ODE 2

AMATH 351

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LaTeXed by Sibelius Peng

Preface

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Sibelius Peng

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Introduction and Review

1.1 Definitions and Terminology

A **differential equation** is any equation involving a function and derivatives of this function.

Ordinary differential equations contain only functions of a single variable, called the independent variable, and derivatives with respect to that variable.

Partial differential equations contain a function of two or more variables and some partial derivatives of this function.

The **order** of a differential equation in the order of the highest derivative in the equation.

A general n-th order ODE has the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.1}$$

where $y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}$ and so on. We assume further it can be written as

$$y^{(n)} = f(x, y', \ldots, y^{(n-1)}). \tag{1.2}$$

Eq. (1.2) is said to be **linear** when f is a linear function of $y, y', \ldots, y^{(n-1)}$. In this case, Eq. (1.2) can be written as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x). \tag{1.3}$$

A differential equation that is not linear is said to be **nonlinear**.

By a **solution** of Eq. (1.2) on an interval I we mean a function $y = \psi(x)$ such that

$$f(x, \psi(x), \psi'(x), \ldots, \psi^{(n-1)(x)})$$

is defined for all x in I and is equal to $\psi^{(n)}(x)$ for all x in I.

A solution in which the dependent variable in expressed only in terms of the independent variable and constants is called an **explicit solution**.

A relation $G(x, y) = 0$ such that there exists at least one function $\psi(x)$ that satisfies the relation and Eq. (1.2) is called an **implicit solution**.

A solution which is free of arbitrary constants is called a **particular solution**.

A solutions that cannot be obtained by specializing any of the parameters in a family of solutions is called a **singular solution**.

Example:

Consider the DE $y' = xy^{1/2}$.

The explicit solution: $y = \left(\frac{x^2}{4} + c\right)^2$

A particular solution is $y = \frac{x^4}{16}$ obtained above for $c = 0$.

A singular solution is $y = 0$ which cannot be obtained from the explicit solution for any choice of constant c.

1.2 Initial-Value Problems

On some interval containing x_0, the problem

Solve

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

subject to the initial conditions

$$y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

where $y_0, \ldots, y_{n-1}$ are arbitrary specified real constants, is called an **initial-value problem** (IVP).

Consider the IVP $y' = f(x, y)$, $y(x_0) = y_0$.

Theorem 1.1: Picard

Let D be a rectangular region in the xy-plane defined by $D = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ and $(x_0, y_0) \in D$ the interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution $y(x)$ defined in an interval I centered at x_0.

1.3 First Order ODE

Separable variables

A first order DE of the form

$$\frac{dy}{dx} = g(x)h(y) \tag{1.4}$$

is said to be **separable** or to have **separable variables**. Solution method:

$$\frac{dy}{h(y)} = g(x)dx$$

Integrate both sides

$$\int \frac{1}{h(y)} dy = \int g(x)dx + C$$

Linear equations

A first order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1.5}$$

is called a **linear equation**.

Solution method:

- Write in its **standard form**
$$\frac{dy}{dx} + p(x) = f(x)$$
- Multiply both sides by the integrating factor $\mu(x) = \exp\left(\int p(x)dx\right)$, and rearrange into the exact form $\frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$
- Integrate both side with respect to x and get the general solution under the form
$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

There are other type of ODEs that you learned how to solve in AMATH 251, such as homogeneous equations, exact equations, Bernouli equations.

2

Theory of Second-Order Linear DEs

2.1 2nd-Order Linear ODEs

The most general 2nd order linear DE is

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

In AMATH 251 we learned how to solve this equation where the coefficients a_2, a_1, a_0 are constants. This equation can be written in several different forms:

1. General form
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{2.1}$$

2. Standard form: If $a_2(x)$ is not identically zero then we obtain
$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.2}$$

3. Associated homogeneous equation: This is the same as the standard form where RHS is zero,
$$y'' + P(x)y' + Q(x)y = 0 \tag{2.3}$$

   If RHS of Eq. (2.2) is non-zero the equation is said to be non-homogeneous or inhomogeneous.

2.2 Existence and Uniqueness

Existence and Uniqueness Before we try and find solutions to the DEs it is usually a good idea to know that a solution exists and it is unique. Otherwise we could be wasting out time. We state a theorem for existence and uniqueness. The ideas of the proof will be presented later when we discuss first-order systems.

Theorem 2.1: Existence and Uniqueness

Let $P(x), Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$ and if $y(x_0)$ and $y'(x_0)$ are any numbers, then Eq. (2.2) has one and only one solution $y(x)$ on the entire interval such that the initial conditions (ICs) are satisfied.

Remark:
If we are looking for a solution to the homogeneous equation with $y(0) = 0, y'(0) = 0$ observe that the trivial solution is an allowable solution. Therefore, by the existence and uniqueness theorem, it must be the only solution.

2.3 General Solutions to 2nd-order DEs

In AMATH 251 for the case of constant coefficients, we learned that the general solution to Eq. (2.2) is a superposition of any particular solution to the non-homogeneous problem and a general solution to the homogeneous one. This also holds true in the case of non-constant coefficients. Therefore, the method of attack is as follows:

1. Find the general solution to the homogeneous problem: In the case of constant coefficients we simply sub $y = e^{rx}$ find the characteristic equation, solve for the characteristic roots, r_1, r_2 form the two independent solutions $y_1(x), y_2(x)$ and get that the general solution is,

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

   where c_1, c_2 are arbitrary constants. In the case of non-constant coefficients we need to do more work to find y_1, y_2. In general we cannot find them explicitly.

2. Find a particular solution to the non-homogeneous problem. There are different methods that we can use.

The following Theorems will help us to find a unique solution of a general second-order scalar equation. First we look at the homogeneous problem and then at the more general DE.

Theorem 2.2: General solutions to 2nd-order homogeneous equations

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation (Eq. (2.3)) on the interval $[a, b]$, then the general solution to the same homogeneous problem is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

for arbitrary constants c_1, c_2.

Proof:
First we can sub y_1, y_2 and their linear superposition into the homogeneous equation to verify they are solutions.

Second we need to verify that this solution can satisfy any set of conditions, say $y(0)$ and $y'(0)$ [a]. We sub in our solution and find,

$$\begin{aligned} c_1 y_1(0) + c_2 y_2(0) &= y(0), \\ c_1 y_1'(0) + c_2 y_2'(0) &= y'(0). \end{aligned}$$

This is a system of two equations and two unknowns c_1, c_2. To be able to solve this for any initial conditions we need that the matrix is non-singular,

$$\det \begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} = y_1(0)y_2'(0) - y_2(0)y_1'(0) \neq 0$$

This motivates the definition of **Wronskian**, $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.

To ensure that our expression is a general solution we need that the initial value of the Wronskian is nonzero, $W(y_1(0), y_2(0)) \neq 0$. □

[a] These should be replaced by $y(x_0) = y_0, y'(x_0) = y_1$ for some $x_0 \in [a, b]$

Also check the alternative proof on page 66 of https://notes.sibeliusp.com/pdfs/1189/amath251.pdf.

Therefore, the above tells us that if the initial value of the Wronskian of the two solutions is non-zero, we have a general solution. Next, we will show that if the Wronskian is non-zero at the initial time it is necessarily non-zero all time. The following theorem states and proves this result.

Lemma 2.3: Uniformity of the Wronskian

If $y_1(x)$ and $y_2(x)$ are any two solutions to Eq. (2.3) on the interval $[a, b]$ then their Wronskian is either identically zero or never zero on $[a, b]$.

Proof:

$$\begin{aligned} W' &= y_1 y_2'' - y_2 y_1'' \\ &= y_1 \left[-P(x)y_2' - Q(x)y_2\right] - y_2 \left[-P(x)y_1' - Q(x)y_1\right] \\ &= -P(x)\left[y_1 y_2' - y_2 y_1'\right] \\ &= -P(x)W \end{aligned}$$

Then $W = W_0 \exp(-\int P(x)dx)$ is either zero everywhere or zero nowhere, depending on its initial values. □

So far we know that the Wronskian is always zero or always non-zero for any two solutions. The next lemma shows the relation between the Wronskian and linear independence.

Lemma 2.4: Linear Dependence and the Wronskian

If $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous equation then they are linearly dependent on this interval if and only if their Wronskian is identically zero.

Proof:

$\Rightarrow$ Let $y_2(x) = cy_1(x)$ then calculate $W(y_1, y_2) = 0$

$\Leftarrow$ Assume Wronskian is zero, then

$$\det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = 0$$

If the determinant of the matrix is zero that means that the matrix is singular and that each column is a scalar multiple of the other. In particular, if we look at the first row we find that $y_2(x)$ is a linear multiple of $y_1(x)$. This is precisely the statement that the functions are linearly dependent.

□

If we combine two lemmas we deduce that if two solutions are linearly independent at the initial time (or any other time for that matter) they are necessarily linear independent for all time. The next example is to give you some practice playing with the Wronskian.

Example:

Can show that $y = c_1 \sin x + c_2 \cos x$ is the general solution to $y'' + y = 0$ on any interval.

Now that we have a grasp on how to find general solutions to homogeneous equations, we can look at solving the non-homogeneous problem.

Theorem 2.5: General solutions to 2nd-order non-homogeneous equations

If $y_h(x)$ is the general solution to the homogeneous problem, Eq. (2.3), and $y_p(x)$ is any particular solution of the non-homogeneous problem, Eq. (2.2), then their superposition, $y(x) = y_h(x) + y_p(x)$, is a general solution to the non-homogeneous problem.

Proof:
To show that we have a general solution to the non homogeneous equation we must shows two things: 1) that it is in fact a solution and 2) we can reproduce any initial condition. With the second condition we can use our uniqueness theorem to guarantee that any two solutions to the DE with the same initial conditions must be equal.

1. skipped.
2. Now we must show that we can reproduce any IC with this solution. We do this as we did before by evaluating our solution and its derivatives at the initial time. First we do as before and define $y_h(x) = c_1y_1(x) + c_2y_2(x)$ and then we get,

$$\begin{aligned} c_1y_1(0) + c_2y_2(0) &= y(0) - y_p(x) \\ c_1y_1'(0) + c_2y_2'(0) &= y'(0) - y_p'(x) \end{aligned}$$

Note that the effect of the particular solution is to offset how we pick our constants c_1 and c_2. By assumption, we have that our homogeneous solution consists of two linearly independent solutions, y_1 and y_2, and so their Wronskian is non-zero and so we can invert this 2×2 system to find a unique solution. This is enough to guarantee that our solution is a general solution to the non-homogeneous system.

□

2.4 BVPs versus IVPs

ODEs can be classified as either **Boundary Value Problems** (BVPs) or **Initial Value Problems** (IVPs). The equations themselves can be the same, what differs are the conditions that are imposed to determine the unknown constants. For IVPs the two conditions are imposed at the same time, for example $y(0) = \alpha, y'(0) = \beta$.

In contrast, in BVPs the two conditions are imposed at different times, or different locations, $y(0) = \alpha, y(1) = \beta$.

As the names suggest, IVPs usually have time as the independent variable and BVPs usually have space as the independent variable.

2.5 Examples of 2nd-Order DEs with non-constant coefficients

There is a long list of DEs with non-constant coefficients. Some of them are particularly famous and have special names. The solutions usually cannot be written in terms of simple functions and we define functions to be the solutions to such equations. They are usually referred to as **special functions**. Many of them arise in looking at solutions to Laplace's equations in different co-ordinate systems. The interested reader is directed to AM 353 for more details on how to obtain these equations.

1. Bessel's equation: p is a constant integer.

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

2. Legendre's equation: p integer.

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

3. Laguerre's equation: constant a

$$xy'' + (1 - x)y' + ay = 0$$

4. Hermite's equation: constant a

$$y'' - 2xy' + 2ay = 0$$

2.6 Reduction of Order

There does not exist a general approach to find a solution for any general 2nd-order ODE. However, if we manage to find one solution to the homogeneous problem, say $y_1(x)$, there is a useful technique that allows us to find a second solution, $y_2(x)$. This is called Reduction of Order. The idea is very simple really. Look for a solution that is a product of some unknown function (that we have to determine) multiplied by the known solution, i.e.

$$y_2(x) = v(x)y_1(x)$$

some intermediate work... Sub it into DE and get

$$v' = \frac{1}{y_1^2}\exp\left(-\int P(x)dx\right)$$

Then

$$v = \int_0^x \frac{1}{y_1^2}\exp\left(-\int P(s)ds\right)\,dx$$

Therefore $y_2 = y_1(x)\int \frac{1}{y_1^2}\exp\left(-\int P(s)ds\right)\,dx$.

Then we can show y_2, y_1 are linearly independent.

> Example:
>
> $y_1 = x^2$ is an exact solution to the homogeneous DE $x^2y'' + xy' - 4y = 0$. Then by the procedure above, $v = -\frac{1}{4x^4}$. Hence $y_2 = \frac{1}{x^2}$. Thus the general solution is $y(x) = c_1x^2 + \frac{c_2}{x^2}$.

2.7 Method of Variation of Parameters

To find the particular solution to a non-homogeneous equation we can always use the method of variation of parameters.

Suppose that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions to the homogeneous problem. Next, we look for a trial solution to the non-homogeneous that is similar to the general solution to the homogeneous problem except that the constant coefficients are replaced with unknown functions, $v_1(x)$, $v_2(x)$, to be determined:

$$y = v_1y_1 + v_2y_2$$

A convenient choice to pick v_1, v_2 is

$$\boxed{v_1'y_1 + v_2'y_2 = 0}$$

Sub them into $y'' + Py' + Qy = R$ and we get in matrix form:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

y_1, y_2 are linearly independent, as we have assumed, that the system is invertible. The solution is,

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{R}{W}\begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

Then solution is thus

$$y = y_1\int^x \frac{-y_2R(x)}{W(y_1,y_2)}dx + y_2\int^x \frac{y_1R(x)}{W(y_1,y_2)}dx$$

In summary, given two linearly independent solutions to the homogeneous problem we can find a particular solution to the non-homogeneous equation where the inhomogeneity is $R(x)$.

Example:

Find a particular solution of $y'' + y = \csc x$.

We know homogeneous solution $y = c_1 \sin x + c_2 \cos x$. Using the formula above, we have

$$y = \sin x \log(\sin x) - x \cos x$$

3

Series Solutions and Special Functions

This Chapter discusses how to construct power series solutions of second-order ODEs with possibly non-constant coefficients. First we review some basic their of power series that is taught in MATH 138, then we show how to construction series solutions to various types of ODEs: ordinary points and singular points. When solving these equations we obtain special functions that arise naturally as solutions to famous equations.

3.1 Review of Power Series

Transcendental Function

These are elementary functions that consist of algebraic functions such as trigonometric, exponential, logarithmic and any of their combinations by addition and multiplication (and their inverses).

Special Functions

These are functions that cannot be expressed in terms of transcendental functions.

Examples of special functions are Bessel functions, which we will study in detail later in this chapter.

Review of Power Series:

1. A power series in x about x_0 is defined to be: $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. Usually we will set $x_0 = 0$ but that is not necessary.
2. A series is said to be **converge** at x if limit of $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists and the sum of the series is the value of the limit.
3. Suppose a power series converges for $|x| < R$ for some $R > 0$ and let's denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

The function $f(x)$ is smooth and we can differentiate term by term, and also integrate it term by term.

4. Given that $f(x)$ is a smooth function for $|x| < R$ with $R > 0$m then we can construct the power series for this function using Taylor's formula:

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

If it is not entirely smooth, then we can use Taylor's formula for the remainder to get a polynomial expansion of the function.

5. A function $f(x)$ has a power series expansion of

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

in a neighborhood of x_0 is said to be **analytic** at x_0. We can recover the coefficient using Taylor's formula,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

to obtain Taylor series expansion of $f(x)$ about x_0.

3.2 Series Solutions of First-Order Equation

Consider $y' = y$. In AM 250 or AM 251 you learned how to solve this equation using various techniques, i.e. separable equations and integrating factors. Even though we already know how to solve it, let's try to solve it with a new technique, that of a power series solution. If we can solve a simple problem with this new method that should guide us along in solving more complicated problems.

Look for a solution of the following form,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

where $a_0, \ldots, a_n, \ldots$ are to be determined.

Differentiate

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and sub into the equation,

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \implies \sum_{n=0}^{\infty} [(n+1) a_{n+1} - a_n] x^n = 0$$

This equation is the power series representation of zero. But implies that each of the coefficients in the series is exactly zero and therefore,

$$a_{n+1} = \frac{a_n}{n+1} \quad \text{for } n = 0, 1, 2, \ldots$$

This is a recursion relation that defines a given coefficient in terms of the previous one:

$$a_n = \frac{a_0}{n!}$$

Therefore, after having done not so much work we have found our general solution to be,

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which is nothing more than a constant times the power series representation of e^x.

3.3 2nd-Order Linear Equations: Ordinary Points

Now that we've seen the basic method in action let's try and tackle the more interesting homogeneous, second-order DE with non-constant coefficients,

$$y'' + P(x)y' + Q(x)y = 0$$

Not too surprisingly, the behaviour of the solution near x_0 is completely determined by the behaviour of $P(x)$ and $Q(x)$ near the same point.

A real number x_0 is called an **ordinary point** if $P(x)$ and $Q(x)$ are analytic at x_0, i.e.,

$$P(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n, \quad |x - x_0| < r \qquad \text{and} \qquad Q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n, \quad |x - x_0| < r$$

for some $r > 0$.

Any point that is not ordinary is referred to as a **singular point**.

Example: Simple harmonic oscillator

To get started finding power series solutions we begin with the simple harmonic oscillator,

$$y'' + \alpha^2 y = 0$$

We begin with a trial solution that is a power series and compute the second derivative:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

We can now substitute our solution into our governing DE and get,

$$\sum_{n=0}^{\infty} \alpha^2 a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = 0$$

$$\sum_{n=0}^{\infty} [\alpha^2 a_n x^n + (n+2)(n+1)a_{n+2} x^n] = 0$$

For this to be zero we need that the coefficient of each term in the series is identically zero:

$$a_{n+2} = -\frac{\alpha^2 a_n}{(n+2)(n+1)}$$

One difference between this and the first-order example is that every coefficient is related with two coefficients before. Thus, to specify a unique solution we must specify a_0 and a_1. Using recursion we can determine all the coefficients in terms of the first two.

$$a_{2n} = (-1)^n \frac{\alpha^{2n} a_0}{(2n)!} \qquad a_{2n+1} = (-1)^n \frac{\alpha^{2n} a_1}{(2n+1)!}$$

The general pattern is readily observed. If we substitute this into the original power series and separate the solution into two linearly independent parts we get,

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n} a_0}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n} a_1}{(2n+1)!}$$

Observe that the first term above is a scalar multiple of $\cos(\alpha x)$ and the second term is a scalar multiple of $\sin(\alpha x)$.

Therefore, even though we used a completely different method, we obtained the same solution we did in AM 250 or AM 251. That shouldn't be surprising it should be expected! Even though there wasn't a need to use this method on this relatively simple equation it was useful and sets up the machinery in how to deal with more general equations with non-constant coefficients.

To illustrate the real power behind power series solutions let's tackle Legendre's equation,

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

Example: Legendre's equation

To get this into the standard form we divide through by $1-x^2$, we then get $P(x)$ and $Q(x)$. We conclude that both of these functions are analytic about $x = 0$ and thus it is an ordinary point. They are not analytic about $x = \pm 1$ and thus they are singular points.

Similarly, we have the recurrence relation:

$$a_{n+2} = -\frac{p(p+1)-n(n+1)}{(n+2)(n+1)}a_n = -\frac{(p-n)(p+n+1)}{(n+2)(n+1)}a_n$$

Then

$$a_{2n} = \frac{(-1)^n[\prod_{i=0}^{n-1}(p-2i)][\prod_{i=1}^{n}(p-1+2i)]}{(2n)!}a_0$$

$$a_{2n+1} = \frac{(-1)^n[\prod_{i=0}^{n-1}(p-1-2i)][\prod_{i=1}^{n}(p+2i)]}{(2n+1)!}a_1$$

We can substitute this into our solution and obtain the following power series solution to Legendre's equation:

$$y = \dots$$

The functions above in the last line following the constants a_0 and a_1 are defined to be Legendre functions of order p. We denote the even and odd Legendre polynomials as $L_p^0(x)$ and $L_p^1(x)$, respectively. For each p we have one such polynomial and another function which is a special function.

The following theorem is taken straight from the textbook by Simmons (Page 180) and is subsequently proven. Here we just state the theorem.

Theorem 3.1: Power Series Solutions at Ordinary Points

Let x_0 be an ordinary point of our standard homogeneous DE and let a_0 and a_1 be arbitrary constants. Then, there exists a unique function $y(x)$ that is analytic at x_0 that is a solution of the DE in a certain neighbourhood of this point and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on the interval $|x - x_0| < R$, with $R > 0$, then the power series expansion of this solution is also valid on the same interval.

3.4 2nd-Order Linear Equations: Singular Points

Let's consider the standard, homogeneous DE,

$$y'' + P(x)y' + Q(x)y = 0$$

We have previous said that x_0 is a singular point if either (or both) $P(x)$ and $Q(x)$ are not analytic at x_0, i.e., they are not continuous there. The following definition of a singular point that is not so bad and easier to work with.

regular/irregular singular

Suppose that x_0 is a singular point of above equation but that

$$(x-x_0)P(x) \quad \text{and} \quad (x-x_0)^2Q(x)$$

are analytic at x_0, i.e.,

$$(x-x_0)P(x) = \sum_{n=0}^{\infty} p_n(x-x_0)^n, \quad |x-x_0| < r \quad \text{and} \quad (x-x_0)Q(x) = \sum_{n=0}^{\infty} q_n(x-x_0)^n, \quad |x-x_0| < r$$

for some $r > 0$, then x_0 is said to be a **regular singular** point of the DE. Otherwise x_0 is an **irregular singular** point.

From this definition we gather that for x_0 to be a regular singular point we need the following:

1. $P(x)$ behaves no worse than $\frac{1}{x-x_0}$ at x_0 and
2. $Q(x)$ behaves no worse that $\frac{1}{(x-x_0)^2}$ at x_0

Example: Legendre's equation

Consider Legendre's equation in standard form,

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

We can show that $x = \pm 1$ are each regular singular points of the equation. For example consider $x = 1$ (the other point is analogous), both $(x-1)P(x)$ and $(x-1)^2Q(x)$ are analytic and thus $x_0 = 1$ is a regular singular point.

Example: Bessel's equation

is usually written as $x^2y'' + xy' + (x^2 - p^2)y = 0$. The standard form:

$$y'' + \frac{1}{x}y' + \frac{x^2-p^2}{x^2}y = 0$$

We look at the limits of the following products of the coefficients as x_0 approaches zero to show that it is a regular singular point:

$$xP(x) = 1$$
$$x^2Q(x) = x^2 - p^2$$

Clearly the RHSs are analytic around $x = 0$ and thus we are done.

To solve DEs about regular singular points consider the related Euler's equation,

$$x^2y'' + p_0xy' + q_0y = 0$$

The general solution is,

$$y = x^r$$

We substitute and obtain the **indicial equation** in terms of r,

$$r(r-1) + p_0r + q_0 = 0$$

This is a quadratic equation that has two solutions and there are three cases:

a) Distinct roots: r_1 and r_2,

$$y(x) = c_1x^{r_1} + c_2x^{r_2}$$

b) Equal roots: $r_1 = r_2$

$$y(x) = c_1x^{r_1} + c_2x^{r_2}\ln x$$

c) Complex roots: $x^r = x^{\alpha \pm i\beta}$

$$x^r = e^{\ln x^r} = e^{r \ln x} = e^{(\alpha \pm i\beta)\ln x} = e^{\alpha \ln x}[\cos(\beta \ln x) + i \sin(\beta \ln x)]$$

The above is for $x > 0$. A similar solution can be found for $x < 0$.

In general the DE for a regular singular point can be thought of as looking like,

$$y'' + \left(\frac{p_0 + p_1 x + p_2 x^2 + \cdots}{x}\right) y' + \left(\frac{q_0 + q_1 x + q_2 x^2 + \cdots}{x^2}\right) y = 0$$

This suggests looking for a more general power series solution of the form,

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad \text{with } a_0 \neq 0$$

This approach is known as the **Method of Frobenius** and the solution is called a **Frobenius Series**.

Example:

$$2x^2 y'' + x(2x+1)y' - y = 0$$

In standard form,

$$y'' + \frac{\frac{1}{2} + x}{x} y' + \frac{-\frac{1}{2}}{x^2} y = 0$$

We first test that $x = 0$ is a regular singular point,

$$xP(x) = -\frac{1}{2}, \quad x^2 Q(x) = \frac{1+x}{2}$$

and

$$p_0 = \lim_{x \to 0} xP(x) = -\frac{1}{2}, \qquad q_0 = \lim_{x \to 0} x^2 Q(x) = \frac{1}{2}$$

The corresponding indicial notation for the associated Euler Equation is thus,

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0 \implies r = 1/2 \text{ or } 1$$

We try our Frobenius series solution:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

and $xy = \sum ..., xy' = \sum ..., 2x^2 y'' = \sum ...$

Sub them into equation and cancel the x^r,

$$a_0[2(r-1)r - r + 1] + \sum_{n=1}^{\infty} \{[2(n+r-1)(n+r) - (n+r-1)]a_n x^n + a_{n-1} x^n\} = 0$$

For $a_0 \neq 0$ we need either $r = 1/2$ or $r = 1$ as found above. For the other powers of x^n we require,

$$a_n = -\frac{1}{(n+r-1)(2(n+r)-1)} a_{n-1}$$

Two cases:

a) For $r = 1$, we have

$$a_n = (-1)^n \frac{1}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!} a_0$$

then $y_1 = x \sum_{n=0}^{\infty} a_n x^n$

b) For $r = \frac{1}{2}$

$$a_n = (-1)^n \frac{1}{n!(1 \cdot 3 \cdot 5 \cdots (2n-1))} a_0$$

then $y_2 = x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n$

Therefore, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

You can check using the ratio test that this is convergent for all x.

3.5 Extended Method of Frobenius

For the general homogeneous DE,

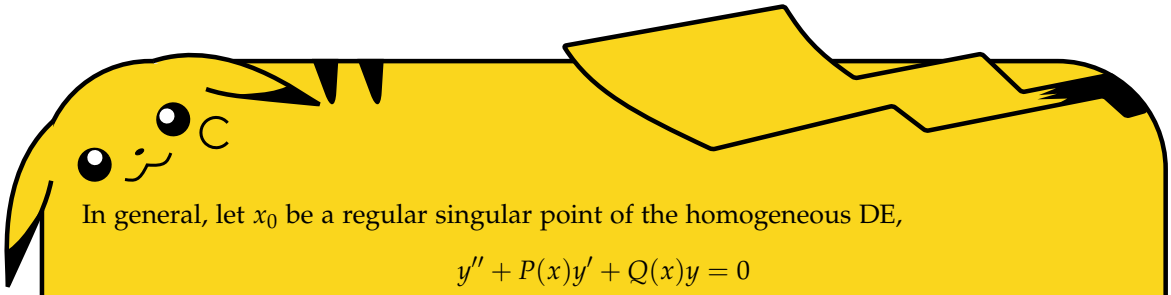
$$x^2 y'' + p(x)xy' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are analytic functions at $x = 0$, the associated Euler equation is

$$x^2 y'' + p(0)xy' + q(0)y = 0$$

The indicial equation obtained after substituting in $y = x^r$ is,

$$r(r-1) + p(0)r + q(0) = 0$$



In general, let x_0 be a regular singular point of the homogeneous DE,

$$y'' + P(x)y' + Q(x)y = 0$$

i.e.,

$$(x - x_0)P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \quad |x - x_0| < r$$

and

$$(x - x_0)^2 Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n, \quad |x - x_0| < r$$

The indicial equation is

$$r(r-1) + p_0 r + q_0 = 0$$

where

$$p_0 = \lim_{x \to x_0} (x - x_0)P(x) \quad q_0 = \lim_{x \to x_0} (x - x_0)^2 Q(x)$$

Assume that the above indicial equation has two real roots, r_1, r_2 with $r_1 \geq r_2$. Then the following is true around the regular singular point x_0:

(a) There is always one Frobenius series solution of the form,

$$y_1 = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{with } a_0 \neq 0,\ 0 < x - x_0 < r$$

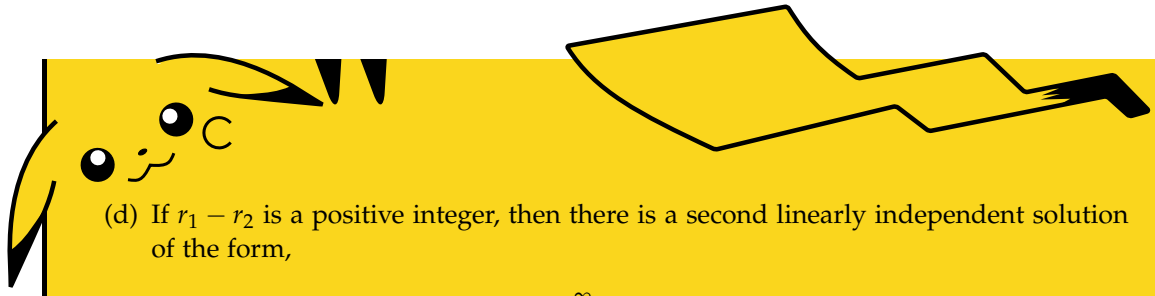
(b) If $r_1 - r_2$ is not an integer, there is a second linearly independent Frobenius series solution of the form,

$$y_2 = (x - x_0)^{r_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n, \quad \text{with } b_0 \neq 0,\ 0 < x - x_0 < r$$

(c) If $r_1 = r_2$, then there is a second linearly independent solution of the form,

$$y_2 = y_1 \log(x - x_0) + (x - x_0)^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad \text{with } c_0 \neq 0,\ 0 < x - x_0 < r$$

This can be obtained from the reduction of order.



(d) If $r_1 - r_2$ is a positive integer, then there is a second linearly independent solution of the form,

$$y_2 = \alpha y_1 \log(x - x_0) + (x - x_0)^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n, \quad \text{with } d_0 \neq 0,\ 0 < x - x_0 < r$$

where α is a constant that may be zero.

3.6 Bessel Functions

Bessel's Equation is

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

The point $x = 0$ is a regular singular point since

$$p_0 = \lim_{x \to 0} xP(x) = \lim_{x \to 0} x\frac{1}{x} = 1$$
$$q_0 = \lim_{x \to 0} x^2 Q(x) = \lim_{x \to 0} x^2 \frac{x^2 - p^2}{x^2} = -p^2$$

we get that the indicial equation is

$$r(r-1) + r - p^2 = 0 \implies r = \pm p$$

To solve this problem we use the Frobenius method and try, $y = \sum_{n=0}^{\infty} a_n x^{n+r}, x^2 y = \sum \ldots, xy' = , x^2 y'' =$

We sub them into Bessel's equation and obtain,

$$\sum_{n=2}^{\infty} ([n(n+2r)]a_n + a_{n-2})x^n + (2r+1)a_1 x = 0$$

where we use the fact that $r^2 = p^2$ and also we divided through by x^r. For both indicial roots, this equation yields,

$$a_1 = 0$$
$$a_n = -\frac{a_{n-2}}{n(n+2r)} \quad \text{for } n = 2, 3, \ldots$$

This implies the odd polynomials all vanish. We can then use the recurrence relation to find the coefficients of the even polynomials.

The general solution for $k \geq 0$ is

$$a_{2k} = \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1+r)(2+r)\cdots(k+r)} a_0$$

Therefore one solution is

$$y = a_0 x^p \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1+p)(2+p)\cdots(k+p)} x^{2k}\right]$$

Now that we have one solution there are difference cases that must be considered to find the second.

(a) If $r_1 - r_2 = 2p$ is not an integer, then use the above analysis and write down the two linearly independent solutions:

$$y_1 = a_0 x^p \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1+p)(2+p)\cdots(k+p)} x^{2k} \right]$$
$$y_2 = a_0 x^{-p} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1-p)(2-p)\cdots(k-p)} x^{2k} \right]$$

(b) If $r_1 = r_2$ then use reduction of order.

(c) If $r_1 - r_2 = 2p$ is a positive integer hen one solution is known, and the same as the above, and the second can be obtained from the previous formula. To see why this is a problem reconsider the recurrence relation before we solved it,

$$n(n+2r)a_n = -a_{n-2} \quad \text{for } n = 2, 3, \ldots$$

For concreteness, say $2p = N$. From the first root to the indicial equation we have,

$$n(n+2p)a_n = -a_{n-2}$$

where we notices that the coefficient on the LHS is nonzero. For the second indicial root we have something very similar,

$$n(n-2p)a_n = -a_{n-2}$$

The problem arises where $n = N$ since in this case the left hand side vanishes and we cannot determine a_n which puts us at a stand still. That is why in this case we go to the reduction of order to find a second linearly independent solution.

3.7 Bessel functions of the first kind of integer order

By convention, to define Bessel function of the first kind of order p, written $J_p(x)$, we use solution y_1 for the particular case where,

$$a_0 = \frac{1}{2^p p!}$$

If we substitute this into our solution we get

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot (k+p)!} \left(\frac{x}{2}\right)^{2k}$$

Looking at the first three terms of the first Bessel functions of the first kind of integer zero and one yields,

$$J_0(x) = \ldots, \quad J_1(x) = \ldots$$

Given this definition we can write down our general solution to Bessel's equation or order p as a linear superposition of Bessel functions of the first kind of order p if this parameter is not an integer,

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x)$$

This is not the convention however. The convention is instead to define the Bessel function or order p of the second kind to be,

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}$$

If p is an integer m then we must take the limit,

$$Y_m(x) = \lim_{p \to m} Y_p(x)$$

and one can show that this limit exists and yields a linearly independent solution.

Therefore, the general solution to Bessel's equation or order p, for any p, is,

$$y(x) = c_1 J_p(x) + c_2 Y_p(x)$$

3.8 Bessel functions of the first kind of arbitrary order

To extend our solution to the case with p non-integer is pretty easy. The only complication with our previous solution is that it involves the factorial function. This function is perfectly well defined for integers but we need to extend this to non-integer values as well. This is done with the Gamma function. It is defined to be,

Gamma function

The **Gamma function** is the generalization of the factorial to the real line and is defined to be,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

It has the following properties:

$$\Gamma(x+1) = x\Gamma(x) \qquad \forall x > 0$$
$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \ldots$$

If we pick $a_0 = \dfrac{1}{2^p \Gamma(p+1)}$, then the Bessel function of the first kind of the p-th order (where p is non-integer) is,

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot \Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k}$$

This works for both p positive and negative. In the case where p is an integer we can still find an expression for $Y_p(x)$ but it is rather lengthy.

3.9 Asymptotic behaviour of the Bessel function of the first kind

From the above expression for $J_p(x)$ we can get an asymptotic expression for large and small values of x. In particular we find that,

$$J_p(x) \approx \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p, \quad \text{as } x \to 0^+$$

It can be shown, but it is not obvious at all, that we also have,

$$J_0(x) \approx \frac{1}{\sqrt{x}}(c_1 \sin x + c_2 \cos x), \quad \text{as } x \to \infty$$

3.10 Bessel functions where p is not an integer

The first solution is given by the above and the second solution is given by,

$$J_{-p}(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot \Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k}$$

Other solutions exist but they are more complicated and will not be dealt with in lecture.

3.11 Gauss's hypergeometric equation

We study, in this section, the famous Gauss hypergemoetric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

where a, b, c are constants.

In its standard form, we see

$$P(x) = \frac{c-(a+b+1)x}{x(1-x)} \quad \text{and} \quad Q(x) = -\frac{ab}{x(1-x)}$$

and $x = 0$ and $x = 1$ are only regular singular points.

Let's first consider its solution near $x = 0$. Since

$$xP(x) = \frac{c-(a+b+1)x}{(1-x)} = [c-(a+b+1)x](1+x+x^2+\ldots) = c+(c-a-b-1)x+\ldots, \quad |x|<1$$

and

$$x^2Q(x) = -\frac{abx}{1-x} = -abx(1+x+x^2+\ldots) = -abx-abx^2-\ldots, \quad |x|<1$$

we see that $p_0 = c$ and $q_0 = 0$, and the indicial equation is

$$m(m-1)+mc = 0$$

which has two roots: $m_1 = 0$ and $m_2 = 1-c$. If $1-c$ is not a positive integer, i.e., if c is not zero or a negative integer, then DE has a solution of the form

$$y = x^0\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n$$

Sub this into the DE,

$$x(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + [c-(a+b+1)x]\sum_{n=1}^{\infty} na_n x^{n-1} - ab\sum_{n=0}^{\infty} a_n x^n = 0$$

which implies

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^n + c\sum_{n=1}^{\infty} na_n x^{n-1} - (a+b+1)\sum_{n=1}^{\infty} na_n x^n - ab\sum_{n=0}^{\infty} a_n x^n = 0$$

Equating the coefficient of x^n gives

$$[n(n+1)+(n+1)]a_{n+1} - [n(n-1)+(a+b+1)n+ab]a_n = 0$$

i.e.,

$$a_{n+1} = \frac{(n+a)(n+b)}{(n+1)(n+c)}a_n$$

Set $a_0 = 1$ and calculate the other a_n in succession, we found

$$a_n = \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}$$

Thus

$$y = 1+\sum_{n=1}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}x^n, \quad |x|<1$$

This is known as **hypergeometric series** and denoted by $F(a,b,c,x)$.

> Remark:
> $F(1,b,b,x) = 1+x+x^2+\ldots = \frac{1}{1-x}, |x|<1$, which is the familiar geometric series.
>
> If a or b is a nonpositive integer, $F(a,b,c,x)$ becomes a polynomial.

If $1-c \neq 0$ or a negative integer, then the DE has a second Frobenius series solution of the form

$$y = x^{1-c}\sum_{n=0}^{\infty} b_n x^n, \quad b_n \neq 0$$

One could compute b_n by sub it directly into DE. But an easier method is the following change of variable.

$$y = x^{1-c}z$$

DE then reduces to

$$x(1-x)z'' + [(2-c) - ([a-c+1] + [b-c+1] + 1)x]z' - (a-c+1)(b-c+1)z = 0$$

which is a hypergeometric equation with a, b, c replaced by $a-c+1, b-c+1$ and $2-c$. Thus

$$z = F(a-c+1, b-c+1, 2-c, x)$$

and hence by the change or variable, the second solution of DE is

$$y = x^{1-c}F(a-c+1, b-c+1, 2-c, x) \qquad 0 < x < 1$$

if $2-c$ is not zero or a negative integer. Thus if c is not an integer, the general solution of DE near $x = 0$ is

$$y = c_1F(a, b, c, x) + c_2x^{1-c}F(a-c+1, b-c+1, 2-c, x) \qquad (*)$$

For the solution of DE near the singular point $x = 1$, we move $x = 1$ to the origin by letting $t = 1 - x$. Then DE becomes

$$t(1-t)y'' + [(a+b-c+1) - (a+b+1)t]\,y' - aby = 0$$

which is again a hypergemoetric equation.

In a view of $(*)$, we see that if $a + b - c$ is not an integer, the general solution of DE near $x = 1$ is

$$y = c_1F(a, b, a+b-c+1, 1-x) + c_2(1-x)^{c-a+b}F(c-b, c-a, c-a-b+1, 1-x)$$

4

Oscillation theory and BVPs

4.1 Qualitative Analysis of ODEs

In this chapter, we will make some qualitative analysis of the second-order linear ODEs with variable coefficients. Let us start by considering the following simple homogeneous ODE with constant coefficients,

$$y'' + y = 0$$

We know that two linearly independent solutions to this equation are $\sin(x)$ and $\cos(x)$. Both of these solutions oscillate out of phase. Let's try and forget what we already know and instead see how we can derive this by simply looking at the DE itself.

1. Define $s(x)$ and $c(x)$ to be two linearly independent solutions to this equation $s(x)$ and $c(x)$ to be two linearly independent solutions with initial conditions $s(0) = 0, s'(0) = 1$ and $c(0) = 1, c'(0) = 0$.

2. The equation tells us the solutions are concave down or up depending on whether the solution is above or below the x-axis.

3. Initially: $s(x)$ starts off at the origin and is increasing but concave downwards. Soon afterwards it will start to decrease and eventually pass the x-axis at which point it will change its concavity and start to turn around. The $c(x)$ curve follows a very similar trajectory but starts at the maximal position. From this we see that both solutions will tend to oscillate about $x = 0$.

4. Observe that if $y(x)$ satisfies the DE than so does $y'(x)$. The initial conditions $s'(0) = 1, s''(0) = 0$ and $c'(0) = 0, c''(0) = -1$ implies that

$$s'(x) = c(x) \qquad \text{and} \qquad c'(x) = -s(x)$$

5. We saw a while ago that the Wronskian is equal to,

$$W = W_0 \exp\left(-\int P(x)dx\right)$$

   In our special equation $P(x) = 0$ and thus in this case the Wronskian is constant everywhere. From the initial conditions we learn that $W_0(s, c) = -1$ and thus we deduce that

$$W(s, c) = s(x)c'(x) - s'(x)c(x) = -1$$

$$s^2(x) + c^2(x) = 1$$

If you recall from Calculus I one way to define the exponential function, e^x is as the unique solution to the DE $y' = y$ with $y(0) = 1$. Similarly, we could define the sine and cosine functions to be the unique

solutions to the above DE with the prescribed initial conditions. This isn't typically done but if we do so we can derive many properties of these equations even without solving the DE.

The basic ideas that we used to analyze the simple harmonic oscillator can be used to study more general DEs. In particular, we can learn information about the zeros of the solutions of

$$y'' + P(x)y' + Q(x)y = 0 \tag{4.1}$$

Theorem 4.1: Sturm separation theorem

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation Eq. (4.1), then the zeros of these solutions are distinct and occur alternatively. By that we mean that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$ and the converse is true.

Proof:
Assume y_1, y_2 are two linearly independent solutions. Lemma 2.3, Lemma 2.4 have shown that the Wronskian of these two functions is necessarily non-zero and thus has a constant sign,

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

1. The above equation shows that the two solutions cannot be zero at the same location for otherwise the Wronskian would be zero, which is a contradiction.
2. Say that x_1 and x_2 are successive zeros of y_2. We need to show that y_1 necessarily vanishes between these points. From above,

$$W(y_1(x_1), y_2(x_1)) = y_1(x_1)y_2'(x_1)$$
$$W(y_1(x_2), y_2(x_2)) = y_1(x_2)y_2'(x_2)$$

   which shows that y_1' and y_2' are both non-zero at x_1, x_2. Moreover, $y_2'(x_1)$ and $y_2'(x_2)$ are opposite in signs, i.e,m $y_2'(x_1)y_2'(x_2) < 0$, since if y_2 is increasing (decreasing) at x_1 it must be decreasing (increasing) at x_2. This implies $y_1(x_1)$ and $y_1(x_2)$ have opposite signs and therefore, by continuity, y_1 must have a zero somewhere in between x_1 and x_2.
3. From this argument that y_1 cannot vanish more than once between x_1 and x_2 but if it did we could deduce that y_2 has a zero between the two zeros, which is a contradiction to what we have supposed.

□

One reason why the simple harmonic oscillator is so easy to understand is because y'' is a scalar multiple of y. This doesn't happen in general because of the y' term. It would be nice to make the y' term disappear so that we can use similar tricks.

Normal form (of the homogeneous equation): If we define $y(x) = u(x)v(x)$ we can easily compute its first two derivatives,

$$y' = vu' + v'u$$
$$y'' = vu'' + 2v'u' + uv''$$

If we then substitute this into the homogeneous equation, Eq. (4.1), we get

$$\left[vu'' + 2v'u' + v''u\right] + P(x)\left[vu' + v'u\right] + Q(x)vu = 0$$
$$vu'' + \left[2v' + P(x)v\right]u' + \left[v'' + P(x)v' + Q(x)v\right]u = 0$$

This equation is much more complicated but appreciate that there is a unique choice of $v(x)$ such that we eliminate the u' term in the above equation. This is done by choosing,

$$2v' + P(x)v = 0 \implies v' = -\frac{1}{2}P(x)v \implies v = \exp\left(-\frac{1}{2}\int P(x)dx\right)$$

If we make this choice than the resulting equation is said to be in **normal form** after we divide through by $v(x)$, as defined above,

$$u'' + q(x)u = 0 \tag{4.2}$$

where we have defined

$$q(x) = Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$$

The original function $y(x)$ is recovered from the following equation,

$$y(x) = \exp\left(-\frac{1}{2}\int P(x)dx\right)u(x)$$

It's clear

$$y(x) = 0 \iff u(x) = 0$$

Thus the oscillatory properties of Eq. (4.1) remain unchanged after the transformation.

The following theorem uses the normal form to determine that some equations have at most one zero.

Theorem 4.2

If $q(x) < 0$ and if $u(x)$ is a nontrivial solution of the normal form Eq. (4.2), then $u(x)$ has at most one zero.

Proof:
Suppose that x_0 is a zero of $u(x)$, which is a non-trivial function. By the property above, we require that $u'(x_0) \neq 0$. Without loss of generality, assume that $u'(x_0) > 0$ By continuity, $u'(x)$ must be positive in a neighborhood of x_0.

Since we have assumed that $q(x) < 0$, we find using the equation that,

$$u''(x) = -q(x)u(x)$$

is positive on the interval to the right of x_0. Therefore, $u'(x)$ is an increasing function on this interval and consequently, there cannot be any zeros to the right of x_0. The same argument holds to the left of x_0 and also in the case where $u'(x_0) < 0$. □

Therefore, if we want to look at oscillating solutions we should consider $q(x) > 0$. This not surprising if we recall what we know about DEs with constant coefficients.

Theorem 4.3

Let $u(x)$ be any nontrivial solution of Eq. (4.2), where $q(x) > 0$ for all $x > 0$. If

$$\int_0^\infty q(x)dx = \infty,$$

then $u(x)$ has s infinitely many zeros on the positive x-axis.

Proof:
See page 30, theorem 1.11.3 in the course notes. □

Theorem 4.4

Let $u(x) \not\equiv 0$ be a solution of Eq. (4.2). Then $u(x)$ has at most a finite number of zeros on a closed interval $[a, b]$.

Proof:
If not, $\exists\{x_n\}_{n=0}^{\infty}$, $x_n \in [a,b]$, such that $u(x_n) = 0, \forall n$. Since $[a,b]$ is closed, $\exists x_0 \in [a,b]$ and $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x_0$ as $k \to \infty$. Then by continuity,

$$u(x_0) = \lim_{k\to\infty} u(x_{n_k}) = 0$$
$$u'(x_0) = \lim_{k\to\infty} \frac{u(x_{n_k}) - u(x_0)}{x_{n_k} - x_0}$$

By uniqueness, we have $u(x) \equiv 0$ which is a contradiction. Thus the proof is complete. □

Remark:
$u(x)$ can have an infinite number of zeros if $u(x)$ is not a solution of Eq. (4.2).

For example:

$$u(x) = \begin{cases} x\sin\frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

$u(x)$ is continuous on $[0,1]$ and has an infinite number of zeros in $[0,1]$.

Example:
Consider the second-order linear ODE

$$y'' - y' + \frac{1}{4}\sin^2 xy = 0 \tag{4.3}$$

Let $u(x) = y(x)e^{\frac{1}{2}\int(-1)dx} = y(x)e^{-\frac{1}{2}x}$. Then DE reduces to

$$u'' + \frac{1}{4}(\sin^2 x - 1)u = 0 \tag{4.4}$$

Since $\frac{1}{4}(\sin^2 x - 1) \le 0$, it follows from Theorem 4.2 that all nonzero solutions of Eq. (4.4), thus of Eq. (4.3) are non-oscillating.

Example:
Consider the second-order linear ODE

$$y'' + \sin xy' + y = 0 \tag{4.5}$$

Let $u(x) = y(x) = e^{-\frac{1}{2}\cos x}$. Then Eq. (4.5) reduces to

$$u'' + \left(1 - \frac{1}{4}\sin^2 x - \frac{1}{2}\cos x\right)u = 0 \tag{4.6}$$

Since $1 - \frac{1}{4}\sin^2 x - \frac{1}{2}\cos x > 0$ and

$$\int_0^x \left(1 - \frac{1}{4}\sin^2 t - \frac{1}{2}\cos t\right)dt \ge \int_0^x \frac{1}{4}dt \longrightarrow \infty \text{ as } x \to \infty$$

It follows $\int_0^\infty \left(1 - \frac{1}{4}\sin^2 t - \frac{1}{2}\cos t\right)dt = \infty$. Thus by Theorem 4.3, all nonzero solutions of Eq. (4.6), thus of Eq. (4.5) are oscillatory.

4.2 The Sturm comparison theorem

The Sturm separation theorem tells us that all solutions of the second-order linear ODE

$$u'' + q(x)u = 0$$

oscillate with essentially the same rapidity. However, we see that solutions of this equation

$$y'' + 4y = 0$$

oscillate more rapidly than those of

$$y'' + y = 0$$

This is in fact a typical behaviour as shown in the following theorem.

Theorem 4.5: Sturm comparison theorem

Consider the second-order linear ODEs

$$y'' + q(x)y = 0 \tag{4.7}$$

$$z'' + r(x)z = 0 \tag{4.8}$$

where $q(x)$ and $r(x)$ are continuous on $(-\infty, \infty)$ and

$$q(x) > r(x) > 0, \quad \forall x \in (-\infty, \infty) \tag{4.9}$$

Then any solution $y(x) \not\equiv 0$ of Eq. (4.7) vanishes at least once between any two successive zeros of a nonzero solution $z(x)$ of Eq. (4.8), i.e., oscillation of solution of Eq. (4.8) implies that of Eq. (4.7).

Proof:

Let $x_1 < x_2$ be such that

$$z(x_1) = z(x_2) = 0, \quad z(x) \neq 0, \quad \forall x \in (x_1, x_2)$$

Suppose, for the sake of contradiction, that

$$y(x) \neq 0, \quad \forall x \in (x_1, x_2)$$

We may assume $z(x) > 0$ and $y(x) > 0, \forall x \in (x_1, x_2)$. Define $m(x) = y(x)z'(x) - z(x)y'(x)$. Then by Eq. (4.9)

$$m'(x) = y(x)z''(x) - z(x)y''(x) = (q(x) - r(x))y(x)z(x) > 0, \quad \forall x \in (x_1, x_2)$$

Thus $m(x_2) > m(x_1) = y(x_1)z'(x_1) > 0$ in view of the fact that $z'(x_1) > 0$ and $z'(x_2) < 0$.

But $m(x_2) = y(x_2)z'(x_2) < 0$, which is a contradiction. Thus we must have $x^* \in (x_1, x_2)$ such that $y(x^*) = 0$. □

Example:

Show that all solutions of the equation

$$u'' + (1 + \sin^2 x)u = 0$$

are oscillatory and the distance between any two successive zeros of a non zero solution of it is bounded by $\frac{\pi}{\sqrt{2}}$ and π.

Solution Let $\epsilon > 0$ be sufficiently small. Then

$$1 - \epsilon < 1 + \sin^2 x < 2 + \epsilon, \quad \forall x \in R$$

Thus by Sturm comparison theorem, the solutions of it oscillate more rapidly than those of

$$y'' + (1 - \epsilon)y = 0 \tag{4.10}$$

and less rapidly than those of

$$z'' + (2 + \epsilon)z = 0 \tag{4.11}$$

Since the distances of any two successive zeros of solutions of Eq. (4.10) and Eq. (4.11) are, respectively $\frac{\pi}{\sqrt{1-\epsilon}}$ and $\frac{\pi}{\sqrt{2+\epsilon}}$ and ϵ is arbitrary, it follows that e distance between any two successive zeros of a nonzero solution of the DE is bounded by $\frac{\pi}{\sqrt{2}}$ and π.

4.3 The vibrating string problem and BVPs

The vibrating string problem. A flexible string is pulled taut on x-axis and fastened at two points $x = 0, x = \pi$. Then it is drawn aside into a curve $y = f(x)$ in xy-plane and released. Let $y(x,t)$ be the motion of the string. Then it can be shown (the details are omitted here), for some positive constant α, that $y(x,t)$ satisfies the partial differential equation

$$\alpha \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{4.12}$$

subject to the initial condition

$$\frac{\partial y}{\partial t} = 0 \quad \text{and} \quad y(x,0) = f(x) \tag{4.13}$$

and the boundary condition

$$y(0,t) = y(\pi,t) = 0 \tag{4.14}$$

Eq. (4.12) - Eq. (4.14) is called the initial-boundary value problem of one-dimensional wave equation (IBVP). To solve the IBVP, we assume $y(x,t) = u(x)v(t)$. Then it can be shown (the details are omitted here), that $u(x)$ satisfies, for some constant λ, the equation

$$u'' + \lambda u = 0 \tag{4.15}$$

subject to the boundary condition

$$u(0) = u(\pi) = 0 \tag{4.16}$$

Eq. (4.15) - Eq. (4.16) are e called boundary value problems which are the prototype of a large class of important BVPs in applied mathematics.

4.4 Eigenvalues and eigenfunctions

In the vibrating string model discussed earlier, we need to solve the BVP

$$u'' + \lambda u = 0 \tag{4.17}$$

$$u(0) = 0, \quad u(\pi) = 0 \tag{4.18}$$

where λ is a parameter to be determined in such a way that Eq. (4.17) - Eq. (4.18) has a nontrivial solution.

If $\lambda \leq 0$, then the BVP Eq. (4.17) - Eq. (4.18) has only trivial solutions since any nonzero solution of Eq. (4.17) has at most one zero. Since we are interested in the nonzero solutions of the BVP Eq. (4.17) - Eq. (4.18), we are restricted to the case $\lambda > 0$, and the general solution of Eq. (4.17) is

$$u(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$$

Condition $u(0) = 0$ yields $c_2 = 0$, thus the solution reduces to

$$u(x) = c_1 \sin \sqrt{\lambda} x$$

This together with the condition $u(\pi) = 0$, implies

$$\lambda = n^2, \quad n = 1, 2, \ldots$$

The values of λ are called **eigenvalues** of the BVP Eq. (4.17) - Eq. (4.18) and the corresponding solutions

$$\sin nx, \quad n = 1, 2, \ldots$$

are called **eigenfunctions**.

We observe the following:

1. Corresponding to each eigenvalue, there are infinite many solutions to BVP Eq. (4.17) - Eq. (4.18)the which forms a 1-dimensional vector space.
2. All eigenvalues form an increasing sequence of positive numbers that approaches ∞.
3. The eigenfunction corresponding to the n-th eigenvalue vanishes at the endpoints of $[0, \pi]$ and has exactly $n-1$ zeros inside this interval.
4. All eigenfunctions are orthogonal on $[0, \pi]$ in the sense

$$\int_0^\pi \sin nx \sin mx \, dx = 0, \quad \text{if } n \neq m \tag{4.19}$$

The formula Eq. (4.19) can be proved by elementary integration, but the following approach given by Euler in 1777 may apply to more general situation.

Proof (Euler 1777):

Let $u_n = \sin nx, u_m = \sin mx$. Then

$$u_n'' + n^2 u_n = 0 \qquad u_m'' + m^2 u_m = 0$$

Combining these two,

$$u_m u_n'' - u_n u_m'' = (m^2 - n^2) u_n u_m$$
$$(u_m u_n' - u_n u_m')' = (m^2 - n^2) u_n u_m$$

Integrating the above equation from 0 to π, we obtain

$$(m^2 - n^2) \int_0^\pi u_n(x) u_m(x) dx = 0$$

which gives Eq. (4.19) since $m \neq n$. □

We shall show later that the above properties can be extended to general Sturm-Liouville BVPs.

4.5 Regular Sturm-Liouville problems

Consider the Sturm-Lioville BVPs

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \lambda q(x) y = 0, \quad y(a) = y(b) = 0$$

where $p(x)$ and $q(x)$ are positive continuous functions on $[a, b]$ and $p(x)$ is continuously differentiable in (a, b).

Theorem 4.6

Then there exists λ_n where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \quad \text{as } n \to \infty$$

such that BVP above has a nontrivial solution if and only if $\lambda = \lambda_n, n = 1, 2, \ldots$. The solution y_{λ_n} is unique except for an arbitrary constant factor, and y_{λ_n} has exactly $n-1$ zeros in the open interval (a, b).

Remark:

λ_n, y_{λ_n} for $n = 1, 2, 3, \ldots$, are the eigenvalues and eigenfunctions of BVP above, respectively.

4.6 General Sturm-Lioville problems

See the lecture notes below (cr. professor).

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \text{ as } n \to \infty$$

such that BVP (4.5.11) has a nontrivial solution if and only if $\lambda = \lambda_n, n = 1, 2, \cdots$. The solution y_{λ_n} is unique except for an arbitrary constant factor, and y_{λ_n} has exactly $n-1$ zeros in the open interval (a, b).

Remark. $\lambda_n, y_{\lambda_n}, n = 1, 2, 3 \cdots$, are the eigenvalues and eigenfunctions of BVP (**??**), respectively.

4.6 General Sturm-Lioville problems

Let us return briefly to the general Sturm-Lioville BVPs of the form

$$\frac{d}{dx}\Big[p(x)\frac{dy}{dx}\Big] + \Big[\lambda q(x) + r(x)\Big]y = 0 \tag{4.6.1}$$

$$c_1 y(a) + c_2 y'(a) = 0, d_1 y(b) + d_2 y'(b) = 0 \tag{4.6.2}$$

where $p(x), q(x)$ and $r(x)$ are continuous on $[a, b]$, $p(x) > 0, q(x) > 0, \forall x \in [a, b], c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. Note that a special case of BVP (4.6.1) - (4.6.2) was discussed in Section 4.5. In the general case, the following result can be proved.

Theorem 4.6.1. There exist real numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \text{ as } n \to \infty \tag{4.6.3}$$

such that BVP (4.6.1) - (4.6.2) has a nonzero solution iff $\lambda = \lambda_n$, $n = 1, 2, \cdots$, and eigenfunctions

$$y_1(x), y_2(x), \cdots, y_n(x), \cdots \tag{4.6.4}$$

are orthogonal on $[a, b]$ with respect to the weight function $q(x)$, i.e.

$$\int_a^b q(x) y_m(x) y_n(x)\, dx = \begin{cases} 0 \text{ if } m \neq n, \\ \alpha_n \neq 0 \text{ if } m = n. \end{cases} \tag{4.6.5}$$

Proof. We shall only prove (4.6.5).
Let

$$m(x) = y_m(x)y_n'(x) - y_n(x)y_m'(x) = det \begin{bmatrix} y_m(x) & y_m'(x) \\ y_n(x) & y_n'(x) \end{bmatrix}.$$

Then from (4.6.2) we have

$$\left\{ \begin{array}{l} c_1 y_m(a) + c_2 y_m'(a) = 0, \\ c_2 y_n(a) + c_2 y_n'(a) = 0, \end{array} \right\} \tag{4.6.6}$$

and

$$\left\{ \begin{array}{l} d_1 y_m(b) + d_2 y_m'(b) = 0, \\ d_a y_n(b) + d_2 y_n'(b) = 0. \end{array} \right\} \tag{4.6.7}$$

Since $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$, it follows from (4.6.6) and (4.6.7) that

$$m(a) = 0 \text{ and } m(b) = 0. \tag{4.6.8}$$

Now consider

$$\begin{array}{r} y_n\left\{\frac{d}{dx}\Big[p\frac{dy_m}{dx}\Big] + \Big[\lambda_m q + r\Big]y_m\right\} = 0 \\ -\ y_m\left\{\frac{d}{dx}\Big[p\frac{dy_n}{dx}\Big] + \Big[\lambda_n q + r\Big]y_n\right\} = 0 \\ \hline y_n\frac{d}{dx}\Big[p\frac{dy_m}{dx}\Big] - y_m\frac{d}{dx}\Big[p\frac{dy_n}{dx}\Big] + (\lambda_m - \lambda_n)qy_m y_n = 0 \end{array}$$

which implies

$$d(\lambda_m - \lambda_n)qy_m y_n = y_m(py_n')' - y_n(py_m')'.$$

Integrating the above equation from a to b and using integration by parts, we obtain

$$\begin{aligned}(\lambda_m - \lambda_n)\int_a^b qy_m y_n\,dx &= \int_a^b y_m(py_n')'\,dx - \int_a^b y_n(py_m')'\,dx \\ &= \left[y_m(py_n')\right]_a^b - \int_a^b y_m'(py_{-n})\,dx - \left[y_n(py_m')\right]_a^b + \int y_n'(py_m')\,dx \\ &= y_m(b)p(b)y_n'(b) - y_m(a)p(a)y_n'(a) - y_n(b)p(b)y_m'(b) + y_n(a)p(a)y_m'(a)\end{aligned}$$

$$\begin{aligned}&= p(b)[y_m(b) - y_n(b)y_m'(b)] - p(a)[y_m(b)y_n'(a) - y_m(a)y_m'(a)] \\ &= p(b)m(b) - p(a)m(a).\end{aligned}$$

Thus from (4.6.8)

$$(\lambda_m - \lambda_n \int_a^b qy_m y_n\,dx = 0$$

which implies (4.6.5).

The significance of property (4.6.5) of the eigenfunctions is that we can obtain expansions of the functions $f(x)$ in terms of the eigenfunctions given by (4.6.4). if we assume

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x), \tag{4.6.9}$$

then multiplying both sides of (4.6.9) by $q(x)y_m(x)$ and integrating term by term from a to b yields

$$\int_a^b f(x)q(x)y_m(x)\,dx = a_m \int_a^b q(x)y_m^2(x)\,dx \tag{4.6.1}$$

which implies

$$a_m = \frac{1}{\alpha_m}\int_a^b f(x)q(x)y_m(x)dx, \quad m = 1, 2, \cdots \tag{4.6.10}$$

Formula (4.6.9) with a_m given in (4.6.10) is called an eigenfunction expansion of $f(x)$.
Remarks:

Remark:

1. We didn't address the convergence of the series (4.6.9) whose study is beyond the scope of this course.
2. We call the BVP (4.6.1)-(4.6.2) a regular Sturm-Liouville problem because the interval $[a, b]$ is finite and the functions $p(x)$ and $q(x)$ are positive and continuous on $[a, b]$. Otherwise, it is called singular, which is considerably more difficult, and therefore not covered by our discussion here.

5

Systems of First-Order Differential Equations

5.1 Introduction

A general first order system of ODEs is of the form

$$\begin{aligned} y_1' &= f_1\left(x, y_1, y_2, \cdots, y_n\right) \\ y_2' &= f_2\left(x, y_1, y_2, \cdots, y_n\right) \\ &\cdots \\ y_n' &= f_n\left(x, y_1, y_2, \cdots, y_n\right) \end{aligned}$$

Vector notation is more compact:

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y})$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \qquad \mathbf{f}(x, \mathbf{y}) = \begin{bmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \\ \vdots \\ f_n(x, \mathbf{y}) \end{bmatrix}$$

A scalar ODE of order n can be written as an n-th order system by change of variables. Consider the following scalar equation,

$$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

let us define the following variables,

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ &\vdots \\ y_n &= y^{(n-1)} \end{aligned}$$

then the equivalent system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, \ldots, y_n) \end{aligned}$$

Example: Lotka-Volterra predator-prey model

Consider the Lotka-Volterra predator-prey model. It describes the evolution of the concentration of

prey x_1 and predators x_2:

$$\frac{dx_1}{dx} = ax_1 - bx_1x_2$$
$$\frac{dx_2}{dx} = -cx_2 + dx_1x_2$$

This is an example of two-first order nonlinear ODEs.

Example: Springs

If we have three springs in a horizontal line with walls at the two ends and two objects of mass m_1 and m_2 in the middle, we can describe the evolution of the position using Newton's second law. We define x_1 and x_2 to be the displacement of the center of mass from the rest position. We assume Hooke's law describes the force of the spring and that there could be additional forcing acting on the system. The two second order ODEs are,

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= k_2\left(x_2 - x_1\right) - k_1x_1 + F_1(t) \\ &= -\left(k_1 + k_2\right)x_1 + k_2x_2 + F_1(t) \\ m_2\frac{d^2x_2}{dt^2} &= -k_3x_2 - k_2\left(x_2 - x_1\right) + F_2(t) \\ &= k_2x_1 - \left(k_2 + k_3\right)x_2 + F_2(t) \end{aligned}$$

This is an example of two-second order ODEs that can be combined in terms of four first-order ODEs. To convert it to a system of first order equations we define,

$$y_1 = x_1, \quad y_2 = \frac{dx_1}{dt}, \quad y_3 = x_2, \quad y_4 = \frac{dx_2}{dt}$$

Following the procedure that we just mentioned we get,

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_1} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ F_1(t) \\ 0 \\ F_2(t) \end{bmatrix}$$

A general **linear system** of first-order DEs is of the form

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} + \mathbf{b}(x)$$

where $\mathbf{A}$ is an $n \times n$ matrix with coefficients $a_{ij}(x)$ and $\mathbf{b}(x)$ is a $1 \times n$ column vector with coefficients $b_i(x), i, j = 1, \ldots, n$

5.2 Existence-Uniqueness of IVPs

5.2.1 The Picard Method

COnsider the IVP

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y}, \quad \mathbf{y}(x_0) = \mathbf{y}_0$$

We can integrate the equation and combine it with the initial condition to get a single equation:

$$\int_{x_0}^{x} \frac{dy}{dx}(s)ds = \int_{x_0}^{x} f(s, y(s))ds$$
$$y(x) - y\left(x_0\right) = \int_{x_0}^{x} f(s, y(s))ds$$
$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds$$

This equation is called the equivalent **integral equation**.

The main idea is to construct a sequence of functions using the above integral equation by iteration. Then prove that the sequence converge to a solution of the IVP. To begin, we use the roughest possible approximation of a solution

$$y_0(x) = y_0$$

This is a horizontal straight line that passes through the point (x_0, y_0). One advantage of this approximation is that it satisfies the initial condition. One disadvantage is that it usually doesn't equal the exact solution at any other points. If you recall from first year calculus that a Taylor series is a means of building up a complicated function using polynomials of all orders, this first guess can be see as the constant term in the Taylor series.

If we substitute this into our integral equation we obtain a new approximate solution,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))ds$$

Because $y_0(x)$ is constant this integral can be evaluated easily to compute $y_1(x)$ explicitly. This is then going to give us a linear function of x, which is like the second term in our Taylor series.

Then given $y_1(x)$ we can also plug it into the integral equation to obtain another approximation,

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))ds$$

This is going to be like the quadratic term in our Taylor series.

If we proceed inductively then at the n-th stage we have,

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))ds$$

This is called **Picard's method of successive approximations**. Because we are integrating it increases the order of the polynomial of the guess by one each time, therefore giving us more freedom to match the solution to the DE.

Example: toy problem

$$y' = y, \quad y(0) = 1$$

which we know has an exact solution $y(x) = e^x$. If we integrate the equation we find that the analogous integral equation is,

$$y(x) = 1 + \int_0^x y(s)ds$$

The n-th step of the Picard Method is of the form,

$$y_n(x) = 1 + \int_0^x y_{n-1}(s)ds$$

We can use this to construct our solution explicitly:

$$\begin{aligned}
y_0 &= 1 \\
y_1(x) &= 1 + \int_0^x y_0(s)ds = 1 + x \\
y_2(x) &= 1 + \int_0^x y_1(s)ds = 1 + x + \frac{x^2}{2} \\
y_3(x) &= 1 + \int_0^x y_2(s)ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \\
&\vdots \\
y_n(x) &= 1 + \int_0^x y_{n-1}(s)ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}
\end{aligned}$$

What we are doing is constructing the Taylor Series expansion of the solution. In this case the successive approximation converges to the exact solution e^x.

Example: slightly more complicated

$$y' = x + y, \quad y(0) = 1$$

The solution to this linear equation (using an integrating factor) is $y(x) = 2e^x - x - 1$.

The integral equation that is equivalent to the above equation is,

$$y(x) = y(0) + \int_0^x [s + y(s)]ds$$

and the n-th term in the approximation is

$$y_n(x) = y(0) + \int_0^x [s + y_{n=1}(s)]ds$$

We can use this to construct our solution as follows:

$$\begin{aligned}
&y_0(x) = 1 \\
&y_1(x) = 1 + \textstyle\int_0^x [s+1]ds = 1 + x + \frac{x^2}{2!} \\
&y_2(x) = 1 + \textstyle\int_0^x \left[1 + 2s + \frac{s^2}{2!},\right] ds = 1 + x + x^2 + \frac{x^3}{3!} \\
&y_3(x) = 1 + \textstyle\int_0^x \left[1 + 2s + s^2 + \frac{s^3}{3!},\right] ds = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!} \\
&y_4(x) = 1 + \textstyle\int_0^x \left[1 + 2s + s^2 + \frac{s^3}{3!} + \frac{s^4}{4!},\right] ds = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\cdot 4} + \frac{x^5}{5!} \\
&\cdots \\
&y_n(x) = 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) + \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

This converges to the following function,

$$1 + x + 2(e^x - x - 1) + 0 = 2e^x - x - 1$$

which is precisely our exact solution.

5.2.2 Picard's Theorem

Theorem 5.1: Picard's Theorem

Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a bounded and closed set D. If (x_0, y_0) is any interior point of D, then there exists a number $h > 0$ with the property that the IVP,

$$y' = f(x,y), \quad y(x_0) = y_0$$

has one and only one solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

Proof:

This proof is rather lengthy and technical and we will simply outline the main ideas or you could check page 37 of https://notes.sibeliusp.com/pdfs/1189/amath251.pdf.

Step 1: There exists a unique solution to the IVP if and only if there exists a unique solution to the associated integral equation. Construct a sequence of functions $\{y_n(x)\}$ by iteration.

Step 2: Our aim is to show that the sequence of functions, $y_n(x)$ converge to the exact solution. If we write down the following telescoping series we see that the convergence of $y_n(x)$ is equivalent to the convergence of the series,

$$y_0 + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)]$$

Step 3: The series above converges to the exact solution $y(x)$.

Step 4: $y(x)$ is a continuous solution of the integral equation.

Step 5: $y(x)$ is the only continuous solution of the integral equation. □

5.3 Linear Systems

5.3.1 Existence and Uniqueness

In this course, we will mainly study the following linear system

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} + \mathbf{b}$$

Example:

Consider the homogeneous second-order DE in general form,

$$y'' + P(x)y' + Q(x)y = 0$$

We can rewrite this as a system by defining,

$$y_1 = y(x), \qquad y_2 = y'(x)$$

then the governing equation is in the form of equation where the governing matrix is

$$\mathbf{A}(x) = \begin{bmatrix} 0 & 1 \\ -Q(x) & -P(x) \end{bmatrix}$$

Previously, we stated Picard's theorem for a general system of non-linear equations. Here we specialize it for the case of inhomogeneous linear systems, since that is what we will need throughout this chapter. Note that in the nonlinear version we needed that both $f(x, y)$ and $\frac{\partial f}{\partial y}$ were continuous. In the case of the linear system we simply need that the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are continuous.

Theorem 5.2: Picard's Theorem for Linear Systems

Let $\mathbf{A}(x)$ and $\mathbf{text}b(x)$ be continuous functions on a closed interval $I = [\alpha, \beta]$, then for any point $x_0 \in I$ and any constant vector $\mathbf{y}_0$, there exists a unique solution $\mathbf{y}(x)$ to the IVP

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} + \mathbf{b}, \quad \mathbf{y}(x_0) = \mathbf{y}_0$$

defined throughout the interval I.

If the RHS of the DE depends on x explicitly, we say the system is **nonautonomous**. Otherwise, we say the system is **autonomous**.

5.3.2 Solution Space for Systems of DE

We begin with a homogeneous system

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y}$$

The solution $\mathbf{y}$ is an $n \times 1$ vector of the form

$$\mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix} \tag{5.1}$$

is an C^1 vector valued function from an interval on the real line, say I, to $\mathbb{R}^n$. Formally we say that the solution is an element of the vector space $C^1(I, \mathbb{R}^n)$. The matrix $\mathbf{A}$ is $n \times n$ and has elements that are denoted with $a_{ij}(x)$, which we assume are continuous single variable functions of x. One special example we will consider is where the matrix $\mathbf{A}$ is constant, which means that each element is constant too.

Next we will show that the solution space of Eq. (5.1) (which is the space of all solutions) is a linear vector space. We go through our check list as we would in linear algebra:

1. Identity Element: $\mathbf{y} = \mathbf{0}$ is a solution, which is an element of the space.
2. Closure: The space is closed under addition and scalar multiplication by the Principle of Superposition. By that we mean that if $\mathbf{y}_1(x)$ and $\mathbf{y}_2(x)$ are each solutions to Eq. (5.1) then so is,
$$\mathbf{y}(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x)$$
for any constants c_1, c_2.

   Proof involves linearity of the derivative and matrix.

   Note that to prove this we needed to use the properties that both the derivative and matrix multiplication are linear operators.

Before we begin constructing a solution we show that the solution space has a dimension of n. See Theorem 2.6 on page 111 of the course notes for a more detailed treatment.

Theorem 5.3

Consider $\mathbf{y}' = \mathbf{A}(x)\mathbf{y}$, where $\mathbf{A}(x)$ is continuous on the interval I. If a solution $\mathbf{y}$ satisfies $\mathbf{y}(x_0) = \mathbf{0}$ for some $x_0 \in I$ then $\mathbf{y}(x) = \mathbf{0}$ for all $x \in I$.

Proof:
The zero solution satisfies both the DE and the IC and thus by existence and uniqueness, it is the only solution. □

Corollary 5.4

Consider $\mathbf{y}' = \mathbf{A}(x)\mathbf{y}$, where $\mathbf{A}(x)$ is continuous on the interval I. If $\mathbf{y}(x)$ is a solution it is either identically zero, $\mathbf{y}(x) = \mathbf{0}$ for all $t \in I$, or never equal to zero, $\mathbf{y}(x) \neq \mathbf{0}$ for all $x \in I$.

Theorem 5.5

Consider $\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y}$, where $\mathbf{A}(x)$ is continuous on the interval I. If the solutions,
$$\{\mathbf{y}_1(x), \ldots, \mathbf{y}_n(x)\}$$
are linearly independent at one point $x_0 \in I$ then they are linearly independent for all x.

Proof:
We prove by contradiction. Suppose for some time $x_* \in I$ the solution at this time is linearly dependent. Then $\exists c_1, \ldots, c_n$ not all zero such that
$$c_1\mathbf{y}_1(x_*) + \cdots + c_n\mathbf{y}_n(x_*) = \mathbf{0}$$
We use these constants to define the following vector-function,
$$\mathbf{y}(x) = c_1\mathbf{y}_1(x) + \cdots + c_n\mathbf{y}_n(x)$$
By superposition, since each term in the sum is a solution to the DE then so must their sum. If we

use the second equation above along with the previous theorem we have that,

$$\mathbf{y}(x) = \mathbf{0} \quad \forall x \in I$$

But if we evaluate this at x_0 we obtain

$$c_1\mathbf{y}_1(x_0) + \cdots + c_n\mathbf{y}_n(x_0) = \mathbf{0}$$

But this is a contradiction since not all the constants are zero but the functions at this time were assumed to be linearly independent. □

Wronskian

We define the **Wronskian** for a system of equation as

$$W[\mathbf{y}_1(x), \cdots, \mathbf{y}_n(x)] = \det[\mathbf{y}_1(x), \cdots, \mathbf{y}_n(x)]$$

Then Theorem 5.5 can be restated to say that the solutions $\{\mathbf{y}_1(x), \cdots, \mathbf{y}_n(x)\}$ are linearly independent if and only if their Wronskian is non-zero.

Theorem 5.6: Dimension of the solution space

The solution space of $\frac{d}{dx}\mathbf{y} = \mathbf{A}(x)\mathbf{y}$ in $\mathbb{R}^n$, where $\mathbf{A}(x)$ is continuous, is an n-dimensional vector space.

Proof:

First, use the notation that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. Since $\mathbf{A}(x)$ is continuous, our theorem for existence and uniqueness (applied to the vector case), implies that a solution must exist for any initial condition. Say that

$$\{\mathbf{y}_1, \ldots, \mathbf{y}_n(x)\}$$

are solutions that satisfy the following initial conditions,

$$\begin{aligned} \mathbf{y}_1(x_0) &= \mathbf{e}_1 \\ &\vdots \\ \mathbf{y}_n(x_0) &= \mathbf{e}_n \end{aligned}$$

Extending our ideas of the Wronskian that if the initial conditions are linearly independent than the solutions are necessarily independent for all times then we must have that our space of functions above are all linearly independent. This means that the space must be of size n. But, we need to show that this is in fact a general solution. By that I mean that any solution can be written in terms of these particular solutions.

To do this consider an arbitrary solution $\mathbf{y}(x)$ that has the following initial condition,

$$\mathbf{y}(x_0) = \mathbf{x}_0$$

Since $\mathbf{x}_0$ is a vector in $\mathbb{R}^n$ then there exists n constants $c_1, \ldots, c_n$ such that we can express this initial condition in terms of our standard basis,

$$\mathbf{y}_0 = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n$$

We take the very same constants and define a new function,

$$\hat{\mathbf{y}}(x) = c_1\mathbf{y}_1(x) + \cdots + c_n\mathbf{y}_n(x)$$

By the principle of superposition we have that this function satisfies the correct initial conditions,

$$\mathbf{y}(x_0) = \mathbf{y}_0$$

Since $\mathbf{y}(x)$ and $\hat{\mathbf{y}}(x)$ satisfy the same initial conditions, the existence and uniqueness theorem implies that they must in fact be identical. Therefore, since we can construct any solution as a superposition or our basis that has n elements the vector space must be n dimensional. □

Basis for the Solution Space

A **basis of the solution space** is a set of n solutions to the DE that are linearly independent on the interval $x \in I$. In other words, the Wronskian of the solutions is always non-zero.

The above theorem implies that if we want to find a basis to the solution space we simply need to find n solutions that have linearly independent initial conditions. To construct a standard basis for the solution space we begin with the standard basis of $\mathbb{R}^n$ which we denote $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. We take the standard basis vectors to be the initial conditions for our DE and say that the standard basis to the DE consists of,

$$\{\mathbf{Y}_1(x), \ldots, \mathbf{Y}_n(x)\}$$

where the initial conditions at x_0 are,

$$\mathbf{Y}_j(x_0) = \mathbf{e}_j \qquad \text{for } j = 1, \ldots, n$$

With this we can define the fundamental matrix.

5.3.3 The Fundamental Matrix

fundamental matrix

The **fundamental matrix** at x_0, denoted by $\mathbf{\Phi}(x, x_0)$, is

$$\mathbf{\Phi}(x, x_0) \equiv [\mathbf{Y}_1(x), \ldots, \mathbf{Y}_n(x)]$$

By that we mean that the fundamental matrix is the matrix formed with the standard basis vectors as the columns of the matrix.

The fundamental matrix satisfies the following properties:

1. Identity property
$$\mathbf{\Phi}(x_0, x_0) = \{\mathbf{Y}_1(x_0), \ldots, \mathbf{Y}_n(x_0)\} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} = \mathbf{I}$$

2. Initial Value Property.

   Let's compute the value of the vector of the fundamental matrix multiplied by any constant vector, say $\mathbf{a}$.
$$\begin{aligned}\mathbf{y}(x) &= \Phi(x, x_0)\,\mathbf{a} \\ &= \{\mathbf{Y}_1(x), \cdots \mathbf{Y}_n(x)\}\,\mathbf{a}, \quad \text{next matrix multiply} \\ &= \mathbf{Y}_1(x)a_1 + \cdots + \mathbf{Y}_n(x)a_n\end{aligned}$$
   where we have defined $\mathbf{a} = (a_1, \ldots, a_n)^T$. Since $\mathbf{y}$ is a linear superposition of solutions, it must itself also be a solution. Moreover, it is the solution that has the following initial condition,
$$\mathbf{y}(x_0) = \mathbf{\Phi}(x_0, x_0)\mathbf{a} = \mathbf{I}\mathbf{a} = \mathbf{a}$$
   Therefore, if we want to construct a solution that has a given initial condition, we simply multiply the fundamental matrix by that initial condition. In some textbooks $\mathbf{\Phi}(x, x_0)$ is called the **propagator**. By that we mean that it advances the solution from x_0 to x.

3. Derivative Property.

Next we look at the derivative of the fundamental matrix. To do this remember that this matrix is composed of column vectors each of which satisfies the DE. Therefore,

$$\frac{d}{dx}\mathbf{\Phi}(x, x_0) = \mathbf{A}(x)\mathbf{\Phi}(x, x_0)$$

To show this differentiate the solution above,

$$\begin{aligned}\frac{d\mathbf{y}}{dx} &= \frac{d}{dx}\Phi\left(x, x_0\right)\mathbf{a} \\ &= \frac{d}{dx}\left[\mathbf{Y}_1(x)a_1 + \cdots + \mathbf{Y}_n(x)a_n\right] \\ &= \mathbf{A}\left[\mathbf{Y}_1(x)a_1 + \cdots + \mathbf{Y}_n(x)a_n\right] \\ &= \mathbf{A}(x)\Phi\left(x, x_0\right)\mathbf{a} \\ &= \mathbf{A}(x)\mathbf{y}\end{aligned}$$

4. Multiplication Property.

   For any x_0, x_1, x_2 in the interval I,

   $$\mathbf{\Phi}(x_2, x_1)\mathbf{\Phi}(x_1, x_0) = \mathbf{\Phi}(x_2, x_0)$$

   Proof:
   If $\mathbf{y}(x)$ is the solution to our linear system with initial condition, $\mathbf{y}_0$ then the solution at position x_1 and x_2 are,

   $$\begin{aligned}\mathbf{y}(x_1) &= \mathbf{\Phi}(x_1, x_0)\mathbf{y}_0 \equiv \mathbf{y}_1 \\ \mathbf{y}(x_2) &= \mathbf{\Phi}(x_2, x_0)\mathbf{y}_0\end{aligned}$$

   But by continuity, if $x_2 > x_1$ we should also be above to write,

   $$\mathbf{y}\left(x_2\right) = \mathbf{\Phi}\left(x_2, x_1\right)\mathbf{y}_1 = \mathbf{\Phi}\left(x_2, x_1\right)\mathbf{\Phi}\left(x_1, x_0\right)\mathbf{y}_0$$

   By comparing this with the equation above and applying existence-uniqueness, we prove the desired result. □

   This statement basically says that if we advance the solution from x_0 to x_1 and then x_2 that is equivalent to advancing it directly from x_0 to x_2. Not surprising.

5. Inverse Property.

   $$[\mathbf{\Phi}(x, x_0)]^{-1} = [\mathbf{\Phi}(x_0, x)]$$

   Proof:
   Apply multiplication property with $x_2 = x_0$:

   $$\mathbf{\Phi}(x_0, x_1)\mathbf{\Phi}(x_1, x_0) = \mathbf{\Phi}(x_0, x_0) = \mathbf{I} \implies [\mathbf{\Phi}(x, x_0)]^{-1} = [\mathbf{\Phi}(x_0, x)]$$

   which reduces to the desired result if we replace x_1 with x. □

   This states that the inverse of going from x_0 to x_2 is going from x_2 to x_0.

6. Time Invariance Property.

   For a time-invariant linear DE where $\mathbf{A}$ is **constant** then the fundamental matrix satisfies,

   $$\mathbf{\Phi}(x, x_0) = \mathbf{\Phi}(x - x_0, 0)$$

   Proof:
   See textbook. □

It is for this reason that in the case where $\mathbf{A}$ is constant (time-invariant), without any ambiguity, we can define the Fundamental Matrix as follows,

$$\boxed{\mathbf{\Phi}(x) \equiv \mathbf{\Phi}(x, 0)}$$

The fundamental matrix in the case of an autonomous system satisfies the following:

1. $\mathbf{\Phi}(0) = \mathbf{\Phi}(0,0) = \mathbf{I}$.
2. $\mathbf{y}(x) = \mathbf{\Phi}(x)\mathbf{a}$ is the unique solution of the IVP $\mathbf{y}' = \mathbf{A}\mathbf{y}, \mathbf{y}(0) = \mathbf{a}$.
3. $\mathbf{\Phi}(x)$ satisfies the matrix DE
4. $\mathbf{\Phi}'(x) = \mathbf{A}\mathbf{\Phi}(x)$
5. $\mathbf{\Phi}(x_1 + x_2) = \mathbf{\Phi}(x_1)\mathbf{\Phi}(x_2)$
6. $\mathbf{\Phi}(-x) = \mathbf{\Phi}(x)^{-1}$

5.4 Nonhomogeneous linear systems

In this section, we learn how to solve inhomogeneous linear systems of the form,

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} + \mathbf{f}(x), \quad \text{with } \mathbf{y}(x_0) = \mathbf{y}_0$$

You can think of $\mathbf{f}(x)$ as an external forcing and $\mathbf{A}(x)$ as the internal dynamics of the system. As present we know that the solution to the homogeneous problem is the fundamental solution matrix, and presumably this will play a role in the nonhomogeneous solution.

Now we recall that the solution to the associated homogeneous problem is,

$$\mathbf{y}_h = \mathbf{\Phi}(x, x_0)\mathbf{a}$$

for any vector $\mathbf{a}$. We use the method of variation of parameters on this solution to find a particular solution. Our particular solution is of the form,

$$\mathbf{y}_p = \mathbf{\Phi}(x, x_0)\mathbf{v}(x)$$

We sub this into the nonhomogeneous equation and simplify,

$$\begin{aligned}
\frac{d\mathbf{y}_p}{dx} = \frac{d\mathbf{\Phi}}{dx}(x, x_0)\,\mathbf{v}(x) + \mathbf{\Phi}(x, x_0)\,\frac{d\mathbf{v}}{dx} &= \mathbf{A}(x)\mathbf{\Phi}(x, x_0)\,\mathbf{v}(x) + \mathbf{f}(x) \\
\mathbf{A}(x)\mathbf{\Phi}(x, x_0)\,\mathbf{v}(x) + \mathbf{\Phi}(x, x_0)\,\frac{d\mathbf{v}}{dx} &= \mathbf{A}(x)\mathbf{\Phi}(x, x_0)\,\mathbf{v}(x) + \mathbf{f}(x), \quad \text{cancel terms} \\
\mathbf{\Phi}(x, x_0)\,\frac{d\mathbf{v}}{dx} &= \mathbf{f}(x), \quad \text{multiply by the inverse} \\
\frac{d\mathbf{v}}{dx} &= [\mathbf{\Phi}(x, x_0)]^{-1}\,\mathbf{f}(x), \quad \text{integrate} \\
\mathbf{v} &= \int_{x_0}^{x} [\mathbf{\Phi}(s, x_0)]^{-1}\,\mathbf{f}(s)ds
\end{aligned}$$

Therefore, the particular solution can be written as

$$\mathbf{y}_p = \mathbf{\Phi}(x, x_0)\int_{x_0}^{x} [\mathbf{\Phi}(s, x_0)]^{-1}\mathbf{f}(s)ds = \int_{x_0}^{x} [\mathbf{\Phi}(x, s)]\mathbf{f}(s)ds$$

and our general solution is the superposition of this and the homogeneous solution. Notice that because this particular solution vanishes at x_0 it is easy to determine the right constant so that we can satisfy the given initial conditions,

$$\mathbf{y} = \mathbf{\Phi}(x, x_0)\mathbf{y}_0 + \int_{x_0}^{x} [\mathbf{\Phi}(x, s)]\mathbf{f}(s)ds$$

This is known as the variation of parameters formulae.

5.5 Finding Solutions using Eigenvalues

Here we explain how we solve linear systems with constant coefficients. When we consider the system of equations,

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}\mathbf{y}$$

it is important to realize that we cannot simply substitute in $e^{\lambda x}$ because this is a scalar and we have a vector equation. It turns out that the correct choice for a system has the same functional form but has a vector as coefficients,

$$\mathbf{y} = \mathbf{v}e^{\lambda x}$$

When we substitute this into the above system we get, using the fact that $\mathbf{v}$ is constant we find

$$\begin{aligned} \frac{d\mathbf{y}}{dx} &= \mathbf{A}\mathbf{y} \\ \lambda \mathbf{v}e^{\lambda x} &= \mathbf{A}\mathbf{v}e^{\lambda x}, \quad \text{divide by } e^{\lambda x} \mathbf{A}\mathbf{v} - \lambda \mathbf{I}\mathbf{v} &= 0, \quad \text{where } \mathbf{I} \text{ is the identity matrix} \\ [\mathbf{A} - \lambda \mathbf{I}]\mathbf{v} &= 0 \end{aligned}$$

From this we gather that to get non-trivial solutions we need that the determinant of the matrix $\mathbf{A}$ is equal to zero. Otherwise, the only possible solution is the trivial solution. That is why we can construct our solution by solving,

$$\det[\mathbf{A} - \lambda \mathbf{I}] = 0$$

for the eigenvalues of the matrix $\mathbf{A}$. The above equation is a polynomial of order n and is also called a **characteristic equation**. After we have the solutions we then need to find the corresponding eigenvectors, $\mathbf{b}$.

To show how this works, consider the following example.

Example:

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = -1$ and the eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Therefore, using the form of our general solution, we find that the two linearly independent solutions are,

$$\mathbf{y}_1 = \mathbf{v}_1 e^{\lambda_1 x}, \quad \mathbf{y}_2 = \mathbf{v}_2 e^{\lambda_2 x}$$

From this we know that the general solution is

$$\mathbf{y}(x) = v_1 \mathbf{y}_1(x) + c_2 \mathbf{y}_2(x) = \begin{pmatrix} c_1 e^{3x} + c_2 e^{-x} \\ 2c_1 e^{3x} - 2c_2 e^{-x} \end{pmatrix}$$

This solution can be used to solve any initial conditions. Given a $\mathbf{y}(0)$ we need to solve for c_1 and c_2. In the previous example, we found a general solution but we did not find the fundamental solution matrix. To compute this matrix we must find the two solutions that have the initial conditions to be the standard basis of $\mathbb{R}^2$. First we must find $\mathbf{Y}_1(x)$ such that $\mathbf{Y}_1(0) = \mathbf{e}_1$. If we evaluate our general solution at the initial time $x = 0$ we find

$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies c_1 = c_2 = \frac{1}{2} \implies \mathbf{Y}_1(x) = \begin{pmatrix} \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} \\ e^{3x} - e^{-x} \end{pmatrix}$$

Similarly, for the second initial condition,

$$\mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies c_1 = \frac{1}{4}, c_2 = -\frac{1}{4} \implies \mathbf{Y}_2(x) = \begin{pmatrix} \frac{1}{4}e^{3x} - \frac{1}{4}e^{-x} \\ \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} \end{pmatrix}$$

Thus our fundamental solution matrix is

$$\mathbf{\Phi}(x,0) = \begin{bmatrix} \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} & \frac{1}{4}e^{3x} - \frac{1}{4}e^{-x} \\ e^{3x} - e^{-x} & \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} \end{bmatrix}$$

Observe that it is true that $\mathbf{\Phi}(0,0) = \mathbf{I}$, as required. The other properties can be checked similarly.

The following Theorem generalizes what we just did above.

Theorem 5.7

Let $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ be eigenpairs for the real $n \times n$ constant matrix $\mathbf{A}$. Assume that $\lambda_1, \ldots, \lambda_n$ are real and that the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Then

$$\{e^{\lambda_1 x}\mathbf{v}_1, \ldots, e^{\lambda_n x}\mathbf{v}_n\}$$

forms the basis of the solution space the system of equation $\frac{d}{dx}\mathbf{y} = \mathbf{A}\mathbf{y}$. The general solution can be written as,

$$\mathbf{y} = c_1 e^{\lambda_1 x}\mathbf{v}_1 + \ldots + c_n e^{\lambda_n x}\mathbf{v}_n$$

Proof:

To show that the general solution is a solution is easy by the superposition principle and, by construction, each term in the sum is a solution.

To show that this solution space is always linear independent we can consider the Wronskian,

$$\begin{aligned} W\left[\mathbf{y}_1, \cdots, \mathbf{y}_n\right] &= \det\left[e^{\lambda_1 x}\mathbf{v}_1, \cdots, e^{\lambda_n x}\mathbf{v}_n\right] \\ &= e^{(\lambda_1 + \cdots + \lambda_n)x} \det\left[\mathbf{v}_1, \cdots, \mathbf{v}_n\right] \end{aligned}$$

Since the exponential is non-zero and determinant is non-zero (because the eigenvectors are linearly independent), we have that the Wronskian is non-zero for all time. □

Using our knowledge of linear algebra we can deduce that if the eigenvalues are distinct, then the eigenvectors are necessarily linearly independent. Therefore, a shorter version of the same theorem is shown below.

Corollary 5.8

Suppose the matrix $\mathbf{A}$ has n eigenpairs $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ such that eigenvalues are real and distinct, then

$$\{e^{\lambda_1 x}\mathbf{v}_1, \ldots, e^{\lambda_n x}\mathbf{v}_n\}$$

forms the basis of the solution space the system of equation $\frac{d}{dx}\mathbf{y} = \mathbf{A}\mathbf{y}$.

Example:

Find the general solution to the system,

$$\frac{d\mathbf{y}}{dx} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -3 & -4 & -3 & 6 \\ 0 & -3 & -2 & 3 \\ -3 & -5 & -3 & 7 \end{pmatrix} \mathbf{y}$$

Eigenvalues $\lambda = -2, -1, 1, 2$. Eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

With this we can write the general solution as,

$$\mathbf{y} = c_1 e^{-2x}\mathbf{v}_1 + c_2 e^{-x}\mathbf{v}_2 + c_3 e^{x}\mathbf{v}_3 + c_4 e^{2x}\mathbf{v}_4$$

To find the fundamental solution matrix we need to solve four systems of equations. To find each column we must determine the constants $c_1, \ldots, c_4$ that produce the standard basis (unit vectors) in $\mathbb{R}^4$. We can determine all the coefficients by computing the inverse of the matrix composed of the eigenvectors. Therefore, we can write the fundamental solution matrix as,

$$\Phi(x) = \begin{pmatrix} e^{-2x} & e^{-x} & e^{x} & 0 \\ 0 & -e^{-x} & 0 & e^{2x} \\ -e^{-2x} & 0 & -e^{x} & 0 \\ 0 & e^{-x} & e^{x} & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 2 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$$

inverse of $\mathbf{A}$

Soon we will find an easier way of constructing this matrix.

5.5.1 Complex Eigenfunctions

In general we know that the eigenvalues that appear can be complex valued. If so then the eigenvalues appear in complex conjugate pairs, say $\lambda = \mu \pm i\nu$. If $\mathbf{v}$ is the complex eigenvector that corresponds to λ then the two eigenpairs are $(\lambda, \mathbf{v})$ and $(\lambda^*, \mathbf{v}^*)$ where * to denote complex conjugate.

$$\mathbf{u}(x) = e^{\lambda x}\mathbf{v}, \quad \mathbf{u}^*(x) = e^{\lambda^* x}\mathbf{v}^*$$

are solutions to the system of DEs.

The matrix $\mathbf{A}$ is assumed to be real and it is convenient to write the solution in real form. To obtain real solutions from our complex one we can simply separate the real and imaginary parts, each of which must be a solution (because the real and imaginary parts are independent). Therefore, we define,

$$\begin{aligned} \mathbf{y}_1(x) &= \text{Real}\{\mathbf{u}(x)\} = e^{\mu x}(\mathbf{a}\cos\nu x - \mathbf{b}\sin\nu x) \\ \mathbf{y}_2(x) &= \text{Imag}\{\mathbf{u}(x)\} = e^{\mu x}(\mathbf{a}\sin\nu x + \mathbf{b}\cos\nu x) \end{aligned}$$

where we have defined $\mathbf{v} = \mathbf{a} + \mathbf{b}i$.

If all the eigenvalues are real and distinct except for two that appear in complex conjugate pairs then we can write the general solution as,

$$\mathbf{y}(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + c_3 e^{\lambda_3 x}\mathbf{v}_3 + \cdots + c_n e^{\lambda_n x}\mathbf{v}_n$$

Example:

See textbook or https://notes.sibeliusp.com/pdfs/1195/amath351.pdf.

5.6 Exponential Matrix

Let's consider the autonomous (time-invariant) system,

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{a}$$

In the scalar case the above system reduces to

$$y' = ay, \quad y(0) = a$$

where the solution is,

$$y(x) = y_0 e^{ax}$$

Motivated by the scalar solution, we rewrite the solution to the autonomous system as,

$$\mathbf{y} = \mathbf{\Phi}(x, 0)\mathbf{a} = e^{x\mathbf{A}}\mathbf{a}$$

This means that the fundamental solution matrix is equal to the matrix exponential. What does it mean to say "the matrix exponential"? To answer this question we again return to the scalar case.

There are several equivalent definitions for the (scalar) exponential function. One is from the Taylor series which states,

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

This suggests that if the matrix $\mathbf{A}$ is constant then the matrix exponential be defined as,

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \dots$$

First, note that the matrix exponential is $n \times n$ if the matrix itself is $n \times n$. Second, this definition only makes sense if the series converges. It is possible to prove theorems about the convergence of this series but this is beyond the scope of the course.

Properties

1. $e^{\mathbf{0}} = \mathbf{I}$
2. $e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}}e^{\mathbf{B}}$ unless $\mathbf{A}$ and $\mathbf{B}$ commute.
3. $e^{-\mathbf{A}} = [e^{\mathbf{A}}]^{-1}$.

These properties bring us with the following facts:

1. Diagonal Matrix:
$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \implies e^{\mathbf{A}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$
2. Upper Triangular:
$$\mathbf{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \implies e^{\mathbf{A}} = \begin{bmatrix} e^a & e^a \\ 0 & e^a \end{bmatrix}$$
Then
$$e^{x\mathbf{A}} = \begin{bmatrix} e^{ax} & xe^{ax} \\ 0 & e^{ax} \end{bmatrix}$$
3. Antisymmetric:
$$\mathbf{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \implies e^{\mathbf{A}} = e^a \begin{bmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{bmatrix}$$

5.7 Linear Systems of ODEs and the Exponential Matrix

Next, we verify that a solution to the standard IVP:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}_0$$

in the case where $\mathbf{A}$ is a constant $n \times n$ matrix is

$$\mathbf{y} = e^{x\mathbf{A}}\mathbf{y}_0$$

To show this, we differentiate term by term:

$$\begin{aligned}
\frac{d\mathbf{y}}{dx} = \frac{d}{dx}e^{x\mathbf{A}}\mathbf{y}_0 &= \frac{d}{dx}\left(\mathbf{I} + x\mathbf{A} + \frac{x^2}{2!}\mathbf{A}^2 + \cdots + \frac{x^n}{n!}\mathbf{A}^n + \cdots\right)\mathbf{y}_0 \\
&= \left(\mathbf{A} + \frac{x}{1!}\mathbf{A}^2 + \cdots + \frac{x^{n-1}}{(n-1)!}\mathbf{A}^n + \cdots\right)\mathbf{y}_0 \\
&= \mathbf{A}\left(\mathbf{I} + \frac{x}{1!}\mathbf{A} + \cdots + \frac{x^{n-1}}{(n-1)!}\mathbf{A}^{n-1} + \cdots\right)\mathbf{y}_0 \\
&= \mathbf{A}\mathbf{y}(x)
\end{aligned}$$

as we hypothesized.

Let's know put together what we've learned about the exponential matrix to solve linear systems with constant coefficients.

Example:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \quad \text{where } A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

From before we know that

$$e^{x\mathbf{A}} = \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{bx} \end{bmatrix}$$

Therefore, if we have an initial condition of $\mathbf{y}(0) = \mathbf{y}_0$ the general solution is

$$\mathbf{y} = e^{x\mathbf{A}}\mathbf{y}_0 = \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{bx} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 e^{ax} \\ a_2 e^{bx} \end{bmatrix}$$

Using his approach is perhaps overkill in that we could have solved each equation independently and obtained the same result. However, this is a method that works generally.

Example:

Example 3.5.2, 3.5.3 from the textbook.

As we said before, these three examples basically cover all the different cases for a 2×2 system since we can classify the eigenvalues of a given matrix in one of the three categories. By that we mean that any matrix can be transferred to one of these three forms using a similarity transformation. Now we can ask ourselves, how do we go from solving 2×2 systems to $n \times n$ systems? In general dealing with the Taylor series is too much to answer. That's why instead, we focus on the eigenvalue decomposition of the matrix $\mathbf{A}$. Recall from linear algebra that any $n \times n$ matrix $\mathbf{A}$ that has the following eigenvalue decomposition:

$$\mathbf{A} = \mathbf{C}\mathbf{B}\mathbf{C}^{-1}$$

In the simplest case where the eigenvalues are real and distinct, say $\lambda_1, \ldots, \lambda_n$, we have

$$\mathbf{B} = \text{diag}[\lambda_1, \ldots, \lambda_n] = \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix}$$

The matrix $\mathbf{C}$ consists of column vectors that are eigenvectors of the original matrix $\mathbf{A}$. If eigenvalue λ_k has the corresponding eigenvector $\mathbf{v}_k$ then the other matrix must be ordered in the same way as $\mathbf{B}$:

$$\mathbf{C} = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$$

In general, if there are repeated eigenvalues we have it in Jordan Normal form. For example if an eigenvalue is repeated twice, $\lambda = \lambda_1 = \lambda_2$ and we can find only one eigenvector for these two, the matrix $\mathbf{C}$ will have the following 2×2 block,

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Conversely, if the two eigenvalues are complex, say $\lambda = a \pm bi$, the corresponding Jordan block is,

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

For an $n \times n$ system, the exponentiation of $\mathbf{B}$ can be constructed using our previous calculations for the 2×2 blocks. If an eigenvalue is repeated more than twice, you need to consider larger Jordan blocks, but the ideas extend naturally.

Given a matrix $\mathbf{A}$ and its eigenvalue decomposition, it is an easy exercise to determine its exponential,

$$\begin{aligned} e^{x\mathbf{A}} &= e^{x\mathbf{CBC}^{-1}} \\ &= \mathbf{I} + x\mathbf{CBC}^{-1} + \frac{x^2}{2!}\left(\mathbf{CBC}^{-1}\right)\left(\mathbf{CBC}^{-1}\right) + \cdots \\ &= \mathbf{C}\left[\mathbf{I} + x\mathbf{B} + \frac{x^2}{2!}\mathbf{B}^2 + \cdots\right]\mathbf{C}^{-1} \\ &= \mathbf{C}e^{x\mathbf{B}}\mathbf{C}^{-1} \end{aligned}$$

We revisit our previous example.

Example:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

Eigenvalue decomposition:

$$\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

Then

$$\begin{aligned} \mathbf{\Phi}(x) = e^{x\mathbf{B}} &= \mathbf{C}e^{x\mathbf{B}}\mathbf{C}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} e^{3x} & 0 \\ 0 & e^{-x} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} & \frac{1}{4}e^{3x} - \frac{1}{4}e^{-x} \\ e^{3x} - e^{-x} & \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} \end{bmatrix} \end{aligned}$$

This is exactly what we previously computed.

5.8 Asymptotic Behaviour of solutions to $\mathbf{y}' = \mathbf{A}\mathbf{y}$

The beauty in the following formula,

$$\mathbf{e}^{x\mathbf{A}} = \mathbf{C}e^{x\mathbf{B}}\mathbf{C}^{-1}$$

is that it is clear that the functional dependency of the solution is deter- mined entirely by the exponential of the eigenvalues (or more generally of the Jordan blocks). If we know the eigenvalues we can determine the long time behaviour even without explicitly knowing the final solution! In particular, it is the real parts of the eigenvalues that determine whether the solution will grow or decay as x increases. The basic idea is that, e^{ax} tends to infinity (zero) as x tends to infinity if the real part of a is positive (negative). The case where $a = 0$ is a bit more complicated. Whether the solution grows or decays depends on whether we have some of the unstable (or stable) eigenvector in our solution.

The following are statements of the stability of the solution (more detail will be given in AMATH 451).

1. If all the eigenvalues of $\mathbf{A}$ have negative real parts, then there are constants $M, k > 0$ such that

   $$\|e^{A(t-s)}\| \leq Me^{-k(t-s)}, \quad t \geq s \geq 0$$

   Hence

   $$\mathbf{y}(x) = e^{x\mathbf{A}}\mathbf{y}_0 \to \mathbf{0},$$

   for all initial conditions. In this case the trivial solution $\mathbf{y} = \mathbf{0}$ is **globally asymptotically stable**.

2. If there is even one eigenvalue of $\mathbf{A}$ that has a positive real part then in general we will have,

   $$\mathbf{y}(x) = e^{x\mathbf{A}}\mathbf{y}_0$$

   is unbounded as $x \to \infty$. This case is said to be **asymptotically unstable**.

3. If there are no eigenvalues with a positive real part then there are two possibilities:

(a) If no eigenvalues are repeated the solution is bounded for all time. This is **neutrally stable**.

(b) If at least one eigenvalue is repeated at least once, then in general the solution will grow algebraically.

The interested students are directed to AMATH 451 "Introduction to Dynamical Systems" for more details on stability of solutions.

To conclude this section, let us revisit the inhomogeneous linear systems of the form,

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} = \mathbf{f}(x), \qquad \mathbf{y}(x_0) = y_0$$

For the special case where the matrix $\mathbf{A}$ is constant, we know that the fundamental solution matrix is equal to the exponential matrix,

$$\mathbf{\Phi}(x, x_0) = \mathbf{\Phi}(x - x_0, 0) = e^{(x-x_0)\mathbf{A}}$$

If we substitute this into the solution that holds in general

$$\mathbf{y} = \mathbf{\Phi}(x, x_0)\mathbf{y}_0 + \int_{x_0}^{x} \mathbf{\Phi}(x, s)\mathbf{f}(s)ds$$

we can simplify it to,

$$\mathbf{y} = \mathbf{e}^{(x-x_0)\mathbf{A}}\mathbf{y}_0 + \int_{x_0}^{x} e^{(x-s)\mathbf{A}}\mathbf{f}(s)ds$$

Observe that this is in exactly the same form as the scalar case. Indeed, we could have derived the solution in this special case by multiplying the original equation by an integrating factor $e^{-x\mathbf{A}}$.

6

Laplace Transform Methods

Laplace Transform is an important tool used to solve both linear ODEs and PDE. They are especially useful when parts of the problem are discontinuous or non-differentiable.

6.1 Review of Laplace Transforms

Laplace Transform

Given a real valued function $y(t)$, the **Laplace Transform** (LT) of y is defined by,

$$\mathcal{L}[y] = Y(s) = \int_0^\infty e^{-st} y(t)dt$$

for all s such that the above improper integral converges.

Example:

$$\mathcal{L}\left[e^{at}\right] = \ldots = \frac{1}{s-a}$$

Note that for the integrand in the first equation above to converge it is necessary that $s > a$.

Example:

If $f(t) = t^n$, $n = 0, 1, 2, \ldots$, then

$$\mathcal{L}[t^n] = \ldots = \frac{n!}{s^{n+1}}, \quad s > 0$$

This is computed through the use of integration by parts n times.

1. Linearity:
$$\mathcal{L}[c_1 f + c_2 g] = c_1 \mathcal{L}[f] + c_2 \mathcal{L}[g]$$
for any constants c_1, c_2 and real valued functions $f(t), g(t)$ whose transforms exist.

2. Existence: If $f(t)$ is piecewise continuous on each interval $[0, b]$ for $b > 0$ and there is a constant α such that $f(t) = O(e^{at})$ as $t \to +\infty$ then $f(t)$ is said to be of **exponetial order** α as $t \to +\infty$. In this case $F(s) = \mathcal{L}[f]$ exists for $s > \alpha$.

3. Differentiation Formulae:

   (a) If f is continuous and f' is piecewise continuous on any interval $[0, b], b > 0$, and f is of

exponential α as $t \to \infty$ then,

$$\mathcal{L}\left[f'\right] = s\mathcal{L}\left[f\right] - f(0), \quad \text{for } s > \alpha$$

(b) Replacing f by f' in the above yields that,

$$\mathcal{L}\left[f''\right] = s^2\mathcal{L}\left[f\right] - sf(0) - f'(0), \quad \text{for } s > \alpha$$

(c)

$$\mathcal{L}\left[f^{(n)}\right] = s^n\mathcal{L}\left[f\right] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) = F(s)$$

4. Shift theorems:

(a) First shift theorem: If $F(s) = \mathcal{L}\left[f\right]$ exists for $s > \alpha \geq 0$, then

$$\mathcal{L}\left[e^{ct}f\right] = F(s-c) \quad \text{for } s - c > \alpha$$

where c is a constant.

(b) Second shift theorem: If $F(s) = \mathcal{L}\left[f\right]$ exists for $s > \alpha \geq 0$, and c is a positive constant, then

$$\mathcal{L}\left[H(t-c)f(t-c)\right] = e^{-cs}F(s) \quad \text{for } s > \alpha$$

where $H(t)$ is the **Heaviside step function** defined by,

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Example:

We review how to use LTs to solve linear second order DEs with constant coefficients. Consider the IVP:

$$y'' + 3y' + 2y = e^x, \quad y(0) = 1, y'(0) = 2$$

Instead of using our standard methods, we take the LT of the DE and use x as our independent variable instead of t,

$$\mathcal{L}\left[y'' + 3y' + 2y\right] = \mathcal{L}\left[e^x\right]$$

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = \mathcal{L}\left[e^x\right]$$

$$(s^2 + 3s + 2)Y(s) - (s+5) = \frac{1}{s-1}$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)} = \frac{7}{2}\frac{1}{s+1} - \frac{8}{3}\frac{1}{s+2} + \frac{1}{6}\frac{1}{s-1}$$

Then

$$y(t) = \frac{7}{2}e^{-x} - \frac{8}{3}e^{-2x} + \frac{1}{6}e^x$$

The general case can be dealt with similarly:

$$y'' + py' + qy = u(t), \quad y(0) = y_0, y'(0) = v_0$$

Then

$$Y(s) = \frac{U(s)}{s^2 + ps + q} + \frac{(s+p)y_0 + v_0}{s^2 + ps + q}$$

where $U(s) = \mathcal{L}\left[u\right]$. The first term is due to the forcing and the second term is due to the initial conditions. For any particular values of p and q we can use the quadratic equation to factor the denominator and the second term can be inverted using partial fractions. This part of the solution by itself solves the homogeneous IVP:

$$y'' + py' + qy = 0, \quad y(0) = y_0, y'(0) = v_0$$

The second term is due to the forcing and thus yields a particular solution. To learn how to invert this term let's rewrite this part as,

$$Y_p(s) = G(s)U(s), \quad \text{where } G(s) = \frac{1}{s^2 + ps + q}$$

As we shall see, $G(s)$ is known as the **transfer function** associated with the homogeneous differential operator in our original problem. Now, we have reduced the problem to,

$$y_p(t) = \mathcal{L}^{-1}\left[G(s)U(s)\right]$$

that is finding the inverse LT of a product of LTs. You might want the inverse LT of the product of LTs to be the product of functions. However this is not the case, i.e.

$$\mathcal{L}\left[fg\right] \neq \mathcal{L}\left[f\right]\mathcal{L}\left[g\right]$$

6.2 Convolution Theorem

convolution

We say that h is the **convolution** of f and g, denoted as $f * g$ and defined as,

$$h(t) = (f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

Properties of the Convolution.

1. Commutativity: $(f * g)(t) = (g * f)(t)$
2. Distributivity: $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. Associativity: $(f * g) * h = f * (g * h)$
4. With zero is zero: $f * 0 = 0 * f = 0$

To prove the first property: make the substitution $u = t - \tau$

Theorem 6.1: Convolution Theorem

If $f(t)$ and $g(t)$ have LTs $F(s)$ and $G(s)$, respectively, for $s > a$, then if we define $H(s) = F(s)G(s)$ we have that

$$h(t) = \mathcal{L}^{-1}\left[H(s)\right] = (f * g)(t)$$

Proof:
See textbook. □

We now have a solid enough foundation to revisit our general problem with constant coefficients. Recall that the IVP is

$$y'' + py' + qy = u(t), \quad y(0) = y_0, y'(0) = v_0$$

After computing the LT of the equation we found that

$$\begin{aligned} Y(s) &= G(s)U(s) + \frac{(s+p)y_0 + v_0}{s^2 + ps + q} \\ &= Y_p(s) + Y_h(s) \end{aligned}$$

where $G(s) = \frac{1}{s^2+ps+q}$. Then y_h can be found by taking inverse transform and

$$y_p(t) = (g * u)(t)$$

where $g(t) = \mathcal{L}^{-1}[G(s)]$.

As an aside, it is worth mentioning that $y_h(t)$ solves the homogeneous DE with nonhomogeneous ICs whereas $y_p(t)$ solves the nonhomogeneous DE with homogeneous ICs,

$$y_p(0) = 0, \quad y_p'(0) = 0$$

Example:

Example 4.2.2 and 4.2.3.

6.3 Linear Systems of First Order DEs

Consider the following system of DEs:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t) \quad \text{with } \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{A}$ is time-invariant. We compute the LT of this vector equation,

$$\mathcal{L}[\mathbf{x}]' = \mathbf{A}\mathcal{L}[\mathbf{x}] + \mathcal{L}[\mathbf{f}(t)]$$

$$\vdots$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s)$$

The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is known as the **transfer function matrix**.

Theorem 6.2

For any constant $n \times n$ matrix $\mathbf{A}$,

$$\mathcal{L}\left[e^{t\mathbf{A}}\right] = (s\mathbf{I} - \mathbf{A})^{-1}$$

for values of s that satisfy $s > \text{Re}(\lambda)$, for all eigenvalues λ of $\mathbf{A}$.

Proof:

See textbook. □

Similarly, we can show

$$\mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s)\right] = \int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{f}(\tau)d\tau$$

Then we found the exactly same solution to the system of equations as we found in the previous chapter.

6.4 Dirac Delta Function

Suppose there is a radioactive compound in a container that decays at a rate of k if left to its own devices. If initially there is a concentration of x_0, we can describe the evolution of the compound using the following IVP:

$$\frac{dx}{dt} = -kx \qquad \text{with } x(0) = x_0$$

Integrate and get solution:

$$x(t) = x_0 e^{-kt}$$

The amount of the compound continues to decrease for all time but it never completely vanishes.

Now, let's further suppose that at an instant, $t = a$, someone adds A units of the compound to the container. Since we had a concentration of $x(a) = x_0 e^{-ka}$ before the addition, we must have $x_0 e^{-ka} + A$ after the addition. This is then a new initial condition and the solution for all subsequent times is,

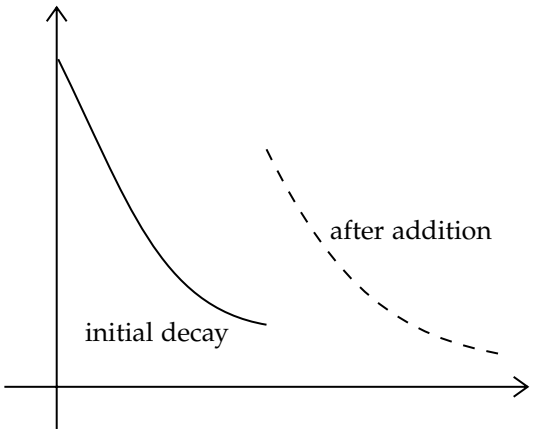
$$x(t) = (x_0 e^{-ka} + A)e^{-kt} \quad \text{for } k \geq a$$

Therefore, if we want to describe the evolution of our radioactive compound as a function of time we can write,

$$x(t) = \begin{cases} x_0 e^{-kt}, & 0 \leq t < a \\ \left(x_0 e^{-ka} + A\right) e^{-k(t-a)}, & t \geq a \end{cases}$$

The solution is discontinuous can be written using the *Heaviside Step Function* as

$$x(t) = x_0 e^{-kt} + A e^{-k(t-a)} H(t-a), \quad \text{for } t \geq 0$$



An interesting question is, what DE can we use to describe the solution above? To determine this we can differentiate the above equation but the problem is that the Heaviside function is not differentiable when its argument is zero. Regardless, we want to find the function, $f(t)$, such that,

$$\frac{dx}{dt} = -kx + f(t)$$

describes our solution. Physically, $f(t)$ describes the instantaneous addition of A units of the compound at time $t = a$. As we will show, this is done using the **Dirac delta function** and the answer to our problem will be $f(t) = A\delta(t-a)$.

This example is said to be forced impulsively because forcing acts in a way to make the solution discontinuous. A somewhat gentler approach is to say that the compound that we add is added gradually until we have reached an extra A units. If this is done over a span of Δ then the constant rate at which the compound is added is A/Δ. This makes the solution continuous but the rate of change is discontinuous. Indeed, the rate of change of the solution is

$$\frac{dx}{dt} = \begin{cases} -kx, & 0 \leq t < a \\ -kx + \frac{A}{\Delta}, & a \leq t \leq a + \Delta \\ -kx, & a + \Delta < t \end{cases}$$

Or this can be written as

$$\frac{dx}{dt} = -kx + f_\Delta(t)$$

with

$$f_\Delta(t) = \frac{A}{\Delta}[H(t-a) - H(t-(a+\Delta))]$$

This function is piecewise constant is if we were to plot it would look like a box function.

Even though the forcing in this equation is not continuous, it is piecewise continuous and thus its LT exists. We can then solve this problem by taking the LT of the equation. This can also be solved using an integrating factor but this is left as an exercise for the student to verify this result. We take the LT of the equation to obtain,

$$sX(s) - x_0 + kX(s) = F_\Delta(s)$$

We solve for $X(s)$,

$$X(s) = \frac{x_0}{s+k} + \frac{1}{x+k} F_\Delta(s)$$

If we use the fact $\mathcal{L}\left[e^{-kt}\right] = \dfrac{1}{s+k}$, then take inverse LT,

$$x(t) = x_0 e^{-kt} + f_\Delta * e^{-kt}$$

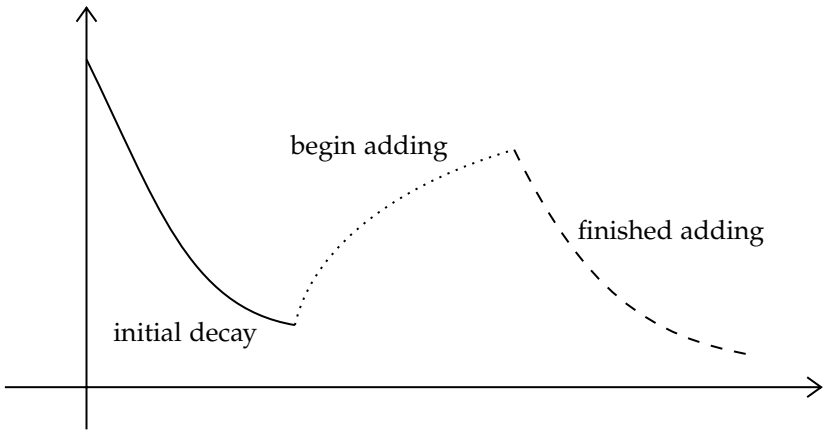
The first term is the homogeneous solution with the given initial condition and the second is the contribution due to the forcing. We can evaluate this convolution, first using the commutativity of this operator,

$$\begin{aligned}
f_\Delta * e^{-kt} &= \int_0^t f_\Delta(\tau) e^{-k(t-\tau)} d\tau && \text{plug in function} \\
&= \frac{A}{\Delta} \int_0^t [H(\tau - a) - H(\tau - (a+\Delta))] e^{-k(t-\tau)} d\tau && \text{evaluate Heaviside function} \\
&= \frac{A}{\Delta} \left[\int_0^t H(\tau - a) e^{-k(t-\tau)} d\tau - \int_0^t H(\tau - (a+\Delta)) e^{-k(t-\tau)} d\tau \right] \\
&\qquad \uparrow \qquad\qquad\qquad\qquad \uparrow \\
&\qquad \tfrac{1}{k}[1 - e^{k(a-t)}] \text{ for } t \geq a \qquad \tfrac{1}{k}[1 - e^{k(a+\Delta-t)}] \text{ for } t \geq a + \Delta
\end{aligned}$$

Therefore, our complete solution is:

$$x(t) = x_0 e^{-kt} + \frac{A}{k\Delta} \left[\left(1 - e^{-k(t-a)}\right) H(t-a) - \left(1 - e^{-k(t-(a+\Delta))}\right) H(t-(a+\Delta)) \right]$$

The picture below shows an example of how the concentration varies when the mass is introduce gradually.



With this expression we should be able to recover the first scenario we considered, where the mass was added impulsively, by taking the limit as $\Delta \to 0^+$: If we approximate this expression using Taylor series for $\Delta \ll 1$, we find,

$$x(a+\Delta) - x(a) \approx \frac{k\Delta}{Ak\Delta} = A$$

as we previously assumed.

Let's consider the forcing function $f_\Delta(t)$ that appears in the DE. By construction, the width of the forcing region is Δ and the height is A/Δ. Therefore, for any value of Δ the area under this curve is A and thus the amount of compound added is always A. The parameter Δ simply changes the rate at which it is added. As we will soon see, when we take the limit of this function as it becomes narrower and narrower, we obtain the famous **Dirac delta function**.

We begin by defining a function $I_\epsilon(t)$ for $\epsilon > 0$ such that,

$$I_\epsilon(t) = \begin{cases} 0 & 0 > t \\ \frac{1}{\epsilon} & 0 \leq t \leq \epsilon \\ 0, & t > \epsilon \end{cases}$$

Observe that this function looks very much like $f_\Delta(t)$ except the width and height are replaced by ϵ and $1/\epsilon$, respectively. Since the integral under the curve is $\epsilon/\epsilon = 1$, for any value of ϵ we know that,

$$\int_{-\infty}^{\infty} I_\epsilon(t)dt = 1, \quad \text{for all } \epsilon > 0$$

Furthermore, if we assume that $f(t)$ is a continuous function on the non- negative real line then,

$$\int_{-\infty}^{\infty} f(t)I_\epsilon(t)dt = \frac{1}{\epsilon}\int_0^\epsilon f(t)dt$$

for all values of ϵ, even ϵ very small. Furthermore, as we assume $f(t)$ is continuous, for all $\epsilon > 0$ there exists (by MVT) a $c_\epsilon \in [0, \epsilon]$ such that

$$\int_0^\epsilon f(t)dt = f(c_\epsilon)\epsilon$$

Thus

$$\int_0^\epsilon f(t)I_\epsilon(t)dt = f(c_\epsilon)$$

However, as $\epsilon \to 0, c_\epsilon \to 0$, because the interval is squeezed to the point 0 and we deduce that

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t)I_\epsilon(t)dt = f(0)$$

From this we extrapolate to realize that more generally, the following is true,

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t)I_\epsilon(t-a)dt = f(a)$$

The formal definition of the Dirac delta function of $I_\epsilon(t)$ as ϵ tends to zero,

Dirac delta function

$$\delta(t) \equiv \lim_{\epsilon\to 0} I_\epsilon(t)$$

We must be a bit careful because the Dirac delta function is not truly a function but rather a **generalized function**[1] or **distribution**[2] that is defined in terms of integrals over continuous functions. It yields the following property,

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

If the zero of the argument of δ is within the domain of the integration, the integral is equal to $f(a)$. If we were to change the bounds of integration to be over any interval that does not contain the origin we would get zero,

$$\int_c^d f(t)\delta(t-a)dt = 0 \quad \text{if } 0 \notin [c,d]$$

Conversely, if we integrate over any interval where zero is in the interior, we get the same value above,

$$\int_c^d f(t)\delta(t-a)dt = f(a) \quad \text{if } 0 \in (c,d)$$

Note:
some people take this integral property to be the definition but the two are equivalent.

Using this work we can easily compute the LT of the Dirac delta function,

$$\mathcal{L}\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa}, \quad \text{if } a \geq 0$$

[1] In mathematics, generalized functions are objects extending the notion of functions (wiki).
[2] Distributions, also known as Schwartz distributions or generalized functions, are objects that generalize the classical notion of functions in mathematical analysis (wiki).

With this knowledge, it is useful to revisit the problem that we originally posed where A units of a compound were added into our container. With out definition of $\delta(t)$ we see that we can reformulate our forcing function as,

$$f_\Delta(t) = AI_\Delta(t)$$

In the limit where the width of the forcing region tends to zero we obtain,

$$f(t) = A\delta(t-a)$$

This means that the DE that governs the evolution of the impulsively forced problem is,

$$\frac{dx}{dt} = -kx + A\delta(t-a)$$

To obtain the final solution we need to compute the convolution of the exponential and the forcing,

$$e^{-kt} * A\delta(t-a) = \int_0^t e^{-k(t-\tau)} A\delta(\tau - a)d\tau$$

The value of this integral depends on whether $a \in [0,t]$. If yes we have that the convolution is equal to $Ae^{-k(t-a)}$, otherwise, it is equal to zero. Therefore, we can evaluate this in terms of the Heaviside step function,

$$x(t) = x_0 e^{-kt} + Ae^{-k(t-a)}H(t-a)$$

which agrees with the solution that we solved piecewise.

Here, we constructed the Dirac delta function as a limit of a rectangular function whose width tends to zero and height tends to infinity. There is not the only choice one can make. Indeed, if you consider the Gaussian function, something that often arises in statistics and PDEs,

$$G_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

the coefficients are chosen such that the integral of the Gaussian over the entire real line for any σ is unit,

$$\int_{-\infty}^{\infty} G_\sigma(\tau)d\tau = 1$$

If you take the limit as $\sigma \to 0$ we see that the function gets narrower and taller and we get that,

$$\lim_{\sigma\to 0} \int_{-\infty}^{\infty} f(t)G_\sigma(t)dt = f(0)$$

6.5 Periodic Application of the Dirac delta function

Let's revisit out chemistry problem once again. Before, we had an initial concentration of radioactive material that decayed and we added a new amount, A, at time t, say. What if instead we add A units periodically at every T time units. We can describe this mathematically as,

$$\frac{dx}{dt} = -kx + f(t), \quad \text{where} \quad f(t) = \sum_{n=1}^{\infty} A\delta(t-nT)$$

To solve for the evolution of this equation we take the LT to obtain,

$$sX(s) - x_0 = -kX(s) + F(s) \quad \text{where} \quad F(s) = \mathcal{L}[f]$$

We can solve for $X(s)$ algebraically and obtain

$$X(s) = \frac{x_0}{k+s} + \frac{F(s)}{k+s}$$

Next, we take the inverse LT and requires we use the convolution theorem,

$$x(t) = x_0 e^{-kt} + (g * f)(t)$$

where $g(t) = \mathcal{L}^{-1}\left[\frac{1}{s+k}\right] = e^{-kt}$.

If we are looking for the long time behaviour than clearly the initial condition does not make any difference because $x_h(t) = x_0 e^{-kt} \to 0$ as $t \to \infty$.

It is only the forcing (that is the periodic addition) that contributes to the long time behaviour. To better understand this part of the solution we evaluate the particular solution,

$$\begin{aligned} x_p(t) = (g * f)(t) &= \int_0^t g(t-\tau) f(\tau) d\tau \\ &= A \sum_{n=1}^{\infty} \int_0^t e^{-k(t-\tau)} \delta(\tau - nT) d\tau \\ &= A e^{-kt} \sum_{n=1}^{\infty} \int_0^t e^{k\tau} \delta(\tau - nT) d\tau \\ &= A e^{-kt} \sum_{n=1}^{\infty} e^{nkT} \\ &= A e^{-kt} \left[e^{kT} + e^{2kT} + \cdots + e^{NkT} \right] \\ &= A e^{k(T-t)} \left[1 + e^{kT} + \cdots + e^{(N-1)kT} \right] \end{aligned}$$

If we say that $t = NT + u$ where $0 \leq u < T$ we can rewrite our particular solution as

$$\begin{aligned} x_p(t) &= A e^{k(-(N-1)T-u)} \left[1 + e^{kT} + \cdots + e^{(N-1)kT} \right] \\ &= A e^{-ku} \left[1 + e^{-kT} + \cdots + e^{-(N-1)kT} \right] \end{aligned}$$

This is a geometric series that we can sum up in the limit as $N \to \infty$. Therefore, in the limit as $t \to \infty$ we have that $N \to \infty$ and thus we get the long time behaviour of

$$x_p \to \frac{A e^{-ku}}{1 - e^{-kT}} \quad \text{as} \quad t \to \infty$$

7

Perturbation Theory

Introduction

It's been remarked many times that the exact solutions of differential equations in terms of elementary functions are very rare, even for linear ODEs, such solutions being limited to some special cases. In Chapter 4, I've shown you a few examples of how, nonetheless, we can extract important information from the DE itself - the so-called qualitative behaviour of the solution - and in Chapter 5 we have seen that the local (i.e. in a neighbourhood) behaviour of the solution can be obtained by power series of their generalization (Frobenius series).

The fact remains, however, that real-life application of mathematics requires explicit solutions which are amenable to numerical computations and comparison with experiments. In such cases one must either make recourse to numerical integration, or to approximate analytic techniques. The latter methods have been well-developed over the past 50 years, and are known collectively as Perturbation Theory of DEs.

We can see the basic idea behind perturbation theory in the following example. Consider the problem

$$y'' - \left(1 + \frac{\epsilon}{1+x^2}\right) y = 0, \quad 0 \leq x < \infty \tag{7.1}$$

$$y(0) = 1, y'(0) = 0, \quad \epsilon \geq 0 \tag{7.2}$$

The ODE is linear, but no closed form solution in terms of elementary functions exists, except for when $\epsilon = 0$. If we just wanted a solution in the neighbourhood of $x = 0$ we could use power series of the type already studied, i.e. $y(x) = \sum a_n x^n$ you can check for yourselves that one gets

$$y(x) = 1 + \frac{(1+\varepsilon)}{2!}x^2 + \frac{(1+\varepsilon^2)}{4!}x^4 + \ldots \tag{7.3}$$

However, as we know already, when such series converge, they do so over a limited interval, so we see that power series solutions of this type do not lead to **global solution**s (i.e. are not valid for all values of x in the given interval).

On the other hand, the DE Eq. (7.1) has a parameter in it, namely, ϵ. So we ask whether it could be possible to take advantage of it to obtain approximate solutions which would be globally valid. In application, the DEs have specific physical meaning; so the parameter ϵ is usually a small parameter that is dimensionless. The idea is then to seek a series solution in powers of ϵ:

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \ldots = \sum_{n=0}^{\infty} \varepsilon^n y_n(x) \tag{7.4}$$

and use the Eq. (7.1) to determine $y_n(x)$.

To see how the method works, sub Eq. (7.4) into Eq. (7.1):

$$y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \ldots - \left(1 + \frac{\varepsilon}{1+x^2}\right)\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots\right) = 0$$

and collect the terms with the same powers of ϵ to get

$$(y_0'' - y_0) + \varepsilon\left(y_1'' - y_1 - \frac{y_0}{1+x^2}\right) + \varepsilon^2\left(y_2'' - y_2 - \frac{y_1}{1+x^2}\right) + \ldots + \varepsilon^n\left(y_n'' - y_n - \frac{y_{n-1}}{1+x^2}\right) + \ldots = 0$$

Now this must be an identity in ϵ, and hence the coefficients of the various powers of ϵ must vanish, giving

$$y_0''(x) - y_0(x) = 0 \tag{7.5}$$

$$y_1''(x) - y_1(x) = \frac{y_0(x)}{1+x^2} \tag{7.6}$$

$$\vdots$$

See what happened? The first DE Eq. (7.5) can be easily solved to give $y_0(x) = c_1 e^x + c_2 e^{-x}$. This is called the **leading order solution** (or zeroth order). Next, since $y_0(x)$ is known, the right-hand side of Eq. (7.6) is known, hence the DE Eq. (7.6) for y_1 is a simple inhomogeneous 2nd order DE with constant coefficients which we can solve. Thus the perturbation solution Eq. (7.4) leads to an infinite sequence of simpler DEs which can be solved in succession. The initial conditions Eq. (7.2) may be treated the same way, namely by substitution of the series Eq. (7.4) into them:

$$1 = y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \cdots \tag{7.7}$$

$$0 = y_0'(0) + \epsilon y_1'(0) + \epsilon^2 y_2'(0) + \cdots \tag{7.8}$$

Therefore

$$y_0(0) = 1 \text{ and } y_0'(0) = 0, \text{ which go with Eq. (7.5)}$$

$$y_1(0) = 0 \text{ and } y_1'(0) = 0, \text{ which go with Eq. (7.6)}$$

$$\ldots \text{ etc.}$$

Note:
The perturbation series Eq. (7.4) is local in ϵ, i.e. is meant to be valid in the neighbourhood of $\epsilon = 0$, but it is global in x, since no restriction on x being small have been imposed.

As said before, ϵ is usually a parameter containing physical quantities, and can be arranged to be very small, in this case we really don't need the entire perturbation series, but expect the solution to be well-approximated by the first two or three terms of the perturbation series.

The perturbation methods alluded to above are not only very powerful, but also very general, in that they apply to all sorts of equations: algebraic, differential, integral-differential, and so on.

7.1 Regular Perturbation

One may wonder what kind of error one makes in approximating the solutions by the first couple of terms of the perturbation series. To analyze the question better, let's consider a simple problem where we can compare the perturbation series with the exact solution.

Example:
Find an approximate solution of the following IVP:

$$y' = 1 + (1+\epsilon)y^2, \epsilon \geq 0 \qquad y(0) = 0 \tag{7.9}$$

We set $y(x) = \sum_{n=0}^{\infty} y_n(x)\epsilon^n$. Sub into the DE and equate like powers of ϵ.

$$\sum_{n=0}^{\infty} y_n'(x)\epsilon^n = 1 + (1+\epsilon)\left(\sum_{n=0}^{\infty} y_n\epsilon^n\right)^2 = 1 + (1+\epsilon)\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} y_{n-k}y_k\right)\epsilon^n$$

Then we have

$$\begin{aligned} y_0' &= 1 + y_0^2 && \text{(zeroth order)} \\ y_1' &= 2y_0y_1 + y_0^2 && \text{(first order)} \\ y_2' &= y_1^2 + 2y_0y_2 + 2y_0y_1 && \text{(second order)} \end{aligned}$$

The initial condition gives

$$0 = y_0(0) + \epsilon y_1(0) + \epsilon^2 y_n(0) + \dots$$

Then $y_n(0) = 0$ for $n = 0, 1, 2, \dots$.

The leading order problem is given by

$$\frac{dy_0}{dx} = 1 + y_0^2, y_0(0) = 0 \Rightarrow y_0(x) = \tan x$$

which is the same as taking $\epsilon = 0$ in the original problem. The known o-th order solution is sub into 1st order equation, which becomes

$$y_1' - (2\tan x)y_1 = \tan^2 x, \quad y_1(0) = 0$$

The integrating factor is $\cos^2 x$. Then the general solution is

$$y_1(x) = c \cdot \cos^2 x + \cos^2 x \int \tan^2 x \sec^2 x dx$$

IC gives $c = 0$, and so

$$y_1(x) = \frac{1}{2}(x\sec^2 x - \tan x)$$

Hence, to order 1,

$$y(x) = \tan x + \frac{\epsilon}{2}(x\sec^2 x - \tan x) + \dots \qquad (*)$$

Higher order terms can be computed in a similar way. Next notice that, although DE is nonlinear, it is separable, and it is not too difficult to find its exact solution, which turns out to be given by

$$y_e(x) = \frac{1}{\sqrt{1+\epsilon}} \tan \sqrt{1+\epsilon}x$$

In order to compare our approximate solution Eq. (*) with this we must expand the exact solution in a series on powers of ϵ.

$$y(x) = y_e(x)|_{\varepsilon=0} + \frac{d}{d\varepsilon}y_e(x)\bigg|_{\varepsilon=0}\varepsilon + \cdots$$

We have

$$y(x) = \tan x + \frac{\epsilon}{2}(x\sec^2 x - \tan x) + \cdots$$

which is exactly Eq. (*).

HOw about the error?

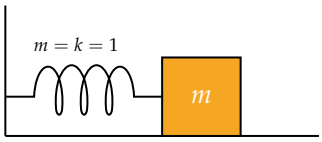
Direct substitution gets very accurate approximation

$$\begin{aligned} y_{\text{exact}} &= \frac{1}{\sqrt{1+\varepsilon}} \tan \sqrt{1+\varepsilon}x = \tan x + \frac{\varepsilon}{2}\left(x\sec^2 x - \tan x\right) + \cdots \\ y_{\text{per}} &= \tan x + \frac{\varepsilon}{2}\left(x\sec^2 x - \tan x\right) + O(\varepsilon^2) \end{aligned}$$

Here are some examples that show how to identify the parameter ϵ.

1. Equation contains a small parameter ϵ

Example: Damped Oscillator

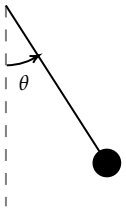


$$x'' + \epsilon x' + x = 0, \quad x(0) = a, x'(0) = 0$$

where $\epsilon x'$ is the friction term, and ϵ is small.

2. Equation does not contain a small parameter ϵ

Example: Pendulum Equation



$$\frac{d^2\theta}{d\tau^2} + \omega_0^2 \sin\theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = 0, \quad \theta_0 \text{ is small}$$

where $\omega_0^2 = \frac{g}{\ell}$, g acceleration of gravity, ℓ the length of pendulum. We introduce a small parameter $\epsilon = \theta_0$ in the equation by letting $\varphi = \frac{\theta}{\epsilon}$, $t = \omega_0 \tau$.

$$\frac{d\theta}{dt} = \varepsilon \frac{d\varphi}{dt}\frac{dt}{d\tau} = \varepsilon\omega_0 \frac{d\varphi}{dt}$$
$$\frac{d^2\theta}{d\tau^2} = \varepsilon\omega_0 \frac{d^2\varphi}{dt^2}\frac{dt}{d\tau} = \varepsilon\omega_0^2 \frac{d^2\varphi}{dt^2}$$

Thus

$$\varepsilon\omega_0^2 \frac{d^2\varphi}{dt^2} + \omega_0^2 \sin\varepsilon\varphi = 0, \quad \varphi(0) = \frac{\theta(0)}{\varepsilon} = \frac{\theta_0}{\varepsilon} = 1, \varphi'(0) = 0$$
$$\frac{d^2\varphi}{dt^2} + \frac{1}{\varepsilon}\sin\varepsilon\varphi = 0, \quad \varphi(0) = 1, \varphi'(0) = 0$$

Perturbation methods take advantage of a small parameter ϵ and look for solution of the form

$$x(t, \delta\epsilon) = \sum_{n=0}^{\infty} x_n(t) = \varepsilon^n$$

How to identify ε (which usually has physical meaning) in the examples above:

1. $\epsilon > 0$ measures the viscosity and is normally very small, i.e. $0 < \varepsilon < 1$.
2. Not small.

The above method, which is called regular perturbation theory, seems so simple and easy in Eq. (7.9). Does it work for all other problems? How do we estimate the error if we don't know the exact solution?

> Remark:
> In real-life problems we never know the exact solution, and the problem of estimating the error made in the perturbation approximations is a delicate one. In general, it requires much more analysis than we have time for, and I must refer you to more advanced courses on a branch of Analysis called Asymptotic Analysis.

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