Continuous Optimization

CO 466

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Preface

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$\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$; $\mathbb{Z}_{++} = \{1, 2, 3, \ldots\}$

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Introduction: Formulations, fundamental background and definitions

Let $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m, h : \mathbb{R}^n \to \mathbb{R}^p$ all continuous.

$$\begin{array}{ll} \inf & f(x) \\ \text{s.t.} & g(x) \leq 0 \\ & h(x) = 0 \end{array} \tag{P}$$

$$S := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}$$

is called the feasible solution set of (P), equivalently feasible region of (P).

Definition 1: global minimizer

$\bar{x} \in \mathbb{R}^n$ is a **global minimizer** of (P) if $x \in S$ and $f(x) \geq f(\bar{x})$ for all $x \in S$.

Sometimes we simply say $\bar{x}$ is a minimizer of (P).

Definition 2: local minimizer

$\bar{x} \in \mathbb{R}^n$ is a **local minimizer** of (P) if $\bar{x} \in S$ and there exists a neighborhood U of $\bar{x}$ such that

$$f(x) \geq f(\bar{x}) \quad \forall x \in S \cap U$$

$\bar{x} \in \mathbb{R}^n$ is a **strict local minimizer** of (P) if $\bar{x} \in S$ and there exists a neighborhood U of $\bar{x}$ such that

$$f(x) > f(\bar{x}) \quad \forall x \in (S \cap U) \setminus \{\bar{x}\}$$

$\bar{x} \in \mathbb{R}^n$ is an **isolated local minimizer** of (P), if $\bar{x} \in S$ and there exists a neighborhood U of $\bar{x}$ such that $\bar{x}$ is the only local minimizer of (P) in $(S \cap U)$.

continuous optimization problem

A **continuous optimization problem** is a problem of optimizing (minimizing or maximizing) a continuous function of finitely many real variables subject to finitely many equations and inequalities on continuous functions of these variables.

What kind of problems can be formulated as Continuous Optimization problems? Almost everything.

Example 3: Fermat's Last Theorem

"There do not exist positive integers x, y, z and an integer $n \geq 3$ such that $x^n + y^n = z^n$."

Consider

$$\begin{aligned}
\inf \quad & f(x) := \left(x_1^{x_4} + x_2^{x_4} - x_3^{x_4}\right)^2 + \sum_{i=1}^{4} \left(\sin(\pi x_i)\right)^2 \\
\text{s.t.} \quad & g_1(x) := 1 - x_1 \leq 0 \\
& g_2(x) := 1 - x_2 \leq 0 \\
& g_3(x) := 1 - x_3 \leq 0 \\
& g_4(x) := 3 - x_4 \leq 0
\end{aligned} \tag{P}$$

The optimal objective value of (P) is zero and attained if and only if FLT is false.

We can show that (P) has a sequence of feasible solutions $\{x^{(k)}\}$ such that $f(x^{(k)}) \searrow 0$. Since $f(x) \geq 0$ for all $x \in \mathbb{R}^4$, the optimal value of (P) is zero.

FLT is true if and only if (P) does not attain its optimal value (of zero).

Even when the number of variables in a continuous optimization problem is very small (e.g., 4) the optimization problem may be notoriously hard. Even discrete structures can be formulated in our framework. $\sin(\pi x_1) = 0 \iff x_1 \in \mathbb{Z}$. In Example 3, we have functions that are "highly nonlinear".

Example 4: Combinatorial Optimization, 0,1 Integer Programming

Let m, n be positive integers, $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ be given. Consider the 0,1 Integer Programming problem.

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b \\
& x \in \{0,1\}^n
\end{aligned} \tag{IP}$$

The first condition can be written as

$$g(x) := Ax - b \leq 0.$$

The second condition can be written as

$$x_j(x_j - 1) = 0 \quad \forall j \in [n] \leftrightarrow h(x) = 0$$

Our continuous optimization problem is only mildly nonlinear.

Some conclusions from Example 3 and 4

Continuous Optimization problems acn be very hard even when the number of variables and constraints are both small, the nonlinearity in f, g, h is very mild.

To successfully solve Continuous Optimization Problems, we must study the problem class at hand, discover special properties and structures and then exploit these special properties & structures.

1.1 Conic Form

Definition 5: cone

A set $K \subseteq \mathbb{R}^n$ is a **cone** if $\forall x \in K, \forall k \in \mathbb{R}_+, \lambda x \in K$.

Definition 6: convex

A set $S \subseteq \mathbb{R}^n$ is **convex** if for every pair of points in S, the line segment joining them lies entirely in S.

That is S is convex if $\forall u, v \in S, \forall \lambda \in [0,1], [\lambda u + (1-\lambda)v] \in S$.

Definition 7: convex cone

A set $K \subseteq \mathbb{R}^n$ is a **convex cone** if it is convex and is a cone.

Let $g : \mathbb{R}^n \to \mathbb{R}^m, f : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. Consider

$$\begin{array}{ll} \inf & f(x) \\ \text{s.t.} & g(x) \preceq_K 0 \end{array}$$

where $K \subseteq \mathbb{R}^m$ is a convex cone and for every $u, v \in \mathbb{R}^m$, $u \succeq_K v$ means $(u-v) \in K$.

This is at least as general as our original (P). Consider $\mathbb{R}^p \ni K := \mathbb{R}^m_+ \oplus \{0\} \ldots$

1.2 Derivatives

Definition 8: directional derivative

The **directional derivative** of $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$ along the direction $d \in \mathbb{R}^n$ is

$$f'(\bar{x}; d) := \lim_{\alpha \searrow 0} \frac{f(\bar{x} + \alpha d) - f(\bar{x})}{\alpha}$$

(Gâteaux (directional) derivative)

Exercise:
What is the directional derivative of $f : \mathbb{R}^n \to \mathbb{R}, f(x) := \|x\|_\infty$, for every $\bar{x}, d \in \mathbb{R}^n$?

Definition 9: differentiable

$f : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\bar{x} \in \mathbb{R}^n$ if $\exists \mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, linear, such that

$$\lim_{\substack{h \to 0 \\ (h \in \mathbb{R}^n)}} \frac{\left\| f(\bar{x} + h) - [f(\bar{x}) + \mathcal{A}(h)] \right\|}{\|h\|} = 0$$

Such $\mathcal{A}$ is called the derivative of f at $\bar{x}$ and is denoted by $Df(\bar{x})$ or $f'(\bar{x})$ (matrix representation of $Df(\bar{x})$). We will also use $\nabla f(\bar{x}) := [f'(x)]^T$.

Suppose $f : \mathbb{E}_1 \to \mathbb{E}_2$, we have

$$\begin{array}{ll} Df(\bar{x}) \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2), & Df : \mathbb{E}_1 \to \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2) \\ D^2 f(\bar{x}) \in \mathcal{L}(\mathbb{E}_1, \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)), & D^2 f : \mathbb{E}_1 \to \mathcal{L}(\mathbb{E}_1, \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)) \end{array}$$

If $f : \mathbb{R}^n \to \mathbb{R}$, then $D^k f(\bar{x})[h^{(1)}, h^{(2)}, \ldots, h^{(k)}]$: k^{th} differential (derivative) along the directions $h^{(1)}, h^{(2)}, \ldots, h^{(k)} \in \mathbb{R}^n$.

Theorem 10: Taylor's Theorem

Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be a $\mathcal{C}^r$ function on U. Let $x, d \in \mathbb{R}^n$. If $x, (x+d)$, and the line segment joining x and $(x+d)$ lie in U, then there exists $z \in (x, x+d)$ such that

$$f(x+d) = f(x) + \sum_{k=1}^{r-1} \frac{1}{k!} D^k f(x) \underbrace{[d, d, \ldots, d]}_{k\text{-times}} + \frac{1}{r!} D^r f(z) \underbrace{[d, d, \ldots, d]}_{r\text{-times}}$$

Definition 11: contraction mapping

Let $U \subseteq \mathbb{R}^n$ be a closed set. $f : U \to U$ is called a **contraction mapping** if there exists $\lambda \in [0, 1)$ such that

$$\|f(x) - f(y)\| \le \lambda \|x - y\| \quad \forall x, y \in U$$

1.3 Fixed Point

Theorem 12: Banach Fixed Point Theorem (1922)

Let $U \subseteq \mathbb{R}^n$ be a closed set and let $f : U \to U$ be a contraction mapping, then

(i) (Existence and uniqueness of solution - fixed point)

the mapping f has a unique fixed point $\bar{x} \in U$.

(ii) (Algorithm and convergence)

For all $x^{(0)} \in U$, the sequence $\{x^{(k)}\}$ generated by $x^{(k+1)} := f\left(x^{(k)}\right)$, $k \in \{0, 1, 2, \ldots\}$ (fixed point iteration) converges to $\bar{x}$. In particular,

$$\|x^{(k)} - \bar{x}\| \le \lambda^k \|x^{(0)} - \bar{x}\| \quad \forall k \in \{0, 1, 2, \ldots\}$$

Proof:
Suppose $U \subseteq \mathbb{R}^n$ is a nonempty closed set, and $f : U \to U$ is a contraction mapping with $\lambda \in [0, 1)$. Let $x^{(k+1)} := f\left(x^{(k)}\right)$ for all $k \in \mathbb{Z}_+$. Then for all $k \in \mathbb{Z}_+$,

$$\|x^{(k+1)} - x^{(k)}\| = \|f(x^{(k)}) - f(x^{(k-1)})\| \le \lambda \|x^{(k)} - x^{(k-1)}\|$$

By induction on k, ... we obtain

$$\|x^{(k)} - x^{(k-1)}\| \le \lambda^k \|x^{(1)} - x^{(0)}\| \quad \forall k \in \mathbb{Z}_+ \tag{$*$}$$

$\forall m \in \mathbb{Z}_{++}, \forall k \in \mathbb{Z}_{++}$,

$$\begin{aligned}
\|x^{(m+k)} - x^{(m)}\| &= \|x^{(m+k)} - x^{(m+k-1)} + x^{(m+k-1)} - x^{(m+k-2)} + \cdots + x^{(m+1)} - x^{(m)}\| \\
&\le \sum_{i=1}^{k} \|x^{(m+i)} - x^{(m+i-1)}\| && \triangle\text{-ineq} \\
&\le (\lambda^{m+k-1} + \lambda^{m+k-2} + \cdots + \lambda^m) \|x^{(1)} - x^{(0)}\| && \text{by } (*) \\
&= \lambda^m (1 + \lambda + \lambda^2 + \cdots + \lambda^{k-1}) \|x^{(1)} - x^{(0)}\| \\
&= \frac{\lambda^m (1 - \lambda^k)}{1 - \lambda} \|x^{(1)} - x^{(0)}\| \\
&\le \frac{\lambda^m}{1 - \lambda} \|x^{(1)} - x^{(0)}\| \to 0 \text{ as } m \to +\infty \text{ (independent of } k)
\end{aligned}$$

Therefore $\{x^{(k)}\}$ os a Cauchy sequence and hence it converges. Let $\bar{x}$ be its limit. $\bar{x} \in U$ (U is closed).

$\forall k \in \mathbb{Z}_+$, we have

$$\|f(\bar{x}) - \bar{x}\| \leq \|f(\bar{x}) - x^{(k)}\| + \|x^{(k)} - \bar{x}\| \leq \lambda \underbrace{\|\bar{x} - x^{(k-1)}\|}_{\to 0} + \underbrace{\|x^{(k)} - \bar{x}\|}_{\to 0}$$

As $k \to +\infty$, RHS $\to 0$. Thus $f(\bar{x}) = \bar{x}$. This proves the existence. Now we prove the uniqueness.

Suppose $\exists \bar{x}, \bar{y} \in U$ such that $f(\bar{x}) = \bar{x}$ and $f(\bar{y}) = \bar{y}$. Then

$$\|\bar{x} - \bar{y}\| = \|f(\bar{x}) - f(\bar{y})\| \leq \lambda\|\bar{x} - \bar{y}\| \implies (1 - \lambda)\|\bar{x} - \bar{y}\| = 0 \underset{\lambda \in [0,1)}{\implies} \bar{x} = \bar{y}$$

Now that we have established existence and uniqueness of $|barx$, for a proof of convergence rate claim, we proceed as in the beginning of the proof. However, we use $\bar{x}$.

$$\|x^{(1)} - \bar{x}\| = \|f(x^{(0)}) - f(\bar{x})\| \leq \lambda\|x^{(0)} - \bar{x}\| \implies \|x^{(2)} - \bar{x}\| \leq \lambda^2\|x^{(0)} - \bar{x}\|$$

By induction on k, we have

$$\|x^{(k)} - \bar{x}\| \leq \lambda^k \|x^{(0)} - \bar{x}\| \quad \forall k \in \mathbb{Z}_k$$

as desired. □

Theorem 13: Brouwer's Fixed Point Theorem (1910)

Let $U \subset \mathbb{R}^n$ be a nonempty, compact and convex set; let $f : U \to U$ continuous such that $f(U) = U$. Then there exists $\bar{x} \in U$ such that $f(\bar{x}) = \bar{x}$.

See the application in https://n.sibp.ro/co/456.

Theorem 14: Kakutani's Fixed Point Theorem (1941)

Let $U \subset \mathbb{R}^n$ be a nonempty, compact convex set and $f : U \to 2^U$ be a set valued map on U. If $\text{Graph}(f) := \left\{\binom{x}{v} \in U \oplus U : v \in f(x)\right\}$ is closed and $f(x)$ is nonempty and convex for every $x \in U$, then there exists $\bar{x} \in U$ such that $\bar{x} \in f(\bar{x})$.

Theorem 15: Borsuk-Ulam Theorem (1930-1933)

Let $f : \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\} \to \mathbb{R}^n$ be continuous. Then there exists $\bar{x} \in \mathbb{R}^{n+1}$ such that $\|\bar{x}\|_2 = 1$ and $f(\bar{x}) = f(-\bar{x})$.

Example:

Let $n := 2$. Assuming temperature and barometric air pressure are continuous functions on the Earth's surface, and Earth's surface is homeomorphic to a sphere, there always exists an antipodal pair of points on Earth with the same temperature & the same air pressure.

1.4 Other

$\mathbb{S}^n := n \times n$ symmetric matrices with real entries.

Theorem 16: Spectral Decomposition Theorem

For every $A \in \mathbb{S}^n$, there exists $Q \in \mathbb{R}^{n \times n}$ orthogonal ($Q^TQ = I$) such that $A = QDQ^T$, where $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix.

In the above theorem, the diagonal matrix D contains all eigenvalues of A, and the columns of Q are the corresponding eigenvectors of A.

Definition 17: positive definite

$A \in \mathbb{R}^{n \times n}$ is **positive semidefinite** if $h^TAh \geq 0$ for all $h \in \mathbb{R}^n$; such A is **positive definite** if $h^TAh > 0$ for all $h \in \mathbb{R}^n \setminus \{0\}$.

If $A \in \mathbb{R}^{n \times n}$ is skew-symmetric ($A = -A^T$), then $h^TAh = (h^TAh)^T = -h^TAh = 0$ for all $h \in \mathbb{R}^n$. Therefore, such A is positive semidefinite but not positive definite.

$$\begin{aligned}
\mathbb{S}^n_+ &:= \text{positive semidefinite matrices in } \mathbb{S}^n, \\
\mathbb{S}^n_{++} &:= \text{positive definite matrices in } \mathbb{S}^n.
\end{aligned}$$

In fact, $\mathbb{S}^n_{++} = \operatorname{int}(\mathbb{S}^n_+)$.

Theorem 18: Choleski Decomposition Theorem

Let $A \in \mathbb{S}^n$, then

(a) A is positive semidefinite if and only if there exists $L \in \mathbb{R}^{n \times n}$ lower triangular such that $A = LL^T$;

(b) A is positive definite if and only if there exists $L \in \mathbb{R}^{n \times n}$ lower triangular and nonsingular such that $A = LL^T$.

Note that Taylor's Theorem (Theorem 10) cannot be completely generalized to functions $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq 2$, even for $r = 1$. However, we have

Theorem 19

Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^m$ be a C^1 on U. Suppose for $\bar{x}, d \in \mathbb{R}^n$, $[\bar{x}, \bar{x} + d] \subset U$. Then

$$f(\bar{x} + d) - f(\bar{x}) = \int_0^1 Df(\bar{x} + \alpha d)d \; (\partial\alpha)$$

A consequence of this result is obtained when $Df(\cdot)$ is Lipschitz continuous on U (in a neighborhood of $[\bar{x}, \bar{x} + d]$ suffices). Let L denote the Lipschitz constant. Then

$$\|Df(x) - Df(y)\| \leq L\|x - y\| \quad \forall x, y \in U$$

Then we have

$$\begin{aligned}
\|f(\bar{x} + d) - f(\bar{x}) - Df(\bar{x})d\|_2 &= \left\| \int_0^1 [Df(\bar{x} + \alpha d) - Df(\bar{x})]d \; (\partial\alpha) \right\|_2 \\
&\leq \int_0^1 \underset{\substack{\uparrow \\ \text{operator} \\ \text{2-norm}}}{\|Df(\bar{x} - \alpha d) - Df(\bar{x})\|_2} \cdot \underset{\substack{\uparrow \\ \text{2-norm} \\ \text{on } \mathbb{R}^n}}{\|d\|_2} (\partial\alpha) \qquad \text{see aside} \\
&\leq \int_0^1 L\|d\|_2 \cdot \|d\|_2 \alpha(\partial\alpha) \\
&= \frac{1}{2}L\|d\|_2^2
\end{aligned}$$

So, if $\|d\|_2 < \epsilon$, then this error in this first-order estimate of $f(\bar{x}+d)$ is bounded above by $\frac{1}{2}L\epsilon^2$.

> Aside:
>
> Let $h := \int_0^1 [Df(\bar{x}+\alpha d) - Df(\bar{x})]d\ (\partial\alpha)$, then
>
> $$\begin{aligned}\|h\|_2^2 = h^T h = h^T \int_0^1 [Df(\bar{x}+\alpha d) - Df(\bar{x})]d\ (\partial\alpha) \\ = \int_0^1 h^T [Df(\bar{x}+\alpha d) - Df(\bar{x})]d\ (\partial\alpha) \\ \leq \int_0^1 \|h\|_2 \left\|[Df(\bar{x}+\alpha d) - Df(\bar{x})]d\right\|_2 (\partial\alpha) \qquad \text{Cauchy-Schwarz}\end{aligned}$$
>
> $$\implies \|h\|_2 \leq \int_0^1 \left\|[Df(\bar{x}+\alpha d) - Df(\bar{x})]d\right\|_2 (\partial\alpha)$$

Note that we may replace f in Theorem 19 by $Df^r(\cdot)$ (assuming $f \in \mathcal{C}^{r+1}$) and apply the same reasoning. Indeed, Theorem 19 can be very useful in the design and analysis of continuous optimization algorithms.

Theorem 20: Inverse Function Theorem

Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ be $\mathcal{C}^1$, $\bar{x} \in U$, $\det(\nabla f(\bar{x})) \neq 0$. Then there exists an open neighborhood V of $\bar{x}$ in U and an open neighborhood W of $f(\bar{x})$ such that

- $f(V) = W$,
- f has a local $\mathcal{C}^1$ inverse $f^{-1} : W \to V$,
- $\forall y \in W$, with $x = f^{-1}(y)$, we have $Df^{-1}(y) = [Df(x)]^{-1}$.

In the above, if f is $\mathcal{C}^r$, then there exists such an $f^{-1} \in \mathcal{C}^r$. Theorem 20 can be proved by utilizing Theorem 12 (in showing that the inverse is well-defined, i.e., one-to-one).

Theorem 21: Implicit Function Theorem

Let $h : \mathbb{R}^n \to \mathbb{R}^p$, $h \in \mathcal{C}^1$ in a neighborhood of $\bar{x} \in \mathbb{R}^n$ where $h(\bar{x}) = 0$. Suppose $h'(\bar{x})$ has full row rank ($\text{rank}(h'(\bar{x})) = p \leq n$). Define a partition $[B|N]$ of columns of $h'(\bar{x})$:

$$h' =: [\underbrace{h'_B(\bar{x})}_{\substack{\in \mathbb{R}^{p\times p},\\ \text{nonsingular},\\ \text{partition}}} | h'_N(\bar{x})]$$

$\bar{x}$ and x with respect to the same $[B|N]$. Then there exist neighborhoods U_B of $\bar{x}_B$ and U_N of $\bar{x}_N$ and a $\mathcal{C}^1$ function $f : U_N \to U_B$ such that

- $f(\bar{x}_N) = \bar{x}_B$,
- $h\binom{x_B}{x_N} = 0 \iff x_B = f(x_N)$ for all $x_B \in U_B, x_N \in U_N$.

Moreover, $f'(x_N) = -[h'_B(\bar{x})]^{-1} h'_N(\bar{x})$.

Recall the very special case (e.g., equality constraints in an LP problem): $A \in \mathbb{R}^{p\times n}$, $\text{rank}(A) = p$ given

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0\end{array}$$

$$\begin{aligned} h(x) &:= Ax - b \implies h'(x) = A \\ \bar{x}_B &= A_B^{-1}b - A_B^{-1}A_N\bar{x}_N \\ x_B &= A_B^{-1}b - A_B^{-1}A_N x_N \\ f(x_N) &:= A_B^{-1}b - A_B^{-1}A_N x_N \end{aligned}$$

In this setting $U_B := \mathbb{R}^p, U_N := \mathbb{R}^{n-p}$.

Lemma 22: Chain Rule

Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be both open sets. $f_1 : U \to \mathbb{R}^m, f_2 : V \to \mathbb{R}^p$ be differentiable on U and V respectively such that $f_1(U) \subseteq V$. Then $(f_2 \circ f_1)$ is differentiable on U and

$$D(f_2 \circ f_1)(\bar{x}) = Df_2(f_1(\bar{x})) \circ Df(\bar{x}) \qquad \forall \bar{x} \in U$$

Example: Line search, directional derivative

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable on $\mathbb{R}^n$. Also, given are a current point $\bar{x} \in \mathbb{R}^n$ and a "search direction" $d \in \mathbb{R}^n$. We define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(\alpha) := f(\bar{x} + \alpha d)$. Then $\phi'(\alpha) = \langle \nabla f(\bar{x} + \alpha d), d\rangle$. If f is $\mathcal{C}^2$, then $\phi''(\alpha) = d^T\nabla^2 f(\bar{x} + \alpha d)d$. Note $\phi'(0) = \langle \nabla f(\bar{x}), d\rangle, \phi''(0) = d^T\nabla^2 f(\bar{x})d$.

Corollary 23

Suppose h and $\bar{x}$ are as in Theorem 21 (Implicit Function Theorem). Also assume $Z \in \mathbb{R}^{n\times p}$ $(q \leq n - p)$ such that $h'(\bar{x})Z = 0$. Then there exists a neighborhood U of $0 \in \mathbb{R}^q$ and a $\mathcal{C}^1$ function $t : U \to \mathbb{R}^n$ such that

- $t(0) = 0$,
- $t'(0) = 0$,
- $h(\bar{x} + Zd_Z + t(d_Z)) = 0$ for all $d_Z \in U$.

So the function t gives us a way of moving away from $\bar{x}$ (a solution of $h(x) = 0$) in a way that keeps feasible with respect to $h(x) = 0$.

Proof:

Let $h, \bar{x}$ and Z be as in the assumptions. Using the partition $[B|N]$, define $z =: \begin{bmatrix} z_B \\ z_N \end{bmatrix}$ (recall $h'(\bar{x}) = [h'_B(\bar{x})|h'_N(\bar{x})]$). Let $U := \{d_Z \in \mathbb{R}^q : (\bar{x}_N + Z_N d_Z) \in \underset{\substack{\uparrow \\ \text{neighborhood of } \bar{x}_N \text{ from Theorem 21}}}{U_N}\}$. Define t by

$$\begin{aligned} t_N(d_Z) &:= 0 \\ t_B(d_Z) &:= f(\bar{x}_N + Z_N d_Z) - \bar{x}_B - Z_B d_Z \end{aligned}$$

Thus,

$$h(\bar{x} + Zd_Z + t(d_Z)) = h\begin{bmatrix} \bar{x}_B + Z_B d_Z + f(\bar{x}_N + Z_N d_Z) - \bar{x}_B - z_B d_Z \\ \bar{x}_N + Z_N d_Z + 0 \end{bmatrix} = h\begin{bmatrix} f_N(\bar{x}_N + Z_N d_Z) \\ \bar{x}_N + Z_N d_Z \end{bmatrix} \underset{\substack{\uparrow \\ \text{By Theorem 21}}}{=} 0$$

Also,

$$t(0) = f(\bar{x}_N) - \bar{x}_B = 0, \qquad t'_N(0) = 0$$

$$t'_B(0) \underset{\substack{\uparrow \\ \text{Chain rule (Lemma 22)}}}{=} f'(\bar{x}_N)Z_N - Z_B = -[h'_B(\bar{x})]^{-1}h'_N(\bar{x})Z_N - Z_B = [h'_B(\bar{x})]^{-1}\underbrace{[-h'_N(\bar{x})Z_N - h'_B(\bar{x})Z_N]}_{=-h'(\bar{x})Z=0} = 0$$

∎

What does the size of the neighborhood depend on?

Note in LPs $t(d_Z) := 0$ for all $d_Z \in \mathbb{R}^q$.

Corollary 24

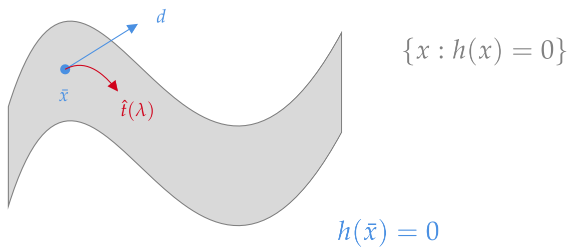
Assume h and $\bar{x}$ are as described in Theorem 21. Let $d \in \mathbb{R}^n$ such that $h'(\bar{x})d = 0$. Then there exists $\bar{\lambda} > 0$ and a $\mathcal{C}^1$ arc (directed curve) $\hat{t}$ with properties:

$$\begin{cases} \hat{t}(0) = \bar{x} \\ h(\hat{t}(\lambda)) = 0 \quad \forall \lambda \in [0, \bar{\lambda}) \\ \hat{t}'(0) = d \end{cases}$$

Proof:
In the statement of Corollary 23, plug in $Z := d$ and then using the resulting t,

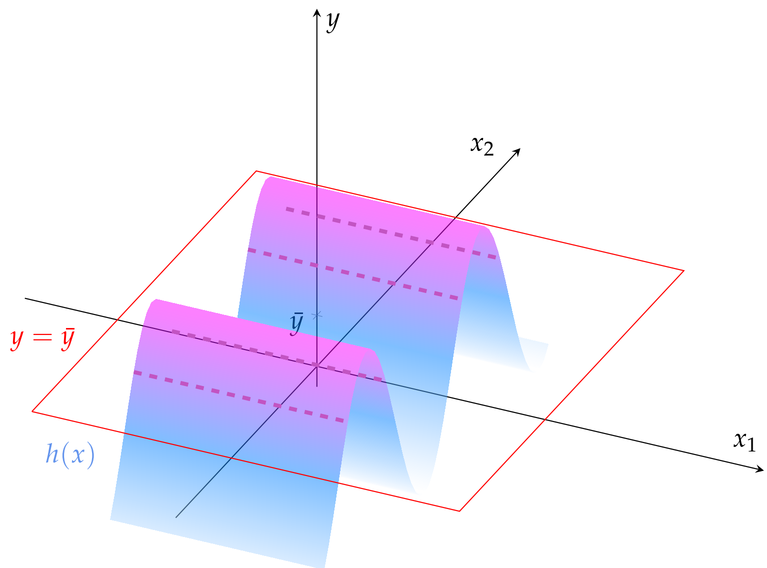
$$\hat{t}(\lambda) := \bar{x} + \underset{\substack{\uparrow \\ d_Z}}{\lambda}\ \underset{\substack{\uparrow \\ Z}}{d} + t(\lambda)$$

$\square$



How applicable are the Theorems 20, 21 and their Corollaries?

Let $h : \mathbb{R}^n \to \mathbb{R}^p$, where $p \leq n$. Call $\bar{x} \in \mathbb{R}^n$ **regular** if $\operatorname{rank}(h'(\bar{x})) = p$; call $\bar{y} \in \mathbb{R}^p$ a **regular value** if $\forall x \in h^{-1}(\bar{y})$ are regular.



Union of these curves is the set $h^{-1}(\bar{y})$.

If h is affine, then $h(x) = Ax - b$ for some given $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$. Let $\bar{y} \in \mathbb{R}^p$ be given. Then $h^{-1}(\bar{y}) = \{x \in \mathbb{R}^n : Ax = \bar{y} + b\}$.

Theorem 25: Sard's Theorem, Morse-Sard Theorem

Let $h : \mathbb{R}^n \to \mathbb{R}^p$, where $p \leq n$, $h \in \mathcal{C}^r$ with $r \geq n - p + 1$. Then the p-dimensional Lebesgue measure of $\{y \in \mathbb{R}^p : y \text{ is not a regular value}\}$ is zero.

Morse (1939) proved the $p = 1$ case, Sard (1942) proved the generalization above. Smale (1965) proved an infinite dimensional version.

2

Unconstrained Continuous Optimization

$f:\mathbb{R}^n \to \mathbb{R}, g:\mathbb{R}^n \to \mathbb{R}^m, h:\mathbb{R}^n \to \mathbb{R}^p$.

$$\begin{array}{ll} \inf & f(x) \\ \text{s.t.} & g(x) \leq 0 \\ & h(x) = 0 \end{array} \tag{P}$$

$S := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}$. Here, we assume $S = \mathbb{R}^n$.

Theorem 26: First-order necessary conditions

Let $f:\mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$ and $S = \mathbb{R}^n$. Then, $\bar{x} \in \mathbb{R}^n$ is a local minimizer for (P) $\implies f'(\bar{x}) = 0$.

$\bar{x}$ is a **stationary point** of f.

Proof:

Suppose $f:\mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1, S = \mathbb{R}^n$, and $\bar{x} \in \mathbb{R}^n$ is a local minimizer for (P). For the sake of seeking a contradiction, suppose $f'(\bar{x}) \neq 0$. Then, there exists $d \in \mathbb{R}^n$ such that $\langle f'(\bar{x}), d\rangle < 0$ (e.g., let $A \in \mathbb{S}^n_{++}$, and set $d := -Af'(\bar{x})$). Consider $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(\alpha) := f(\bar{x} + \alpha d)$. Then, $\phi'(0) = \langle f'(\bar{x}), d\rangle < 0$. Thus, for all sufficiently small, positive α, $f(\bar{x} + \alpha d) < f(\bar{x})$. Therefore, $\bar{x}$ is not a local minimizer for (P). $\square$

Optimality conditions are widely used in algorithm design. E.g., for many software $\|\nabla f(x^{(k)})\| < \epsilon$ is a part of the stopping criteria.

Definition 27

$d \in \mathbb{R}^n$ is a **decent direction** for f at $\bar{x} \in \mathbb{R}^n$, if $\langle f'(\bar{x}), d\rangle < 0$.

$d \in \mathbb{R}^n$ is an **improving direction** for f at $\bar{x}$, if $f(\bar{x} + \alpha d) < f(\bar{x})$ $\forall \alpha > 0$ and sufficiently small.

Theorem 28: Second-order necessary conditions

Let $f:\mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^2$ and $S = \mathbb{R}^n$. If $\bar{x} \in \mathbb{R}^n$ a local minimizer for (P), then $f'(\bar{x}) = 0$ and $\nabla^2 f(\bar{x}) \in \mathbb{S}^n_{+}$.

Proof:

Suppose $\bar{x}$ is a local minimizer for (P). Since f is $\mathcal{C}^2$ by Theorem 27, $f'(\bar{x}) = 0$. Suppose for the sake of contradiction that $\nabla^2 f(\bar{x}) \notin \mathbb{S}^n_{+}$. Since $f \in \mathcal{C}^2, \nabla^2 f(\bar{x}) \in \mathbb{S}^n$. Therefore, there exists $d \in \mathbb{R}^n$

such that $d^T\nabla^2 f(\bar{x})d < 0$. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(\alpha) := f(\bar{x} + \alpha d)$. Then $\phi'(0) = \langle \nabla f(\bar{x}), d\rangle = 0$, $\phi''(0) = d^T\nabla^2 f(\bar{x})d < 0$. Therefore, for all $\epsilon > 0$ and sufficiently small $f(\bar{x} + \epsilon d) < f(\bar{x})$ which contradicts the fact that $\bar{x}$ is a local minimizer for (P). □

Definition 29: direction of negative curvature

$d \in \mathbb{R}^n$ is called a **direction of negative curvature** for f at $\bar{x}$ if $d^T\nabla^2 f(\bar{x})d < 0$.

Theorem 30: Taylor's Theorem - implicit remainder version

Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be $\mathcal{C}^r$ on U. Let $\bar{x}, d \in \mathbb{R}^n$, assume $[\bar{x}, \bar{x} + d] \subset U$. Then,

$$f(\bar{x} + d) = f(\bar{x}) + \sum_{k=1}^{r} \frac{1}{k!} D^k f(\bar{x})[\underbrace{d, \ldots, d}_{k\text{-times}}] + \mathcal{R}(\bar{x}, d),$$

where $\mathcal{R}(\bar{x}, \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{\mathcal{R}(\bar{x}, h)}{\|h\|^r} = 0.$$

Theorem 31: Second order sufficient conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^2$, $S = \mathbb{R}^n$. Let $\bar{x} \in \mathbb{R}^n$. If $f'(\bar{x}) = 0$ and $\nabla^2 f(\bar{x}) \in \mathbb{S}^n_{++}$, then $\bar{x}$ is a strict local minimizer for (P).

Proof:
Let $\bar{x} \in \mathbb{R}^n$ such that $f'(\bar{x}) = 0$ and $\nabla^2 f(\bar{x}) \in \mathbb{S}^n_{++}$,

$$\underset{\substack{\uparrow \\ \lambda_n(\nabla^2 f(\bar{x}))}}{\delta} := \min\{d^T\nabla^2 f(\bar{x})d : \|d\|_2 = 1\} > 0$$

By Theorem 30, for all $d \in \mathbb{R}^n$, $\|d\|_2 = 1$, and $\alpha > 0$ and small enough, we have

$$f(\bar{x} + \alpha d) = f(\bar{x}) + \underbrace{\alpha\langle \nabla f(\bar{x}), d\rangle}_{=0} + \frac{\alpha^2}{2} d^T\nabla^2 f(\bar{x})d + o(\alpha^2) \geq f(\bar{x}) + \frac{\delta}{2}\alpha^2 + o(\alpha^2)$$

Choose a neighborhood U of $\bar{x}$ such that $\frac{\delta}{2}\alpha^2 > |o(\alpha^2)|$. Then for all $x \in U \setminus \{\bar{x}\}$, $f(x) > f(\bar{x})$. Therefore, $\bar{x}$ is a strict local minimizer for (P). □

How applicable is this last theorem?

Proposition 32

Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^2$ and consider $\tilde{f}(x) := f(x) + c^T x$, where $c \in \mathbb{R}^n$ is given. Then for almost all $c \in \mathbb{R}^n$, $\tilde{f}(\bar{x}) = 0 \implies \nabla^2 f(\bar{x})$ is nonsingular.

Proof:
Apply Sard's Theorem (Theorem 25) to $g(x) := f'(x)$, with $r := 1$ and $p := n$. □

What if f has some nice structure, can we say more?

Definition 33: convex function

$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is **convex** if $\operatorname{epi}(f) := \left\{ \begin{pmatrix} \mu \\ x \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n : f(x) \leq \mu \right\}$ is convex.

Here $\operatorname{epi}(f)$ denotes the epi graph of f.

Theorem 34

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $S := \mathbb{R}^n$. Then every local minimizer of (P) is a global minimizer of (P). If in addition, f is differentiable on $\mathbb{R}^n$, then every stationary point of f is a global minimizer of (P).

2.1 Affine Subspace Constraints

One of the most popular form of continuous optimization problems is

$$\begin{array}{ll} \inf & f(x) \\ \text{s.t.} & Ax = b \end{array}$$

where $A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p$ are given.

At a first glance (and strictly speaking), (P) does not belong to the class of unconstrained continuous optimization problems. We may assume $\operatorname{rank}(A) = p$; otherwise

- we easily prove $Ax = b$ has no solution, which implies (P) is infeasible, or
- we easily find all redundant equations and $\bar{x} \in \mathbb{R}^n$ such that $A\bar{x} = b$.

So, $\operatorname{rank}(A) = p$. Find a basis B of A and form the partitions

$$[A_B | A_N] := A, \quad \left[\begin{array}{c} x_B \\ \hline x_N \end{array}\right] := x.$$

Then,

$$Ax = b \iff x_B = A_B^{-1} b - A_B^{-1} A_N x_N.$$

Therefore, for every $x \in S$,

$$f(x) = f\begin{pmatrix} A_B^{-1} b - A_B^{-1} A_N x_N \\ x_N \end{pmatrix}.$$

We define $\tilde{f} : \mathbb{R}^{n-p} \to \mathbb{R}$ by

$$\tilde{f}(x_N) := f\begin{pmatrix} A_B^{-1} b - A_B^{-1} A_N x_N \\ x_N \end{pmatrix}.$$

Thus (P) is equivalent to

$$\inf_{x \in \mathbb{R}^{n-p}} \tilde{f}(x) \tag{$\tilde{\text{P}}$}$$

and we can start any algorithm from any starting point $x^{(0)} \in \mathbb{R}^{n-p}$.

Another equivalent approach:

Let $\bar{x} \in S$ (i.e., $A\bar{x} = b$). Then, $S = \{\bar{x} + u : u \in \operatorname{Null}(A)\}$.

Let columns of $Z \in \mathbb{R}^{n \times (n-p)}$ form a basis for $\operatorname{Null}(A)$. Then (P) is also equivalent to

$$\inf_{v \in \mathbb{R}^{n-p}} \hat{f}(v),$$

where $\hat{f} : \mathbb{R}^{n-p} \to \mathbb{R}$ is defined as $\hat{f}(v) := f(\bar{x} + Zv)$.

In applications, with either of these two approaches, we must be very careful about exploiting sparsity as well as making sure we can efficiently and accurately evaluate all ingredients of the algorithms we choose to use on such problems.

2.2 Applications

Some other ways of dealing with constrained optimization problems using unconstrained optimization algorithms: Form the Lagrangian for (P):

$$\mathcal{L}(x, v) := f(x) + v^T(b - Ax),$$

where $v \in \mathbb{R}^p$ represents the Lagrange multipliers (dual variables corresponding to the constraints).

Use a penalty function (penalizing any violation of the constraints):

$$\rho(x, \eta) := f(x) + \eta \|Ax - b\|_\beta^\gamma,$$

where $\beta, \gamma \in \mathbb{R}$ suitably defined, $\eta \in \mathbb{R}_{++}$ a penalty parameter.

In compressed sensing and related applications, one seeks a solution of

$$\inf \{f(x) + \eta \|x\|_0 : Ax = b\},$$

where $\|x\|_0 :=$ number of nonzero entries of x. As an approximation, many researches and practitioners work with

$$\inf \{f(x) + \eta_1 \|x\|_1 + \eta_2 \|Ax = b\|_2^\gamma\},$$

where $\eta_1, \eta_2, \gamma \in \mathbb{R}$, usually fixed.

We can generalize such approaches to matrix variables. Very many interesting applications in Machine Learning, AI and modern Data Science. In many of these applications, we want to find a low-rank solution.

> Example:
>
> $\min\{\text{rank}(X) : A(X) = b\}$, where $A : \mathbb{R}^{m \times n} \to \mathbb{R}^p$ linear, $b \in \mathbb{R}^p$, both A, b are given.

2.3 Prototype low-rank approximation problem

Given $A \in \mathbb{R}_+^{m \times n}$ (both m and n are huge). We want to find matrices $U \in \mathbb{R}_+^{m \times k}$, $V \in \mathbb{R}_+^{n \times k}$ such that $A = UV^T$ and k is as small as possible.

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