1, 1+1, 1+1+1, \ldots$ are distinct, because if $\underbrace{1+\cdots+1}_{a} = \underbrace{1+\cdots+1}_{b}$ where $a < b$, then $(\underbrace{1+\cdots+1}_{b}) - (\underbrace{1+\cdots+1}_{a}) = 1+\cdots+1_{b-a} = 0$, contradicting $\mathrm{char}(F) = 0$. □

Theorem 2.3

Let F be a field with $\mathrm{char}(F) = m \neq 0$. Then m is prime.

Proof:
Suppose m is composite, say $m = ab$ where $2 \leq a, b \leq m-1$. Let $s = \underbrace{1+\cdots+1}_{a}$ and $t = \underbrace{1+\cdots+1}_{b}$; note that $s, t \neq 0$. Then $s \cdot t = (\underbrace{1+\cdots+1}_{a}) \cdot (\underbrace{1+\cdots+1}_{b}) = \underbrace{1+\cdots+1}_{ab=m} = 0$. Thus

$$s \cdot t \cdot t^{-1} = s \cdot 1 = s = 0,$$

a contradiction. We then conclude that m is prime. □

Let F be a finite field of characteristic p. Consider the subset of elements of F:

$$E = \{0, 1, 1+1, 1+1, \ldots, \underbrace{1+\cdots+1}_{p-1}\}.$$

The elements of E are distinct. One can verify that E is a field, using the same operations as F. E is a **subfield** of F. If we identify the elements of E with the elements of $\mathbb{Z}_p$ in the natural way, then E is essentially the same field as $\mathbb{Z}_p$. We have proven:

Theorem 2.4

LEt F be a finite field of char p. Then $\mathbb{Z}_p$ is a subfield of F.

Finite fields as vector spaces

Let F be a finite field of characteristic p. Identify:

$$\begin{array}{rcl} \text{vectors} & \leftrightarrow & \text{elements of } F \\ \text{scalars} & \leftrightarrow & \text{elements of } \mathbb{Z}_p \\ \text{vector addition} & \leftrightarrow & \text{addition of } F \\ \text{scalar multiplication} & \leftrightarrow & \text{multiplication of } F \end{array}$$

Then F is a vector space over $\mathbb{Z}_p$ (i.e., the axioms of what it means to be a vector space are satisfied).

2.2 Finite fields: Non-existence

Theorem 2.5

Let F be a finite field of characteristic p. Then the order of F is p^n, for some positive integer n.

Proof:
Let the dimension of F as a vector space over $\mathbb{Z}_p$ be n. Let $\alpha_1, \ldots, \alpha_n$ be a basis for F over $\mathbb{Z}_p$. Then each element $\beta \in F$ can be written uniquely in the form $\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$, where $c_i \in \mathbb{Z}_p$. Thus $F = \left\{ \sum_{i=1}^{n} c_i\alpha_i : c_i \in \mathbb{Z}_p \right\}$, so $|F| = p^n$. □

For example, there do not exist finite fields of order $6, 10, 12, 14, 15, \ldots$

Do finite fields of orders $4, 8, 9, 16, 25, 27, \ldots$ exist?

2.3 Existence of finite fields

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