Using Linear Algebra to Study Game Theory

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Cover Letter

Hello! Us Math 22 students, namely, Sibi Raja, Damon Phan, and Arleen Saini are beyond delighted to present this final project to you. This project was a result of the combined efforts of ourselves, our peers, and our teachers. In particular, we would like to extend our gratitude toward Howard Huang, Maya Robinson, Amy Kaniper, Benjamin Langman, Grayson Kemplin, Xuanthe Nguyen, and Mattie Hung. Their feedback, in addition to the guidance we received from our TF, Kai Xu, was absolutely invaluable in the development of our final product. The primary suggestions our reviewers had were to reduce the number of definitions we had in our introduction, provide examples to solidify the concept of a stochastic vector, supplement the paper with a succinct conclusion, and to improve the general formatting of the piece.

We worked to resolve any format-relevant issues by way of reorganizing existing content and numbering each definition and theorem, which supported each proof and improved the flow of the paper. Additionally, we defined terms throughout the paper as needed instead of defining everything in the introduction. The addition of a conclusion provided the necessary wrap-up to concretely tie everything together.

Minor issues with regard to clarity were rectified by adding a subsection on the Nash Equilibrium which further explained the intricacies of what it means to be in Nash Equilibrium whilst engaging in a non-cooperative game. For each separate proof, we explained the purpose, ensuring that every minute detail was made as clear as possible. One example of a significant alteration that we made was with the skew-symmetric theorem.

While allocating workload, we ensured that each group member got the opportunity to engage entirely with both linear algebra and economics-relevant content. We collectively worked toward establishing definitions and background information in the introduction for the topic of game theory. Sibi worked on the second section to explain the concepts of expected value and Nash equilibrium. Damon and Arleen both worked on the third section as each of them took charge of the skew-symmetric matrices and diagonal matrices subsections, respectively. This plan of action allowed us to each write a comprehensive mathematical proof.

Finally, the most important part of the paper was the Examples of Game Theory Problems, featuring Horse Betting, The Prisoner's Dilemma, Beach Day Coordination, and the One Finger Two Fingers Game. We anticipated that putting our pre-established theorems into practice would be an excellent way in which to demonstrate the powers of game theory and linear algebra in the application thereof. We each established 1-2 relevant examples in section four to go alongside our particular subsections earlier in the paper. Furthermore, we thought that coming up with unique, interesting questions could unify all of the material and work to clarify the relationship between the sections. Finally, we worked together to write a cohesive conclusion and once more to write this cover letter. A final person we'd like to thank is Dusty: thank you for your continued dedication to teaching us the material in a fun and approachable way! Thank you for reading our paper. We've had a wonderful time in Math 22A and we look forward to hearing your thoughts on our paper.

1 Introduction

Tic Tac Toe, Matching Pennies, and Rock Paper Scissors. What do these three have in common? They are all games that most have encountered at some point in their childhood. For centuries, people have battled with the conflicting incentives of cooperation and self-interest, while attempting to "game the system" and make optimal decisions. However, the science of strategy is one that is nuanced and complex. Despite its importance in board games, cards, and even war, this sub-field of economics, otherwise known as game theory, remains obscure.

Game theory is a theoretical framework through which we can anticipate how interacting choices between economic agents, driven by their respective preferences and utilities, affects outcomes. This framework capitalizes on mathematical modeling in order to effectively capture the possible decisions that players can make in the pursuit of winning. Where does game theory come into play? Whenever rational people make decisions under strictly delineated constraints and rules, and where each player gets a payoff based on the decisions made, we have a game!

Here are a few foundational definitions that set the preface for how linear algebra can be used in game theory as well as how game theory can be represented.

Definition 1.1 (Payoff Matrix)**.** *A payoff matrix is a matrix that can be used to represent the different players, the decisions available to each of them, and the outcomes associated with those decisions. All entries are with respect to Player 1. For instance, consider the following payoff matrix, which represents a game between Player 1 and Player 2:*

		Player 2	
		A	B
Player 1	A	(5)	(-4)
	B	(-2)	(7)

Table 1: We can see here that Player 1 and Player 2 each have 2 decisions available to them: Decision A and Decision B. The entries in the matrix represent the outcomes for every possible decision combination that Players 1 and 2 take, with each entry being in relation to Player 1's gain or loss (since Player 1's gain is equivalent to Player 2's loss and vice versa). Thus, positive entries represent gains for Player 1 and negative entries represent gains for Player 2.

Definition 1.2 (Player)**.** *An economic agent who competes with other agents and has a specific set of choices they can feasibly make*

Definition 1.3 (Player's Strategy)**.** *Any of the options which the player chooses in a setting where the outcome depends not only on their own actions but on the actions of others*

Definition 1.4 (Row Player/Column Player)**.** *In matrix payoff games, the row player chooses a row in a matrix, whereas the column player simultaneously chooses a column. Conventionally, Player 1 is the row player, and Player 2 is the column player, which is also the convention we will use going forward.*

These definitions and concepts in game theory allow scholars to have a complex analysis of risk and reward in situations. With this paper, we attempt to explore and answer the following questions: what is game theory from an economic and mathematical standpoint? Are there certain characteristics in game theory problems that can be connected to special properties in linear algebra? Can such properties be used to optimize problem-solving strategies?

2 General Mathematics behind Game Theory

In this section, we present some fundamental definitions that will be soon used in theorems that are crucial to understanding game theory from a mathematical perspective. To begin, the concept of expected value will help us understand a way of analyzing payoff matrices.

2.1 Expected Value

Definition 2.1 (Expected Value/Utility/Payoff). *As its name suggests, this is the expected payoff you expect to get per game in the long run. The payoff is the value of the game minus the cost of the game. The expected payoff can be calculated in relation to a specific player or the overall game itself.*

Expected value is often a topic of interest in games known as *Two-Person Zero-Sum Games*, which is a basic model in game theory where one player aims to maximize their payoff while the other player aims to minimize this payoff. In a Two-Person Zero Sum Game, the gain of a player is exactly equivalent to the loss of the other player. An important theorem in Game Theory that relates to Two-Person Zero-Sum Games is the Minimax Theorem, (also referred to as the Fundamental Theorem of Two-Person Zero-Sum Games).

Theorem 2.1 (Minimax Theorem). *In every finite two-person zero-sum game, there exist optimal strategies for each player that produce an expected utility. Both the expected utilities are the same, which can be interpreted as the expected value of the entire game when added together. (This theorem was proved true by John von Neumann in 1928. For the purpose of this paper, the proof of this theorem was not included, but is available in various sources online).*

The Minimax theorem allows us to understand where the expected value of a game comes from. Now, let us look into how we can go about calculating the expected value of a game using stochastic vectors.

Definition 2.2 (Stochastic Vector). *A vector $\vec{x}$ is a stochastic vector if it contains non-negative entries that represent probabilities of different events occurring, which all sum up to one:* $\vec{x} = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix}$, *(where* $\sum_{n=1}^{m} x_n = 1$*)*

For example, consider the following two vectors $\vec{y} = [0.6 \;\; 0.5 \;\; 0.2]$ and $\vec{z} = [0.3 \;\; 0.7]$. Summing over the entries of $\vec{y}$, we get $\sum_{n=1}^{3} y_n = 1.3$, while doing the same for $\vec{z}$, we get $\sum_{n=1}^{2} z_n = 1$. Because $1.3 \neq 1$, $\vec{y}$ is not a stochastic vector. However, $\vec{z}$ is in fact a stochastic vector since its entries do in fact sum up to exactly 1.

Theorem 2.2 (Calculation of Expected Value of a Game). *The expected value of a game can be calculated by multiplying each entry in a payoff matrix by the respective probability of that payoff occurring.*

Proof. Consider the following m x n payoff matrix A (where $m, n > 0$ and m and n can be equal to each other) that represents the various payoffs within a given game, stochastic row vector x that represents the probabilities of Player 1 (the row player) choosing a given row as their strategy, and stochastic column vector y that represents the probabilities of Player 2 (the column player) choosing a given column as their strategy.

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

The row vector represents the probability of Player 1 choosing a strategy from row 1 to row m, while the column vector represents the probability of Player 2 choosing a strategy from column 1 to column n.

To find the expected value of this game v, we can use Theorem 2.2 and compute as follows:

$v = (a_{11} \cdot x_1 \cdot y_1) + ... + (a_{m1} \cdot \; x_m \cdot y_1) + ... + (a_{m1} \cdot x_m \cdot y_1) + ... + (a_{mn} \cdot x_m \cdot y_n)$

Thus, this technique will allow us to compute the expected value of any game after representing it with a payoff matrix and using its corresponding stochastic vectors.

$\square$

2.2 Nash Equilibrium

Another characteristic of payoff matrices that we can analyze is the Nash equilibrium.

Definition 2.3 (Nash Equilibrium)**.** *A Nash equilibrium is the set of strategies for both players that allows each player to achieve the best outcome without any need to change their strategies throughout the game. Each strategy within the set does not get changed, no matter what a player's opponent does.*

While proving the existence of a Nash equilibrium in a payoff matrix is an interesting feat, the task involves mathematical content that is outside the scope of this paper. However, we will go over how exactly to find a Nash equilibrium in a payoff matrix. After this example, the notion of a Nash equilibrium should be more clear.

Finding a Nash Equilibrium

Suppose we have the following payoff matrix:

		Player 2	
		A	B
Player 1	A	$(1, 1)$	$(5, -2)$
	B	$(-2, 5)$	$(4, 4)$

Table 2: Note that this matrix does not represent the typical payoff matrix that we have used thus far because each entry has two values. In this representation, the first value in each entry corresponds to the row player's payoffs, which in this case is Player 1. Conversely, the second value in each entry corresponds to the column player's payoffs, which in this case is Player 2.

To find the Nash equilibrium of this payoff matrix, we compare the different payoffs for each player's choices individually while holding assumptions about the other player. Doing this allows us to find which strategies are optimal and will result in the greatest gain for each player.

Recall that Player 1 can choose between options A and B. Assuming that Player 2 will choose option A, Player 1 can choose option A or B themselves. The payoff with option A is 1, while the payoff with option B is -2. Thus, Player 1's optimal strategy in this case would be to choose option A as it would generate the greater gain ($1 > -2$). Note that the first optimal strategy of Player 1 occurs in the top left cell. Now, assume that Player 2 will choose option B. Player 1 can again choose option A or option B for themselves. If Player 1 chooses option A, the payoff is 5, while choosing option B would only result in a payoff of 4. Thus, Player 1's optimal strategy in this case is to choose option A if Player 2 chooses option B, which signifies the second optimal strategy of Player 1 now occurs in the top right cell of the payoff matrix.

Next, let's compare the different payoffs for each of Player 2's choices to find their optimal strategies. Assuming that Player 1 will choose option A, Player 2 can choose between option A or option B, which have payoffs of 1 and -2, respectively. Because a payoff of 1 is greater than a payoff of -2, Player 2's first optimal strategy would be to choose option A if Player 1 chooses option A, which signifies the top left cell in the matrix. Now, let's assume that Player 1 will choose option B. Now Player 2 can again choose between options A and B, which generate payoffs of 5 and 4, respectively. Because a payoff of 5 is more desired than a Payoff of 4, Player 2's optimal strategy is to choose option A if Player 1 chooses option B. Thus, Player 2's second optimal strategy occurs in the bottom left cell of the payoff matrix.

At this point, we have finished finding the optimal strategies for each of the two players. But where is the Nash equilibrium? Well, it is important to note that both players have an optimal strategy that overlaps in the top left cell of the matrix. Because of this, that entry in the matrix is the Nash equilibrium of the game at hand! Intuitively, it is best to think of the Nash equilibrium as a point in a payoff matrix in which both players' optimal strategies overlap. As you can tell by now, the Nash equilibrium is extremely useful for analyzing games because it can show where the greatest payoffs occur by factoring into account the decisions of *both* players.

With these definitions and theorems in mind, we can understand the underlying connection between game theory and linear algebra: matrices can be used to represent payoffs for players in various situations. Furthermore, the Nash equilibrium of each situation can be easily found after using a matrix to model a given game theory problem, and the linear combinations present within matrices can be used to calculate the expected payoff values for players.

3 Characteristics of Matrices in Game Theory

In this section, we explore effective strategies to solve game theory problems, which include recognizing if a matrix model is diagonal and skew-symmetric. Doing these can tell us a great deal of information about the situation at hand in an efficient manner.

3.1 Skew-Symmetric Matrices

Recall the following definition of a skew-symmetric matrix from linear algebra:

Definition 3.1 (Skew-Symmetric Matrix)**.** *A skew-symmetric matrix is a matrix whose transposed matrix is equivalent to the negative of that matrix.* ($A^T = -A$)

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix},\ A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \\ a_{1m} & & a_{mm} \end{bmatrix},\ -A^T = \begin{bmatrix} -a_{11} & \dots & -a_{m1} \\ \vdots & \ddots & \\ -a_{1m} & & -a_{mm} \end{bmatrix}$$

As we can see from the matrix representations, matrix A is skew-symmetric if its rows and columns are negatives of each other. Another interesting takeaway one may have is that the entries of the main diagonal of A must be all zeros, as that is the only way of getting relations such as $a_{11} = -a_{11}$ and $a_{mm} = -a_{mm}$. But, the proofs for these ideas can be found elsewhere, as they are not within the scope of this paper. However, an interesting property of skew-symmetric matrices arises in game theory when they are used to represent payoff matrices.

When a zero-sum game is symmetric, players have no advantage over each other, so if the row player and the column player had the same strategy, both players can expect a payoff of at least 0. A symmetric game also means that the payoff matrix is skew-symmetric, which we can prove.

Let's walk through a scenario and prove just that.

Imagine two players are playing a game represented by a 2x2 matrix where there is a row player and a column player. The row player tries to maximize winnings whereas the column player tries to minimize losses.

Theorem 3.1. *The expected value of a skew-symmetric matrix A is equivalent to the negative expected value of negative transpose matrix A ($v(A) = -v(-A^T)$).*

Proof. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Using expected value formulas:

$$D = (a+d) - (b+c), v = \frac{(ad - bc)}{D}$$

So,

$$v(A) = \frac{(ad - bc)}{(a+d) - (b+c)}$$

$$v(A) = -v(-A^T)$$

$$v(A) = -v(\begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix})$$

$$v(A) = \frac{-[(-a)(-d)-(-c)(-b)]}{[(-a)+(-d)]-[(-c)+(-b)]}$$

$$v(A) = \frac{[(-ad)+(cb)]}{-a-d+c+b} \times \frac{-1}{-1}$$

$$v(A) = \frac{(ad-bc)}{(a+d)-(b+c)}$$

$$v(A) = -v(-A^T)$$

□

3.2 Diagonal Matrices

Recall the definition of a diagonal matrix from linear algebra:

Definition 3.2 (Diagonal Matrices). *If A is an $n \times n$ matrix, A is a diagonal matrix if all of the entries outside the main diagonal are zero.*

Theorem 3.2. *For diagonal matrices, the value of the game is proportional to the reciprocals of the diagonal elements.*

Proof. Let $p = (p_1, ..., p_m)$ be any optimal strategy for Player I and $q = (q_1, .., q_n)$ be any optimal strategy for Player II. Consider diagonal matrix A.

$$A = \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & & d_m \end{bmatrix}$$

Suppose all diagonal terms are positive such that $d_j > 0$ for all j. Then, $V = p_j d_j$ for $j = 1, .., m$.

Now we can solve for p_j.

$p_j = \frac{V}{d_j}$ for $j = 1, ..., m$

By symmetry,

$q_j = \frac{V}{d_j}$ for $j = 1, ..., m$

To find the value of the game, V, we will sum both sides over j.

$V \sum_{j=1}^{m} \frac{1}{d_j} = 1$ which can be rewritten as $V = (\sum_{j=1}^{m} \frac{1}{d_j})^{-1}$

As shown, V is proportional to the sum of the reciprocals of the diagonal. Hence, the proof is complete.

□

4 Examples of Game Theory Problems

With keeping the mathematical techniques we just discussed, we will begin to solve concrete game theory problems in this section, starting with a simple Two-Person Zero Sum Game.

4.1 Horse Betting

Consider the following scenario: Steve and Mary are betting on a horse race. There are two horses participating in the race: Horse A and Horse B. Suppose that both Steve and Mary choose a horse to bet on at random, with Steve having a 20% probability of choosing Horse A and an 80% probability of choosing Horse B. On the other hand, Mary has a 60% probability of choosing Horse A and a 40% probability of choosing Horse B.

Given the following payoff matrix, how can we find the expected payoff of this situation?

		Mary	
		Horse A	*Horse B*
Steve	*Horse A*	(5)	(-4)
	Horse B	(-2)	(7)

Solution:

We can begin by constructing the stochastic vectors $\vec{s}$ and $\vec{m}$ that correspond to the choices available to Steve and Mary, respectively. Populating each stochastic vector starting with the probability of choosing Horse A and ending with the probability of choosing Horse B, we have the following:

$$\vec{s} = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix}, \vec{m} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

Note that Steve is the row player, and Mary is the column player. Because of this, the stochastic vector $\vec{s}$ is a row vector and the stochastic vector $\vec{m}$ is a column vector. Also note that the entries in each stochastic vector all sum up to 1, which satisfies our definition of a stochastic vector.

Applying Theorem 2.2, we can calculate the expected payoff of this game, v as follows:

$v = (5 \cdot 0.2 \cdot 0.6) + (-4 \cdot \ 0.2 \cdot 0.4) + (-2 \cdot 0.8 \cdot 0.6) + (7 \cdot 0.8 \cdot 0.4) = 5.32$

Thus, we have found the expected payoff of the betting game to be 5.32. Note that this is a positive value, so it represents a gain of 5.32 for Steve (which is equivalent to a loss of 5.32 for Mary).

4.2 Nash Equilibrium in "The Prisoner's Dilemma"

Here, we tackle the iconic Prisoner's Dilemma problem in Game Theory to understand why exactly there is a dilemma.

Consider the following scenario: Josh and Sarah are two criminals that murdered the mayor of their city. The police have taken Josh and Sarah into custody, but they don't know for certain if Josh and Sarah are the true murderers. Thus, the police take Josh and Sarah into two separate rooms and give them each two options: either to confess or remain silent. The police tell them that if they both remain silent, then they each get sentenced to 1 year in prison. However, if one of them confesses while the other one remains silent, then the one who confessed gets let go while the one who remained silent has to face 4 years in prison. Finally, if they both confess, then they both will face 3 years in prison. Thus, we can represent the situation in a matrix as follows:

		Sarah	
		Silent	*Confess*
Josh	*Silent*	$(-1, -1)$	$(-4, 0)$
	Confess	$(0, -4)$	$(-3, -3)$

Table 3: Again, note that this matrix has two values in each entry. Josh's payoffs correspond to the first value in each entry as he is the row player while Sarah's payoffs correspond to the second value in each entry as she is the column player.

So, what is the Nash equilibrium of this matrix?

Solution:

We can begin by comparing the different payoffs for each of Josh's choices. Josh can either remain silent or confess. Assuming Sarah will remain silent, Josh can either remain silent and spend 1 year in jail or confess and not spend any time in jail. Thus, Josh's optimal choice in this case is to confess. Now assuming Sarah will confess, Josh can choose to either confess or remain silent. If Josh remains silent, he spends 4 years in jail while confessing would allow him to only spend 3 years in jail. Again, Josh's optimal choice would be to confess. Thus, Josh's optimal strategy is to confess regardless of what Sarah does.

Now let's compare the different payoffs for each of Sarah's choices. Sarah can either remain silent or confess. Assuming Josh will remain silent, Sarah can either remain silent and spend 1 year in jail or confess and not spend any time in jail. Thus, Sarah's optimal choice in this case is to confess. Now assuming Josh will confess, Sarah can choose to either confess or remain silent. If Sarah remains silent, she spends 4 years in jail while confessing would allow her to only spend 3 years in jail. Again, Sarah's optimal choice would be to confess. Thus, Sarah's optimal strategy is also to confess regardless of what Josh does.

We know that both players have the same optimal strategy: to confess. Thus, we have found the Nash equilibrium of this matrix. However, note that if Josh and Sarah both confess, then they will each spend 3 years in prison. Instead of doing this, they could have both just chosen to remain silent and only spend 1 year in prison each, which is the much better choice. Thus, this is exactly the dilemma that is present in the problem – the Nash equilibrium point (which recall is the point that represents the optimal strategy for both players) makes both players worse off than another strategy.

4.3 Beach Day Coordination

Consider two roommates, Kait and Dustin. They both enjoy each other's company but have lost their phones and neither can communicate with the other before deciding whether to stay at home (where they will still see each other) or go to the beach (where they will once again see each other). Both prefer going to the beach but even more so love being with each other. This game can be represented by the following matrix form, representing their combined utilities for each combination:

		Dustin	
		Beach	Home
Kait	Beach	(4)	(0)
	Home	(0)	(3)

So, what is the best way to find the expected value of this matrix?

Solution:

Let us rewrite this in matrix form.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

The first step is to recognize that this is a diagonal matrix, which is convenient as we know exactly how to work with these types of matrices. We know from Section 3.2 that the optimal strategy for Dustin and Kait is equal, with $p_j = \frac{V}{d_j}$ for $j = 1, ..., m$ and $q_j = \frac{V}{d_j}$ for $j = 1, ..., m$.

First, let us compute $V = (\sum_{j=1}^{m} \frac{1}{d_j})^{-1}$.

The sum of the reciprocals of the diagonal elements of matrix A is: $\frac{1}{4} + \frac{1}{3} = \frac{7}{12}$. Therefore, the value, V of the game is $\frac{7}{12}$.

Likewise, we have that the optimal strategies are $p_j = q_j = (\frac{\frac{7}{12}}{4}, \frac{\frac{7}{12}}{3}) = (\frac{7}{48}, \frac{7}{36})$. In other words, Kait's average payoff is at least V no matter what column j Dustin chooses.

4.4 One Finger Two Fingers

Imagine two players are playing a game where the row player has to hold up one or two fingers, and the column player has to hold up one or two fingers. The various combinations of their choices is represented by the matrix below:

		Column Player	
		1 Finger	2 Fingers
Row Player	1 Finger	0	2
	2 Fingers	-2	0

If both players hold up 1 finger, then both players win nothing, while if both players hold up 2 fingers, both players also win nothing. Moreover, if the Row Player holds up 2 fingers while the Column Player holds up 1 finger, then the Row Player loses 2 dollars; the Row Player will win 2 dollars if the Column Player holds up 2 fingers while the Row Player holds up only 1 finger.

What are the important takeaways we can get from this situation?

Solution:

You may notice that this is a skew-symmetric matrix. Thus, we can apply Theorem 3.1 here

Recall that Theorem 3.1 states that the expected value of matrix A is equivalent to the negative expected value of negative transpose matrix A ($v(A) = -v(-A^T)$). This is equivalent to $-A^T = A$.

Let

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, -A^T = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

Using the Same Expected Value Formulas Again:

$$D = (a + d) - (b + c), v = \frac{(ad - bc)}{D}$$

We want to test that $v(A) = -v(-A^T)$

$$v(A) = \frac{(0)(0) - (2)(-2)}{(0 + 0) - (2 - 2)}$$

$$v(A) = \frac{4}{0}$$

Now, solving for $-v(-A^T)$

$$-v(-A^T) = \frac{-[(0)(0) - (2)(-2)]}{(0 + 0) - (2 - 2)} \times \frac{-1}{-1}$$

$$-v(-A^T) = \frac{4}{0}$$

$$v(A) = -v(-A^T)$$

As you might notice, there is a 0 in the denominator of each calculation. Mathematically speaking, this means that the expected value is undefined, but in a game-theory context, this can be interpreted as the expected value is equal to zero. Thus, the game is completely fair and symmetric, so neither player can expect to gain or lose from partaking in the game. What an interesting situation! While situations like these can be unusual to encounter, it is important to note that they do in fact occur in the real world.

5 Conclusion

Following the past sections, we hope to have provided significant insight as to how Linear Algebra can be applied to better understand Game Theory and its economic implications. As we have established, game theory is a framework through which we can anticipate how interacting choices between economic agents affect their respective outcomes. Like the examples we developed in the paper, game theory can equally so be applied to games like Tic Tac Toe, Matching Pennies, and Rock Paper Scissors. Beyond mere childhood games, however, the possibilities of game theory are practically endless. One such example is chess. In the game of chess, individual players hope to find Nash equilibria in order to maximize their payoff in the game. In the short run, this may be sacrificing one piece to take one from the opponent. Longer term, it is picking a strategy to get checkmate. Another example is game theory for defense (military) applications. It has a long history of being used to study possible strategies of cooperation and aggression. Why is game theory significant? The framework takes advantage of mathematical models to effectively capture the possible decisions that players can make while playing a game. Clearly, it has stood the test of time by way of its tangible real-world implications.

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