



Partial Differential Equations 2

AMATH 453



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Preface

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Waves and Diffusions

1.1 The wave equation

We already know the wave equation ($c > 0$):

$$u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty,$$

and the general solution is of the form

$$u(x, t) = f(x + ct) + g(x - ct).$$

With initial conditions imposed, we have the IVP

$$u_{tt} - c^2 u_{xx} = 0, \quad \begin{cases} u(x, 0) = \phi(x), \\ u_t(x, 0) = \psi(x). \end{cases}$$

The solution to IVP is then

$$u(x) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds.$$

To interpret the integral, we can let $\psi(x) = \mu'(x)$, then the integral becomes

$$\int_{x-ct}^{x+ct} \psi(s) ds = \mu(x + ct) - \mu(x - ct).$$

1.2 Conservation laws

Given a wave equation, we multiply by u_t :

$$\begin{aligned} u_t u_{tt} - c^2 u_t u_{xx} &= 0 \\ \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 \right) - c^2 \left[\frac{\partial}{\partial x} (u_t u_x) - u_{tx} u_x \right] &= 0 \\ \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) - \frac{\partial}{\partial x} (c^2 u_t u_x) &= 0 \end{aligned}$$

Then the conservation law states that

$$\frac{\partial R}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where $R \in (-\infty, +\infty)$, and $F \rightarrow 0$ with $x \rightarrow \pm\infty$.

1.3 The Diffusion Equation & Maximum principle

The diffusion equation is given by

$$u_t = ku_{xx}, \quad -\infty < x < \infty$$

with diffusion constant $k > 0$.

We define

$$R = (a, b) \times (0, \infty)$$

$$R_T = (a, b) \times (0, T]$$

$$\overline{R_T} = [a, b] \times [0, T]$$

$$C_T = \{a \leq x \leq b, t = 0\} \cup \{a, 0 \leq t \leq T\} \cup \{b, 0 \leq t \leq T\}$$

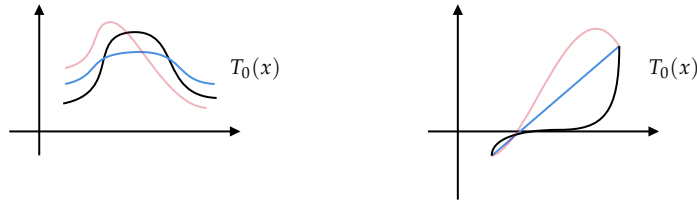
Theorem 1.1: Maximum principle

If $u \in C(\overline{R_T}) \cap C^2(R_T)$ is a solution of the diffusion equation, then $u(x, t) \leq \max_{C_T} \{u\}$ for all $(x, t) \in R_T, T > 0$. Here C_T is called the parabolic boundary of R_T .

Remark:

1. We can replace $u_t - ku_{xx} = 0$ with $u_t - ku_{xx} \leq 0$.
2. A stronger version of the theorem exists which says that $u(x, t) < \max_{C_T} \{u\}$ unless u is constant.
3. Same result applies to the minimum of u by replacing u with $-u$. However, in this case, (1) doesn't apply. Now we need $u_t - ku_{xx} \geq 0$.

Here are some intuitions. Consider a rod lying on $[a, b]$ with initial non-constant temperature $T_0(x)$. Then as time goes, only blue T is possible, not red T .



Proof:

Let $M = \max_{C_T} u$. Note that M exists since u is continuous on C_T , and C_T is a closed boundary. We need to show that $u \leq M$ on $\overline{R_T}$.

Let

$$v(x, t) = u(x, t) + \epsilon x^2, \quad \epsilon > 0$$

Let $r = \max\{|a|, |b|\}$. Then $v(x, t) \leq M + \epsilon r^2$ on C_T . Now we prove that $v \leq M + \epsilon r^2$ on R_T .

On R_T , we have

$$u = v - \epsilon x^2 \leq M + \epsilon(r^2 - x^2)$$

Now if we take the derivative,

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon = -2k\epsilon < 0 \quad (*)$$

- (i) Suppose $v(x, t)$ has a maximum at an interior point (x_0, t_0) , i.e., $(x_0, t_0) \in (a, b) \times (0, T)$. Then

$v_t(x_0, t_0) = 0$. Moreover, $v_{xx}(x_0, t_0) \leq 0$. Then

$$v_t(x_0, t_0) - kv_{xx}(x_0, t_0) = -kv_{xx}(x_0, t_0) \geq 0$$

contradicting (*), thus there are no interior max.

(ii) Suppose $v(x, t)$ has a maximum at an interior point of the upper boundary. $v_t(x_0, T) \geq 0$. Then

$$v_t(x_0, T) - kv_{xx}(x_0, T) \geq 0$$

contradicting (*), thus there are no maximum along the upper boundary.

But v is continuous on $\overline{R_T}$, thus it has a maximum value which we now know must occur on C_T . Hence $v \leq M + \epsilon r^2$ on $\overline{R_T}$. Letting $\epsilon \rightarrow 0$, we have $u \leq M$ on R_T . \square

1.4 Uniqueness of the Dirichlet Problem

$$\begin{aligned} u_t - ku_{xx} &= f(x, t) & a < x < b, 0 < t < \infty \\ u(x, 0) &= \phi(x) \\ u(a, t) &= g(t) \\ u(b, t) &= h(t) \end{aligned} \tag{1.1}$$

Theorem 1.2

The solution of (1.1) is unique.

Proof:

Suppose there are two solutions $u_1(x, t)$ and $u_2(x, t)$. Let $w(x, t) = u_1 - u_2$. Now we calculate

$$\begin{aligned} w_t - kw_{xx} &= (u_{1t} - ku_{1xx}) - (u_{2t} - ku_{2xx}) = f - f = 0 \\ w(x, 0) &= u_1(x, 0) - u_2(x, 0) = \phi - \phi = 0 \\ w(a, t) &= w(b, t) = 0 \end{aligned}$$

By maximum principle, we have $w \leq 0$ on the boundary, and by minimum principle, $w \geq 0$, since $\max_{C_T} \{w\} = \min_{C_T} \{w\} = 0$. Then we conclude that $w \equiv 0$. \square

Now we present a second proof using energy method:

Proof:

Given $w_t - kw_{xx} = 0$, multiply both sides by w :

$$0 = ww_t - kw_{xx} = \frac{\partial}{\partial t} \left(\frac{1}{2} w^2 \right) - k \frac{\partial}{\partial x} (w w_x) + k w_x^2$$

If we integrate both sides,

$$\frac{d}{dt} \int_a^b \frac{1}{2} w^2 dx = k \int_a^b (w w_x)_x dx - k \int_a^b w_x^2 dx = k w w_x \Big|_a^b - k \int_a^b w_x^2 dx$$

Thus

$$\frac{d}{dt} \int_a^b \frac{1}{2} w^2 dx = -k \int_a^b w_x^2 dx$$

Then

$$\int_a^b \frac{1}{2} w^2 dx = 0 \quad \text{for all the time}$$

Then $w \equiv 0$ on $a \leq x \leq b, 0 \leq t \leq T$. \square

Now let's examine stability. Consider

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(a, t) &= u(b, t) = 0 \end{aligned}$$

and let $u_j(x, t)$ be the solution for $u(x, 0) = \phi_j(x)$ for $j = 1, 2$.

Let $w = u_1 - u_2$. Proceeding as before (energy method) we have

$$\int_a^b (u_1 - u_2)^2 dx \leq \int_a^b (\phi_1 - \phi_2)^2 dx$$

This tells us $\|u_1 - u_2\|_2 \rightarrow 0$ as $\|\phi_1 - \phi_2\|_2 \rightarrow 0$. This is called **stability in the square integrable sense**.

Alternatively, by maximum principle,

$$\max |u_1 - u_2| \leq \max |\phi_1 - \phi_2|$$

using maximum & minimum principle, i.e.,

$$\begin{aligned} \max\{u_1 - u_2\} &\leq \max\{\phi_1 - \phi_2\} \\ \min\{u_1 - u_2\} &\geq \min\{\phi_1 - \phi_2\} \end{aligned}$$

This is called **stability in the uniform sense**.

1.5 Diffusion on the Whole Line

Consider the initial value problem

$$u_t - ku_{xx} = 0 \quad \text{on } -\infty < x < \infty, \quad 0 < t < \infty \quad (1.2)$$

$$u(x, 0) = \phi(x) \quad (1.3)$$

If $s(x, t)$ is a solution of (1.2), then so is

$$u(x, t) = \int_{-\infty}^{\infty} s(x - y, t) g(y) dy \quad (1.4)$$

for any function $g(y)$. We can find u_t, u_x, u_{xx} and take it into (1.2):

$$u_t - ku_{xx} = \int_{-\infty}^{\infty} [s_t(x - y, t) - ks_{xx}(x - y, t)] g(y) dy = 0$$

So we now find a solution of (1.2) with the property that $s(x, 0) = \delta(x)$, i.e., solve

$$\begin{aligned} s_t - ks_{xx} &= 0 \\ s(x, 0) &= \delta(x) \end{aligned}$$

To do this, consider the problem:

$$\begin{aligned} v_t - kv_{xx} &= 0 \\ v(x, 0) &= v_0 H(x) \\ H &= \text{Heaviside function} \end{aligned} \quad (1.5)$$

v_0 carries the dimension of v , thus $H(x)$ is dimensionless.

Similarity solution of (1.5)

Let $Q = \frac{v}{v_0}$ which is dimensionless, then the original problem gets transformed to

$$\begin{aligned} Q_t &= kQ_{xx} \\ Q(x, 0) &= H(x) \end{aligned}$$

The solution can only be a function of x, t and k : $Q = F(x, t, k)$. Then we can apply dimensionless analysis. This means Q can only depend on dimensionless combinations of x, t and k . We have

$$\begin{aligned} [x] &= L \\ [t] &= T \\ [k] &= \frac{L^2}{T} \end{aligned}$$

Then

$$[x^a t^b k^c] = L^a T^b \frac{L^{2c}}{T^c} \implies b = c, 2c = -a$$

This tells us

$$Q = f(\theta) \quad \text{where } \theta = \frac{x}{\sqrt{kt}}$$

By chain rule, we have

$$\begin{aligned} Q_t &= f'(\theta) \cdot \theta_t = -\frac{1}{2} \frac{\theta}{t} f'(\theta) \\ Q_x &= f'(\theta) \cdot \theta_x = \frac{1}{\sqrt{kt}} f'(\theta) \\ Q_{xx} &= \frac{1}{kt} f''(\theta) \end{aligned}$$

Then

$$\begin{aligned} Q_t - kQ_{xx} &= -\frac{\theta}{2t} f' - \frac{k}{kt} f'' = 0 \\ f''(\theta) &= -\frac{1}{2} \theta f'(\theta) \\ f'(\theta) &= A e^{-\frac{\theta^2}{4}} \\ f(\theta) &= A \int_{-\infty}^{\theta} e^{-s^2/4} ds + C \end{aligned}$$

As $x \rightarrow +\infty, \theta \rightarrow +\infty$, and $Q(x, t) = f(\theta) \rightarrow 1$. Then $\lim_{\theta \rightarrow +\infty} f(\theta) = 1$.

As $x \rightarrow -\infty, \theta \rightarrow -\infty$ and $Q(x, t) = f(\theta) \rightarrow 0$, $\lim_{\theta \rightarrow -\infty} f(\theta) = 0$.

Therefore, C must be 0, and $A \int_{-\infty}^{\infty} e^{-s^2/4} ds = 1$. Using the change of variable $\eta = \frac{s}{2}$:

$$\int_{-\infty}^{\theta} e^{-s^2/4} ds = 2 \int_{-\infty}^{\theta/2} e^{-\eta^2} d\eta = 2 \int_{-\infty}^{x/\sqrt{4kt}} e^{-\eta^2} d\eta$$

So if we take $\theta = \frac{x}{\sqrt{4kt}}$ at the beginning, we get $\tilde{A} = 2A$ and

$$\tilde{A} \int_{-\infty}^{\infty} e^{-s^2} ds = 1 \implies \tilde{A} = \frac{1}{\sqrt{\pi}}$$

Thus we get

$$Q = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-s^2} ds$$

Note that for $x > 0$, as $t \rightarrow 0^+$, $\frac{x}{\sqrt{4kt}} \rightarrow +\infty$ and $Q(x, t) \rightarrow \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 1$.

And for $x < 0$ as $t \rightarrow 0^+$, $Q \rightarrow 0$.