Computational Methods for DEs

AMATH 342

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Preface

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Euler's Method and Beyond

1.1 Ordinary differential equations and the Lipschitz condition

In ODE, we solve for unknown function which depends on only a single independent variable. For example,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -ku(t),$$

where u denotes the temperature, t is the time.

In PDE, the unknown function depends on two or more independent variables. For example,

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2},$$

where u is temperature, x is space, and t is time.

In this course, approximately 8 weeks will be spent on numerical method for ODEs.

Why numerical methods? Consider an example:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -ku, \qquad t \geq t_0.$$

The exact solution is $u(t) = c\exp(-kt)$ (one can verify this solution by subbing it into the DE). To uniquely determine the constant c we need an initial condition:

$$u(t_0) = u_0,$$

then $u(t) = u_0 \exp(-kt)$.

> Example:
>
> Find u, v, w such that
>
> $$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= v - w \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= u(1-w) - v \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= uv - w \end{aligned}$$

This is a coupled system, and non-linear. If we denote

$$\vec{y} := \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{R}^3,$$

then the system can be rewritten as

$$\vec{y}'(t) = \vec{f}(t, \vec{y}) = \begin{bmatrix} v - w \\ u(1-w) - v \\ uv - w \end{bmatrix}.$$

Our goal ($\sim$ 8 weeks) is to approximate the solution to $\vec{y}'(t) = \vec{f}(t, \vec{y})$, $t \geq t_0, \vec{y}(t_0) = \vec{y}_0$, where

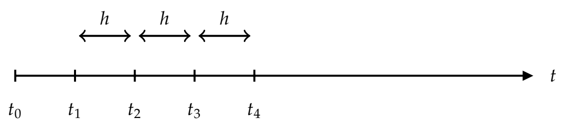
- $\vec{y} = \mathbb{R}^d$ (unknown),
- $\vec{y}_0 \in \mathbb{R}^d$, given vector,
- $\vec{f}(t, \vec{y}(t)) \in \mathbb{R}^d$ is a "well-behaved" function.

"Well-behaved" here means $\vec{f}(t, \vec{y})$ satisfies the Lipschitz condition:

$$\left\| \vec{f}(t, \vec{x}) - \vec{f}(t, \vec{y}) \right\| \leq \lambda \|\vec{x} - \vec{y}\|, \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^d, \text{ for } t \geq t_0.$$

Here $\lambda > 0$ is called Lipschitz constant.

1.2 Euler's method



We have a sequence: $t_0, t_1 = t_0 + h, t_2 = t_0 + 2h, \ldots$, where h is the time step.

Given $\vec{y}$ at t_0, can we guess the value of $\vec{y}$ at t_1?

We know $\vec{y}'(t) = \vec{f}(t, \vec{y}(t))$, then we have

$$\int_{t_0}^{t_1} \vec{y}'(t)dt = \int_{t_0}^{t_1} \vec{f}(t, \vec{y}(t))dt.$$

We can make the approximation: $\vec{f}(t, \vec{y}(t)) \approx \vec{f}(t_0, \vec{y}(t_0))$ on the interval $[t_0, t_1]$. Then

$$\int_{t_0}^{t_1} \vec{y}'(t)dt \approx \int_{t_0}^{t_1} \vec{f}(t_0, \vec{y}(t_0))dt$$

$$\vec{y}(t_1) \approx \vec{y}(t_0) + (t_1 - t_0)\vec{f}(t_0, \vec{y}(t_0))$$

$$\boxed{\vec{y}(t_1) \approx \vec{y}(t_0) + h\vec{f}(t_0, \vec{y}(t_0))}$$

Here we use the notation: $\vec{y}_n$ is the *numerical approximation* to the exact solution $\vec{y}(t_n)$. So

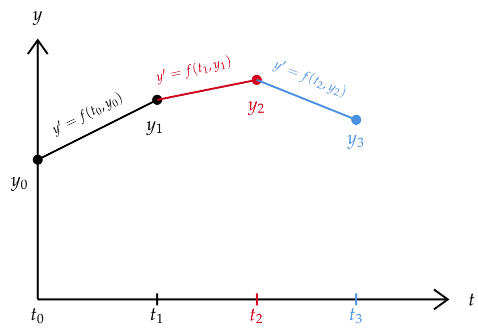
$$\vec{y}(t_1) \approx \vec{y}(t_0) + h\vec{f}(t_0, \vec{y}(t_0))$$

and

$$\vec{y}_1 = \vec{y}_0 + h\vec{f}(t_0, \vec{y}_0).$$

Note:
At $t = t_0$, $\vec{y}(t_0) = \vec{y}_0$, but for $n > 0$, $\vec{y}_n \approx \vec{y}(t_n)$.

Let's visualize in the scalar case:



We obtain the recursive scheme:

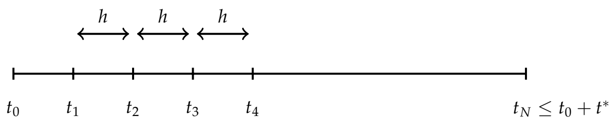
$$\boxed{\vec{y}_{n+1} = \vec{y}_n + h\vec{f}(t_n, \vec{y}_n), \quad n = 0, 1, 2, \dots}$$

which is known as **Euler's Method**.

How good is Euler's method in approximating the solution to an ODE of the form $\vec{y}' = f(t, \vec{y})$?

Notation + Definitions

Time interval: $t \in [t_0, t_0 + t^*]$



Here $N = \lfloor t^*/h \rfloor$ = number of time steps.

Every h results in a different grid. The smaller h, the more time steps.

The numerical solution $\vec{y}_n$ on a grid with time step h will sometimes be denoted by

$$\vec{y}_{n,h},\ n = 0, 1, 2, \dots, \lfloor t^*/h \rfloor.$$

Important A numerical method is **convergent** if for any ODE problem $\vec{y}' = \vec{f}(t, \vec{y})$, $t \geq t_0$, $\vec{y}(t_0) = \vec{y}_0$ with Lipschitz, it holds that

$$\lim_{h \to 0} \max_{n=0,1,\dots,N} \|\vec{y}_{n,h} - \vec{y}(t_n)\| = 0.$$

If a numerical method does not converge, it is **useless**!

Does Euler's converge? Yes.

Proof:

Assume that the function $\vec{f}$ and $\vec{y}$ are analytic. Define the numerical error:

$$\vec{e}_{n,h} = \vec{y}_{n,h} - \vec{y}(t_n).$$

We need to show that

$$\lim_{h \to 0} \max_{n=0,\dots,N} \|\vec{e}_{n,h}\| = 0.$$

By Taylor series, around $t = t_n$, we have

$$\begin{aligned} \vec{y}(t_{n+1}) &= \vec{y}(t_n) + (t_{n+1} - t_n)\vec{y}'(t_n) + O\big((t_{n+1} - t_n)^2\big) \\ &= \vec{y}(t_n) + h\vec{f}(t_n, \vec{y}(t_n)) + O(h^2) \end{aligned} \tag{1.1}$$

because $\vec{y}'(t_n) = \vec{f}(t_n, \vec{y}(t_n))$.

From Euler's method, we have

$$\vec{y}_{n+1,h} = \vec{y}_{n,h} + h\vec{f}(t_n, \vec{y}_{n,h}) \tag{1.2}$$

Subtract (1.1) from (1.2),

$$\begin{aligned}\vec{y}_{n+1,h} - \vec{y}(t_{n+1}) &= \vec{y}_{n,h} - \vec{y}(t_n) + h\big[\vec{f}(t_n, \vec{y}_{n,h}) - \vec{f}(t_n, \vec{y}(t_n))\big] + O(h^2)\\ \vec{e}_{n+1,h} &= \vec{e}_{n,h} + h\big[\vec{f}(t_n, \vec{y}(t_n) + \vec{e}_{n,h}) - \vec{f}(t_n, \vec{y}(t_n))\big] + O(h^2)\end{aligned}$$

$$\begin{aligned}\|\vec{e}_{n+1,h}\| &= \Big\|\vec{e}_{n,h} + h\big[\vec{f}(t_n, \vec{y}(t_n) + \vec{e}_{n,h}) - \vec{f}(t_n, \vec{y}(t_n))\big] + O(h^2)\Big\|\\ &\le \|\vec{e}_{n,h}\| + h\Big\|\big[\vec{f}(t_n, \vec{y}(t_n) + \vec{e}_{n,h}) - \vec{f}(t_n, \vec{y}(t_n))\big]\Big\| + \|O(h^2)\|\end{aligned}$$

Since $\vec{y}$ is analytic: $\|O(h^2)\| \le ch^2$ where $c > 0$ constant.

As $\vec{f}$ is Lipschitz,

$$\Big\|\big[\vec{f}(t_n, \vec{y}(t_n) + \vec{e}_{n,h}) - \vec{f}(t_n, \vec{y}(t_n))\big]\Big\| \le \lambda\|\vec{e}_{n,h}\|.$$

This implies

$$\|e_{n+1,h}\| \le (1 + h\lambda)\|e_{n,h}\| + ch^2 \quad \text{for } n = 0, 1, 2, \ldots, N-1.$$

We claim

$$\|\vec{e}_{n,h}\| \le \frac{ch}{\lambda}[(1 + h\lambda)^n - 1], \quad n = 0, 1, 2, \ldots.$$

If $n = 0$, $\|\vec{e}_{0,h}\| \le 0$ is true because $\vec{e}_{0,h} = 0$.

When $n > 0$, assume claim is true up to n. We want to show the claim holds for $n + 1$.

$$\begin{aligned}\|\vec{e}_{n+1,h}\| &\le (1 + h\lambda)\|\vec{e}_{n,h}\| + ch^2\\ &\le (1 + h\lambda)\frac{ch}{\lambda}[(1 + h\lambda)^n - 1] + ch^2\\ &= \frac{ch}{\lambda}[(1 + h\lambda)^{n+1} - 1].\end{aligned}$$

Thus the claim is proved to be true using induction.

Consider the Taylor series expansion of $\exp(x)$ at $x = 0$: $\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, thus for $x > 0$, we have $1 + x < \exp(x)$. Then in our case, as $h\lambda > 0$, we have

$$1 + h\lambda < \exp(h\lambda).$$

$$(1 + h\lambda)^n < \exp(nh\lambda), \quad n = 1, 2, \ldots, \lfloor t^*/h \rfloor.$$

As $nh < t^*$, we then have

$$(1 + h\lambda)^n < \exp(t^*\lambda).$$

Therefore,

$$\|\vec{e}_{n,h}\| \le \frac{ch}{\lambda}[(1 + h\lambda)^n - 1] \le \frac{ch}{\lambda}[\exp(t^*\lambda) - 1],$$

for $n = 0, 1, \ldots, N$. Then we conclude that

$$\lim_{\substack{h\to 0\\ 0\le nh\le t^*}} \|\vec{e}_{n,h}\| \le \lim_{\substack{h\to 0\\ 0\le nh\le t^*}} \underbrace{\left[\frac{c}{\lambda}(\exp(t^*\lambda) - 1)\right]}_{\text{indep. of } n \text{ or } h} h = \lim_{h\to 0} \tilde{c}h = 0.$$

Hence, Euler's method is convergent! □

Now it's time to see an example!

Solve the following ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky, \quad y(0) = 1, \quad t \in [0, 1]$$

where $k = 5$.