Deterministic OR Models

CO 370

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Preface

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What is operations research (OR)? There's no standard definitions for it. One particular definition: use of mathematical models to make complex decisions for real life problems. The origin is British military in WW2. OR is actually everywhere today. Key milestone: Simplex algorithm (1947).

Recall optimization problem is of the form:

$$\begin{aligned} \max \quad & f(x) \\ \text{s.t.} \quad & \text{a set of constraints} \end{aligned}$$

There are some applications: mail delivery, machine scheduling, inventory problem, network design, facility location, class scheduling, portfolio optimization, surgery planning, sensor location.

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Part I:

Formulations

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LP formulations

1.1 Production problem

Products $J = \{1, \ldots, n\}$
Resources $I = \{1, \ldots, m\}$

Data:

- $\forall j \in J : c_j =$ value of unit of product j
- $\forall i \in I : b_i =$ number of units of resource i available
- $\forall i \in I, \forall j \in J : a_{ij} =$ number of units of resource i going to product j

Goal: maximize values of product made subject to available resources

Var: $x_j =$ number of units of product j produced

Then problem is

$$\begin{array}{rll} \max & \sum_{j \in J} c_j x_j & \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \leq b_i & (i \in I) \\ & x_j \geq 0 & (j \in J) \end{array}$$

Now let's generalize this problem to have more than one period.

Products $J = \{1, \ldots, n\}$
Resources $I = \{1, \ldots, m\}$
Periods $K = \{1, \ldots, p\}$

Then we have **data**

- $\forall j \in J, k \in K$: $c_{jk} =$ unit value of product j in period k
- $\forall i \in I, k \in K$: $w_{ik} =$ unit price for resource i in period k
- $\forall i \in I, j \in J$: $a_{ij} =$ number of units of resource i going into product j

and the **goal**: decide how much of each resource to buy & how much of each product to make during each period, to maximize total profit. Unused resources are available at next time period.

Var:

- p_{ik} = number of units of resource i, purchased at start of period k
- x_{jk} = number of units of product j made in period k
- z_{ik} = number of units of resource i at the end of period k

$$\text{Profit} = \sum_{k\in K}\left[\sum_{j\in J} c_{jk}x_{jk} - \sum_{i\in I} w_{ik}p_{ik}\right] \tag{1.1}$$

Then we keep track of resources: for $i \in I$, $k \in K$

$$z_{ik} = z_{i(k-1)} + p_{ik} - \sum_{j\in J} a_{ij}x_{jk} \tag{1.2}$$

and we define for $i \in I$,

$$z_{i0} = 0 \tag{1.3}$$

Thus the optimization problem is

$$\begin{aligned} \max \quad & (1.1) \\ \text{s.t.} \quad & (1.2), (1.3) \\ & p, x, z \geq 0 \end{aligned} \tag{P}$$

Remark:
If (P) has a feasible solution of value that is bigger than 0, then (P) is unbounded. So we are missing some assumptions, maybe? For example, b_{ik} = amount of resource i that can be bound during period k. Then we can add constraints: $p_{ik} \leq b_{ik}$.

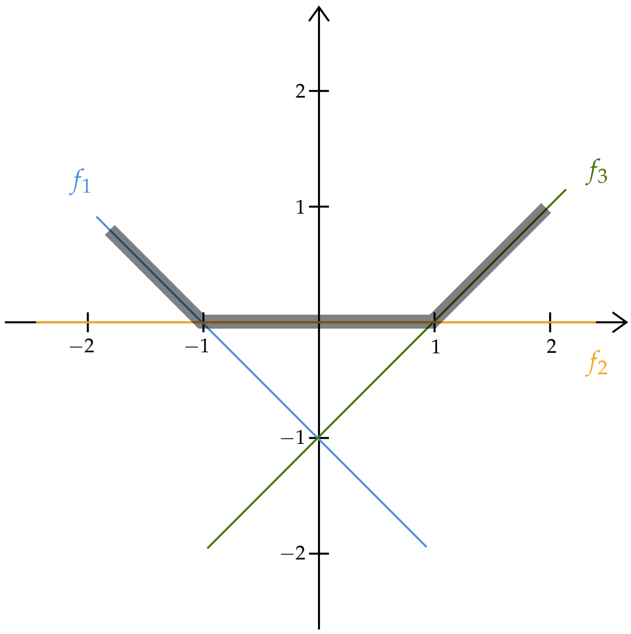
1.2 Minimax

Consider the problem of the form

$$\begin{aligned} &\min_x \max\{f_1(x), \ldots, f_k(x)\} := g(x) \\ &\text{s.t.} \ \cdots \end{aligned}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$.

Example:
$f_1(x) = -x - 1, f_2(x) = 0, f_3(x) = x - 1$. Then $\max\{f_1(x), f_2(x), f_3(x)\}$ is as follows



A motivation

- $\forall i \in [k]$, $f_i(x) =$ completion time for task i.
- Project consists of task $1, \ldots, k$.
- $g(x) =$ completion time of entire project

Note that minimax is not an optimization problem as we defined it. We can revise it as follows

$$\begin{array}{rl} \min & y \\ \text{s.t.} & y \geq f_1(x) \\ & y \geq f_2(x) \\ & \vdots \\ & y \geq f_k(x) \\ & \cdots \end{array}$$

An application minimize a piece-wise linear convex function using linear programming.

2

Flows

digraph

A directed graph (digraph) is a pair (V, E) where

- V is a set of vertices,
- E is a set of ordered pairs of vertices called arcs.

Notation Let $q \in V$, then

$$\delta^+(q) = \{e \in E \mid e \text{ leaves } q\}$$
$$\delta^-(q) = \{e \in E \mid e \text{ arrives at } q\}$$

2.1 Max st-flow model

Given

1. digraph $G = (V, E)$,
2. two vertices $s, t \in V$, and $s \neq t$,
3. $\forall e \in E$, arc e has capacity $u_e \geq 0$.

Now we construct an LP.

For every arc e, we will have a variable x_e, and x_e will be called the **flow** on arc e.

Notation Let $q \in V$:

$$f_x(q) := \sum_{e \in \delta^+(q)} x_e - \sum_{e \in \delta^-(q)} x_e$$

The maximization problem is then

$$\begin{array}{rll} \max & f_x(s) & \\ \text{s.t.} & f_x(q) = 0 & (q \in V, q \neq s, q \neq t) \\ & 0 \leq x_e \leq u_e & (e \in E) \end{array} \tag{P}$$

A feasible solution to (P) is a flow. An optimal solution to (P) is a maximum flow. The value of a flow x is $f_x(S)$.

Remark:
(P) is always feasible. It is not unbounded, so there always exists a maximum flow.

Application computer network. Suppose we have

- computers s and t ($s \neq t$)
- capacity u_e (gb/s) for every link e

The goal is to computer number gb that can be sent from s to t across network. Then x_e is the amount of information across e. $f_x(q) = 0$ means no information lost.

Magic property

If u is integer, then there exists an optimal solution to (P) that is integer.

Remark:
We need the condition "u is integer" so that the property is still true. Also, an optimal solution to (P) is not necessarily integers.

Generalize max st-flows

We can add lower bounds to arcs: ℓ_e $\forall e \in E$

$$\begin{array}{rll} \max & f_x(s) & \\ \text{s.t.} & f_x(q) = 0 & (q \in V, q \neq s, q \neq t) \\ & \ell_e \leq x_e \leq u_e & (e \in E) \end{array} \tag{Q}$$

Magic property - revised

If ℓ, u is integer, and there exists an optimal solution to (Q), then there exists an optimal solution to (Q) that is integer.

Example: Consistent rounding
The goal is to round all entries to nearest up/down integer, so that row sums & column sums still hold.

Any feasible solutions give consistent rounding.

2.2 Min cost flow model

Given

- Digraph $G = (V, E)$
- Capacities $u_e \geq 0$ ($e \in E$)
- Costs c_e ($e \in E$)
- Supply/demands b_q ($q \in V$)

Then the model is

$$\begin{array}{lll} \min & \sum_{e\in E} c_e x_e & \\ \text{s.t.} & f_x(q) = b_q & (q \in V) \\ & 0 \leq x_e \leq u_e & (e \in E) \end{array} \tag{P}$$

Similarly, feasible solution to (P) is a flow. An optimal solution to (P) is a min cost flow.

Magic property - min cost flows

Suppose u, b are integer, then if there exists a min cost flow, then there exists a min cost flow that is integer.

Similarly, we can add a lower bound:

$$\begin{array}{lll} \min & \sum_{e\in E} c_e x_e & \\ \text{s.t.} & f_x(q) = b_q & (q \in V) \\ & \ell_e \leq x_e \leq u_e & (e \in E) \end{array} \tag{P}$$

Magic property - min cost flows (revised)

Suppose u, b, ℓ are integer, then if there exists a min cost flow, then there exists a min cost flow that is integer.

What is a necessary condition for b_q so that there exists a flow? $\sum_{q\in V} b_q = 0$.

Example: Staffing problem

hours	# of employees required
1-2	2
2-3	3
3-4	3
4-5	2

and we have cost of hiring a single employee between

hours	cost
1-5	6
1-4	4
3-5	5
2-4	3

The goal is to minimize cost of hiring employees while meeting staff needs.

3

IP formulations

3.1 IP tricks

Imaging we are forcing variable to take some prescribed set of values: $x \in \{5, 9, 13, 36\}$. Then we can introduce variables $z_1, z_2, z_3, z_4 \in \{0, 1\}$ so that we have two constraints:

$$\begin{aligned} 1 &= z_1 + z_2 + z_3 + z_4 \\ x &= 5z_1 + 9z_2 + 13z_3 + 36z_4 \end{aligned}$$

3.2 Modeling piece-wise linear function

Let $f : \mathbb{R} \to \mathbb{R}$ be piecewise linear. Given $a_1, \ldots, a_k$ and $f_1, \ldots, f_k$ such that $f(a_i) = f_i$. The goal is to write IP constraints with variables x, y such that $y = f(x)$, $x \in \underbrace{[a_1, a_k]}_{\uparrow\ \mathrm{dom}(f)}$.

To generalize,

1. $\lambda_1, \ldots, \lambda_k \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$
2. $x = \sum_{i=1}^{k} \lambda_i a_i$
3. $y = \sum_{i=1}^{k} \lambda_i f_i$
4. $z_0, \ldots, z_k \in \{0, 1\}$, $z_0 = z_k = 0$
5. $\sum_{i=0}^{k} z_i = 1$
6. $\forall p \in [k]$, $\lambda_p \leq z_{p-1} + z_p$

By (4) and (5), we may assume that $z_p = 1$ and $z_j = 0\ \forall j \neq p$.

Claim λ_p, λ_{p+1} are the only non-zero λ variables.

Proof:
Pick $j \neq p, j \neq p + 1$. As

$$0 \leq \lambda_j \leq z_{j-1} + z_j = 0 + 0 = 0$$

□

With this claim, we can simplify (2) and (3).

3.3 Modeling union of polyhedra

Let

$$P_1 = \{x \mid A^1 x \leq b^1\}$$
$$P_2 = \{x \mid A^2 x \leq b^2\}$$

Goal: write condition: $x \in P_1 \cup P_2$ as part of IP.

Hypothesis: If $x \in P_1 \cup P_2$, then $0 \leq x \leq U$ for some U.

Constraints:

1. $y_1, y_2 \in \{0, 1\}$
2. $y_1 + y_2 = 1$
3. $x = x^1 + x^2$
4. $A^i x^i \leq y_i b^i$, $i = 1, 2$
5. $0 \leq x^i \leq y_i U$, $i = 1, 2$

To show that $x \in P_1 \cup P_2 \iff \exists x^1, x^2, y_1, y_2$ such that all 5 conditions hold.

Proof:
First let's assume (1) - (5) hold. We may assume $y_1 = 1, y_2 = 0$. Then (5) tells us $x^2 = 0$. (3) tells us $x = x^1$. (4) implies

$$A^1 x^1 \leq y_1 b^1 \implies A^1 x \leq b^1 \implies x \in P_1 \subseteq P_1 \cup P_2$$

Now suppose $x \in P_1 \cup P_2$. We may assume $x \in P_1$. Then we set $y_1 = 1, y_2 = 0, x^1 = x, x^2 = 0$. Then we can verify that all 5 conditions hold. $\square$

3.4 Perfect formulations

basis

Let A be a matrix with column indices $\{1, \ldots, n\}$. Then $B \subseteq \{1, \ldots, n\}$ is a basis if

1. A_B square,
2. A_B non-singular.

Remark:
A has a basis $\iff$ rows of A are independent.

basic solution

Let A be a matrix with column indices $1, \ldots, n$. Consider

$$Ax = b \tag{$*$}$$

Pick B as basis of A. Then x is a basic solution of $(*)$ if

1. $Ax = b$.
2. $x_j = 0$ if $j \notin B$.

x is a basic solution for $(*)$ if x is a basic solution for some basis B.

standard equality form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and rows of A are independent.

The correctness of simplex algorithm implies the following theorem:

Theorem 3.1

If an LP is SEF has an optimal solution, then it has an optimal solution that is basic.

Let A be matrix, b vector with same number of entries as rows of A. Then $A \leftarrow_j b$ denotes matrix obtained from A by replacing column j by b.

Theorem 3.2: Cramer's rule

Let M be a non-singular matrix and consider $Mx = b$.

$$\bar{x}_j = \frac{\det(M \leftarrow_j b)}{\det(M)} \quad \forall j$$

then $M\bar{x} = b$.

Proposition 3.3

Let M be square matrix with $\det(M) = \pm 1$, and M, b are integer. Then there exists a unique solution to $Mx = b$ is integer.

Proof:
Directly from Cramer's rule. □

Totally unimodular matrix

A matrix A is totally unimodular if every square submatrix N of A, $\det(N) \in \{0, +1, -1\}$.

Proposition 3.4

Let A be TU, b integer. Then every basic solution of $Ax = b$ is integer.

Proof:
Suppose $\bar{x}$ is a basic solution for basis B. If $j \notin B$, $\bar{x}_j = 0$, which is integer. The basic variables $\bar{x}_B$ are the unique solution to $A_B x_B = b$. Since A is TU, then $\det(A_B) \in \{0, -1, +1\}$. Since B is basis, $\det(A_B) \neq 0$. Then $\det(A_B) = \pm 1$. A_B is integer as it is TU. Then by Proposition 3.3, $\bar{x}_B$ is integer. □

Theorem 3.5

If an LP is SEF has an optimal solution, it has an optimal solution that is basic.

Theorem 3.6: main theorem

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b, x \geq 0 \end{aligned} \tag{P}$$

Suppose A is TU and b integer. Then if (P) has an optimal solution, it has an optimal solution that is integer.

Proof:
We may assume rows of A are linearly independent. Then (P) is SEF. Thus if (P) has an optimal solution, then by Theorem 3.5, it has an optimal basic solution $\bar{x}$. By Proposition 3.4, every basic solution is integer. In particular, so is $\bar{x}$. □

Constructing TU matrices

Proposition 3.7

Let A be a $0, -1, +1$ matrix such that for every column

1. we have at most one $+1$,
2. we have at most one -1.

Then A is TU.

Proof:
We need to show every $k \times k$ submatrix M of A has $\det(M) \in \{0, -1, +1\}$. We can prove it by induction on k. Discuss by cases.

- M has a column of zeros. Then $\det(M) = 0$.
- Every column has exactly one $+1$ and one -1, then row sum is zero, then rows are linearly dependent, thus $\det(M) = 0$.
- One column has a unique non-zero entry M_{ij}. Applying cofactor expansion,

$$\det(M) = M_{ij}(-1)^{i+j} \det\left(M'_{ij}\right)$$

Then by induction, the statement holds.

□

Constructing TU matrices from TU matrices

Proposition 3.8

Let A be TU. Then

1. permutation of A,
2. A^T,
3. any matrix obtained from A by multiplying rows/columns by -1,
4. any matrix obtained by adding a unit vector to thw rows/columns

are TU.

Theorem 3.9

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b, x \geq 0 \end{aligned} \tag{P}$$

Suppose A is TU, b integer. Then if (P) has an optimal solution, then it has one that is integer.

Theorem 3.10

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b, 0 \leq x \leq U \end{aligned} \tag{Q}$$

Suppose A is TU, b and U are integer. Then if (Q) has optimal solution, then it has one that is integer.

Proof:
We can rewrite (Q) in SEF:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \begin{bmatrix} A & 0 \\ I & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} b \\ U \end{bmatrix} \\ & x, s \geq 0 \end{aligned} \tag{P'}$$

We denote the first constraint by $A'x' = b'$. Now note that b' is integer as b, U are integer. It's clear that A' is TU. Then apply Theorem 3.9. □

3.5 Application to flows

vertex-arc incidence matrix

M is the vertex-arc incidence matrix of digraph $G = (V, E)$ if

1. rows of M correspond to V,
2. columns of M correspond to E,
3. for column uv we have $+1$ for every u, -1 for entry v and 0 otherwise.

Remark:
Let M be vertex-arc incidence matrix of $G = (V, E)$. Pick $v \in V$: for every entry e of row v, if $e \in \delta^+(v)$, then we have $+1$; if $e \in \delta^-(v)$, we have -1, otherwise we have 0.

Magic property - min cost flow

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & f_x(q) = b_q \quad (q \in V) \\ & 0 \leq x \leq U \end{aligned} \tag{P}$$

Suppose b, U integer, and there exists an optimal solution. Then there exists an integer optimal solution.

Proof:
Let A be the vertex-arc matrix.

Claim $f_x(q) = b_q \iff \text{row}_q(A)x = b_q$.

Proof of the claim:
Using the remark, we know that

$$\text{row}_q(A)x = \sum_{e \in \delta^+(q)} x_e - \sum_{e \in \delta^-(q)} x_e := f_x(q)$$

□

We can write (P) as

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & 0 \leq x \leq U \end{aligned}$$

We proved A is TU. Then it follows by Theorem 3.10. □

4

Cone programming

This is generalization of

1. linear programming
2. second-order cone programming
3. semi-definite programming

We have good algorithms for these problems.

cone

$C \subseteq \mathbb{R}^m$ is a cone if

1. C is convex,
2. $\forall \lambda \geq 0$ and $a \in C$, $\lambda a \in C$.

The second condition: C is closed under (non-negative scaling).

pointed cone

A cone is pointed if it does not contain an infinite line.

4.1 Examples of cones

$\mathbb{R}^m_+ = \{u \in \mathbb{R}^m : u \geq 0\}$

Second Order Cone (SOC):

$$L^m = \{u \in \mathbb{R}^m : \|(u_1, \ldots, u_{m-1})\| \leq u_m\}$$

semi-definite matrix

An $n \times n$ matrix M is semi-definite if $\forall x \in \mathbb{R}^n$: $x^T M x \geq 0$.

Semi-definite cone is

$$S^n_+ = \{n \times n \text{ semi-definite matrices } M\}$$

Direct sum

Let $A \subseteq \mathbb{R}^p$, $B \subseteq \mathbb{R}^q$. The direct sum and A and B is

$$A \oplus B := \{(a,b) : a \in A, b \in B\} \subseteq \mathbb{R}^{p+q}$$

Proposition 4.1

Let A, B be cones, then $A \oplus B$ is a cone.

4.2 Cone programming model

Let $c \in \mathbb{R}^n$, A $m \times n$ matrix, $b \in \mathbb{R}^m$, K pointed cone.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax - b \in K \end{aligned} \tag{P}$$

Cases:

1. $K = \mathbb{R}^m_+$. Then $Ax - b \in \mathbb{R}^m_+ \iff Ax \geq b$. Then it's an LP.
2. $K = L^m$. $Ax - b \in L^m$ can be rewritten as
$$\begin{bmatrix} A' \\ d^T \end{bmatrix} x - \begin{bmatrix} b' \\ p \end{bmatrix} \in L^m$$
$$\|A'x - b\| \leq d^T x - p$$
and this is restricted SOC program.
3. $K = S^n_+$. This is semi-definite program.

Example: Least square problem

$$\min_x \|Ax - b\|$$

or

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \|Ax - b\| \leq t \end{aligned}$$

which is restricted SOC program

4.3 (General) SOC program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \|A_i x - b_i\| \leq d_i^T x - p_i \quad (i = 1, \dots, k) \end{aligned}$$

Is this still a cone program? In other words, is there A, B, K such that the above constraint is equivalent to $Ax - b \in K$? Yes. It's direct sum of pointed cones.

5

Robust optimization

$$\begin{aligned} \min \quad & \cdots \\ \text{s.t.} \quad & a^T x \leq \beta \\ & \cdots \end{aligned}$$

This particular constraint might be important. And a is part of data that is given, but a here might have some imprecision. This brings us the uncertainty.

5.1 Model uncertainty for a

Given: $\bar{a} \in \mathbb{R}^n$ estimate, $\epsilon \subseteq \mathbb{R}^n$ uncertainty. True a satisfies $a - \bar{a} \in \epsilon$.

Then we can write the condition

$$a^T x \leq \beta \quad (a - \bar{a} \in \epsilon)$$

We have different choices for ϵ, or think it for different ways.

For example, the ball:

$$a^T x \leq \beta \quad (a - \bar{a} \in \{u : \|u\| \leq r\} := \epsilon_1) \tag{5.1}$$

the cube:

$$a^T x \leq \beta \quad (a - \bar{a} \in \{u : -r \leq u_i \leq r, \forall i\} := \epsilon_2) \tag{5.2}$$

Let

$$\begin{aligned} S_1 &= \{x \mid x \text{ satisfies } (5.1)\} \\ S_2 &= \{x \mid x \text{ satisfies } (5.2)\} \end{aligned}$$

What's the relationship between S_1 and S_2? We know that $\epsilon_1 \subseteq \epsilon_2$. The constraints in (5.1) form a subset of constraints in (5.2). Thus $S_2 \subseteq S_1$.

We will see: (5.1) can be replaced by a single SOC constraint: $\|f(x)\| \leq g(x)$ where f, g affine. (5.2) can be replaced by a finite number of linear constraints (with additional variables).

5.2 Modeling uncertainty by a ball

Given data $r > 0$, $\bar{a} \in \mathbb{R}^n$, $\beta \in \mathbb{R}$. We then have $\epsilon = \{u \in \mathbb{R}^n \mid \|U\| \leq r\}$.

$$a^T x \leq \beta \qquad (a - \bar{a} \in \epsilon) \tag{$*$}$$

As $(*)$ has infinitely many of inequalities, the goal is to replace $(*)$ with a single inequality. The key idea is to fix x. Note that

$$(*) \iff \max\{a^T x : a - \bar{a} \in \epsilon\} \le \beta$$
$$\bar{a}^T x + \max\{a^T x - \bar{a}^T x : a - \bar{a} \in \epsilon\} \le \beta$$
$$\bar{a}^T x + \max\{(a - \bar{a})^T x : a - \bar{a} \in \epsilon\} \le \beta$$
$$\bar{a}^T x + \max\{u^T x : u \in \epsilon\} \le \beta$$

Note that $\max\{u^T x : u \in \epsilon\} = r\|x\|$ given ϵ is a radius r ball. Then

$$\bar{a}^T x + r\|x\| \le \beta$$

and this is exactly SOC constraint.

ellipsoid

An ellipsoid is a set of the form $\{Pu : \|u\| \le 1\}$ and P is a non-singular matrix.

For example, let $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

This is generalization of the ball.

5.3 Modeling uncertainty by a hypercube

Recall the constraint

$$a^T x \le \beta \quad (a - \bar{a} \in \epsilon)$$

where $\epsilon = \{U \mid -r \le v_j \le r, \forall j\}$. We need to show

1. ( ) is equivalent to single constraint, involving absolute values.
2. ( ) is equivalent to finite number of linear constraints.

$$a^T x \ge \beta \quad (a - \bar{a} \in \epsilon)$$

Fix x. Then the above constraint is equivalent to

$$\min\{a^T x \mid a - \bar{a} \in \epsilon\} \ge \beta$$

Then similarly,

$$\bar{a}^T x + \min\{u^T x \mid u \in \epsilon\} \ge \beta$$

Note that assume u is optimal, then

$$\begin{aligned} u^T x &= \sum_j u_j x_j \\ &= \sum_{j:x_j \ge 0} u_j x_j + \sum_{j:x_j < 0} u_j x_j \\ &= \sum_{j:x_j \ge 0} -r x_j + \sum_{j:x_j < 0} r x_j \\ &= -r \sum_j |x_j| \end{aligned}$$

Proposition 5.1

$$a^T x \geq \beta \quad (a - \bar{a} \in \{u \mid -r \leq u_j \leq r, \forall j\})$$

is equivalent to

$$\bar{a}^T x - r \sum_j |x_j| \geq \beta$$

Corollary 5.2

$$a^T x \leq \beta \quad (a - \bar{a} \in \{u \mid -r \leq u_j \leq r, \forall j\})$$

is equivalent to

$$\bar{a}^T x + r \sum_j |x_j| \leq \beta$$

Proposition 5.3

1. $\exists x$ such that $\bar{a}^T x + r \sum_j |x_j| \leq \beta$
2. $\exists x, y$ such that $\bar{a}^T x + r \sum_j y_j \leq \beta$, $y_j \geq x_j$, $y_j \geq -x_j$.

are equivalent.

Proposition 5.4

$$a^T x \leq \beta \quad (a - \bar{a} \in \{u \mid -r \leq u_j \leq r, \forall j\})$$

is equivalent to

$$\bar{a}^T x + r \sum_j y_j \leq \beta$$

$$y_j \geq x_j, y_j \geq -x_j, \ \forall j$$

This can be generalized to be a rectangle. Given $\Delta \geq 0$. Then

$$\epsilon = \{i \mid -\delta_j \leq u_j \leq \delta_j, \ \forall j\}$$

For example, $\Delta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Part II:

Interpretations of optimal solutions

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Duality review