Discrete Models in Applied Mathematics

AMATH 343

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Preface

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Contents

1 Introduction

Discrete models are used to analyze or predict properties of a system over discrete time units t_k, $k = 1, 2, \ldots$, as opposed to analyzing it over a continuous time variable $t \in \mathbb{R}$.

In a simple example, where we model the population of a particular species of perennial plant in a given ecosystem, we can let $p(n)$ be the number of plants in this system n years past this year, where $n \geq 0$. We can also use p_n for simplicity. Then the populations p_n may be viewed as elements of a sequence $\mathbf{p} = \{p_0, p_1, \ldots\}$.

1.1 Radioactive Decay

Imagine a rock containing a radioactive element "X". We denote $T_{1/2}$ the radioactive "half-life" of X. Then we have

> If our sample contains a units of X at some time t, then only one-half the original amount, $\frac{1}{2}a$ units are present at time $t + T_{1/2}$.

We let x_k be the amount of X in our sample at $t_k = kT_{1/2}$, for $k = 0, 1, 2, \ldots$.

The half-life property gives us

$$x_k = \frac{1}{2} x_{k-1}, \qquad k = 1, 2, 3, \ldots \tag{1.1}$$

(1.1) is an example of a **difference equation** in the variables x_k, $k = 0, 1, 2, \ldots$. We abbreviate as "**d.e.**" in this course.

The expression $x_k = (1/2)^k x_0$ is the solution to (1.1) with initial condition x_0.

We often interested in the **long-term** or **asymptotic** behavior of the sequence, i.e., $\{x_k\}$ with $k \to \infty$.

If now we assume $x(t)$ is continuous, then x_k is the result of sampling at times t_k. In this and other applications, the sampling can be viewed as a "stroboscopic" examination of a certain physical property $x(t)$ of a physical or biological system that evolves over time. Here we can use the true "radioactive decay low", i.e.,

[Rate of decay] proportional to [amount of radioactive substance present]

This leads to differential equation with decay constant $k > 0$ specific to X:

$$\frac{dx}{dt} = -kx,$$

and the solution to this DE satisfying the initial condition $x(0) = x_0$ is $x(t) = x_0 e^{-kt}$.

1.2 Population growth

The model of "Malthusian growth" is as follows:

[Rate of population growth] is proportional to [population at time t]

This yields the following DE

$$\frac{dx}{dt} = ax, \qquad a > 0$$

The solution to this DE satisfying the initial condition $x(0) = x_0$ is

$$x(t) = x_0 e^{at}.$$

This DE represents a continuous dynamical model of population evolution.

The propagation of annual plants is better described by discrete models:

$$x_{n+1} = cx_n,$$

for c some constant.

General questions regarding discrete mathematical models

Q1 Given $x_0, \ldots, x_n$ for some $n > 0$,can we determine x_{n+1} uniquely? How many of previous values do we need?

The simplest type of model is $x_n = f(x_{n-1})$ for $n = 1, 2, \ldots$. We typically require f not only continuous in x but also increasing in x (for population model). Also we need $f(0) = 0$. This leads to the simplest case $f(x) = cx$.

Later in this course, the term **discrete dynamical system** will be used to refer to such models.

Q2 What's the behavior of the sequence $\{x_n\}$ as $n \to \infty$?

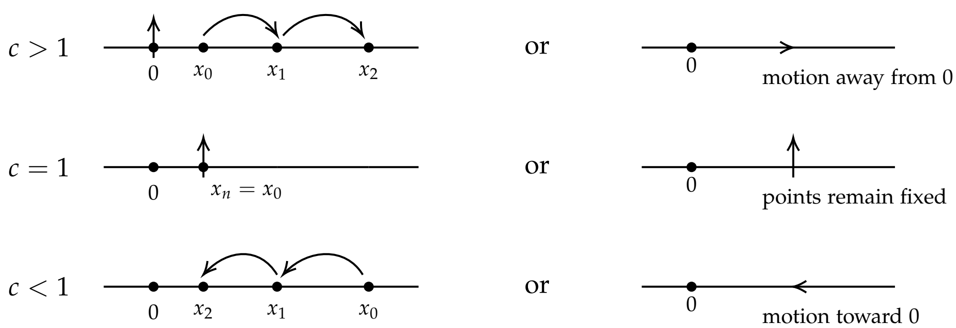
So here we analyze the asymptotic behavior of sequences $x_n = cx_{n-1}$, $n \geq 1$. The solution to DE with initial condition x_0 is $x_n = c^n x_0, n \geq 0$.

Case 1: $c > 1$, population grows monotonically without bound.

Case 2: $c = 1$, population remains constant.

Case 3: $0 < c < 1$, population decreases monotonically with limit 0.

We can depict as follows:



These sets of diagrams are known as **phase potraits** of the dynamic system.

We use $f^{\circ n}$ to denote n-fold composition of f with itself. For example, $x_2 = f(f(x_0)) = f^{\circ 2}(x_0)$.

The solution to the dynamic system is $x_n = c^n x_0$. If $x_0 = 0$, then $x_n = 0$ for all n. The point $x = 0$ is a **fixed point** of the function $f(x) = cx$.

Then we discuss the behavior of the sequences.

- $c > 0$
  - $0 < c < 1$. $x = 0$ is an **attractive fixed point**.
  - $c > 1$. $x = 0$ is a **repulsive fixed point**.
  - $c = 1$, each x is a fixed point. $x = 0$ is neither attractive or repulsive fixed point. In many books, it is called **neutral fixed point** or **indifferent fixed point**. Note that fixed points here are unstable, because if we perturb the initial condition a bit, unlike the other two cases, the long term result/behavior is different.
- $c < 0$, x_n and x_{n-1} alternate in sign.

1.3 Applications

The discrete models above, or discrete dynamic systems, have the following relation in general: for some $n \geq 1$,

$$x_k = f(k, x_{k-1}, x_{k-2}, \ldots, x_{k-n}), \qquad k \geq n. \tag{1.2}$$

(1.2) represents the general form of **difference equation of order** n**.** It also can be called **recursion relations**.

Index