T10 $H_0 \sim p_0(x) = 1\{(0;1)\}$

$H_1 \sim p_1(x) = \dfrac{e^{1-x}}{e-1}\{(0;1)\}$

a) $n=1$ α

$$l = \frac{L_1}{L_0} = \frac{p_1(x)}{p_0(x)} = \frac{e^{1-x}}{(e-1)1} \geq C$$ Теорема Неймана - Пирсона

Разрешаем относительно x

$1-x \geq \ln(C)(e-1)$

$x \leq \underbrace{1-(e-1)\ln C}_{A}$

$G_{кр}: x \leq A$

$P(x \leq A \mid H_0) = \alpha$

$\int_0^A p_0(x)\,dx = \alpha$ $\alpha = A$ $G_{крит}: x \leq \alpha$

$\alpha_1 = \alpha$ - ошибка 1-го рода

Ищем мощность:

$$W = P(x \leq \overset{\alpha}{\overset{\shortparallel}{A}} \mid H_1) = \int_0^\alpha p_1(x)\,dx = \int_0^\alpha \frac{e^{1-x}}{e-1}dx = -\frac{e^{1-x}}{e-1}\Big|_0^\alpha = \frac{e}{e-1}(1-e^{-\alpha})$$

$\alpha_2 = 1 - W$

b) $n=2$

$$l = \frac{L_1}{L_0} = \frac{p_1(x_1)p_1(x_2)}{p_0(x_1)p_0(x_2)} = \frac{e^{1-x_1}e^{1-x_2}}{(e-1)^2\,1\cdot 1} \geq C$$

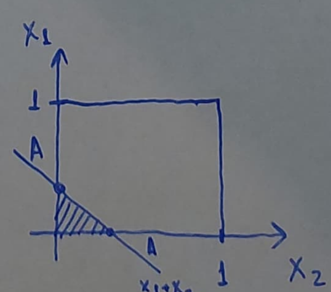
Опять разрешаем отн. x_1 и x_2

$$e^{-x_1-x_2} \geq \frac{C(e-1)^2}{e^2}$$

$-x_1 - x_2 \geq \underbrace{\ln(\ldots)}_{A}$

$G_{крит}: x_1 + x_2 \leq A$

Находим A: $P(x_1 + x_2 \leq A \mid H_0) \leq \alpha$



$$P(x_1+x_2 \leq A \mid H_0) = \iint\limits_{x_1+x_2 \leq A} 1\cdot 1 \cdot dx_1 dx_2 = \alpha$$

$\frac{1}{2}A^2 = \alpha$

$A = \sqrt{2\alpha}$

$$W = P(x_1 + x_2 \le A \mid H_1) = \iint\limits_{x_1+x_2\le A} \frac{e^{1-x_1} e^{1-x_2}}{(e-1)^2} dx_1 dx_2 = \frac{e^2}{(e-1)^2} \int_0^A dx_1 \int_0^{A-x} e^{-x_1} e^{-x_2} dx_2 =$$

$$= \frac{e^2}{(e-1)^2} \int_0^A e^{-x_1} (1 - e^{-A+x_1}) dx_1 = \frac{e^2}{(e-1)^2} \left[1 - e^{-A} - e^{-A} A\right]$$

можно подставить $A = \sqrt{2\alpha}$

$$\alpha_2 = 1 - W$$

c) Ассимпт. критерий

$$l = \frac{L_1}{L_0} = \frac{\prod p_1(x_i)}{\prod p_0(x_i)} = \prod \frac{p_1(x_i)}{p_0(x_i)} \ge C$$

$$\sum_{i=1}^n \underbrace{\ln \frac{p_1(x_i)}{p_0(x_i)}}_{\eta_i} \ge \ln C \qquad \text{ЦПТ} \quad \frac{\sum \eta_i - n M\eta_i}{\sqrt{n D\eta_i}} \rightsquigarrow N(0;1)$$

$$\eta_i = \ln \frac{e^{1-x_i}}{e-1} = \ln \frac{e}{e-1} - x_i$$

$$H_0: \ M\eta_i = M\left[\ln\frac{e}{e-1} - x_i\right] = \ln\frac{e}{e-1} - \frac{1}{2}$$

$$D\eta_i = D\left[\ln\frac{e}{e-1} - x_i\right] = Dx_i = \frac{1}{12}$$

$$P(\ln l \ge \ln C \mid H_0) = \alpha$$

$$P\left(\sum \eta_i \ge \ln C \mid H_0\right) = P\left(\underbrace{\frac{\sum\eta_i - n\left(\ln\frac{e}{e-1} - \frac{1}{2}\right)}{\sqrt{n\,^1/_{12}}}}_{=\ \varkappa\ \rightsquigarrow\ N(0;1)} \ge \underbrace{\frac{\ln C - n\left(\ln\frac{e}{e-1} - ^1/_2\right)}{\sqrt{^n/_{12}}}}_{=\ A}\right)$$

$$\int_A^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \alpha \qquad A = U_{1-\alpha}$$

Найдем $G_{кр}$.

$$\ln l \ge \ln C$$

$$\sum \eta_i = n\ln\frac{e}{e-1} - \sum x_i$$

$$\frac{\ln C - n\left(\ln\frac{e}{e-1}\right) + ^n/_2}{\sqrt{^n/_{12}}} = U_{1-\alpha}$$

$$\ln C = n\ln\frac{e}{e-1} - \frac{n}{2} + U_{1-\alpha}\sqrt{^n/_{12}}$$

$$n\ln\frac{e}{e-1} - \sum x_i \ge n\ln\frac{e}{e-1} - \frac{n}{2} + U_{1-\alpha}\sqrt{\frac{n}{12}}$$

$G_{кр}: \ \bar{x} \le \frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}}$

$W = P(\ln \ell \ge \ln C \mid H_0)$

$\ln \ell = \sum \eta_i$

Воспользуемся ЦПТ:

$$W = P\left(\frac{\sum \eta_i - nM\eta_i}{\sqrt{nD\eta_i}} \ge \underbrace{\frac{\ln C - nM\eta_i}{\sqrt{nD\eta_i}}}_{B}\right) = \int_B^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$

$$H_1: \ M\eta_i = \ln\frac{e}{e-1} - Mx_i = \ln\frac{e}{e-1} - \int_0^1 \frac{e^{1-x}}{e-1}\,dx = \ln\frac{e}{e-1} - \frac{e-2}{e-1}$$

$$D\eta_i = \frac{e^2-3e+1}{(e-1)^2}$$

$$G_{кр}: \ \bar{x} \le \frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}}$$

$\alpha_2 = 1 - W$

$\alpha_1 = \alpha$

$$B = \frac{n\overbrace{\left(\frac{e-2}{e-1} - \frac{1}{2}\right)}^{-0{,}082} + u_{1-\alpha}\sqrt{\frac{n}{12}}}{\sqrt{n\frac{e^2-3e+1}{(e-1)^2}}} = E + \overset{<0}{F}\sqrt{n} \quad \Rightarrow B \xrightarrow[n\to+\infty]{} -\infty$$

E, F

$$W = \int_B^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \xrightarrow[n\to\infty]{} \int_{-\infty}^{\infty} e^{-x^2/2}\frac{1}{\sqrt{2\pi}} = 1$$

d) $G_{кр}: x_{min} < C$ $\qquad \alpha$ - уровень значимости

$\alpha_1 = P(\vec{x}_n \in G_{кр} \mid H_0) = P(x_{min} < C \mid H_0) = \alpha$

$x_{min} \sim 1 - (1 - F(x))^n$

$H_0: \xi \sim p_0(x) = 1\{(0;1)\}$ $\qquad \xi \sim F_0(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$

$\alpha = P(x_{min} < C \mid H_0) = 1 - (1 - C)^n \Rightarrow C = 1 - \sqrt[n]{1-\alpha}$

$G_{кр}: \ x_{min} \le 1 - \sqrt[n]{1-\alpha}$

$\alpha_1 = \alpha$

$W = P(\vec{x}_n \in G_{кр} \mid H_1) = P(x_{min} < C \mid H_1) = 1 - (1 - F_1(x))^n$

$H_1: \ \xi \sim p_1(x) = \frac{e^{1-x}}{e-1}\{(0;1)\}$

$$F_1(x) = \int_{-\infty}^{x} p_1(t)\,dt = \frac{e}{e-1}\int_0^x e^{-t}\,dt = \frac{e}{e-1}(1-e^{-x})$$

$$W = 1-(1-F_1(c))^n = 1-\left(1-\frac{e}{e-1}(1-e^{-c})\right)^n = 1-\left(\frac{-1}{e-1}+\frac{e^{\sqrt[n]{1-\alpha}}}{e-1}\right)^n$$

$$\alpha_2 = 1-W = \left(\frac{-1}{e-1}+\frac{e^{\sqrt[n]{1-\alpha}}}{e-1}\right)^n$$

Проверка на состоятельность:

$$e^{\sqrt[n]{1-\alpha}} = e^{e^{\frac{1}{n}\ln(1-\alpha)}} = e^{\left(1+\frac{\ln(1-\alpha)}{n}+o\left(\frac{1}{n}\right)\right)} = e\left[1+\frac{\ln(1-\alpha)}{n}+o\left(\frac{1}{n}\right)\right]$$

$$W = 1-\left(\frac{1}{e-1}\left(-1+e^{\sqrt[n]{1-\alpha}}\right)\right)^n = 1-\left(\frac{1}{e-1}\left(-1+e\left[1+\frac{\ln(1-\alpha)}{n}+o\left(\frac{1}{n}\right)\right]\right)\right)^n =$$

$$= 1-\left(1+\frac{\ln(1-\alpha)}{n(e-1)}+o\left(\frac{1}{n}\right)\right)^n \xrightarrow[n\to\infty]{} 1-e^{\frac{\ln(1-\alpha)}{e-1}} = 1-(1-\alpha)^{\frac{1}{e-1}}$$

Возьмем $\alpha = 0{,}05$ (стандартное)

$W = 0{,}029 \underset{n\to\infty}{\not\to} 1$ не явл. сост.

T11 $n=2$ $\alpha=0,2$ $W-?$

$\alpha_1 = P(\vec{x}_n \in G_{кр} \mid H_0) = 0,2$

$G_{кр}: \ell \geq C$

H_0:

$x_2 \backslash x_1$	1	2	3	4
1	1/16	1/16	1/24	1/12
2	1/16	1/16	1/24	1/12
3	1/24	1/24	1/36	1/18
4	1/12	1/12	1/18	1/9

$p(1)=\frac{1}{4}$ $p(2)=\frac{1}{4}$ $p(3)=\frac{1}{6}$ $p(4)=\frac{1}{3}$

H_1:

$x_2 \backslash x_1$	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

ℓ:

$x_2 \backslash x_1$	1	2	3	4
1	1	1	3/2	3/4
2	1	1	3/2	3/4
3	3/2	3/2	9/4	9/8
4	3/4	3/4	9/8	9/16

$P(\ell(\vec{x}_n) \geq C \mid H_0) = \alpha \longrightarrow$ выбираем C

$\alpha = 0,2 \Rightarrow C = \frac{3}{2}$

$\Rightarrow G_{кр}: \ell \geq \frac{3}{2}$

$\alpha_1 = P(\ell \geq \frac{3}{2} \mid H_0) = 7/36 \approx 0,2$

$W = P(\ell \geq \frac{3}{2} \mid H_1) = 5/16 \approx 0,3125$

$\alpha_2 = 1 - W = 0,6875$

Ответ: $W = 0,3125$

№ 12. $\xi \sim N(\theta_1, \theta_2^2)$ - коэфф. трения $\theta_1 \in \mathbb{R}$ $\theta_2 > 0$

$D\xi = 0{,}1$ - в ходе эксперимента

$n = 25 \longrightarrow D\xi = 0{,}2$ - оценка

$H_0: \theta_2^2 = 0{,}1$

$H_1: \theta_2^2 > 0{,}1$

Т. Фишера

$$\frac{\bar{x} - \theta_1}{\theta_2}\sqrt{n} \sim N(0;1) \qquad \frac{\bar{x} - \theta_1}{s}\sqrt{n} \sim t(n-1) \qquad \underbrace{\frac{s^2(n-1)}{\theta_2^2} \sim \chi^2(n-1)}_{\text{берем это, так. как тут } \theta_2^2}$$

$$H_0: \Delta = \frac{s^2(n-1)}{\theta_2^2}$$

$$\tilde{\Delta} = \frac{0{,}2 \cdot 24}{0{,}1} = 48$$

$$p\text{-value} = 2\,P(\Delta \geq \tilde{\Delta}) = 2\int_{48}^{\infty} q_{\chi^2(24)}(x)\,dx = 0{,}005$$

Т.к. χ^2 не симметричное, то сравниваем с $\frac{\alpha}{2}$ и $1 - \frac{\alpha}{2}$

$$0{,}025 < 0{,}005 < 0{,}975$$

$\Rightarrow$ отвергаем гипотезу H_0

$$P(\vec{x}_n \in G_{кр} \mid H_0) = P(\Delta \geq C \mid H_0) = \alpha$$

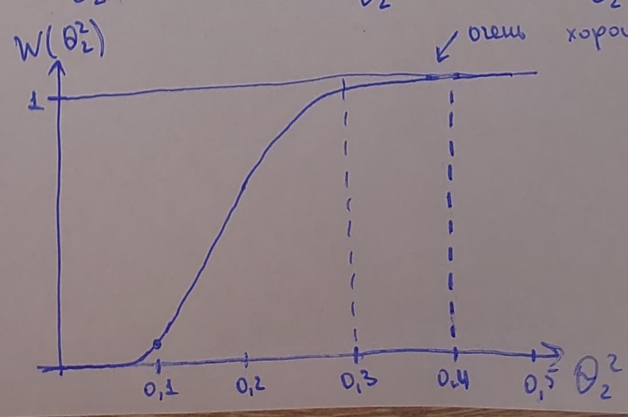
$$W = P(\vec{x}_n \in G_{кр} \mid H_1) - ?$$

$$\int_C^{+\infty} q_{\chi^2(24)}(x)\,dx = \alpha = 0{,}05 \Rightarrow C = 36{,}41 \Rightarrow G_{кр}: \Delta \geq 36{,}41$$

$$W = P(\vec{x}_n \in G_{кр} \mid H_1) = P(\Delta \geq C \mid H_1) = P\left(\frac{s^2(n-1)}{a^2} \geq 36{,}41 \mid H_1\right) =$$

$$= P\left(\frac{s^2(n-1)}{a^2} \cdot \frac{a^2}{\theta_2^2} \geq 36{,}41 \cdot \frac{a^2}{\theta_2^2} \mid H_1\right) = P\left(\frac{s^2(n-1)}{\theta_2^2} \geq 36{,}41 \cdot \frac{a^2}{\theta_2^2} \mid H_1\right) =$$

$$= \int_{36{,}41\frac{a^2}{\theta_2^2}}^{\infty} q_{\chi^2(24)}(x)\,dx = W(\theta_2^2)$$



Т 14. $\xi \sim N(a; \sigma_x^2 = 2)$ $\vec{x}_n$ – выборка

$\eta \sim N(b, \sigma_y = 1)$ $\vec{y}_m$ – выборка

$x = \{-1.11; -6.10; 2.42\}$ $\bar{x} = -1{,}6$ $n = 3$

$y = \{-2.29, -2.91\}$ $\bar{y} = -2{,}6$ $m = 2$

$H_0: a = b$

$H_1: a \neq b; \ a > b; \ a < b$ альтернативн.

$$\bar{x} - \bar{y} - (a - b) \sim N\left(0; \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}\right)$$

$$H_0: \underbrace{\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}}}_{\Delta} \sim N(0; 1)$$

$$\tilde{\Delta} = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{2}{n} + \frac{1}{m}}} = \frac{-1{,}6 + 2{,}6}{\sqrt{\frac{2m + n}{nm}}} = 0{,}9258$$

1) $H_1: \ a > b$

$p\text{-value} = P(\Delta \geq |\tilde{\Delta}| \mid H_0) = \int_{0,9258}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0{,}17$ => нет осн. отвергать H_0 (если $\alpha = 0{,}05$)

$a < b$

$p\text{-value} = P(\Delta \leq -|\tilde{\Delta}| \mid H_0) = \int_{-\infty}^{-0,9258} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} dx = 0{,}17$ => нет осн. отверг. H_0

$a \neq b$

$p\text{-value} = P(|\Delta| \geq |\tilde{\Delta}| \mid H_0) = \int_{-\infty}^{-0,9256} + \int_{0,9256}^{+\infty} = 0{,}35$ => нет осн. отвергать H_0