T4.

a) $\xi \sim \rho(x, \theta) = \frac{1}{\theta}\{[\theta ; 2\theta]\}$ $\theta \in [\theta ; 2\theta]$

$\vec{x}_n$ - выборка $F(x) = \frac{x}{\theta} - 1$

Метод моментов:

$\alpha_k = M[\xi^k] = \int\limits_{-\infty}^{\infty} x^k \rho(x, \theta)\, dx$ $\tilde{\alpha}_k = \frac{1}{n}\sum\limits_{i=1}^{n} x_i^k$

$M[\xi] = \int\limits_{\theta}^{2\theta} x \frac{1}{\theta}\, dx = \frac{3}{2}\theta$ $\tilde{\alpha}_1 = \bar{x}$ $\frac{3}{2}\theta = \bar{x} \Rightarrow \hat{\theta}_1 = \frac{2}{3}\bar{x}$

Метод максимального правдоподобия:

$L(x, \theta) = \prod\limits_{i=1}^{n} \rho(x_i : \theta) = \left(\frac{1}{\theta}\right)^n$ $\ln L = n \ln(\theta)$

$\frac{\partial \ln L}{\partial \theta} = \frac{-n}{\theta}$ sup на $[\theta : 2\theta]$ при $\theta \to \theta_{max}$

$\begin{cases} 2\theta \geq x_{max} \\ \theta \geq x_{max} \end{cases} \rightarrow \frac{1}{2} x_{max} \leq \theta \leq x_{min} \longrightarrow$ sup при $\hat{\theta}_2 = \frac{1}{2} x_{max}$

b) $M[\bar{\theta}_1] = M\left[\frac{2}{3n}\sum\limits_{i=1}^{n} x_i\right] = \frac{2}{3} M\xi = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta$ - несмещ

$D[\theta_1] = \frac{4}{9n^2}\sum D\xi$ $M[\xi^2] = \int\limits_{\theta}^{2\theta} x^2 \frac{1}{\theta}\, dx = \frac{7\theta^2}{3}$

$D\xi = \left(\frac{7}{3}\theta^2 - \frac{9}{4}\theta^2\right) n = \frac{\theta^2}{12} n$

$D[\theta_1] = \frac{4}{9n^2}\cdot n \cdot \frac{\theta^2}{12} = \frac{\theta^2}{27n} \xrightarrow[n \to \infty]{} 0$ - состоятел

$M[\bar{\theta}_2] = M\left[\frac{1}{2} x_{max}\right] = \frac{1}{2}\int\limits^{2\theta} x \cdot n \left(\frac{x}{\theta} - 1\right)^{n-1} \frac{1}{\theta}\, dx =$

$= \left| t = \frac{x}{\theta} - 1 \right| = \theta \int\limits_{t_1}^{t_2} n (t+1) \cdot t^{n-1}\, dt = \frac{2n+1}{2n+2}\theta$

$q(y) = n(F(y))^{n-1} F'(y) = n\left(\frac{x}{\theta} - 1\right)^{n-1}\frac{1}{\theta}$ смещенная

$$\tilde{\Theta}^* = \frac{1}{2} x_{max} \cdot \frac{2n+2}{2n+1} = \frac{n+1}{2n+1} x_{max}$$ – несмещенная

$$M[x_{max}] = \int_{\Theta}^{2\Theta} x n \left(\frac{x}{\Theta} - 1\right)^{n-1} \frac{1}{\Theta} dx = \frac{2n+1}{n+1} \Theta$$

$$M[x_{max}^2] = \int_{0}^{2\Theta} x^2 n \left(\frac{x}{\Theta} - 1\right)^{n-1} \frac{1}{\Theta} dx = \frac{n(4n^2+8n+2)\Theta^2}{n(n+1)(n+2)}$$

$$D[\Theta_2^*] = \frac{(n+1)^2}{(2n+1)^2} \cdot D[x_{max}] = \frac{(n+1)^2}{(2n+1)^2} \left[\frac{4n^2+8n+2}{(n+1)(n+2)} - \frac{(2n+1)^2}{(n+1)^2}\right] \Theta^2$$

$$= \frac{\Theta^2 n}{4n^3+12n^2+9n+2} = \frac{\Theta^2 n}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$ Θ_2^* – состоят.

c) $I(\Theta) = M\left[\left(\frac{1}{\Theta}\right)^2\right] = \frac{1}{\Theta^2} > 0 \quad \forall \Theta \in [\Theta; 2\Theta]$

$g'(\Theta) = \frac{-1}{\Theta^2}$ $\quad$ f. непр. дифф по Θ на Θ

$$0 = \int_{\Theta}^{2\Theta} \frac{\partial}{\partial\Theta} f(x, \Theta) dx = \int_{\Theta}^{2\Theta} -\frac{1}{\Theta^2} dx = -\frac{1}{\Theta} \neq 0$$ модель нерегулярна

$$n D[\tilde{\Theta}_1] = \frac{\Theta^2}{27} \xrightarrow[n\to\infty]{} \frac{\Theta^2}{27}$$

$$n D[\tilde{\Theta}_2^*] = \frac{\Theta^2 n^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

d) $P(t_1 < f(x_n; \Theta) < t_2) = \beta$

$t_2 - t_1 \to \min$

$f(x_n, \Theta) = \frac{x_{max}}{\Theta}$ $\quad$ с $F_f(x) = (F(\Theta; x))^n = (x-1)^n$

$t_1 = q_{\frac{1-\beta}{2}}$ $\quad$ $t_2 = q_{\frac{1+\beta}{2}}$ $\quad$ $t_1 < \frac{x_{max}}{\Theta} < t_2$

$$\frac{x_{max}}{t_1} < \theta < \frac{x_{max}}{t_2} \qquad t_2 = \sqrt[n]{\frac{1+\beta}{2}} + 1 \qquad t_1 = \sqrt[n]{\frac{1-\beta}{2}} + 1$$

$t_2 - \theta < t_1$ — точный довер. интер.

г) По ОМП не знаем (модель не регулярна)

Ищем по ОММ

$$f(\tilde{\alpha}) = \tilde{\theta} = \frac{2}{3}\bar{x} \qquad \nabla f = \frac{2}{3}$$

$$\theta^2 = \sqrt{n}\,\frac{g(\tilde{\alpha}) - g(\alpha)}{\sigma(\alpha)} \sim N(0,1)$$

$$f(\alpha) = \theta = \frac{2}{3}\alpha_1 \quad \alpha_1 = \bar{x} \quad \alpha_2 = \overline{x^2}$$

$$K = \overline{x^2} - \bar{x}^2 \qquad \sigma(\alpha) = \frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2}$$

$$\frac{\tilde{\theta} - \theta}{\frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2}} \sim N(0;1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad u_{\frac{1+\beta}{2}} = -u_{\frac{1-\beta}{2}}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u_{\frac{1-\beta}{2}}} e^{-\frac{x^2}{2}}\,dx = \frac{1-\beta}{2} = \frac{1}{2}\left(\operatorname{erf}\left(\frac{u_2}{u_1}\right) + 1\right) = \frac{1-\beta}{2}$$

$$\operatorname{erf}\left(\frac{u_{\frac{1-\beta}{2}}}{\sqrt{2}}\right) = -\beta \longrightarrow \operatorname{erf}^{-1}(-\beta) = \frac{u_{\frac{1-\beta}{2}}}{\sqrt{2}}$$

$$-\sqrt{2}\operatorname{erf}^{-1}(-\beta) > \sqrt{n}\,\frac{\tilde{\theta} - \theta}{\frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2}} > \sqrt{2}\operatorname{erf}^{-1}(-\beta) \qquad \tilde{\theta} = \frac{2}{3}\bar{x}$$

$$-\frac{1}{\sqrt{n}}\operatorname{erf}^{-1}(-\beta)\cdot\frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2} + \frac{2}{3}\bar{x} > \theta > \frac{\sqrt{2}}{\sqrt{n}}\operatorname{erf}^{-1}(-\beta)\cdot\frac{2}{3}\sqrt{\overline{x^2} - \bar{x}^2} + \frac{2}{3}\bar{x}$$

асимпт. довер. интервал.

T5.

а) Метод МП

$$L(x,\theta) = \prod_{i=1}^{n} p(x,\theta) = \frac{(\theta-1)^n}{\prod_{i=1}^{n} x_i^{\theta}} \qquad \ln L = n\ln(\theta-1) - \theta\sum_{i=1}^{n}\ln x_i$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{n}{\theta-1} - \sum_{i=1}^{n}\ln x_i = 0 \qquad \sum_{i=1}^{n}\ln x_i = \frac{n}{\theta-1}$$

$$\tilde{\theta} = \frac{n}{\sum_{i=1}^{n}\ln x_i} + 1$$

б) med - медиана

$$\int_{-\infty}^{med} p(x)\,dx = \frac{1}{2} \qquad \int_{1}^{med}(\theta-1)\frac{1}{x^{\theta}}\,dx = -x^{-\theta+1}\Big|_{1}^{med} = \frac{1}{2}$$

$$med = 2^{\frac{1}{\theta-1}} \qquad \frac{f(\tilde{\theta}) - f(\theta)}{G(\theta)}\sqrt{n} \rightsquigarrow N(0,1)$$

$$f(\theta) = 2^{\frac{1}{\theta-1}} \to \nabla f = 2^{\frac{1}{\theta-1}}\ln 2\cdot\left(\frac{1}{(\theta-1)^2}\right)$$

$$f(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}} \qquad \tilde{\theta} = \frac{n}{\sum_{i=1}^{n}\ln x_i} + 1$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = \int_1^{\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2\frac{\theta-1}{x^{\theta}}\,dx = \frac{1}{(\theta-1)^2}$$

$$G(\theta) = \left[\frac{1}{2^{\theta-1}}\ln 2\cdot\left(\frac{1}{(\tilde{\theta}-1)^2}\right)\cdot(\tilde{\theta}-1)^2\cdot 2^{\frac{1}{\tilde{\theta}-1}}\ln 2\cdot\frac{1}{(\tilde{\theta}-1)^2}\right]^{1/2} =$$

$$= 2^{\frac{1}{\tilde{\theta}-1}}\ln 2\cdot\frac{1}{(\tilde{\theta}-1)^2}$$

$$-\sqrt{2}\,\mathrm{erf}^{-1}(-\beta) > \frac{2^{\frac{1}{\theta-1}} - med}{2^{\frac{1}{\theta-1}} \ln 2 \, \frac{1}{(\theta-1)^2}} \sqrt{n} > \sqrt{2}\,\mathrm{erf}^{-1}(-\beta)$$

$$\frac{\sqrt{2}}{\sqrt{n}} \mathrm{erf}^{-1}(-\beta) \cdot 2^{\frac{1}{\hat\theta-1}} \cdot \ln 2 \cdot \frac{1}{(\hat\theta-1)^2} = EP$$

$$EP + \frac{1}{2^{\hat\theta-1}} < med < -EP + \frac{1}{2^{\hat\theta-1}} \qquad \text{где } \hat\theta = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

Проверяем, что модель регулярна: ρ непр. дифф. по θ в Θ

$$0 = \int_1^{\infty} \frac{\partial}{\partial\theta} \rho(x,\theta)\, dx = \int_1^{\infty} \frac{x^\theta - x^\theta \ln x\,(\theta-1)}{x^{2\theta}}\, dx = x^{1-\theta} \ln \Big|_1^{\infty} = 0$$

модель регулярна, все хорошо!

c) По ОМП $\quad f(\theta) = \theta \quad \nabla f = 1$

$$f(\hat\theta) = \hat\theta \qquad I^{-1} = (\theta-1)^2 \qquad G = \theta - 1$$

$$G(\hat\theta) = \hat\theta - 1$$

$$-\sqrt{2}\,\mathrm{erf}^{-1}(-\beta) > \frac{\hat\theta - \theta}{\hat\theta - 1} \sqrt{n} > \sqrt{2}\,\mathrm{erf}^{-1}(-\beta)$$

$$\frac{\sqrt{2}}{\sqrt{n}} (\hat\theta - 1)\,\mathrm{erf}^{-1}(-\beta) + \hat\theta < \theta < -\frac{\sqrt{2}}{\sqrt{n}}\,\mathrm{erf}^{-1}(-\beta)(\hat\theta - 1) + \hat\theta,$$

где $\hat\theta = \dfrac{n}{\sum_{i=1}^{n} \ln x_i} + 1$

Асимп. дов. интер. по ОМП

По ОММ

оценка $\alpha_1 = M\xi = \int\limits^{\infty} x \frac{\theta - 1}{x^\theta} dx = \frac{\theta - 1}{\theta - 2}$ для $\theta > 2$

$\tilde{\alpha}_1 = \bar{x}$ $\quad 1 + \frac{1}{\theta - 2} = \tilde{x}^1 \to \tilde{\theta} = \frac{1}{\bar{x} - 1} + 2$ при $\theta > 2$

$f(\tilde{\alpha}_1) = \tilde{\theta} = \frac{1}{\bar{x} - 1} + 2 \qquad \nabla f(\tilde{\alpha}_1) = \frac{-1}{(\bar{x} - 1)^2}$; $k = \tilde{\alpha}_2 - \tilde{\alpha}_1^2 = \overline{x^2} - \bar{x}^2$

$$-\sqrt{2}\, \mathrm{erf}^{-1}(-\beta) > \frac{\frac{1}{\bar{x}-1} + 2 - \theta}{\frac{1}{(\bar{x}-1)^2}\sqrt{(\overline{x^2} - \bar{x}^2)}} \sqrt{n} > \sqrt{2}\, \mathrm{erf}^{-1}(-\beta)$$

$$\frac{\sqrt{2}}{\sqrt{n}} \mathrm{erf}^{-1}(-\beta) \frac{\sqrt{\overline{x^2} - \bar{x}^2}}{(\bar{x} - 1)^2} + \frac{1}{\bar{x} - 1} + 2 < \theta < -\frac{\sqrt{2}}{\sqrt{n}} \mathrm{erf}^{-1}(-\beta) \frac{\sqrt{\overline{x^2} - \bar{x}^2}}{(\bar{x} - 1)^2} + \frac{1}{\bar{x} - 1} + 2$$

асимп. дов. интер. по ОММ.

Проверим, что модель сильно регулярна

$$0 = \frac{\partial^2}{\partial \theta^2} \int\limits_1^{\infty} \rho(x, \theta)\, dx = \int\limits_1^{\infty} \frac{\partial^2}{\partial \theta^2} \rho(x, \theta) = \int\limits_1^{\infty} \frac{\ln x((\theta - 1)\ln x - 2)}{x^\theta} = 0$$

все работает!